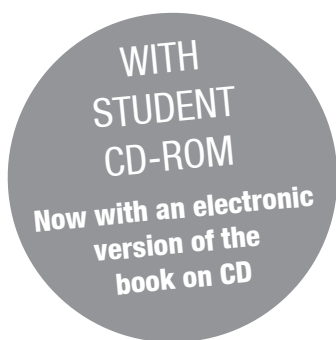




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12

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Mathematics

3unit

BILL PENDER

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Preface

This textbook has been written for students in Years 11 and 12 taking the course previously known as ‘3 Unit Mathematics’, but renamed in the new HSC as two courses, ‘Mathematics’ (previously called ‘2 Unit Mathematics’) and ‘Mathematics, Extension 1’. The book develops the content at the level required for the 2 and 3 Unit HSC examinations. There are two volumes — the present volume is roughly intended for Year 12, and the previous volume for Year 11. Schools will, however, differ in their choices of order of topics and in their rates of progress.

Although these Syllabuses have not been rewritten for the new HSC, there has been a gradual shift of emphasis in recent examination papers.

- The interdependence of the course content has been emphasised.
- Graphs have been used much more freely in argument.
- Structured problem solving has been expanded.
- There has been more stress on explanation and proof.

This text addresses these new emphases, and the exercises contain a wide variety of different types of questions.

There is an abundance of questions in each exercise — too many for any one student — carefully grouped in three graded sets, so that with proper selection the book can be used at all levels of ability. In particular, both those who subsequently drop to 2 Units of Mathematics, and those who in Year 12 take 4 Units of Mathematics, will find an appropriate level of challenge. We have written a separate book, also in two volumes, for the 2 Unit ‘Mathematics’ course alone.

We would like to thank our colleagues at Sydney Grammar School and Newington College for their invaluable help in advising us and commenting on the successive drafts, and for their patience in the face of some difficulties in earlier drafts. We would also like to thank the Headmasters of Sydney Grammar School and Newington College for their encouragement of this project, and Peter Cribb and the team at Cambridge University Press, Melbourne, for their support and help in discussions. Finally, our thanks go to our families for encouraging us, despite the distractions it has caused to family life.

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How to Use This Book

This book has been written so that it is suitable for the full range of 3 Unit students, whatever their abilities and ambitions. The book covers the 2 Unit and 3 Unit content without distinction, because 3 Unit students need to study the 2 Unit content in more depth than is possible in a 2 Unit text. Nevertheless, students who subsequently move to the 2 Unit course should find plenty of work here at a level appropriate for them.

The Exercises: No-one should try to do all the questions! We have written long exercises so that everyone will find enough questions of a suitable standard — each student will need to select from them, and there should be plenty left for revision. The book provides a great variety of questions, and representatives of all types should be selected.

Each chapter is divided into a number of sections. Each of these sections has its own substantial exercise, subdivided into three groups of questions:

FOUNDATION: These questions are intended to drill the new content of the section at a reasonably straightforward level. There is little point in proceeding without mastery of this group.

DEVELOPMENT: This group is usually the longest. It contains more substantial questions, questions requiring proof or explanation, problems where the new content can be applied, and problems involving content from other sections and chapters to put the new ideas in a wider context. Later questions here can be very demanding, and Groups 1 and 2 should be sufficient to meet the demands of all but exceptionally difficult problems in 3 Unit HSC papers.

EXTENSION: These questions are quite hard, and are intended principally for those taking the 4 Unit course. Some are algebraically challenging, some establish a general result beyond the theory of the course, some make difficult connections between topics or give an alternative approach, some deal with logical problems unsuitable for the text of a 3 Unit book. Students taking the 4 Unit course should attempt some of these.

The Theory and the Worked Exercises: The theory has been developed with as much rigour as is appropriate at school, even for those taking the 4 Unit course. This leaves students and their teachers free to choose how thoroughly the theory is presented in a particular class. It can often be helpful to learn a method first and then return to the details of the proof and explanation when the point of it all has become clear.

The main formulae, methods, definitions and results have been boxed and numbered consecutively through each chapter. They provide a summary only, and

represent an absolute minimum of what should be known. The worked examples have been chosen to illustrate the new methods introduced in the section, and should be sufficient preparation for the questions of the following exercise.

The Order of the Topics: We have presented the topics in the order we have found most satisfactory in our own teaching. There are, however, many effective orderings of the topics, and the book allows all the flexibility needed in the many different situations that apply in different schools (apart from the few questions that provide links between topics).

The time needed for the work on polynomials in Chapter Four, on Euclidean geometry in Chapters Eight and Nine, and on the first few sections of probability in Chapter Ten, will depend on students' experiences in Years 9 and 10. The Study Notes at the start of each chapter make further specific remarks about each topic.

We have left Euclidean geometry, polynomials and elementary probability until Year 12 for two reasons. First, we believe as much calculus as possible should be developed in Year 11, ideally including the logarithmic and exponential functions and the trigonometric functions. These are the fundamental ideas in the course, and it is best if Year 12 is used then to consolidate and extend them (and students subsequently taking the 4 Unit course particularly need this material early). Secondly, the Years 9 and 10 Advanced Course already develops elementary probability in the Core, and much of the work on polynomials and Euclidean geometry in Options recommended for those proceeding to 3 Unit, so that revisiting them in Year 12 with the extensions and greater sophistication required seems an ideal arrangement.

The Structure of the Course: Recent examination papers have included longer questions combining ideas from different topics, thus making clear the strong inter-connections amongst the various topics. Calculus is the backbone of the course, and the two processes of differentiation and integration, inverses of each other, dominate most of the topics. We have introduced both processes using geometrical ideas, basing differentiation on tangents and integration on areas, but the subsequent discussions, applications and exercises give many other ways of understanding them. For example, questions about rates are prominent from an early stage.

Besides linear functions, three groups of functions dominate the course:

THE QUADRATIC FUNCTIONS: These functions are known from earlier years. They are algebraic representations of the parabola, and arise naturally in situations where areas are being considered or where a constant acceleration is being applied. They can be studied without calculus, but calculus provides an alternative and sometimes quicker approach.

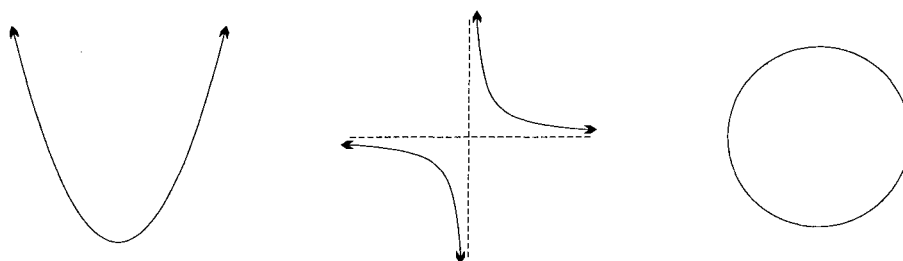
THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS: Calculus is essential for the study of these functions. We have chosen to introduce the logarithmic function first, using definite integrals of the reciprocal function $y = 1/x$. This approach is more satisfying because it makes clear the relationship between these functions and the rectangular hyperbola $y = 1/x$, and because it gives a clear picture of the new number e . It is also more rigorous. Later, however, one can never overemphasise the fundamental property that the exponential

function with base e is its own derivative — this is the reason why these functions are essential for the study of natural growth and decay, and therefore occur in almost every application of mathematics.

Arithmetic and geometric sequences arise naturally throughout the course. They are the values, respectively, of linear and exponential functions at integers, and these interrelationships need to be developed, particularly in the context of applications to finance.

THE TRIGONOMETRIC FUNCTIONS: Again, calculus is essential for the study of these functions, whose definition, like the associated definition of π , is based on the circle. The graphs of the sine and cosine functions are waves, and they are essential for the study of all periodic phenomena — hence the detailed study of simple harmonic motion in Year 12.

Thus the three basic functions of the course — x^2 , e^x and $\sin x$ — and the related numbers e and π are developed from the three most basic degree 2 curves — the parabola, the rectangular hyperbola and the circle. In this way, everything in the course, whether in calculus, geometry, trigonometry, coordinate geometry or algebra, is easily related to everything else.



The geometry of the circle is mostly studied using Euclidean methods, and the highly structured arguments used here contrast with the algebraic arguments used in the coordinate geometry approach to the parabola. In the 4 Unit course, the geometry of the rectangular hyperbola is given special consideration in the context of a coordinate geometry treatment of general conics.

Polynomials constitute a generalisation of quadratics, and move the course a little beyond the degree 2 phenomena described above. The particular case of the binomial theorem then becomes the bridge from elementary probability using tree diagrams to the binomial distribution with all its practical applications. Unfortunately, the power series that link polynomials with the exponential and trigonometric functions are too sophisticated for a school course. Projective geometry and calculus with complex numbers are even further removed, so it is not really possible to explain that exponential and trigonometric functions are the same thing, although there are many clues.

Algebra, Graphs and Language: One of the chief purposes of the course, stressed in recent examinations, is to encourage arguments that relate a curve to its equation. Being able to predict the behaviour of a curve given only its equation is a constant concern of the exercises. Conversely, the behaviour of a graph can often be used to solve an algebraic problem. We have drawn as many sketches in the book as space allowed, but as a matter of routine, students should draw diagrams for almost every problem they attempt. It is because sketches can so easily be drawn that this type of mathematics is so satisfactory for study at school.

This course is intended to develop simultaneously algebraic agility, geometric intuition, and rigorous language and logic. Ideally then, any solution should display elegant and error-free algebra, diagrams to display the situation, and clarity of language and logic in argument.

Theory and Applications: Elegance of argument and perfection of structure are fundamental in mathematics. We have kept to these values as far as is reasonable in the development of the theory and in the exercises. The application of mathematics to the world around us is equally fundamental, and we have given many examples of the usefulness of everything in the course. Calculus is particularly suitable for presenting this double view of mathematics.

We would therefore urge the reader sometimes to pay attention to the details of argument in proofs and to the abstract structures and their interrelationships, and at other times to become involved in the interpretation provided by the applications.

Limits, Continuity and the Real Numbers: This is a first course in calculus, geometrically and intuitively developed. It is not a course in analysis, and any attempt to provide a rigorous treatment of limits, continuity or the real numbers would be quite inappropriate. We believe that the limits required in this course present little difficulty to intuitive understanding — really little more is needed than $\lim_{x \rightarrow \infty} 1/x = 0$ and the occasional use of the sandwich principle in proofs. Characterising the tangent as the limit of the secant is a dramatic new idea, clearly marking the beginning of calculus, and quite accessible. Continuity and differentiability need only occasional attention, given the well-behaved functions that occur in the course. The real numbers are defined geometrically as points on the number line, and provided that intuitive ideas about lines are accepted, everything needed about them can be justified from this definition. In particular, the intermediate value theorem, which states that a continuous function can only change sign at a zero, is taken to be obvious.

These unavoidable gaps concern only very subtle issues of ‘foundations’, and we are fortunate that everything else in the course can be developed rigorously so that students are given that characteristic mathematical experience of certainty and total understanding. This is the great contribution that mathematics brings to all our education.

Technology: There is much discussion, but little agreement yet, about what role technology should play in the mathematics classroom or what machines or software may be effective. This is a time for experimentation and diversity. We have therefore given only a few specific recommendations about technology, but we encourage such investigation, and the exercises give plenty of scope for this. The graphs of functions are at the centre of the course, and the more experience and intuitive understanding students have, the better able they are to interpret the mathematics correctly. A warning here is appropriate — any machine drawing of a curve should be accompanied by a clear understanding of why such a curve arises from the particular equation or situation.

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The Book of Nature is written in the language of Mathematics.

— The seventeenth century Italian scientist Galileo

It is more important to have beauty in one's equations than to have them fit experiment.

— The twentieth century English physicist Dirac

Even if there is only one possible unified theory, it is just a set of rules and equations. What is it that breathes fire into the equations and makes a universe for them to describe? The usual approach of science of constructing a mathematical model cannot answer the questions of why there should be a universe for the model to describe.

— Steven Hawking, *A Brief History of Time*

The Inverse Trigonometric Functions

A proper understanding of how to solve trigonometric equations requires a theory of inverse trigonometric functions. This theory is complicated by the fact that the trigonometric functions are periodic functions — they therefore fail the horizontal line test quite seriously, in that some horizontal lines cross their graphs infinitely many times. Understanding inverse trigonometric functions therefore requires further discussion of the procedures for restricting the domain of a function so that the inverse relation is also a function. Once the functions are established, the usual methods of differential and integral calculus can be applied to them.

This theory gives rise to primitives of two purely algebraic functions

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x \quad (\text{or } -\cos^{-1} x) \quad \text{and} \quad \int \frac{1}{1+x^2} dx = \tan^{-1} x,$$

which are similar to the earlier primitive $\int \frac{1}{x} dx = \log x$ in that in all three cases, a purely algebraic function has a primitive which is non-algebraic.

STUDY NOTES: Inverse relations and functions were first introduced in Section 2H of the Year 11 volume. That material is summarised in Section 1A in preparation for more detail about restricted functions, but some further revision may be necessary. Sections 1B–1E then develop the standard theory of inverse trigonometric functions and their graphs, and the associated derivatives and integrals. In Section 1F these functions are used to establish some formulae for the general solutions of trigonometric equations.

1A Restricting the Domain

Section 2H of the Year 11 volume discussed how the inverse relation of a function may or may not be a function, and briefly mentioned that if the inverse is not a function, then the domain can be restricted so that the inverse of this restricted function is a function. This section revisits those ideas and develops a more systematic approach to restricting the domain.

Inverse Relations and Inverse Functions: First, here is a summary of the basic theory of inverse functions and relations. The examples given later will illustrate the various points. Suppose that $f(x)$ is a function whose inverse relation is being considered.

INVERSE FUNCTIONS AND RELATIONS:

1

- The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.
- The inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.
- The domain and range of the inverse relation are the range and domain respectively of the original function.
- To find the equations and conditions of the inverse relation, write x for y and y for x every time each variable occurs.
- If the inverse relation is also a function, the inverse function is written as $f^{-1}(x)$. Then the composition of the function and its inverse, in either order, leaves every number unchanged:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

- If the inverse is not a function, then the domain of the original function can be restricted so that the inverse of the restricted function is a function.

The following worked exercise illustrates the fourth and fifth points above.

WORKED EXERCISE: Find the inverse function of $f(x) = \frac{x-2}{x+2}$. Then show directly that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

SOLUTION: Let $y = \frac{x-2}{x+2}$.

Then the inverse relation is $x = \frac{y-2}{y+2}$ (writing y for x and x for y)

$$\begin{aligned} xy + 2x &= y - 2 \\ y(x - 1) &= -2x - 2 \\ y &= \frac{2 + 2x}{1 - x}. \end{aligned}$$

Since there is only one solution for y , the inverse relation is a function,

and
$$f^{-1}(x) = \frac{2 + 2x}{1 - x}.$$

$$\begin{aligned} \text{Then } f(f^{-1}(x)) &= f\left(\frac{2 + 2x}{1 - x}\right) & \text{and } f^{-1}(f(x)) &= f^{-1}\left(\frac{x - 2}{x + 2}\right) \\ &= \frac{\frac{2 + 2x}{1 - x} - 2}{\frac{2 + 2x}{1 - x} + 2} \times \frac{1 - x}{1 - x} & &= \frac{2 + \frac{2(x - 2)}{x + 2}}{1 - \frac{x - 2}{x + 2}} \times \frac{x + 2}{x + 2} \\ &= \frac{(2 + 2x) - 2(1 - x)}{(2 + 2x) + 2(1 - x)} & &= \frac{2(x + 2) + 2(x - 2)}{(x + 2) - (x - 2)} \\ &= \frac{4x}{4} & &= \frac{4x}{4} \\ &= x, \text{ as required.} & &= x \text{ as required.} \end{aligned}$$

Increasing and Decreasing Functions: Increasing means getting bigger, and we say that a function $f(x)$ is an *increasing function* if $f(x)$ increases as x increases:

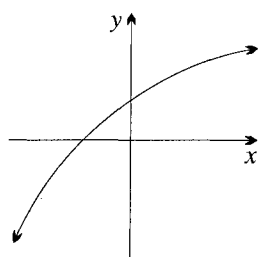
$$f(a) < f(b), \text{ whenever } a < b.$$

For example, if $f(x)$ is an increasing function, then provided $f(x)$ is defined there, $f(2) < f(3)$, and $f(5) < f(10)$. In the language of coordinate geometry, this means that every chord slopes upwards, because the ratio $\frac{f(b) - f(a)}{b - a}$ must be positive, for all pairs of distinct numbers a and b . Decreasing functions are defined similarly.

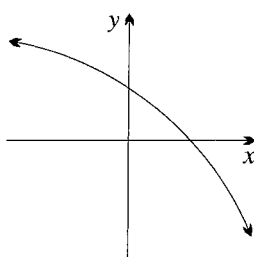
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INCREASING AND DECREASING FUNCTIONS: Suppose that $f(x)$ is a function.

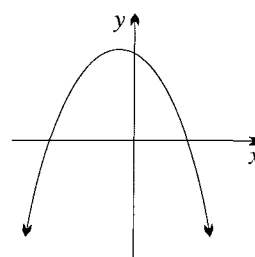
- $f(x)$ is called an *increasing function* if every chord slopes upwards, that is, $f(a) < f(b)$, whenever $a < b$.
- $f(x)$ is called a *decreasing function* if every chord slopes downwards, that is, $f(a) > f(b)$, whenever $a < b$.



An increasing function



A decreasing function



Neither of these

NOTE: These are *global* definitions, looking at the graph of the function as a whole. They should be contrasted with the *pointwise* definitions introduced in Chapter Ten of the Year 11 volume, where a function $f(x)$ was called *increasing at $x = a$* if $f'(a) > 0$, that is, if the tangent slopes upwards at the point.

Throughout our course, a tangent describes the behaviour of a function at a particular point, whereas a chord relates the values of the function at two different points.

The exact relationship between the global and pointwise definitions of *increasing* are surprisingly difficult to state, as the examples in the following paragraphs demonstrate, but in this course it will be sufficient to rely on the graph and common sense.

The Inverse Relation of an Increasing or Decreasing Function: When a horizontal line crosses a graph twice, it generates a horizontal chord. But every chord of an increasing function slopes upwards, and so an increasing function cannot possibly fail the horizontal line test. This means that the inverse relation of every increasing function is a function. The same argument applies to decreasing functions.

3

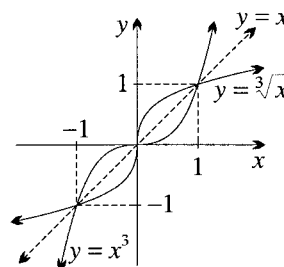
INCREASING OR DECREASING FUNCTIONS AND THE INVERSE RELATION:

- The inverse of an increasing or decreasing function is a function.
- The inverse of an increasing function is increasing, and the inverse of a decreasing function is decreasing.

To justify the second remark, notice that reflection in $y = x$ maps lines sloping upwards to lines sloping upwards, and maps lines sloping downwards to lines sloping downwards.

Example — The Cube and Cube Root Functions: The function $f(x) = x^3$ and its inverse function $f^{-1}(x) = \sqrt[3]{x}$ are graphed to the right.

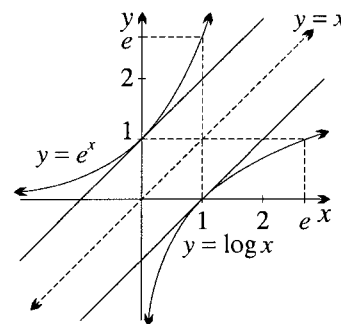
- $f(x) = x^3$ is an increasing function, because every chord slopes upwards. Hence it passes the horizontal line test, and its inverse is a function, which is also increasing.
- $f(x)$ is not, however, increasing at every point, because the tangent at the origin is horizontal. Correspondingly, the tangent to $y = \sqrt[3]{x}$ at the origin is vertical.
- For all x , $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{x})^3 = x$.



Example — The Logarithmic and Exponential Functions:

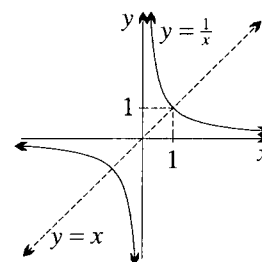
The two functions $f(x) = e^x$ and $f^{-1}(x) = \log x$ provide a particularly clear example of a function and its inverse.

- $f(x) = e^x$ is an increasing function, because every chord slopes upwards. Hence it passes the horizontal line test, and its inverse is a function, which is also increasing.
- $f(x) = e^x$ is also increasing at every point, because its derivative is $f'(x) = e^x$ which is always positive.
- For all x , $\log e^x = x$, and for $x > 0$, $e^{\log x} = x$.



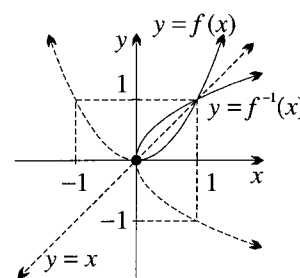
Example — The Reciprocal Function: The function $f(x) = 1/x$ is its own inverse, because the reciprocal of the reciprocal of any nonzero number is always the original number. Correspondingly, its graph is symmetric in $y = x$.

- $f(x) = 1/x$ is neither increasing nor decreasing, because chords joining points on the same branch slope downwards, and chords joining points on different branches slope upwards. Nevertheless, it passes the horizontal line test, and its inverse (which is itself) is a function.
- $f(x) = 1/x$ is decreasing at every point, because its derivative is $f'(x) = -1/x^2$, which is always negative.



Restricting the Domain — The Square and Square Root Functions: The two functions $y = x^2$ and $y = \sqrt{x}$ give our first example of restricting the domain so that the inverse of the restricted function is a function.

- $y = x^2$ is neither increasing nor decreasing, because some of its chords slope upwards, some slope downwards, and some are horizontal. Its inverse $x = y^2$ is not a function — for example, the number 1 has two square roots, 1 and -1 .
- Define the restricted function $f(x)$ by $f(x) = x^2$, where $x \geq 0$. This is the part of $y = x^2$ shown undotted in the diagram on the right. Then $f(x)$ is an increasing function, and so has an inverse which is written as $f^{-1}(x) = \sqrt{x}$, and which is also increasing.
- For all $x > 0$, $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$.



Further Examples of Restricting the Domain: These two worked exercises show the process of restricting the domain applied to more general functions. Since $y = x$ is the mirror exchanging the graphs of a function and its inverse, and since points on a mirror are reflected to themselves, it follows that if the graph of the function intersects the line $y = x$, then it intersects the inverse there too.

WORKED EXERCISE: Explain why the inverse relation of $f(x) = (x-1)^2 + 2$ is not a function. Define $g(x)$ to be the restriction of $f(x)$ to the largest possible domain containing $x = 0$ so that $g(x)$ has an inverse function. Write down the equation of $g^{-1}(x)$, then sketch $g(x)$ and $g^{-1}(x)$ on one set of axes.

SOLUTION: The graph of $y = f(x)$ is a parabola with vertex $(1, 2)$.

This fails the horizontal line test, so the inverse is not a function.

(Alternatively, $f(0) = f(2) = 3$, so $y = 3$ meets the curve twice.)

Restricting $f(x)$ to the domain $x \leq 1$ gives the function

$$g(x) = (x-1)^2 + 2, \text{ where } x \leq 1,$$

which is sketched opposite, and includes the value at $x = 0$.

Since $g(x)$ is a decreasing function, it has an inverse with equation

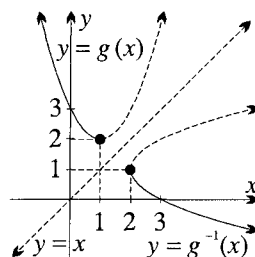
$$x = (y-1)^2 + 2, \text{ where } y \leq 1.$$

Solving for y , $(y-1)^2 = x-2$, where $y \leq 1$,

$$y = 1 + \sqrt{x-2} \text{ or } 1 - \sqrt{x-2}, \text{ where } y \leq 1.$$

Hence

$$g^{-1}(x) = 1 - \sqrt{x-2}, \text{ since } y \leq 1.$$



WORKED EXERCISE: Use calculus to find the turning points and points of inflexion of $y = (x-2)^2(x+1)$, then sketch the curve. Explain why the restriction $f(x)$ of this function to the part of the curve between the two turning points has an inverse function. Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on one set of axes, and write down an equation satisfied by the x -coordinate of the point M where the function and its inverse intersect.

SOLUTION: For $y = (x-2)^2(x+1) = x^3 - 3x^2 + 4$,

$$y' = 3x^2 - 6x = 3x(x-2),$$

and

$$y'' = 6x - 6 = 6(x-1).$$

So there are zeroes at $x = 2$ and $x = -1$, and (after testing) turning points at $(0, 4)$ (a maximum) and $(2, 0)$ (a minimum), and a point of inflexion at $(1, 2)$.

The part of the curve between the turning points is decreasing,

so the function $f(x) = (x-2)^2(x+1)$, where $0 \leq x \leq 2$,

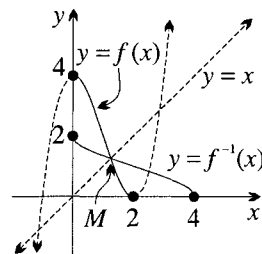
has an inverse function $f^{-1}(x)$, which is also decreasing.

The curves $y = f(x)$ and $y = f^{-1}(x)$ intersect on $y = x$,

and substituting $y = x$ into the function,

$$x = x^3 - 3x^2 + 4,$$

so the x -coordinate of M satisfies the cubic $x^3 - 3x^2 - x + 4 = 0$.



Exercise 1A

- Consider the functions $f = \{(0, 2), (1, 3), (2, 4)\}$ and $g = \{(0, 2), (1, 2), (2, 2)\}$.
 - Write down the inverse relation of each function.
 - Graph each function and its inverse relation on a number plane, using separate diagrams for f and g .
 - State whether or not each inverse relation is a function.

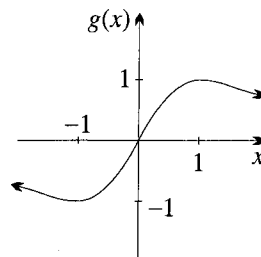
2. The function $f(x) = x + 3$ is defined over the domain $0 \leq x \leq 2$.
 - (a) State the range of $f(x)$.
 - (b) State the domain and range of $f^{-1}(x)$.
 - (c) Write down the rule for $f^{-1}(x)$.
 3. The function F is defined by $F(x) = \sqrt{x}$ over the domain $0 \leq x \leq 4$.
 - (a) State the range of $F(x)$.
 - (c) Write down the rule for $F^{-1}(x)$.
 - (b) State the domain and range of $F^{-1}(x)$.
 - (d) Graph F and F^{-1} .
 4. Sketch the graph of each function. Then use reflection in the line $y = x$ to sketch the inverse relation. State whether or not the inverse is a function, and find its equation if it is. Also, state whether $f(x)$ and $f^{-1}(x)$ (if it exists) are increasing, decreasing or neither.
 - (a) $f(x) = 2x$
 - (c) $f(x) = \sqrt{1 - x^2}$
 - (e) $f(x) = 2^x$
 - (b) $f(x) = x^3 + 1$
 - (d) $f(x) = x^2 - 4$
 - (f) $f(x) = \sqrt{x - 3}$
 5. Consider the functions $f(x) = 3x + 2$ and $g(x) = \frac{1}{3}(x - 2)$.
 - (a) Find $f(g(x))$ and $g(f(x))$.
 - (b) What is the relationship between $f(x)$ and $g(x)$?
 6. Each function $g(x)$ is defined over a restricted domain so that $g^{-1}(x)$ exists. Find $g^{-1}(x)$ and write down its domain and range. (Sketches of g and g^{-1} will prove helpful.)
 - (a) $g(x) = x^2, x \geq 0$
 - (b) $g(x) = x^2 + 2, x \leq 0$
 - (c) $g(x) = -\sqrt{4 - x^2}, 0 \leq x \leq 2$
 7. (a) Write down $\frac{dy}{dx}$ for the function $y = x^3 - 1$.
 - (b) Make x the subject and hence find $\frac{dx}{dy}$.
 - (c) Hence show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.
 8. Repeat the previous question for $y = \sqrt{x}$.
- DEVELOPMENT**
9. The function $F(x) = x^2 + 2x + 4$ is defined over the domain $x \geq -1$.
 - (a) Sketch the graphs of $F(x)$ and $F^{-1}(x)$ on the same diagram.
 - (b) Find the equation of $F^{-1}(x)$ and state its domain and range.
 10. (a) Solve the equation $1 - \ln x = 0$.
 - (b) Sketch the graph of $f(x) = 1 - \ln x$ by suitably transforming the graph of $y = \ln x$.
 - (c) Hence sketch the graph of $f^{-1}(x)$ on the same diagram.
 - (d) Find the equation of $f^{-1}(x)$ and state its domain and range.
 - (e) Classify $f(x)$ and $f^{-1}(x)$ as increasing, decreasing or neither.
 11. (a) Carefully sketch the function defined by $g(x) = \frac{x + 2}{x + 1}$, for $x > -1$.
 - (b) Find $g^{-1}(x)$ and sketch it on the same diagram. Is $g^{-1}(x)$ increasing or decreasing?
 - (c) Find any values of x for which $g(x) = g^{-1}(x)$. [HINT: The easiest way is to solve $g(x) = x$. Why does this work?]
 12. The previous question seems to imply that the graphs of a function and its inverse can only intersect on the line $y = x$. This is not always the case.
 - (a) Find the equation of the inverse of $y = -x^3$.
 - (b) At what points do the graphs of the function and its inverse meet?
 - (c) Sketch the situation.

13. (a) Explain how the graph of $f(x) = x^2$ must be transformed to obtain the graph of $g(x) = (x + 2)^2 - 4$.
 (b) Hence sketch the graph of $g(x)$, showing the x and y intercepts and the vertex.
 (c) What is the largest domain containing $x = 0$ for which $g(x)$ has an inverse function?
 (d) Let $g^{-1}(x)$ be the inverse function corresponding to the domain of $g(x)$ in part (c). What is the domain of $g^{-1}(x)$? Is $g^{-1}(x)$ increasing or decreasing?
 (e) Find the equation of $g^{-1}(x)$, and sketch it on your diagram in part (b).
 (f) Classify $g(x)$ and $g^{-1}(x)$ as either increasing, decreasing or neither.
14. (a) Show that $F(x) = x^3 - 3x$ is an odd function.
 (b) Sketch the graph of $F(x)$, showing the x -intercepts and the coordinates of the two stationary points. Is $F(x)$ increasing or decreasing?
 (c) What is the largest domain containing $x = 0$ for which $F(x)$ has an inverse function?
 (d) State the domain of $F^{-1}(x)$, and sketch it on the same diagram as part (b).
15. (a) State the domain of $f(x) = \frac{e^x}{1 + e^x}$. (b) Show that $f'(x) = \frac{e^x}{(1 + e^x)^2}$.
 (c) Hence explain why $f(x)$ is increasing for all x .
 (d) Explain why $f(x)$ has an inverse function, and find its equation.
16. (a) Sketch $y = 1 + x^2$ and hence sketch $f(x) = \frac{1}{1 + x^2}$. Is $f(x)$ increasing or decreasing?
 (b) What is the largest domain containing $x = -1$ for which $f(x)$ has an inverse function?
 (c) State the domain of $f^{-1}(x)$, and sketch it on the same diagram as part (a).
 (d) Find the rule for $f^{-1}(x)$.
 (e) Is $f^{-1}(x)$ increasing or decreasing?
17. (a) Show that any linear function $f(x) = mx + b$ has an inverse function if $m \neq 0$.
 (b) Does the constant function $F(x) = b$ have an inverse function?
18. The function $f(x)$ is defined by $f(x) = x - \frac{1}{x}$, for $x > 0$.
 (a) By considering the graphs of $y = x$ and $y = \frac{1}{x}$ for $x > 0$, sketch $y = f(x)$.
 (b) Sketch $y = f^{-1}(x)$ on the same diagram.
 (c) By completing the square or using the quadratic formula, show that

$$f^{-1}(x) = \frac{1}{2} \left(x + \sqrt{4 + x^2} \right).$$

19. The diagram shows the function $g(x) = \frac{2x}{1 + x^2}$, whose domain is all real x .

- (a) Show that $g\left(\frac{1}{a}\right) = g(a)$, for all $a \neq 0$.
 (b) Hence explain why the inverse of $g(x)$ is not a function.
 (c) (i) What is the largest domain of $g(x)$ containing $x = 0$ for which $g^{-1}(x)$ exists?
 (ii) Sketch $g^{-1}(x)$ for this domain of $g(x)$.
 (iii) Find the equation of $g^{-1}(x)$ for this domain of $g(x)$.
 (d) Repeat part (c) for the largest domain of $g(x)$ that does not contain $x = 0$.
 (e) Show that the two expressions for $g^{-1}(x)$ in parts (c) and (d) are reciprocals of each other. Why could we have anticipated this?



20. Consider the function $f(x) = \frac{1}{6}(x^2 - 4x + 24)$.
- Sketch the parabola $y = f(x)$, showing the vertex and any x - or y -intercepts.
 - State the largest domain containing only positive numbers for which $f(x)$ has an inverse function $f^{-1}(x)$.
 - Sketch $f^{-1}(x)$ on your diagram from part (a), and state its domain.
 - Find any points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.
 - Let N be a negative real number. Find $f^{-1}(f(N))$.
21. (a) Prove, both geometrically and algebraically, that if an odd function has an inverse function, then that inverse function is also odd.
- (b) What sort of even functions have inverse functions?
22. [The hyperbolic sine function] The function $\sinh x$ is defined by $\sinh x = \frac{1}{2}(e^x - e^{-x})$.
- State the domain of $\sinh x$.
 - Find the value of $\sinh 0$.
 - Show that $y = \sinh x$ is an odd function.
 - Find $\frac{d}{dx}(\sinh x)$ and hence show that $\sinh x$ is increasing for all x .
 - To which curve is $y = \sinh x$ asymptotic for large values of x ?
 - Sketch $y = \sinh x$, and explain why the function has an inverse function $\sinh^{-1} x$.
 - Sketch the graph of $\sinh^{-1} x$ on the same diagram as part (f).
 - Show that $\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$, by treating the equation $x = \frac{1}{2}(e^y - e^{-y})$ as a quadratic equation in e^y .
 - Find $\frac{d}{dx}(\sinh^{-1} x)$, and hence find $\int \frac{dx}{\sqrt{1+x^2}}$.

EXTENSION

23. Suppose that f is a one-to-one function with domain D and range R . Then the function g with domain R and range D is the inverse of f if

$$f(g(x)) = x \text{ for every } x \text{ in } R \quad \text{and} \quad g(f(x)) = x \text{ for every } x \text{ in } D.$$

Use this characterisation to prove that the functions

$$f(x) = -\frac{2}{3}\sqrt{9-x^2}, \text{ where } 0 \leq x \leq 3, \quad \text{and} \quad g(x) = \frac{3}{2}\sqrt{4-x^2}, \text{ where } -2 \leq x \leq 0,$$

are inverse functions.

24. THEOREM: If f is a differentiable function for all real x and has an inverse function g , then $g'(x) = \frac{1}{f'(g(x))}$, provided that $f'(g(x)) \neq 0$.

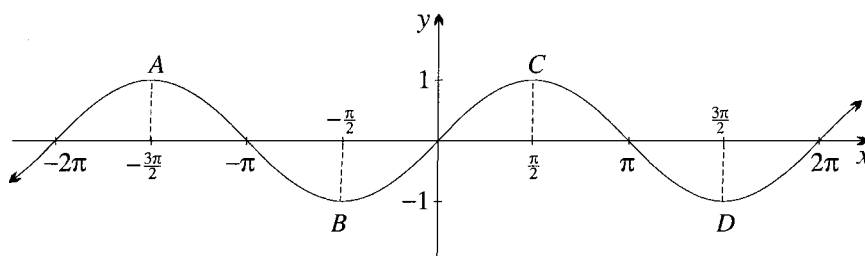
- (a) It is known that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and that $y = e^x$ is the inverse function of $y = \ln x$.

Use this information and the above theorem to prove that $\frac{d}{dx}(e^x) = e^x$.

- (b) (i) Show that the function $f(x) = x^3 + 3x$ is increasing for all real x , and hence that it has an inverse function, $f^{-1}(x)$. (ii) Use the theorem to find the gradient of the tangent to the curve $y = f^{-1}(x)$ at the point $(4, 1)$.
- (c) Prove the theorem in general.

1B Defining the Inverse Trigonometric Functions

Each of the six trigonometric functions fails the horizontal line test completely, in that there are horizontal lines which cross each of their graphs infinitely many times. For example, $y = \sin x$ is graphed below, and clearly every horizontal line between $y = 1$ and $y = -1$ crosses it infinitely many times.

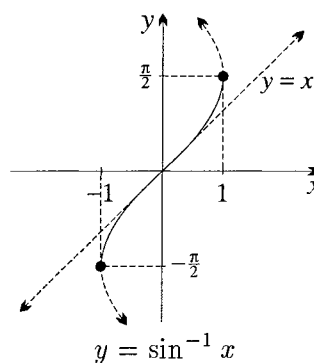
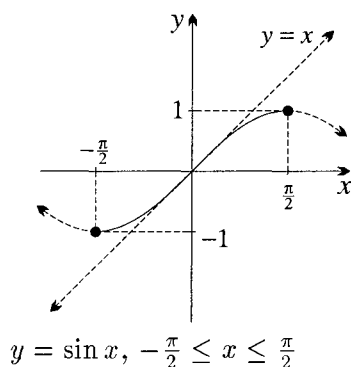


To create an inverse function from $y = \sin x$, we need to restrict the domain to a piece of the curve between two turning points. For example, the pieces AB , BC and CD all satisfy the horizontal line test. Since acute angles should be included, the obvious choice is the arc BC from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

The Definition of $\sin^{-1} x$: The function $y = \sin^{-1} x$ (which is read as ‘inverse sine ex’) is accordingly defined to be the inverse function of the restricted function

$$y = \sin x, \text{ where } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

The two curves are sketched below. Notice, when sketching the graphs, that $y = x$ is a tangent to $y = \sin x$ at the origin. Thus when the graph is reflected in $y = x$, the line $y = x$ does not move, and so it is also the tangent to $y = \sin^{-1} x$ at the origin. Notice also that $y = \sin x$ is horizontal at its turning points, and hence $y = \sin^{-1} x$ is vertical at its endpoints.



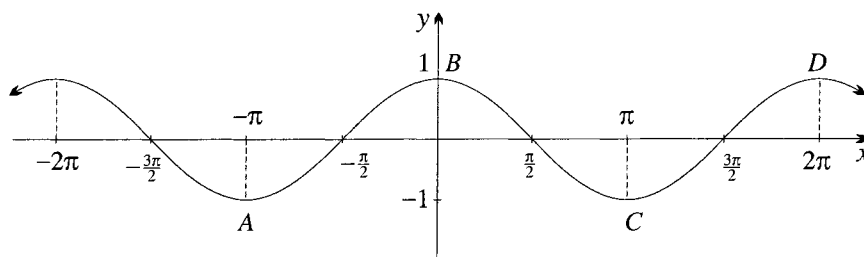
THE DEFINITION OF $y = \sin^{-1} x$:

- $y = \sin^{-1} x$ is not the inverse relation of $y = \sin x$, it is the inverse function of the restriction of $y = \sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- $y = \sin^{-1} x$ has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- $y = \sin^{-1} x$ is an increasing function.
- $y = \sin^{-1} x$ has tangent $y = x$ at the origin, and is vertical at its endpoints.

NOTE: In this course, radian measure is used exclusively when dealing with the inverse trigonometric functions. Calculations using degrees should be avoided, or at least not included in the formal working of problems.

5 **RADIAN MEASURE:** Use radians when dealing with inverse trigonometric functions.

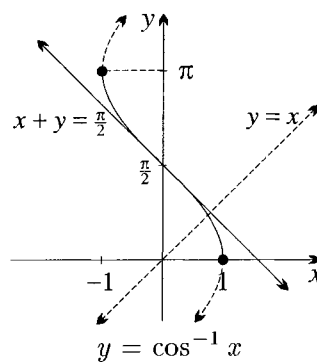
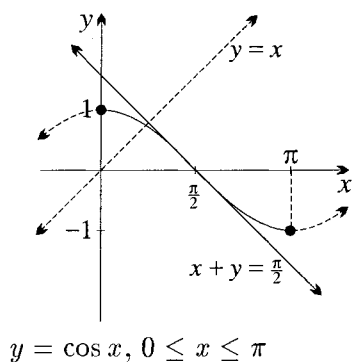
The Definition of $\cos^{-1} x$: The function $y = \cos x$ is graphed below. To create a satisfactory inverse function from $y = \cos x$, we need to restrict the domain to a piece of the curve between two turning points. Since acute angles should be included, the obvious choice is the arc BC from $x = 0$ to $x = \pi$.



Thus the function $y = \cos^{-1} x$ (read as 'inverse cos ex') is defined to be the inverse function of the restricted function

$$y = \cos x, \text{ where } 0 \leq x \leq \pi,$$

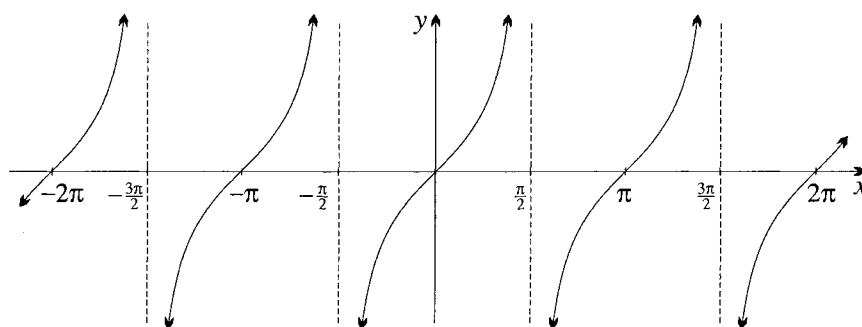
and the two curves are sketched below. Notice that the tangent to $y = \cos x$ at its x -intercept $(\frac{\pi}{2}, 0)$ is the line $t: x + y = \frac{\pi}{2}$ with gradient -1 . Reflection in $y = x$ reflects this line onto itself, so t is also the tangent to $y = \cos^{-1} x$ at its y -intercept $(0, \frac{\pi}{2})$. Like $y = \sin^{-1} x$, the graph is vertical at its endpoints.



THE DEFINITION OF $y = \cos^{-1} x$:

- 6**
- $y = \cos^{-1} x$ is not the inverse relation of $y = \cos x$, it is the inverse function of the restriction of $y = \cos x$ to $0 \leq x \leq \pi$.
 - $y = \cos^{-1} x$ has domain $-1 \leq x \leq 1$ and range $0 \leq y \leq \pi$.
 - $y = \cos^{-1} x$ is a decreasing function.
 - $y = \cos^{-1} x$ has gradient -1 at its y -intercept, and is vertical at its endpoints.

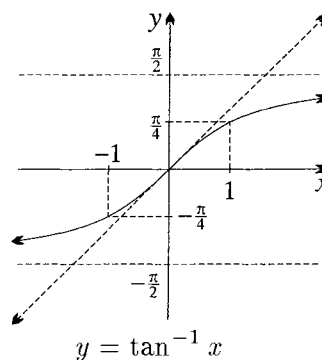
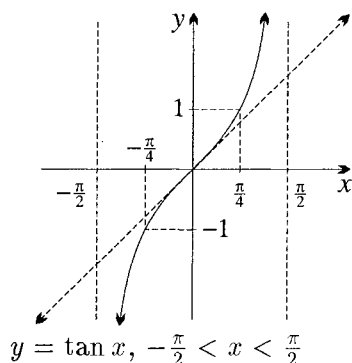
The Definition of $\tan^{-1} x$: The graph of $y = \tan x$ on the next page consists of a collection of disconnected branches. The most satisfactory inverse function is formed by choosing the branch in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



Thus the function $y = \tan^{-1} x$ is defined to be the inverse function of

$$y = \tan x, \text{ where } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The line of reflection $y = x$ is the tangent to both curves at the origin. Notice also that the vertical asymptotes $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are reflected into the horizontal asymptotes $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.



THE DEFINITION OF $y = \tan^{-1} x$:

- 7
- $y = \tan^{-1} x$ is not the inverse relation of $y = \tan x$, it is the inverse function of the restriction of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
 - $y = \tan^{-1} x$ has domain the real line and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
 - $y = \tan^{-1} x$ is an increasing function.
 - $y = \tan^{-1} x$ has gradient 1 at its y -intercept.
 - The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

Inverse Functions of cosec x , sec x and cot x : It is not convenient in this course to define the functions $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ because of difficulties associated with discontinuities. Extension questions in Exercises 1C and 1D investigate these situations.

Calculations with the Inverse Trigonometric Functions: The key to calculations is to include the restriction every time an expression involving the inverse trigonometric functions is rewritten using trigonometric functions.

INTERPRETING THE RESTRICTIONS:

- 8
- $y = \sin^{-1} x$ means $x = \sin y$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
 - $y = \cos^{-1} x$ means $x = \cos y$ where $0 \leq y \leq \pi$.
 - $y = \tan^{-1} x$ means $x = \tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

WORKED EXERCISE: Find: (a) $\cos^{-1}(-\frac{1}{2})$ (b) $\tan^{-1}(-1)$

SOLUTION:

(a) Let $\alpha = \cos^{-1}(-\frac{1}{2})$.
Then $\cos \alpha = -\frac{1}{2}$, where $0 \leq \alpha \leq \pi$.
Hence α is in the second quadrant,
and the related angle is $\frac{\pi}{3}$,
so $\alpha = \frac{2\pi}{3}$.

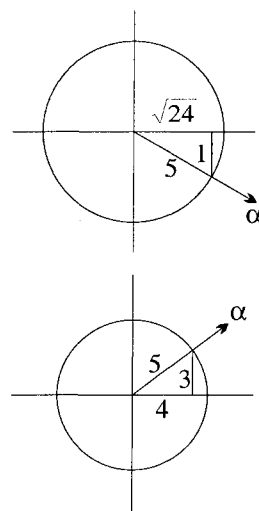
(b) Let $\alpha = \tan^{-1}(-1)$.
Then $\tan \alpha = -1$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.
Hence α is in the fourth quadrant,
and the related angle is $\frac{\pi}{4}$,
so $\alpha = -\frac{\pi}{4}$.

WORKED EXERCISE: Find: (a) $\tan \sin^{-1}(-\frac{1}{5})$ (b) $\sin(2 \cos^{-1} \frac{4}{5})$

SOLUTION:

(a) Let $\alpha = \sin^{-1}(-\frac{1}{5})$.
Then $\sin \alpha = -\frac{1}{5}$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.
Hence α is in the fourth quadrant,
and $\tan \alpha = \frac{-1}{\sqrt{24}}$
 $= -\frac{1}{12}\sqrt{6}$.

(b) Let $\alpha = \cos^{-1} \frac{4}{5}$.
Then $\cos \alpha = \frac{4}{5}$, where $0 \leq \alpha \leq \pi$.
Hence α is in the first quadrant,
and $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
 $= 2 \times \frac{3}{5} \times \frac{4}{5}$
 $= \frac{24}{25}$.



Exercise 1B

1. Read off the graph the values of the following correct to two decimal places:

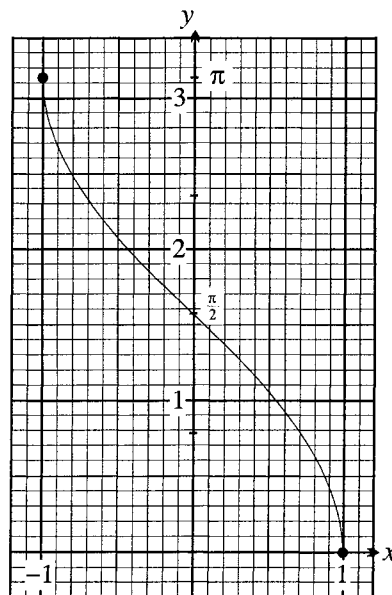
- (a) $\cos^{-1} 0.4$
(b) $\cos^{-1} 0.8$
(c) $\cos^{-1} 0.25$
(d) $\cos^{-1}(-0.1)$
(e) $\cos^{-1}(-0.4)$
(f) $\cos^{-1}(-0.75)$

2. Find the exact value of each of the following:

- (a) $\sin^{-1} 0$ (e) $\sin^{-1}(-1)$ (i) $\sin^{-1}(-\frac{\sqrt{3}}{2})$
(b) $\sin^{-1} \frac{1}{2}$ (f) $\cos^{-1} 0$ (j) $\cos^{-1}(-\frac{1}{\sqrt{2}})$
(c) $\cos^{-1} 1$ (g) $\tan^{-1} 0$ (k) $\tan^{-1}(-\frac{1}{\sqrt{3}})$
(d) $\tan^{-1} 1$ (h) $\tan^{-1}(-1)$ (l) $\cos^{-1}(-1)$

3. Use your calculator to find, correct to three decimal places, the value of:

- (a) $\cos^{-1} 0.123$ (c) $\sin^{-1} \frac{2}{3}$ (e) $\tan^{-1} 5$
(b) $\cos^{-1}(-0.123)$ (d) $\sin^{-1}(-\frac{2}{3})$ (f) $\tan^{-1}(-5)$



4. Find the exact value of:

(a) $\sin^{-1}(-\frac{1}{2}) + \cos^{-1}(-\frac{1}{2})$	(d) $\cos^{-1}(\sin \frac{\pi}{3})$	(g) $\tan^{-1}(-\tan \frac{\pi}{6})$
(b) $\sin(\cos^{-1} 0)$	(e) $\sin(\cos^{-1} \frac{1}{2}\sqrt{3})$	(h) $\cos(2 \tan^{-1}(-1))$
(c) $\tan(\tan^{-1} 1)$	(f) $\cos^{-1}(\cos \frac{3\pi}{4})$	(i) $\tan^{-1}(\sqrt{6} \sin \frac{\pi}{4})$

DEVELOPMENT

5. Find the exact value of:

(a) $\sin^{-1}(\sin \frac{4\pi}{3})$	(c) $\tan^{-1}(\tan \frac{5\pi}{6})$	(e) $\sin^{-1}(2 \sin(-\frac{\pi}{6}))$
(b) $\cos^{-1}(\cos(-\frac{\pi}{4}))$	(d) $\cos^{-1}(\cos \frac{5\pi}{4})$	(f) $\tan^{-1}(3 \tan \frac{7\pi}{6})$

6. (a) In each part use a right-angled triangle within a quadrants diagram to help find the exact value of:

(i) $\sin(\cos^{-1} \frac{3}{5})$	(iii) $\cos(\sin^{-1} \frac{2}{3})$	(v) $\cos(\tan^{-1}(-\frac{1}{3}))$
(ii) $\tan(\sin^{-1} \frac{5}{13})$	(iv) $\sin(\cos^{-1}(-\frac{15}{17}))$	(vi) $\tan(\cos^{-1}(-\frac{3}{4}))$

(b) Use a right-angled triangle in each part to show that:

(i) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$	(ii) $\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

7. Use an appropriate compound-angle formula and the techniques of the previous question where necessary to find the exact value of:

(a) $\sin(\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{12}{13})$	(c) $\tan(\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{3}{5})$
(b) $\cos(\tan^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{4})$	(d) $\tan(\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{12}{13})$

8. Use an appropriate double-angle formula to find the exact value of:

(a) $\cos(2 \cos^{-1} \frac{1}{3})$	(b) $\sin(2 \cos^{-1} \frac{6}{7})$	(c) $\tan(2 \tan^{-1}(-2))$

9. (a) If $\alpha = \tan^{-1} \frac{1}{2}$ and $\beta = \tan^{-1} \frac{1}{3}$, show that $\tan(\alpha + \beta) = 1$.

(b) Hence find the exact value of $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$.

10. Use a technique similar to that in the previous question to show that:

(a) $\sin^{-1} \frac{1}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{10}} = \frac{\pi}{4}$	(c) $\cos^{-1} \frac{3}{11} - \sin^{-1} \frac{3}{4} = \sin^{-1} \frac{19}{44}$
(b) $\tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{2}{9}$	(d) $\sin^{-1} \frac{1}{3} + \cos^{-1} \frac{1}{3} = \frac{\pi}{2}$

11. (a) If $\theta = \sin^{-1} \frac{3}{5}$, show that $\cos 2\theta = \frac{7}{25}$.

(b) Hence show that $\cos^{-1} \frac{7}{25} = 2 \sin^{-1} \frac{3}{5}$.

12. Use techniques similar to that in the previous question to prove that:

(a) $\tan^{-1} \frac{3}{4} = 2 \tan^{-1} \frac{1}{3}$	(b) $2 \cos^{-1} x = \cos^{-1}(2x^2 - 1)$, for $0 \leq x \leq 1$
(c) $2 \tan^{-1} 2 = \pi - \cos^{-1} \frac{3}{5}$ [HINT: Use the fact that $\tan(\pi - x) = -\tan x$.]	

13. (a) Explain why $\sin^{-1}(\sin 2) \neq 2$. (b) Sketch the curve $y = \sin x$ for $0 \leq x \leq \pi$, and use symmetry to explain why $\sin 2 = \sin(\pi - 2)$.

(c) What is the exact value of $\sin^{-1}(\sin 2)$?

14. Let x be a positive number and let $\theta = \tan^{-1} x$.

(a) Simplify $\tan(\frac{\pi}{2} - \theta)$. (b) Show that $\tan^{-1} \frac{1}{x} = \frac{\pi}{2} - \theta$.

(c) Hence show that $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$, for $x > 0$.

(d) Use the fact that $\tan^{-1} x$ is odd to find $\tan^{-1} x + \tan^{-1} \frac{1}{x}$, for $x < 0$.

15. (a) If $\alpha = \tan^{-1} x$ and $\beta = \tan^{-1} 2x$, write down an expression for $\tan(\alpha + \beta)$ in terms of x .

(b) Hence solve the equation $\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3$.

16. Using an approach similar to that in the previous question, solve for x :

(a) $\tan^{-1} x + \tan^{-1} 2 = \tan^{-1} 7$

(b) $\tan^{-1} 3x - \tan^{-1} x = \tan^{-1} \frac{1}{2}$

17. (a) If $\alpha = \sin^{-1} x$, $\beta = \tan^{-1} x$ and $\alpha + \beta = \frac{\pi}{2}$, show that $\cos(\alpha + \beta) = \frac{\sqrt{1-x^2}-x^2}{\sqrt{1+x^2}}$.

(b) Hence show that $x^2 = \frac{\sqrt{5}-1}{2}$.

18. (a) If $u = \tan^{-1} \frac{1}{3}$ and $v = \tan^{-1} \frac{1}{5}$, show that $\tan(u+v) = \frac{4}{7}$.

(b) Show that $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$.

19. Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{2}{5} + \tan^{-1} \frac{8}{9} = \frac{\pi}{2}$.

20. Solve $\tan^{-1} \frac{x}{x+1} + \tan^{-1} \frac{x}{1-x} = \tan^{-1} \frac{6}{7}$.

EXTENSION

21. Prove by mathematical induction that for all positive integer values of n ,

$$\tan^{-1} \frac{1}{2 \times 1^2} + \tan^{-1} \frac{1}{2 \times 2^2} + \cdots + \tan^{-1} \frac{1}{2n^2} = \frac{\pi}{4} - \tan^{-1} \frac{1}{2n+1}.$$

22. Given that $a^2 + b^2 = 1$, prove that the expression $\tan^{-1} \left(\frac{ax}{1-bx} \right) - \tan^{-1} \left(\frac{x-b}{a} \right)$ is independent of x .

23. (a) Show that $\frac{x^2}{x^4 + x^2 + 1} \leq \frac{1}{3}$, for all real x .

(b) Determine the range of $y = \tan^{-1} \left(\frac{1}{1+x^2} \right)$ and the range of $y = \tan^{-1} \left(\frac{x^2}{1+x^2} \right)$.

(c) Show that $\tan^{-1} \left(\frac{1}{1+x^2} \right) + \tan^{-1} \left(\frac{x^2}{1+x^2} \right) = \tan^{-1} \left(1 + \frac{x^2}{1+x^2+x^4} \right)$.

(d) Hence determine the range of $y = \tan^{-1} \left(\frac{1}{1+x^2} \right) + \tan^{-1} \left(\frac{x^2}{1+x^2} \right)$.

1C Graphs Involving Inverse Trigonometric Functions

This section deals mostly with graphs that can be obtained using transformations of the graphs of the three inverse trigonometric functions. Graphs requiring calculus will be covered in the next section.

Graphs Involving Shifting, Reflecting and Stretching: The usual transformation processes can be applied, but substitution of key values should be used to confirm the graph. In the case of $\tan^{-1} x$, it is wise to take limits so as to confirm the horizontal asymptotes.

WORKED EXERCISE: Sketch, stating the domain and range:

(a) $y = 2 \sin^{-1}(x - 1)$

(b) $y = \pi - \tan^{-1} 3x$

SOLUTION:

- (a) $y = 2 \sin^{-1}(x - 1)$ is $y = \sin^{-1} x$ shifted right 1 unit, then stretched vertically by a factor of 2. This should be confirmed by making the following three substitutions:

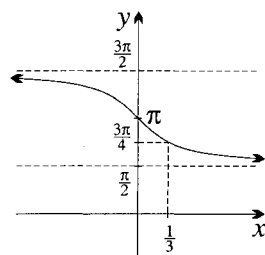
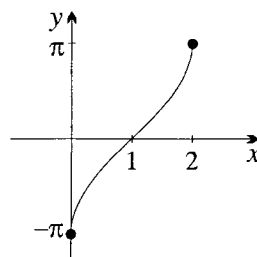
x	0	1	2
y	$-\pi$	0	π

The domain is $0 \leq x \leq 2$, and the range is $-\pi \leq y \leq \pi$.

- (b) $y = \pi - \tan^{-1} 3x$ is $y = \tan^{-1} x$ stretched horizontally by a factor of $\frac{1}{3}$, then reflected in the y -axis, then shifted upwards by π . This should be confirmed by the following table of values and limits:

x	$\rightarrow -\infty$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\rightarrow \infty$
y	$\rightarrow \frac{3\pi}{2}$	$\frac{5\pi}{4}$	π	$\frac{3\pi}{4}$	$\rightarrow \frac{\pi}{2}$

The domain is all real numbers, and the range is $\frac{\pi}{2} < y < \frac{3\pi}{2}$.



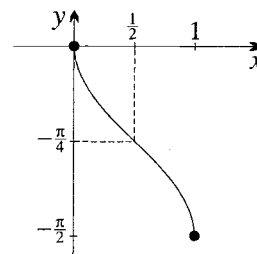
More Complicated Transformations: A curve like $y = -\frac{1}{2} \cos^{-1}(1 - 2x)$ could be obtained by transformations. But the situation is so complicated that the best approach is to construct an appropriate table of values, combined with knowledge of the general shape of the curve.

WORKED EXERCISE: Sketch $y = -\frac{1}{2} \cos^{-1}(1 - 2x)$, and state its domain and range.

SOLUTION: Using a table of values:

x	0	$\frac{1}{2}$	1
y	0	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$

The domain is $0 \leq x \leq 1$ and the range is $-\frac{\pi}{2} \leq y \leq 0$.



Symmetries of the Inverse Trigonometric Functions: The two functions $y = \sin^{-1} x$ and $y = \tan^{-1} x$ are both odd, but $y = \cos^{-1} x$ has odd symmetry about its y -intercept $(0, \frac{\pi}{2})$.

SYMMETRIES OF THE INVERSE TRIGONOMETRIC FUNCTIONS:

9

- $y = \sin^{-1} x$ is odd, that is, $\sin^{-1}(-x) = -\sin^{-1} x$.
- $y = \tan^{-1} x$ is odd, that is, $\tan^{-1}(-x) = -\tan^{-1} x$.
- $y = \cos^{-1} x$ has odd symmetry about its y -intercept $(0, \frac{\pi}{2})$, that is,

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

Only the last identity needs proof.

PROOF: Let $\alpha = \cos^{-1}(-x)$.

Then $-x = \cos \alpha$, where $0 \leq \alpha \leq \pi$,

so $\cos(\pi - \alpha) = x$, since $\cos(\pi - \alpha) = -\cos \alpha$,

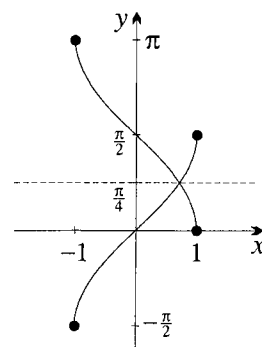
$$\begin{aligned}\pi - \alpha &= \cos^{-1} x, \text{ since } 0 \leq \pi - \alpha \leq \pi, \\ \alpha &= \pi - \cos^{-1} x, \text{ as required.}\end{aligned}$$

The Identity $\sin^{-1} x + \cos^{-1} x = \pi/2$: The graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ are reflections of each other in the horizontal line $y = \frac{\pi}{4}$. Hence adding the graphs pointwise, it should be clear that

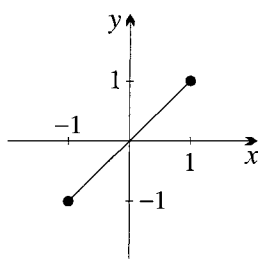
10 **COMPLEMENTARY ANGLES:** $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

This is really only another form of the complementary angle identity $\cos(\frac{\pi}{2} - \theta) = \sin \theta$ — here is an algebraic proof which makes this relationship clear.

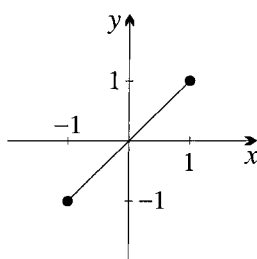
PROOF: Let $\alpha = \cos^{-1} x$.
 Then $x = \cos \alpha$, where $0 \leq \alpha \leq \pi$,
 $\sin(\frac{\pi}{2} - \alpha) = x$, since $\sin(\frac{\pi}{2} - \alpha) = \cos \alpha$,
 $\sin^{-1} x = \frac{\pi}{2} - \alpha$, since $-\frac{\pi}{2} \leq \frac{\pi}{2} - \alpha \leq \frac{\pi}{2}$,
 $\sin^{-1} x + \alpha = \frac{\pi}{2}$, as required.



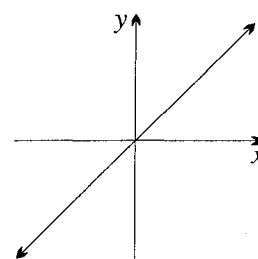
The Graphs of $\sin \sin^{-1} x$, $\cos \cos^{-1} x$ and $\tan \tan^{-1} x$: The composite function defined by $y = \sin \sin^{-1} x$ has the same domain as $\sin^{-1} x$, that is, $-1 \leq x \leq 1$. Since it is the function $y = \sin^{-1} x$ followed by the function $y = \sin x$, the composite is therefore the identity function $y = x$ restricted to $-1 \leq x \leq 1$.



$$y = \sin \sin^{-1} x$$



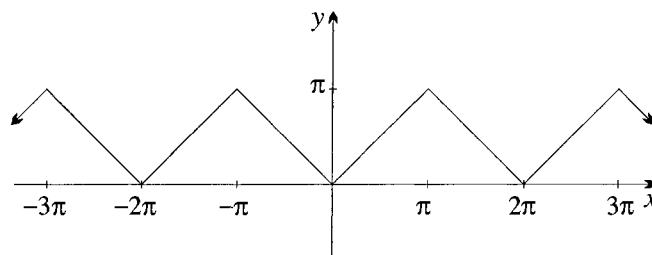
$$y = \cos \cos^{-1} x$$



$$y = \tan \tan^{-1} x$$

The same remarks apply to $y = \cos \cos^{-1} x$ and $y = \tan \tan^{-1} x$, except that the domain of $y = \tan \tan^{-1} x$ is the whole real number line.

The Graph of $\cos^{-1} \cos x$: The domain of this function is the whole real number line, and the graph is far more complicated. Constructing a simple table of values is probably the surest approach, but under the graph is an argument based on symmetries.



A. For $0 \leq x \leq \pi$, $\cos^{-1} \cos x = x$, and the graph follows $y = x$.

B. Since $\cos x$ is an even function, the graph in the interval $-\pi \leq x \leq 0$ is the reflection of the graph in the interval $0 \leq x \leq \pi$.

C. We now have the shape of the graph in the interval $-\pi \leq x \leq \pi$. Since the graph has period 2π , the rest of the graph is just a repetition of this section.

The exercises deal with the other confusing functions $\sin^{-1} \sin x$ and $\tan^{-1} \tan x$, and also with functions like $y = \sin^{-1} \cos x$.

Exercise 1C

- Sketch each function, stating the domain and range and whether it is even, odd or neither:
 - $y = \tan^{-1} x$
 - $y = \cos^{-1} x$
 - $y = \sin^{-1} x$
- Sketch each function, using appropriate translations of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$. State the domain and range, and whether it is even, odd or neither.
 - $y = \sin^{-1}(x - 1)$
 - $y = \cos^{-1}(x + 1)$
 - $y - \frac{\pi}{2} = \tan^{-1} x$
- Sketch each function by stretching $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ horizontally or vertically as appropriate. State the domain and range, and whether it is even, odd or neither.
 - $y = 2 \sin^{-1} x$
 - $y = \cos^{-1} 2x$
 - $y = \frac{1}{2} \tan^{-1} x$
- Sketch each function by reflecting in the x - or y -axis as appropriate. State the domain and range, and whether it is even, odd or neither.
 - $y = -\cos^{-1} x$
 - $y = \tan^{-1}(-x)$
 - $y = -\sin^{-1}(-x)$
- Sketch each function, stating the domain and range, and whether it is even, odd or neither:
 - $y = 3 \sin^{-1} 2x$
 - $y = \tan^{-1}(x - 1) - \frac{\pi}{2}$
 - $\frac{1}{2}y = 2 \cos^{-1}(x - 2)$
 - $y = \frac{1}{2} \cos^{-1} 3x$
 - $3y = 2 \sin^{-1} \frac{x}{2}$
 - $y = \frac{1}{4} \cos^{-1}(-x)$
- Consider the function $y = 4 \sin^{-1}(2x + 1)$.
 - Solve $-1 \leq 2x + 1 \leq 1$ to find the domain.
 - Solve $-\frac{\pi}{2} \leq \frac{y}{4} \leq \frac{\pi}{2}$ to find the range.
 - Hence sketch the graph of the function.
 - Use similar steps to find the domain and range of each function, and hence sketch it:
 - $y = 3 \cos^{-1}(2x - 1)$
 - $y = \frac{1}{2} \sin^{-1}(3x + 2)$
 - $y = 2 \tan^{-1}(4x - 1)$

DEVELOPMENT

- Sketch the graphs of $y = \cos^{-1} x$ and $y = \sin^{-1} x - \frac{\pi}{2}$ on the same set of axes.
 - Hence show graphically that $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$.
 - Use a graphical approach to show that:
 - $\tan^{-1}(-x) = -\tan^{-1} x$
 - $\cos^{-1} x + \cos^{-1}(-x) = \pi$
- Determine the domain and range of $y = \sin^{-1}(1 - x)$.
 - Complete the table to the right, and hence sketch the graph of the function.
 - About which line are the graphs of $y = \sin^{-1}(1 - x)$ and $y = \sin^{-1} x$ symmetrical?
- Find the domain and range, draw up a table of values if necessary, and then sketch:
 - $y = 2 \cos^{-1}(1 - x)$
 - $y = \tan^{-1}(\sqrt{3} - x)$
 - $y = \frac{1}{3} \sin^{-1}(2 - 3x)$

x	0	1	2
y			

10. Sketch the graph of each function using the methods of this section:
- (a) $-y = \sin^{-1}(x+1)$ (b) $y = -\tan^{-1}(1-x)$ (c) $y + \frac{\pi}{2} = \frac{1}{2} \cos^{-1}(-x)$
11. Sketch these graphs, stating whether each function is even, odd or neither:
- (a) $y = \sin(\sin^{-1} 2x)$ (b) $y = \cos(\cos^{-1} \frac{x}{2})$ (c) $y = \tan(\tan^{-1}(x-1))$
12. Consider the function $f(x) = \sin 2x$.
- (a) Sketch the graph of $f(x)$, for $-\pi \leq x \leq \pi$.
- (b) What is the largest domain containing $x = 0$ for which $f(x)$ has an inverse function?
- (c) Sketch the graph of $f^{-1}(x)$ by reflecting in the line $y = x$.
- (d) Find the equation of $f^{-1}(x)$, and state its symmetry.
13. (a) What is the domain of $y = \sin \cos^{-1} x$? Is it even, odd or neither?
- (b) By considering the range of $\cos^{-1} x$, explain why $\sin \cos^{-1} x \geq 0$, for all x in its domain.
- (c) By squaring both sides of $y = \sin \cos^{-1} x$ and using the identity $\sin^2 \theta + \cos^2 \theta = 1$, show that $y = \sqrt{1-x^2}$.
- (d) Hence sketch $y = \sin \cos^{-1} x$.
- (e) Use similar methods to sketch the graph of $y = \cos \sin^{-1} x$.
14. Consider the function $y = \tan^{-1} \tan x$.
- (a) State its domain and range, and whether it is even, odd or neither.
- (b) Simplify $\tan^{-1} \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
- (c) What is the period of the function?
- (d) Use the above information and a table of values if necessary to sketch the function.
15. In a worked exercise, $y = \cos^{-1} \cos x$ is sketched. Use $\sin^{-1} t = \frac{\pi}{2} - \cos^{-1} t$ and simple transformations to sketch $y = \sin^{-1} \cos x$. State its symmetry.

EXTENSION

16. Consider the function $y = \sin^{-1} \sin x$.
- (a) State its domain, range and period, and whether it is even, odd or neither.
- (b) Simplify $\sin^{-1} \sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and sketch the function in this region.
- (c) Use the symmetry of $\sin x$ in $x = \frac{\pi}{2}$ to continue the sketch for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.
- (d) Use the above information and a table of values if necessary to sketch the function.
- (e) Hence sketch $y = \cos^{-1} \sin x$ by making use of the fact that $\cos^{-1} t = \frac{\pi}{2} - \sin^{-1} t$.
17. (a) Sketch $f(x) = \cos x$, for $0 \leq x \leq 2\pi$.
- (b) What is the largest domain containing $x = \frac{3\pi}{2}$ for which $f(x)$ has an inverse function?
- (c) Sketch the graph of $f^{-1}(x)$ by reflection in $y = x$.
- (d) Show that $\cos(2\pi - x) = \cos x$, and that if $\pi \leq x \leq 2\pi$, then $0 \leq 2\pi - x \leq \pi$.
- (e) Hence find the equation of $f^{-1}(x)$.
18. One way (and a rather bizarre way!) to define the function $y = \sec^{-1} x$ is as the inverse of the restriction of $y = \sec x$ to the domain $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$.
- (a) Sketch the graph of the function $y = \sec^{-1} x$ as defined above.
- (b) Find the value of: (i) $\sec^{-1} 2$ (ii) $\sec^{-1}(-2)$
- (c) Show that $\tan(\sec^{-1} x) = \sqrt{x^2 - 1}$.

1D Differentiation

Having formed the three inverse trigonometric functions, we can now apply the normal processes of calculus to them. This section is concerned with their derivatives and its usual applications to curve-sketching and maximisation.

Differentiating $\sin^{-1} x$ and $\cos^{-1} x$: The functions $y = \sin^{-1} x$ and $y = \cos^{-1} x$ can be differentiated by changing to the inverse function and using the known derivatives of the sine and cosine functions — this same procedure was used in Section 13B of the Year 11 volume when the derivative of $y = e^x$ was found by changing to the inverse function $x = \log y$. In this case, however, we need to keep track of the restrictions to the domain, which are needed later in the working to make a significant choice between positive and negative square roots.

A. Let $y = \sin^{-1} x$.

Then $x = \sin y$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$,

$$\text{so } \frac{dx}{dy} = \cos y.$$

Since y is in the first or fourth quadrant, $\cos y$ is positive,

$$\text{so } \cos y = +\sqrt{1 - \sin^2 y} \\ = \sqrt{1 - x^2}.$$

$$\text{Thus } \frac{dx}{dy} = \sqrt{1 - x^2}$$

$$\text{and } \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

$$\text{Hence } \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.$$

B. Let $y = \cos^{-1} x$.

Then $x = \cos y$, where $0 \leq y \leq \pi$,

$$\text{so } \frac{dx}{dy} = -\sin y.$$

Since y is in the first or second quadrant, $\sin y$ is positive,

$$\text{so } \sin y = +\sqrt{1 - \cos^2 y} \\ = \sqrt{1 - x^2}.$$

$$\text{Thus } \frac{dx}{dy} = -\sqrt{1 - x^2}$$

$$\text{and } \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

$$\text{Hence } \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

Differentiating $\tan^{-1} x$: The problem of which square root to choose does not arise when differentiating $y = \tan^{-1} x$.

Let $y = \tan^{-1} x$.

Then $x = \tan y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$\text{so } \frac{dx}{dy} = \sec^2 y \\ = 1 + \tan^2 y.$$

$$\text{Hence } \frac{dx}{dy} = 1 + x^2$$

$$\text{and } \frac{dy}{dx} = \frac{1}{1 + x^2}, \text{ giving the standard form } \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.$$

STANDARD FORMS FOR DIFFERENTIATION:

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}} \\ \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

WORKED EXERCISE: Differentiate the functions:

(a) $y = x \tan^{-1} x$

(b) $y = \sin^{-1}(ax + b)$

SOLUTION:

(a) $y = x \tan^{-1} x$

$$y' = vu' + uv'$$

$$= \tan^{-1} x \times 1 + x \times \frac{1}{1+x^2}$$

$$= \tan^{-1} x + \frac{x}{1+x^2}$$

(b) $y = \sin^{-1}(ax + b)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-(ax+b)^2}} \times a$$

$$= \frac{a}{\sqrt{1-(ax+b)^2}}$$

Let $u = x$

and $v = \tan^{-1} x$.

Then $u' = 1$

and $v' = \frac{1}{1+x^2}$.

Let $u = ax + b$,

then $y = \sin^{-1} u$.

Hence $\frac{du}{dx} = a$

and $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$.

Linear Extensions: The method used in part (b) above can be applied to all three inverse trigonometric functions, giving a further set of standard forms.

FURTHER STANDARD FORMS FOR DIFFERENTIATION:

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$$\frac{d}{dx} \sin^{-1}(ax + b) = \frac{a}{\sqrt{1-(ax+b)^2}}$$

$$\frac{d}{dx} \cos^{-1}(ax + b) = -\frac{a}{\sqrt{1-(ax+b)^2}}$$

$$\frac{d}{dx} \tan^{-1}(ax + b) = \frac{a}{1+(ax+b)^2}$$

WORKED EXERCISE:

(a) Find the points A and B on the curve $y = \cos^{-1}(x-1)$ where the tangent has gradient -2 .

(b) Sketch the curve, showing these points.

SOLUTION:

(a) Differentiating, $y' = -\frac{1}{\sqrt{1-(x-1)^2}}$. (b)

Put $y' = -2$.

Then $-\frac{1}{\sqrt{1-(x-1)^2}} = -2$

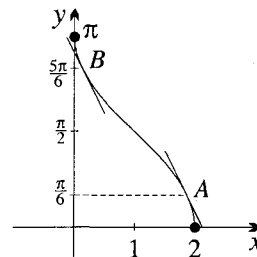
$$1-(x-1)^2 = \frac{1}{4}$$

$$(x-1)^2 = \frac{3}{4}$$

$$x-1 = \frac{1}{2}\sqrt{3} \text{ or } -\frac{1}{2}\sqrt{3}$$

$$x = 1 + \frac{1}{2}\sqrt{3} \text{ or } 1 - \frac{1}{2}\sqrt{3},$$

so the points are $A(1 + \frac{1}{2}\sqrt{3}, \frac{\pi}{6})$ and $B(1 - \frac{1}{2}\sqrt{3}, \frac{5\pi}{6})$.



Functions whose Derivatives are Zero are Constants: Several identities involving inverse trigonometric functions can be obtained by showing that some derivative is zero, and hence that the original function must be a constant. The following identity is the clearest example — it has been proven already in Section 1C using symmetry arguments.

WORKED EXERCISE:

- (a) Differentiate $\sin^{-1} x + \cos^{-1} x$.
 (b) Hence prove the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

SOLUTION:

$$(a) \quad \frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$$

- (b) Hence $\sin^{-1} x + \cos^{-1} x = C$, for some constant C .
 Substitute $x = 0$, then $0 + \frac{\pi}{2} = C$,
 so $C = \frac{\pi}{2}$, and $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, as required.

Curve Sketching Using Calculus: The usual methods of curve sketching can now be extended to curves whose equations involve the inverse trigonometric functions. The following worked example applies calculus to sketching the curve $y = \cos^{-1} \cos x$, which was sketched without calculus in the previous section.

WORKED EXERCISE: Use calculus to sketch $y = \cos^{-1} \cos x$.

SOLUTION: The function is periodic with the same period as $\cos x$, that is, 2π .

A simple table of values gives some key points:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	...
y	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	...

The shape of the curve joining these points can be obtained by calculus.

Differentiating using the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin x}{\sqrt{1-\cos^2 x}} \\ &= \frac{\sin x}{\sqrt{\sin^2 x}}. \end{aligned}$$

When $\sin x$ is positive, $\sqrt{\sin^2 x} = \sin x$,

$$\text{so } \frac{dy}{dx} = 1.$$

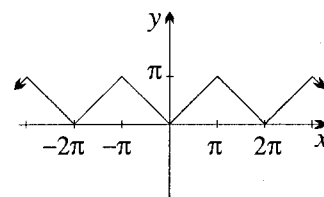
When $\sin x$ is negative, $\sqrt{\sin^2 x} = -\sin x$,

$$\text{so } \frac{dy}{dx} = -1.$$

$$\text{Hence } \frac{dy}{dx} = \begin{cases} 1, & \text{for } x \text{ in quadrants 1 and 2,} \\ -1, & \text{for } x \text{ in quadrants 3 and 4.} \end{cases}$$

This means that the graph consists of a series of intervals, each with gradient 1 or -1.

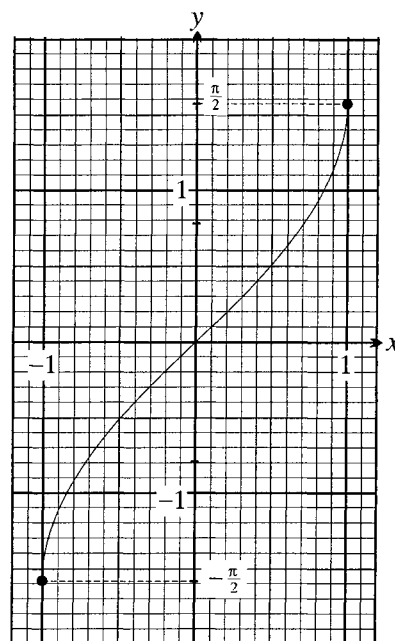
$$\begin{aligned} \text{Let } u &= \cos x, \\ \text{then } y &= \cos^{-1} u. \\ \text{Hence } \frac{du}{dx} &= -\sin x \\ \text{and } \frac{dy}{du} &= -\frac{1}{\sqrt{1-u^2}}. \end{aligned}$$



Exercise 1D

1. (a) Photocopy the graph of $y = \sin^{-1} x$ shown to the right. Then carefully draw a tangent at each x value in the table. Then, by measurement and calculation of rise/run, find the gradient of each tangent to two decimal places and fill in the second row of the table.

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\frac{dy}{dx}$									

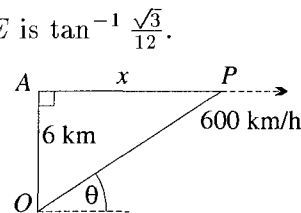
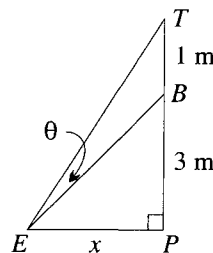


- (b) Check your gradients using $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.
2. Differentiate with respect to x :
- | | | |
|---------------------|--------------------------|------------------------------|
| (a) $\cos^{-1} x$ | (g) $\sin^{-1} x^2$ | (m) $\sin^{-1} \frac{1}{5}x$ |
| (b) $\tan^{-1} x$ | (h) $\tan^{-1} x^3$ | (n) $\tan^{-1} \frac{1}{4}x$ |
| (c) $\sin^{-1} 2x$ | (i) $\tan^{-1}(x+2)$ | (o) $\cos^{-1} \sqrt{x}$ |
| (d) $\tan^{-1} 3x$ | (j) $\cos^{-1}(1-x)$ | (p) $\tan^{-1} \sqrt{x}$ |
| (e) $\cos^{-1} 5x$ | (k) $x \sin^{-1} x$ | (q) $\tan^{-1} \frac{1}{x}$ |
| (f) $\sin^{-1}(-x)$ | (l) $(1+x^2)\tan^{-1} x$ | |
3. Find the gradient of the tangent to each curve at the point indicated:
- | | |
|---|---|
| (a) $y = 2 \tan^{-1} x$, at $x = 0$ | (c) $y = \tan^{-1} 2x$, at $x = -\frac{1}{2}$ |
| (b) $y = \sqrt{3} \sin^{-1} x$, at $x = \frac{1}{2}$ | (d) $y = \cos^{-1} \frac{x}{2}$, at $x = \sqrt{3}$ |
4. Find, in the form $y = mx + b$, the equation of the tangent and the normal to each curve at the point indicated:
- | | |
|---------------------------------------|---|
| (a) $y = 2 \cos^{-1} 3x$, at $x = 0$ | (b) $y = \sin^{-1} \frac{x}{2}$, at $x = \sqrt{2}$ |
|---------------------------------------|---|
5. (a) Show that $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$.
- (b) Hence explain why $\sin^{-1} x + \cos^{-1} x$ must be a constant function, and use any convenient value of x in its domain to find the value of the constant.
6. Use the method of the previous question to show that each of these functions is a constant function, and find the value of the constant.
- | | |
|-----------------------------------|--|
| (a) $\cos^{-1} x + \cos^{-1}(-x)$ | (b) $2 \sin^{-1} \sqrt{x} - \sin^{-1}(2x-1)$ |
|-----------------------------------|--|

DEVELOPMENT

7. (a) If $f(x) = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)$, show that $f''(x) = \frac{1}{1+x^2}$.
- (b) Is the graph of $y = f(x)$ concave up or concave down at $x = -1$?
8. Show that the gradient of the curve $y = \frac{\sin^{-1} x}{x}$ at the point where $x = \frac{1}{2}$ is $\frac{2}{3}(2\sqrt{3} - \pi)$.
9. Find the derivative of each function in simplest form:
- | | | |
|------------------------------------|-------------------------------|-------------------------------------|
| (a) $x \cos^{-1} x - \sqrt{1-x^2}$ | (d) $\tan^{-1} \frac{1}{1-x}$ | (g) $\sin^{-1} \sqrt{\log x}$ |
| (b) $\sin^{-1} e^{3x}$ | (e) $\sin^{-1} e^x$ | (h) $\sqrt{x} \sin^{-1} \sqrt{1-x}$ |
| (c) $\sin^{-1} \frac{1}{4}(2x-3)$ | (f) $\log \sqrt{\sin^{-1} x}$ | (i) $\tan^{-1} \frac{x+2}{1-2x}$ |

10. (a) (i) If $y = (\sin^{-1} x)^2$, show that $y'' = \frac{2 + \frac{2x \sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}$.
 (ii) Hence show that $(1-x^2)y'' - xy' - 2 = 0$.
 (b) Show that $y = e^{\sin^{-1} x}$ satisfies the differential equation $(1-x^2)y'' - xy' - y = 0$.
11. Consider the function $f(x) = \cos^{-1} x^2$.
 (a) What is the domain of $f(x)$?
 (b) About which line is the graph of $y = f(x)$ symmetrical?
 (c) Find $f'(x)$. (d) Show that $y = f(x)$ has a maximum turning point at $x = 0$.
 (e) Show that $f'(x)$ is undefined at the endpoints of the domain. What is the geometrical significance of this?
 (f) Sketch the graph of $y = f(x)$.
12. A picture 1 metre tall is hung on a wall with its bottom edge 3 metres above the eye E of a viewer. Let the distance EP be x metres, and let θ be the angle that the picture subtends at E .
 (a) Show that $\theta = \tan^{-1} \frac{4}{x} - \tan^{-1} \frac{3}{x}$.
 (b) Show that θ is maximised when the viewer is $2\sqrt{3}$ metres from the wall.
 (c) Show that the maximum angle subtended by the picture at E is $\tan^{-1} \frac{\sqrt{3}}{12}$.
13. A plane P at an altitude of 6 km and at a constant speed of 600 km/h is flying directly away from an observer at O on the ground. A is the point on the path of the plane directly above O , and the distance AP is x km. The angle of elevation of the plane from the observer is θ .
 (a) Show that $\theta = \tan^{-1} \frac{6}{x}$.
 (b) Show that $\frac{d\theta}{dt} = \frac{-3600}{x^2 + 36}$ radians per hour.
 (c) Hence find, in radians per second, the rate at which θ is decreasing at the instant when the distance AP is 3 km.
14. (a) State the domain of $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$, and its symmetry.
 (b) Show that $f'(x) = 0$ for all values of x in the domain.
 (c) Show that $f(x) = \begin{cases} \frac{\pi}{2}, & \text{for } x > 0, \\ -\frac{\pi}{2}, & \text{for } x < 0, \end{cases}$ and hence sketch the graph of $f(x)$.
15. Find $\frac{dy}{dx}$ in terms of t , given that:
 (a) $x = \sin^{-1} \sqrt{t}$ and $y = \sqrt{1-t}$ (b) $x = \ln(1+t^2)$ and $y = t - \tan^{-1} t$
16. Consider the function $f(x) = \cos^{-1} \frac{1}{x}$.
 (a) State the domain of $f(x)$. [HINT: Think about it rather than relying on algebra.]
 (b) Recalling that $\sqrt{x^2} = |x|$, show that $f'(x) = \frac{1}{|x|\sqrt{x^2-1}}$.
 (c) Comment on $f'(1)$ and $f'(-1)$.
 (d) Use the expression for $f'(x)$ in part (b) to write down separate expressions for $f'(x)$ when $x > 1$ and when $x < -1$.



- (e) Explain why $f(x)$ is increasing for $x > 1$ and for $x < -1$.
 (f) Find: (i) $\lim_{x \rightarrow \infty} f(x)$ (ii) $\lim_{x \rightarrow -\infty} f(x)$ (g) Sketch the graph of $y = f(x)$.
17. The function $f(x)$ is defined by the rule $f(x) = \sin^{-1} \sin x$.
 (a) State the domain and range of $f(x)$, and whether it is even, odd or neither.
 (b) Show that $f'(x) = \frac{\cos x}{|\cos x|}$. (c) Is $f'(x)$ defined whenever $\cos x = 0$?
 (d) What are the only two values that $f'(x)$ takes if $\cos x \neq 0$, and when does each of these values occur?
 (e) Sketch the graph of $f(x)$ using the above information and a table of values if necessary. [NOTE: This function was sketched in the previous exercise using a different approach. Look back and compare.]

EXTENSION

18. (a) What is the domain of $g(x) = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$?
 (b) Show that $g'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{x}{|x|\sqrt{1 - x^2}}$.
 (c) Hence determine the interval over which $g(x)$ is constant, and find this constant.
19. In question 9(i), you proved that $\frac{d}{dx} \tan^{-1} \frac{x+2}{1-2x}$ was $\frac{1}{1+x^2}$, which is also the derivative of $\tan^{-1} x$. What is going on?
20. Find $\frac{dy}{dx}$ by differentiating implicitly:
 (a) $\sin^{-1}(x+y) = 1$ (b) $\cos^{-1} xy = x^2$ (c) $\tan^{-1} \frac{y}{x} = \log \sqrt{x^2 + y^2}$
21. [The inverse cosecant function] The most straightforward way to define $\operatorname{cosec}^{-1} x$ is as the inverse function of the restriction of $y = \operatorname{cosec} x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, excluding $x = 0$.
 (a) Graph $y = \operatorname{cosec}^{-1} x$, and state its domain, range and symmetry.
 (b) Show that $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2 - 1}}$ (except at endpoints).
 (c) Show that $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$, for $x \geq 1$ or $x \leq -1$.
22. [The inverse cotangent function] The function $y = \cot^{-1} x$ can be defined as the inverse function of the restriction of $y = \cot x$:
 (i) to $0 < x < \pi$, or (ii) to $-\frac{\pi}{2} < x \leq \frac{\pi}{2}$, excluding $x = 0$.
 (a) Graph both functions, and state their domains, ranges and symmetries.
 (b) Show that in both cases, $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$ (except at endpoints).
 (c) Is it true that in both cases $\cot^{-1} x = \tan^{-1} \frac{1}{x}$, for $x \neq 0$?
 (d) What are the advantages of each definition?
23. [The inverse secant function] In the previous exercise, the function $y = \sec x$ was restricted to the domain $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$, to produce an inverse function, $y = \sec^{-1} x$.
 (a) Starting with $\sec y = x$, show that $\frac{dy}{dx} = \frac{1}{\sec y \tan y}$.
 (b) Hence show that $\frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$. (c) Find $\frac{d}{dx} (\sec^{-1} \sqrt{x^2 - 1})$.
 (d) The more straightforward definition of $\sec^{-1} x$ restricts $\sec x$ to $0 \leq x \leq \pi$, excluding $x = \frac{\pi}{2}$. Graph this version of $\sec^{-1} x$, and state its domain, range and symmetry.

1E Integration

This section deals with the integrals associated with the inverse trigonometric functions, and with the standard applications of those integrals to areas, volumes and the calculation of functions whose derivatives are known.

The Basic Standard Forms: Differentiation of the two inverse trigonometric functions

$\sin^{-1} x$ and $\tan^{-1} x$ yields the purely algebraic functions $\frac{1}{\sqrt{1-x^2}}$ and $\frac{1}{1+x^2}$.

This is a remarkable result, and is a sure sign that trigonometric functions are very closely related to algebraic functions associated with squares and square roots — a fact that was already clear when the trigonometric functions were defined using the circle, whose equation $x^2 + y^2 = r^2$ is purely algebraic. This section concerns integration, and we begin by reversing the previous standard forms for differentiation:

13

$$\text{STANDARD FORMS: } \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \quad \text{or} \quad -\cos^{-1} x + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

Thus the inverse trigonometric functions are required for the integration of purely algebraic functions. These standard forms should be compared with the standard form $\int \frac{1}{x} dx = \log x$, where the logarithmic function was required for the integration of the algebraically defined function $y = 1/x$.

The Functions $y = \frac{1}{\sqrt{1-x^2}}$ and $y = \frac{1}{1+x^2}$: The primitives of both these functions

have now been obtained, and they should therefore be regarded as reasonably standard functions whose graphs should be known. The sketch of each function and some important definite integrals associated with them are developed in questions 18 and 19 in the following exercise.

WORKED EXERCISE: Evaluate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx$ using both standard forms.

SOLUTION:

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx &= \left[\sin^{-1} x \right]_0^{\frac{1}{2}} & \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx &= \left[-\cos^{-1} x \right]_0^{\frac{1}{2}} \\ &= \sin^{-1} \frac{1}{2} - \sin^{-1} 0 & &= -\cos^{-1} \frac{1}{2} + \cos^{-1} 0 \\ &= \frac{\pi}{6} - 0 & &= -\frac{\pi}{3} + \frac{\pi}{2} \\ &= \frac{\pi}{6} & &= \frac{\pi}{6} \end{aligned}$$

WORKED EXERCISE: Evaluate exactly or correct to four significant figures:

$$(a) \int_0^1 \frac{1}{1+x^2} dx \qquad (b) \int_0^4 \frac{1}{1+x^2} dx$$

SOLUTION:

$$\begin{aligned} (a) \int_0^1 \frac{1}{1+x^2} dx &= \left[\tan^{-1} x \right]_0^1 & (b) \int_0^4 \frac{1}{1+x^2} dx &= \left[\tan^{-1} x \right]_0^4 \\ &= \tan^{-1} 1 - \tan^{-1} 0 & &= \tan^{-1} 4 - \tan^{-1} 0 \\ &= \frac{\pi}{4} & &\doteq 1.329 \end{aligned}$$

More General Standard Forms: When constants are involved, the calculation of the primitive becomes fiddly. The standard integrals given in the HSC papers are:

14 **STANDARD FORMS WITH ONE CONSTANT:** For some constant C ,

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C \quad \text{or} \quad -\cos^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

PROOF:

A. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{a\sqrt{1 - (\frac{x}{a})^2}} dx$

$$= \int \frac{1}{\sqrt{1 - (\frac{x}{a})^2}} \times \frac{1}{a} dx$$

$$= \sin^{-1} \frac{x}{a} + C$$

B. $\int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a^2(1 + (\frac{x}{a})^2)} dx$

$$= \frac{1}{a} \int \frac{1}{1 + (\frac{x}{a})^2} \times \frac{1}{a} dx$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

Let $u = \frac{x}{a}$.
Then $\frac{du}{dx} = \frac{1}{a}$.
 $\int \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} dx = \sin^{-1} u$

Let $u = \frac{x}{a}$.
Then $\frac{du}{dx} = \frac{1}{a}$.
 $\int \frac{1}{1 + u^2} \frac{du}{dx} dx = \tan^{-1} u$

WORKED EXERCISE: Here are four indefinite integrals. In parts (a) and (b), the formulae can be applied immediately, but in parts (c) and (d), the coefficients of x^2 need to be taken out first.

(a) $\int \frac{2}{\sqrt{9 - x^2}} dx = 2 \sin^{-1} \frac{x}{3} + C$

(b) $\int \frac{1}{8 + x^2} dx = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x}{2\sqrt{2}} + C$

(c) $\int \frac{6}{49 + 25x^2} dx$

$$= \frac{6}{25} \int \frac{1}{\frac{49}{25} + x^2} dx$$

$$= \frac{6}{25} \times \frac{5}{7} \tan^{-1} \frac{x}{7/5} + C$$

$$= \frac{6}{35} \tan^{-1} \frac{5x}{7} + C$$

(d) $\int \frac{1}{\sqrt{5 - 3x^2}} dx$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\frac{5}{3} - x^2}} dx$$

$$= \frac{1}{3}\sqrt{3} \sin^{-1} x\sqrt{\frac{3}{5}} + C$$

Because manipulating the constants in parts (c) and (d) is still difficult, some prefer to remember these fuller versions of the standard forms:

15 **STANDARD FORMS WITH TWO CONSTANTS:**

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}} dx = \frac{1}{b} \sin^{-1} \frac{bx}{a} + C \quad \text{or} \quad -\frac{1}{b} \cos^{-1} \frac{bx}{a} + C$$

$$\int \frac{1}{a^2 + b^2x^2} dx = \frac{1}{ab} \tan^{-1} \frac{bx}{a} + C$$

These forms can be proven in the same manner as the forms with a single constant, or they can be developed from those forms in the same way as was done in parts (c) and (d) above (and they are proven by differentiation in the following exercise). With these more general forms, parts (c) and (d) can be written down without any intermediate working.

Reverse Chain Rule: In the usual way, the standard forms can be extended to give forms appropriate for the reverse chain rule.

16
 THE REVERSE CHAIN RULE:

$$\int \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} dx = \sin^{-1} u + C \quad \text{or} \quad -\cos^{-1} u + C$$

$$\int \frac{1}{1+u^2} \frac{du}{dx} dx = \tan^{-1} u + C$$

WORKED EXERCISE: Find a primitive of $\frac{x}{1+x^4}$.

SOLUTION:
$$\begin{aligned} \int \frac{x}{1+x^4} dx &= \frac{1}{2} \int \frac{2x}{1+x^4} dx \\ &= \frac{1}{2} \tan^{-1} x^2 + C, \text{ for some constant } C. \end{aligned}$$

Let $u = x^2$.
 Then $u' = 2x$.

$$\int \frac{1}{1+u^2} \frac{du}{dx} dx = \tan^{-1} u$$

Given a Derivative, Find an Integral: As always, the result of a product-rule differentiation can be used to obtain an integral. In particular, this allows the primitives of the inverse trigonometric functions to be obtained.

WORKED EXERCISE:

- (a) Differentiate $x \sin^{-1} x$, and hence find a primitive of $\sin^{-1} x$.
 (b) Find the shaded area under the curve $y = \sin^{-1} x$ from $x = 0$ to $x = 1$.

SOLUTION:

- (a) Let $y = x \sin^{-1} x$.

Using the product rule with $u = x$ and $v = \sin^{-1} x$,

$$\frac{dy}{dx} = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}.$$

Hence
$$\begin{aligned} \int \sin^{-1} x dx + \int \frac{x}{\sqrt{1-x^2}} dx &= x \sin^{-1} x \\ \int \sin^{-1} x dx &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx. \end{aligned}$$

Using the reverse chain rule,

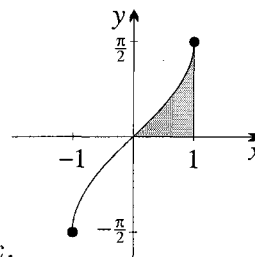
$$\begin{aligned} - \int \frac{x}{\sqrt{1-x^2}} dx &= \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx \\ &= \frac{1}{2} \times (1-x^2)^{\frac{1}{2}} \times \frac{2}{1} \\ &= \sqrt{1-x^2}, \end{aligned}$$

Let $u = 1 - x^2$.
 Then $\frac{du}{dx} = -2x$.

$$\int u^{-\frac{1}{2}} \frac{du}{dx} dx = u^{\frac{1}{2}} \times 2.$$

so
$$\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C,$$

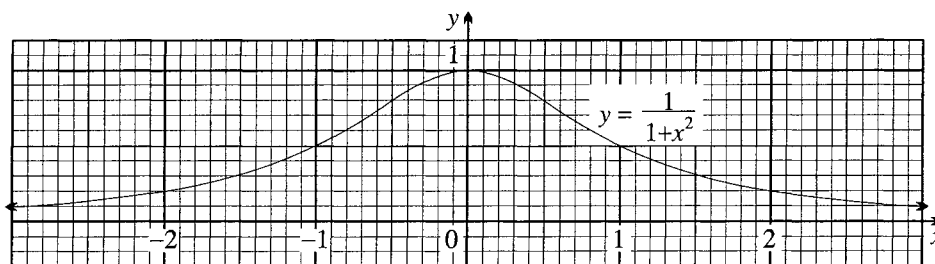
and
$$\begin{aligned} \int_0^1 \sin^{-1} x dx &= \left[x \sin^{-1} x + \sqrt{1-x^2} \right]_0^1 \\ &= (1 \times \frac{\pi}{2} + 0) - (0 + 1) \\ &= \frac{\pi}{2} - 1 \text{ square units.} \end{aligned}$$



NOTE: We have already established in Section 14I of the Year 11 volume that the area under $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ is exactly 1 square unit. This means that the area between $y = \sin^{-1} x$ and the y -axis is 1, and subtracting this area from the rectangle of area $\frac{\pi}{2}$ in the diagram above gives the same value $\frac{\pi}{2} - 1$ for the shaded area.

Exercise 1E

1. (a)



Find each of the following to two decimal places from the graph by counting the number of little squares in the region under the curve:

(i) $\int_0^1 \frac{1}{1+x^2} dx$

(iii) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1+x^2} dx$

(ii) $\int_0^2 \frac{1}{1+x^2} dx$

(iv) $\int_{-3}^{-1} \frac{1}{1+x^2} dx$

(b) Check your answers to (a) by using the fact that $\tan^{-1} x$ is a primitive of $\frac{1}{1+x^2}$.

2. Find:

(a) $\int \frac{-1}{\sqrt{1-x^2}} dx$

(c) $\int \frac{1}{9+x^2} dx$

(e) $\int \frac{1}{2+x^2} dx$

(b) $\int \frac{1}{\sqrt{4-x^2}} dx$

(d) $\int \frac{1}{\sqrt{\frac{4}{9}-x^2}} dx$

(f) $\int \frac{-1}{\sqrt{5-x^2}} dx$

3. Find the exact value of:

(a) $\int_0^3 \frac{1}{\sqrt{9-x^2}} dx$

(c) $\int_0^1 \frac{1}{\sqrt{2-x^2}} dx$

(e) $\int_{\frac{1}{6}\sqrt{3}}^{\frac{1}{6}} \frac{-1}{\sqrt{\frac{1}{9}-x^2}} dx$

(b) $\int_0^2 \frac{1}{4+x^2} dx$

(d) $\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\frac{1}{2}}{\frac{1}{4}+x^2} dx$

(f) $\int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\sqrt{\frac{9}{4}-x^2}} dx$

4. Find the equation of the curve, given that:

(a) $y' = (1-x^2)^{-\frac{1}{2}}$ and the curve passes through the point $(0, \pi)$.

(b) $y' = 4(16+x^2)^{-1}$ and the curve passes through the point $(-4, 0)$.

5. (a) If $y' = \frac{1}{\sqrt{36-x^2}}$ and $y = \frac{\pi}{6}$ when $x = 3$, find the value of y when $x = 3\sqrt{3}$.

(b) Given that $y' = \frac{2}{4+x^2}$ and that $y = \frac{\pi}{3}$ when $x = 2$, find y when $x = \frac{2}{\sqrt{3}}$.

DEVELOPMENT

6. Find:

(a) $\int \frac{1}{\sqrt{1-4x^2}} dx$

(c) $\int \frac{-1}{\sqrt{1-2x^2}} dx$

(e) $\int \frac{1}{25+9x^2} dx$

(b) $\int \frac{1}{1+16x^2} dx$

(d) $\int \frac{1}{\sqrt{4-9x^2}} dx$

(f) $\int \frac{-1}{\sqrt{3-4x^2}} dx$

7. Find the exact value of:

(a) $\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}} dx$

(c) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-3x^2}} dx$

(e) $\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2} dx$

(b) $\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1+4x^2} dx$

(d) $\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9-4x^2}} dx$

(f) $\int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5+2x^2} dx$

8. By differentiating each RHS, prove the extended standard forms with two constants given in Box 15 of the text:

(a) $\int \frac{1}{\sqrt{a^2-b^2x^2}} dx = \frac{1}{b} \sin^{-1} \frac{bx}{a} + C$

(b) $\int \frac{1}{a^2+b^2x^2} dx = \frac{1}{ab} \tan^{-1} \frac{bx}{a} + C$

9. (a) Shade the region bounded by $y = \sin^{-1} x$, the x -axis and the vertical line $x = \frac{1}{2}$.(b) Show that $\frac{d}{dx}(x \sin^{-1} x + \sqrt{1-x^2}) = \sin^{-1} x$.

(c) Hence find the exact area of the region.

10. (a) Shade the region bounded by the curve $y = \sin^{-1} x$, the y -axis and the line $y = \frac{\pi}{6}$.

(b) Find the exact area of this region.

(c) Hence use an alternative approach to confirm the area in the previous question.

11. (a) Show that $\frac{d}{dx}(\cos^{-1}(2-x)) = \frac{1}{\sqrt{4x-x^2-3}}$. (b) Hence find $\int_1^2 \frac{1}{\sqrt{4x-x^2-3}} dx$.12. (a) Differentiate $\tan^{-1} \frac{1}{2}x^3$. (b) Hence find $\int \frac{x^2}{4+x^6} dx$.13. (a) The portion of the curve $y = \frac{1}{\sqrt{7+x^2}}$ from $x = 0$ to $x = \sqrt{7}$ is rotated about the x -axis through a complete revolution. Find exactly the volume generated.(b) Find the volume of the solid formed when the region between $y = (1-16x^2)^{-\frac{1}{4}}$ and the x -axis from $x = -\frac{1}{8}$ to $x = \frac{1}{8}\sqrt{3}$ is rotated about the x -axis.14. (a) Show that $x^2+6x+10 = (x+3)^2+1$. (b) Hence find $\int \frac{1}{x^2+6x+10} dx$.15. (a) Differentiate $x \tan^{-1} x$. (b) Hence find $\int_0^1 \tan^{-1} x dx$.

16. Without finding any primitives, use symmetry arguments to evaluate:

(a) $\int_{-\frac{1}{3}}^{\frac{1}{3}} \sin^{-1} x dx$

(c) $\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x dx$

(e) $\int_{-3}^3 \frac{x}{1+x^2} dx$

(b) $\int_{-5}^5 \tan^{-1} x dx$

(d) $\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{x}{\sqrt{1-x^2}} dx$

(f) $\int_{-6}^6 \sqrt{36-x^2} dx$

17. (a) Given that $f(x) = \frac{x}{1+x^2} - \tan^{-1} x$: (i) find $f(0)$, (ii) show that $f'(x) = \frac{-2x^2}{(1+x^2)^2}$.

(b) Hence: (i) explain why $f(x) < 0$ for all $x > 0$, (ii) find $\int_0^1 \frac{x^2}{(1+x^2)^2} dx$.

18. Consider the function $f(x) = \frac{1}{\sqrt{4-x^2}}$.

- (a) Sketch the graph of $y = \sqrt{4-x^2}$. (b) Hence sketch the graph of $y = f(x)$.
 (c) Write down the domain and range of $f(x)$, and describe its symmetry.
 (d) Find the area between the curve and the x -axis from $x = -1$ to $x = 1$.
 (e) Find the total area between the curve and the x -axis. [NOTE: This is an example of an unbounded region having a finite area.]

19. Consider the function $f(x) = \frac{4}{x^2+4}$.

- (a) What is the axis of symmetry of $y = f(x)$? (b) What are the domain and range?
 (c) Show that the graph of $f(x)$ has a maximum turning point at $(0, 1)$.
 (d) Find $\lim_{x \rightarrow \infty} f(x)$, and hence sketch $y = f(x)$. On the same axis, sketch $y = \frac{1}{4}(x^2 + 4)$.
 (e) Calculate the area bounded by the curve and the x -axis from $x = -2\sqrt{3}$ to $x = \frac{2}{3}\sqrt{3}$.
 (f) Find the exact area between the curve and the x -axis from $x = -a$ to $x = a$, where a is a positive constant.
 (g) By letting a tend to infinity, find the total area between the curve and the x -axis. [NOTE: This is another example of an unbounded region having a finite area.]

20. Show that $\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2} dx = \frac{\pi}{4}$.

21. (a) Show that $\frac{d}{dx} (\tan^{-1}(\frac{3}{2} \tan x)) = \frac{6}{5 \sin^2 x + 4}$.

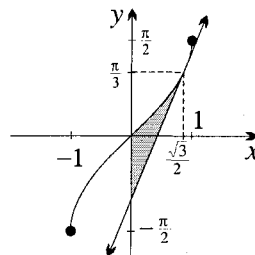
(b) Hence find, correct to three significant figures, the area bounded by the curve $y = \frac{1}{5 \sin^2 x + 4}$ and the x -axis from $x = 0$ to $x = 7$.

22. (a) Use Simpson's rule with five points to approximate $I = \int_0^1 \frac{1}{1+x^2} dx$, expressing your answer in simplest fraction form.

(b) Find the exact value of I , and hence show that $\pi \div \frac{8011}{2550}$. To how many decimal places is this approximation accurate?

23. The diagram shows the region bounded by $y = \sin^{-1} x$, the y -axis and the tangent to the curve at the point $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$.

- (a) Show that the area of the region is $\frac{1}{4}$ unit².
 (b) Show that the volume of the solid formed when the region is rotated about the y -axis is $\frac{\pi}{24}(9\sqrt{3} - 4\pi)$ unit³.



24. Find, using the reverse chain rule:

(a) $\int \frac{1}{\sqrt{x}(1+x)} dx$

(b) $\int_0^1 \frac{1}{e^{-x} + e^x} dx$

EXTENSION

25. [The power series for $\tan^{-1} x$] Suppose that x is a positive real number.

- (a) Find the sum of the geometric series $1 - t^2 + t^4 - t^6 + \dots + t^{4n}$, and hence show that for $0 < t < x$,

$$\frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \dots + t^{4n}.$$

- (b) Find $1 - t^2 + t^4 - t^6 + \dots + t^{4n} - t^{4n+2}$, and hence show that for $0 < t < x$,

$$1 - t^2 + t^4 - t^6 + \dots + t^{4n} < \frac{1}{1+t^2} + t^{4n+2}.$$

- (c) By integrating the inequalities of parts (a) and (b) from $t = 0$ to $t = x$, show that

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3}.$$

- (d) By taking limits as $n \rightarrow \infty$, show that for $0 \leq x \leq 1$,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots.$$

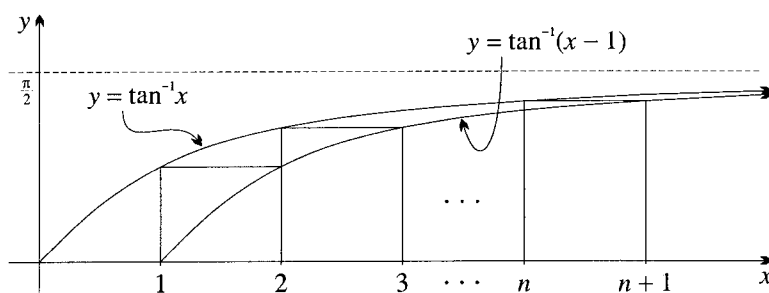
- (e) Use the fact that $\tan^{-1} x$ is an odd function to prove this identity for $-1 \leq x < 0$.

- (f) [Gregory's series] Use a suitable substitution to prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

- (g) By combining the terms in pairs, show that $\frac{\pi}{8} = \frac{1}{1 \times 3} + \frac{1}{5 \times 7} + \frac{1}{9 \times 11} + \dots$, and use the calculator to find how close an approximation to π can be obtained by taking 10 terms.

26. [A sandwiching argument]



In the diagram, n rectangles are constructed between the two curves $y = \tan^{-1} x$ and $y = \tan^{-1}(x-1)$ in the interval $1 \leq x \leq n+1$.

- (a) Write down an expression for S_n , the sum of the areas of the n rectangles.
 (b) Differentiate $x \tan^{-1} x$ and hence find a primitive of $\tan^{-1} x$.
 (c) Show that for all $n \geq 1$,

$$n \tan^{-1} n - \frac{1}{2} \ln(n^2 + 1) < S_n < (n+1) \tan^{-1}(n+1) - \frac{1}{2} \ln\left(\frac{n^2}{2} + n + 1\right) - \frac{\pi}{4}$$

- (d) Deduce that $1562 < \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 + \dots + \tan^{-1} 1000 < 1565$.

1F General Solutions of Trigonometric Equations

Using the inverse trigonometric functions, we can write down general solutions to trigonometric equations of the type $\sin x = a$ and $\sin x = \sin \alpha$.

Solving Trigonometric Equations Without Restrictions: Because each of the trigonometric functions is periodic, any unrestricted trigonometric equation that has one solution must have infinitely many. This section will later develop formulae for those general solutions, but they can always be found using the methods already established, as is demonstrated in the following worked exercise. The key to general solutions is provided by the periods of the trigonometric functions:

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PERIODS OF THE TRIGONOMETRIC FUNCTIONS:

$\sin x$ and $\cos x$ have period 2π , $\tan x$ has period π .

WORKED EXERCISE: Find the general solution, in radians, of:

(a) $\cos x = \frac{1}{2}$

(b) $\tan x = 1$

(c) $\sin x = \frac{1}{2}\sqrt{3}$

SOLUTION:

- (a) Since $\cos x$ is positive, x must be in the 1st or 4th quadrants.

Also, the related acute angle is $\frac{\pi}{3}$.

Hence $x = \frac{\pi}{3}$ and $x = -\frac{\pi}{3}$ are the solutions within a revolution.

Since $\cos x$ has period 2π , the general solution is

$$x = \frac{\pi}{3} + 2n\pi \text{ or } -\frac{\pi}{3} + 2n\pi, \text{ where } n \text{ is an integer.}$$

- (b) Since $\tan x$ is positive, x must be in the 1st or 3rd quadrants.

Also, the related angle is $\frac{\pi}{4}$.

Hence $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ are the solutions within a revolution.

Since $\tan x$ has period π , the general solution is

$$x = \frac{\pi}{4} + n\pi, \text{ where } n \text{ is an integer.}$$

(Notice that this includes the other solution $x = \frac{5\pi}{4}$, which is obtained by putting $n = 1$.)

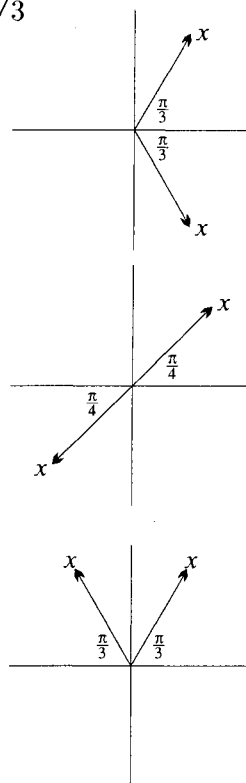
- (c) Since $\sin x$ is positive, x must be in the 1st or 2nd quadrants.

Also, the related acute angle is $\frac{\pi}{3}$.

Hence $x = \frac{\pi}{3}$ and $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ are both solutions.

Since $\sin x$ has period 2π , the general solution is

$$x = \frac{\pi}{3} + 2n\pi \text{ or } \frac{2\pi}{3} + 2n\pi, \text{ where } n \text{ is an integer.}$$



The Equation $\cos x = a$: More generally, suppose that $\cos x = a$, where $-1 \leq a \leq 1$.

First, $x = \cos^{-1} a$ is a solution.

Secondly, $x = -\cos^{-1} a$ is a solution, because $\cos x$ is an even function.

This gives two solutions within a revolution, so the general solution is

$x = \cos^{-1} a + 2n\pi$ or $x = -\cos^{-1} a + 2n\pi$, where n is an integer.

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THE GENERAL SOLUTION OF $\cos x = a$: The general solution of $\cos x = a$ is

$$x = \cos^{-1} a + 2n\pi \text{ or } x = -\cos^{-1} a + 2n\pi, \text{ where } n \text{ is an integer.}$$

The Equation $\tan x = a$: Suppose that $\tan x = a$, where a is a constant.

One solution is $x = \tan^{-1} a$.

But $\tan x$ has period π , and only one solution within each period, so the general solution is $x = \tan^{-1} a + n\pi$, where n is an integer.

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THE GENERAL SOLUTION OF $\tan x = a$: The general solution of $\tan x = a$ is

$$x = \tan^{-1} a + n\pi, \text{ where } n \text{ is an integer.}$$

The Equation $\sin x = a$: Suppose that $\sin x = a$, where $-1 \leq a \leq 1$.

First, $x = \sin^{-1} a$ is a solution.

Also, if $\sin \theta = a$, then $\sin(\pi - \theta) = a$, so $x = \pi - \sin^{-1} a$ is a solution.

This gives two solutions within each revolution, so the general solution is $x = \sin^{-1} a + 2n\pi$ or $x = (\pi - \sin^{-1} a) + 2n\pi$, where n is an integer.

[Alternatively, we can write $x = 2n\pi + \sin^{-1} a$ or $x = (2n + 1)\pi - \sin^{-1} a$.

The first can be written as $x = m\pi + \sin^{-1} a$, where m is even,

and the second can be written as $x = m\pi - \sin^{-1} a$, where m is odd.

Using the switch $(-1)^m$, which changes sign according as m is even or odd,

we can write both families together as $x = (-1)^m \sin^{-1} a + m\pi$, where m is an integer.]

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THE GENERAL SOLUTION OF $\sin x = a$: The general solution of $\sin x = a$ is

$$x = \sin^{-1} a + 2n\pi \quad \text{or} \quad x = (\pi - \sin^{-1} a) + 2n\pi, \text{ where } n \text{ is an integer.}$$

[Alternatively, we can write these two families together using the switch $(-1)^m$:

$$x = (-1)^m \sin^{-1} a + m\pi, \text{ where } m \text{ is an integer.}]$$

NOTE: The alternative notation for solving $\sin^{-1} x = a$ is very elegant, and is very quick if properly applied, but it is not at all easy to use or to remember. In this text, we will enclose it in square brackets when it is used.

WORKED EXERCISE: Use these formulae to find the general solution of:

(a) $\cos x = -\frac{1}{2}$

(b) $\sin x = \frac{1}{2}\sqrt{2}$

(c) $\tan x = -2$

SOLUTION:

(a) $x = \cos^{-1}(-\frac{1}{2}) + 2n\pi$ or $-\cos^{-1}(-\frac{1}{2}) + 2n\pi$, where n is an integer,
 $= \frac{2\pi}{3} + 2n\pi$ or $-\frac{2\pi}{3} + 2n\pi$.

(b) $x = \sin^{-1} \frac{1}{2}\sqrt{2} + 2n\pi$ or $(\pi - \sin^{-1} \frac{1}{2}\sqrt{2}) + 2n\pi$, where n is an integer,
 $= \frac{\pi}{4} + 2n\pi$ or $\frac{3\pi}{4} + 2n\pi$.

[Alternatively, $x = (-1)^m \frac{\pi}{4} + m\pi$, where m is an integer.]

(c) $x = \tan^{-1}(-2) + n\pi$, where n is an integer,
 $= -\tan^{-1} 2 + n\pi$, which can be approximated if required.

The Equations $\sin x = \sin \alpha$, $\cos x = \cos \alpha$ and $\tan x = \tan \alpha$: Using similar methods, the general solutions of these three equations can be written down.

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GENERAL SOLUTIONS OF $\sin x = \sin \alpha$, $\cos x = \cos \alpha$ and $\tan x = \tan \alpha$:

The general solution of $\cos x = \cos \alpha$ is

$$x = \alpha + 2n\pi \quad \text{or} \quad x = -\alpha + 2n\pi, \text{ where } n \text{ is an integer.}$$

The general solution of $\tan x = \tan \alpha$ is

$$x = \alpha + n\pi, \text{ where } n \text{ is an integer.}$$

The general solution of $\sin x = \sin \alpha$ is

$$x = \alpha + 2n\pi \quad \text{or} \quad x = (\pi - \alpha) + 2n\pi, \text{ where } n \text{ is an integer.}$$

[Alternatively, $x = m\pi + (-1)^m \alpha$, where m is an integer.]

PROOF:

A. One solution of $\cos x = \cos \alpha$ is $x = \alpha$.

Also, $\cos \alpha = \cos(-\alpha)$, since cosine is even, so $x = -\alpha$ is also a solution.

This gives the required two solutions within a single period of 2π ,

so the general solution is $x = \alpha + 2n\pi$ or $x = -\alpha + 2n\pi$, where n is an integer.

B. One solution of $\tan x = \tan \alpha$ is $x = \alpha$.

This gives the required one solution within a single period of π ,

so the general solution is $x = \alpha + n\pi$, where n is an integer.

C. One solution of $\sin x = \sin \alpha$ is $x = \alpha$.

Also, $\sin \alpha = \sin(\pi - \alpha)$, so $x = \pi - \alpha$ is also a solution.

This gives the required two solutions within a single period of 2π ,

so the general solution is $x = \alpha + 2n\pi$ or $x = (\pi - \alpha) + 2n\pi$, where n is an integer.

WORKED EXERCISE: Use these formulae to find the general solution of $\sin x = \sin \frac{\pi}{5}$.

SOLUTION: $x = \frac{\pi}{5} + 2n\pi$ or $x = (\pi - \frac{\pi}{5}) + 2n\pi$, where n is an integer,
 $x = \frac{\pi}{5} + 2n\pi$ or $x = \frac{4\pi}{5} + 2n\pi$.

[Alternatively, $x = (-1)^m \frac{\pi}{5} + m\pi$, where n is an integer.]

WORKED EXERCISE: Solve: (a) $\cos 4x = \cos x$ (b) $\sin 4x = \cos x$

SOLUTION:

(a) Using the general solution of $\cos x = \cos \alpha$ from Box 21,

$$4x = x + 2n\pi \quad \text{or} \quad 4x = -x + 2n\pi, \text{ where } n \in \mathbf{Z},$$

$$3x = 2n\pi \quad \text{or} \quad 5x = 2n\pi, \text{ where } n \in \mathbf{Z},$$

$$x = \frac{2}{3}n\pi \quad \text{or} \quad x = \frac{2}{5}n\pi, \text{ where } n \in \mathbf{Z}.$$

(b) First, $\sin 4x = \sin(\frac{\pi}{2} - x)$, using the identity $\cos x = \sin(\frac{\pi}{2} - x)$.

Hence, using the general solution of $\sin x = \sin \alpha$ from Box 21,

$$4x = \frac{\pi}{2} - x + 2n\pi \quad \text{or} \quad 4x = \pi - (\frac{\pi}{2} - x) + 2n\pi, \text{ where } n \in \mathbf{Z},$$

$$5x = (2n + \frac{1}{2})\pi \quad \text{or} \quad 4x = x + \frac{\pi}{2} + 2n\pi, \text{ where } n \in \mathbf{Z},$$

$$x = (4n + 1)\frac{\pi}{10} \quad \text{or} \quad 3x = (2n + \frac{1}{2})\pi, \text{ where } n \in \mathbf{Z},$$

$$x = (4n + 1)\frac{\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

Exercise 1F

- Consider the equation $\tan x = 1$.
 - Draw a diagram showing x in its two possible quadrants, and show the related angle.
 - Write down the first six positive solutions.
 - Write down the first six negative solutions.
 - Carefully observe that each of these twelve solutions can be written as an integer multiple of π plus $\frac{\pi}{4}$, and hence write down a general solution of $\tan x = 1$.
 - Sketch the graphs of $y = \tan x$ (for $-2\pi \leq x \leq 2\pi$) and $y = 1$ on the same diagram and show as many of the above solutions as possible.
- Consider the equation $\cos x = \frac{1}{2}$.
 - Draw a diagram showing x in its two possible quadrants, and show the related angle.
 - Write down the first six positive solutions.
 - Write down the first six negative solutions.
 - Carefully observe that each of these twelve solutions can be written either as an integer multiple of 2π plus $\frac{\pi}{3}$ or as an integer multiple of 2π minus $\frac{\pi}{3}$, and hence write down a general solution of $\cos x = \frac{1}{2}$.
 - Sketch the graphs of $y = \cos x$ (for $-2\pi \leq x \leq 2\pi$) and $y = \frac{1}{2}$ on the same diagram and show as many of the above solutions as possible.
- Consider the equation $\sin x = \frac{1}{2}$.
 - Draw a diagram showing x in its two possible quadrants, and show the related angle.
 - Write down the first six positive solutions.
 - Write down the first six negative solutions.
 - Carefully observe that each of these twelve solutions can be written either as a multiple of 2π plus $\frac{\pi}{6}$ or as a multiple of 2π plus $\frac{5\pi}{6}$, and hence write down a general solution of $\sin x = \frac{1}{2}$.
 - Sketch the graphs of $y = \sin x$ (for $-2\pi \leq x \leq 2\pi$) and $y = \frac{1}{2}$ on the same diagram and show as many of the above solutions as possible.
- Write down a general solution of:

(a) $\tan x = \sqrt{3}$	(c) $\sin x = \frac{1}{2}\sqrt{3}$	(e) $\cos x = -\frac{1}{2}$
(b) $\cos x = \frac{1}{2}\sqrt{2}$	(d) $\tan x = -1$	(f) $\sin x = -\frac{1}{2}$
- Write down a general solution of:

(a) $\cos \theta = \cos \frac{\pi}{6}$	(c) $\sin \theta = \sin \frac{\pi}{5}$	(e) $\tan \theta = \tan(-\frac{\pi}{3})$
(b) $\tan \theta = \tan \frac{\pi}{4}$	(d) $\sin \theta = \sin \frac{4\pi}{3}$	(f) $\cos \theta = \cos \frac{5\pi}{6}$
- Write down a general solution for each of the following by referring to the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.

(a) $\sin x = 0$	(c) $\tan x = 0$	(e) $\sin x = 1$
(b) $\cos x = 1$	(d) $\cos x = 0$	(f) $\sin x = -1$

DEVELOPMENT

- In each case: (i) find a general solution, (ii) write down all solutions in $-\pi \leq x \leq \pi$.

(a) $\cos 2x = 1$	(e) $\cos(x + \frac{\pi}{6}) = -\frac{1}{2}\sqrt{2}$	(i) $\tan 4x = \tan \frac{\pi}{3}$
(b) $\sin \frac{1}{2}x = \frac{1}{2}\sqrt{2}$	(f) $\tan(2x - \frac{\pi}{6}) = -\sqrt{3}$	(j) $\tan(x + \frac{\pi}{4}) = \tan \frac{5\pi}{8}$
(c) $\tan 3x = \frac{1}{3}\sqrt{3}$	(g) $\cos 2x = \cos \frac{\pi}{5}$	(k) $\cos(x - \frac{\pi}{7}) = \cos \frac{4\pi}{7}$
(d) $\sin(x - \frac{\pi}{4}) = 0$	(h) $\sin 3x = \sin \frac{\pi}{2}$	(l) $\sin(2x + \frac{3\pi}{10}) = \sin(-\frac{\pi}{10})$

8. In each case: (i) find a general solution, (ii) write down all solutions in $-\pi \leq \theta \leq \pi$.
- (a) $\sin^2 \theta + \sin \theta = 0$ (c) $\cot^2(\theta - \frac{\pi}{6}) = 3$ (e) $\sin 2\theta + \sqrt{3} \cos 2\theta = 0$
 (b) $\sin 2\theta = \cos \theta$ (d) $2\sin^2 \theta = 3 + 3\cos \theta$ (f) $\sec^2 2\theta = 1 + \tan 2\theta$
9. Consider the equation $\tan 4x = \tan x$.
- (a) Show that $4x = n\pi + x$. (b) Hence show that $x = \frac{n\pi}{3}$, where $n \in \mathbf{Z}$.
 (c) Hence write down all solutions in the domain $0 \leq x \leq 2\pi$.
10. Consider the equation $\sin 3x = \sin x$.
- (a) Show that $3x = x + 2n\pi$ or $3x = (\pi - x) + 2n\pi$. Hence show that $x = n\pi$ or $x = (2n + 1)\frac{\pi}{4}$. [Alternatively, show that $3x = n\pi + (-1)^n x$, and hence show that if n is even, $x = \frac{n\pi}{2}$, and if n is odd, $x = \frac{n\pi}{4}$.]
 (b) Hence write down all solutions in the domain $0 \leq x \leq 2\pi$.
11. Consider the equation $\cos 3x = \sin x$.
- (a) Show that $3x = 2n\pi + (\frac{\pi}{2} - x)$ or $2n\pi - (\frac{\pi}{2} - x)$.
 (b) Hence show that $x = n\pi - \frac{\pi}{4}$ or $\frac{n\pi}{2} + \frac{\pi}{8}$, where $n \in \mathbf{Z}$.
 (c) Hence write down all solutions in the domain $0 \leq x \leq 2\pi$.
12. Using methods similar to those in the previous two questions, solve for $0 \leq x \leq \pi$:
- (a) $\sin 5x = \sin x$ (c) $\sin 5x = \cos x$
 (b) $\cos 5x = \cos x$ (d) $\cos 5x = \sin x$

EXTENSION

13. Sketch the graphs of the following relations:
- (a) $\cos y = \cos x$ (c) $\cos y = \sin x$ (e) $\cot y = \tan x$
 (b) $\sin y = \sin x$ (d) $\tan y = \tan x$ (f) $\sec y = \sec x$
- Which graphs are symmetric in the x -axis, which in the y -axis, and which in $y = x$?

CHAPTER TWO

Further Trigonometry

We have now established the basic calculus of the trigonometric functions and their inverse functions. Along the way, there has been much work on trigonometric equations, and on the application of trigonometry to problems in two dimensions. This chapter will give a systematic account of trigonometric identities and equations, and then extend the applications of trigonometry to problems in three dimensions. Much of this material will be used when the methods of calculus are consolidated and developed further in Chapter Three on motion and in Chapter Six on further calculus.

Although the sine and cosine waves are not so prominent here, it is important to keep in mind that they are the impulse for most of the trigonometry in this course. Remember that the tangent and cotangent functions are the ratios of the heights of the two waves, and that the secant and cosecant functions arise when these tangent and cotangent functions are differentiated.

STUDY NOTES: Trigonometric identities and equations are closely linked, because the solution of trigonometric equations so often comes down to the application of some identity. Sections 2A–2C deal systematically with identities, with particular emphasis on compound angles. Section 2D applies these identities to the solutions of trigonometric equations. Section 2E deals with the sum of sine and cosine waves in preparation for simple harmonic motion in Chapter Three. Section 2F is an extension on some 4 Unit identities called *sums to products* and *products to sums* that are best studied in the context of this chapter by those taking the 4 Unit course. Finally, Sections 2G and 2H develop the application of trigonometry to problems in three-dimensional space — they require the new ideas of the angle between a line and a plane, and the angle between two planes.

2A Trigonometric Identities

Developing fluency in trigonometric identities is the purpose of the first three sections. Most of the identities have been established already, and are listed again here for reference. But the triple-angle formulae in this section are new (although it is not intended that they be memorised), and so are the t -formulae in the next section.

Identities Relating the Six Trigonometric Functions: Four groups of identities relating the six trigonometric functions were developed in Chapter Four of the Year 11 volume, and are listed here for reference.

1

THE RECIPROCAL IDENTITIES:

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

THE RATIO IDENTITIES:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

THE PYTHAGOREAN IDENTITIES:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

THE COMPLEMENTARY IDENTITIES:

$$\cos(90^\circ - \theta) = \sin \theta$$

$$\cot(90^\circ - \theta) = \tan \theta$$

$$\operatorname{cosec}(90^\circ - \theta) = \sec \theta$$

Each of these identities holds provided both LHS and RHS are well defined.

WORKED EXERCISE: Show that $\tan \theta + \cot \theta = \sec \theta \operatorname{cosec} \theta$.

SOLUTION: LHS = $\tan \theta + \cot \theta$

$$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}, \text{ using the ratio identities,}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}, \text{ using a common denominator,}$$

$$= 1 \times \operatorname{cosec} \theta \sec \theta, \text{ using the Pythagorean and reciprocal identities,}$$

$$= \text{RHS, as required.}$$

The Compound-Angle Formulae: These formulae were developed in Chapter Fourteen of the Year 11 volume.

2

THE COMPOUND-ANGLE FORMULAE:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

WORKED EXERCISE:

(a) Find $\tan 75^\circ$. (b) Use small-angle theory to approximate $\sin 61^\circ$.

SOLUTION:

$$\begin{aligned} \text{(a) } \tan 75^\circ &= \tan(45^\circ + 30^\circ) \\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \\ &= \frac{1}{2}(4 + 2\sqrt{3}) \\ &= 2 + \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{(b) } \sin 61^\circ &= \sin(60^\circ + 1^\circ) \\ &= \sin 60^\circ \cos 1^\circ + \cos 60^\circ \sin 1^\circ \end{aligned}$$

For small angles, $\cos \theta \doteq 1$,
and $\sin \theta \doteq \theta$, where θ is in radians.
Since $1^\circ = \frac{\pi}{180}$ radians,

$$\sin 61^\circ \doteq \frac{1}{2}\sqrt{3} \times 1 + \frac{1}{2} \times \frac{\pi}{180}$$

$$\doteq \frac{1}{360}(180\sqrt{3} + \pi).$$

Double-Angle Formulae: These formulae are reviewed from Chapter Fourteen of the Year 11 volume — they follow immediately from the compound-angle formulae by setting α and β equal to θ . There are three forms of the $\cos 2\theta$ formula because $\sin^2 \theta$ and $\cos^2 \theta$ are easily related to each other by the Pythagorean identities.

3

THE DOUBLE-ANGLE FORMULAE:

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta & \cos 2\theta &= \cos^2 \theta - \sin^2 \theta & \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ & & &= 2 \cos^2 \theta - 1 \\ & & &= 1 - 2 \sin^2 \theta\end{aligned}$$

Expressing $\sin^2 \theta$ and $\cos^2 \theta$ in terms of $\cos 2\theta$: The second and third forms of the $\cos 2\theta$ formula above are important because they allow the squares $\sin^2 \theta$ and $\cos^2 \theta$ to be expressed in terms of the simple trigonometric function $\cos 2\theta$.

From $\cos 2\theta = 2 \cos^2 \theta - 1$,

$$2 \cos^2 \theta = 1 + \cos 2\theta$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta.$$

From $\cos 2\theta = 1 - 2 \sin^2 \theta$,

$$2 \sin^2 \theta = 1 - \cos 2\theta$$

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta.$$

4

EXPRESSING $\sin^2 \theta$ AND $\cos^2 \theta$ IN TERMS OF $\cos 2\theta$:

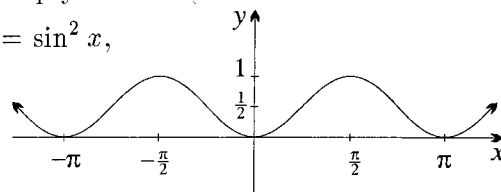
$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \qquad \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

Notice that $\cos^2 \theta + \sin^2 \theta = (\frac{1}{2} + \frac{1}{2} \cos 2\theta) + (\frac{1}{2} - \frac{1}{2} \cos 2\theta) = 1$, in accordance with the Pythagorean identities. This observation may help you to memorise them.

WORKED EXERCISE: Without using calculus, sketch $y = \sin^2 x$, and state its amplitude, period and range.

SOLUTION: Using the identities above,

$$y = \frac{1}{2} - \frac{1}{2} \cos 2x.$$



This is the graph of $y = \cos 2x$ turned upside down, then stretched vertically by the factor $\frac{1}{2}$, then shifted up $\frac{1}{2}$. Its period is π , and its amplitude is $\frac{1}{2}$. Since it oscillates around $\frac{1}{2}$ rather than 0, its range is $0 \leq y \leq 1$.

The Triple-Angle Formulae: Memorisation of triple-angle formulae is not required in the course, but their proof and their application can reasonably be required. Here are the three formulae, followed by the proof of the $\sin 3\theta$ formula — the proofs of the other two are left to the following exercise.

5

THE TRIPLE-ANGLE FORMULAE:

(Memorisation is not required.)

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

PROOF OF THE FORMULA FOR $\sin 3\theta$:

$$\sin 3\theta = \sin(2\theta + \theta)$$

$$= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta, \text{ using the formula for } \sin(\alpha + \beta),$$

$$= 2 \sin \theta \cos^2 \theta + (1 - 2 \sin^2 \theta) \sin \theta, \text{ using the double-angle formulae,}$$

$$= 2 \sin \theta (1 - \sin^2 \theta) + (1 - 2 \sin^2 \theta) \sin \theta, \text{ since } \cos^2 \theta = 1 - \sin^2 \theta,$$

$$= 3 \sin \theta - 4 \sin^3 \theta, \text{ after expanding and collecting terms.}$$

Exercise 2A

1. Simplify, using the compound-angle results:

- (a) $\cos 3\theta \cos \theta + \sin 3\theta \sin \theta$ (d) $\cos 15^\circ \cos 55^\circ - \sin 15^\circ \sin 55^\circ$
 (b) $\sin 50^\circ \cos 10^\circ - \cos 50^\circ \sin 10^\circ$ (e) $\sin 4\alpha \cos 2\alpha + \cos 4\alpha \sin 2\alpha$
 (c) $\frac{\tan 41^\circ + \tan 9^\circ}{1 - \tan 41^\circ \tan 9^\circ}$ (f) $\frac{1 + \tan 2\theta \tan \theta}{\tan 2\theta - \tan \theta}$

2. Simplify, using the double-angle results:

- (a) $2 \sin 2\theta \cos 2\theta$ (c) $2 \cos^2 3\alpha - 1$ (e) $1 - 2 \sin^2 25^\circ$
 (b) $\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x$ (d) $\frac{2 \tan 35^\circ}{1 - \tan^2 35^\circ}$ (f) $\frac{2 \tan 4x}{1 - \tan^2 4x}$

3. Given that the angles A and B are acute, and that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find:

- (a) $\cos A$ (c) $\cos(A + B)$ (e) $\tan 2A$
 (b) $\cos 2A$ (d) $\sin 2B$ (f) $\tan(B - A)$

4. (a) By writing 75° as $45^\circ + 30^\circ$, show that:

- (i) $\sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ (ii) $\cos 75^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$
 (b) Hence show that:
 (i) $\sin 75^\circ \cos 75^\circ = \frac{1}{4}$ (iii) $\sin^2 75^\circ - \cos^2 75^\circ = \sin 60^\circ$
 (ii) $\sin 75^\circ - \cos 75^\circ = \sin 45^\circ$ (iv) $\sin^2 75^\circ + \cos^2 75^\circ = 1$

5. Use the compound-angle and double-angle results to find the exact value of:

- (a) $2 \sin 15^\circ \cos 15^\circ$ (g) $2 \cos^2 \frac{7\pi}{12} - 1$
 (b) $\cos 35^\circ \cos 5^\circ + \sin 35^\circ \sin 5^\circ$ (h) $\frac{\tan \frac{25\pi}{18} - \tan \frac{\pi}{18}}{1 + \tan \frac{25\pi}{18} \tan \frac{\pi}{18}}$
 (c) $\frac{\tan 110^\circ + \tan 25^\circ}{1 - \tan 110^\circ \tan 25^\circ}$ (i) $\frac{\sin 105^\circ \cos 105^\circ}{\cos^2 67\frac{1}{2}^\circ - \sin^2 67\frac{1}{2}^\circ}$
 (d) $1 - 2 \sin^2 22\frac{1}{2}^\circ$ (j) $\frac{2 \cos^2 \frac{2\pi}{5} - 1}{1 - 2 \sin^2 \frac{\pi}{10}}$
 (e) $\cos \frac{\pi}{12} \sin \frac{\pi}{12}$
 (f) $\sin \frac{8\pi}{9} \cos \frac{2\pi}{9} - \cos \frac{8\pi}{9} \sin \frac{2\pi}{9}$

6. Simplify the following using the three double-angle results $\sin A \cos A = \frac{1}{2} \sin 2A$, $1 - \cos 2A = 2 \sin^2 A$ and $1 + \cos 2A = 2 \cos^2 A$:

- (a) $\sin \frac{\theta}{2} \cos \frac{\theta}{2}$ (c) $\frac{1}{2}(1 + \cos 4x)$ (e) $\sqrt{\frac{1}{2}(1 + \cos 40^\circ)}$ (g) $\sqrt{\frac{1}{2}(1 - \cos 10x)}$
 (b) $\frac{1}{2}(1 - \cos 2x)$ (d) $1 - \cos 6\theta$ (f) $1 + \cos \alpha$ (h) $\sin^2 \alpha \cos^2 \alpha$

7. Suppose that θ is an acute angle and $\cos \theta = \frac{4}{5}$. Using the results $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, find the exact value of:

- (a) $\cos \frac{1}{2}\theta$ (b) $\sin \frac{1}{2}\theta$ (c) $\tan \frac{1}{2}\theta$

8. Prove each of the following identities:

- (a) $(\sin \alpha - \cos \alpha)^2 = 1 - \sin 2\alpha$ (g) $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta$
 (b) $\cos^4 x - \sin^4 x = \cos 2x$ (h) $\frac{\sin 2x}{1 + \cos 2x} = \tan x$
 (c) $\cos A - \sin 2A \sin A = \cos A \cos 2A$ (i) $\frac{1 - \cos 2\alpha}{1 + \cos 2\alpha} = \tan^2 \alpha$
 (d) $\sin 2\theta(\tan \theta + \cot \theta) = 2$ (j) $\tan 2A(\cot A - \tan A) = 2$,
 (e) $\cot \alpha \sin 2\alpha - \cos 2\alpha = 1$ (provided $\cot A \neq \tan A$)
 (f) $\frac{1}{1 - \tan A} - \frac{1}{1 + \tan A} = \tan 2A$

DEVELOPMENT

9. (a) If $\sin \theta = \frac{1}{3}$ and $\cos \theta < 0$, find the exact value of $\tan 2\theta$.
(b) If $\frac{3\pi}{2} < \theta < 2\pi$ and $\cos \theta = \frac{31}{50}$, find $\cos \frac{\theta}{2}$.
10. (a) By writing 3θ as $2\theta + \theta$ and using appropriate compound-angle and double-angle results, prove that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$.
(b) Hence show that $\cos 40^\circ$ is a root of the equation $8x^3 - 6x + 1 = 0$.
(c) Show also that $\cos 3\theta = \frac{5}{9}\sqrt{3}$, if $\tan \theta = \sqrt{2}$ and $\pi < \theta < \frac{3\pi}{2}$.
11. (a) Show that $\sin 3x = 3\sin x - 4\sin^3 x$.
(b) Use the identities for $\cos 3x$ (see the previous question) and $\sin 3x$ to show that

$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

- 12.** If θ is acute and $\cos \theta = \frac{2}{3}$, find the exact value of:

- (a) $\sin \theta$ (c) $\sin 2\theta$ (e) $\sin 4\theta$ (g) $\tan 3\theta$ (i) $\cos \frac{1}{2}\theta$
 (b) $\cos 2\theta$ (d) $\sin 3\theta$ (f) $\cos 4\theta$ (h) $\tan 4\theta$ (j) $\tan \frac{1}{2}\theta$

- 13.** Prove each of the following identities:

- $$\begin{array}{ll}
 \text{(a)} \quad \cot 2\alpha + \tan \alpha = \operatorname{cosec} 2\alpha & \text{(f)} \quad \frac{1 + \sin 2\alpha}{1 + \cos 2\alpha} = \frac{1}{2}(1 + \tan \alpha)^2 \\
 \text{(b)} \quad \frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4 \cos 2A & \text{(g)} \quad \cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \\
 \text{(c)} \quad \frac{2 \sin^3 \theta + 2 \cos^3 \theta}{\sin \theta + \cos \theta} = 2 - \sin 2\theta & \text{(h)} \quad 8 \cos^4 x = 3 + 4 \cos 2x + \cos 4x \\
 & \quad [\text{HINT: } \cos^4 x = \left(\frac{1}{2}(1 + \cos 2x)\right)^2] \\
 \text{(d)} \quad \tan 2x \cot x = 1 + \sec 2x & \text{(i)} \quad \operatorname{cosec} 4A + \cot 4A = \frac{1}{2}(\cot A - \tan A) \\
 \text{(e)} \quad \frac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta - 1} = \tan\left(\theta + \frac{\pi}{4}\right) & \text{(j)} \quad \tan\left(\frac{\pi}{4} + x\right) = \sec 2x + \tan 2x \\
 \text{(k)} \quad \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta = \frac{1}{2}(\cos 2\alpha + \cos 2\beta) \\
 \text{(l)} \quad (\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4 \cos^2 \frac{1}{2}(A - B) \\
 \text{(m)} \quad \frac{\sin 2\alpha + \cos 2\alpha}{2 \cos \alpha + \sin \alpha - 2(\cos^3 \alpha + \sin^3 \alpha)} = \operatorname{cosec} \alpha \\
 \text{(n)} \quad (\tan \theta + \tan 2\theta)(\cot \theta + \cot 3\theta) = 4 \quad [\text{HINT: Use the } \tan 3x \text{ identity in question 11.}]
 \end{array}$$

14. Use the compound-angle results and small-angle theory (see the appropriate worked exercise in the notes) to show that:

- $$\begin{array}{ll} \text{(a)} \quad \cos 46^\circ \div \frac{1}{360}\sqrt{2}(180 - \pi) & \text{(c)} \quad \sin 59^\circ \div \frac{1}{360}(180\sqrt{3} - \pi) \\ \text{(b)} \quad \tan 61^\circ \div \frac{180\sqrt{3} + \pi}{180 - \pi\sqrt{3}} & \text{(d)} \quad \sec 29^\circ \div \frac{360}{180\sqrt{3} + \pi} \end{array}$$

- 15.** Eliminate θ from each pair of parametric equations:

- (a) $x = 2 + \cos \theta$, $y = \cos 2\theta$ (c) $x = 2 \tan \frac{1}{2}\theta$, $y = \cos \theta$
 (b) $x = \tan \theta + 1$, $y = \tan 2\theta$ (d) $x = 3 \sin \theta$, $y = 6 \sin 2\theta$

- 16.** (a) Write down the exact value of $\cos 45^\circ$.

- (b) Hence show that: (i) $\cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2+\sqrt{2}}$ (ii) $\cos 11\frac{1}{4}^\circ = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

17. (a) Show that $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$. (b) Show that $\tan 165^\circ = \sqrt{3} - 2$.
(c) Hence show that $\tan 82\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$.

EXTENSION

18. (a) Eliminate θ from the equations $\cos \theta + \sin \theta = a$, $\cos 2\theta = b$.
 (b) Eliminate θ and ϕ from $\sin \theta + \sin \phi = a$, $\cos \theta + \cos \phi = b$ and $\cos(\theta - \phi) = c$.
19. (a) Explain why $\sin 54^\circ = \cos 36^\circ$. (b) Prove that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.
 (c) Hence show that $4 \sin^3 18^\circ - 2 \sin^2 18^\circ - 3 \sin 18^\circ + 1 = 0$.
 (d) Hence show that $\sin 18^\circ$ is a root of the equation $4x^2 + 2x - 1 = 0$, and find its value.
 (e) Show that: (i) $\sin 54^\circ = \frac{\sqrt{5} + 1}{4}$ (ii) $\cos 54^\circ = \frac{1}{4} \sqrt{10 - 2\sqrt{5}}$
 (f) Show that $\sqrt{8 + 2\sqrt{10 - 2\sqrt{5}}} = \sqrt{5 + \sqrt{5}} + \sqrt{3 - \sqrt{5}}$.
 (g) Hence show that $\cos 27^\circ = \frac{1}{4} \left(\sqrt{5 + \sqrt{5}} + \sqrt{3 - \sqrt{5}} \right)$.
20. (a) Prove by induction that $\cos \frac{90^\circ}{2^n} = \frac{1}{2} \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}}}_{n \text{ terms}}$.
 (b) Find $\sin \frac{90^\circ}{2^n}$. Hence find an expression converging to π , and investigate it as a means of approximating π .

2B The t -Formulae

The t -formulae express $\sin \theta$, $\cos \theta$ and $\tan \theta$ as algebraic functions of the single trigonometric function $\tan \frac{1}{2}\theta$. In the proliferation of trigonometric identities, this can sometimes provide a systematic approach that does not rely on seeing some clever trick.

The t -Formulae: The first of the t -formulae is a restatement of the double-angle formula for the tangent function. The other two formulae follow quickly from it.

6	THE t-FORMULAE: Let $t = \tan \frac{1}{2}\theta$. Then: $\sin \theta = \frac{2t}{1 + t^2} \qquad \cos \theta = \frac{1 - t^2}{1 + t^2} \qquad \tan \theta = \frac{2t}{1 - t^2}$

PROOF:

Let $t = \tan \frac{1}{2}\theta$. We seek to express $\sin \theta$, $\cos \theta$ and $\tan \theta$ in terms of t .

First, $\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}$, by the double-angle formula,

$$= \frac{2t}{1 - t^2}. \quad (1)$$

Secondly, $\cos \theta = \cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta$, by the double-angle formula,

$$= \frac{\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta}{\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta}, \text{ by the Pythagorean identity,}$$

$$= \frac{1 - \tan^2 \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}, \text{ dividing through by } \cos^2 \frac{1}{2}\theta,$$

$$= \frac{1 - t^2}{1 + t^2}. \quad (2)$$

Thirdly, $\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$, by the double-angle formula,

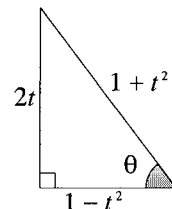
$$\begin{aligned} &= \frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta}, \text{ by the Pythagorean identity,} \\ &= \frac{2 \tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}, \text{ dividing through by } \cos^2 \frac{1}{2}\theta, \\ &= \frac{2t}{1 + t^2}. \end{aligned} \quad (3)$$

NOTE: The proofs given above for these identities rely heavily on the idea of expressions that are *homogeneous* of degree 2 in $\sin x$ and $\cos x$, meaning that the sum of the indices of $\sin x$ and $\cos x$ in each term is 2 — such expressions are easily converted into expressions in $\tan^2 x$ alone. Homogeneous equations will be reviewed in Section 2D.

An Algebraic Identity, and a Way to Memorise the t -formulae:

On the right is a right triangle which demonstrates the relationship amongst the three formulae when θ is acute. The three sides are related by Pythagoras' theorem, and the algebra rests on the quadratic identity

$$(1 - t^2)^2 + (2t)^2 = (1 + t^2)^2.$$



This diagram may help to memorise the t -formulae.

WORKED EXERCISE: Use the t -formulae to prove:

$$(a) \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} \quad (b) \sec 2x + \tan 2x = \tan\left(x + \frac{\pi}{4}\right)$$

SOLUTION:

$$(a) \text{ Let } t = \tan \frac{1}{2}x.$$

$$\begin{aligned} \text{LHS} &= \left(1 - \frac{1 - t^2}{1 + t^2}\right) \div \frac{2t}{1 + t^2} \\ &= \frac{1 + t^2 - 1 + t^2}{1 + t^2} \times \frac{1 + t^2}{2t} \\ &= \frac{2t^2}{2t} \\ &= t \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{2t}{1 + t^2} \times \left(1 + \frac{1 - t^2}{1 + t^2}\right)^{-1} \\ &= \frac{2t}{1 + t^2} \times \frac{1 + t^2}{1 + t^2 + 1 - t^2} \\ &= \frac{2t}{2} \\ &= t \\ &= \text{LHS} \end{aligned}$$

Notice that we have proven the further identity

$$\frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} = \tan \frac{1}{2}\theta.$$

$$(b) \text{ Let } t = \tan x.$$

$$\begin{aligned} \text{LHS} &= \frac{1 + t^2}{1 - t^2} + \frac{2t}{1 - t^2} \\ &= \frac{1 + 2t + t^2}{(1 + t)(1 - t)} \\ &= \frac{(1 + t)^2}{(1 + t)(1 - t)} \\ &= \frac{1 + t}{1 - t} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{\tan x + 1}{1 - \tan x \times 1} \\ &= \frac{1 + t}{1 - t} \\ &= \text{LHS} \end{aligned}$$

Exercise 2B

1. Write in terms of t , where $t = \tan \frac{1}{2}\theta$:

(a) $\sin \theta$

(c) $\tan \theta$

(e) $1 - \cos \theta$

(b) $\cos \theta$

(d) $\sec \theta$

(f) $\frac{1 - \cos \theta}{\sin \theta}$

2. Write in terms of t , where $t = \tan \theta$:

(a) $\cos 2\theta$

(b) $1 - \sin 2\theta$

(c) $\tan 2\theta + \sec 2\theta$

3. Use the $t = \tan \frac{1}{2}\theta$ results to simplify:

(a) $\frac{2 \tan 10^\circ}{1 - \tan^2 10^\circ}$

(c) $\frac{1 - \tan^2 10^\circ}{1 + \tan^2 10^\circ}$

(e) $\frac{2 \tan 2x}{1 - \tan^2 2x}$

(b) $\frac{2 \tan 10^\circ}{1 + \tan^2 10^\circ}$

(d) $\frac{2 \tan 2x}{1 + \tan^2 2x}$

(f) $\frac{1 - \tan^2 2x}{1 + \tan^2 2x}$

4. Use the $t = \tan \frac{1}{2}\theta$ results to find the exact value of:

(a) $\frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}$

(c) $\frac{1 - \tan^2 75^\circ}{1 + \tan^2 75^\circ}$

(e) $\frac{1 - \tan^2 \frac{3\pi}{8}}{1 + \tan^2 \frac{3\pi}{8}}$

(b) $\frac{2 \tan 15^\circ}{1 + \tan^2 15^\circ}$

(d) $\frac{2 \tan 112\frac{1}{2}^\circ}{1 + \tan^2 112\frac{1}{2}^\circ}$

(f) $\frac{2 \tan \frac{11\pi}{12}}{1 - \tan^2 \frac{11\pi}{12}}$

DEVELOPMENT

5. Prove each of the following identities using the $t = \tan \frac{1}{2}\theta$ results:

(a) $\cos \theta (\tan \theta - \tan \frac{1}{2}\theta) = \tan \frac{1}{2}\theta$

(e) $\frac{\tan \theta \tan \frac{1}{2}\theta}{\tan \theta - \tan \frac{1}{2}\theta} = \sin \theta$

(b) $\frac{1 - \cos 2x}{\sin 2x} = \tan x$

(f) $\frac{\cos \theta + \sin \theta - 1}{\cos \theta - \sin \theta + 1} = \tan \frac{1}{2}\theta$

(c) $\frac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{1}{2}\theta$

(g) $\frac{\tan 2\alpha + \cot \alpha}{\tan 2\alpha - \tan \alpha} = \cot^2 \alpha$

(d) $\frac{1 + \operatorname{cosec} \theta}{\cot \theta} = \frac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta}$

(h) $\tan(\frac{1}{2}x + \frac{\pi}{4}) + \tan(\frac{1}{2}x - \frac{\pi}{4}) = 2 \tan x$

6. (a) Given that $t = \tan 112\frac{1}{2}^\circ$, show that $\frac{2t}{1 - t^2} = 1$.(b) (i) Hence show that $\tan 112\frac{1}{2}^\circ = -\sqrt{2} - 1$.

(ii) What does the other root of the equation represent?

7. Use the method of the previous question to show that:

(a) $\tan 15^\circ = 2 - \sqrt{3}$

(b) $\tan \frac{7\pi}{8} = 1 - \sqrt{2}$

8. Suppose that $\tan \alpha = -\frac{1}{3}$ and $\frac{\pi}{2} < \alpha < \pi$. Find the exact value of:

(a) $\tan 2\alpha$

(b) $\sin 2\alpha$

(c) $\cos 2\alpha$

(d) $\tan \frac{1}{2}\alpha$

9. [Alternative derivations of the t -formulae] Let $t = \tan \frac{1}{2}\theta$.(a) (i) Express $\cos \theta$ in terms of $\cos \frac{1}{2}\theta$.(ii) Write $\cos^2 \frac{1}{2}\theta$ as $\frac{1}{\sec^2 \frac{1}{2}\theta}$, and hence show that $\cos \theta = \frac{1 - t^2}{1 + t^2}$.(b) (i) Write $\sin \theta$ in terms of $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$.(ii) Write $\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$ as $\frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} \cos^2 \frac{1}{2}\theta$, and hence show that $\sin \theta = \frac{2t}{1 + t^2}$.

10. (a) If $x = \tan \theta + \sec \theta$, use the t -formulae to show that $\frac{x^2 - 1}{x^2 + 1} = \sin \theta$.

(b) If $x = \cos 2\theta$, use the t -formulae to show that $\sqrt{\frac{1+x}{1-x}} = |\cot \theta|$.

11. If $\cos x = \frac{5 \cos y - 3}{5 - 3 \cos y}$, prove that $\tan^2 \frac{1}{2}x = 4 \tan^2 \frac{1}{2}y$.

[HINT: Let $t_1 = \tan \frac{1}{2}x$ and $t_2 = \tan \frac{1}{2}y$.]

EXTENSION

12. Consider the integral $I = \int \frac{1}{1 + \cos x} dx$, and let $t = \tan \frac{1}{2}x$.

(a) Show that $\frac{dx}{dt} = \frac{2}{1+t^2}$.

(b) By writing dx as $\frac{dx}{dt} dt$, show that $I = \int dt = \tan \frac{1}{2}x + C$.

13. Use the same approach as in the previous question to show that

$$\int \operatorname{cosec} x \, dx = \log_e(\tan \frac{1}{2}x) + C.$$

14. (a) Show that $\frac{1}{(x+2)(2x+1)} = \frac{1}{3} \left(\frac{2}{2x+1} - \frac{1}{x+2} \right)$.

(b) Hence show that $\int_0^1 \frac{1}{(x+2)(2x+1)} dx = \frac{1}{3} \log 2$.

(c) Using the approach of question 12, deduce that $\int_0^{\frac{\pi}{2}} \frac{3}{4 + 5 \sin x} dx = \log 2$.

2C Applications of Trigonometric Identities

The exercise of this section contains further examples of trigonometric identities, but it also seeks to relate the trigonometric identities of the previous two sections to geometric situations and to calculus.

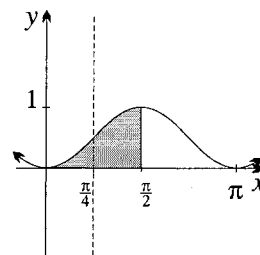
The Integration of $\cos^2 x$ and $\sin^2 x$: The identities expressing $\sin^2 \theta$ and $\cos^2 \theta$ in terms of $\cos 2\theta$ provide the standard way of finding primitives of $\sin^2 x$ and $\cos^2 x$.

WORKED EXERCISE: Find: (a) $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ (b) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

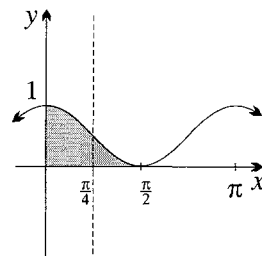
Explain from their graphs why these integrals are equal.

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \int_0^{\frac{\pi}{2}} \sin^2 x \, dx &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{4} - \frac{1}{4} \sin \pi \right) - \left(0 - \frac{1}{4} \sin 0 \right) \\ &= \frac{\pi}{4} \end{aligned}$$



$$\begin{aligned}
 \text{(b)} \quad \int_0^{\frac{\pi}{2}} \cos^2 x \, dx &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \left[\frac{1}{2}x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \\
 &= \left(\frac{\pi}{4} + \frac{1}{4} \sin \pi \right) - \left(0 + \frac{1}{4} \sin 0 \right) \\
 &= \frac{\pi}{4}
 \end{aligned}$$



Since $\cos^2 x = \sin^2(\frac{\pi}{2} - x)$, the regions represented by the two integrals are reflections of each other in this vertical line $x = \frac{\pi}{4}$, and so have the same area. Also, the answer $\frac{\pi}{4}$ can easily be seen by taking advantage of the symmetry of each graph to cut and paste the shaded region to form a rectangle.

Geometric Configurations and Trigonometric Identities: There is an endless variety of geometric configurations in which trigonometric identities play a role. The worked exercise below involves the expansion of $\sin 2\theta$ and the range of $\cos \theta$.

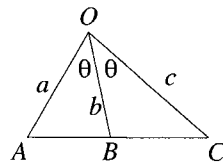
WORKED EXERCISE: Three sticks of lengths a , b and c extend from a point O so that their endpoints A , B and C respectively are collinear, and so that OB bisects $\angle AOC$. Let $\theta = \angle AOB = \angle BOC$.

(a) Find the areas of $\triangle AOB$, $\triangle BOC$ and $\triangle AOC$ in terms of a , b , c and θ .

(b) Hence show that $\cos \theta = \frac{b(a+c)}{2ac}$.

(c) Show that the middle stick OB cannot be the longest stick.

(d) If $a = b$ and $c = 2b$, find the area of $\triangle AOC$ in terms of b .



SOLUTION:

(a) Using the formula for the area of a triangle, area $\triangle AOB = \frac{1}{2}ab \sin \theta$,

$$\text{area } \triangle BOC = \frac{1}{2}bc \sin \theta,$$

$$\text{area } \triangle AOC = \frac{1}{2}ac \sin 2\theta.$$

(b) Since the area of $\triangle AOC$ is the sum of the areas of $\triangle AOB$ and $\triangle BOC$,

$$\frac{1}{2}ac \sin 2\theta = \frac{1}{2}ab \sin \theta + \frac{1}{2}bc \sin \theta$$

$$2ac \sin \theta \cos \theta = ab \sin \theta + bc \sin \theta$$

$$\cos \theta = \frac{b(a+c)}{2ac}, \text{ as required.}$$

$$\begin{aligned}
 \text{(c) From part (b), } \cos \theta &= \frac{b(a+c)}{2ac} \\
 &= \frac{b}{2c} + \frac{b}{2a}.
 \end{aligned}$$

If b were the longest stick, then both terms would be greater than $\frac{1}{2}$, and so $\cos \theta$ would be greater than 1, which is impossible.

$$\begin{aligned}
 \text{(d) From part (b), } \cos \theta &= \frac{b(a+c)}{2ac} \\
 &= \frac{b \times 3b}{2b \times 2b} \\
 &= \frac{3}{4}
 \end{aligned}$$

so

$$\sin \theta = \frac{1}{4}\sqrt{7}.$$

From part (a), area $\triangle AOC = ac \sin \theta \cos \theta$

$$\begin{aligned}
 &= b \times 2b \times \frac{1}{4}\sqrt{7} \times \frac{3}{4} \\
 &= \frac{3}{8}b^2\sqrt{7}.
 \end{aligned}$$

Exercise 2C

1. Find, using appropriate compound-angle results:

(a) $\sin \angle BAC$

(b) $\cos \angle BAC$

2. In the diagram opposite, suppose that
- $\tan \beta = \frac{1}{3}$
- .

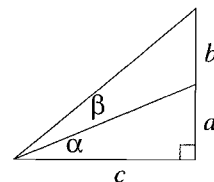
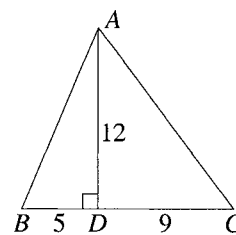
- (a) Write down an expression for
- $\tan \alpha$
- .

- (b) Use an appropriate compound-angle formula to show

that $\tan(\alpha + \beta) = \frac{3a + c}{3c - a}$.

- (c) Write down an alternative expression for
- $\tan(\alpha + \beta)$
- .

- (d) Hence show that
- $b = \frac{a^2 + c^2}{3c - a}$
- .



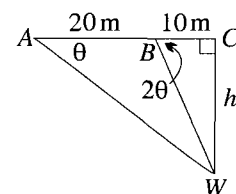
3. Points
- A, B, C
- and
- W
- lie in the same vertical plane. A bird at
- A
- observes a worm at
- W
- at an angle of depression
- θ
- . After flying 20 metres horizontally to
- B
- , the angle of depression of the worm is
- 2θ
- . If the bird flew another 10 metres horizontally it would be directly above the worm. Let
- $WC = h$
- metres.

- (a) Write
- $\tan 2\theta$
- in terms of
- $\tan \theta$
- .

- (b) Use the two right-angled triangles to write two equations in
- h
- and
- θ
- .

- (c) Use parts (a) and (b) to show that
- $\frac{h}{10} = \frac{60h}{900 - h^2}$
- .

- (d) Hence show that
- $h = 10\sqrt{3}$
- metres.

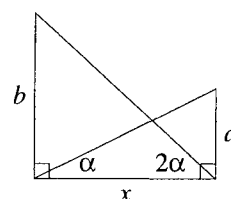


4. (a) Using the diagram opposite, write down expressions for
- $\tan \alpha$
- and
- $\tan 2\alpha$
- .

- (b) Use the double-angle formula for
- $\tan 2\alpha$
- to show that

$$\frac{b}{x} = \frac{2ax}{x^2 - a^2}.$$

- (c) Hence show that
- $x = a\sqrt{\frac{b}{b - 2a}}$
- . (d) Why is it necessary to assume that
- $b > 2a$
- ?



DEVELOPMENT

5. Use the results
- $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$
- and
- $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$
- to find:

(a) $\int_0^{\pi} \sin^2 x \, dx$

(c) $\int_0^{\frac{\pi}{6}} \sin^2 \frac{1}{2}x \, dx$

(e) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(x + \frac{\pi}{12}) \, dx$

(b) $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

(d) $\int_0^{\frac{\pi}{16}} \cos^2 2x \, dx$

(f) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2(x - \frac{\pi}{6}) \, dx$

6. (a) Sketch the graph of
- $y = \cos 2x$
- , for
- $0 \leq x \leq 2\pi$
- .

- (b) Hence sketch, on the same diagram,
- $y = \frac{1}{2}(1 + \cos 2x)$
- and
- $y = \frac{1}{2}(1 - \cos 2x)$
- .

- (c) Hence show graphically that
- $\cos^2 x + \sin^2 x = 1$
- .

7. Explain why
- $\cos x \sin x$
- cannot exceed
- $\frac{1}{2}$
- .

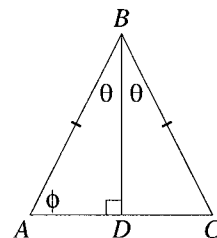
8. (a) If
- $\tan \theta = \frac{x \sin \phi}{1 - x \cos \phi}$
- , show that
- $x = \frac{\sin \theta}{\sin(\theta + \phi)}$
- .

- (b) If also
- $\tan \phi = \frac{y \sin \theta}{1 - y \cos \theta}$
- , find
- $\frac{x}{y}$
- in simplest form in terms of
- θ
- and
- ϕ
- .

9. $\triangle ABC$ is isosceles with $AB = CB$, and D lies on AC with $BD \perp AC$. Let $\angle ABD = \angle CBD = \theta$, and $\angle BAD = \phi$.

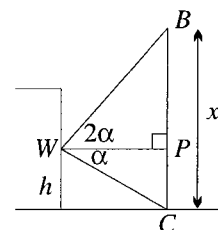
- (a) Show that $\sin \phi = \cos \theta$.
 (b) Use the sine rule in $\triangle ABC$ to show that $\sin 2\theta = 2 \sin \theta \cos \theta$.
 (c) If $0 < \theta < \frac{\pi}{2}$, show that

$$\sin 2\theta + \sin 2\theta \cos^2 \theta + \sin 2\theta \cos^4 \theta + \cdots = 2 \cot \theta.$$



10. An office-worker is looking out a window W of a building standing on level ground. From W , a car C has an angle of depression α , while a balloon B directly above the car has an angle of elevation 2α . The height of the balloon above the car is x , and the height of the window above the ground is h .

- (a) Show that $\frac{\tan \alpha}{h} = \frac{\tan 2\alpha}{x - h}$.
 (b) Hence show that $h = \frac{1 - \tan^2 \alpha}{3 - \tan^2 \alpha} x$.



11. In $\triangle ABC$, $\frac{b}{c} = \frac{4}{3}$.

- (a) If $B = 2C$, show that $\cos C = \frac{2}{3}$. (b) If $B = 3C$, show that $\sin C = \frac{1}{6}\sqrt{15}$.

12. (a) By writing $\sin^4 x$ as $(\sin^2 x)^2$, show that $\sin^4 x = \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$.

- (b) Find a similar result for $\cos^4 x$.

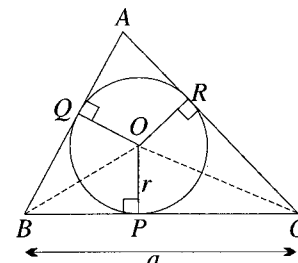
- (c) Hence find: (i) $\int_0^\pi \sin^4 x \, dx$ (ii) $\int_0^{\frac{\pi}{4}} \cos^4 x \, dx$

13. The diagram shows a circle with centre O and radius r inscribed in a triangle ABC .

- (a) Prove that $\angle OBP = \angle OBQ$.

- (b) Prove that $\frac{a}{r} = \cot \frac{1}{2} B + \cot \frac{1}{2} C$.

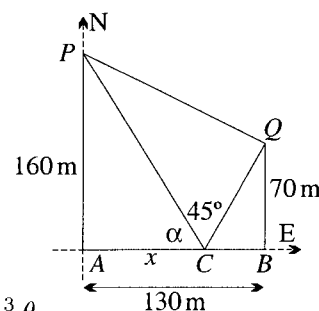
- (c) Hence prove that $\frac{a}{r} = \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} B \sin \frac{1}{2} C}$.



14. In the diagram opposite, P and Q are landmarks which are 160 metres and 70 metres due north of points A and B respectively. A and B lie 130 metres apart on a west-east road. C is a point on the road between A and B and $\angle PCQ = 45^\circ$. Let $AC = x$ and $\angle ACP = \alpha$.

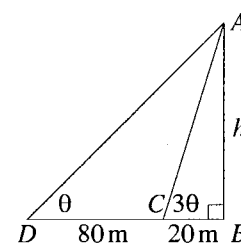
- (a) Show that $\tan(135^\circ - \alpha) = \frac{70}{130 - x}$.

- (b) Hence show that $AC = 120$ metres.



15. (a) By expressing 3θ as $2\theta + \theta$, show that $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$.

- (b) A tower AB has height h metres. The angle of elevation of the top of the tower at a point C 20 metres from its base is three times the angle of elevation at a point D 80 metres further away from its base. Use the identity in part (a) to show that $h = \frac{100}{7}\sqrt{7}$ metres.



16. Define $F(x) = \int_0^x \sin^2 t \, dt$, where $0 \leq x \leq 2\pi$.
- Show that $F(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$.
 - Explain why $F'(x) = \sin^2 x$. Hence state the values of x in the given domain for which $F(x)$ is: (i) stationary, (ii) increasing, (iii) decreasing.
 - Explain why $F(x)$ never differs from $\frac{1}{2}x$ by more than $\frac{1}{4}$.
 - Find any points of inflexion of $F(x)$ in the given domain.
 - Sketch, on the same diagram, the graphs of $y = F(x)$ and $y = F'(x)$ over the given domain, and observe how they are related.
 - (i) For what value of k is $\int_0^k \sin^2 x \, dx = \frac{3\pi}{2}$?
 (ii) For what values of k is $\int_0^k \sin^2 x \, dx = \frac{n\pi}{2}$, where n is an integer?

————— EXTENSION —————

17. The lengths of the sides of a triangle form an arithmetic progression and the largest angle of the triangle exceeds the smallest by 90° . Show that the lengths of the sides of the triangle are in the ratio $\sqrt{7} - 1 : \sqrt{7} : \sqrt{7} + 1$. [HINT: One possible approach makes use of both the sine and cosine rules.]
18. [Harmonic conjugates] In $\triangle ABC$, the bisectors of the internal and external angles at A meet BC produced at P and Q respectively. Prove that Q divides BC externally in the same ratio as that in which P divides BC internally.
19. Suppose that $\tan^2 x = \tan(x - \alpha)\tan(x - \beta)$. Show that $\tan 2x = \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$.

2D Trigonometric Equations

Trigonometric equations occur whenever trigonometric functions are being analysed, and careful study of them is essential. This section presents a systematic approach to their solution, and begins with the account given in Chapter Four of the Year 11 volume when the compound-angle formulae were not yet available.

Simple Trigonometric Equations: More complicated trigonometric equations eventually reduce to equations like

$$\cos x = -1, \quad \text{or} \quad \tan x = -\sqrt{3}, \quad \text{for } -2\pi \leq x \leq 2\pi,$$

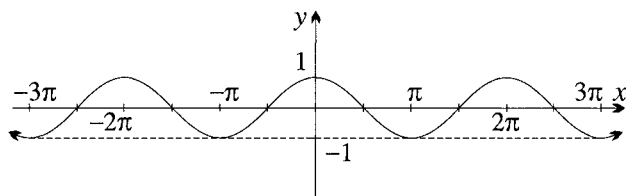
where there may or may not be a restriction on the domain. The methods here should be familiar by now.

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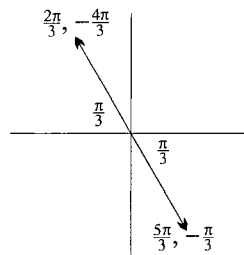
SIMPLE TRIGONOMETRIC EQUATIONS: If a trigonometric equation involves angles at the boundaries of quadrants, read the solutions off a sketch of the graph. Otherwise, draw a quadrants diagram, and read the solutions off it.

WORKED EXERCISE: Solve: (a) $\cos x = -1$ (b) $\tan x = -\sqrt{3}$, for $-2\pi \leq x \leq 2\pi$

SOLUTION:



- (a) Reading from the graph of $y = \cos x$,
 $x = \pi, -\pi, 3\pi, -3\pi, \dots$
 $x = (2n+1)\pi$, where n is an integer.



- (b) The related angle is $\frac{\pi}{3}$,
 and x is in quadrants 2 or 4,
 so $x = \frac{2\pi}{3}, \frac{5\pi}{3}, -\frac{\pi}{3}$ or $-\frac{4\pi}{3}$.

Simple Trigonometric Equations with a Compound Angle: Most troubles are avoided by substitution for the compound angle. Any given restrictions on the original angle must then be carried through to restrictions on the compound angle.

8

SIMPLE EQUATIONS WITH A COMPOUND ANGLE: Let u be the compound angle. From the given restrictions on x , find the resulting restrictions on u .

WORKED EXERCISE: Solve $\sin(3x + \frac{5\pi}{4}) = 1$, for $-\pi \leq x \leq \pi$.

SOLUTION: Let

$$u = 3x + \frac{5\pi}{4}.$$

Then

$$-3\pi \leq 3x \leq 3\pi$$

$$+ \frac{5\pi}{4}$$

$$-\frac{7\pi}{4} \leq 3x + \frac{5\pi}{4} \leq \frac{17\pi}{4}.$$

Hence

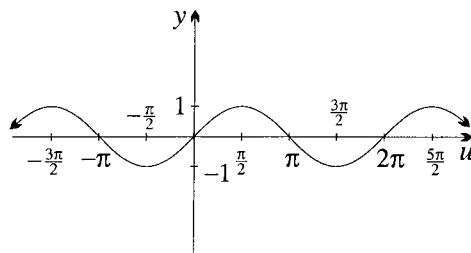
$$\sin u = 1, \text{ for } -\frac{7\pi}{4} \leq u \leq \frac{17\pi}{4}$$

$$u = -\frac{3\pi}{2}, \frac{\pi}{2} \text{ or } \frac{5\pi}{2}$$

$$3x + \frac{5\pi}{4} = -\frac{3\pi}{2}, \frac{\pi}{2} \text{ or } \frac{5\pi}{2}$$

$$3x = -\frac{11\pi}{4}, -\frac{3\pi}{4} \text{ or } \frac{5\pi}{4}$$

$$x = -\frac{11\pi}{12}, -\frac{\pi}{4} \text{ or } \frac{5\pi}{12}.$$



Equations Requiring Algebraic Substitutions: If there are powers or reciprocals of the one trigonometric function present, it is usually best to make a substitution for that trigonometric function.

9

ALGEBRAIC SUBSTITUTION FOR A TRIGONOMETRIC FUNCTION: Substitute u for the trigonometric function, solve the resulting algebraic equation, then solve each of the resulting trigonometric equations.

WORKED EXERCISE: Solve $2 \cos x = 1 + \sec x$, for $0 \leq x \leq 2\pi$.

SOLUTION: Let

$$u = \cos x.$$

Then

$$2u = 1 + \frac{1}{u}$$

$$2u^2 - u - 1 = 0$$

$$(2u+1)(u-1) = 0$$

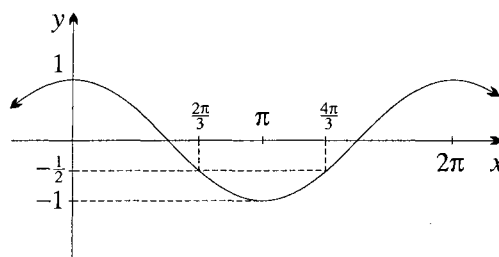
$$u = 1 \text{ or } u = -\frac{1}{2},$$

so

$$\cos x = 1 \text{ or } \cos x = -\frac{1}{2}.$$

Hence

$$x = 0, 2\pi, \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}.$$



Equations with More than One Trigonometric Function, but the Same Angle: This is where trigonometric identities come into play.

10 EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION: Trigonometric identities can usually be used to produce an equation in only one trigonometric function.

WORKED EXERCISE: Solve the equation $2 \tan \theta = \sec \theta$, for $0^\circ \leq \theta \leq 360^\circ$:

- (a) using the ratio identities, (b) by squaring both sides.

SOLUTION:

(a) $2 \tan \theta = \sec \theta$

$$\frac{2 \sin \theta}{\cos \theta} = \frac{1}{\cos \theta}$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = 30^\circ \text{ or } 150^\circ$$

(b) Squaring, $4 \tan^2 \theta = \sec^2 \theta$

$$4 \sec^2 \theta - 4 = \sec^2 \theta$$

$$\sec^2 \theta = \frac{4}{3}$$

$$\cos \theta = \frac{1}{2} \sqrt{3} \text{ or } -\frac{1}{2} \sqrt{3}$$

$$\theta = 30^\circ, 150^\circ, 210^\circ \text{ or } 330^\circ.$$

Checking each solution, $\theta = 30^\circ$ or 150° .

The Dangers of Squaring an Equation: Squaring an equation is to be avoided if possible, because squaring may introduce extra solutions, as it did in part (b) above. If an equation does have to be squared, each solution must be checked in the original equation to see whether it is a solution or not. Here are two very simple equations, both purely algebraic, where the effect of squaring can easily be seen.

(a) Suppose that $x = 3$.

Squaring, $x^2 = 9$

so $x = 3$ or $x = -3$.

Here $x = -3$ is a spurious solution.

(b) Suppose that $\sqrt{x} = -5$.

Squaring, $x = 25$.

But $\sqrt{25} = 5$, not -5 .

In fact, there are no solutions.

Equations Involving Different Angles: When different angles are involved in the same trigonometric equation, the usual approach is to use compound-angle identities to change all the trigonometric functions to functions of the one angle.

11 EQUATIONS INVOLVING DIFFERENT ANGLES: Use compound-angle identities to change all the trigonometric functions to functions of the one angle.

Frequently such an equation can be solved by more than one method.

WORKED EXERCISE: Solve $\cos 2x = 4 \sin^2 x - 14 \cos^2 x$, for $0 \leq x \leq 2\pi$:

- (a) by changing all the angles to x , (b) by changing all the angles to $2x$.

SOLUTION:

(a) $\cos 2x = 4 \sin^2 x - 14 \cos^2 x$

$$\cos^2 x - \sin^2 x = 4 \sin^2 x - 14 \cos^2 x$$

$$15 \cos^2 x = 5 \sin^2 x$$

$$\tan x = \sqrt{3} \text{ or } -\sqrt{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

(b) $\cos 2x = 4 \sin^2 x - 14 \cos^2 x$

$$\cos 2x = 4\left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) - 14\left(\frac{1}{2} + \frac{1}{2} \cos 2x\right)$$

$$10 \cos 2x = -5$$

$$\cos 2x = -\frac{1}{2}$$

$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3} \text{ or } \frac{10\pi}{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

Homogeneous Equations: Equations homogeneous in $\sin x$ and $\cos x$ were mentioned earlier as a special case of the application of trigonometric identities.

12

HOMOGENEOUS EQUATIONS: An equation is called *homogeneous* in $\sin x$ and $\cos x$ if the sum of the indices of $\sin x$ and $\cos x$ in each term is the same.
To solve an equation homogeneous in $\sin x$ and $\cos x$, divide through by a power of $\cos x$ to produce an equation in $\tan x$ alone.

The expansions of $\sin 2x$ and $\cos 2x$ are homogeneous of degree 2 in $\sin x$ and $\cos x$. Also, $1 = \sin^2 x + \cos^2 x$ can be regarded as being homogeneous of degree 2.

WORKED EXERCISE: Solve $\sin 2x + \cos 2x = \sin^2 x + 1$, for $0 \leq x \leq 2\pi$.

SOLUTION: Expanding, $2 \sin x \cos x + (\cos^2 x - \sin^2 x) = \sin^2 x + (\sin^2 x + \cos^2 x)$

$$3 \sin^2 x - 2 \sin x \cos x = 0$$

$\div \cos^2 x$

$$3 \tan^2 x - 2 \tan x = 0$$

$$\tan x(3 \tan x - 2) = 0$$

$$\tan x = 0 \text{ or } \tan x = \frac{2}{3}.$$

Hence $x = 0, \pi$ or 2π , or $x \doteq 0.588$ or 3.730 .

The Equations $\sin x = \sin \alpha$, $\cos x = \cos \alpha$ and $\tan x = \tan \alpha$: The methods associated with general solutions of trigonometric equations from the last chapter can often be very useful.

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THE GENERAL SOLUTIONS OF $\sin x = \sin \alpha$, $\cos x = \cos \alpha$ AND $\tan x = \tan \alpha$:

- If $\tan x = \tan \alpha$, then $x = n\pi + \alpha$, where n is an integer.
- If $\cos x = \cos \alpha$, then $x = 2n\pi + \alpha$ or $2n\pi - \alpha$, where n is an integer.
- If $\sin x = \sin \alpha$, then $x = 2n\pi + \alpha$ or $(2n + 1)\pi - \alpha$, where n is an integer.

WORKED EXERCISE: Solve $\tan 4x = -\tan 2x$:

(a) using the $\tan 2\theta$ formula,

(b) using solutions of $\tan \alpha = \tan \beta$.

SOLUTION:

(a) $\tan 4x = -\tan 2x$

Let $t = \tan 2x$.

Then $\frac{2t}{1-t^2} = -t$

$$2t = -t + t^3$$

$$t^3 - 3t = 0$$

$$t(t^2 - 3) = 0.$$

Hence $\tan 2x = 0$ or $\tan 2x = \sqrt{3}$ or $\tan 2x = -\sqrt{3}$

$$2x = k\pi \text{ or } \frac{\pi}{3} + k\pi \text{ or } -\frac{\pi}{3} + k\pi, \text{ where } k \text{ is an integer,}$$

$$x = \frac{1}{6}n\pi, \text{ where } n \text{ is an integer.}$$

(b) $\tan 4x = -\tan 2x$.

Since $\tan \theta$ is an odd function,

$$\tan 4x = \tan(-2x)$$

$$4x = -2x + n\pi, \text{ where } n \text{ is an integer}$$

$$6x = n\pi$$

$$x = \frac{1}{6}n\pi.$$

Another Approach to Trigonometric Functions of Multiples of 18° : In Chapter Four of the Year 11 volume, we used a construction within a pentagon to generate trigonometric functions of some multiples of 18° . Here is another approach through alternative solutions of trigonometric equations.

WORKED EXERCISE: Solve $\sin 3x = \cos 2x$, for $0^\circ \leq x \leq 360^\circ$:

(a) graphically,

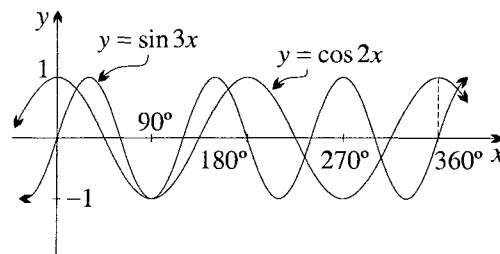
(b) using solutions of $\cos \alpha = \cos \beta$.

Begin solving using compound-angle formulae, and hence find $\sin 18^\circ$ and $\sin 54^\circ$.

[HINT: Use the factorisation $4u^3 - 2u^2 - 3u + 1 = (u - 1)(4u^2 + 2u - 1)$.]

SOLUTION:

(a) The graphs of the two functions are sketched opposite. They make it clear that there are five solutions, and from the graph, one solution is 90° , and the other four are approximately 20° , 160° , 230° and 310° .



(b)

$$\sin 3x = \cos 2x$$

$$\cos(90^\circ - 3x) = \cos 2x, \text{ since } \sin \theta = \cos(90^\circ - \theta),$$

$$2x = 90^\circ - 3x + 360n^\circ$$

$$\text{or } 2x = -90^\circ + 3x + 360n^\circ$$

$$5x = 90^\circ, 450^\circ, 810^\circ, 1170^\circ, 1530^\circ$$

$$\text{or } x = 90^\circ.$$

Hence

$$x = 18^\circ, 90^\circ, 162^\circ, 234^\circ \text{ or } 306^\circ.$$

Alternatively,

$$\sin 3x = \cos 2x$$

$$3\sin x - 4\sin^3 x = 1 - 2\sin^2 x, \text{ using compound-angle identities.}$$

Let

$$u = \sin x.$$

$$\text{Then } 4u^3 - 2u^2 - 3u + 1 = 0$$

$$(u - 1)(4u^2 + 2u - 1) = 0, \text{ by the given factorisation.}$$

The quadratic has discriminant 20, so the three solutions of the cubic are

$$u = 1 \text{ or } u = \frac{1}{4}(-1 + \sqrt{5}) \text{ or } u = \frac{1}{4}(-1 - \sqrt{5}).$$

Now $\sin x = 1$ has the one solution $x = 90^\circ$,

and $\sin x = \frac{1}{4}(-1 + \sqrt{5})$ and $\sin x = \frac{1}{4}(-1 - \sqrt{5})$ each have two solutions.

From part (b), we conclude that $\sin 18^\circ = \sin 162^\circ = \frac{1}{4}(-1 + \sqrt{5})$.

Also, $\sin 234^\circ = \sin 306^\circ = \frac{1}{4}(-1 - \sqrt{5})$, so $\sin 54^\circ = \frac{1}{4}(1 + \sqrt{5})$.

NOTE: From these results, the values of all the trigonometric functions at 18° , 36° , 54° and 72° can be calculated. See the Extension to the following exercise.

Exercise 2D

1. Solve each equation for $0 \leq x \leq 2\pi$:

(a) $\sqrt{2} \sin x = 1$

(c) $\cot x = \sqrt{3}$

(e) $4 \cos^2 x - 3 = 0$

(b) $2 \cos x + 1 = 0$

(d) $\sin^2 x = 1$

(f) $\sec^2 x - 2 = 0$

2. Solve each equation for $0^\circ \leq \alpha \leq 360^\circ$:

(a) $\tan 2\alpha = \sqrt{3}$

(b) $\cos 2\alpha = 1$

(c) $\sin 3\alpha = \frac{1}{2}$

(d) $\tan 3\alpha = -1$

3. Solve, for $0 \leq \theta \leq 2\pi$:

(a) $\sin(\theta - \frac{\pi}{6}) = \frac{1}{2}\sqrt{3}$

(c) $\sin(2\theta - \frac{\pi}{2}) = \frac{1}{2}\sqrt{2}$

(b) $\cos(\theta + \frac{\pi}{4}) = -\frac{1}{2}\sqrt{3}$

(d) $\cos(2\theta + \frac{\pi}{6}) = -\frac{1}{2}$

4. Use the basic trigonometric identities such as $\frac{\sin x}{\cos x} = \tan x$ to solve, for $0 \leq x \leq 2\pi$:

(a) $\sin x - \sqrt{3} \cos x = 0$

(c) $4 \cos 2x = 3 \sec 2x$

(b) $4 \sin x = \operatorname{cosec} x$

(d) $\sin^2 \frac{1}{2}x = 3 \cos^2 \frac{1}{2}x$

5. Use Pythagorean identities where necessary (such as $\sin^2 x + \cos^2 x = 1$) and factoring to solve the following for $0^\circ \leq \alpha \leq 360^\circ$. Give answers to the nearest minute where necessary.

(a) $\sin^2 \alpha = \sin \alpha$	(f) $2 \sin^2 \alpha + 3 \cos \alpha = 3$
(b) $\sec^2 \alpha = 2 \sec \alpha$	(g) $\sec^2 \alpha - \tan \alpha - 3 = 0$
(c) $\cos^2 \alpha = \sin \alpha \cos \alpha$	(h) $\cos^2 2\alpha - 2 \sin 2\alpha + 2 = 0$
(d) $\tan^2 \alpha - 3 \tan \alpha - 4 = 0$	(i) $\operatorname{cosec}^3 2\alpha = 4 \operatorname{cosec} 2\alpha$
(e) $2 \sin^2 \alpha = \sin \alpha + 1$	(j) $\sqrt{3} \operatorname{cosec}^2 \frac{1}{2}\alpha + \cot \frac{1}{2}\alpha = \sqrt{3}$

6. Use compound-angle formulae to solve, for $0 \leq \theta \leq 2\pi$:

(a) $\sin(\theta + \frac{\pi}{6}) = 2 \sin(\theta - \frac{\pi}{6})$	(c) $\cos 4\theta \cos \theta + \sin 4\theta \sin \theta = \frac{1}{2}$
(b) $\cos(\theta - \frac{\pi}{6}) = 2 \cos(\theta + \frac{\pi}{6})$	(d) $\cos 3\theta = \cos 2\theta \cos \theta$

[HINT: In part (d), write $\cos 3\theta$ as $\cos(2\theta + \theta)$.]

7. Use double-angle formulae to solve, for $0 \leq x \leq 2\pi$:

(a) $\sin 2x = \sin x$	(c) $\tan 2x + \tan x = 0$
(b) $\cos 2x = \sin x$	(d) $\sin 2x = \tan x$

8. Use a sketch of the LHS in each case to help solve, for $0 \leq x \leq 2\pi$:

(a) $\sin x > 0$	(c) $\cos x \leq \frac{1}{2}$	(e) $\cos(x - \frac{\pi}{4}) \leq \frac{1}{2}$
(b) $\sin 2x > 0$	(d) $\cos 2x \leq \frac{1}{2}$	(f) $\tan 2x \geq 1$

DEVELOPMENT

9. Solve, for $0^\circ \leq A \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:

(a) $2 \sin^2 A - 5 \cos A - 4 = 0$	(f) $\tan^2 A + 3 \cot^2 A = 4$
(b) $\tan^2 A = 3(\sec A - 1)$	(g) $2(\cos A - \sec A) = \tan A$
(c) $3 \tan A - \cot A = 2$	(h) $\cot A + 3 \tan A = 5 \operatorname{cosec} A$
(d) $\sqrt{3} \operatorname{cosec}^2 A = 4 \cot A$	(i) $\sin^2 A - 2 \sin A \cos A - 3 \cos^2 A = 0$
(e) $2 \cos 2A + \sec 2A + 3 = 0$	(j) $\tan^2 A + 8 \cos^2 A = 5$

10. Solve, for $0^\circ \leq \theta \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:

(a) $2 \sin 2\theta + \cos \theta = 0$	(g) $10 \cos \theta + 13 \cos \frac{1}{2}\theta = 5$
(b) $2 \cos^2 \theta + \cos 2\theta = 0$	(h) $\tan \theta = 3 \tan \frac{1}{2}\theta$
(c) $2 \cos 2\theta + 4 \cos \theta = 1$	(i) $\cos^2 2\theta = \sin^2 \theta$
(d) $8 \sin^2 \theta \cos^2 \theta = 1$	[HINT: Use $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$.]
(e) $3 \cos 2\theta + \sin \theta = 1$	(j) $\cos 2\theta + 3 = 3 \sin 2\theta$
(f) $\cos 2\theta = 3 \cos^2 \theta - 2 \sin^2 \theta$	[HINT: Write 3 as $3 \cos^2 \theta + 3 \sin^2 \theta$.]

11. (a) Show that $\sqrt{3}u^2 - (1 + \sqrt{3})u + 1 = (\sqrt{3}u - 1)(u - 1)$.

(b) Hence solve the homogeneous equation $\sqrt{3} \sin^2 x + \cos^2 x = (1 + \sqrt{3}) \sin x \cos x$, for $0 \leq x \leq 2\pi$. [HINT: Divide both sides by $\cos^2 x$.]

(c) Similarly, solve $\sin^2 x = (\sqrt{3} - 1) \sin x \cos x + \sqrt{3} \cos^2 x$, for $0 \leq x \leq 2\pi$.

12. Find general solutions of:

(a) $\cos 2x = \cos x$	(c) $\sin x + \cos 2x = 1$
(b) $2 \sin 2x \cos x = \sqrt{3} \sin 2x$	(d) $\sin(x + \frac{\pi}{4}) = 2 \cos(x - \frac{\pi}{4})$

13. Consider the equation $\cos 3x = \cos 2x$.

(a) Show that $x = \frac{2}{5}\pi n$, where n is an integer.

(b) Find all solutions in the domain $0 \leq x \leq 2\pi$.

14. Find the x -coordinates of any stationary points on each of the following curves in the interval $0 \leq x \leq 2\pi$.
- (a) $y = e^{\tan x - 4x}$ (d) $y = \sin x - \frac{1}{2} \cos 2x$
 (b) $y = \ln \cos x + \tan x - x$ (e) $y = \sin x + \frac{1}{2} \sin 2x$
 (c) $y = x + \cos(2x - \frac{\pi}{3})$ (f) $y = 2x - \sin 2x + 2 \sin^2 x$
15. Consider the equation $\tan(\frac{\pi}{4} + \theta) = 3 \tan(\frac{\pi}{4} - \theta)$.
- (a) Show that $\tan^2 \theta - 4 \tan \theta + 1 = 0$.
 (b) Hence use the quadratic formula to solve the equation for $0 \leq \theta \leq \pi$.
16. Given the equation $2 \cos x - 1 = 2 \cos 2x$:
- (a) Show that $\cos x = \frac{1}{4}(1 + \sqrt{5})$ or $\cos x = \frac{1}{4}(1 - \sqrt{5})$.
 (b) Hence solve the equation for $0 \leq x \leq 360^\circ$, using the calculator.
17. (a) Show that $\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$.
 (b) Hence solve the equation $\sin^2 3\theta - \sin^2 \theta = \sin 2\theta$, for $0 \leq \theta \leq \pi$.
18. Use sketches to help solve, for $0 \leq x \leq 2\pi$:
- (a) $\sin^2 x \geq \frac{1}{2}$ (c) $\sin^2 x \geq \cos^2 x$ (e) $2 \cos^2 x \geq \sin x + 2$
 (b) $\tan^2 x < \tan x$ (d) $2 \cos^2 x + \cos x \leq 1$ (f) $\sec^2 x \geq 1 + \sqrt{3} \tan x$
19. Find the values of k for which:
- (a) $\int_0^k \sin^2 x \, dx = \int_0^k \cos^2 x \, dx$ (b) $\int_0^k \sin^2 x \, dx > \int_0^k \cos^2 x \, dx$
20. Sketch the curve $y = e^{\cos x}$, for $0 \leq x \leq 2\pi$, after finding the stationary point and the two inflexion points (approximately).
21. Sketch the curve $y = \frac{\sin x}{1 + \tan x}$, for $0 \leq x \leq 2\pi$, after finding the x -intercepts, the vertical asymptotes and the stationary points. Why are there open circles at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$?
22. (a) Show that the function $y = e^x \tan x$ is increasing for all x in its domain.
 (b) Find the x -intercepts for $-\frac{\pi}{2} < x < \frac{3\pi}{2}$ and the gradient at each x -intercept.
 (c) Show that the curve is concave up at each x -intercept.
 (d) Sketch the curve, for $-\frac{\pi}{2} < x < \frac{3\pi}{2}$.

EXTENSION

23. It was proven in the notes that $\sin 18^\circ = \frac{1}{4}(-1 + \sqrt{5})$ and $\sin 54^\circ = \frac{1}{4}(1 + \sqrt{5})$. Use these results to find the sine, cosine and tangent of 18° , 36° , 54° and 72° .
24. Consider the equation $\sin \theta + \cos \theta = \sin 2\theta$, for $0^\circ \leq \theta \leq 360^\circ$.
- (a) By squaring both sides, show that $\sin^2 2\theta - \sin 2\theta - 1 = 0$.
 (b) Hence solve for θ over the given domain, giving solutions to the nearest minute.
 [HINT: Beware of the fact that squaring can create invalid solutions.]
25. (a) Show that $\cos 3x = 4 \cos^3 x - 3 \cos x$.
 (b) By substituting $x = 2 \cos \theta$, show that the equation $x^3 - 3x - 1 = 0$ has roots $2 \cos 20^\circ$, $-2 \sin 10^\circ$ and $-2 \cos 40^\circ$.
 (c) Use a similar technique to find, correct to three decimal places, the three real roots of the equation $x^3 - 12x = 8\sqrt{3}$.

26. (a) If $t = \tan x$, show that $\tan 4x = \frac{4t(1-t^2)}{1-6t^2+t^4}$.
- (b) If $\tan 4x \tan x = 1$, show that $5t^4 - 10t^2 + 1 = 0$.
- (c) Show that $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ and that $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$.
- (d) Hence show that $\frac{\pi}{10}$ and $\frac{3\pi}{10}$ both satisfy $\tan 4x \tan x = 1$.
- (e) Hence write down, in trigonometric form, the four real roots of the polynomial equation $5x^4 - 10x^2 + 1 = 0$.

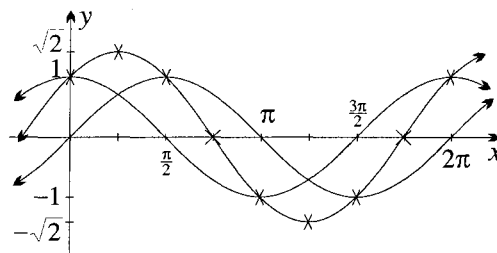
2E The Sum of Sine and Cosine Functions

The sine and cosine curves are the same, except that the sine wave is the cosine wave shifted right by $\frac{\pi}{2}$. This section analyses what happens when the sine and cosine curves are added, and, more generally, when multiples of the two curves are added. The surprising result is that $y = a \sin x + b \cos x$ is still a sine or cosine wave, but shifted sideways so that the zeroes no longer lie on multiples of $\frac{\pi}{2}$.

These forms for $a \sin x + b \cos x$ give a systematic method of solving any equation of the form $a \cos x + b \sin x = c$. Later in the section, an alternative method of solution using the t -formulae is developed.

Sketching $y = \sin x + \cos x$ by Graphical Methods:

The diagram to the right shows the two graphs of $y = \sin x$ and $y = \cos x$. From these two graphs, the sum function $y = \sin x + \cos x$ has been drawn on the same diagram — the crosses represent obvious points to mark on the graph of the sum.



- The new graph has the same period as $y = \sin x$ and $y = \cos x$, that is, 2π . It looks like a wave, and within $0 \leq x \leq 2\pi$ there are zeroes at the two values $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, where $\sin x$ and $\cos x$ take opposite values.
- The new amplitude is bigger than 1. The value at $x = \frac{\pi}{4}$ is $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$, so if the maximum occurs there, as seems likely, the amplitude is $\sqrt{2}$.

This would indicate that the resulting sum function is $y = \sqrt{2} \sin(x + \frac{\pi}{4})$, since it is the stretched sine curve shifted left $\frac{\pi}{4}$. Checking this by expansion:

$$\begin{aligned} \sqrt{2} \sin(x + \frac{\pi}{4}) &= \sqrt{2} (\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}) \\ &= \sin x + \cos x, \text{ as expected, since } \cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}. \end{aligned}$$

The General Algebraic Approach — The Auxiliary Angle: It is true in general that any function of the form $f(x) = a \sin x + b \cos x$ can be written as a single wave function. There are four possible forms in which the wave can be written, and the process is done by expanding the standard form and equating coefficients of $\sin x$ and $\cos x$.

AUXILIARY-ANGLE METHOD:

- Any function of the form $f(x) = a \sin x + b \cos x$, where a and b are constants (not both zero), can be written in any one of the four forms:

$$y = R \sin(x - \alpha) \quad y = R \cos(x - \alpha)$$

$$y = R \sin(x + \alpha) \quad y = R \cos(x + \alpha)$$

where $R > 0$ and $0^\circ \leq \alpha < 360^\circ$. The constant $R = \sqrt{a^2 + b^2}$ is the same for all forms, but the *auxiliary angle* α depends on which form is chosen.

- To begin the process, expand the standard form and equate coefficients of $\sin x$ and $\cos x$. Be careful to identify the quadrant in which α lies.

14

The following worked exercise continues with the example given at the start of the section, and shows the systematic algorithm used to obtain the required form.

WORKED EXERCISE: Express $y = \sin x + \cos x$ in the two forms:

- (a) $R \sin(x + \alpha)$, (b) $R \cos(x + \alpha)$,

where, in each case, $R > 0$ and $0 \leq \alpha < 2\pi$. Then sketch the curve, showing all intercepts and turning points in the interval $0 \leq x \leq 2\pi$.

SOLUTION:

- (a) Expanding, $R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$,
so for all x , $\sin x + \cos x = R \sin x \cos \alpha + R \cos x \sin \alpha$.

Equating coefficients of $\sin x$, $R \cos \alpha = 1$, (1)

equating coefficients of $\cos x$, $R \sin \alpha = 1$. (2)

Squaring and adding, $R^2 = 2$

and since $R > 0$, $R = \sqrt{2}$.

From (1), $\cos \alpha = \frac{1}{\sqrt{2}}$, (1A)

and from (2), $\sin \alpha = \frac{1}{\sqrt{2}}$, (2A)

so α is in the 1st quadrant, with related angle $\frac{\pi}{4}$.

Hence $\sin x + \cos x = \sqrt{2} \sin(x + \frac{\pi}{4})$.

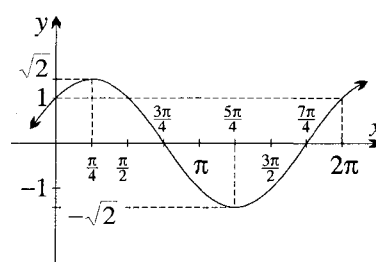
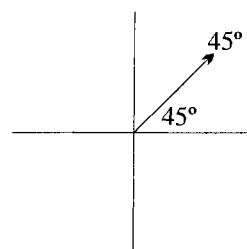
The graph is $y = \sin x$ shifted left by $\frac{\pi}{4}$

and stretched vertically by a factor of $\sqrt{2}$.

Thus the x -intercepts are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$,

there is a maximum of $\sqrt{2}$ when $x = \frac{\pi}{4}$,

and a minimum of $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.



- (b) Expanding, $R \cos(x + \alpha) = R \cos x \cos \alpha - R \sin x \sin \alpha$,
so for all x , $\sin x + \cos x = R \cos x \cos \alpha - R \sin x \sin \alpha$.

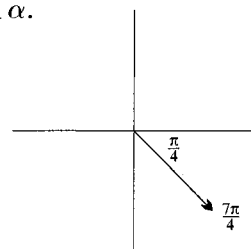
Equating coefficients of $\cos x$, $R \cos \alpha = 1$, (1)

equating coefficients of $\sin x$, $R \sin \alpha = -1$. (2)

Squaring and adding, $R^2 = 2$

and since $R > 0$, $R = \sqrt{2}$.

From (1), $\cos \alpha = \frac{1}{\sqrt{2}}$, (1A)



and from (2),
$$\sin \alpha = -\frac{1}{\sqrt{2}}, \quad (2A)$$

so α is in the 4th quadrant, with related angle $\frac{\pi}{4}$.

Hence
$$\sin x + \cos x = \sqrt{2} \cos\left(x + \frac{7\pi}{4}\right).$$

The graph above could equally well be obtained from this.

It is $y = \cos x$ shifted left by $\frac{7\pi}{4}$ and stretched vertically by a factor of $\sqrt{2}$.

Approximating the Auxiliary Angle: Unless special angles are involved, the auxiliary angle will need to be approximated on the calculator. Degrees or radian measure may be used, but the next worked exercise uses degrees to make the working a little clearer.

WORKED EXERCISE:

- (a) Express $y = 3 \sin x - 4 \cos x$ in the form $y = R \cos(x - \alpha)$, where $R > 0$ and $0^\circ \leq \alpha < 360^\circ$, giving α correct to the nearest degree.
- (b) Sketch the curve, showing, correct to the nearest degree, all intercepts and turning points in the interval $-180^\circ \leq x \leq 180^\circ$.

SOLUTION:

(a) Expanding,
$$R \cos(x - \alpha) = R \cos x \cos \alpha + R \sin x \sin \alpha,$$

so for all x ,
$$3 \sin x - 4 \cos x = R \cos x \cos \alpha + R \sin x \sin \alpha.$$

Equating coefficients of $\cos x$, $R \cos \alpha = -4$, (1)

equating coefficients of $\sin x$, $R \sin \alpha = 3$. (2)

Squaring and adding,
$$R^2 = 25$$

and since $R > 0$,
$$R = 5.$$

From (1),
$$\cos \alpha = -\frac{4}{5}, \quad (1A)$$

and from (2),
$$\sin \alpha = \frac{3}{5}, \quad (2A)$$

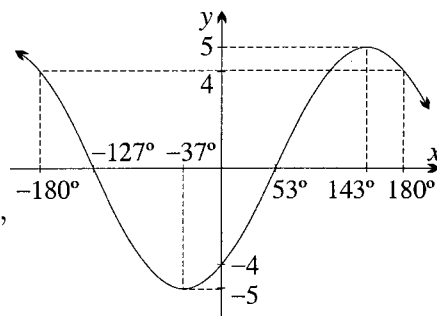
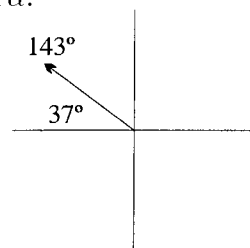
so α is in the 2nd quadrant,

with related angle about 37° .

Hence $3 \sin x - 4 \cos x = 5 \cos(x - \alpha)$,

where $\alpha \doteq 143^\circ$.

- (b) The graph is $y = \cos x$ shifted right by $\alpha \doteq 143^\circ$ and stretched vertically by a factor of 5. Thus the x -intercepts are $x \doteq 53^\circ$ and $x \doteq -127^\circ$, there is a maximum of 5 when $x \doteq 143^\circ$, and a minimum of -5 when $x \doteq -37^\circ$.



A Note on the Calculator and Approximations for the Auxiliary Angle: In the previous worked exercise, the exact value of α is $\alpha = 180^\circ - \sin^{-1} \frac{3}{5}$ (or $\alpha = 180^\circ - \cos^{-1} \frac{4}{5}$), because α is in the second quadrant. It is this value which is obtained on the calculator, and if there are subsequent calculations to do, as in the equation solved below, this value should be stored in memory and used whenever the auxiliary angle is required. Re-entry of the approximation may lead to rounding errors.

Solving Equations of the Form $a \sin x + b \cos x = c$, and Inequations: Once the LHS has been put in one of the four standard forms, the solutions can easily be obtained. It is always important to keep track of the restriction on the compound angle. The worked exercise below continues with the previous example.

WORKED EXERCISE:

- (a) Using the previous worked exercise, solve the equation $3 \sin x - 4 \cos x = -2$, for $-180^\circ \leq x \leq 180^\circ$, correct to the nearest degree.
- (b) Hence use the graph to solve $3 \sin x - 4 \cos x \leq -2$, for $-180^\circ \leq x \leq 180^\circ$.

SOLUTION:

- (a) Using $3 \sin x - 4 \cos x = 5 \cos(x - \alpha)$, where $\alpha \doteq 143^\circ$,
 $5 \cos(x - \alpha) = -2$, where $-323^\circ \leq x - \alpha \leq 37^\circ$
 $\cos(x - \alpha) = -\frac{2}{5}$.

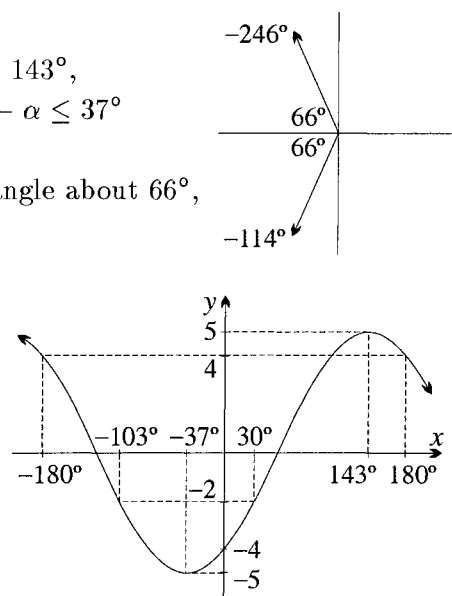
Hence $x - \alpha$ is in quadrant 2 or 3, with related angle about 66° ,so $x - \alpha \doteq -114^\circ$ or -246°

$$x \doteq 30^\circ \text{ or } -103^\circ.$$

Be careful to use the calculator's memory here.

Never re-enter approximations of the angles.

- (b) The graph to the right shows the previously drawn graph of $y = 3 \sin x - 4 \cos x$ with the horizontal line $y = -2$ added. This roughly verifies the two answers obtained in part (a). It also shows that the solution to the inequality $3 \sin x - 4 \cos x \leq -2$ is $-103^\circ \leq x \leq 30^\circ$.



Using the t -formulae to Solve $a \sin x + b \cos x = c$: The t -formulae provide a quite different method of solution by substituting $t = \tan \frac{1}{2}x$. The advantage of this method is that only a single approximation is involved. There are two disadvantages. First, the intuition about the LHS being a shifted wave function is lost. Secondly, if $x = 180^\circ$ happens to be a solution, it will not be found by this method, because $\tan \frac{1}{2}x$ is not defined at $x = 180^\circ$.

WORKED EXERCISE: Solve $3 \sin x - 4 \cos x = -2$, for $-180^\circ \leq x \leq 180^\circ$, correct to the nearest minute, using the substitution $t = \tan \frac{1}{2}x$.

SOLUTION: Using $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$, the equation becomes

$$\frac{6t}{1+t^2} - \frac{4-4t^2}{1+t^2} = -2, \text{ provided that } x \neq 180^\circ,$$

$$6t - 4 + 4t^2 = -2 - 2t^2$$

$$6t^2 + 6t - 2 = 0$$

$$3t^2 + 3t - 1 = 0, \text{ which has discriminant } \Delta = 21,$$

$$\tan \frac{1}{2}x = -\frac{1}{2} + \frac{1}{6}\sqrt{21} \text{ or } -\frac{1}{2} - \frac{1}{6}\sqrt{21}.$$

Since $-180^\circ \leq x \leq 180^\circ$, the restriction on $\frac{1}{2}x$ is $-90^\circ \leq \frac{1}{2}x \leq 90^\circ$,

so $\frac{1}{2}x = 14.775\,961\dots^\circ$ or $-51.645\,859\dots^\circ$

$$x \doteq 29^\circ 33' \text{ or } -103^\circ 18'.$$

The Problem when $x = 180^\circ$ is a Solution: The substitution $t = \tan \frac{1}{2}x$ fails when $x = 180^\circ$, because $\tan 90^\circ$ is undefined. One must always be aware, therefore, of this possibility, and be prepared to add this answer to the final solution. The situation can easily be recognised in either of the following ways:

- The terms in t^2 cancel out, leaving a linear equation in t .
- The coefficient of $\cos x$ is the opposite of the constant term.

WORKED EXERCISE: Solve $7 \sin x - 4 \cos x = 4$, for $0^\circ \leq x \leq 360^\circ$, by using the substitution $t = \tan \frac{1}{2}x$.

SOLUTION: Substituting $t = \tan \frac{1}{2}x$ gives

$$\frac{14t}{1+t^2} - \frac{4-4t^2}{1+t^2} = 4, \text{ provided that } x \neq 180^\circ,$$

$$14t - 4 + 4t^2 = 4 + 4t^2$$

$$14t = 8.$$

[WARNING: The terms in t^2 cancelled out — check $t = 180^\circ$!]

$$\text{Hence } \tan \frac{1}{2}x = \frac{4}{7}$$

$$x \doteq 59^\circ 29'.$$

But $x = 180^\circ$ is also a solution, since then $\text{LHS} = 7 \times 0 - 4 \times (-1) = \text{RHS}$,
so $x = 180^\circ$ or $x \doteq 59^\circ 29'$.

A Summary of Methods of Solving $a \sin x + b \cos x = c$: Here then is a summary of the two approaches to the solution.

SOLVING EQUATIONS OF THE FORM $a \sin x + b \cos x = c$:

- **THE AUXILIARY-ANGLE METHOD:** Get the LHS into one of the forms

$$R \sin(x + \alpha), \quad R \sin(x - \alpha), \quad R \cos(x + \alpha) \quad \text{or} \quad R \cos(x - \alpha),$$

15 then solve the resulting equation.

- **USING THE t -FORMULAE:** Substitute $t = \tan \frac{1}{2}x$ and then solve the resulting quadratic in t . Be aware that $x = 180^\circ$ will also be a solution if:
 - * the terms in t^2 cancel out, leaving a linear equation in t , or equivalently,
 - * the coefficient of $\cos x$ is the opposite of the constant term.

Exercise 2E

- Find R and α exactly, if $R > 0$ and $0 \leq \alpha < 2\pi$, and:
 - $R \sin \alpha = \sqrt{3}$ and $R \cos \alpha = 1$,
 - $R \sin \alpha = 3$ and $R \cos \alpha = 3$.
- Find R (exactly) and α (correct to the nearest minute), if $R > 0$ and $0^\circ \leq \alpha < 360^\circ$, and:
 - $R \sin \alpha = 5$ and $R \cos \alpha = 12$,
 - $R \cos \alpha = 2$ and $R \sin \alpha = 4$.
- If $\cos x - \sin x = A \cos(x + \alpha)$, show that $A \cos \alpha = 1$ and $A \sin \alpha = 1$.
 - Find the positive value of A by squaring and adding.
 - Find α , if $0 \leq \alpha < 2\pi$.
 - State the maximum and minimum values of $\cos x - \sin x$ and the first positive values of x for which they occur.
 - Solve the equation $\cos x - \sin x = -1$, for $0 \leq x \leq 2\pi$.
 - Write down the amplitude and period of $\cos x - \sin x$. Hence sketch $y = \cos x - \sin x$, for $0 \leq x \leq 2\pi$. Indicate on your sketch the line $y = -1$ and the solutions to the equation in part (e).
- Sketch $y = \cos x$ and $y = \sin x$ on one set of axes. Then, by taking differences of heights, sketch $y = \cos x - \sin x$. Compare your sketch with that in the previous question.

5. (a) If $\sqrt{3} \cos x - \sin x = B \cos(x + \theta)$, show that $B \cos \theta = \sqrt{3}$ and $B \sin \theta = 1$.
 (b) Find B , if $B > 0$, by squaring and adding.
 (c) Find θ , if $0 \leq \theta < 2\pi$.
 (d) State the greatest and least possible values of $\sqrt{3} \cos x - \sin x$ and the values of x closest to $x = 0$ for which they occur.
 (e) Solve the equation $\sqrt{3} \cos x - \sin x = 1$, for $0 \leq x \leq 2\pi$.
 (f) Sketch $y = \sqrt{3} \cos x - \sin x$, for $0 \leq x \leq 2\pi$. On the same diagram, sketch the line $y = 1$. Indicate on your diagram the solutions to the equation in part (e).
6. Let $4 \sin x - 3 \cos x = A \sin(x - \alpha)$, where $A > 0$ and $0^\circ \leq \alpha < 360^\circ$.
 (a) Show that $A \cos \alpha = 4$ and $A \sin \alpha = 3$.
 (b) Show that $A = 5$ and $\alpha = \tan^{-1} \frac{3}{4}$.
 (c) Hence solve the equation $4 \sin x - 3 \cos x = 5$, for $0^\circ \leq x \leq 360^\circ$. Give the solution(s) correct to the nearest minute.
7. Consider the equation $2 \cos x + \sin x = 1$.
 (a) Let $2 \cos x + \sin x = B \cos(x - \theta)$, where $B > 0$ and $0^\circ \leq \theta < 360^\circ$. Show that $B = \sqrt{5}$ and $\theta = \tan^{-1} \frac{1}{2}$.
 (b) Hence find, correct to the nearest minute where necessary, the solutions of the equation, for $0^\circ \leq x \leq 360^\circ$.
8. Let $\cos x - 3 \sin x = D \cos(x + \phi)$, where $D > 0$ and $0^\circ \leq \phi < 360^\circ$.
 (a) Show that $D = \sqrt{10}$ and $\phi = \tan^{-1} 3$.
 (b) Hence solve $\cos x - 3 \sin x = 3$, for $0^\circ \leq x \leq 360^\circ$. Give the solutions correct to the nearest minute where necessary.
9. Consider the equation $\sqrt{5} \sin x + 2 \cos x = -2$.
 (a) Transform the LHS into the form $C \sin(x + \alpha)$, where $C > 0$ and $0^\circ \leq \alpha < 360^\circ$.
 (b) Find, correct to the nearest minute where necessary, the solutions of the equation, for $0^\circ \leq x \leq 360^\circ$.
10. Solve each equation, for $0^\circ \leq x \leq 360^\circ$, by transforming the LHS into a single-term sine or cosine function. Give solutions correct to the nearest minute.
 (a) $3 \sin x + 5 \cos x = 4$ (c) $7 \cos x - 2 \sin x = 5$
 (b) $6 \sin x - 5 \cos x = 7$ (d) $9 \cos x + 7 \sin x = 3$
11. Consider the equation $\cos x - \sin x = 1$, where $0 \leq x \leq 2\pi$.
 (a) Using the substitutions $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$, where $t = \tan \frac{1}{2}x$, show that the equation can be written as $t^2 + t = 0$.
 (b) Hence show that $\tan \frac{1}{2}x = 0$ or -1 , where $0 \leq \frac{1}{2}x \leq \pi$.
 (c) Hence solve the given equation for x .
12. Consider the equation $\sqrt{3} \sin x + \cos x = 1$.
 (a) Show that the equation can be written as $t^2 = \sqrt{3}t$, where $t = \tan \frac{1}{2}x$.
 (b) Hence solve the equation, for $0 \leq x \leq 2\pi$.
13. (a) Show that the equation $4 \cos x + \sin x = 1$ can be written as $(5t + 3)(t - 1) = 0$, where $t = \tan \frac{1}{2}x$.
 (b) Hence solve the equation, for $0^\circ \leq x \leq 360^\circ$. Give the solutions correct to the nearest minute where necessary.

14. (a) Show that the equation $3 \sin x - 2 \cos x = 2$ can be written as $3t - 2 = 0$, where $t = \tan \frac{1}{2}x$.
 (b) Hence solve the equation for $0^\circ \leq x \leq 360^\circ$, giving solutions correct to the nearest minute where necessary. (Remember to check $x = 180^\circ$ as a possible solution, given that the resulting equation in t is linear.)
15. (a) Show that the equation $6 \sin x - 4 \cos x = 5$ can be written as $t^2 - 12t + 9 = 0$, where $t = \tan \frac{1}{2}x$.
 (b) Show that $\tan \frac{1}{2}x = 6 + 3\sqrt{3}$ or $6 - 3\sqrt{3}$.
 (c) Hence show that $77^\circ 35'$ and $169^\circ 48'$ are the solutions (to the nearest minute) of the given equation over the domain $0^\circ \leq x \leq 360^\circ$.
16. Solve each equation, for $0^\circ \leq x \leq 360^\circ$, by using the $t = \tan \frac{1}{2}x$ results. Give solutions correct to the nearest minute where necessary.
- | | |
|-------------------------------|--------------------------------|
| (a) $5 \sin x + 4 \cos x = 5$ | (c) $3 \sin x - 2 \cos x = 1$ |
| (b) $7 \cos x - 6 \sin x = 2$ | (d) $5 \cos x + 6 \sin x = -5$ |

————— DEVELOPMENT —————

17. Find A and α exactly, if $A > 0$ and $0 \leq \alpha < 2\pi$, and:
 (a) $A \sin \alpha = 1$ and $A \cos \alpha = -\sqrt{3}$, (b) $A \cos \alpha = -5$ and $A \sin \alpha = -5$.
18. Find A (exactly) and α (correct to the nearest minute), if $A > 0$ and $0^\circ \leq \alpha < 360^\circ$, and:
 (a) $A \cos \alpha = 5$ and $A \sin \alpha = -4$, (b) $A \sin \alpha = -11$ and $A \cos \alpha = -2$.
19. (a) (i) Express $\sqrt{3} \cos x + \sin x$ in the form $A \cos(x + \theta)$, where $A > 0$ and $0 < \theta < 2\pi$.
 (ii) Hence solve $\sqrt{3} \cos x + \sin x = 1$, for $0 \leq x < 2\pi$.
 (b) (i) Express $\cos x - \sin x$ in the form $B \sin(x + \alpha)$, where $B > 0$ and $0 < \alpha < 2\pi$.
 (ii) Hence solve $\cos x - \sin x = 1$, for $0 \leq x < 2\pi$.
 (c) (i) Express $\sin x - \sqrt{3} \cos x$ in the form $C \sin(x + \beta)$, where $C > 0$ and $0 < \beta < 2\pi$.
 (ii) Hence solve $\sin x - \sqrt{3} \cos x = -1$, for $0 \leq x < 2\pi$.
 (d) (i) Express $-\cos x - \sin x$ in the form $D \cos(x - \phi)$, where $D > 0$ and $0 < \phi < 2\pi$.
 (ii) Hence solve $-\cos x - \sin x = 1$, for $0 \leq x < 2\pi$.
20. (a) (i) Express $2 \cos x - \sin x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 360^\circ$. (Write α to the nearest minute.)
 (ii) Hence solve $2 \cos x - \sin x = 1$, for $0^\circ \leq x < 360^\circ$. Give the solutions correct to the nearest minute where necessary.
 (b) (i) Express $-3 \sin x - 4 \cos x$ in the form $S \cos(x - \beta)$, where $S > 0$ and $0 < \beta < 2\pi$. (Write β to four decimal places.)
 (ii) Hence solve $-3 \sin x - 4 \cos x = 2$, for $0 \leq x < 2\pi$. Give the solutions correct to two decimal places.
21. (a) (i) Show that $\sin x - \cos x = \sqrt{2} \sin(x - \frac{\pi}{4})$.
 (ii) Hence sketch the graph of $y = \sin x - \cos x$, for $0 \leq x \leq 2\pi$.
 (iii) Use your sketch to determine the values of x in the domain $0 \leq x \leq 2\pi$ for which $\sin x - \cos x > 1$.
 (b) Use a similar approach to that in part (a) to solve, for $0 \leq x \leq 2\pi$:
 (i) $\sin x + \sqrt{3} \cos x \leq 1$ (iii) $|\sqrt{3} \sin x + \cos x| < 1$
 (ii) $\sin x - \sqrt{3} \cos x < -1$ (iv) $\cos x - \sin x \geq \frac{1}{2}\sqrt{2}$

22. Solve, for $0 \leq x \leq 2\pi$:

(a) $\sin x - \cos x = \sqrt{1.5}$ (b) $\sqrt{3} \sin 2x - \cos 2x = 2$ (c) $\sin 4x + \cos 4x = 1$

23. Solve, for $0^\circ \leq x \leq 360^\circ$. Give solutions correct to the nearest minute:

(a) $2 \sec x - 2 \tan x = 5$ (b) $2 \operatorname{cosec} x + 5 \cot x = 3$

24. Suppose that $a \cos x = 1 + \sin x$, where $0^\circ < x < 90^\circ$:

(a) Prove that $\frac{a-1}{a+1} = t$, where $t = \tan \frac{1}{2}x$.

(b) Hence find, to the nearest minute, the acute angle x that satisfies $2 \cos x - \sin x = 1$.

25. Solve the equation $\sin \theta + \cos \theta = \cos 2\theta$, for $0 \leq \theta \leq 2\pi$.

26. (a) Show that $(\sqrt{3} + 1) \cos 2x + (\sqrt{3} - 1) \sin 2x = 2\sqrt{2} \cos(2x - \frac{\pi}{12})$.

(b) Hence find the general solution of $(\sqrt{3} + 1) \cos 2x + (\sqrt{3} - 1) \sin 2x = 2$.

EXTENSION

27. (a) Prove that: (i) $\sin \theta = \cos(\theta - \frac{\pi}{2})$ (ii) $\cos \theta = \sin(\theta + \frac{\pi}{2})$

(b) Use the result $\sin x + \sqrt{3} \cos x = 2 \sin(x + \frac{\pi}{3})$ to express $\sin x + \sqrt{3} \cos x$ in each of the other three standard forms.

(c) Repeat part (b) using the result $\cos x - \sin x = \sqrt{2} \cos(x + \frac{\pi}{4})$. Then sketch the functions $y = \sqrt{2} \cos x$ and $y = \sqrt{2} \sin x$.

28. (a) Prove that $\sin(\theta + \pi) = -\sin \theta$.

(b) Given that $\sqrt{3} \sin x + \cos x = 2 \sin(x + \frac{\pi}{6})$, use appropriate reflections in the x - and y -axes, the fact that $\sin x$ is odd and $\cos x$ is even, and part (a) to prove that:

(i) $-\sqrt{3} \sin x + \cos x = 2 \sin(x + \frac{5\pi}{6})$ (ii) $-\sqrt{3} \sin x - \cos x = 2 \sin(x + \frac{7\pi}{6})$

(iii) $\sqrt{3} \sin x - \cos x = 2 \sin(x - \frac{\pi}{6})$

29. Consider the equation $a \cos x + b \sin x = c$, where a , b and c are constants.

(a) Show that the equation can be written in the form $(a+c)t^2 - 2bt - (a-c) = 0$, where $t = \tan \frac{1}{2}x$. (b) Show that the root(s) of the equation are real if $c^2 \leq a^2 + b^2$.

(c) Suppose that $\tan \frac{1}{2}\alpha$ and $\tan \frac{1}{2}\beta$ are distinct real roots of the quadratic equation in part (a). Prove that $\tan \frac{1}{2}(\alpha + \beta) = b/a$.

30. Consider the equation $a \cos x + b \sin x = c$, where a , b and c are positive constants. Let $a = r \cos \theta$ and $b = r \sin \theta$, where θ is acute.

(a) Show that $a^2 + b^2 = r^2$ and that $\tan \theta = \frac{b}{a}$.

(b) Show that the equation becomes $\cos(x - \theta) = \frac{c}{r}$, and hence write down the general solution.

(c) Show that there are no real roots if $c > \sqrt{a^2 + b^2}$.

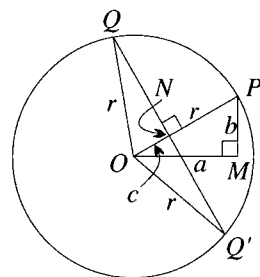
(d) In the diagram opposite, the circle has centre O and radius $OP = r$. Suppose that $OM = a$, $MP = b$ and $MP \perp OM$. Suppose also that $ON = c$ and the chord QNQ' is drawn perpendicular to OP .

(i) State the size of $\angle MOP$.

(ii) Show that $\cos \angle NOQ = \cos \angle NOQ' = c/r$.

(iii) State which part of the general solution in part (b) contains $\angle MOQ$, and which part contains $\angle MOQ'$.

(e) What condition corresponds geometrically to the condition $c > \sqrt{a^2 + b^2}$, for which the equation has no real roots?



2F Extension — Products to Sums and Sums to Products

This section concerns a set of identities that convert the sum of two sine or cosine functions to the product of two sine or cosine functions, and vice versa. For example, we shall show that

$$\sin 3x + \sin 11x = 2 \sin 7x \cos 4x.$$

The product form on the right is important for purposes such as finding the zeroes of the function. The sum form on the left is important, for example, when integrating the function.

These identities form part of the 4 Unit course, but are not required in the 3 Unit course. The worked exercises give only the examples mentioned above of their use, but the exercise following gives a fuller range of their applications.

Products to Sums: We begin with the four compound-angle formulae involving sine and cosine:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad (1A)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad (1B)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (1C)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (1D)$$

Adding and subtracting equations (1A) and (1B), then adding and subtracting equations (1C) and (1D), gives the four products-to-sums formulae:

16	PRODUCTS TO SUMS: $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$ $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$ $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$ $-2 \sin A \sin B = \cos(A + B) - \cos(A - B)$

WORKED EXERCISE: Find $\int_0^{\frac{\pi}{3}} \sin 7x \cos 4x \, dx$.

SOLUTION:

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sin 7x \cos 4x \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{3}} (\sin 3x + \sin 11x) \, dx \\ &= \left[-\frac{1}{6} \cos 3x - \frac{1}{22} \cos 11x \right]_0^{\frac{\pi}{3}} \\ &= -\frac{1}{6} \cos 0 + \frac{1}{6} \cos \pi - \frac{1}{22} \cos 0 + \frac{1}{22} \cos \frac{11\pi}{3} \\ &= -\frac{1}{6} - \frac{1}{6} - \frac{1}{22} + \frac{1}{44} \\ &= -\frac{47}{132} \end{aligned}$$

Sums to Products: The previous formulae can be reversed to become formulae for sums to products by making a simple pair of substitutions. Let

$$S = A + B \quad \text{and} \quad T = A - B.$$

Then adding and subtracting these formulae gives

$$A = \frac{1}{2}(S + T) \quad \text{and} \quad B = \frac{1}{2}(S - T).$$

Substituting these into the products-to-sums formulae, and reversing them:

SUMS TO PRODUCTS:**17**

$$\sin S + \sin T = 2 \sin \frac{1}{2}(S + T) \cos \frac{1}{2}(S - T)$$

$$\sin S - \sin T = 2 \cos \frac{1}{2}(S + T) \sin \frac{1}{2}(S - T)$$

$$\cos S + \cos T = 2 \cos \frac{1}{2}(S + T) \cos \frac{1}{2}(S - T)$$

$$\cos S - \cos T = -2 \sin \frac{1}{2}(S + T) \sin \frac{1}{2}(S - T)$$

WORKED EXERCISE: Solve $\sin 3x + \sin 11x = 0$, for $0 \leq x \leq \pi$:

(a) using sums to products,

(b) using solutions to $\sin \alpha = \sin \beta$.**SOLUTION:**(a) $\sin 3x + \sin 11x = 0$ $2 \sin 7x \cos 4x = 0$, using sums to products,

$\sin 7x = 0$

or $\cos 4x = 0$

$7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi$ or $4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$,

so $x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$ or $\frac{7\pi}{8}$.

(b) Alternatively, $\sin 11x = \sin(-3x)$, since $\sin \theta$ is odd,so, using the solutions of $\sin \alpha = \sin \beta$,

$11x = -3x + 2n\pi$ or $11x = 3x + (2n + 1)\pi$, where n is an integer,

$7x = n\pi$ or $4x = (n + \frac{1}{2})\pi$, giving the same answers as before.

Exercise 2F

1. (a) Establish the following identities by expanding the RHS:

(i) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

(b) Hence express as a sum or difference of trigonometric functions:

(i) $2 \cos 35^\circ \cos 15^\circ$

(iii) $2 \sin 3\alpha \cos \alpha$

(ii) $2 \cos 48^\circ \sin 32^\circ$

(iv) $2 \sin(x + y) \sin(x - y)$

(c) Use the products-to-sums identities to prove that

$$2 \sin 3\theta \cos 2\theta + 2 \cos 6\theta \sin \theta = \sin 7\theta + \sin \theta.$$

2. (a) Let $P = A + B$ and $Q = A - B$ in the identities in part (a) of the previous question to establish these identities:

(i) $\sin P + \sin Q = 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$

(ii) $\sin P - \sin Q = 2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$

(iii) $\cos P + \cos Q = 2 \cos \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$

(iv) $\cos P - \cos Q = -2 \sin \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$

(b) Hence express as products:

(i) $\cos 16^\circ + \cos 12^\circ$

(iii) $\sin 6x + \sin 4x$

(ii) $\sin 56^\circ - \sin 20^\circ$

(iv) $\cos(2x + 3y) - \cos(2x - 3y)$

(c) Use the sums-to-products identities to prove that:

(i) $\sin 35^\circ + \sin 25^\circ = \sin 85^\circ$

(ii) $\frac{\sin 3\theta + \sin \theta}{\cos 3\theta + \cos \theta} = \tan 2\theta$

3. (a) (i) Show that $\sin 3x + \sin x = 2 \sin 2x \cos x$.
 (ii) Hence solve the equation $\sin 3x + \sin x = 0$, for $0 \leq x \leq \pi$.
 (b) Using a similar method, solve $\cos 3x + \cos x = 0$, for $0 \leq x \leq \pi$.
4. (a) (i) Show that $2 \sin 3x \cos x = \sin 4x + \sin 2x$.
 (ii) Hence find $\int 2 \sin 3x \cos x \, dx$.
 (b) Using a similar method, find $\int 2 \cos 3x \cos x \, dx$.

DEVELOPMENT

5. Prove the following identities:

(a) $\frac{\cos 6\alpha - \cos 4\alpha + \cos 2\alpha}{\sin 6\alpha - \sin 4\alpha + \sin 2\alpha} = \cot 4\alpha$

(b) $4 \cos 4x \cos 2x \cos x = \cos 7x + \cos 5x + \cos 3x + \cos x$

6. Evaluate: (a) $\int_0^{\frac{\pi}{12}} \cos 4x \sin 2x \, dx$ (b) $\int_0^{\frac{\pi}{24}} \sin 5x \sin x \, dx$

7. [These identities are known as the *orthogonality relations*.]

- (a) If m and n are positive integers, use the products-to-sums identities to prove:

(i) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

(ii) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & \text{for } m \neq n, \\ \pi, & \text{for } m = n. \end{cases}$

(iii) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{for } m \neq n, \\ \pi, & \text{for } m = n. \end{cases}$

- (b) The functions $f(x)$ and $g(x)$ are defined by $f(x) = \sin x + 2 \sin 2x + 3 \sin 3x + 4 \sin 4x$, and $g(x) = \cos x + 2 \cos 2x + 3 \cos 3x + 4 \cos 4x$. Use parts (a)(i), (ii) and (iii) to find:

(i) $\int_{-\pi}^{\pi} f(x)g(x) \, dx$ (ii) $\int_{-\pi}^{\pi} (f(x))^2 \, dx$ (iii) $\int_{-\pi}^{\pi} (g(x))^2 \, dx$

8. (a) (i) Use the result $2 \sin B \cos A = \sin(A+B) - \sin(A-B)$ to show that

$$2 \sin x (\cos 2x + \cos 4x + \cos 6x) = \sin 7x - \sin x.$$

- (ii) Deduce that $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$ and hence $\cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7} = \frac{1}{2}$.

- (b) Use the result $\sin A \sin B = \frac{1}{2} (\cos(A-B) - \cos(A+B))$ to prove that

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}.$$

9. (a) Express $\sin 3x + \sin x$ as a product.

- (b) Hence solve the equation $\sin 3x + \sin 2x + \sin x = 0$, for $0 \leq x < 2\pi$.

10. Solve each of the following equations, for $0 \leq x \leq \pi$:

(a) $\cos 5x + \cos x = 0$

(d) $\cos 4x + \cos 2x = \cos 3x + \cos x$

(b) $\sin 4x - \sin x = 0$

(e) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

(c) $\cos 3x + \cos 5x = \cos 4x$

(f) $\sin 5x \cos 4x = \sin 3x \cos 2x$

11. (a) Solve the equation $\cos 5x = \sin x$, for $0 \leq x \leq \pi$. [Write $\sin x$ as $\cos(\frac{\pi}{2} - x)$.]

- (b) Find general solutions of the equation $\sin 3x = \cos 2x$. [Write $\cos 2x$ as $\sin(\frac{\pi}{2} - 2x)$.]

EXTENSION

12. [The three angles of a triangle] If A , B and C are the three angles of any triangle, prove:

- (a) $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$
- (b) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- (c) $\sin^2 B + \sin^2 C - \sin^2 A = 2 \cos A \sin B \sin C$
- (d) $\frac{\sin 2A - \sin 2B + \sin 2C}{\sin^2 A - \sin^2 B + \sin^2 C} = 2 \cot A \tan B \cot C$

13. Consider the definite integral $D = \int_{-\pi}^{\pi} \cos \lambda x \cos nx \, dx$, where n is a positive integer, and λ is any positive real number.

- (a) Show that $D = \begin{cases} \pi, & \text{for } \lambda = n, \\ 0, & \text{for } \lambda \text{ a positive integer, } \lambda \neq n, \\ \frac{(-1)^n 2\lambda \sin \lambda\pi}{\lambda^2 - n^2}, & \text{for } \lambda \text{ not an integer.} \end{cases}$
- (b) Show that when $0 < \lambda < n$, $|D| < \pi$.
- (c) Show that when $\lambda > n + \frac{1}{2}$, $|D| < 3$.
- (d) Is π the maximum value of $|D|$, for all positive integers n and all positive reals λ ?

2G Three-Dimensional Trigonometry

Trigonometry, in its application to mensuration problems, essentially deals with triangles, which are two-dimensional objects. Hence when trigonometry is applied to a three-dimensional problem, the diagram must be broken up into a collection of triangles in space, and trigonometry used for each in turn.

Three-dimensional work, however, requires two new ideas about angles — the angle between a line and a plane, and the angle between two planes — and these angles will need to be defined and discussed.

Trigonometry and Pythagoras' Theorem in Three Dimensions: As remarked above, every three-dimensional problem in trigonometry requires a careful sketch showing the triangles where trigonometry and Pythagoras' theorem are to be applied.

TRIGONOMETRY AND PYTHAGORAS' THEOREM IN THREE DIMENSIONS:

1. Draw a careful sketch of the situation.
2. Note carefully all the triangles in the figure.
3. Mark all right angles in these triangles.
4. Always state which triangle you are working with.

WORKED EXERCISE: The rectangular prism $ABCDEFGH$ sketched below has sides of length $AB = 5$ cm, $BC = 4$ cm and $AE = 3$ cm.

- (a) Find the lengths of the three diagonals AC , AF and FC .
- (b) Find the angle $\angle CAF$ between the diagonals AC and AF .
- (c) Find the length of the space diagonal AG .
- (d) Find the angle between the space diagonal AG and the edge AB .

SOLUTION:

(a) In $\triangle ABC$, $AC^2 = 5^2 + 4^2$, using Pythagoras,

so $AC = \sqrt{41}$ cm.

In $\triangle ABF$, $AF^2 = 5^2 + 3^2$, using Pythagoras,

so $AF = \sqrt{34}$ cm.

In $\triangle FBC$, $FC^2 = 3^2 + 4^2$, using Pythagoras,

so $FC = 5$ cm.

$$(b) \text{ In } \triangle CAF, \cos \angle CAF = \frac{41 + 34 - 25}{2 \times \sqrt{41} \times \sqrt{34}}, \text{ using the cosine rule,}$$

$$= \frac{50}{2 \times \sqrt{41} \times \sqrt{34}},$$

so $\angle CAF \doteq 47^\circ 58'$.

(c) In $\triangle ACG$, $AG^2 = AC^2 + CG^2$, since $AC \perp CG$,

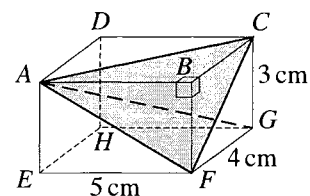
$= 41 + 9,$

so $AG = 5\sqrt{2}$.

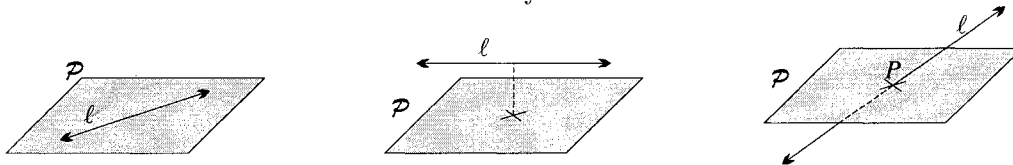
(d) In $\triangle BAG$, $\cos \angle BAG = \frac{5}{5\sqrt{2}}$, since $AB \perp BG$,

$= \frac{1}{\sqrt{2}},$

so $\angle BAG = 45^\circ$.



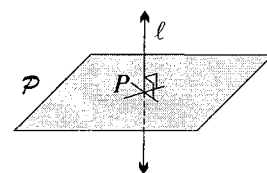
The Angle Between a Line and a Plane: In three-dimensional space, a plane $\mathcal{P}$ and a line ℓ can be related in three different ways:



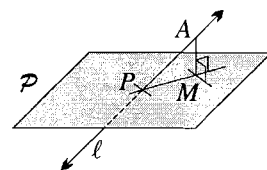
In the first diagram above, the line lies wholly within the plane. In the second diagram, the line never meets the plane, and the plane and the line are said to be *parallel*. In the third diagram, the line meets the plane in a single point P called the *intersection* of $\mathcal{P}$ and ℓ .

When the line ℓ meets the plane $\mathcal{P}$ in the single point P , it can do so in two distinct ways.

In the upper diagram to the right, the line ℓ is perpendicular to every line in the plane through P , and we say that ℓ is *perpendicular to the plane* $\mathcal{P}$.



In the lower diagram to the right, ℓ is not perpendicular to $\mathcal{P}$, and we define the angle θ between the line and the plane as follows. Choose any other point A on ℓ , and then construct the point M in the plane $\mathcal{P}$ so that $AM \perp \mathcal{P}$. Then $\angle APM$ is defined to be the angle between the plane and the line.



WORKED EXERCISE: Find the angle between a slant edge and the base in a square pyramid of height 8 metres whose base has side length 12 metres.

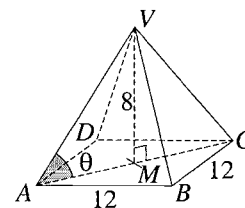
SOLUTION: Using Pythagoras' theorem in the base $ABCD$, the diagonal AC has length $12\sqrt{2}$ metres.

The perpendicular from the vertex V to the base meets the base at the midpoint M of the diagonal AC .

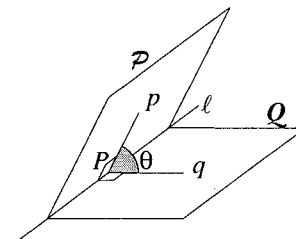
$$\begin{aligned}\text{In } \triangle MAV, \tan \angle MAV &= \frac{8}{6\sqrt{2}} \\ &= \frac{2}{3}\sqrt{2},\end{aligned}$$

$$\text{so } \angle MAV \doteq 43^\circ 19',$$

and this is the angle between the edge and the base.



The Angle Between Two Planes: In three-dimensional space, any two planes that are not parallel meet in a line ℓ , called the *line of intersection of the two planes*. Take any point P on this line of intersection, and construct the lines p and q perpendicular to this line of intersection and lying in the planes $\mathcal{P}$ and $\mathcal{Q}$ respectively. The angle between the planes $\mathcal{P}$ and $\mathcal{Q}$ is defined to be the angle between these two lines.



- 19 THE ANGLE BETWEEN TWO PLANES:** Suppose that the line p in the plane $\mathcal{P}$ and the line q in the plane $\mathcal{Q}$ meet at the point P on the line ℓ of intersection of the planes, and are both perpendicular to ℓ . Then the angle between the planes is the angle between the lines p and q .

WORKED EXERCISE: In the pyramid of the previous worked exercise, find the angle between an oblique face of the pyramid and the base.

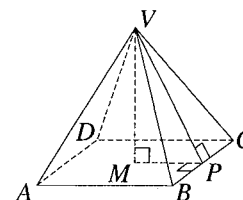
SOLUTION: Let P be the midpoint of the edge BC .

Then $VP \perp BC$ and $MP \perp BC$,

so $\angle VPM$ is the angle required.

$$\text{Now } \tan \angle VPM = \frac{8}{6}$$

$$\text{so } \angle VPM \doteq 53^\circ 8'.$$



WORKED EXERCISE: [A harder question] A 2 metre $\times$ 3 metre rectangular sheet of metal leans lengthwise against a corner of a room, with its top vertices equidistant from the corner and 2 metres above the ground.

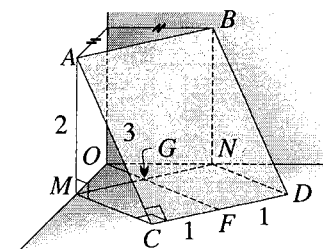
- What is the angle between the sheet of metal and the floor.
- How far is the bottom edge of the sheet from the corner of the floor?

SOLUTION:

- The diagram shows the piece of metal $ABCD$ and the corner O of the floor. The vertical line down the wall from A meets the floor at M . Notice that $AC \perp CD$ and $MC \perp CD$, so $\angle ACM$ is the angle between the sheet and the floor.

$$\text{In } \triangle ACM, \sin \angle ACM = \frac{2}{3},$$

$$\text{so } \angle ACM \doteq 41^\circ 49'.$$



- Let the vertical line down the wall from B meet the floor at N . Let F be the midpoint of CD , and G be the midpoint of MN . Then OGF is the closest distance between the bottom edge CD of the sheet and the corner O of the floor.

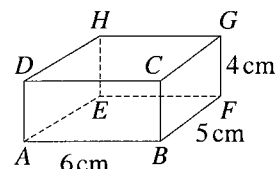
First, $\triangle OMN$ is an isosceles right triangle with hypotenuse $MN = 2$,
so the altitude OG of $\triangle OMN$ has length 1.

Secondly, in $\triangle AMC$, $MC^2 = 3^2 - 2^2$
 $= 5$,
 $MC = \sqrt{5}$.

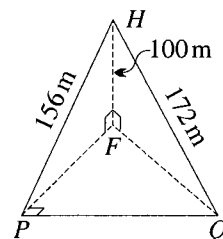
Since $MNDC$ is a rectangle, $OF = 1 + \sqrt{5}$ metres.

Exercise 2G

- The diagram shows a box in the shape of a rectangular prism.
 - Find, correct to the nearest minute, the angle that the diagonal plane $AEGC$ makes with the face $BCGF$.
 - Find the length of the diagonal AG of the box, correct to the nearest millimetre.
 - Find, correct to the nearest minute, the angle that the diagonal AG makes with the base $AEFB$.



- A helicopter H is hovering 100 metres above the level ground below. Two observers P and Q on the ground are 156 metres and 172 metres respectively from H . The helicopter is due north of P , while Q is due east of P .
 - Find the angles of elevation of the helicopter from P and Q , correct to the nearest minute,
 - Find the distance between the two observers P and Q , correct to the nearest metre.

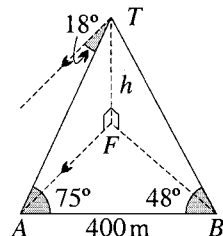


- The points A and B are 400 metres apart in a horizontal plane. The angle of depression of A from the top T of a vertical tower standing on the plane is 18° . Also, $\angle TAB = 75^\circ$ and $\angle TBA = 48^\circ$.

(a) Show that $TA = \frac{400 \sin 48^\circ}{\sin 57^\circ}$.

- (b) Hence find the height h of the tower, correct to the nearest metre.

- (c) Find, correct to the nearest degree, the angle of depression of B from T .



- The diagram shows a cube $ABCDEFGH$. The diagonals AG and CE meet at P . Q is the midpoint of the diagonal EG of the top face. Suppose that $2x$ is the side length of the cube and α is the acute angle between the diagonals AG and CE .

- (a) State the length of PQ .

(b) Show that $EQ = \sqrt{2}x$.

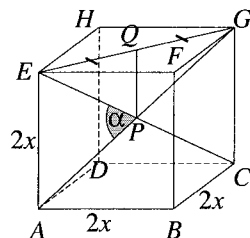
(c) Hence show that $EP = \sqrt{3}x$.

(d) Hence show that $\cos \angle EPQ = \frac{1}{3}\sqrt{3}$.

(e) By using an appropriate double-angle formula, deduce that $\cos \angle EPG = -\frac{1}{3}$, and hence that $\cos \alpha = \frac{1}{3}$.

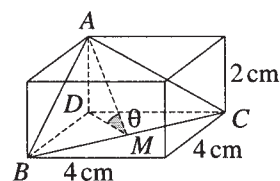
(f) Confirm the fact that $\cos \alpha = \frac{1}{3}$ by using the cosine rule in $\triangle APE$.

- (g) Find, correct to the nearest minute, the angle that the diagonal AG makes with the base $ABCD$ of the cube.



5. The prism in the diagram has a square base of side 4 cm and its height is 2 cm. ABC is a diagonal plane of the prism. Let θ be the acute angle between the diagonal plane and the base of the prism.

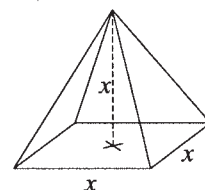
- (a) Show that $MD = 2\sqrt{2}$ cm.
 (b) Hence find θ , correct to the nearest minute.



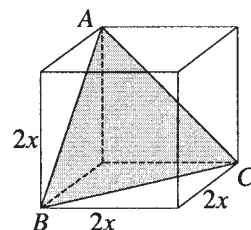
DEVELOPMENT

6. The diagram shows a square pyramid whose perpendicular height is equal to the side of the base. Find, correct to the nearest minute:

- (a) the angle between an oblique face and the base,
 (b) the angle between a slant edge and the base,
 (c) the angle between an opposite pair of oblique faces.

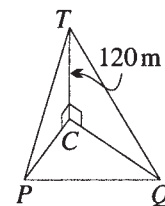


7. The diagram shows a cube of side $2x$ in which a diagonal plane ABC is drawn. Find, correct to the nearest minute, the angle between this diagonal plane and the base of the cube.



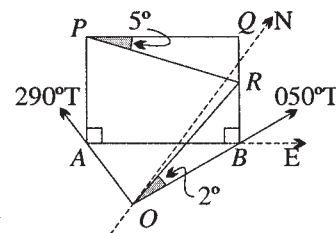
8. Two boats P and Q are observed from the top T of a vertical cliff CT of height 120 metres. P is on a bearing of 195° from the cliff and its angle of depression from T is 22° . Q is on a bearing of 161° from the cliff and its angle of depression from T is 27° .

- (a) Show that $\angle PCQ = 34^\circ$.
 (b) Use the cosine rule to show that the boats are approximately 166 metres apart.



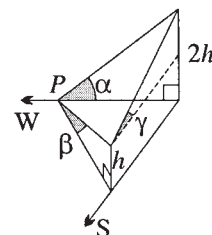
9. A plane is flying along the path PR . Its constant speed is 300 km/h. It flies directly over landmarks A and B , where B is due east of A . An observer at O first sights the plane when it is over A at a bearing of 290° T, and then, ten minutes later, he sights the plane when it is over B at a bearing of 50° T and with an angle of elevation of 2° .

- (a) Show that the plane has travelled 50 km in the ten minutes between observations.
 (b) Show that $\angle AOB = 120^\circ$.
 (c) Prove that the observer is 19 670 metres, correct to the nearest ten metres, from landmark B .
 (d) Find the height h of the plane, correct to the nearest 10 metres, when it was directly above A .



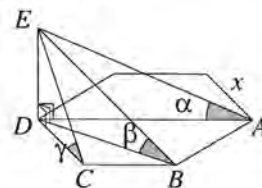
10. Two towers of height $2h$ and h stand on a horizontal plane. The shorter tower is due south of the taller tower. From a point P due west of the taller tower, the angles of elevation of the tops of the taller and shorter towers are α and β respectively. The angle of elevation of the top of the taller tower from the top of the shorter tower is γ . Show that

$$4 \cot^2 \alpha = \cot^2 \beta - \cot^2 \gamma.$$



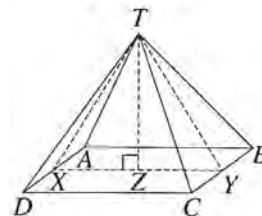
11. A, B, C and D are four of the vertices of a horizontal regular hexagon of side length x . DE is vertical and subtends angles of α, β and γ at A, B and C respectively.

- (a) Show that each interior angle of a regular hexagon is 120° .
 (b) Show that $\angle BAD = 60^\circ$ and $\angle ABD = 90^\circ$.
 (c) Show that $BD = \sqrt{3}x$ and $AD = 2x$.
 (d) Hence show that $\cot^2 \alpha = \cot^2 \beta + \cot^2 \gamma$.



12. The diagram shows a rectangular pyramid. X and Y are the midpoints of AD and BC respectively and T is directly above Z . $TX = 15$ cm, $TY = 20$ cm, $AB = 25$ cm and $BC = 10$ cm.

- (a) Show that $\angle XTY = 90^\circ$.
 (b) Hence show that T is 12 cm above the base.
 (c) Hence find, correct to the nearest minute, the angle that the front face DCT makes with the base.



13. A plane is flying due east at 600 km/h at a constant altitude. From an observation point P on the ground, the plane is sighted on a bearing of 320° . One minute later, the bearing of the plane is 75° and its angle of elevation is 25° .

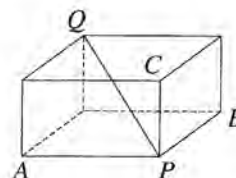
- (a) How far has the plane travelled between the two sightings?
 (b) Draw a diagram to represent the given information.
 (c) Show that the altitude h metres of the plane is given by $h = \frac{10\,000 \sin 50^\circ \tan 25^\circ}{\sin 65^\circ}$, and hence find the altitude, correct to the nearest metre.
 (d) Find, correct to the nearest degree, the angle of elevation of the plane from P when it was first sighted.

14. (a) The diagonal PQ of the rectangular prism in the diagram makes angles of α, β and γ respectively with the edges PA, PB and PC .

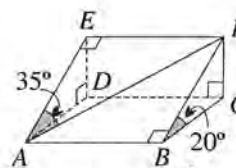
- (i) Prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
 (ii) What is the two-dimensional version of this result?

- (b) Suppose that the diagonal PQ makes angles of θ, ϕ and ψ with the three faces of the prism that meet at P .

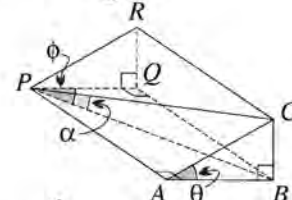
- (i) Prove that $\sin^2 \theta + \sin^2 \phi + \sin^2 \psi = 1$.
 (ii) What is the two-dimensional version of this result?



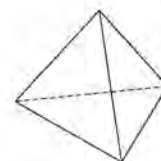
15. The diagram shows a hill inclined at 20° to the horizontal. A straight road AF on the hill makes an angle of 35° with a line of greatest slope. Find, correct to the nearest minute, the inclination of the road to the horizontal.



16. The plane surface $APRC$ is inclined at an angle θ to the horizontal plane $APQB$. Both $APRC$ and $APQB$ are rectangles. PR is a line of greatest slope on the inclined plane. $\angle BPQ = \phi$ and $\angle BPC = \alpha$. Show that $\tan \alpha = \tan \theta \cos \phi$.



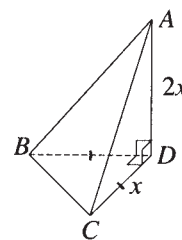
17. The diagram shows a triangular pyramid, all of whose faces are equilateral triangles — such a solid is called a *regular tetrahedron*. Suppose that the slant edges are inclined at an angle θ to the base. Show that $\cos \theta = \frac{1}{3}\sqrt{3}$.



18. A square pyramid has perpendicular height equal to the side length of its base.
- Show that the angle between a slant edge and a base edge it meets is $\cos^{-1} \frac{1}{6}\sqrt{6}$.
 - Show that the angle between adjacent oblique faces is $\cos^{-1}(-\frac{1}{5})$.

EXTENSION

19. A cube has one edge AB of its base inclined at an angle θ to the horizontal and another edge AC of its base horizontal. The diagonal AP of the cube is inclined at angle ϕ to the horizontal.
- Show that the height h of the point P above the horizontal plane containing the edge AC is given by $h = x \cos \theta (1 + \tan \theta)$, where x is the side length of the cube.
 - Hence show that $\cos^2 \phi = \frac{2}{3}(1 - \sin \theta \cos \theta)$.
20. The diagram shows a triangular pyramid $ABCD$. The horizontal base BCD is an isosceles triangle whose equal sides BD and CD are at right angles and have length x units. The edge AD has length $2x$ units and is vertical.
- Let α be the acute angle between the front face ABC and the base BCD . Show that $\alpha = \cos^{-1} \frac{1}{3}$.
 - Let θ be the acute angle between the front face ABC and a side face (that is, either ABD or ACD). Show that $\theta = \cos^{-1} \frac{2}{3}$.



2H Further Three-Dimensional Trigonometry

This section continues with somewhat harder three-dimensional problems. Some trigonometric problems are difficult simply because the diagram is complicated to visualise or because the necessary calculations are intricate. But other problems are difficult because no triangle in the figure can be solved — in such cases, an equation must be formed and solved in the required pronumeral.

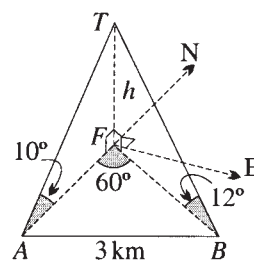
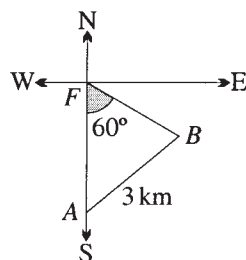
Three-dimensional Problems in which No Triangle can be Solved: In the following classic problem, there are four triangles forming a tetrahedron, but no triangle can be solved, because no more than two measurements are known in any one of these triangles. The method is to introduce a pronumeral for the height, then work around the figure until *four* measurements are known in terms of h in the base triangle — at this point an equation in h can be formed and solved.

WORKED EXERCISE: A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10° . After driving 3 km further in a straight line, the tower is in the direction $N60^\circ W$, with angle of elevation 12° .

(a) How high is the tower?

(b) In what direction is she driving?

SOLUTION:



Let the tower be TF , and let the motorist be driving from A to B .
There are four triangles, none of which can be solved.

(a) Let h be the height of the tower.

In $\triangle TAF$, $AF = h \cot 10^\circ$.

In $\triangle TBF$, $BF = h \cot 12^\circ$.

We now have expressions for four measurements in $\triangle ABF$,
so we can use the cosine rule to form an equation in h .

In $\triangle ABF$, $3^2 = h^2 \cot^2 10^\circ + h^2 \cot^2 12^\circ - 2h^2 \cot 10^\circ \cot 12^\circ \times \cos 60^\circ$

$$9 = h^2(\cot^2 10^\circ + \cot^2 12^\circ - \cot 10^\circ \cot 12^\circ)$$

$$h^2 = \frac{9}{\cot^2 10^\circ + \cot^2 12^\circ - \cot 10^\circ \cot 12^\circ},$$

so the tower is about 571 metres high.

(b) Let $\theta = \angle FAB$.

In $\triangle AFB$, $\frac{\sin \theta}{h \cot 12^\circ} = \frac{\sin 60^\circ}{3}$

$$\sin \theta = h \cot 12^\circ \times \frac{\sqrt{3}}{6}$$

$$\theta \doteq 51^\circ,$$

so her direction is about N51°E.

The General Method of Approach: Here is a summary of what has been said about three-dimensional problems (apart from the ideas of angles between lines and planes and between planes and planes).

THREE-DIMENSIONAL TRIGONOMETRY:

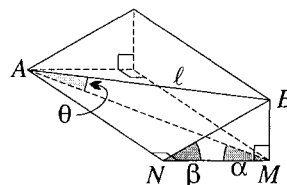
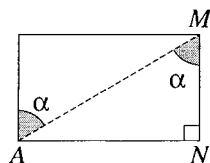
20

1. Draw a careful diagram of the situation, marking all right angles.
2. A plan diagram, looking down, is usually a great help.
3. Identify every triangle in the diagram, to see whether it can be solved.
4. If one triangle can be solved, then work from it around the diagram until the problem is solved.
5. If no triangle can be solved, assign a pronumeral to what is to be found, then work around the diagram until an equation in that pronumeral can be formed and solved.

Problems Involving Pronumerals: When a problem involves pronumerals, there is little difference in the methods used. The solution will usually require working around the diagram, beginning with a triangle in which expressions for three measurements are known, until an equation can be formed.

WORKED EXERCISE: [A harder example] A hillside is a plane of gradient m facing due south. A map shows a straight road on the hillside going in the direction α east of north. Find the gradient of the road in terms of m and α .

SOLUTION:



The diagrams above show a piece AB of the road of length ℓ .

Let $\theta = \angle BAM$ be the angle of inclination of the road,

and let $\beta = \angle BNM$ be the angle of inclination of the hillside.

In $\triangle ABM$, $BM = \ell \sin \theta$,

and $AM = \ell \cos \theta$.

In $\triangle AMN$, $MN = AM \cos \alpha$
 $= \ell \cos \theta \cos \alpha$.

In $\triangle BMN$, $BM = MN \tan \beta$
 $\ell \sin \theta = \ell \cos \theta \cos \alpha \tan \beta$
 $\tan \theta = \cos \alpha \tan \beta$.

But $\tan \theta$ and $\tan \beta$ are the gradients of the road and hillside respectively, so the gradient of the road is $m \cos \alpha$.

Exercise 2H

1. A balloon B is due north of an observer P and its angle of elevation is 62° . From another observer Q 100 metres from P , the balloon is due west and its angle of elevation is 55° . Let the height of the balloon be h metres and let C be the point on the level ground vertically below B .

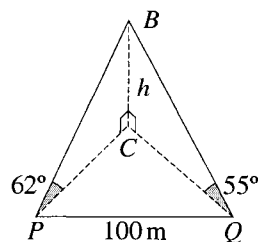
(a) Show that $PC = h \cot 62^\circ$, and write down a similar expression for QC .

(b) Explain why $\angle PCQ = 90^\circ$.

(c) Use Pythagoras' theorem in $\triangle CPQ$ to show that

$$h^2 = \frac{100^2}{\cot^2 62^\circ + \cot^2 55^\circ}.$$

(d) Hence find h , correct to the nearest metre.



2. From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20° . From a point Q situated 40 metres from P and due east of the tower, the angle of elevation is 35° . Let h metres be the height of the tower.

(a) Draw a diagram to represent the situation.

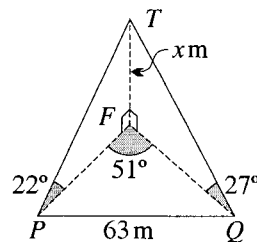
(b) Show that $h = \frac{40}{\sqrt{\tan^2 70^\circ + \tan^2 55^\circ}}$, and evaluate h , correct to the nearest metre.

3. In the diagram, TF represents a vertical tower of height x metres standing on level ground. From P and Q at ground level, the angles of elevation of T are 22° and 27° respectively. $PQ = 63$ metres and $\angle PFQ = 51^\circ$.

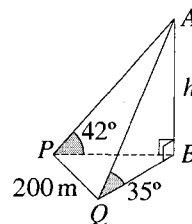
(a) Show that $PF = x \cot 22^\circ$ and write down a similar expression for QF .

(b) Use the cosine rule to show that $x^2 = \frac{63^2}{\cot^2 22^\circ + \cot^2 27^\circ - 2 \cot 22^\circ \cot 27^\circ \cos 51^\circ}$.

(c) Use a calculator to show that $x \div 32$.



4. The points P , Q and B lie in a horizontal plane. From P , which is due west of B , the angle of elevation of the top of a tower AB of height h metres is 42° . From Q , which is on a bearing of 196° from the tower, the angle of elevation of the top of the tower is 35° . The distance PQ is 200 metres.



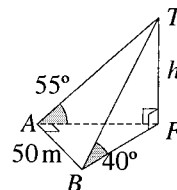
(a) Explain why $\angle PBQ = 74^\circ$.

(b) Show that
$$h^2 = \frac{200^2}{\cot^2 42^\circ + \cot^2 35^\circ - 2 \cot 35^\circ \cot 42^\circ \cos 74^\circ}.$$

(c) Hence find the height of the tower, correct to the nearest metre.

DEVELOPMENT

5. The diagram shows a tower of height h metres standing on level ground. The angles of elevation of the top T of the tower from two points A and B on the ground nearby are 55° and 40° respectively. The distance AB is 50 metres and the interval AB is perpendicular to the interval AF , where F is the foot of the tower.



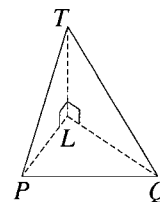
(a) Find AT and BT in terms of h .

(b) What is the size of $\angle BAT$?

(c) Use Pythagoras' theorem in $\triangle BAT$ to show that
$$h = \frac{50 \sin 55^\circ \sin 40^\circ}{\sqrt{\sin^2 55^\circ - \sin^2 40^\circ}}.$$

(d) Hence find the height of the tower, correct to the nearest metre.

6. The diagram shows two observers P and Q 600 metres apart on level ground. The angles of elevation of the top T of a landmark TL from P and Q are 9° and 12° respectively. The bearings of the landmark from P and Q are 32° and 306° respectively. Let $h = TL$ be the height of the landmark.



(a) Show that $\angle PLQ = 86^\circ$.

(b) Find expressions for PL and QL in terms of h .

(c) Hence show that $h \div 79$ metres.

7. PQ is a straight level road. Q is x metres due east of P . A vertical tower of height h metres is situated due north of P . The angles of elevation of the top of the tower from P and Q are α and β respectively.

(a) Draw a diagram representing the situation.

(b) Show that $x^2 + h^2 \cot^2 \alpha = h^2 \cot^2 \beta$.

(c) Hence show that
$$h = \frac{x \sin \alpha \sin \beta}{\sqrt{\sin(\alpha + \beta) \sin(\alpha - \beta)}}.$$

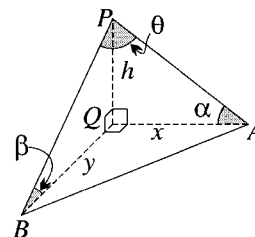
8. In the diagram of a triangular pyramid, $AQ = x$, $BQ = y$, $PQ = h$, $\angle APB = \theta$, $\angle PAQ = \alpha$ and $\angle PBQ = \beta$. Also, there are three right angles at Q .

(a) Show that $x = h \cot \alpha$ and write down a similar expression for y .

(b) Use Pythagoras' theorem and the cosine rule to show

$$\text{that } \cos \theta = \frac{h^2}{\sqrt{(x^2 + h^2)(y^2 + h^2)}}.$$

(c) Hence show that $\sin \alpha \sin \beta = \cos \theta$.



9. A man walking along a straight, flat road passes by three observation points P , Q and R at intervals of 200 metres. From these three points, the respective angles of elevation of the top of a vertical tower are 30° , 45° and 45° . Let h metres be the height of the tower.

(a) Draw a diagram representing the situation.

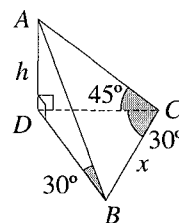
(b) (i) Find, in terms of h , the distances from P , Q and R to the foot F of the tower.

(ii) Let $\angle FRQ = \alpha$. Find two different expressions for $\cos \alpha$ in terms of h , and hence find the height of the tower.

10. $ABCD$ is a triangular pyramid with base BCD and perpendicular height AD .

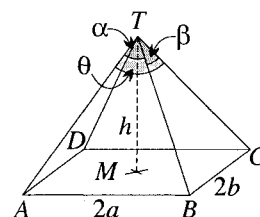
- (a) Find BD and CD in terms of h .
 (b) Use the cosine rule to show that $2h^2 = x^2 - \sqrt{3}hx$.
 (c) Let $u = \frac{h}{x}$. Write the result of the previous part as a quadratic equation in u , and hence show that

$$\frac{h}{x} = \frac{\sqrt{11} - \sqrt{3}}{4}.$$



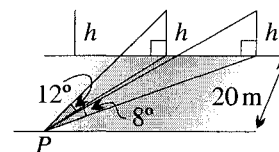
11. The diagram shows a rectangular pyramid. The base $ABCD$ has sides $2a$ and $2b$ and its diagonals meet at M . The perpendicular height TM is h . Let $\angle ATB = \alpha$, $\angle BTC = \beta$ and $\angle ATC = \theta$.

- (a) Use Pythagoras' theorem to find AC , AM and AT in terms of a , b and h .
 (b) Use the cosine rule to find $\cos \alpha$, $\cos \beta$ and $\cos \theta$ in terms of a , b and h .
 (c) Show that $\cos \alpha + \cos \beta = 1 + \cos \theta$.



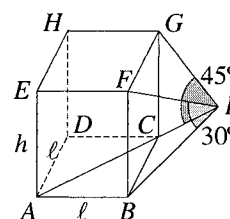
12. The diagram shows three telegraph poles of equal height h metres standing equally spaced on the same side of a straight road 20 metres wide. From an observer at P on the other side of the road directly opposite the first pole, the angles of elevation of the tops of the other two poles are 12° and 8° respectively. Let x metres be the distance between two adjacent poles.

- (a) Show that $h^2 = \frac{x^2 + 20^2}{\cot^2 12^\circ}$.
 (b) Hence show that $x^2 = \frac{20^2(\cot^2 8^\circ - \cot^2 12^\circ)}{4\cot^2 12^\circ - \cot^2 8^\circ}$.
 (c) Hence calculate the distance between adjacent poles, correct to the nearest metre.



13. A building is in the shape of a square prism with base edge ℓ metres and height h metres. It stands on level ground. The diagonal AC of the base is extended to K , and from K , the respective angles of elevation of F and G are 30° and 45° .

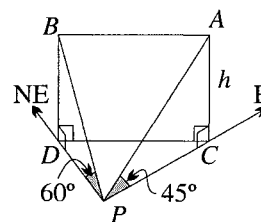
- (a) Show that $BK^2 = h^2 + \ell^2 + \sqrt{2}h\ell$.
 (b) Hence show that $2h^2 - \ell^2 = \sqrt{2}h\ell$.
 (c) Deduce that $\frac{h}{\ell} = \frac{\sqrt{2} + \sqrt{10}}{4}$.



14. From a point P on level ground, a man observes the angle of elevation of the summit of a mountain due north of him to be 18° . After walking 3 km in a direction $N50^\circ E$ to a point Q , the man finds that the angle of elevation of the summit is now 13° .

- (a) Show that $(\cot^2 13^\circ - \cot^2 18^\circ)h^2 + (6000 \cot 18^\circ \cos 50^\circ)h - 3000^2 = 0$, where h metres is the height of the mountain.
 (b) Hence find the height, correct to the nearest metre.

15. A plane is flying at a constant height h , and with constant speed. An observer at P sighted the plane due east at an angle of elevation of 45° . Soon after it was sighted again in a north-easterly direction at an angle of elevation of 60° .



- ## EXTENSION

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CHAPTER THREE

Motion

Anyone watching objects in motion can see that they often make patterns with a striking simplicity and predictability. These patterns are related to the simplest objects in geometry and arithmetic. A thrown ball traces out a parabolic path. A cork bobbing in flowing water traces out a sine wave. A rolling billiard ball moves in a straight line, rebounding symmetrically off the table edge. The stars and planets move in more complicated, but highly predictable, paths across the sky. The relationship between physics and mathematics, logically and historically, begins with these and many similar observations.

Mathematics and physics, however, remain quite distinct disciplines. Physics asks questions about the nature of the world and is based on experiment, but mathematics asks questions about logic and logical structures, and proceeds by thought, imagination and argument alone, its results and methods quite independent of the nature of the world. This chapter will begin the application of mathematics to the description of a moving object. But because this is a mathematics course, our attention will not be on the nature of space and time, but on the new insights that the physical world brings to the mathematical objects already developed earlier in the course. We will be applying the well-known linear, quadratic, exponential and trigonometric functions. Our principal goal will be to produce a striking alternative interpretation of the first and second derivatives as the physical notions of velocity and acceleration so well known to our senses.

STUDY NOTES: The first three sections set up the basic relationship between calculus and the three functions for displacement, velocity and acceleration. Simple harmonic motion is then discussed in Section 3D in terms of the time equations. Section 3E deals with situations where velocity or acceleration are known as functions of displacement rather than time, and this allows a second discussion of simple harmonic motion in Section 3F, based on its characteristic differential equation. The last two Sections 3G and 3H pass from motion in one dimension to the two-dimensional motion of a projectile, briefly introducing vectors.

Students without a good background in physics may benefit from some extra experimental work, particularly in simple harmonic motion and projectile motion, so that some of the motions described here can be observed and harmonised with the mathematical description. Although forces and their relationship with acceleration are only introduced in the 4 Unit course, some physical understanding of Newton's second law $F = m\ddot{x}$ would greatly aid understanding of what is happening.

3A Average Velocity and Speed

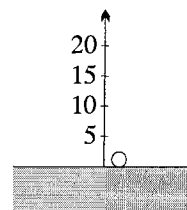
This first section sets up the mathematical description of motion in one dimension, using a function to describe the relationship between time and the position of an object in motion. Average velocity is described as the gradient of the chord on this displacement–time graph. This will lead, in the next section, to the description of instantaneous velocity as the gradient of a tangent.

Motion in One Dimension: When a particle is moving in one dimension along a line, its position is varying over time. We can specify that position at any time t by a single number x , called the *displacement*, and the whole motion can be described by giving x as a function of the time t .

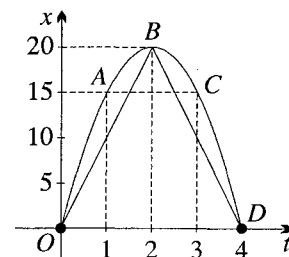
For example, suppose that a ball is thrown vertically upwards from ground level, and lands 4 seconds later in the same place. Its motion can be described approximately by the following equation and table of values:

$x = 5t(4 - t)$	t	0	1	2	3	4
	x	0	15	20	15	0

Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line. This vertical line has been made into a number line, with the origin at the ground, upwards as the positive direction, and metres as the units of distance. The origin of time is when the ball is thrown, and the units of time are seconds.



The displacement–time graph is sketched to the right — this graph must not be mistaken as a picture of the ball's path. The curve is a section of a parabola with vertex at $(2, 20)$, which means that the ball achieves a maximum height of 20 metres after 2 seconds. When $t = 4$, the height is zero, and the ball is back on the ground. The equation of motion therefore has quite restricted domain and range:



$$0 \leq t \leq 4 \quad \text{and} \quad 0 \leq x \leq 20.$$

Most equations of motion have this sort of restriction on the domain of t . In particular, it is a convention of this course that negative values of time are excluded unless the question specifically allows it.

1

MOTION IN ONE DIMENSION: Motion in one dimension is specified by giving the displacement x on the number line as a function of time t after time zero. Negative values of time are excluded unless otherwise stated.

WORKED EXERCISE: In the example above, where $x = 5t(4 - t)$, at what times is the ball $8\frac{3}{4}$ metres above the ground?

SOLUTION: Put $x = 8\frac{3}{4}$. Then $8\frac{3}{4} = 5t(4 - t)$
 $\frac{35}{4} = 20t - 5t^2$

$$\boxed{\times 4} \quad 20t^2 - 80t + 35 = 0$$

$$\boxed{\div 5} \quad 4t^2 - 16t + 7 = 0$$

$$(2t - 1)(2t - 7) = 0$$

$$t = \frac{1}{2} \text{ or } 3\frac{1}{2}.$$

Hence the ball is $8\frac{3}{4}$ metres high after $\frac{1}{2}$ seconds and again after $3\frac{1}{2}$ seconds.

Average Velocity: During its ascent, the ball in the example above moved 20 metres upwards. This is a change in displacement of +20 metres in 2 seconds, giving an average velocity of 10 metres per second. The *average velocity* is thus the gradient of the chord OB on the displacement–time graph (be careful, because there are different scales on the two axes). Hence the formula for average velocity is the familiar gradient formula.

2

AVERAGE VELOCITY: Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$. Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

That is, on the displacement–time graph,

$$\text{average velocity} = \text{gradient of the chord}.$$

During its descent, the ball moved 20 metres downwards in 2 seconds, which is a change in displacement of $0 - 20 = -20$ metres. The average velocity is therefore -10 metres per second, and is equal to the gradient of the chord BD .

WORKED EXERCISE: Find the average velocities of the ball during the first second and during the third second.

SOLUTION: Velocity during 1st second

$$\begin{aligned} &= \frac{x_2 - x_1}{t_2 - t_1} \\ &= \frac{15 - 0}{1 - 0} \\ &= 15 \text{ m/s.} \end{aligned}$$

This is the gradient of OA .

Velocity during 3rd second

$$\begin{aligned} &= \frac{x_2 - x_1}{t_2 - t_1} \\ &= \frac{15 - 20}{3 - 2} \\ &= -5 \text{ m/s.} \end{aligned}$$

This is the gradient of BC .

Distance Travelled: The change in displacement can be positive, negative or zero. *Distance*, however, is always positive or zero. In our previous example, the change in displacement during the third and fourth seconds is -20 metres, but the distance travelled is 20 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the distance travelled by the ball is $20 + 20 = 40$ metres, even though the ball's change in displacement over the first 4 seconds is zero because the ball is back at its original position.

3

DISTANCE TRAVELLED: *Distance travelled* is always positive or zero, and takes into account any journey and return.

Average Speed: The *average speed* is the distance travelled divided by the time taken. Speed, unlike velocity, can never be negative.

4 **AVERAGE SPEED:** $\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$

During the 4 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 40 metres, so

$$\text{average velocity} = \frac{0}{4} = 0 \text{ m/s}, \quad \text{average speed} = \frac{40}{4} = 10 \text{ m/s}.$$

WORKED EXERCISE: Find the average velocity and average speed of the ball:

- (a) during the fourth second, (b) during the last three seconds.

SOLUTION:

- (a) During the fourth second, change in displacement = -15 metres,
so average velocity = -15 m/s.
Distance travelled = 15 metres,
so average speed = 15 m/s.
- (b) From $t = 1$ to $t = 4$, change in displacement = -15 metres,
so average velocity = -5 m/s.
Distance travelled = 25 metres,
so average speed = $8\frac{1}{3}$ m/s.

Exercise 3A

1. A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

- (a) Copy and complete the table to the right of values of the displacement at certain times.

t	0	1	2	3
x				

- (b) Hence find the average velocity:

- (i) during the first second, (iii) during the first three seconds,
(ii) during the first two seconds, (iv) during the third second.

- (c) Sketch the displacement–time graph, and add the chords corresponding to the average velocities calculated in part (b).

2. A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance is in centimetres and time is in seconds.

- (a) Copy and complete the table to the right.

t					
x	0	2	4	6	8

- (b) Hence find the average velocity as the particle moves:

- (i) from $x = 0$ to $x = 2$, (iii) from $x = 4$ to $x = 6$,
(ii) from $x = 2$ to $x = 4$, (iv) from $x = 0$ to $x = 6$.

- (c) Sketch the displacement–time graph, and add the chords corresponding to the average velocities calculated in part (b). What does the equality of the answers to parts (ii) and (iv) of part (b) tell you about the corresponding chords?

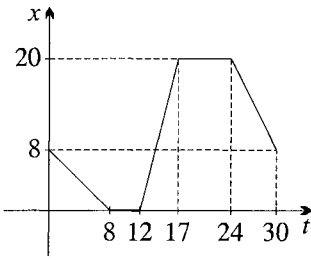
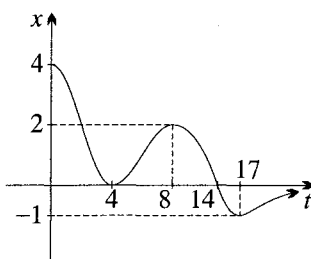
3. A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

- (a) Copy and complete the table to the right.

t	0	1	2	3	4
x					

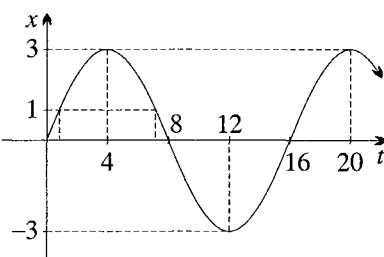
- (b) Hence find the average velocity as the particle moves:

- (i) from $t = 0$ to $t = 2$, (ii) from $t = 2$ to $t = 4$, (iii) from $t = 0$ to $t = 4$.

- (c) Sketch the displacement–time graph, and add the chords corresponding to the average velocities calculated in part (b).
- (d) Find the total distance travelled during the first 4 seconds, and the average speeds over the time intervals specified in part (b).
4. Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x metres from the front gate after t seconds.
- 
- (a) What is her velocity: (i) during the first 8 seconds, (ii) while she is driving forwards, (iii) while she is reversing the second time?
- (b) Find the total distance travelled, and the average speed, over the 30 seconds.
- (c) Find the change in displacement, and the average velocity, over the 30 seconds.
- (d) Find her average speed if she had not paused at the gate and at the garage.
5. Michael the mailman rides 1 km up a hill at a constant speed of 10 km/hr, and then rides 1 km down the other side of the hill at a constant speed of 30 km/hr.
- (a) How many minutes does he take to ride: (i) up the hill, (ii) down the hill?
- (b) Draw a displacement–time graph, with the time axis in minutes.
- (c) What is his average speed over the total 2 km journey?
- (d) What is the average of the speeds up and down the hill?
6. Sadie the snail is crawling up a 6-metre-high wall. She takes an hour to crawl up 3 metres, then falls asleep for an hour and slides down 2 metres, repeating the cycle until she reaches the top of the wall.
- (a) Sketch the displacement–time graph. (b) How long does Sadie take to reach the top?
- (c) What is her average speed? (d) Which places does she visit exactly three times?
7. A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The diagram graphs the height x in metres of the weight above the water as a function of time t after she first drops it.
- 
- (a) How many times is the weight: (i) at $x = 3$, (ii) at $x = 1$, (iii) at $x = -\frac{1}{2}$?
- (b) At what times is the weight: (i) at the water surface, (ii) above the water surface?
- (c) How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?
- (d) What is its greatest depth under the water, and when does it occur?
- (e) What happens to the weight eventually?
- (f) What is its average velocity: (i) during the first 4 seconds, (ii) from $t = 4$ to $t = 8$, (iii) from $t = 8$ to $t = 17$?
- (g) What distance does it travel: (i) over the first 4 seconds, (ii) over the first 8 seconds, (iii) over the first 17 seconds, (iv) eventually?
- (h) What is its average speed over the first: (i) 4, (ii) 8, (iii) 17 seconds?

DEVELOPMENT

8. A particle is moving according to $x = 3 \sin \frac{\pi}{8}t$, in units of centimetres and seconds. Its displacement–time graph is sketched opposite.



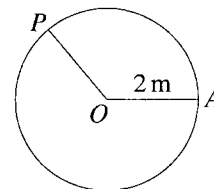
- (a) Use $T = \frac{2\pi}{n}$ to confirm that the period is 16 seconds.
 - (b) Find the first two times when the displacement is maximum.
 - (c) When, during the first 20 seconds, is the particle on the negative side of the origin?
 - (d) Find the total distance travelled during the first 16 seconds, and the average speed.
 - (e) (i) Find, correct to three significant figures, the first two positive solutions of the trigonometric equation $\sin \frac{\pi}{8}t = \frac{1}{3}$ [HINT: Use radian mode on the calculator.]
(ii) Hence find, correct to three significant figures, the first two times when $x = 1$. Then find the total distance travelled between these two times, and the average speed during this time.
9. A particle moves according to $x = 10 \cos \frac{\pi}{12}t$, in units of metres and seconds.
- (a) Find the amplitude and period of the motion.
 - (b) Sketch the displacement–time graph over the first 60 seconds.
 - (c) What is the maximum distance the particle reaches from its initial position, and when, during the first minute, is it there?
 - (d) How far does the particle move during the first minute, and what is its average speed?
 - (e) When, during the first minute, is the particle 10 metres from its initial position?
 - (f) Use the fact that $\cos \frac{\pi}{3} = \frac{1}{2}$ to copy and complete this table of values:

t	4	8	12	16	20	24
x						

- (g) From the table, find the average velocity during the first 4 seconds, the second 4 seconds, and the third 4 seconds.
 - (h) Use the graph and the table of values to find when the particle is more than 15 metres from its initial position.
10. A particle is moving on a horizontal number line according to the equation $x = 4 \sin \frac{\pi}{6}t$, in units of metres and seconds.
- (a) Sketch the displacement–time graph.
 - (b) How many times does the particle return to the origin by the end of the first minute?
 - (c) Find at what times it visits $x = 4$ during the first minute.
 - (d) Find how far it travels during the first 12 seconds, and its average speed in that time.
 - (e) Find the values of x when $t = 0$, $t = 1$ and $t = 3$. Hence show that its average speed during the first second is twice its average speed during the next 2 seconds.
11. A balloon rises so that its height h in metres after t minutes is $h = 8000(1 - e^{-0.06t})$.
- (a) What height does it start from, and what happens to the height as $t \rightarrow \infty$?
 - (b) Copy and complete the table to the right, correct to the nearest metre.
 - (c) Sketch the displacement–time graph of the motion.

t	0	10	20	30
h				

- (d) Find the balloon's average velocity during the first 10 minutes, the second 10 minutes and the third 10 minutes, correct to the nearest metre per minute.
- (e) Show that the solution of $1 - e^{-0.06t} = 0.99$ is $t = \frac{\log 100}{0.06}$.
- (f) Hence find how long (correct to the nearest minute) the balloon takes to reach 99% of its final height.
12. A toy train is travelling anticlockwise on a circular track of radius 2 metres and centre O . At time zero the train is at a point A , and t seconds later it is at the point P distant $x = 4\log(t+1)$ metres around the track.
- (a) Sketch the graph of x as a function of t .
- (b) Find, when $t = 2$, the position of the point P , the average speed from A to P , the size of $\angle AOP$ and the length of the chord AP (in exact form, then correct to four significant figures).
- (c) More generally, find $\angle AOP$ as a function of t . Hence find, in exact form, and then correct to the nearest second, the first three times when the train returns to A .
- (d) Explain whether the train will return to A finitely or infinitely many times.
13. Two engines, Thomas and Henry, move on close parallel tracks. They start at the origin, and are together again at time $t = e - 1$. Thomas' displacement-time equation, in units of metres and minutes, is $x = 300\log(t+1)$, and Henry's is $x = kt$, for some constant k .
- (a) Sketch the two graphs.
- (b) Show that $k = \frac{300}{e-1}$.
- (c) Use calculus to find the maximum distance between Henry and Thomas during the first $e - 1$ minutes, and the time when it occurs (in exact form, and then correct to the nearest metre or the nearest second).



EXTENSION

14. [The arithmetic mean, the geometric mean and the harmonic mean] The *harmonic mean* of two numbers a and b is defined to be the number h such that

$$\frac{1}{h} \text{ is the arithmetic mean of } \frac{1}{a} \text{ and } \frac{1}{b}.$$

Suppose that town B lies on the road between town A and town C , and that a cyclist rides from A to B at a constant speed U , and then rides from B to C at a constant speed V .

- (a) Prove that if town B lies midway between towns A and C , then the cyclist's average speed W over the total distance AC is the harmonic mean of U and V .
- (b) Now suppose that the distances AB and BC are not equal.
- (i) Show that if W is the arithmetic mean of U and V , then

$$AB : BC = U : V.$$

- (ii) Show that if W is the geometric mean of U and V , then

$$AB : BC = \sqrt{U} : \sqrt{V}.$$

3B Velocity and Acceleration as Derivatives

If I drive the 160 km from Sydney to Newcastle in 2 hours, my average velocity is 80 km per hour. However, my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. Just as an average velocity corresponds to the gradient of a chord on the displacement–time graph, so an instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous Velocity and Speed: From now on, the words *velocity* and *speed* alone will mean instantaneous velocity and instantaneous speed.

INSTANTANEOUS VELOCITY: The *instantaneous velocity* v of the particle is the derivative of the displacement with respect to time:

5
$$v = \frac{dx}{dt} \quad \left(\text{This derivative } \frac{dx}{dt} \text{ can also be written as } \dot{x}. \right)$$

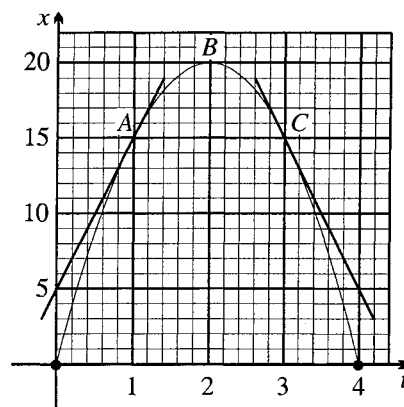
That is, v = gradient of the tangent on the displacement–time graph.

The *instantaneous speed* is the absolute value $|v|$ of the velocity.

The notation $\dot{x}$ is yet another way of writing the derivative. The dot over the x , or over any symbol, stands for differentiation with respect to time t , so that v , dx/dt and $\dot{x}$ are alternative symbols for velocity.

WORKED EXERCISE: Here again is the displacement–time graph of the ball moving with equation $x = 20t - 5t^2$.

- Differentiate to find the equation of the velocity v , draw up a table of values at 1-second intervals, and sketch the velocity–time graph.
- Measure the gradients of the tangents that have been drawn at A , B and C on the displacement–time graph, and compare your answers with the table of values in part (a).
- With what velocity was the ball originally thrown?
- What is its impact speed when it hits the ground?



SOLUTION:

- (a) The equation of motion is $x = 5t(4 - t)$

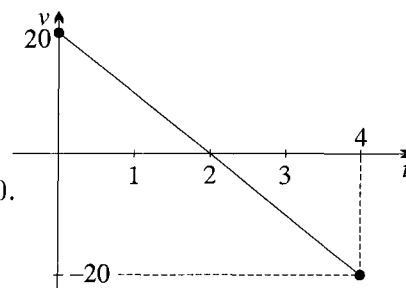
$$x = 20t - 5t^2.$$

Differentiating,

$$v = 20 - 10t,$$

which is linear, with v -intercept 20 and gradient -10 .

t	0	1	2	3	4
v	20	10	0	-10	-20



- These values agree with the measurements of the gradients of the tangents at A where $x = 1$, at B where $x = 2$, and at C where $x = 3$. (Be careful to take account of the different scales on the two axes.)
- When $t = 0$, $v = 20$, so the ball was originally thrown upwards at 20 m/s.
- When $t = 4$, $v = -20$, so the ball hits the ground at 20 m/s.

Vector and Scalar Quantities: Displacement and velocity are *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards, and a negative displacement would mean it was below ground level. Distance and speed, however, are called *scalar quantities* — they measure only the magnitude of displacement and velocity respectively, and therefore cannot be negative.

Stationary Points: A particle is stationary when its velocity is zero, that is, when $\frac{dx}{dt} = 0$. This is the origin of the word ‘stationary point’, introduced in Chapter Ten of the Year 11 volume to describe a point on a graph where the derivative is zero. For example, the thrown ball was stationary for an instant at the top of its flight when $t = 2$, because the velocity was zero at the instant when the motion changed from upwards to downwards.

6

STATIONARY POINTS: To find when a particle is *stationary* (meaning momentarily at rest), put $v = 0$ and solve for t .

WORKED EXERCISE: A particle is moving according to the equation $x = 2 \sin \pi t$.

- Find the equation for its velocity, and graph both equations.
- Find when the particle is at the origin, and its speed then.
- Find when and where the particle is stationary.
- Briefly describe the motion.

SOLUTION: We are given that

$$x = 2 \sin \pi t.$$

- Differentiating,
and the graphs are drawn opposite.

$$v = 2\pi \cos \pi t,$$

- When the particle is at the origin, $x = 0$
 $2 \sin \pi t = 0$

and since conventionally $t \geq 0$,

$$t = 0, 1, 2, 3, \dots$$

When $t = 0, 2, \dots$,

$$v = 2\pi,$$

and when $t = 1, 3, \dots$,

$$v = -2\pi.$$

Hence the particle is at the origin when $t = 0, 1, 2, \dots$,
and the speed then is always 2π .

- When the particle is stationary, $v = 0$
 $2\pi \cos \pi t = 0$

$$t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \dots$$

When $t = \frac{1}{2}, 2\frac{1}{2}, \dots$,

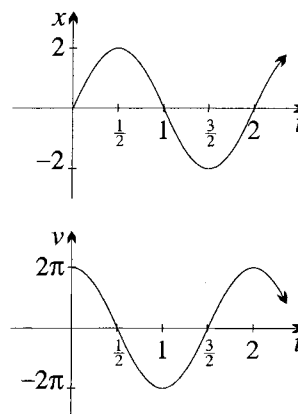
$$x = 2,$$

and when $t = 1\frac{1}{2}, 3\frac{1}{2}, \dots$,

$$x = -2.$$

Hence the particle is stationary when $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \dots$,
and is alternately 2 units right and left of the origin.

- The particle oscillates for ever between $x = -2$ and $x = 2$, with period 2, beginning at the origin, and moving first to $x = 2$.



Limiting Values of Displacement and Velocity: Sometimes a question will ask what happens to the particle ‘eventually’, or ‘as time goes on’. This simply means take the limit as $t \rightarrow \infty$.

WORKED EXERCISE: A particle moves so that its height x metres above the ground t seconds after time zero is $x = 2 - e^{-3t}$.

- (a) Find displacement and velocity initially, and eventually.
 (b) Briefly describe the motion and sketch the graphs of displacement and velocity.

SOLUTION:

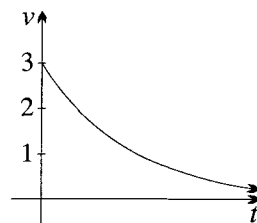
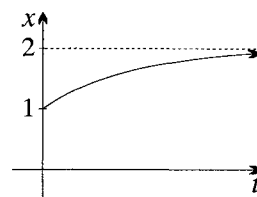
- (a) We are given that $x = 2 - e^{-3t}$.

Differentiating, $v = 3e^{-3t}$.

When $t = 0$, $x = 1$ and $v = 3$.

As $t \rightarrow \infty$, $x \rightarrow 2$ and $v \rightarrow 0$.

- (b) Hence the particle starts 1 metre above the ground with initial velocity 3 m/s upwards, and moves towards its limiting position at height 2 metres with speed tending to 0.



Acceleration: A particle whose velocity is changing is said to be *accelerating*, and the value of the acceleration is defined to be the rate of change of the velocity. Thus the acceleration is $\dot{v}$, meaning the derivative $\frac{dv}{dt}$ with respect to time.

But the velocity is itself the derivative of the displacement, so the acceleration is the second derivative $\frac{d^2x}{dt^2}$ of displacement, and can therefore be written as $\ddot{x}$.

7

ACCELERATION AS A SECOND DERIVATIVE: Acceleration is the first derivative of velocity with respect to time, and the second derivative of displacement:

$$\text{acceleration} = \dot{v} = \ddot{x}.$$

WORKED EXERCISE: In the previous worked exercise, $x = 2 - e^{-3t}$ and $v = 3e^{-3t}$.

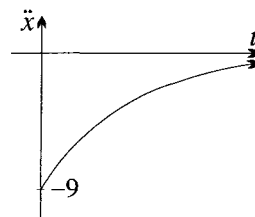
- (a) Find the acceleration function, and sketch the acceleration–time graph.
 (b) In what direction is the particle accelerating?
 (c) What happens to the acceleration eventually?

SOLUTION:

- (a) Since $v = 3e^{-3t}$,
 $\ddot{x} = -9e^{-3t}$.

- (b) The acceleration is always negative, so the particle is accelerating downwards. (Since it is always moving upwards, this means that it is always slowing down.)

- (c) Since $e^{-3t} \rightarrow 0$ as $t \rightarrow \infty$, the acceleration tends to zero as time goes on.



WORKED EXERCISE: The height x of a ball thrown in the air is given by $x = 5t(4 - t)$, in units of metres and seconds.

- (a) Show that its acceleration is a constant function, and sketch its graph.
 (b) State when the ball is speeding up and when it is slowing down, explaining why this can happen when the acceleration is constant.

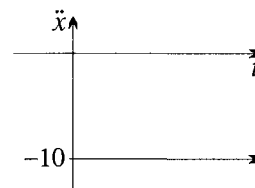
SOLUTION:

$$\begin{aligned} \text{(a) Differentiating, } x &= 20t - 5t^2, \\ \dot{x} &= 20 - 10t \\ \ddot{x} &= -10. \end{aligned}$$

Hence the acceleration is always 10 m/s^2 downwards.

- (b) During the first two seconds, the ball has positive velocity, meaning that it is rising, and the ball is slowing down by 10 m/s every second.

During the third and fourth seconds, however, the ball has negative velocity, meaning that it is falling, and the ball is speeding up by 10 m/s every second.



Units of Acceleration: In the previous example, the ball's velocity was decreasing by 10 m/s every second, and we therefore say that the ball is accelerating at ' -10 metres per second per second', written shorthand as -10 m/s^2 or as -10 ms^{-2} .

The units correspond with the indices of the second derivative $\frac{d^2x}{dt^2}$.

Acceleration should normally be regarded as a vector quantity, that is, it has a direction built into it. The ball's acceleration should therefore be given as -10 m/s^2 , or as 10 m/s^2 *downwards* if the question is using the convention of upwards as positive.

Extension — Newton's Second Law of Motion: Newton's second law of motion — a law of physics, not of mathematics — says that when a force is applied to a body free to move, the body accelerates with an acceleration proportional to the force, and inversely proportional to the mass of the body. Written symbolically,

$$F = m\ddot{x},$$

where m is the mass of the body, and F is the force applied. (The units of force are chosen to make the constant of proportionality 1 — in units of kilograms, metres and seconds, the units of force are, appropriately, called *newtons*.) This means that acceleration is felt in our bodies as a force, as we all know when a motor car accelerates away from the lights, or comes to a stop quickly. In this way, the second derivative becomes directly observable to our senses as a force, just as the first derivative, velocity, is observable to our sight.

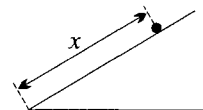
Although these things are only treated in the 4 Unit course, it is helpful to have an intuitive idea that force and acceleration are closely related.

Exercise 3B

NOTE: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The mathematics should be well-known, but the physical interpretations can be confusing.

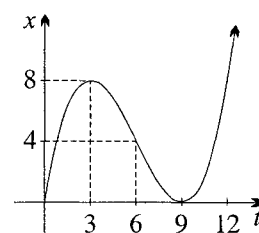
1. A particle moves according to the equation $x = t^2 - 8t$, in units of metres and seconds.
 - (a) Differentiate to find the functions v and $\ddot{x}$, and show that the acceleration is constant.
 - (b) What are the displacement, velocity and acceleration after 5 seconds?
 - (c) When is the particle stationary, and where is it then?

2. A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2 - t + 2$.
 - (a) Differentiate to find v and $\ddot{x}$ as functions of t .
 - (b) Where is the particle initially, and what are its speed and acceleration?
 - (c) At time $t = 3$: (i) Is the particle left or right of the origin? (ii) Is it travelling to the left or to the right? (iii) In what direction is it accelerating?
 - (d) When is the particle's acceleration zero, and what is its speed then?
3. Find the functions v and $\ddot{x}$ for a particle P moving according to $x = 2 \sin \pi t$.
 - (a) Show that P is at the origin when $t = 1$, and find its velocity and acceleration then.
 - (b) In what direction is the particle: (i) moving, (ii) accelerating, when $t = \frac{1}{3}$?
4. If $x = e^{-4t}$, find the functions v and $\ddot{x}$.
 - (a) Explain why neither x nor v nor $\ddot{x}$ can ever change sign, and state their signs.
 - (b) Where is the particle: (i) initially, (ii) eventually?
 - (c) What are the particle's velocity and acceleration: (i) initially, (ii) eventually?
5. A cricket ball is thrown vertically upwards, and its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.
 - (a) Find v and $\ddot{x}$ as functions of t , and show that the ball is always accelerating downwards. Then sketch graphs of x , v and $\ddot{x}$ against t .
 - (b) Find the speed at which the ball was thrown, find when it returns to the ground, and show that its speed then is equal to the initial speed.
 - (c) Find its maximum height above the ground, and the time to reach this height.
 - (d) Find the acceleration at the top of the flight, and explain why the acceleration can be nonzero when the ball is stationary.
 - (e) When is the ball's height 15 metres, and what are its velocities then?
6. A particle moves according to $x = t^2 - 8t + 7$, in units of metres and seconds.
 - (a) Find v and $\ddot{x}$ as functions of t , then sketch graphs of x , v and $\ddot{x}$ against t .
 - (b) When is the particle: (i) at the origin, (ii) stationary?
 - (c) What is the maximum distance from the origin, and when does it occur: (i) during the first 2 seconds, (ii) during the first 6 seconds, (iii) during the first 10 seconds?
 - (d) What is the particle's average velocity during the first 7 seconds? When and where is its instantaneous velocity equal to this average?
 - (e) How far does it travel during the first 7 seconds, and what is its average speed?
7. A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.
 - (a) Find the functions v and $\ddot{x}$, and sketch x and v .
 - (b) In which direction is the ice moving, and in which direction is it accelerating: (i) when $t = 2$, (ii) when $t = 4$?
 - (c) When is the ice stationary, for how long is it stationary, where is it then, and is it accelerating then?
 - (d) Find the average velocity over the first 2 seconds, and the time and place where the instantaneous velocity equals this average velocity.
 - (e) Show that the average speed during the first 3 seconds, the next 3 seconds and the first 6 seconds are all the same.



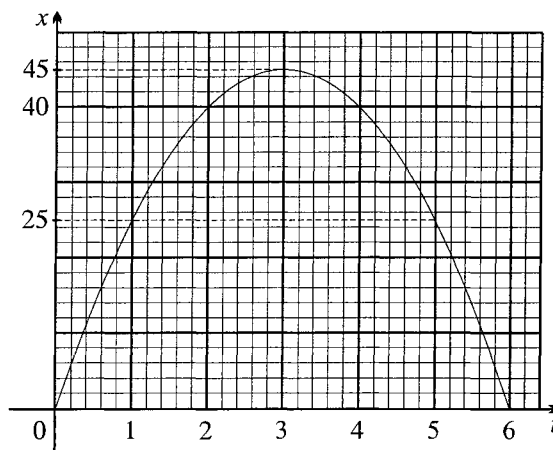
DEVELOPMENT

8. A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.



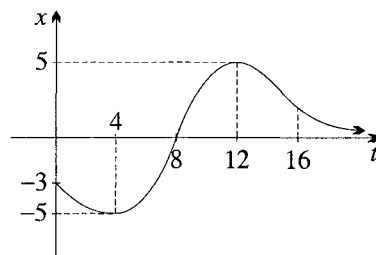
- In the first 10 seconds, what is its maximum distance from the origin, and when does it occur?
- When is the particle: (i) stationary, (ii) moving to the right, (iii) moving to the left?
- When does it return to the origin, what is its velocity then, and in which direction is it accelerating?
- When is its acceleration zero, where is it then, and in what direction is it moving?
- During what time is its acceleration negative?
- At about what times is: (i) the displacement, (ii) the velocity, (iii) the speed, about the same as at $t = 2$?
- Sketch (roughly) the graphs of v and $\ddot{x}$.

9. A stone was thrown vertically upwards, and the graph to the right shows its height x metres at time t seconds after it was thrown.



- What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?
 - Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0, 1, 2, 3, 4, 5$ and 6 .
 - For what length of time was the stone stationary at the top of its flight?
 - The graph is concave down everywhere. How is this relevant to the motion?
 - Draw a graph of the instantaneous velocity of the stone from $t = 0$ to $t = 6$. What does the graph tell you about what happened to the velocity during these 6 seconds?
10. A particle is moving according to $x = 4 \cos \frac{\pi}{4}t$, in units of metres and seconds.
- Find v and $\ddot{x}$, and sketch graphs of x , v and $\ddot{x}$ against t , for $0 \leq t \leq 8$.
 - What are the particle's maximum displacement, velocity and acceleration, and when, during the first 8 seconds, do they occur?
 - How far does it travel during the first 20 seconds, and what is its average speed?
 - When, during the first 8 seconds, is: (i) $x = 2$, (ii) $x < 2$?
 - When, during the first 8 seconds, is: (i) $v = \frac{\pi}{2}$, (ii) $v > \frac{\pi}{2}$?
11. A particle is oscillating on a spring so that its height is $x = 6 \sin 2t$ cm at time t seconds.
- Find v and $\ddot{x}$ as function of t , and sketch graphs of x , v and $\ddot{x}$, for $0 \leq t \leq 2\pi$.
 - Show that $\ddot{x} = -kx$, for some constant k , and find k .
 - When, during the first π seconds, is the particle:
 - at the origin, (ii) stationary, (iii) moving with zero acceleration?
 - When, during the first π seconds, is the particle:
 - below the origin, (ii) moving downwards, (iii) accelerating downwards?
 - Find the first time the particle has: (i) displacement $x = 3$, (ii) speed $|v| = 6$.

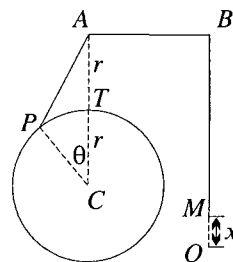
12. A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



- At what times is this particle: (i) below the origin, (ii) moving downwards, (iii) accelerating downwards?
 - At about what time is its speed greatest?
 - At about what times is: (i) distance from the origin, (ii) velocity, (iii) speed, about the same as at $t = 3$?
 - How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?
 - How far will the particle eventually travel?
 - Sketch the graphs of v and $\ddot{x}$ as functions of time.
13. A large stone is falling through a layer of mud, and its depth x metres below ground level at time t minutes is given by $x = 12 - 12e^{-0.5t}$.
- Find v and $\ddot{x}$ as functions of t , and sketch graphs of x , v and $\ddot{x}$.
 - In which direction is the stone: (i) travelling, (ii) accelerating?
 - What happens to the position, velocity and acceleration of the particle as $t \rightarrow \infty$?
 - Find when the stone is halfway between the origin and its final position. Show that its speed is then half its initial speed, and its acceleration is half its initial acceleration.
 - How long, correct to the nearest minute, will it take for the stone to reach within 1 mm of its final position?
14. Two particles A and B are moving along a horizontal line, with their distances x_A and x_B to the right of the origin O at time t given by $x_A = 4te^{-t}$ and $x_B = -4t^2e^{-t}$. The particles are joined by a piece of elastic, whose midpoint M has position x_M at time t .
- Explain why $x_M = 2e^{-t}(t - t^2)$, find when M returns to the origin, and find its speed and direction at this time.
 - Find at what times M is furthest right and furthest left of O .
 - What happens to A , B and M eventually? (d) When are A and B furthest apart?

EXTENSION

15. The diagram to the right shows a point P that is rotating anticlockwise in a circle of radius r and centre C at a steady rate. A string passes over fixed pulleys at A and B , where A is distant r above the top T of the circle, and connects P to a mass M on the end of the string. At time zero, P is at T , and the mass M is at the point O . Let x be the height of the mass above the point O at time t seconds later, and θ be the angle $\angle TCP$ through which P has moved.



- Show that $x = -r + r\sqrt{5 - 4\cos\theta}$, and find the range of x .
- Find $\frac{dx}{d\theta}$, and find for what values of θ the mass M is travelling:
 - upwards,
 - downwards.
- Show that $\frac{d^2x}{d\theta^2} = -\frac{2r(2\cos^2\theta - 5\cos\theta + 2)}{(5 - 4\cos\theta)^{\frac{3}{2}}}$. Find for what values of θ the speed of M is maximum, and find $\frac{dx}{d\theta}$ at these values of θ .

- (d) Explain geometrically why these values of θ give the maximum speed, and why they give the values of $\frac{dx}{d\theta}$ they do.
16. [This question will require resolution of forces.] At what angle α should the surface in question 7 be inclined to the horizontal to produce these equations?

3C Integrating with Respect to Time

The inverse process of differentiation is integration. Therefore if the acceleration function is known, integration will generate the velocity function, and integration of the velocity function will generate the displacement function.

Initial or Boundary Conditions: Taking the primitive of a function always involves an arbitrary constant. Hence one or more boundary conditions are required to determine the motion completely.

WORKED EXERCISE: The velocity of a particle initially at the origin is $v = \sin \frac{1}{4}t$.

- (a) Find the displacement function. (b) Find the acceleration function.
 (c) Find the values of displacement, velocity and acceleration when $t = 4\pi$.
 (d) Briefly describe the motion, and sketch the displacement–time graph.

SOLUTION: Given: $v = \sin \frac{1}{4}t$. (1)

- (a) Integrating, $x = -4 \cos \frac{1}{4}t + C$, for some constant C ,
 and substituting $x = 0$ when $t = 0$:

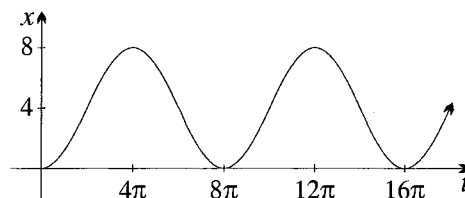
$$0 = -4 \times 1 + C,$$

$$\text{so } C = 4, \text{ and } x = 4 - 4 \cos \frac{1}{4}t. \quad (2)$$

- (b) Differentiating, $\ddot{x} = \frac{1}{4} \cos \frac{1}{4}t$.

- (c) When $t = 4\pi$, $x = 4 - 4 \times \cos \pi = 8$ metres,
 $v = \sin \pi = 0$ m/s,
 and $\ddot{x} = \frac{1}{4} \cos \pi = -\frac{1}{4}$ m/s². (3)

- (d) The particle oscillates between $x = 0$ and $x = 8$ with period 8π seconds.



WORKED EXERCISE: The acceleration of a particle is given by $\ddot{x} = e^{-2t}$, and the particle is initially stationary at the origin.

- (a) Find the velocity function. (b) Find the displacement function.
 (c) Find the displacement when $t = 10$.
 (d) Briefly describe the velocity of the particle as time goes on.

SOLUTION: Given: $\ddot{x} = e^{-2t}$. (1)

- (a) Integrating, $v = -\frac{1}{2}e^{-2t} + C$.

$$\text{When } t = 0, v = 0, \text{ so } 0 = -\frac{1}{2} + C,$$

$$\text{so } C = \frac{1}{2}, \text{ and } v = -\frac{1}{2}e^{-2t} + \frac{1}{2}. \quad (2)$$

- (b) Integrating again, $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t + D$.

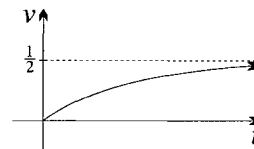
$$\text{When } t = 0, x = 0, \text{ so } 0 = \frac{1}{4} + D,$$

$$\text{so } D = -\frac{1}{4}, \text{ and } x = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}. \quad (3)$$

(c) When $t = 10$,
$$x = \frac{1}{4}e^{-20} + 5 - \frac{1}{4}$$

$$= 4\frac{3}{4} + \frac{1}{4}e^{-20} \text{ metres.}$$

(d) The velocity is initially zero, and increases with limit $\frac{1}{2}$ m/s.



The Acceleration Due to Gravity: Since the time of Galileo, it has been known that near the surface of the Earth, a body free to fall accelerates downwards at a constant rate, whatever its mass, and whatever its velocity (neglecting air resistance). This acceleration is called the *acceleration due to gravity*, and is conventionally given the symbol g . The value of this acceleration is about 9.8 m/s^2 , or in round figures, 10 m/s^2 .

WORKED EXERCISE: A stone is dropped from the top of a high building. How far has it travelled, and how fast is it going, after 5 seconds? (Take $g = 9.8 \text{ m/s}^2$.)

SOLUTION: Let x be the distance travelled t seconds after the stone is dropped. This puts the origin of space at the top of the building and the origin of time at the instant when the stone is dropped, and makes downwards positive.

Then $\ddot{x} = 9.8$ (given).

Integrating, $v = 9.8t + C$, for some constant C .

Since the stone was dropped, its initial speed was zero,

and substituting, $0 = 0 + C$,

so $C = 0$, and $v = 9.8t$.

Integrating again, $x = 4.9t^2 + D$, for some constant D .

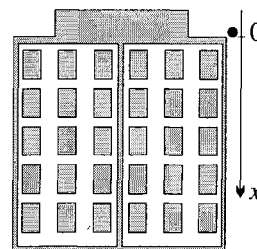
Since the initial displacement of the stone was zero,

$$0 = 0 + D,$$

so $D = 0$, and $x = 4.9t^2$.

When $t = 5$, $v = 49$ and $x = 122.5$.

Hence the stone has fallen 122.5 metres and is moving downwards at 49 m/s.



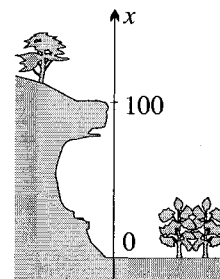
(1)

(2)

(3)

Making a Convenient Choice of the Origin and the Positive Direction: Physical problems do not come with origins and directions attached, and it is up to us to choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible. The previous worked exercise made reasonable choices, but the following worked exercise makes quite different choices. In all such problems, the physical interpretation of negatives and displacements is the responsibility of the mathematician, and the final answer should be free of them.

WORKED EXERCISE: A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of 40 m/s, and it falls back past the lookout onto the valley floor below. How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10 \text{ m/s}^2$.)



SOLUTION: Let x be the distance above the valley floor t seconds after the stone is thrown. This puts the origin of space at the valley floor and the origin of time

at the instant when the stone is thrown. It also makes upwards positive, so that $\ddot{x} = -10$, because the acceleration is downwards.

As discussed, $\ddot{x} = -10$. (1)

Integrating, $v = -10t + C$, for some constant C .

Since $v = 40$ when $t = 0$, $40 = 0 + C$,

so $C = 40$, and $v = -10t + 40$. (2)

Integrating again, $x = -5t^2 + 40t + D$, for some constant D .

Since $x = 100$ when $t = 0$, $100 = 0 + 0 + D$,

so $D = 100$, and $x = -5t^2 + 40t + 100$. (3)

The stone hits the ground when $x = 0$, that is,

$$-5t^2 + 40t + 100 = 0$$

$$t^2 - 8t - 20 = 0$$

$$(t - 10)(t + 2) = 0$$

$$t = 10 \text{ or } -2.$$

Since the ball was not in flight at $t = -2$, the ball hits the ground after 10 seconds.

At that time, $v = -100 + 40 = -60$, so it hits the ground at 60 m/s.

Formulae from Physics Cannot be Used: This course requires that even problems where the acceleration is constant, such as the two above, must be solved by integration of the acceleration function. Many readers will know of three very useful equations for motion with constant acceleration a :

$$v = u + at \quad \text{and} \quad s = ut + \frac{1}{2}at^2 \quad \text{and} \quad v^2 = u^2 + 2as.$$

These equations automate the integration process, and so cannot be used in this course. Questions in Exercises 3C and 3E develop proper proofs of these results.

Using Definite Integrals to find Changes of Displacement and Velocity: The change in displacement during some period of time can be found quickly using a definite integral of the velocity. This avoids evaluating the constant of integration, and is therefore useful when no boundary conditions have been given. The disadvantage is that the displacement–time function remains unknown. Change in velocity can be calculated similarly, using a definite integral of the acceleration.

USING DEFINITE INTEGRALS TO FIND CHANGES IN DISPLACEMENT AND VELOCITY:

Given velocity v as a function of time, then from $t = t_1$ to $t = t_2$,

$$\text{change in displacement} = \int_{t_1}^{t_2} v \, dt.$$

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Given acceleration $\ddot{x}$ as a function of time, then from $t = t_1$ to $t = t_2$,

$$\text{change in velocity} = \int_{t_1}^{t_2} \ddot{x} \, dt.$$

WORKED EXERCISE: In these questions, the units are metres and seconds.

- (a) Given $v = 4 - e^{4-t}$, find the change in displacement during the third second.
 (b) Given $\ddot{x} = 12 \sin 2t$, find the change in velocity during the first $\frac{\pi}{2}$ seconds.

SOLUTION:

(a) Change in displacement

$$\begin{aligned}
 &= \int_2^3 (4 - e^{4-t}) dt \\
 &= \left[4t + e^{4-t} \right]_2^3 \\
 &= (12 + e) - (8 + e^2) \\
 &= 4 + e - e^2 \text{ metres.}
 \end{aligned}$$

(b) Change in velocity

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{2}} 12 \sin 2t dt \\
 &= -6 \left[\cos 2t \right]_0^{\frac{\pi}{2}} \\
 &= -6(-1 - 1) \\
 &= 12 \text{ m/s.}
 \end{aligned}$$

Exercise 3C

1. Find the velocity and displacement functions of a particle whose initial velocity and displacement are zero if:

(a) $\ddot{x} = -4$	(c) $\ddot{x} = e^{\frac{1}{2}t}$	(e) $\ddot{x} = 8 \sin 2t$	(g) $\ddot{x} = \sqrt{t}$
(b) $\ddot{x} = 6t$	(d) $\ddot{x} = e^{-3t}$	(f) $\ddot{x} = \cos \pi t$	(h) $\ddot{x} = 12(t+1)^{-2}$

2. Find the acceleration and displacement functions of a particle whose initial displacement is -2 if:

(a) $v = -4$	(c) $v = e^{\frac{1}{2}t}$	(e) $v = 8 \sin 2t$	(g) $v = \sqrt{t}$
(b) $v = 6t$	(d) $v = e^{-3t}$	(f) $v = \cos \pi t$	(h) $v = 12(t+1)^{-2}$

3. A stone is dropped from a lookout 80 metres high. Take $g = 10 \text{ m/s}^2$, and downwards as positive, so that $\ddot{x} = 10$.

- (a) Using the lookout as the origin, find the velocity and displacement as functions of t . [HINT: When $t = 0$, $v = 0$ and $x = 0$.]
 (b) Find: (i) the time the stone takes to fall, (ii) its impact speed.
 (c) Where is it, and what is its speed, halfway through its flight time?
 (d) How long does it take to go halfway down, and what is its speed then?

4. A stone is thrown downwards from the top of a 120-metre building, with an initial speed of 25 m/s . Take $g = 10 \text{ m/s}^2$, and take upwards as positive, so that $\ddot{x} = -10$.

- (a) Using the ground as the origin, find the acceleration, velocity and height x of the stone t seconds after it is thrown. [HINT: When $t = 0$, $v = -25$ and $x = 120$.] Hence find:
 (i) the time it takes to reach the ground, (ii) the impact speed.
 (b) Rework part (a) with the origin at the top of the building, and downwards positive.

5. A particle is moving with acceleration $\ddot{x} = 12t$. Initially it has velocity -24 m/s , and is 20 metres on the positive side of the origin.

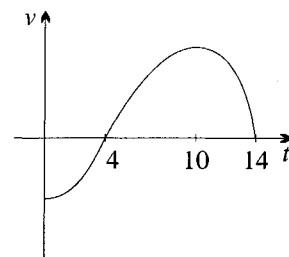
- (a) Find the velocity and displacement functions.
 (b) When does the particle return to its initial position, and what is its speed then?
 (c) What is the minimum displacement, and when does it occur?
 (d) Find x when $t = 0, 1, 2, 3$ and 4 , and sketch the displacement–time graph.

DEVELOPMENT

6. A car moves along a straight road from its front gate, where it is initially stationary. During the first 10 seconds, it has a constant acceleration of 2 m/s^2 , it has zero acceleration during the next 30 seconds, and it decelerates at 1 m/s^2 for the final 20 seconds.

- (a) What is the maximum speed, and how far does the car go altogether?
 (b) Sketch the graphs of acceleration, velocity and distance from the gate.

7. Write down a definite integral for each quantity to be calculated below. If possible, evaluate it exactly. Otherwise, use the trapezoidal rule with three function values in part (a), and Simpson's rule with five function values in part (b), giving your answers correct to three significant figures.
- (a) Find the change in displacement during the 2nd second of motion of a particle whose velocity is:
- (i) $v = \frac{4}{t+1}$ (ii) $v = \frac{4}{\log(t+1)}$
- (b) Find the change in velocity during the 2nd second of motion of a particle whose acceleration is:
- (i) $\ddot{x} = \sin \pi t$ (ii) $\ddot{x} = t \sin \pi t$
8. A body is moving with its acceleration proportional to the time elapsed. When $t = 1$, $v = -6$, and when $t = 2$, $v = 3$.
- (a) Find the functions $\ddot{x}$ and v . [HINT: Let $\ddot{x} = kt$, where k is the constant of proportionality. Then integrate, using the usual constant C of integration. Then find C and k by substituting the two given values of t .]
- (b) When does the body return to its original position?
9. [A proof of three constant-acceleration formulae from physics — not to be used elsewhere]
- (a) A particle moves with constant acceleration a . Its initial velocity is u , and at time t it is moving with velocity v and is distant s from its initial position. Show that:
- (i) $v = u + at$ (ii) $s = ut + \frac{1}{2}at^2$ (iii) $v^2 = u^2 + 2as$
- (b) Solve questions 3 and 4 using formulae (ii) and (i), and again using (iii) and (i).
10. A body falling through air experiences an acceleration $\ddot{x} = -40e^{-2t}$ m/s² (we are taking upwards as positive). Initially, it is thrown upwards with speed 15 m/s.
- (a) Taking the origin at the point where it is thrown, find the functions v and x , and find when the body is stationary.
- (b) Find its maximum height, and the acceleration then.
- (c) Describe the velocity of the body as $t \rightarrow \infty$.
11. If a particle moves from $x = -1$ with velocity $v = \frac{1}{t+1}$, how long does it take to get to the origin, and what are its speed and acceleration then? Describe its subsequent motion.
12. A mouse emerges from his hole and moves out and back along a line. His velocity at time t seconds is $v = 4t(t-3)(t-6) = 4t^3 - 36t^2 + 72t$ cm/s.
- (a) When does he return to his original position, and how fast is he then going?
- (b) How far does he travel during this time, and what is his average speed?
- (c) What is his maximum speed, and when does it occur?
- (d) If a video of these 6 seconds were played backwards, could this be detected?
13. The graph to the right shows a particle's velocity–time graph.
- (a) When is the particle moving forwards?
- (b) When is the acceleration positive?
- (c) When is it furthest from its starting point?
- (d) When is it furthest in the negative direction?
- (e) About when does it return to its starting point?
- (f) Sketch the graphs of acceleration and displacement, assuming that the particle starts at the origin.



14. A particle is moving with velocity $v = 16 - 4t$ cm/s on a horizontal number line.
- Find $\ddot{x}$ and x . (The function x will have a constant of integration.)
 - When does it return to its original position, and what is its speed then?
 - When is the particle stationary? Find the maximum distances right and left of the initial position during the first 10 seconds, and the corresponding times and accelerations.
 - How far does it travel in the first 10 seconds, and what is its average speed?
15. A moving particle is subject to an acceleration of $\ddot{x} = -2 \cos t$ m/s². Initially, it is at $x = 2$, moving with velocity 1 m/s, and it travels for 2π seconds.
- Find the functions v and x .
 - When is the acceleration positive?
 - When and where is the particle stationary, and when is it moving backwards?
 - What are the maximum and minimum velocities, and when and where do they occur?
 - Find the change in displacement and the average velocity.
 - Sketch the displacement–time graph, and hence find the distance travelled and the average speed.
16. Particles P_1 and P_2 move with velocities $v_1 = 6 + 2t$ and $v_2 = 4 - 2t$, in units of metres and seconds. Initially, P_1 is at $x = 2$ and P_2 is at $x = 1$.
- Find x_1 , x_2 and the difference $D = x_1 - x_2$.
 - Prove that the particles never meet, and find the minimum distance between them.
 - Prove that the midpoint M between the two particles is moving with constant velocity, and find its distance from each particle after 3 seconds.
17. Once again, the trains Thomas and Henry are on parallel tracks, level with each other at time zero. Thomas is moving with velocity $v_T = \frac{20}{t+1}$ and Henry with velocity $v_H = 5$.
- Who is moving faster initially, and by how much?
 - Find the displacements x_T and x_H of the two trains, if they start at the origin.
 - Use your calculator to find during which second the trains are level, and find the speed at which the trains are drawing apart at the end of this second.
 - When is Henry furthest behind Thomas, and by how much (to the nearest metre)?
18. A ball is dropped from a lookout 180 metres high. At the same time, a stone is fired vertically upwards from the valley floor with speed V m/s. Take $g = 10$ m/s².
- Find for what values of V a collision in the air will occur. Find, in terms of V , the time and the height when collision occurs, and prove that the collision speed is V m/s.
 - Find the value of V for which they collide halfway up the cliff, and the time taken.

EXTENSION

19. A falling body experiences both the gravitational acceleration g and air resistance that is proportional to its velocity. Thus a typical equation of motion is $\ddot{x} = -10 - 2v$ m/s². Suppose that the body is dropped from the origin.
- By writing $\ddot{x} = \frac{dv}{dt}$ and taking reciprocals, find t as a function of v , and hence find v as a function of t . Then find x as a function of t .
 - Describe the motion of the particle.

3D Simple Harmonic Motion — The Time Equations

As has been mentioned before, some of the most common physical phenomena around us fluctuate — sound waves, light waves, tides, heartbeats — and are therefore governed by sine and cosine functions. The simplest such phenomena are governed by a single sine or cosine function, and accordingly, our course makes a detailed study of motion governed by such a function, called *simple harmonic motion*. This section approaches the topic through the displacement–time equation, but the topic will be studied again in Section 3F using the motion’s characteristic acceleration–displacement equation.

Simple Harmonic Motion: *Simple harmonic motion* (or *SHM* for short) is any motion whose displacement–time equation, apart from the constants, is a single sine or cosine function. More precisely:

SIMPLE HARMONIC MOTION — THE DISPLACEMENT–TIME EQUATION: A particle is said to be moving in *simple harmonic motion with centre the origin* if

$$x = a \sin(nt + \alpha) \quad \text{or} \quad x = a \cos(nt + \alpha),$$

where a , n and α are constants, with a and n positive.

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- The constant a is called the *amplitude* of the motion, and the particle is confined in the interval $-a \leq x \leq a$. The origin is called the *centre* of the motion, because it is the midpoint between the two extremes of the motion, $x = -a$ and $x = a$.
- The *period* T of the motion is given by $T = \frac{2\pi}{n}$.
- At any time t , the quantity $nt + \alpha$ is called the *phase*. In particular, the phase at time $t = 0$ is α , and therefore α is called the *initial phase*.

Since $\cos \theta = \sin(\theta + \frac{\pi}{2})$, either the sine or the cosine function can be used for any particular motion. If the question allows a choice, it is best to choose the function with zero initial phase, because the algebra is easier when $\alpha = 0$.

Simple Harmonic Motion about Other Centres: The motion of a particle oscillating about the point $x = x_0$ rather than the origin can be described simply by adding the constant x_0 .

SIMPLE HARMONIC MOTION ABOUT $x = x_0$: A particle is said to be moving in *simple harmonic motion with centre $x = x_0$* if

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$$x = x_0 + a \sin(nt + \alpha) \quad \text{or} \quad x = x_0 + a \cos(nt + \alpha).$$

The amplitude of the motion is still a , and the particle is confined to the interval $x_0 - a \leq x \leq x_0 + a$ with centre at the midpoint $x = x_0$.

WORKED EXERCISE: A particle is moving in simple harmonic motion according to the equation $x = 2 + 4 \cos(2t + \frac{\pi}{3})$.

- Find the centre, period, amplitude and extremes of the motion.
- What is the initial phase, and where is the particle at $t = 0$?

(c) Find the first time when the particle is at:

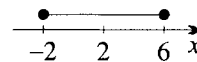
- (i) the centre of motion, (iii) the maximum displacement,
(ii) the origin, (iv) the minimum displacement.

SOLUTION:

(a) The equation has the correct form for SHM.

The centre is $x = 2$, the amplitude is 4 and the period is $\frac{2\pi}{2} = \pi$.

The motion therefore lies in the interval $-2 \leq x \leq 6$.



(b) The initial phase is $\frac{\pi}{3}$. When $t = 0$, $x = 2 + 4 \cos \frac{\pi}{3} = 4$.

(c) (i) Put $2 + 4 \cos(2t + \frac{\pi}{3}) = 2$.

Then $\cos(2t + \frac{\pi}{3}) = 0$

$$2t + \frac{\pi}{3} = \frac{\pi}{2}$$

$$t = \frac{\pi}{12}.$$

(iii) Put $2 + 4 \cos(2t + \frac{\pi}{3}) = 6$.

Then $\cos(2t + \frac{\pi}{3}) = 1$

$$2t + \frac{\pi}{3} = 2\pi$$

$$t = \frac{5\pi}{6}.$$

(ii) Put $2 + 4 \cos(2t + \frac{\pi}{3}) = 0$.

Then $\cos(2t + \frac{\pi}{3}) = -\frac{1}{2}$

$$2t + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$t = \frac{\pi}{6}.$$

(iv) Put $2 + 4 \cos(2t + \frac{\pi}{3}) = -2$.

Then $\cos(2t + \frac{\pi}{3}) = -1$

$$2t + \frac{\pi}{3} = \pi$$

$$t = \frac{\pi}{3}.$$

Finding Acceleration and Velocity: Velocity and acceleration are found by differentiation in the usual way. Doing this in the case when the centre is at the origin results in a most important relationship between acceleration and displacement:

Let $x = a \sin(nt + \alpha)$.

Then $v = an \cos(nt + \alpha)$

and $\ddot{x} = -an^2 \sin(nt + \alpha)$.

Hence $\ddot{x} = -n^2 x$.

Let $x = a \cos(nt + \alpha)$.

Then $v = -an \sin(nt + \alpha)$

and $\ddot{x} = -an^2 \cos(nt + \alpha)$.

Hence $\ddot{x} = -n^2 x$.

In both cases, $\ddot{x} = -n^2 x$, meaning that the acceleration is proportional to the displacement, but acts in the opposite direction. This equation is characteristic of simple harmonic motion, and can be used to test whether a given motion is simple harmonic. It is called a *second-order differential equation* because it involves the second derivative of the function.

THE DIFFERENTIAL EQUATION FOR SIMPLE HARMONIC MOTION: If a particle is moving in simple harmonic motion with centre the origin and period $\frac{2\pi}{n}$, then

$$\ddot{x} = -n^2 x.$$

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This means that the acceleration is proportional to the displacement, but acts in the opposite direction.

This equation is usually the most straightforward way to test whether a given motion is simple harmonic with centre the origin.

In Section 3F, we will use this as the starting point for our second discussion of simple harmonic motion.

WORKED EXERCISE: Suppose that a particle is moving according to $x = 2 \sin(3t + \frac{3\pi}{4})$.

- Write down the amplitude, centre, period and initial phase of the motion.
- Find the times and positions when the velocity is first: (i) zero, (ii) maximum.
- Find the times and positions when $\ddot{x}$ is first: (i) zero, (ii) maximum.
- Express the acceleration $\ddot{x}$ as a multiple of the displacement x .

SOLUTION:

- The amplitude is 2, the centre is $x = 0$, the period is $\frac{2\pi}{3}$, and the initial phase is $\frac{3\pi}{4}$.

- Differentiating, $v = 6 \cos(3t + \frac{3\pi}{4})$, so the maximum velocity is 6.

(This is because $\cos(3t + \frac{3\pi}{4})$ has a maximum of 1.)

$$\begin{array}{ll} \text{(i) When } v = 0, \cos(3t + \frac{3\pi}{4}) = 0 & \text{(ii) When } v = 6, 6 \cos(3t + \frac{3\pi}{4}) = 6 \\ 3t + \frac{3\pi}{4} = \frac{3\pi}{2} & 3t + \frac{3\pi}{4} = 2\pi \\ t = \frac{\pi}{4}. & t = \frac{5\pi}{12}. \end{array}$$

$$\begin{array}{l} \text{When } t = \frac{\pi}{4}, x = 2 \sin \frac{3\pi}{2} \\ = -2. \end{array}$$

$$\begin{array}{l} \text{When } t = \frac{5\pi}{12}, x = 2 \sin 2\pi \\ = 0. \end{array}$$

- Differentiating, $\ddot{x} = -18 \sin(3t + \frac{3\pi}{4})$, so the maximum acceleration is 18.

(This is because $\sin(3t + \frac{3\pi}{4})$ has a maximum of 1.)

$$\begin{array}{ll} \text{(i) When } \ddot{x} = 0, \sin(3t + \frac{3\pi}{4}) = 0 & \text{(ii) When } \ddot{x} = 18, -18 \sin(3t + \frac{3\pi}{4}) = 18 \\ 3t + \frac{3\pi}{4} = \pi & 3t + \frac{3\pi}{4} = \frac{3\pi}{2} \\ t = \frac{\pi}{12}. & t = \frac{\pi}{4}. \end{array}$$

$$\begin{array}{l} \text{When } t = \frac{\pi}{12}, x = 2 \sin \pi \\ = 0. \end{array}$$

$$\begin{array}{l} \text{When } t = \frac{\pi}{4}, x = 2 \sin \frac{3\pi}{2} \\ = -2. \end{array}$$

- Since $x = 2 \sin(3t + \frac{3\pi}{4})$ and $\ddot{x} = -18 \sin(3t + \frac{3\pi}{4})$, it follows that $\ddot{x} = -9x$.

Since $n = 3$, this agrees with the general result $\ddot{x} = -n^2x$.

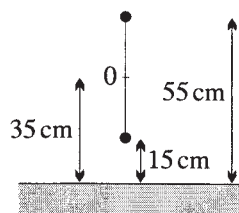
Choosing Convenient Origins of Space and Time: Many questions on simple harmonic motion do not specify a choice of axes. In these cases, the reader should set up the axes for displacement and time, and choose the function, to make the equations as simple as possible. First, choose the centre of motion as the origin of displacement — this makes $x_0 = 0$, so that the constant term disappears.

Secondly, the function and the origin of time should, if possible, be chosen so that the initial phase $\alpha = 0$. The key to this is that $\sin t$ is initially zero and rising, and $\cos t$ is initially maximum.

CHOOSING THE ORIGIN OF TIME AND THE FUNCTION: Try to make the initial phase zero:

- Use $x = a \cos nt$ if the particle starts at the positive extreme of its motion, and use $x = -a \cos nt$ if the particle starts at the negative extreme.
- Use $x = a \sin nt$ if the particle starts at the middle of its motion with positive velocity, and use $x = -a \sin nt$ if the particle starts at the middle of its motion with negative velocity.
- If the particle starts anywhere else, try to change the origin of time. Otherwise, use $x = a \cos(nt + \alpha)$ or $x = a \sin(nt + \alpha)$, and then substitute the boundary conditions to find α and a .

WORKED EXERCISE: A weight hanging from the roof on an elastic string is moving in simple harmonic motion. It takes 4 seconds to move from the bottom of its motion, 15 cm above the floor, to the top of its motion, 55 cm above the floor.



- Find where it is 3 seconds after rising through the centre of motion.
- Find its speed at the centre of motion.
- Find the maximum acceleration.

SOLUTION: Choose the origin at the centre of motion, 35 cm above the ground.

Choose the origin of time at the instant it passes through the centre, moving upwards.

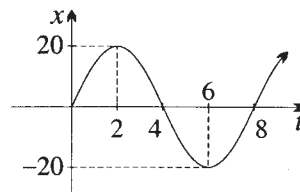
Then the amplitude is 20 cm and the period is $2 \times 4 = 8$ seconds,

$$\text{so } n = \frac{2\pi}{8} = \frac{\pi}{4}, \text{ and } x = 20 \sin \frac{\pi}{4}t.$$

$$\begin{aligned} \text{(a) When } t = 3, \quad x &= 20 \sin \frac{3\pi}{4} \\ &= 10\sqrt{2}, \end{aligned}$$

so the weight is $35 + 10\sqrt{2}$ cm above the ground.

$$\begin{aligned} \text{(b) Differentiating, } \quad v &= 5\pi \cos \frac{\pi}{4}t. \\ \text{The weight is at the centre when } t &= 0, \\ \text{and then } \quad \text{speed} &= 5\pi \cos 0 \\ &= 5\pi \text{ cm/s.} \end{aligned}$$



$$\begin{aligned} \text{(c) Differentiating again, } \quad \ddot{x} &= -\frac{5\pi^2}{4} \sin \frac{\pi}{4}t, \\ \text{so the maximum acceleration is } &\frac{5\pi^2}{4} \text{ cm/s}^2. \end{aligned}$$

WORKED EXERCISE: [Tides can be modelled by simple harmonic motion] On a certain day the depth of water in a harbour at low tide at 3:30 am is 5 metres. At the following high tide at 9:45 am the depth is 15 metres. Assuming the rise and fall of the surface of the water to be simple harmonic motion, find between what times during the morning a ship may safely enter the harbour if a minimum depth of $12\frac{1}{2}$ metres of water is required.

SOLUTION: Let x be the number of metres by which the water depth exceeds 10 metres at time t hours after 3:30 am. (This places the origin of displacement at mean tide, and the origin of time at low tide.) The motion is simple harmonic with amplitude 5 metres and period $12\frac{1}{2}$ hours.

$$\text{Hence } n = \frac{2\pi}{T} = \frac{2\pi}{12\frac{1}{2}} = \frac{4\pi}{25},$$

$$\text{and so the height is } x = -5 \cos \frac{4\pi}{25}t.$$

$$\text{The ship may enter safely when } x \geq 2\frac{1}{2}.$$

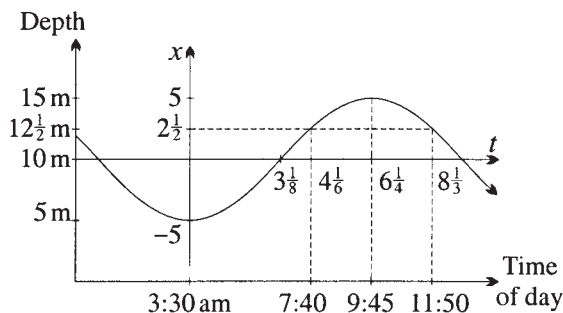
$$\text{Solving first } x = 2\frac{1}{2},$$

$$-5 \cos \frac{4\pi}{25}t = 2\frac{1}{2}$$

$$\cos \frac{4\pi}{25}t = -\frac{1}{2}$$

$$\frac{4\pi}{25}t = \pi - \frac{\pi}{3} \text{ or } \pi + \frac{\pi}{3}$$

$$t = 4\frac{1}{6} \text{ or } 8\frac{1}{3}.$$



$$\text{Hence from the graph, the ship may enter when } 4\frac{1}{6} \leq t \leq 8\frac{1}{3},$$

that is, between 7:40 am and 11:50 am (remembering that $t = 0$ is 3:30 am).

The Graphs of $x = a \cos(nt + \alpha)$ and $x = a \sin(nt + \alpha)$: When the initial phase α is nonzero, the graphs can be sketched by shifting the graphs of $x = a \cos nt$ and $x = a \sin nt$. The key step here is to take out the factor of n and write

$$x = a \cos n(t + \alpha/n) \quad \text{and} \quad x = a \sin n(t + \alpha/n),$$

These graphs are $x = a \cos nt$ and $x = a \sin nt$ shifted left by α/n .

THE GRAPHS OF $x = a \cos(nt + \alpha)$ AND $x = a \sin(nt + \alpha)$: Write $nt + \alpha = n(t + \alpha/n)$.

Then the equations become

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$$x = a \cos n(t + \alpha/n) \quad \text{and} \quad x = a \sin n(t + \alpha/n),$$

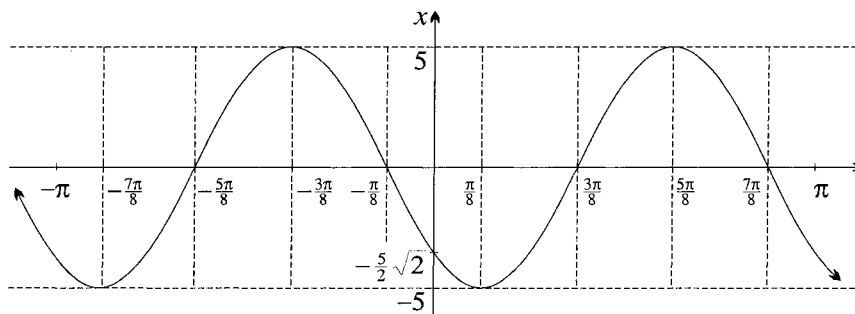
which are $x = a \cos nt$ and $x = a \sin nt$ shifted left by α/n .

WORKED EXERCISE: Sketch $x = 5 \cos(2t + \frac{3\pi}{4})$, for $-\pi \leq t \leq \pi$.

SOLUTION: $x = 5 \cos(2t + \frac{3\pi}{4}) = 5 \cos 2(t + \frac{3\pi}{8})$

This is a cosine wave with amplitude 5 and period π , shifted left by $\frac{3\pi}{8}$ units.

Also when $t = 0$, $x = 5 \cos \frac{3\pi}{4} = -\frac{5}{2}\sqrt{2}$.



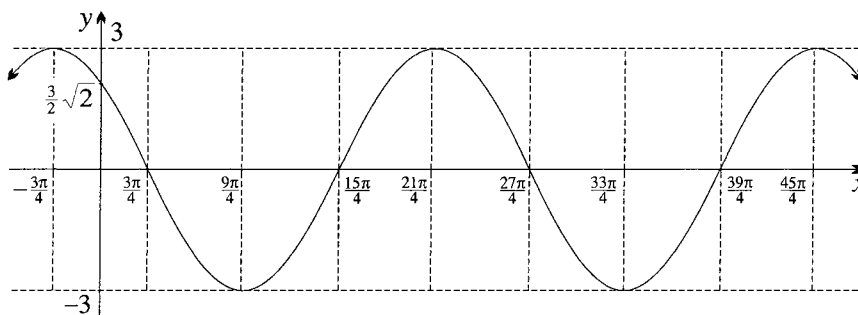
WORKED EXERCISE: Sketch $y = -3 \sin(\frac{1}{3}x - \frac{\pi}{4})$.

SOLUTION: $y = -3 \sin(\frac{1}{3}x - \frac{\pi}{4}) = -3 \sin \frac{1}{3}(x - \frac{3\pi}{4})$

This is $y = 3 \sin \frac{1}{3}(x - \frac{3\pi}{4})$ reflected in the x -axis.

This is a sine wave with amplitude 3, and period 6π , shifted right by $\frac{3\pi}{4}$.

Also when $x = 0$, $y = -3 \sin(-\frac{\pi}{4}) = \frac{3}{2}\sqrt{2}$.



WORKED EXERCISE: A weight on a spring is moving in simple harmonic motion with a period of $\frac{2\pi}{5}$ seconds. A laser observation at a certain instant shows it to be 15 cm below the origin, moving upwards at 60 cm/s.

- (a) Find the displacement x of the weight above the origin as a function of the time t after the laser observation. Use the form $x = a \sin(nt - \alpha)$.

- (b) Find how long the weight takes to reach the origin (three significant figures).
 (c) [A harder question] Find when the weight returns to where it was first observed. Use the $\sin A = \sin B$ approach to solve the trigonometric equation.

SOLUTION:

- (a) We know $n = 2\pi/T$ and $T = \frac{2\pi}{5}$, so $n = 5$.

Let $x = a \sin(5t - \alpha)$, where $a > 0$ and $0 \leq \alpha < 2\pi$,

then differentiating, $v = 5a \cos(5t - \alpha)$.

When $t = 0$, $x = -15$, so $-15 = a \sin(-\alpha)$,

and since $\sin \theta$ is odd, $15 = a \sin \alpha$. (1)

When $t = 0$, $v = 60$, so $60 = 5a \cos(-\alpha)$,

and since $\cos \theta$ is even, $12 = a \cos \alpha$. (2)

Squaring and adding, $369 = a^2$,

$$a = 3\sqrt{41} \quad (\text{since } a > 0).$$

Substituting, $\sin \alpha = 5/\sqrt{41}$, (1A)

and $\cos \alpha = 4/\sqrt{41}$. (2A)

Hence α is acute with $\alpha = \tan^{-1} \frac{5}{4} \doteq 0.896 \dots$ (store in memory)

and so $x = 3\sqrt{41} \sin(5t - \alpha)$.

- (b) When $x = 0$, $\sin(5t - \alpha) = 0$,

so for the first positive solution, $5t - \alpha = 0$

$$t = \frac{1}{5}\alpha$$

$$\doteq 0.179 \text{ seconds.}$$

- (c) To find when the weight returns to its starting place,
 we need the first positive solution of $x(t) = x(0)$.

That is, $\sin(5t - \alpha) = \sin(-\alpha)$

$$5t - \alpha = \pi - (-\alpha) \quad (\text{using solutions to } \sin A = \sin B)$$

$$5t = \pi + 2\alpha$$

$$t = \frac{\pi}{5} + \frac{2}{5}\alpha$$

$$\doteq 0.987 \text{ seconds.}$$

Using the Standard Form $x = b \sin nt + c \cos nt$: We know from Section 2E that functions of the form $x = a \sin(nt + \alpha)$ or $x = \cos(nt + \alpha)$ are equivalent to functions of the form $x = b \sin nt + c \cos nt$. This standard form is often easier to use. First, it avoids the difficulties with the calculation of the auxiliary angle. Secondly, it makes substitution of the initial displacement and velocity particularly easy.

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THE STANDARD FORM $x = b \sin nt + c \cos nt$ FOR SIMPLE HARMONIC MOTION: When a particle starts neither at the origin nor at one extreme, it may be more convenient to use the standard form

$$x = b \sin nt + c \cos nt.$$

[This becomes $x = x_0 + b \sin nt + c \cos nt$ if the centre is not at the origin.]

Provided that the centre is at the origin, the displacement still satisfies the differential equation $\ddot{x} = -n^2x$. To check this:

$$x = b \sin nt + c \cos nt$$

$$\dot{x} = nb \cos nt - nc \sin nt$$

$$\begin{aligned}\ddot{x} &= -n^2b \sin nt - n^2c \cos nt \\ &= -n^2x, \text{ as required.}\end{aligned}$$

This is hardly surprising, since the function is the same function, but written in a different form.

WORKED EXERCISE: Repeat the previous worked exercise using the standard form $x = b \sin nt + c \cos nt$. Use the t -formulae to solve part (c).

SOLUTION:

(a) Let $x = b \sin 5t + c \cos 5t$.

Differentiating, $v = 5b \cos 5t - 5c \sin 5t$.

When $t = 0$, $x = -15$, so $-15 = 0 + c$

$$c = -15.$$

When $t = 0$, $v = 60$, so $60 = 5b + 0$

$$b = 12$$

Hence $x = 12 \sin 5t - 15 \cos 5t$.

(b) When $x = 0$, $12 \sin 5t = 15 \cos 5t$

$$\tan 5t = \frac{5}{4},$$

so the first positive solution is $t = \frac{1}{5} \tan^{-1} \frac{5}{4}$
 $\doteq 0.179$ seconds.

(c) Put $12 \sin 5t - 15 \cos 5t = -15$

$$4 \sin 5t - 5 \cos 5t = -5.$$

Let $T = \tan \frac{5}{2}t$. (Here $\theta = 5t$, so $\frac{1}{2}\theta = \frac{5}{2}t$.)

Then $\frac{8T}{1+T^2} - \frac{5(1-T^2)}{1+T^2} = -5$

$$8T - 5 + 5T^2 = -5 - 5T^2$$

$$10T^2 + 8T = 0$$

$$2T(5T + 4) = 0,$$

so $\tan \frac{5}{2}t = 0$ or $\tan \frac{5}{2}t = -\frac{4}{5}$.

Hence the first positive solution is $t = \frac{2}{5}(\pi - \tan^{-1} \frac{4}{5})$
 $\doteq 0.987$ seconds.

Exercise 3D

- A particle is moving in simple harmonic motion with displacement $x = \frac{4}{\pi} \sin \pi t$, in units of metres and seconds.
 - Differentiate to find v and $\ddot{x}$ as functions of time, and show that $\ddot{x} = -\pi^2x$.
 - What are the amplitude, period and centre of the motion?
 - What are the maximum speed, acceleration and distance from the origin?
 - Sketch the graphs of x , v and $\ddot{x}$ against time.
 - Find the next two times the particle is at the origin, and the velocities then.
 - Find the first two times the particle is stationary, and the accelerations then.

2. A particle is moving in simple harmonic motion with period 4 seconds and centre the origin, and starts from rest 12 cm on the positive side of the origin.
 - (a) Find x as a function of t . [HINT: Since it starts at the maximum, this is a cosine function, so put $x = a \cos nt$. Now find a and n from the data.]
 - (b) Differentiate to find v and $\ddot{x}$ as functions of t , and show that $\ddot{x} = -n^2x$.
 - (c) How long is it between visits to the origin?
3. A particle moving in simple harmonic motion has speed 12 m/s at the origin. Find the displacement–time equation if it is known that for positive constants a and n :
 - (a) $x = a \sin 3t$ (b) $x = 2 \sin nt$ (c) $x = a \cos 8t$ (d) $x = 16 \cos nt$
 [HINT: Start by differentiating the given equation to find the equation of v . Then use the fact that the speed at the origin is the maximum value of $|v|$.]
4. [HINT: Since each particle starts from the origin, moving forwards, its displacement–time equation is a sine function. Thus put $x = a \sin nt$, then find a and n from the data.]
 - (a) A particle moving in simple harmonic motion with centre the origin and period π seconds starts from the origin with velocity 4 m/s. Find x and v as functions of time, and the interval within which it moves.
 - (b) A particle moving in simple harmonic motion with centre the origin and amplitude 6 metres starts from the origin with velocity 4 m/s. Find x and v as functions of time, and the period of its motion.
5. (a) A particle's displacement is given by $x = b \sin nt + c \cos nt$, where $n > 0$. Find v and $\ddot{x}$ as functions of t . Then show that $\ddot{x} = -n^2x$, and hence that the motion is simple harmonic.
 - (b) By substituting into the expressions for x and v , find b and c if initially the particle is at rest at $x = 3$.
 - (c) Find b , c and n , and the first time the particle reaches the origin, if the particle is initially at rest at $x = 5$, and the period is 1 second.
6. A particle's displacement is $x = 12 - 2 \cos 3t$, in units of centimetres and seconds.
 - (a) Differentiate to find v and $\ddot{x}$ as functions of t , show that the particle is initially stationary at $x = 10$, and sketch the displacement–time graph.
 - (b) What are the amplitude, period and centre of the motion?
 - (c) In what interval is the particle moving, and how long does it take to go from one end to the other?
 - (d) Find the first two times after time zero when the particle is closest to the origin, and the speed and acceleration then.
 - (e) Find the first two times when the particle is at the centre, and the speed and acceleration then.
7. A particle is moving in simple harmonic motion according to $x = 6 \sin(2t + \frac{\pi}{2})$.
 - (a) What are the amplitude, period and initial phase?
 - (b) Find $\dot{x}$ and $\ddot{x}$, and show that $\ddot{x} = -n^2x$, for some $n > 0$.
 - (c) Find the first two times when the particle is at the origin, and the velocity then.
 - (d) Find the first two times when the velocity is maximum, and the position then.
 - (e) Find the first two times the particle returns to its initial position, and its velocity and acceleration then.

8. (a) Explain why $\sin(t + \frac{\pi}{2}) = \cos t$, and $\cos(t - \frac{\pi}{2}) = \sin t$:
 (i) algebraically, (ii) by shifting.
 (b) Simplify $x = \sin(t - \frac{\pi}{2})$ and $x = \cos(t + \frac{\pi}{2})$: (i) algebraically, (ii) by shifting.

DEVELOPMENT

9. A particle is travelling in simple harmonic motion about the origin with period 24 seconds and amplitude 120 metres. Initially it is at the origin, moving forwards.
 (a) Write down the functions x and v , and state the maximum speed.
 (b) What is the first time when it is 30 metres: (i) to the right of the origin, (ii) to the left of the origin? (Answer correct to four significant figures.)
 (c) Find the first two times its speed is half its maximum speed.
10. A particle moves in simple harmonic motion about the origin with period $\frac{\pi}{2}$ seconds. Initially the particle is at rest 4 cm to the right of O .
 (a) Write down the displacement–time and velocity–time functions.
 (b) Find how long the particle takes to move from its initial position to: (i) a point 2 cm to the right of O , (ii) a point 2 cm on the left of O .
 (c) Find the first two times when the speed is half the maximum speed.
11. The equation of motion of a particle is $x = \sin^2 t$. Use trigonometric identities to put the equation in the form $x = x_0 - a \cos nt$, and state the centre, amplitude, range and period of the motion.
12. A particle moves according to $x = 3 - 2 \cos^2 2t$, in units of centimetres and seconds.
 (a) Use trigonometric identities to put the equation in the form $x = x_0 - a \cos nt$.
 (b) Find the centre of motion, the amplitude, the range of the motion and the period.
 (c) What is the maximum speed of the particle, and when does it first occur?
13. A particle's displacement is given by $x = b \sin nt + c \cos nt$, where $n > 0$. Find v as a function of t . Then find b , c and n , and the first two times the particle reaches the origin, if:
 (a) the period is 4π , the initial displacement is 6 and the initial velocity is 3,
 (b) the period is 6, $x(0) = -2$ and $\dot{x}(0) = 3$.
14. By taking out the coefficient of t , state the amplitude, period and natural shift left or right of each graph. Hence sketch the curve in a domain showing at least one full period. Show the coordinates of all intercepts. [HINT: For example, the first function is $x = 4 \cos 2(t - \frac{\pi}{6})$, which has amplitude 4, period π , and is $x = 4 \cos 2t$ shifted right by $\frac{\pi}{6}$.]
 (a) $x = 4 \cos(2t - \frac{\pi}{3})$ (c) $x = -3 \cos(\frac{1}{3}t + \pi)$
 (b) $x = \frac{1}{3} \sin(\frac{1}{2}t + \frac{\pi}{4})$ (d) $x = -2 \sin(4t - \pi)$
 How many times is each particle at the origin during the first 2π seconds?
15. Use the functions in the previous question to sketch these graphs. Show all intercepts.
 (a) $x = 4 + 4 \cos(2t - \frac{\pi}{3})$ (c) $x = -3 - 3 \cos(\frac{1}{3}t + \pi)$
 (b) $x = -1 + \frac{1}{3} \sin(\frac{1}{2}t + \frac{\pi}{4})$ (d) $x = 3 - 2 \sin(4t - \pi)$
 How many times is each particle at the origin during the first 2π seconds?
16. Given that $x = a \sin(nt + \alpha)$ (in units of metres and seconds), find v as a function of time. Find a , n and α if $a > 0$, $n > 0$, $0 \leq \alpha < 2\pi$ and:
 (a) the period is 6 seconds, and initially $x = 0$ and $v = 5$,
 (b) the period is 3π seconds, and initially $x = -5$ and $v = 0$,
 (c) the period is 2π seconds and initially $x = 1$ and $v = -1$.

17. Given $x = a \cos(2t - \varepsilon)$, find the function v . Find a and ε if $a > 0$, $0 \leq \varepsilon < 2\pi$ and:
- (a) initially $x = 0$ and $v = 6$, (b) initially $x = 1$ and $v = -2\sqrt{3}$.
18. A particle is moving in simple harmonic motion according to $x = a \cos(\frac{\pi}{8}t + \alpha)$, where $a > 0$ and $0 \leq \alpha < 2\pi$. When $t = 2$ it passes through the origin, and when $t = 4$ its velocity is 4 cm/s in the negative direction. Find the amplitude a and the initial phase α .
19. A particle is moving in simple harmonic motion with period 8π seconds according to $x = a \sin(nt + \alpha)$, where x is the displacement in metres, and $a > 0$ and $0 \leq \alpha < 2\pi$. When $t = 1$, $x = 3$ and $v = -1$. Find a and α correct to four significant figures.
20. A particle moving in simple harmonic motion has period $\frac{\pi}{2}$ seconds. Initially the particle is at $x = 3$ with velocity $v = 16$ m/s.
- (a) Find x as a function of t in the form $x = b \sin nt + c \cos nt$.
- (b) Find x as a function of t in the form $x = a \cos(nt - \varepsilon)$, where $a > 0$ and $0 \leq \varepsilon < 2\pi$.
- (c) Find the amplitude and the maximum speed of the particle.
- (d) Find the first time the particle is at the origin, using each of the above displacement functions in turn. Prove that the two answers obtained are the same.
21. A particle moves in simple harmonic motion with period 8π . Initially, it is at the point P where $x = 4$, moving with velocity $v = 6$. Find, correct to three significant figures, how long it takes to return to P :
- (a) by expressing the motion in the form $x = b \sin nt + c \cos nt$, and using the t -formulae.
- (b) by expressing the motion in the form $x = a \cos(nt - \alpha)$, and using the solutions to $\cos A = \cos B$. [HINT: You will find that $\frac{1}{\sqrt{37}} = \cos \alpha$.]
22. A particle moves on a line, and the table below shows some observations of its positions at certain times:

t (in seconds)	0	7	9	11	18
x (in metres)	0			2	0

- (a) Complete the table if the particle is moving with constant acceleration.
- (b) Complete the table if the particle is moving in simple harmonic motion with centre the origin and period 12 seconds.
23. The temperature at each instant of a day can be modelled by a simple harmonic function oscillating between 9° at 4:00 am and 19° at 4:00 pm. Find, correct to the nearest minute, the times between 4:00 am and 4:00 pm when the temperature is:
- (a) 14° (b) 11° (c) 17°
24. The rise and fall in sea level due to tides can be modelled by simple harmonic motion. On a certain day, a channel is 10 metres deep at 9:00 am when it is low tide, and 16 metres deep at 4:00 pm when it is high tide. If a ship needs 12 metres of water to sail down a channel safely, at what times (correct to the nearest minute) between 9:00 am and 9:00 pm can the ship pass through?
25. (a) Express $x = -4 \cos 3\pi t + 2 \sin 3\pi t$ in the form $x = a \cos(3\pi t - \varepsilon)$, where $a > 0$ and $0 \leq \varepsilon < 2\pi$, giving ε to four significant figures. (The units are cm and seconds.)
- (b) Hence find, correct to the nearest 0.001 seconds:
- (i) when the particle is first 3 cm on the positive side of the origin.
- (ii) when the particle is first moving with velocity -1 cm/s.

26. A particle is moving in simple harmonic motion with period $2\pi/n$, centre the origin, initial position $x(0)$ and initial velocity $v(0)$. Find its displacement–time equation in the form $x = b \sin nt + c \cos nt$, and write down its amplitude.
27. Show that for any particle moving in simple harmonic motion, the ratio of the average speed over one oscillation to the maximum speed is $2 : \pi$.

————— EXTENSION —————

28. [Sums to products and products to sums are useful in this question.]
- (a) Express $\sin nt + \sin(nt + \alpha)$ in the form $a \sin(nt + \varepsilon)$, and hence show that
- $$\sin nt + \sin(nt + \pi) \equiv 0.$$
- (b) Show that $\sin nt + \sin(nt + \alpha) + \sin(nt + 2\alpha) \equiv (1 + 2 \cos \alpha) \sin(nt + \alpha)$, and hence
- $$\sin nt + \sin\left(nt + \frac{2\pi}{3}\right) + \sin\left(nt + \frac{4\pi}{3}\right) \equiv 0.$$
- (c) Prove $\sin nt + \sin(nt + \alpha) + \sin(nt + 2\alpha) + \sin(nt + 3\alpha) \equiv (2 \cos \frac{1}{2}\alpha + 2 \cos \frac{3}{2}\alpha) \sin\left(nt + \frac{3}{2}\alpha\right)$. Hence show that
- $$\sin nt + \sin\left(nt + \frac{\pi}{2}\right) + \sin(nt + \pi) + \sin\left(nt + \frac{3\pi}{2}\right) \equiv 0.$$
- (d) Generalise these results to $\sin nt + \sin(nt + \alpha) + \sin(nt + 2\alpha) + \cdots + \sin(nt + (k-1)\alpha)$, and show that if $\alpha = \frac{2\pi}{k}$, then
- $$\sin nt + \sin(nt + \alpha) + \sin(nt + 2\alpha) + \cdots + \sin(nt + (k-1)\alpha) \equiv 0.$$

3E Motion Using Functions of Displacement

In many physical situations, the acceleration or velocity of the particle is more naturally understood as a function of where it is (the displacement x) than of how long it has been travelling (the time t). For example, the acceleration of a body being drawn towards a magnet depends on how far it is from the magnet.

In such situations, the function must be integrated with respect to x rather than t , because x is the variable in the function. This section deals with the necessary mathematical techniques.

Velocity as a Function of Displacement: Suppose that the velocity is given as a function of displacement, for example $v = e^{-x}$. All that is required here is to take reciprocals of both sides, because the reciprocal of $v = \frac{dx}{dt}$ is $\frac{dt}{dx}$.

15 **VELOCITY AS A FUNCTION OF DISPLACEMENT:** If the velocity is given as a function of displacement, take the reciprocal to give $\frac{dt}{dx}$ as a function of x , and then integrate with respect to x .

WORKED EXERCISE: Suppose that a particle is initially at the origin, and moves according to $v = e^{-x}$ m/s. Find x , v and $\ddot{x}$ in terms of t , and find how long it takes for the particle to travel 1 metre. Briefly describe the subsequent motion.

SOLUTION:

$$\begin{aligned}\text{Given } \frac{dx}{dt} &= e^{-x}, \\ \frac{dt}{dx} &= e^x \\ t &= e^x + C.\end{aligned}$$

When $t = 0$, $x = 0$,

$$0 = 1 + C,$$

so $C = -1$, and $t = e^x - 1$,and it takes $e - 1$ seconds to go 1 metre.

$$\begin{aligned}\text{Solving for } x, \quad e^x &= t + 1 \\ x &= \log(t + 1).\end{aligned}$$

$$\text{Differentiating, } v = \frac{1}{t + 1}$$

$$\text{and } \ddot{x} = -\frac{1}{(t + 1)^2}.$$

The particle moves to infinity.

Its velocity remains positive, but decreases with limit zero.

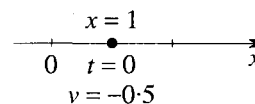
WORKED EXERCISE: A particle moves so that its velocity is proportional to its displacement from the origin O . Initially it is 1 cm to the right of the origin, moving to the left with a speed of 0.5 cm/s. Find the displacement and velocity as functions of time, and briefly describe its motion.

SOLUTION: $v = kx$, for some constant k .When $x = 1$, $v = -\frac{1}{2}$, so $-\frac{1}{2} = k \times 1$,so $k = -\frac{1}{2}$, and $v = -\frac{1}{2}x$.

$$\text{Taking reciprocals, } \frac{dt}{dx} = -\frac{2}{x}$$

and integrating, $t = -2\log x + C$, for some constant C .When $x = 1$, $t = 0$, so $0 = 0 + C$,so $C = 0$, and $t = -2\log x$.Solving for x , $x = e^{-\frac{1}{2}t}$,and differentiating, $v = -\frac{1}{2}e^{-\frac{1}{2}t}$.

Thus the particle continues to move to the left, its speed decreasing with limit zero, and the origin being its limiting position.



Acceleration as a Function of Displacement: Acceleration has been defined as the rate of change of velocity with respect to time, that is as $\ddot{x} = \frac{dv}{dt}$. Dealing with situations where acceleration is a function of displacement requires the following alternative form for acceleration.

ACCELERATION AS A DERIVATIVE WITH RESPECT TO DISPLACEMENT:

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The acceleration is given by $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$.

PROOF: [Examinable]

First, using the chain rule:

$$\begin{aligned}\frac{d}{dx}(\frac{1}{2}v^2) &= \frac{d}{dv}(\frac{1}{2}v^2) \times \frac{dv}{dx} \\ &= v \frac{dv}{dx}.\end{aligned}$$

Secondly, using the chain rule again,

$$\begin{aligned}v \frac{dv}{dx} &= \frac{dx}{dt} \times \frac{dv}{dx} \\ &= \frac{dv}{dt} \\ &= \ddot{x}.\end{aligned}$$

The method of solving such problems is now clear:

- 17 ACCELERATION AS A FUNCTION OF DISPLACEMENT:** If the acceleration is given as a function of displacement, use the form $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ for acceleration, and then integrate with respect to x .

NOTE: The intermediate step in the proof above shows that $\ddot{x} = v \frac{dv}{dx}$ is yet another form of the acceleration. This form is very useful when acceleration is a function of velocity — air resistance is a good example of this, because the resistance offered by the air to a projectile moving through it is a function of the projectile's speed. Such equations are a topic in the 4 Unit course, not the 3 Unit course, but a couple of these questions are offered in the Extension section of the following exercise.

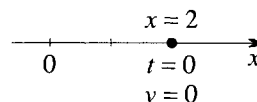
WORKED EXERCISE: Suppose that a ball attached to the ceiling by a long spring will hang at rest at the point $x = 0$. The ball is lifted 2 metres above $x = 0$ and dropped, and subsequently moves according to the equation $\ddot{x} = -4x$. Find its speed as a function of x , and show that it comes to rest 2 metres below $x = 0$. Find its maximum speed and the place where this occurs.

SOLUTION: We know that

$$\ddot{x} = -4x,$$

so

$$\frac{d}{dx}(\frac{1}{2}v^2) = -4x.$$



Integrating with respect to x , $\frac{1}{2}v^2 = -2x^2 + \frac{1}{2}C$, for some constant C ,
 $v^2 = -4x^2 + C$.

(NOTE: It is easier to work with v^2 as the subject, so it is easier to take the constant of integration as $\frac{1}{2}C$ rather than C .)

When $x = 2$, $v = 0$, so

$$0 = -16 + C$$

so $C = 16$, and

$$v^2 = 16 - 4x^2.$$

Hence $v = 0$ when $x = -2$, as required.

The maximum speed is 4 m/s when $x = 0$.

Acceleration as a Function of Displacement — The Second Integration: Integrating using $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ will straightforwardly yield v^2 as a function of x . Further integration, however, requires taking the square root of v^2 , and this will be blocked or very complicated if the sign of v cannot be determined easily.

- 18 ACCELERATION AS A FUNCTION OF DISPLACEMENT — THE SECOND INTEGRATION:** The first integration will give v^2 as a function of x . If the sign of v can be determined, then take square roots to give v as a function of x , and proceed as before.

WORKED EXERCISE: A particle is moving with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.

- Find v^2 as a function of displacement.
- Assuming that v is never positive, find the displacement as a function of time, and briefly describe the motion, mentioning what happens as $t \rightarrow \infty$.
- [A harder question] Explain why the velocity can never be positive.

SOLUTION:

(a) Since acceleration is given as a function of displacement, we write:

$$\frac{d}{dx}(\tfrac{1}{2}v^2) = 3x^2$$

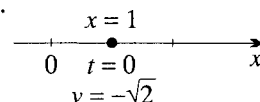
$$\tfrac{1}{2}v^2 = x^3 + \tfrac{1}{2}C, \text{ for some constant } C.$$

$$v^2 = 2x^3 + C, \text{ for some constant } C.$$

$$\text{When } x = 1, v = -\sqrt{2}, \text{ so } 2 = 2 + C,$$

$$\text{so } C = 0, \text{ and}$$

$$v^2 = 2x^3.$$

(b) Taking square roots, $v = -\sqrt{2}x^{\frac{3}{2}}$, assuming that v is never positive.

$$\text{Taking reciprocals, } \frac{dt}{dx} = -\tfrac{1}{2}\sqrt{2}x^{-\frac{3}{2}}$$

$$t = \sqrt{2}x^{-\frac{1}{2}} + D, \text{ for some constant } D.$$

$$\text{When } t = 0, x = 1, \text{ so } 0 = \sqrt{2} + D,$$

$$\text{so } D = -\sqrt{2}, \text{ and}$$

$$t = \sqrt{2}x^{-\frac{1}{2}} - \sqrt{2}$$

$$x^{-\frac{1}{2}} = \frac{t + \sqrt{2}}{\sqrt{2}}$$

$$x = \frac{2}{(t + \sqrt{2})^2}.$$

Hence the particle begins at $x = 1$, and moves backwards towards the origin.As $t \rightarrow \infty$, its speed has limit zero, and its limiting position is $x = 0$.

(c) Initially, v is negative. Since $v^2 = 2x^3$, it follows that v can only be zero at the origin; but since $\ddot{x} = 3x^2$ the acceleration at the origin would also be zero. Hence if the particle ever arrived at the origin it would then be permanently at rest. Thus the velocity can never change from negative to positive.

Exercise 3E

1. In each case, v is given as a function of x , and it is known that $x = 1$ when $t = 0$. Express: (i) t in terms of x , (ii) x in terms of t . [HINT: Start by taking reciprocals of both sides, which gives dt/dx as a function of x . Then integrate with respect to x .]

(a) $v = 6$

(c) $v = 2x - 1$

(e) $v = -6x^3$

(g) $v = 1 + x^2$

(b) $v = -6x^{-2}$

(d) $v = -6x^2$

(f) $v = e^{-2x}$

(h) $v = \cos^2 x$

2. In each motion of the previous question, find $\ddot{x}$ using the formula $\ddot{x} = \frac{d}{dx}(\tfrac{1}{2}v^2)$.

3. In each case, the acceleration $\ddot{x}$ is given as a function of x . By replacing $\ddot{x}$ by $\frac{d}{dx}(\tfrac{1}{2}v^2)$ and integrating, express v^2 in terms of x , given that $v = 0$ when $x = 0$.

(a) $\ddot{x} = 6x^2$

(c) $\ddot{x} = 6$

(e) $\ddot{x} = \sin 6x$

(b) $\ddot{x} = \frac{1}{e^x}$

(d) $\ddot{x} = \frac{1}{2x+1}$

(f) $\ddot{x} = \frac{1}{4+x^2}$

4. A stone is dropped from a lookout 500 metres above the valley floor. Take $g = 10 \text{ m/s}^2$, ignore air resistance, take downwards as positive, and use the lookout as the origin of displacement — the equation of motion is then $\ddot{x} = 10$.

(a) Replace $\ddot{x}$ by $\frac{d}{dx}(\tfrac{1}{2}v^2)$ and show that $v^2 = 20x$. Hence find the impact speed.

- (b) Explain why, during the fall, $v = \sqrt{20x}$ rather than $v = -\sqrt{20x}$.
- (c) Integrate to find the displacement–time function, and find how long it takes to fall.
5. [An alternative approach to the worked exercise in Section 3B] A ball is thrown vertically upwards at 20 m/s^2 . Take $g = 10 \text{ m/s}$, ignore air resistance, take upwards as positive, and use the ground as the origin of displacement — the equation of motion is then $\ddot{x} = -10$.
- (a) Show that $v^2 = 400 - 20x$, and find the maximum height.
- (b) Explain why $v = \sqrt{400 - 20x}$ while the ball is rising.
- (c) Integrate to find the displacement–time function, and find how long it takes the ball to reach maximum height.
6. [A formula from physics — not to be used in this course] A particle moves with constant acceleration a , so that its equation of motion is $\ddot{x} = a$. Its initial velocity is u . After t seconds, its velocity is v and its displacement is s .
- (a) Use $\frac{d}{dx}(\frac{1}{2}v^2)$ for acceleration to show that $v^2 = u^2 + 2as$.
- (b) Verify the impact speed in the previous question using this formula.
7. The acceleration of a particle P is given by $\ddot{x} = -2x$ (in units of centimetres and seconds), and the particle starts from rest at $x = 2$.
- (a) Find the speed of P when it first reaches $x = 1$, and explain whether it must then be moving backwards or forwards.
- (b) In what interval is the motion confined, and what is the maximum speed?
8. A particle moves according to $v = \frac{8}{3}x^{-2}$, where $t \geq 1$. When $t = 1$, the particle is at $x = 2$.
- (a) Find t as a function of x , and x as a function of t .
- (b) Hence find v and $\ddot{x}$ as functions of t .
- (c) Use $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ to find $\ddot{x}$ as a function of x .
9. (a) Prove that $\frac{d}{dx}(x \log x) = \log x + 1$.
- (b) A particle moves according to $\ddot{x} = 1 + \log x$. Initially it is stationary at $x = 1$. Find v^2 as a function of x .
- (c) Explain why v is always positive for $t > 0$, and find v when $x = e^2$.
10. A particle moves according to $\ddot{x} = \frac{1}{36 + x^2}$, and is initially at rest at O .
- (a) Find v^2 as a function of x , and explain why v is always positive for $t > 0$.
- (b) Find: (i) the velocity at $x = 6$, (ii) the velocity as $t \rightarrow \infty$.

DEVELOPMENT

11. A plane lands on a runway at 100 m/s . It then brakes with a constant deceleration until it stops 2 km down the runway.
- (a) Explain why the equation of motion is $\ddot{x} = -k$, for some positive constant k . By integrating with respect to x , find k , and find v^2 as a function of x .
- (b) Find: (i) the velocity after 1 km , (ii) where it is when the velocity is 50 m/s .
- (c) Explain why, during the braking, $v = \sqrt{10\,000 - 5x}$ rather than $v = -\sqrt{10\,000 - 5x}$.
- (d) Integrate to find the displacement–time function, and find how long it takes to stop.

12. (a) A particle has acceleration $\ddot{x} = e^{-x}$, and initially $v = 2$ and $x = 0$. Find v^2 as a function of x , and explain why v is always positive and at least 2. Then briefly explain what happens as time goes on.
- (b) Another particle has the same acceleration $\ddot{x} = e^{-x}$, and initially is also at $x = 0$. Find what the initial velocity V was if the particle first goes backwards, but turns around at $x = -1$. What happens to the velocity as time goes on?
13. The velocity of a particle starting at the origin is $v = \cos^2 2x$.
- (a) Explain why the particle can never be in the same place at two different times.
- (b) Find x and v as functions of t , and find the limiting position as $t \rightarrow \infty$.
- (c) Show that $\ddot{x} = -4\cos^3 2x \sin 2x$, and find t , v and $\ddot{x}$ when $x = \frac{\pi}{8}$.
14. Suppose that $v = 6 - 2x$, and that initially, the particle is at the origin.
- (a) Find the acceleration at the origin.
- (b) Show that $t = -\frac{1}{2} \log(1 - \frac{1}{3}x)$, and find x as a function of t .
- (c) Describe the behaviour of the particle as $t \rightarrow \infty$.
15. The velocity of a particle at displacement x is given by $v = x^2 e^{-x^2}$, and initially the particle is at $x = \frac{1}{2}$.
- (a) Explain why the particle can never be on the negative side of $x = \frac{1}{2}$. Then find the acceleration as a function of x , and hence find the maximum velocity and where it occurs.
- (b) Explain why the time T for the particle to travel to $x = 1$ is $T = \int_{\frac{1}{2}}^1 \frac{e^{x^2}}{x^2} dx$. Then use Simpson's rule with three function values to approximate T , giving your answer correct to four significant figures.
16. A particle moves with acceleration $-\frac{1}{2}e^{-x} \text{ m/s}^2$.
- (a) Initially it is at the origin with velocity 1 m/s . Find an expression for v^2 .
- (b) Explain why v is always positive for $t > 0$. Hence find the displacement as a function of time, and describe what happens to the particle as $t \rightarrow \infty$.
17. A particle's acceleration is $\ddot{x} = 2x - 1$, and initially the particle is at rest at $x = 5$.
- (a) Find v^2 , and explain why the particle can never be at the origin.
- (b) Find where $|v| = 2\sqrt{5}$, justifying your answer, and describe the subsequent motion.
18. Two particles A and B are moving towards the origin from the positive side with equations $v_A = -(16 + x^2)$ and $v_B = -4\sqrt{16 - x^2}$. If A is released from $x = 4$, where should B be released from, if they are to be released together and reach the origin together?
19. A particle moves with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.
- (a) Find v^2 as a function of displacement.
- (b) Explain why the velocity can never be positive. Then find the displacement-time function, and briefly describe the motion.
20. For a particle moving on the x -axis, $v^2 = 14x - x^2$.
- (a) By completing the square, find the section of the number line where the particle is confined, then find its maximum speed and where this occurs.
- (b) Where is $|\ddot{x}| \leq 3$?

21. A particle's acceleration is given by $\ddot{x} = x(3x - 14)$, and initially it is at the origin with velocity $6\sqrt{2}$.
- Show that $v^2 = 2(x + 2)(x - 3)(x - 6)$, and sketch the graph of v^2 .
 - Find the velocity and acceleration at $x = 3$. In which direction does the particle move off from $x = 3$?
 - Find the maximum speed, and where it occurs. Describe the motion of the particle.
22. An electron is fired with initial velocity 10^7 m/s into an alternating force field so that its acceleration x metres from its point of entry is $k\pi \sin \pi x$, for some positive constant k .
- Find v^2 in terms of x and k , explain why its velocity never drops below its initial velocity, and find where the electron will have this minimum velocity and where it will have maximum velocity.
 - If the electron's maximum velocity is 2×10^7 m/s, find k , and hence find the maximum acceleration and where it occurs.
23. The velocity of a particle is proportional to its displacement. When $t = 0$, $x = 2$, and when $t = 10$, $x = 4$. Find the displacement–time function, and find the displacement when $t = 25$.

EXTENSION

24. Newton's law of gravitation says that an object falling towards a planet has acceleration $\ddot{x} = -kx^{-2}$, for some positive constant k , where x is the distance from the centre of the planet. Show that if the body starts from rest at a distance D from the centre, then its speed at a distance x from the centre is $\sqrt{\frac{2k(D-x)}{Dx}}$.
25. A projectile is fired vertically upwards with speed V from the surface of the Earth.
- Assuming the same equation of motion as in the previous question, and ignoring air resistance, show that $k = gR^2$, where R is the radius of the Earth.
 - Find v^2 in terms of x and hence find the maximum height of the projectile.
 - [The escape velocity from the Earth] Given that $R = 6400$ km and $g = 9.8$ m/s², find the least value of V so that the projectile will never return.
26. Assume that a bullet, fired at 1 km/s, moves through water with deceleration proportional to the square of the velocity, so that $\ddot{x} = -kv^2$, for some positive constant k .
- If the velocity after 100 metres is 10 m/s, start with $\ddot{x} = v \frac{dv}{dx}$ and find where the bullet is when its velocity is 1 m/s.
 - If the velocity after 1 second is 10 m/s, use $\ddot{x} = \frac{dv}{dt}$ to find at what time the bullet has velocity 1 m/s.
27. Another type of bullet, when fired under water, moves with deceleration proportional to its velocity, so that $\ddot{x} = -kv$, for some positive constant k . Its initial speed is $\frac{1}{2}$ km/s, and its speed after it has gone 50 metres is 250 m/s.
- Use $\ddot{x} = v \frac{dv}{dx}$ to find v as a function of x , then find x as a function of t .
 - Show that it takes $\frac{1}{5} \log 2$ seconds to go the first 50 metres, and describe the subsequent motion of the bullet.

3F Simple Harmonic Motion — The Differential Equation

In Section 3D, we showed that a particle in simple harmonic motion with the centre of motion at the origin satisfies the differential equation

$$\ddot{x} = -n^2x.$$

In this section, we shall use this differential equation as the basis of a further study of simple harmonic motion. Since $\ddot{x}$ is now given as a function of displacement rather than time, we will need the techniques of the previous section, which used the identity $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ before performing the integration.

An Alternative Definition of Simple Harmonic Motion: For a particle moving in simple harmonic motion, the acceleration has a particularly simple form $\ddot{x} = -n^2x$ when it is expressed as a function of displacement — this is a linear function of x with acceleration proportional to x but oppositely directed. As with many motions, it is this acceleration–displacement function that can be measured accurately by measuring the force at various places on the number line.

For these reasons, it is convenient to introduce an alternative definition of simple harmonic motion as motion satisfying this differential equation.

SIMPLE HARMONIC MOTION — THE DIFFERENTIAL EQUATION: The motion of a particle is called *simple harmonic motion* if its displacement from some origin satisfies

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$$\ddot{x} = -n^2x, \text{ where } n \text{ is a positive constant.}$$

The acceleration is thus proportional to displacement, but oppositely directed. This equation is usually the most straightforward way to test whether a given motion is simple harmonic with centre the origin.

Having two definitions of the one thing may be convenient, but it does require a theorem proving that the two definitions are equivalent. First, we proved in Section 3D that motion satisfying $x = a \cos(nt + \alpha)$ or $x = a \sin(nt + \alpha)$ satisfied the differential equation $\ddot{x} = -n^2x$. Conversely, Extension questions in the following exercise prove that the differential equation has no other solutions. Although this converse is intuitively obvious, its proof is rather difficult and is not required in the course, so the following theorem can be assumed without proof whenever it is required in an exercise.

THE SOLUTIONS OF $\ddot{x} = -n^2x$: If a particle's motion satisfies $\ddot{x} = -n^2x$, then its displacement–time equation has the form

20
$$x = a \sin(nt + \alpha) \quad \text{or} \quad x = a \cos(nt + \alpha),$$

where a , n and α are constants, with $n > 0$ and $a > 0$. In particular, the period of the motion is $T = \frac{2\pi}{n}$.

Alternatively, the solution of the differential equation can be written as

$$x = b \sin nt + c \cos nt, \text{ where } b \text{ and } c \text{ are constants.}$$

The following worked exercise extracts the period from the differential equation.

WORKED EXERCISE: A particle is moving so that $\ddot{x} = -4x$, in units of centimetres and seconds. Initially, it is stationary at $x = 6$.

- (a) Write down the period and amplitude, and the displacement–time function.
 (b) Find the position, velocity and acceleration of the particle at $t = \frac{\pi}{3}$.

SOLUTION:

- (a) Since $\ddot{x} = -4x$, we know $n^2 = 4$, so $n = 2$ and the period is $\frac{2\pi}{2} = \pi$.

Since it starts stationary at $x = 6$, we know that $a = 6$.

Hence $x = 6 \cos 2t$ (cosine starts at the maximum).

- (b) Differentiating, $v = -12 \sin 2t$,

and $\ddot{x} = -24 \cos 2t$.

When $t = \frac{\pi}{3}$, $x = 6 \cos \frac{2\pi}{3} = -3$,

and $v = -12 \sin \frac{2\pi}{3} = -6\sqrt{3}$ cm/s,

and $\ddot{x} = -24 \cos \frac{2\pi}{3} = 12$ cm/s².

Notice that at $x = \frac{\pi}{3}$, $\ddot{x} = -4x$, as given by the differential equation.

Integrating the Differential Equation: It is quite straightforward to integrate the differential equation once, using the methods of the previous section. This integration gives v^2 as function of x .

WORKED EXERCISE: In the previous worked exercise, $\ddot{x} = -4x$, and the particle was initially stationary at $x = 6$.

- (a) Find v^2 as a function of x .
 (b) Verify this using the previous expressions for displacement and velocity.
 (c) Find the velocity and acceleration when the particle is at $x = 3$.

SOLUTION:

- (a) Replacing $\ddot{x}$ by $\frac{d}{dx}(\frac{1}{2}v^2)$, $\frac{d}{dx}(\frac{1}{2}v^2) = -4x$.

Then integrating, $\frac{1}{2}v^2 = -2x^2 + \frac{1}{2}C$, for some constant C
 $v^2 = -4x^2 + C$.

When $x = 6$, $v = 0$, so $0 = -144 + C$,

so $C = 144$, and $v^2 = 144 - 4x^2$

$$v^2 = 4(36 - x^2).$$

- (b) This can be confirmed by substituting $x = 6 \cos 2t$ and $v = -12 \sin 2t$:

$$\begin{aligned} \text{RHS} &= 4(36 - 36 \cos^2 2t) & \text{LHS} &= 12^2 \sin^2 2t \\ &= 4 \times 36 \sin^2 2t & &= \text{RHS} \end{aligned}$$

- (c) When $x = 3$, $\ddot{x} = -4 \times 3$ Also, $v^2 = 4(36 - 9)$
 $= -12$ cm/s². $= 4 \times 27$,

$$\text{so } v = 6\sqrt{3} \text{ or } -6\sqrt{3}.$$

This integration can easily be done in the general case, but the result should be derived by integration each time, and not quoted as a known result. The proof of the following result is left to the exercises.

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THE FIRST INTEGRATION OF $\ddot{x} = -n^2x$: If a particle is moving in simple harmonic motion with amplitude a , then

$$v^2 = n^2(a^2 - x^2).$$

This result should be derived by integration each time, and not quoted.

The second integration is blocked — taking the square root of v^2 requires cases, because v is positive half the time and negative the other half — and should not be attempted. Instead, quote the solutions of the differential equation, as explained.

The Five Functions of Simple Harmonic Motion: We now have x , v and $\ddot{x}$ as functions of t , and $\ddot{x}$ and v^2 as functions of x . This gives altogether five functions.

WORKED EXERCISE: A particle P moves so that its acceleration is proportional to its displacement x from a fixed point O and opposite in direction. Initially the particle is at the origin, moving with velocity 12 m/s, and the particle is stationary when $x = 4$.

- Find $\ddot{x}$ and v^2 as functions of x .
- Find x , v and $\ddot{x}$ as functions of t .
- Find the displacement, acceleration and times when the particle is at rest.
- Find the velocity, acceleration and times when the displacement is zero.
- Find the displacement, velocity and acceleration when $t = \frac{4\pi}{9}$.
- Find the acceleration and velocity and times when $x = 2$.

SOLUTION:

- We know that $\ddot{x} = -n^2x$,
where $n > 0$ is a constant of proportionality.

Hence
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x.$$

Integrating,
$$\begin{aligned}\frac{1}{2}v^2 &= -\frac{1}{2}n^2x^2 + \frac{1}{2}C \\ v^2 &= -n^2x^2 + C.\end{aligned}$$

When $x = 0$, $v = 12$, so $144 = 0 + C$, hence $C = 144$.

When $x = 4$, $v = 0$, so $0 = -n^2 \times 16 + 144$,

so $n^2 = 9$ and $n = 3$, since $n > 0$.

Hence $\ddot{x} = -9x$ (1)

and $v^2 = 9(16 - x^2)$. (2)

This can also be written as $\frac{x^2}{16} + \frac{v^2}{144} = 1$,

which is the unit circle stretched by a factor of 4 in the x -direction,
and by a factor of 12 in the v -direction.

- Also, since the amplitude a is 4 and $n = 3$,

$$x = 4 \sin 3t \quad (\text{sine starts at the origin, moving up}). \quad (3)$$

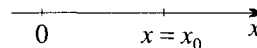
Differentiating, $v = 12 \cos 3t$, (4)

and $\ddot{x} = -36 \sin 3t$. (5)

- (c) Substituting $v = 0$,
 from (2), $x = 4$ or -4 ,
 from (1), $\ddot{x} = -36$ or 36 ,
 from (4), $t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \dots$
- (d) Substituting $x = 0$,
 from (2), $v = 12$ or -12 ,
 from (1), $\ddot{x} = 0$,
 from (3), $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \dots$
- (e) Substituting $t = \frac{4\pi}{9}$,
 from (3), $x = -2\sqrt{3}$,
 from (4), $v = -6$,
 from (5), $\ddot{x} = 18\sqrt{3}$.
- (f) Substituting $x = 2$,
 from (2), $v = 6\sqrt{3}$ or $-6\sqrt{3}$,
 from (1), $\ddot{x} = -18$,
 from (3), $t = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \dots$

Moving the Origin of Space: So far in this section, only simple harmonic motion with the centre at the origin has been considered. Now both speed and acceleration are independent of what origin is chosen, so the velocity and acceleration functions are unchanged if the centre of motion is shifted from the origin. This means that if the origin is shifted from $x = 0$ to $x = x_0$, then x will be replaced by $x - x_0$ in the equations of motion, but v and $\ddot{x}$ will be unchanged.

Hence the equation of simple harmonic motion with centre $x = x_0$ becomes $\ddot{x} = -n^2(x - x_0)$, and acceleration is now proportional to the displacement from $x = x_0$ but oppositely directed.



22 **SIMPLE HARMONIC MOTION WITH CENTRE AT $x = x_0$:** A particle is moving in simple harmonic motion about $x = x_0$ if its acceleration is proportional to its displacement $x - x_0$ from $x = x_0$ but oppositely directed, that is, if

$$\ddot{x} = -n^2(x - x_0), \text{ where } n \text{ is a positive constant.}$$

WORKED EXERCISE: [This question involves a situation where the centre of motion is at first unknown, and must be found by expressing $\ddot{x}$ in terms of x .] A particle's motion satisfies the equation $v^2 = -x^2 + 7x - 12$.

- (a) Show that the motion is simple harmonic, and find the centre, period and amplitude of the motion.
- (b) Find where the particle is when its speed is half the maximum speed.

SOLUTION:

(a) Differentiating, $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$

$$= -x + 3\frac{1}{2},$$

so

$$\ddot{x} = -(x - 3\frac{1}{2}),$$

which is in the form $\ddot{x} = -n^2(x - x_0)$, with $x_0 = 3\frac{1}{2}$ and $n = 1$.

Hence the motion is simple harmonic, with centre $x = 3\frac{1}{2}$ and period 2π .

Put $v = 0$ (to find where the particle stops at its extremes)

$$-x^2 + 7x - 12 = 0$$

$$x = 3 \text{ or } x = 4,$$

so the extremes of the motion are $x = 3$ and $x = 4$, and so the amplitude is $\frac{1}{2}$.

- (b) The maximum speed occurs at the centre $x = 3\frac{1}{2}$.

Substituting into $v^2 = -(x-3)(x-4)$, $v^2 = \frac{1}{4}$, and so speed $|v| = \frac{1}{2}$.

To find where the particle is when it has half that speed, put $v = \frac{1}{4}$:

$$-x^2 + 7x - 12 = \frac{1}{16}$$

$$16x^2 - 112x + 193 = 0$$

$$\text{so } \Delta = 8^2 \times 3, \text{ and } x = 3\frac{1}{2} + \frac{1}{4}\sqrt{3} \text{ or } 3\frac{1}{2} - \frac{1}{4}\sqrt{3}.$$

Exercise 3F

- A particle is moving according to $x = 3 \cos 2t$ (in units of metres and seconds).
 - Derive expressions for v and $\ddot{x}$ as functions of t , and for v^2 and $\ddot{x}$ in terms of x .
 - Find the speed and acceleration of the particle at $x = 2$.
- A particle is oscillating according to the equation $\ddot{x} = -9x$ (in units of metres and seconds), and is stationary when $x = 5$.
 - Integrate this equation to find an equation for v^2 .
 - Find the velocity and acceleration when $x = 3$.
 - What is the speed at the origin, and what is the period?
- A particle is oscillating according to the equation $\ddot{x} = -16x$ (in units of centimetres and seconds), and its speed at the origin is 24 cm/s.
 - Integrate this equation to find an equation for v^2 .
 - What are the amplitude and the period?
 - Find the speed and acceleration when $x = 2$.
- A particle is moving with amplitude 6 metres according to $\ddot{x} = -4x$ (the units are metres and seconds).
 - Find the velocity–displacement equation, the period and the maximum speed.
 - Find the simplest form of the displacement–time equation if initially the particle is:
 - stationary at $x = 6$,
 - stationary at $x = -6$,
 - at the origin with positive velocity,
 - at the origin with negative velocity.
- A ball on the end of a spring moves according to $\ddot{x} = -256x$ (in units of centimetres and minutes). The ball is pulled down 2 cm from the origin and released. Find the speed at the centre of motion.
 - Another ball on a spring moves according to $\ddot{x} + \frac{1}{4}x = 0$ (in units of centimetres and seconds), and its speed at the equilibrium position is 4 cm/s. How far was it pulled down from the origin before it was released?
- [In these questions, the differential equation will need to be formed first.]
 - A particle moving in simple harmonic motion has period $\frac{\pi}{2}$ minutes, and it starts from the mean position with velocity 4 m/min. Find the amplitude, then find the displacement and velocity as functions of time.
 - The motion of a buoy floating on top of the waves can be modelled as simple harmonic motion with period 3 seconds. If the waves rise and fall 2 metres about their mean position, find the buoy's greatest speed and acceleration.

DEVELOPMENT

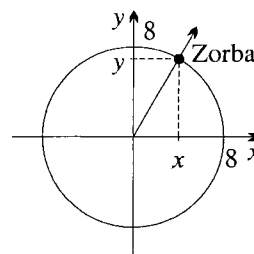
7. A particle oscillates between two points A and B 20 cm apart, moving in simple harmonic motion with period 8 seconds. Let O be the midpoint of AB .
 - (a) Find the maximum speed and acceleration, and the places where they occur.
 - (b) Find the speed and acceleration when the particle is 6 cm from O .
8. The amplitude of a particle moving in simple harmonic motion is 5 metres, and its acceleration when 2 metres from its mean position is 4 m/s^2 . Find the speed of the particle at the mean position and when it is 4 metres from the mean position.
9. (a) A particle is moving with simple harmonic motion of period π seconds and maximum velocity 8 m/s. If the particle started from rest at $x = a$, find a , then find the velocity when the particle is distant 3 metres from the mean position.
 - (b) A point moves with period π seconds so that its acceleration is proportional to its displacement x from O and oppositely directed. It passes through O with speed 5 m/s. Find its speed and acceleration 1.5 metres from O .
10. (a) A particle moving in simple harmonic motion on a horizontal line has amplitude 2 metres. If its speed passing through the centre O of motion is 15 m/s, find v^2 as a function of the displacement x to the right of O , and find the velocity and the acceleration of the particle when it is $\frac{2}{3}$ metres to the right of O .
 - (b) A particle moves so that its acceleration is proportional to its displacement x from the origin O . When 4 cm on the positive side of O , its velocity is 20 cm/s and its acceleration is $-6\frac{2}{3}\text{ cm/s}^2$. Find the amplitude of the motion.
11. [The general integral] Suppose that a particle is moving in simple harmonic motion with amplitude a and equation of motion $\ddot{x} = -n^2x$, where $n > 0$.
 - (a) Prove that $v^2 = n^2(a^2 - x^2)$.
 - (b) Find expressions for: (i) the speed at the origin, (ii) the speed and acceleration halfway between the origin and the maximum displacement.
12. A particle moving in simple harmonic motion starts at the origin with velocity V . Prove that the particle first comes to rest after travelling a distance V/n .
13. A particle moves in simple harmonic motion with centre O , and passes through O with speed $10\sqrt{3}\text{ cm/s}$. By integrating $\ddot{x} = -n^2x$, calculate the speed when the particle is halfway between its mean position and a point of instantaneous rest.
14. (a) A particle moving in a straight line obeys $v^2 = -9x^2 + 18x + 27$. Prove that the motion is simple harmonic, and find the centre of motion, the period and the amplitude.
 - (b) Repeat part (a) for:

(i) $v^2 = 80 + 64x - 16x^2$	(iii) $v^2 = -2x^2 - 8x - 6$
(ii) $v^2 = -9x^2 + 108x - 180$	(iv) $v^2 = 8 - 10x - 3x^2$
15. (a) Show that the motion $x = \sin^2 5t$ (in units of metres and minutes) is simple harmonic by showing that it satisfies $\ddot{x} = n^2(x_0 - x)$, for some x_0 and some $n > 0$:
 - (i) by first writing the displacement function as $x = \frac{1}{2} - \frac{1}{2} \cos 10t$,
 - (ii) by differentiating x directly without any use of double-angle identities.
 - (b) Find the centre, range and period of the motion, and the next time it visits the origin.

16. A particle moves in simple harmonic motion according to $\ddot{x} = -9(x - 7)$, in units of centimetres and seconds. Its amplitude is 7 cm.
- Find the centre of motion, and hence explain why the velocity at the origin is zero.
 - Integrate to find v^2 as a function of x , complete the square in this expression, and hence find the maximum speed.
 - Explain how, although the particle is stationary at the origin, it is nevertheless able to move away from the origin.
17. A particle is moving according to $x = 4 \cos 3t - 6 \sin 3t$.
- Prove that the acceleration is proportional to the displacement but oppositely directed, and hence that the motion is simple harmonic.
 - Find the period, amplitude and maximum speed of the particle, and find the acceleration when the particle is halfway between its mean position and one of its extreme positions.
18. The motion of a particle is given by $x = 3 + \sin 4t + \sqrt{3} \cos 4t$.
- Prove that $\ddot{x} = 16(3 - x)$, and write down the centre and period of the motion.
 - Express the motion in the form $x = x_0 + a \sin(4t + \alpha)$, where $a > 0$ and $0 \leq \alpha < 2\pi$.
 - At what times is the particle at the centre, and what is its speed there?
19. A particle moves according to the equation $x = 10 + 8 \sin 2t + 6 \cos 2t$.
- Prove that the motion is simple harmonic, and find the centre of motion, the period and the amplitude.
 - Find, correct to four significant figures, when the particle first reaches the origin.

EXTENSION

20. [Simple harmonic motion is the projection of circular motion onto a diameter.] A Ferris wheel of radius 8 metres mounted in the north-south plane is turning anticlockwise at 1 revolution per minute. At time zero, Zorba is level with the centre of the wheel and north of it.



- Let x and y be Zorba's horizontal distance north of the centre and height above the centre respectively. Show that $x = 8 \cos 2\pi t$ and $y = 8 \sin 2\pi t$.
 - Find expressions for $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$, and show that $\ddot{x} = -4\pi^2 x$ and $\ddot{y} = -4\pi^2 y$.
 - Find how far (in radians) the wheel has turned during the first revolution when:
 - $x : y = \sqrt{3} : 1$
 - $\dot{x} : \dot{y} = -\sqrt{3} : 1$
 - $\dot{x} = \dot{y}$
21. A particle moves in simple harmonic motion according to $\ddot{x} = -n^2 x$.
- Prove that $v^2 = n^2(a^2 - x^2)$, where a is the amplitude of the motion.
 - The particle has speeds v_1 and v_2 when the displacements are x_1 and x_2 respectively. Show that the period T is given by

$$T = 2\pi \sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}},$$

and find a similar expression for the amplitude.

- The particle has speeds of 8 cm/s and 6 cm/s when it is 3 cm and 4 cm respectively from O . Find the amplitude, the period and the maximum speed of the particle.

22. A particle moving in simple harmonic motion has amplitude a and maximum speed V . Find its velocity when $x = \frac{1}{2}a$, and its displacement when $v = \frac{1}{2}V$. Prove also the more general results

$$|v| = V\sqrt{1 - x^2/a^2} \quad \text{and} \quad |x| = a\sqrt{1 - v^2/V^2}.$$

23. Two balls on elastic strings are moving vertically in simple harmonic motion with the same period 2π and with centres level with each other. The second ball was set in motion α seconds later, where $0 \leq \alpha < 2\pi$, with twice the amplitude, so their equations are

$$x_1 = \sin t \quad \text{and} \quad x_2 = 2 \sin(t - \alpha).$$

Let $x = \sin t - 2 \sin(t - \alpha)$ be the height of the first ball above the second.

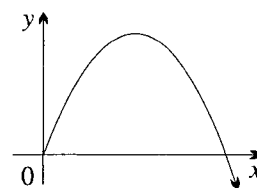
- Show that $\ddot{x} = -x$, and hence that x is also simple harmonic with period 2π .
 - Show that the greatest vertical difference A between the balls is $A = \sqrt{5 - 4 \cos \alpha}$. What are the maximum and minimum values of A , and what form does x then have?
 - Show that the balls are level when $\tan t = \frac{4T}{1 - 3T^2}$, where $T = \tan \frac{1}{2}\alpha$. How many times are they level in the time interval $0 \leq t < 2\pi$?
 - For what values of α is the vertical distance between the balls maximum at $t = 0$, and what form does x then have?
24. [This is a proof that there are no more solutions of the differential equation $\ddot{x} = -n^2x$.] Suppose that $\ddot{x} = -n^2x$, where $n > 0$, and let $a = x(0)$ and $bn = \dot{x}(0)$.
- Let $u = x - (a \cos nt + b \sin nt)$. Show that $u(0) = 0$ and $\dot{u}(0) = 0$.
 - Find $\ddot{u}$, and show that $\ddot{u} = -n^2u$.
 - Write $\ddot{u} = \frac{d}{du}(\frac{1}{2}\dot{u}^2)$, then integrate to show that $\dot{u}^2 = -n^2u^2$.
 - Hence show that $u = 0$ for all t , and hence that $x = a \cos nt + b \sin nt$.
25. [An alternative proof] Suppose that x and y are functions of t satisfying
- $$\ddot{x} = -n^2x, \quad \ddot{y} = -n^2y, \quad x(0) = y(0), \quad \text{and} \quad \dot{x}(0) = \dot{y}(0),$$
- where n is a positive constant, and $x(0)$ and $\dot{x}(0)$ are not both zero.
- Show that $\frac{d}{dt}(\dot{x}y - x\dot{y}) = 0$, and hence that $\dot{x}y = x\dot{y}$, for all t .
 - Show that $\frac{d}{dt}\left(\frac{x}{y}\right) = 0$, and hence that $y = x$, for all t .
 - Hence show that $x = a \cos nt + b \sin nt$, where $a = x(0)$ and $b = \frac{\dot{x}(0)}{n}$.

3G Projectile Motion — The Time Equations

In these final sections, we shall consider one case of motion in two dimensions — the motion of a projectile, like a thrown ball or a shell fired from a gun.

A *projectile* is something that is thrown or fired into the air, and subsequently moves under the influence of gravity alone. Notice that missiles and aeroplanes are not projectiles, because they have motors on them that keep pushing them forwards. We shall ignore any effects of air resistance, so we will not be dealing with things like leaves or pieces of paper where air resistance has a large effect. Everyone can see that a projectile moves in a parabolic path. Our task is to set up the equations that describe this motion.

The Coordinates of Displacement and Time: The diagram on the right shows the sort of path we would expect a projectile to move in. The two-dimensional space in which it moves has been made into a number plane by choosing an origin — in this case the point from which the projectile was fired — and measuring horizontal distance x and vertical distance y from this origin. We could put time t on the graph, but this would require a third dimension for the t -axis. But we can treat t as a parameter, because every point on the path corresponds to a unique time after projection.



These pronumerals x , y and t for horizontal distance, vertical distance and time respectively will be used without further introduction in this section.

Velocity and the Resolution of Velocity: When an object is moving through the air, we can describe its velocity by giving its speed and the angle at which it is moving. For example, a ball may at some instant be moving at 12 m/s with an *angle of inclination* of 60° or -60° . This angle of inclination is always measured from the horizontal, and is taken as negative if the object is travelling downwards.

SPECIFYING MOTION BY SPEED AND ANGLE OF INCLINATION: The velocity of a projectile can be specified by giving its speed and angle of inclination.

- 23** The *angle of inclination* is the acute angle between the path and the horizontal. It is positive if the object is travelling upwards, and negative if the object is travelling downwards.

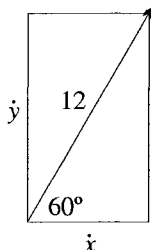
But we can also specify the velocity at that instant by giving the rates $\dot{x}$ and $\dot{y}$ at which the horizontal displacement x and the vertical displacement y are changing. The conversion from one system of measurement to the other requires a *velocity resolution diagram* like those in the worked exercises below.

WORKED EXERCISE: Find the horizontal and vertical components of the velocity of a projectile moving with speed 12 m/s and angle of inclination:

- (a) 60°

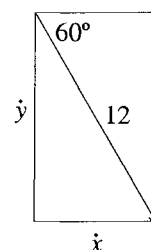
SOLUTION:

$$\begin{aligned} \text{(a)} \quad \dot{x} &= 12 \cos 60^\circ \\ &= 6 \text{ m/s} \\ \dot{y} &= 12 \sin 60^\circ \\ &= 6\sqrt{3} \text{ m/s} \end{aligned}$$



- (b) -60°

$$\begin{aligned} \text{(b)} \quad \dot{x} &= 12 \cos 60^\circ \\ &= 6 \text{ m/s} \\ \dot{y} &= -12 \sin 60^\circ \\ &= -6\sqrt{3} \text{ m/s} \end{aligned}$$



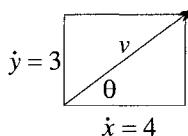
WORKED EXERCISE: Find the speed v and angle of inclination θ (correct to the nearest degree) of a projectile for which:

- (a) $\dot{x} = 4 \text{ m/s}$ and $\dot{y} = 3 \text{ m/s}$,

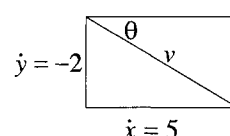
- (b) $\dot{x} = 5 \text{ m/s}$ and $\dot{y} = -2 \text{ m/s}$.

SOLUTION:

$$\begin{aligned} \text{(a)} \quad v^2 &= 4^2 + 3^2 \\ v &= 5 \text{ m/s} \\ \tan \theta &= \frac{3}{4} \\ \theta &\doteq 37^\circ \end{aligned}$$



$$\begin{aligned} \text{(b)} \quad v^2 &= 5^2 + 2^2 \\ v &= \sqrt{29} \text{ m/s} \\ \tan \theta &= -\frac{2}{5} \\ \theta &\doteq -22^\circ \end{aligned}$$



RESOLUTION OF VELOCITY: To convert between velocity given in terms of speed v and angle of inclination θ , and velocity given in terms of horizontal and vertical components $\dot{x}$ and $\dot{y}$, use a *velocity resolution diagram*.

- 24 Alternatively, use the conversion equations

$$\begin{cases} \dot{x} = v \cos \theta \\ \dot{y} = v \sin \theta \end{cases} \quad \text{and} \quad \begin{cases} v^2 = \dot{x}^2 + \dot{y}^2 \\ \tan \theta = \dot{y}/\dot{x} \end{cases}$$

The Independence of the Vertical and Horizontal Motion: We have already seen that gravity affects every object free to move by accelerating it downwards with the same constant acceleration g , where g is about 9.8 m/s^2 , or 10 m/s^2 in round figures. Because this acceleration is downwards, it affects the vertical component $\dot{y}$ of the velocity according to $\ddot{y} = -g$. It has no effect, however, on the horizontal component $\dot{x}$, and thus $\ddot{x} = 0$. Every projectile motion is governed by this same pair of equations.

THE FUNDAMENTAL EQUATIONS OF PROJECTILE MOTION: Every projectile motion is governed by the pair of equations

25
$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g.$$

Unless otherwise indicated, every question on projectile motion should begin with these equations. This will involve four integrations and four substitutions of the boundary conditions.

WORKED EXERCISE: A ball is thrown with initial velocity 40 m/s and angle of inclination 30° from the top of a stand 25 metres above the ground.

- Using the stand as the origin and $g = 10 \text{ m/s}^2$, find the six equations of motion.
- Find how high the ball rises, how long it takes to get there, what its speed is then, and how far it is horizontally from the stand.
- Find the flight time, the horizontal range, and the impact speed and angle.

SOLUTION: Initially, $x = y = 0$, and $\dot{x} = 40 \cos 30^\circ$ $\dot{y} = 40 \sin 30^\circ$
 $= 20\sqrt{3},$ $= 20.$

(a) To begin, $\ddot{x} = 0.$ (1) Integrating, $\dot{x} = C_1.$ When $t = 0$, $\dot{x} = 20\sqrt{3}$ $20\sqrt{3} = C_1,$ so $\dot{x} = 20\sqrt{3}.$ (2) Integrating, $x = 20t\sqrt{3} + C_2.$ When $t = 0$, $x = 0$ $0 = C_2,$ so $x = 20t\sqrt{3}.$ (3)	To begin, $\ddot{y} = -10.$ (4) Integrating, $\dot{y} = -10t + C_3.$ When $t = 0$, $\dot{y} = 20$ $20 = C_3,$ so $\dot{y} = -10t + 20.$ (5) Integrating, $y = -5t^2 + 20t + C_4.$ When $t = 0$, $y = 0$ $0 = C_4,$ so $y = -5t^2 + 20t.$ (6)

- At the top of its flight, the vertical component of the ball's velocity is zero, so put $\dot{y} = 0.$
 From (5), $-10t + 20 = 0$

$t = 2$ seconds (the time taken).

When $t = 2$, from (6), $y = -20 + 40$
 $= 20$ metres (the maximum height).

When $t = 2$, from (3), $x = 40\sqrt{3}$ metres (the horizontal distance).

Because the vertical component of velocity is zero, the speed there is $\dot{x} = 20\sqrt{3}$ m/s.

(c) It hits the ground when it is 25 metres below the stand,

so put $y = -25$.

From (6), $-5t^2 + 20t = -25$

$$t^2 - 4t - 5 = 0$$

$$(t - 5)(t + 1) = 0$$

so it hits the ground when $t = 5$ ($t = -1$ is inadmissible).

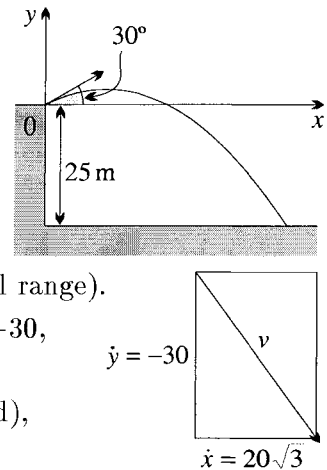
When $t = 5$, from (3), $x = 100\sqrt{3}$ metres (the horizontal range).

Also, $\dot{x} = 20\sqrt{3}$ and $\dot{y} = -50 + 20 = -30$,

so $v^2 = 1200 + 900$

$v = 10\sqrt{21}$ m/s (the impact speed),

and $\tan \theta = -\frac{30}{20\sqrt{3}}$
 $\theta \doteq -40^\circ 54'$, and the impact angle is about $40^\circ 54'$.



Using Pronumerals for Initial Velocity and Angle of Inclination: Many problems in projectile motion require the initial velocity or angle of inclination to be found so that the projectile behaves in some particular fashion. Often the muzzle speed of a gun will be fixed, but the angle at which it is fired can be easily altered — in such situations there are usually two solutions, corresponding to a low-flying shot and a ‘lobbed’ shot that goes high in the air.

WORKED EXERCISE: A gun at O fires shells with an initial speed of 200 m/s but a variable angle of inclination α . Take $g = 10 \text{ m/s}^2$.

- Find the two possible angles at which the gun can be set so that it will hit a fortress F 2 km away on top of a mountain 1000 metres high.
- Show that the two angles are equally inclined to OF and to the vertical.
- Find the corresponding flight times and the impact speeds and angles.

SOLUTION: Place the origin at the gun, so that initially, $x = y = 0$.

Resolving the initial velocity, $\dot{x} = 200 \cos \alpha$, $\dot{y} = 200 \sin \alpha$.

To begin, $\ddot{x} = 0$. (1)

Integrating, $\dot{x} = C_1$.

When $t = 0$, $\dot{x} = 200 \cos \alpha$

$$200 \cos \alpha = C_1,$$

so $\dot{x} = 200 \cos \alpha$. (2)

Integrating, $x = 200t \cos \alpha + C_2$.

When $t = 0$, $x = 0$

$$0 = C_2,$$

so $x = 200t \cos \alpha$. (3)

To begin, $\ddot{y} = -10$. (4)

Integrating, $\dot{y} = -10t + C_3$.

When $t = 0$, $\dot{y} = 200 \sin \alpha$

$$200 \sin \alpha = C_3,$$

so $\dot{y} = -10t + 200 \sin \alpha$. (5)

Integrating, $y = -5t^2 + 200t \sin \alpha + C_4$.

When $t = 0$, $y = 0$

$$0 = C_4,$$

so $y = -5t^2 + 200t \sin \alpha$. (6)

(a) Since the fortress is 2 km away, $x = 2000$
 so from (3), $200t \cos \alpha = 2000$

$$t = \frac{10}{\cos \alpha}.$$

Since the mountain is 1000 metres high, $y = 1000$
 so from (6), $-5t^2 + 200t \sin \alpha = 1000.$

Hence $-\frac{500}{\cos^2 \alpha} + \frac{2000 \sin \alpha}{\cos \alpha} - 1000 = 0$

$$\sec^2 \alpha - 4 \tan \alpha + 2 = 0.$$

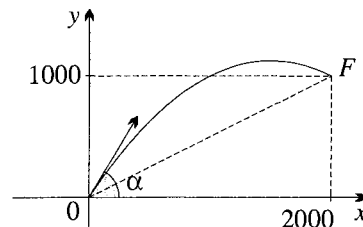
But $\sec^2 \alpha = \tan^2 \alpha + 1$,

so $\tan^2 \alpha - 4 \tan \alpha + 3 = 0$

$(\tan \alpha - 3)(\tan \alpha - 1) = 0$

$\tan \alpha = 1$ or 3

$\alpha = 45^\circ$ or $\tan^{-1} 3 [\doteq 71^\circ 34'].$



- (b) $\angle OFX = \tan^{-1} \frac{1}{2} \doteq 26^\circ 34'$, so the 45° shot is inclined at $18^\circ 26'$ to OF ,
 and the $71^\circ 34'$ shot is inclined at $18^\circ 26'$ to the vertical.
 (This calculation can also be done using exact values.)

- (c) When $\alpha = 45^\circ$, from (a), $t = \frac{10}{\cos \alpha} = 10\sqrt{2}$ seconds,
 and when $t = 10\sqrt{2}$, from (5), $\dot{y} = -100\sqrt{2} + 200 \times \frac{1}{2}\sqrt{2} = 0$,
 so from (2), the shell hits horizontally at $100\sqrt{2}$ m/s.

When $\alpha = \tan^{-1} 3$, $\cos \alpha = \frac{1}{\sqrt{10}}$ and $\sin \alpha = \frac{3}{\sqrt{10}},$

so from (a), $t = \frac{10}{\cos \alpha} = 10\sqrt{10}$ seconds,

and when $t = 10\sqrt{10}$, from (5), $\dot{y} = -100\sqrt{10} + 60\sqrt{10} = -40\sqrt{10},$

and from (2), $\dot{x} = 20\sqrt{10},$

so $v^2 = 16\,000 + 4000 = 20\,000$

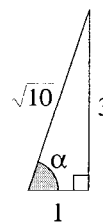
$v = 100\sqrt{2}$ m/s,

and

$\tan \theta = \dot{y}/\dot{x} = -2,$

$\theta = -\tan^{-1} 2 [\doteq -63^\circ 26'],$

so the shell hits at $100\sqrt{2}$ m/s at about $63^\circ 26'$ to the horizontal.



Exercise 3G

1. Use a velocity resolution diagram to find $\dot{x}$ and $\dot{y}$, given that the projectile's speed v and angle of inclination θ are:

(a) $v = 12$
 $\theta = 30^\circ$

(b) $v = 8$
 $\theta = -45^\circ$

(c) $v = 20$
 $\theta = \tan^{-1} \frac{4}{3}$

2. Use a velocity resolution diagram to find the speed v and angle of inclination θ of a projectile, given that $\dot{x}$ and $\dot{y}$ are:

(a) $\dot{x} = 6$
 $\dot{y} = 6$

(b) $\dot{x} = 7$
 $\dot{y} = -7\sqrt{3}$

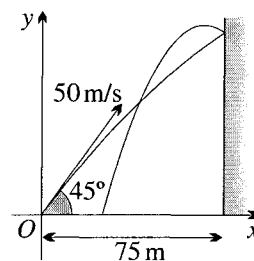
(c) $\dot{x} = 5$
 $\dot{y} = 7$

3. A stone is projected from a point on level ground with velocity $10\sqrt{2}$ m/s at an angle of elevation of 45° . Let x and y be the respective horizontal and vertical components of the displacement of the stone from the point of projection, and take $g = 10$ m/s².
- Use a velocity resolution diagram to determine the initial values of $\dot{x}$ and $\dot{y}$.
 - Beginning with $\ddot{x} = 0$ and $\ddot{y} = -10$, integrate each equation twice, substituting boundary conditions each time, to find the equations of $\dot{x}$, x , $\dot{y}$ and y in terms of t .
 - By substituting $\dot{y} = 0$, find the greatest height and the time taken to reach it.
 - By substituting $y = 0$, find the horizontal distance travelled, and the flight time.
 - Find $\dot{x}$, x , $\dot{y}$ and y when $t = 0.5$.
 - Hence find how far the stone is from the point of projection when $t = 0.5$.
 - Use a velocity resolution diagram to find its speed (to the nearest m/s) and its direction of motion (as an angle of elevation to the nearest degree) when $t = 0.5$.
4. Steve tosses an apple to Adam who is sitting near him. Adam catches the apple at exactly the same height that Steve released it. Suppose that the initial speed of the apple is $V = 5$ m/s, and the initial angle α of elevation is given by $\tan \alpha = 2$.
- Use a velocity resolution diagram to find the initial values of $\dot{x}$ and $\dot{y}$.
 - Find $\dot{x}$, x , $\dot{y}$ and y by integrating $\ddot{x} = 0$ and $\ddot{y} = -10$, taking the origin at Steve's hands.
 - Show by substitution into y that the apple is in the air for less than 1 second.
 - Find the greatest height above the point of release reached by the apple.
 - Show that the flight time is $\frac{2}{5}\sqrt{5}$ seconds, and hence find the horizontal distance travelled by the apple.
 - Find $\dot{x}$ and $\dot{y}$ at the time Adam catches the apple. Then use a velocity resolution diagram to show that the final speed equals the initial speed, and the final angle of inclination is the opposite of the initial angle of elevation.
 - The path of the apple is a parabolic arc. By eliminating t from the equations for x and y , find its equation in Cartesian form.
5. A projectile is fired with velocity $V = 40$ m/s on a horizontal plane at an angle of elevation $\alpha = 60^\circ$. Take $g = 10$ m/s², and let the origin be the point of projection.
- Show that $\dot{x} = 20$ and $\dot{y} = -10t + 20\sqrt{3}$, and find x and y .
 - Find the flight time, and the horizontal range of the projectile.
 - Find the maximum height reached, and the time taken to reach it.
 - An observer claims that the projectile would have had a greater horizontal range if its angle of projection had been halved. Investigate this claim by reworking the question with $\alpha = 30^\circ$.
6. A pebble is thrown from the top of a vertical cliff with velocity 20 m/s at an angle of elevation of 30° . The cliff is 75 metres high and overlooks a river.
- Derive expressions for the horizontal and vertical components of the displacement of the pebble from the top of the cliff after t seconds. (Take $g = 10$ m/s².)
 - Find the time it takes for the pebble to hit the water and the distance from the base of the cliff to the point of impact.
 - Find the greatest height that the pebble reaches above the river.
 - Find the values of $\dot{x}$ and $\dot{y}$ at the instant when the pebble hits the water. Hence use a velocity resolution diagram to find the speed (to the nearest m/s) and the acute angle (to the nearest degree) at which the pebble hits the water.

- (e) The path of the pebble is a parabolic arc. By eliminating t from the equations for x and y , find its equation in Cartesian form.
7. A plane is flying horizontally at 363.6 km/h and its altitude is 600 metres. It is to drop a food parcel onto a large cross marked on the ground in a remote area.
- Convert the speed of the plane into metres per second.
 - Derive expressions for the horizontal and vertical components of the food parcel's displacement from the point where it was dropped. (Take $g = 10 \text{ m/s}^2$.)
 - Show that the food parcel will be in the air for $2\sqrt{30}$ seconds.
 - Find the speed and angle at which the food parcel will hit the ground.
 - At what horizontal distance from the cross, correct to the nearest metre, should the plane drop the food parcel?
8. Jeffrey the golfer hit a ball which was lying on level ground. Two seconds into its flight, the ball just cleared a 28-metre-tall tree which was exactly $24\sqrt{5}$ metres from where the ball was hit. Let $V \text{ m/s}$ be the initial velocity of the ball, and let θ be the angle to the horizontal at which the ball was hit. Take $g = 10 \text{ m/s}^2$.
- Show that the horizontal and vertical components of the displacement of the ball from its initial position are $x = Vt \cos \theta$ and $y = -5t^2 + Vt \sin \theta$.
 - Show that $V \cos \theta = 12\sqrt{5}$ and $V \sin \theta = 24$.
 - By squaring and adding, find V . Then find θ , correct to the nearest minute.
 - Find, correct to the nearest metre, how far Jeffrey hit the ball.

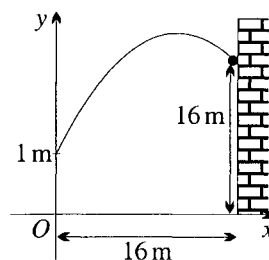
DEVELOPMENT

9. [The general case] A gun at $O(0,0)$ fires a shell across level ground with muzzle speed V and angle α of elevation.
- Derive, from $\ddot{x} = 0$ and $\ddot{y} = -g$, the other four equations of motion.
 - Find the maximum height H , and the time taken to reach it.
 - If V is constant and α varies, find the greatest value of H and the corresponding value of α . What value of α gives half this maximum value?
 - Find the range R and flight time T .
 - If V is constant and α varies, find the greatest value of R and the corresponding value of α . What value of α gives half this maximum value?
10. Gee Ming the golfer hits a ball from level ground with an initial speed of 50 m/s and an initial angle of elevation of 45° . The ball rebounds off an advertising hoarding 75 metres away. Take $g = 10 \text{ m/s}^2$.
- Show that the ball hits the hoarding after $\frac{3}{2}\sqrt{2}$ seconds at a point 52.5 metres high.
 - Show that the speed v of the ball when it strikes the hoarding is $5\sqrt{58} \text{ m/s}$ at an angle of elevation α to the horizontal, where $\alpha = \tan^{-1} \frac{2}{5}$.
 - Assuming that the ball rebounds off the hoarding at an angle of elevation α with a speed of 20% of v , find how far from Gee Ming the ball lands.



11. Antonina threw a ball with velocity 20 m/s from a point exactly one metre above the level ground she was standing on. The ball travelled towards a wall of a tall building 16 metres away. The plane in which the ball travelled was perpendicular to the wall. The ball struck the wall 16 metres above the ground. Take $g = 10 \text{ m/s}^2$.

- (a) Let the origin be the point on the ground directly below the point from which the ball was released. Show that, t seconds after the ball was thrown, $x = 20t \cos \theta$ and $y = -5t^2 + 20t \sin \theta + 1$, where θ is the angle to the horizontal at which the ball was originally thrown.

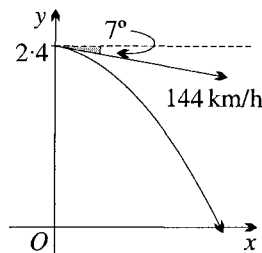


- (b) The ball hit the wall after T seconds. Show that $4 = 5T \cos \theta$ and $3 = 4T \sin \theta - T^2$.

- (c) Hence show that $16 \tan^2 \theta - 80 \tan \theta + 91 = 0$.

- (d) Hence find the two possible values of θ , correct to the nearest minute.

12. Glenn the fast bowler runs in to bowl and releases the ball 2.4 metres above the ground with speed 144 km/h at an angle of 7° below the horizontal. Take the origin to be the point where the ball is released, and take $g = 10 \text{ m/s}^2$.



- (a) Show that the coordinates of the ball t seconds after its release are given by

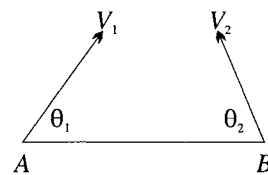
$$x = 40t \cos 7^\circ, \quad y = 2.4 - 40t \sin 7^\circ - 5t^2.$$

- (b) How long will it be (to the nearest 0.01 seconds) before the ball hits the pitch?

- (c) Calculate the angle (to the nearest degree) at which the ball will hit the pitch.

- (d) The batsman is standing 19 metres from the point of release. If the ball lands more than 5 metres in front of him, it will be classified as a 'short-pitched' delivery. Is this particular delivery short-pitched?

13. Two particles P_1 and P_2 are projected simultaneously from the points A and B , where AB is horizontal. The motion takes place in the vertical plane through A and B . The initial velocity of P_1 is V_1 at an angle θ_1 to the horizontal, and the initial velocity of P_2 is V_2 at an angle θ_2 to the horizontal. You may assume that the equations of motion of a particle projected with velocity V at an angle θ to the horizontal are $x = Vt \cos \theta$ and $y = -\frac{1}{2}gt^2 + Vt \sin \theta$.



- (a) Show that the condition for the particles to collide is $V_1 \sin \theta_1 = V_2 \sin \theta_2$.

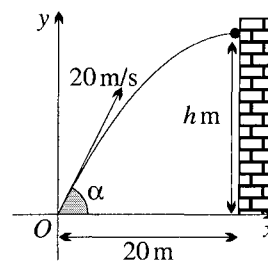
- (b) Suppose that $AB = 200$ metres, $V_1 = 30 \text{ m/s}$, $\theta_1 = \sin^{-1} \frac{4}{5}$, $\theta_2 = \sin^{-1} \frac{3}{5}$, $g = 10 \text{ m/s}^2$ and that the particles collide.

- (i) Show that $V_2 = 40 \text{ m/s}$, and that the particles collide after 4 seconds.

- (ii) Find the height of the point of collision above AB .

- (iii) Find, correct to the nearest degree, the obtuse angle between the directions of motion of the particles at the instant they collide.

14. A cricketer hits the ball from ground level with a speed of 20 m/s and an angle of elevation α . It flies towards a high wall 20 metres away on level ground. Take the origin at the point where the ball was hit, and take $g = 10 \text{ m/s}^2$.



- (a) Show that the ball hits the wall when $h = 20 \tan \alpha - 5 \sec^2 \alpha$.

- (b) Show that $\frac{d}{d\alpha}(\sec \alpha) = \sec \alpha \tan \alpha$.

- (c) Show that the maximum value of h occurs when $\tan \alpha = 2$.

- (d) Find the maximum height.

- (e) Find the speed and angle (to the nearest minute) at which the ball hits the wall.

15. A stone is propelled upwards at an angle θ to the horizontal from the top of a vertical cliff 40 metres above a lake. The speed of propulsion is 20 m/s. Take $g = 10 \text{ m/s}^2$.

(a) Show that $x(t)$ and $y(t)$, the horizontal and vertical components of the stone's displacement from the top of the cliff, are given by

$$x(t) = 20t \cos \theta, \quad y(t) = -5t^2 + 20t \sin \theta.$$

(b) If the stone hits the lake at time T seconds, show that

$$(x(T))^2 = 400T^2 - (5T^2 - 40)^2.$$

(c) Hence find, by differentiation, the value of T that maximises $(x(T))^2$, and then find the value of θ that maximises the distance between the foot of the cliff and the point where the stone hits the lake.

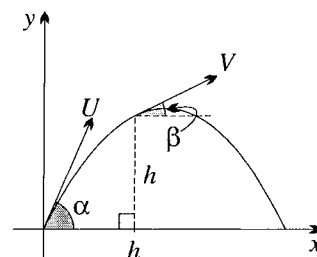
16. A particle P_1 is projected from the origin with velocity V at an angle of elevation θ .

(a) Assuming the usual equations of motion, show that the particle reaches a maximum height of $\frac{V^2 \sin^2 \theta}{2g}$.

(b) A second particle P_2 is projected from the origin with velocity $\frac{3}{2}V$ at an angle $\frac{1}{2}\theta$ to the horizontal. The two particles reach the same maximum height.

(i) Show that $\theta = \cos^{-1} \frac{1}{8}$. (ii) Do the two particles take the same time to reach this maximum height? Justify your answer.

17. A projectile was fired from the origin with velocity U at an angle of α to the horizontal. At time T_1 on its ascent, it passed with velocity V through a point whose horizontal and vertical distances from the origin are equal, and its direction of motion at that time was at an angle of β to the horizontal. At time T_2 the projectile returned to the horizontal plane from which it was fired.



(a) (i) Show that $T_1 = \frac{2U}{g} (\sin \alpha - \cos \alpha)$.

(ii) Hence show that $T_2 = \frac{T_1}{1 - \cot \alpha}$. (iii) Deduce that $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

(b) (i) Explain why $V \cos \beta = U \cos \alpha$.

(ii) If $\beta = \frac{1}{2}\alpha$, show that $\beta = \cos^{-1} \left(\frac{V + \sqrt{V^2 + 8U^2}}{4U} \right)$.

18. (a) Consider the function $y = 2 \sin(x - \theta) \cos x$.

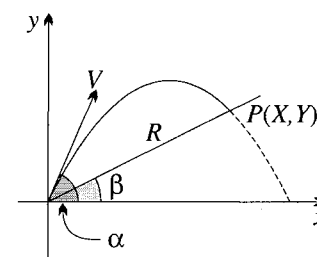
(i) Show that $\frac{dy}{dx} = 2 \cos(2x - \theta)$. (ii) Hence, or otherwise, show that

$$2 \sin(x - \theta) \cos x = \sin(2x - \theta) - \sin \theta.$$

(b) A projectile is fired from the origin with velocity V at an angle of α to the horizontal up a plane inclined at β to the horizontal. Assume that the horizontal and vertical components of the projectile's displacement are given by $x = Vt \cos \alpha$ and $y = Vt \sin \alpha - \frac{1}{2}gt^2$.

(i) If the projectile strikes the plane at (X, Y) , show that

$$X = \frac{2V^2 \cos^2 \alpha (\tan \alpha - \tan \beta)}{g}.$$



(ii) Hence show that the range R of the projectile up the plane is given by

$$R = \frac{2V^2 \cos \alpha \sin(\alpha - \beta)}{g \cos^2 \beta}.$$

(iii) Use part (a)(ii) to show that the maximum possible value of R is $\frac{V^2}{g(1 + \sin \beta)}$.

(iv) If the angle of inclination of the plane is 14° , at what angle to the horizontal should the projectile be fired in order to attain the maximum possible range?

EXTENSION

19. A tall building stands on level ground. The nozzle of a water sprinkler is positioned at a point P on the ground at a distance d from a wall of the building. Water sprays from the nozzle with speed V and the nozzle can be pointed in any direction from P .

(a) If $V > \sqrt{gd}$, prove that the water can reach the wall above ground level.

(b) Suppose that $V = 2\sqrt{gd}$. Show that the portion of the wall that can be sprayed with water is a parabolic segment of height $\frac{15}{8}d$ and area $\frac{5}{2}d^2\sqrt{15}$.

3H Projectile Motion — The Equation of Path

The formulae for x and y in terms of t give a parametric equation of the physical path of the projectile through the x - y plane. Eliminating t will give the Cartesian equation of the path, which is simply an upside-down parabola. Many questions are solved more elegantly by consideration of the equation of path. Unless the question gives it, however, the equation of path must be derived each time.

The General Case: The following working derives the equation of path in the general case of a projectile fired from the origin with initial speed V and angle of elevation α .

Resolving the initial velocity, $\dot{x} = V \cos \alpha$ and $\dot{y} = V \sin \alpha$.

To begin, $\ddot{x} = 0$. (1)

Integrating, $\dot{x} = C_1$.

When $t = 0$, $\dot{x} = V \cos \alpha$

$$V \cos \alpha = C_1,$$

so $\dot{x} = V \cos \alpha$. (2)

Integrating, $x = Vt \cos \alpha + C_2$.

When $t = 0$, $x = 0$

$$0 = C_2,$$

so $x = Vt \cos \alpha$. (3)

From (3), $t = \frac{x}{V \cos \alpha}$.

Substituting into (6), $y = -\frac{gx^2}{2V^2 \cos^2 \alpha} + \frac{Vx \sin \alpha}{V \cos \alpha}$,

which becomes $y = -\frac{gx^2}{2V^2} (1 + \tan^2 \alpha) + x \tan \alpha$,

using the Pythagorean identity $\frac{1}{\cos^2 \alpha} = \sec^2 \alpha = 1 + \tan^2 \alpha$.

To begin, $\ddot{y} = -g$. (4)

Integrating, $\dot{y} = -gt + C_3$.

When $t = 0$, $\dot{y} = V \sin \alpha$

$$V \sin \alpha = C_3,$$

so $\dot{y} = -gt + V \sin \alpha$. (5)

Integrating, $y = -\frac{1}{2}gt^2 + Vt \sin \alpha + C_4$.

When $t = 0$, $y = 0$

$$0 = C_4,$$

so $y = -\frac{1}{2}gt^2 + Vt \sin \alpha$. (6)

This working must always be shown unless the equation is given in the question.

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THE EQUATION OF PATH (not to be memorised): The path of a projectile fired from the origin with initial speed V and angle of elevation α is

$$y = -\frac{gx^2}{2V^2} \sec^2 \alpha + x \tan \alpha. \quad [\text{NOTE: } \sec^2 \alpha = 1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha}.]$$

This equation is quadratic in x , $\tan \alpha$ and V , and linear in g and y .

Differentiation of the equation of path gives the gradient of the path for any value of x , and thus is an alternative approach to finding the angle of inclination of a projectile in flight.

WORKED EXERCISE: Use the equation of path above in these questions.

- (a) Show that the range on level ground is $\frac{V^2}{g} \sin 2\alpha$, and hence find the maximum range for a given initial speed V and variable angle α of elevation.
- (b) Arrange the equation of path as a quadratic in $\tan \alpha$, and hence show that with a given initial speed V and variable angle α of elevation, a projectile can be fired through the point $P(x, y)$ if and only if

$$2V^2gy \leq V^4 - g^2x^2.$$

SOLUTION:

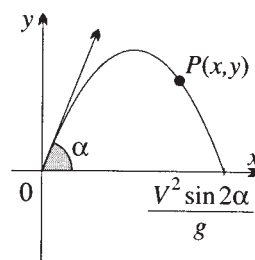
- (a) Put $y = 0$, then $\frac{gx^2 \sec^2 \alpha}{2V^2} = x \tan \alpha$,

$$\begin{aligned} \text{so } x = 0 \text{ or } \frac{gx}{2V^2 \cos^2 \alpha} &= \frac{\sin \alpha}{\cos \alpha} \\ x &= \frac{2V^2}{g} \cos \alpha \sin \alpha \\ x &= \frac{V^2}{g} \sin 2\alpha. \end{aligned}$$

Hence the projectile lands $\frac{V^2}{g} \sin 2\alpha$ away from the origin.

Since $\sin 2\alpha$ has a maximum value of 1 when $\alpha = 45^\circ$,

the maximum range is $\frac{V^2}{g}$ when $\alpha = 45^\circ$.



- (b) Multiplying both sides of the equation of path by $2V^2$,

$$\begin{aligned} 2V^2y &= -gx^2(1 + \tan^2 \alpha) + 2V^2x \tan \alpha \\ 2V^2y + gx^2 + gx^2 \tan^2 \alpha - 2V^2x \tan \alpha &= 0 \\ gx^2 \tan^2 \alpha - 2V^2x \tan \alpha + (2V^2y + gx^2) &= 0. \end{aligned}$$

The equation has now been written as a quadratic in $\tan \alpha$.

It will have a solution for $\tan \alpha$ provided that

$$\begin{aligned} \Delta &\geq 0, \\ \text{that is, } (2V^2x)^2 - 4 \times gx^2 \times (2V^2y + gx^2) &\geq 0 \\ 4V^4x^2 - 8x^2V^2gy - 4g^2x^4 &\geq 0 \\ \boxed{\div 4x^2} \quad 2V^2gy &\leq V^4 - g^2x^2. \end{aligned}$$

Exercise 3H

1. A cricket ball is thrown from the origin on level ground, and the equation of the path of its motion is $y = x - \frac{1}{40}x^2$, where x and y are in metres.
 - (a) Find the horizontal range of the ball.
 - (b) Find the greatest height.
 - (c) Find the gradient of the tangent at $x = 0$, and hence find the angle of projection α .
 - (d) Find, by substitution, whether the ball goes above or below the point $A(10, 8)$.
 - (e) The general equation of path is

$$y = -\frac{gx^2}{2V^2 \cos^2 \alpha} + x \tan \alpha,$$

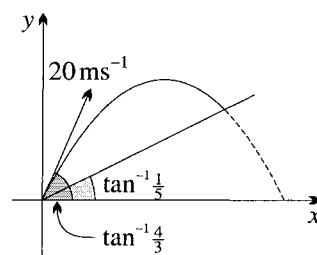
where V is the initial velocity. Taking $g = 10 \text{ m/s}^2$, find V .

2. A stone is fired on a level floor with initial speed $V = 10 \text{ m/s}$ and angle of elevation 45° .
 - (a) Find $\dot{x}$, x , $\dot{y}$ and y by integration from $\dot{x} = 0$ and $\dot{y} = -10$. Then, by eliminating t , show that the equation of path is $y = -\frac{1}{10}x^2 + x = \frac{1}{10}x(10 - x)$.
 - (b) Use the theory of quadratics to find the range and the maximum height.
 - (c) Suppose first that the stone hits a wall 8 metres away.
 - (i) Find how far up the wall the stone hits.
 - (ii) Differentiate the equation of path, and hence find the angle of inclination when the stone hits the wall.
 - (d) Suppose now that the stone hits a ceiling 2.1 metres high.
 - (i) Find the horizontal distance before impact.
 - (ii) Find the angle at which the stone hits the ceiling.
3. A particle is projected from the origin at time $t = 0$ seconds and follows a parabolic path with parametric equations $x = 12t$ and $y = 9t - 5t^2$ (where x and y are in metres).
 - (a) Show that the Cartesian equation of the path is $y = \frac{3}{4}x - \frac{5}{144}x^2$.
 - (b) Find the horizontal range R and the greatest height H .
 - (c) Find the gradient at $x = 0$, and hence find the initial angle of projection.
 - (d) Find $\dot{x}$ and $\dot{y}$ when $t = 0$. Hence use a velocity resolution diagram to find the initial velocity, and to confirm the initial angle of projection.
 - (e) Find when the particle is 4 metres high, and the horizontal displacement then.
4. A bullet is fired horizontally at 200 m/s from a window 45 metres above the level ground below. It doesn't hit anything and falls harmlessly to the ground.
 - (a) Write down the initial values of $\dot{x}$ and $\dot{y}$.
 - (b) Taking $g = 10 \text{ m/s}^2$ and the origin at the window, find $\dot{x}$, x , $\dot{y}$ and y . Hence find the Cartesian equation of path.
 - (c) Find the horizontal distance that the bullet travels. [HINT: Put $y = -45$.]
 - (d) Find, correct to the nearest minute, the angle at which the bullet hits the ground.

DEVELOPMENT

5. A ball is thrown on level ground at an initial speed of $V \text{ m/s}$ and at an angle of projection α . Assume that, t seconds after release, the horizontal and vertical displacements are given by $x = Vt \cos \alpha$ and $y = Vt \sin \alpha - \frac{1}{2}gt^2$.
 - (a) Show that the trajectory has Cartesian equation $y = \frac{x}{\cos^2 \alpha} \left(\sin \alpha \cos \alpha - \frac{gx}{2V^2} \right)$.

- (b) Hence show that the horizontal range is $\frac{V^2 \sin 2\alpha}{g}$.
- (c) When $V = 30$ m/s, the ball lands 45 metres away. Take $g = 10$ m/s².
- Find the two possible values of α .
 - A 2-metre-high fence is placed 40 metres from the thrower. Examine each trajectory to see whether the ball will still travel 45 metres.
6. A gun can fire a shell with a constant initial speed V and a variable angle of elevation α . Assume that t seconds after being fired, the horizontal and vertical displacements x and y of the shell from the gun are given by the same equations as in the previous question.
- (a) Show that the Cartesian equation of the shell's path may be written as
- $$gx^2 \tan^2 \alpha - 2xV^2 \tan \alpha + (2yV^2 + gx^2) = 0.$$
- (b) Suppose that $V = 200$ m/s, $g = 10$ m/s² and the shell hits a target positioned 3 km horizontally and 0.5 km vertically from the gun. Show that $\tan \alpha = \frac{4 \pm \sqrt{3}}{3}$, and hence find the two possible values of α , correct to the nearest minute.
7. A ball is thrown with initial velocity 20 m/s at an angle of elevation of $\tan^{-1} \frac{4}{3}$.
- (a) Show that the parabolic path of the ball has parametric equations $x = 12t$ and $y = 16t - 5t^2$.
- (b) Hence find the horizontal range of the ball, and its greatest height.
- (c) Suppose that, as shown opposite, the ball is thrown up a road inclined at $\tan^{-1} \frac{1}{5}$ to the horizontal. Show that:
- the ball is about 9 metres above the road when it reaches its greatest height,
 - the time of flight is 2.72 seconds, and find, correct to the nearest tenth of a metre, the distance the ball has been thrown up the road.
8. Talia is holding the garden hose at ground level and pointing it obliquely so that it sprays water in a parabolic path 2 metres high and 8 metres long. Find, using $g = 10$ m/s², the initial speed and angle of elevation, and the time each droplet of water is in the air. Where is the latus rectum of the parabola?
9. A boy throws a ball with speed V m/s at an angle of 45° to the horizontal.
- (a) Derive expressions for the horizontal and vertical components of the displacement of the ball from the point of projection.
- (b) Hence show that the Cartesian equation of the path of the ball is $y = x - \frac{gx^2}{V^2}$.
- (c) The boy is now standing on a hill inclined at an angle θ to the horizontal. He throws the ball at the same angle of elevation of 45° and at the same speed of V m/s. If he can throw the ball 60 metres down the hill but only 30 metres up the hill, use the result in part (b) to show that



$$\tan \theta = 1 - \frac{30g \cos \theta}{V^2} = \frac{60g \cos \theta}{V^2} - 1,$$

and hence that $\theta = \tan^{-1} \frac{1}{3}$.

10. A particle is projected from the origin with velocity V m/s at an angle of α to the horizontal.

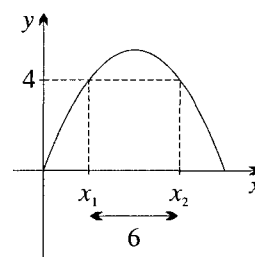
(a) Assuming that the coordinates of the particle at time t are $(Vt \cos \alpha, Vt \sin \alpha - \frac{1}{2}gt^2)$, prove that the horizontal range R of the particle is $\frac{V^2 \sin 2\alpha}{g}$.

(b) Hence prove that the path of the particle has equation $y = x \left(1 - \frac{x}{R}\right) \tan \alpha$.

(c) Suppose that $\alpha = 45^\circ$ and that the particle passes through two points 6 metres apart and 4 metres above the point of projection, as shown in the diagram. Let x_1 and x_2 be the x -coordinates of the two points.

(i) Show that x_1 and x_2 are the roots of the equation $x^2 - Rx + 4R = 0$.

(ii) Use the identity $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_2x_1$ to find R .



11. A projectile is fired from the origin with velocity V and angle of elevation α , where α is acute. Assume the usual equations of motion.

(a) Let $k = \frac{V^2}{2g}$. Show that the Cartesian equation of the parabolic path of the projectile can be written as

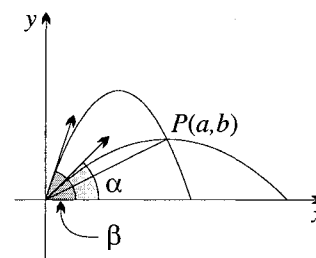
$$x^2 \tan^2 \alpha - 4kx \tan \alpha + (4ky + x^2) = 0.$$

- (b) Show that the projectile can pass through the point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2 only if $X^2 < 4k^2 - 4kY$.
- (c) Suppose that $\tan \alpha_1$ and $\tan \alpha_2$ are the two real roots of the quadratic equation in $\tan \alpha$ in part (a). Show that $\tan \alpha_1 \tan \alpha_2 > 1$, and hence explain why it is impossible for α_1 and α_2 both to be less than 45° .

EXTENSION

12. A gun at $O(0,0)$ has a fixed muzzle speed and a variable angle of elevation.

- (a) If the gun can hit a target at $P(a,b)$ with two different angles of elevation α and β , show that the angle between OP and α equals the angle between β and the vertical.
- (b) If the gun is firing up a plane of angle of elevation ψ , show that the maximum range is obtained when the gun is fired at the angle that bisects the angle between the plane and vertical.



13. [Some theorems about projectile motion] Ferdinand is feeding his pet bird, Rinaldo, who is sitting on the branch of a tree, by firing pieces of meat to him with a meat-firing device.

- (a) Ferdinand aims the device at the bird and fires. At the same instant, Rinaldo drops off his branch and falls under gravity. Prove that Rinaldo will catch the meat.
- (b) Rinaldo returns to his perch, and Ferdinand fires a piece of meat so that it will hit the bird. At the same instant, Rinaldo flies off horizontally away from Ferdinand at a constant speed. The meat rises twice the height of the perch, and Rinaldo catches it in flight as it descends. What is the ratio of the horizontal component of the meat's speed to Rinaldo's speed?

- (c) Again Rinaldo returns to his perch, and Ferdinand fires some more meat, intending that the bird will catch it as it descends, having risen twice the height of the perch. At the same instant, Rinaldo flies off horizontally towards Ferdinand at a constant speed, and catches the meat in flight as it ascends. What is the ratio this time?
- (d) What is the ratio of Rinaldo's constant speeds in parts (b) and (c)?
14. [The focus and directrix of the path] A gun at $O(0,0)$ has a fixed muzzle speed V and a variable angle α of elevation. Find the vertex, focus and directrix of the parabola by completing the square in the equation of path $y = x \tan \alpha - \frac{x^2}{4k \cos^2 \alpha}$, where $k = v^2/2g$.
- (a) Prove that the directrix is independent of α , and that its height is the maximum height when the projectile is fired vertically upwards.
- (b) Find the locus of the focus S and the vertex.
- (c) Prove that the initial angle of projection bisects the angle between OS and the vertical.
- (d) [A relationship with the definition of the parabola] Let d be a fixed tangent to a fixed circle C , and let $\mathcal{P}$ be any parabola whose focus lies on C and whose directrix is d . Use the definition of the parabola to show that $\mathcal{P}$ passes through the centre.
15. A projectile is fired up an inclined plane with a fixed muzzle velocity and variable angle of projection. Show that the following four statements are logically equivalent (meaning that if any one of them is true, then the other three are also true).
- The range up the plane is maximum.
 - The focus of the parabolic path lies on the plane.
 - The angle of projection is at right angles to the angle of flight at impact.
 - The angle of projection bisects the angle between the plane and the vertical.
16. [For those taking physics] Let v be the speed at time t of a projectile fired with initial velocity V and initial angle of elevation α .
- (a) Prove that at any time t during the flight, the quantity $gy + \frac{1}{2}v^2$ is independent of time and independent of α .
- (b) Explain the interpretation given to this quantity in physics.

Polynomial Functions

The primitive of a linear function is a quadratic, the primitive of a quadratic function is a cubic, and so on, so ultimately the study even of linear functions must involve the study of polynomial functions of arbitrary degree. In this course, linear and quadratic functions are studied in great detail, but this chapter begins the systematic study of polynomials of higher degree. The intention is first to study the interrelationships between their factorisation, their graphs, their zeroes and their coefficients, and secondly, to reinterpret all these ideas geometrically by examining curves defined by algebraic equations.

STUDY NOTES: After the terminology of polynomials has been introduced in Section 4A, graphs are drawn in Section 4B — as always, machine drawing of some of these examples may illuminate the wide variety of possible curves generated by polynomials. Sections 4C–4E concern the division of polynomials, the remainder and factor theorems, and their consequences. The work in these sections may have been introduced at a more elementary level in earlier years. Section 4F, however, which deals with the relationship between the zeroes and the coefficients, is probably quite new except in the context of quadratics. The problem of factoring a given polynomial is common to Sections 4B–4F, and a variety of alternative approaches are developed through these sections. The final Section 4G applies the methods of the chapter to geometrical problems about polynomial curves, circles and rectangular hyperbolas.

4A The Language of Polynomials

Polynomials are expressions like the quadratic $x^2 - 5x + 6$ or the quartic $3x^4 - \frac{2}{3}x^3 + 4x + 7$. They have occurred routinely throughout the course so far, but in order to speak about polynomials in general, our language and notation needs to be a little more systematic.

1

POLYNOMIALS: A *polynomial function* is a function that can be written as a sum:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients $a_0, a_1, \dots, a_n$ are constants, and n is a cardinal number.

The term a_0 is called the *constant term*. This is the value of the polynomial at $x = 0$, and so is the y -intercept of the graph. The constant term can also be written as $a_0 x^0$, so that every term is then a multiple $a_k x^k$ of a power of x in which the index k is a cardinal number. This allows sigma notation to be used, and we can write

$$P(x) = \sum_{k=0}^n a_k x^k.$$

Such notation is very elegant, but it can also be confusing, and questions involving sigma notation are usually best converted into the longer notation before proceeding. In the next chapter, however, we will need such notation.

NOTE: Careful readers may notice that $a_0 x^0$ is undefined at $x = 0$. This means that rewriting the quadratic $x^2 + 3x + 2$ as $x^2 + 3x^1 + 2x^0$ causes a problem at $x = 0$. To overcome this, the convention is made that the term $a_0 x^0$ is interpreted as a_0 before any substitution is performed.

Leading Term and Degree: The term of highest index with nonzero coefficient is called the *leading term*. Its coefficient is called the *leading coefficient* and its index is called the *degree*. For example, the polynomial

$$P(x) = -5x^6 - 3x^4 = 2x^3 + x^2 - x + 9$$

has leading term $-5x^6$, leading coefficient -5 and degree 6, which is written as ' $\deg F(x) = 6$ '.

A *monic polynomial* is a polynomial whose leading coefficient is 1; for example, $P(x) = x^3 - 2x^2 - 3x + 4$ is monic. Notice that every polynomial is a multiple of a monic polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n \left(x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{a_0}{a_n} \right).$$

Some Names of Polynomials: Polynomials of low degree have standard names.

- The *zero polynomial* $Z(x) = 0$ is a special case. It has a constant term 0. But it has no term with a nonzero coefficient, and therefore has no leading term, no leading coefficient, and most importantly, no degree. It is also quite exceptional in that its graph is the x -axis, so that every real number is a zero of the zero polynomial.
- A *constant polynomial* is a polynomial whose only term is the constant term, for example,

$$P(x) = 4, \quad Q(x) = -\frac{3}{5}, \quad R(x) = \pi, \quad Z(x) = 0.$$

Apart from the zero polynomial, all constant polynomials have degree 0, and are equal to their leading term and to their leading coefficient.

- A *linear polynomial* is a polynomial whose graph is a straight line:

$$P(x) = 4x - 3, \quad Q(x) = -\frac{1}{2}x, \quad R(x) = 2, \quad Z(x) = 0.$$

Linear polynomials have degree 1 when the coefficient of x is nonzero, and are constant polynomials when the coefficient of x is zero.

- A polynomial of degree 2 is called a *quadratic polynomial*:

$$P(x) = 3x^2 + 4x - 1, \quad Q(x) = -\frac{1}{2}x - x^2, \quad R(x) = 9 - x^2.$$

Notice that the coefficient of x^2 must be nonzero for the degree to be 2.

- Polynomials of higher degree are called *cubics* (degree 3), *quartics* (degree 4), *quintics* (degree 5), and so on.

Addition and Subtraction: Any two polynomials can be added or subtracted, and the results are again polynomials:

$$(5x^3 - 4x + 3) + (3x^2 - 3x - 2) = 5x^3 + 3x^2 - 7x + 1$$

$$(5x^3 - 4x + 3) - (3x^2 - 3x - 2) = 5x^3 - 3x^2 - x + 5$$

The zero polynomial $Z(x) = 0$ is the neutral element for addition, in the sense that $P(x) + 0 = P(x)$, for all polynomials $P(x)$. The *opposite polynomial* $-P(x)$ of any polynomial $P(x)$ is obtained by taking the opposite of every coefficient. Then the sum of $P(x)$ and $-P(x)$ is the zero polynomial; for example,

$$(4x^4 - 2x^2 + 3x - 7) + (-4x^4 + 2x^2 - 3x + 7) = 0.$$

The degree of the sum or difference of two polynomials is normally the maximum of the degrees of the two polynomials, as in the first example above, where the two polynomials had degrees 2 and 3 and their sum had degree 3. If, however, the two polynomials have the same degree, then the leading terms may cancel out and disappear, for example,

$$(x^2 - 3x + 2) + (9 + 4x - x^2) = x + 11, \text{ which has degree 1,}$$

or the two polynomials may be opposites so that their sum is zero.

2

DEGREE OF THE SUM AND DIFFERENCE: Suppose that $P(x)$ and $Q(x)$ are nonzero polynomials of degree n and m respectively.

- If $n \neq m$, then $\deg(P(x) + Q(x)) = \text{maximum of } m \text{ and } n$.
- If $n = m$, then $\deg(P(x) + Q(x)) \leq n$ or $P(x) + Q(x) = 0$.

Multiplication: Any two polynomials can be multiplied, giving another polynomial:

$$\begin{aligned} (3x^3 + 2x + 1) \times (x^2 - 1) &= (3x^5 + 2x^3 + x^2) - (3x^3 + 2x + 1) \\ &= 3x^5 - x^2 + x^2 - 2x - 1 \end{aligned}$$

The constant polynomial $I(x) = 1$ is the neutral element for multiplication, in the sense that $P(x) \times 1 = P(x)$, for all polynomials $P(x)$. Multiplication by the zero polynomial on the other hand always gives the zero polynomial.

If two polynomials are nonzero, then the degree of their product is the sum of their degrees, because the leading term of the product is always the product of the two leading terms.

3

DEGREE OF THE PRODUCT: If $P(x)$ and $Q(x)$ are nonzero polynomials, then

$$\deg(P(x) \times Q(x)) = \deg P(x) + \deg Q(x).$$

Factorisation of Polynomials: The most important problem of this chapter is the factorisation of a given polynomial. For example,

$$x(x+2)^2(x-2)^2(x^2+x+1) = x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x$$

is a reasonably routine expansion of a factored polynomial, but it is not clear how to move from the expanded form back to the factored form.

Identically Equal Polynomials: We need to be quite clear what is meant by saying that two polynomials are the same.

IDENTICALLY EQUAL POLYNOMIALS: Two polynomials $P(x)$ and $Q(x)$ are called *identically equal*, written as $P(x) \equiv Q(x)$, if they are equal for all values of x :

$$4 \quad P(x) \equiv Q(x) \quad \text{means} \quad P(x) = Q(x), \text{ for all } x.$$

For two polynomials to be equal, the corresponding coefficients in the two polynomials must all be equal.

WORKED EXERCISE: Find a, b, c, d and e if $ax^4 + bx^3 + cx^2 + dx + e \equiv (x^2 - 3)^2$.

SOLUTION: Expanding, $(x^2 - 3)^2 = x^4 - 6x^2 + 9$.

Now comparing coefficients, $a = 1, b = 0, c = -6, d = 0$ and $e = 9$.

Polynomial Equations: If $P(x)$ is a polynomial, then the equation formed by setting $P(x) = 0$ is a *polynomial equation*. For example, using the polynomial in the previous paragraph, we can form the equation

$$x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x = 0.$$

Solving polynomial equations and factoring polynomial functions are very closely related. For example, using the factoring of the previous paragraph,

$$x(x+2)^2(x-2)^2(x^2+x+1) = 0,$$

so the solutions are $x = 0, x = 2$ and $x = -2$. Notice that the quadratic factor $x^2 + x + 1$ has no zeroes, because its discriminant is $\Delta = -3$.

The solutions of a polynomial equation are called *roots*, whereas the zeroes of a polynomial function are the values of x where the value of the polynomial is zero. The distinction between the words is not always strictly observed.

Exercise 4A

1. State whether or not the following are polynomials.

(a) $3x^2 - 7x$	(e) $\sqrt{3}x^2 + \sqrt{5}x$	(i) $\log_e x$
(b) $\frac{1}{x^2} + x$	(f) $2^x - 1$	(j) $\frac{4}{3}x^3 - ex^2 + \pi x$
(c) $\sqrt{x} - 2$	(g) $(x+1)^3$	(k) 5
(d) $3x^{\frac{2}{3}} - 5x + 11$	(h) $\frac{7x^{13} + 3x}{4}$	(l) $\frac{x-2}{x+1}$

2. For each polynomial, state: (i) the degree, (ii) the leading coefficient, (iii) the leading term, (iv) the constant term, (v) whether or not the polynomial is monic.

Expand the polynomial first where necessary.

(a) $4x^3 + 7x^2 - 11$	(d) x^{12}	(g) 0
(b) $10 - 4x - 6x^3$	(e) $x^2(x-2)$	(h) $x(x^3 - 5x + 1) - x^2(x^2 - 2)$
(c) 2	(f) $(x^2 - 3x)(1 - x^3)$	(i) $6x^7 - 4x^6 - (2x^5 + 1)(5 + 3x^2)$

3. If $P(x) = 5x + 2$ and $Q(x) = x^2 - 3x + 1$, find:

(a) $P(x) + Q(x)$	(c) $P(x) - Q(x)$	(e) $P(x)Q(x)$
(b) $Q(x) + P(x)$	(d) $Q(x) - P(x)$	(f) $Q(x)P(x)$

4. If $P(x) = 5x + 2$, $Q(x) = x^2 - 3x + 1$ and $R(x) = 2x^2 - 3$, show, by expanding separately the LHS and RHS, that:
- $P(x)(Q(x) + R(x)) = P(x)Q(x) + P(x)R(x)$
 - $(P(x)Q(x))R(x) = P(x)(Q(x)R(x))$
 - $(P(x) + Q(x)) + R(x) = P(x) + (Q(x) + R(x))$
5. Express each of the following polynomials as a multiple of a monic polynomial:
- $2x^2 - 3x + 4$
 - $3x^3 - 6x^2 - 5x + 1$
 - $-2x^5 + 7x^4 - 4x + 11$
 - $\frac{2}{3}x^3 - 4x + 16$

DEVELOPMENT

6. Factor the following polynomials completely, and state all the zeroes.
- $x^3 - 8x^2 - 20x$
 - $2x^4 - x^3 - x^2$
 - $x^4 - 5x^2 - 36$
 - $x^3 - 8$
 - $x^4 - 81$
 - $x^6 - 1$
7. (a) The polynomials $P(x)$ and $Q(x)$ have degrees p and q respectively, and $p \neq q$. What is the degree of: (i) $P(x)Q(x)$, (ii) $P(x) + Q(x)$?
- (b) What differences would it make if $P(x)$ and $Q(x)$ both had the same degree p ?
- (c) Give an example of two polynomials, both of degree 2, which have a sum of degree 0.
8. Write down the monic polynomial whose degree, leading coefficient, and constant term are all equal.
9. Find the values of a , b and c if:
- $ax^2 + bx + c \equiv 3x^2 - 4x + 1$
 - $(a - b)x^2 + (2a + b)x \equiv 7x - x^2$
 - $a(x - 1)^2 + b(x - 1) + c \equiv x^2$
 - $a(x + 2)^2 + b(x + 3)^2 + c(x + 4)^2 \equiv 2x^2 + 8x + 6$
10. For the polynomial $(a - 4)x^7 + (2 - 3b)x^3 + (5c - 1)$, find the values of a , b and c if it is:
- of degree 3,
 - of degree 0,
 - of degree 7 and monic,
 - the zero polynomial.
11. Suppose that $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ and $P(3x) \equiv P(x)$.
- Show that $81ax^4 + 27bx^3 + 9cx^2 + 3dx + e \equiv ax^4 + bx^3 + cx^2 + dx + e$.
 - Hence show that $P(x)$ is a constant polynomial.
12. (a) Show that if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.
- (b) Show that if $Q(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.
- (c) Give a general statement of the situation in parts (a) and (b).
13. $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are polynomials. Indicate whether the following statements are true or false, giving reasons for your answers.
- If $P(x)$ is even, then $P'(x)$ is odd.
 - If $Q'(x)$ is even, then $Q(x)$ is odd.
 - If $R(x)$ is odd, then $R'(x)$ is even.
 - If $S'(x)$ is odd, then $S(x)$ is even.

EXTENSION

14. Real numbers a and b are said to be *multiplicative inverses* if $ab = ba = 1$.
- What can be said about two polynomials if they are multiplicative inverses.
 - Explain why a polynomial of degree ≥ 1 cannot have a multiplicative inverse.
15. We have assumed in the notes above that if two polynomials $P(x)$ and $Q(x)$ are equal for all values of x (that is, if their graphs are the same), then their degrees are equal and their corresponding coefficients are equal. Here is a proof using calculus.
- Explain why substituting $x = 0$ proves that the constant terms are equal.
 - Explain why differentiating k times and substituting $x = 0$ proves that the coefficients of x^k are equal.

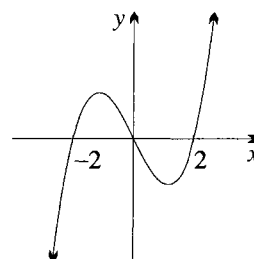
4B Graphs of Polynomial Functions

A lot of work has already been done on sketching polynomial functions. We know already that the graph of any polynomial function will be a continuous and differentiable curve, whose domain is all real numbers, and which possibly intersects the x -axis at one or more points. This section will concentrate on two main concerns. First, how does the graph behave for large positive and negative values of x ? Secondly, given the full factorisation of the polynomial, how does the graph behave near its various x -intercepts? We will not be concerned here with further questions about turning points and inflexions which are not zeroes.

The Graphs of Polynomial Functions: It should be intuitively obvious that for large positive and negative values of x , the behaviour of the curve is governed entirely by the sign of its leading term. For example, the cubic graph sketched on the right below is

$$P(x) = x^3 - 4x = x(x-2)(x+2).$$

For large positive values of x , the degree 1 term $-4x$ is negative, but is completely swamped by the positive values of the degree 3 term x^3 . Hence $P(x) \rightarrow \infty$ as $x \rightarrow \infty$. On the other hand, for large negative values of x , the term $-4x$ is positive, but is negligible compared with the far bigger negative values of the term x^3 . Hence $P(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.



In the same way, every polynomial of odd degree has a graph that disappears off diagonally opposite corners. Being continuous, it must therefore be zero somewhere. Our example actually has three zeroes, but however much it were raised or lowered or twisted, only two zeroes could ever be removed. Here is the general situation.

- BEHAVIOUR OF POLYNOMIALS FOR LARGE x :** Suppose that $P(x)$ is a polynomial of degree at least 1 with leading term $a_n x^n$.

5

 - As $x \rightarrow \infty$, $P(x) \rightarrow \infty$ if a_n is positive, and $P(x) \rightarrow -\infty$ if a_n is negative.
 - As $x \rightarrow -\infty$, $P(x)$ behaves the same as when $x \rightarrow \infty$ if the degree is even, but $P(x)$ behaves in the opposite way if the degree is odd.
 - It follows that every polynomial of odd degree has at least one zero.

PROOF: Clearly the leading term dominates proceedings as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, but here is a more formal proof, should it be required.

A. Let
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

then
$$\frac{P(x)}{x^n} = a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}.$$

As $x \rightarrow \infty$ or $x \rightarrow -\infty$,
$$\frac{P(x)}{x^n} \rightarrow a_n.$$

Hence for large positive x , $P(x)$ has the same sign as a_n . For large negative x , $P(x)$ has the same sign as a_n when n is even, and the opposite sign to a_n when n is odd.

- B. If $P(x)$ is a polynomial of odd degree, then $P(x) \rightarrow \infty$ on either the left or right side, and $P(x) \rightarrow -\infty$ on the other side. Hence, being a continuous function, $P(x)$ must cross the x -axis somewhere.

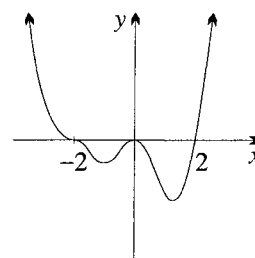
Zeroes and Sign: If the polynomial can be completely factored, then its zeroes can be read off very quickly, and our earlier methods would then have called for a table of test values to decide its sign. Here, for example, is the table of test values and the sketch of

$$P(x) = (x + 2)^3 x^2 (x - 2).$$

x	-3	-2	-1	0	1	2	3
y	45	0	-9	0	-27	0	1125

The function changes sign around $x = -2$ and $x = 2$, where the associated factors $(x + 2)^3$ and $(x - 2)$ have odd degrees, but not around $x = 0$ where the factor x^2 has even degree.

The curve has a horizontal inflexion on the x -axis at $x = -2$ corresponding to the factor $(x + 2)^3$ of odd degree, and a turning point on the x -axis at $x = 0$ corresponding to the factor x^2 of even degree — proving this will require calculus, although the result is fairly obvious by comparison with the known graphs of $y = x^2$, $y = x^3$ and $y = x^4$.



Multiple Zeroes: Some machinery is needed to describe the situation. The zero $x = -2$ of the polynomial $P(x) = (x + 2)^3 x^2 (x - 2)$ is called a *triple zero*, the zero $x = 0$ is called a *double zero*, and the zero $x = 2$ is called a *simple zero*.

6

MULTIPLE ZEROES: Suppose that $x - \alpha$ is a factor of a polynomial $P(x)$, and

$$P(x) = (x - \alpha)^m Q(x), \text{ where } Q(x) \text{ is not divisible by } x - \alpha.$$

Then $x = \alpha$ is called a *zero of multiplicity m* .

A zero of multiplicity 1 is called a *simple zero*, and a zero of multiplicity greater than 1 is called a *multiple zero*.

Behaviour at Simple and Multiple Zeroes: In general:

7

MULTIPLE ZEROES AND THE SHAPE OF THE CURVE: Suppose that $x = \alpha$ is a zero of a polynomial $P(x)$.

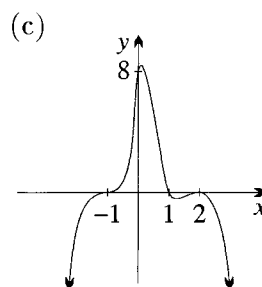
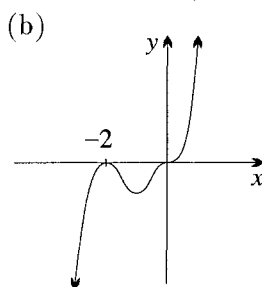
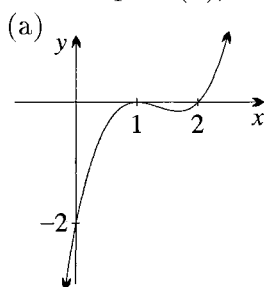
- If $x = \alpha$ has even multiplicity, the curve is tangent to the x -axis at $x = \alpha$, and does not cross the x -axis there.
- If $x = \alpha$ has odd multiplicity at least 3, the curve has a point of inflexion on the x -axis at $x = \alpha$.
- If $x = \alpha$ is a simple zero, then the curve crosses the x -axis at $x = \alpha$ and is not tangent to the x -axis there.

Because the proof relies on the factor theorem, it cannot be presented until Section 4E (where it is proven as Consequence G of the factor theorem). Sketching the curves at the outset seems more appropriate than maintaining logical order.

WORKED EXERCISE: Sketch, showing the behaviour near any x -intercepts:

- $P(x) = (x - 1)^2(x - 2)$
- $Q(x) = x^3(x + 2)^4(x^2 + x + 1)$
- $R(x) = -2(x - 2)^2(x + 1)^5(x - 1)$

SOLUTION: In part (b), $x^2 + x + 1$ is irreducible, because $\Delta = 1 - 4 < 0$.



Exercise 4B

1. Without the aid of calculus, sketch graphs of the following linear polynomials, clearly indicating all intercepts with the axes:

(a) $P(x) = 2$ (b) $P(x) = x$ (c) $P(x) = x - 4$ (d) $P(x) = 3 - 2x$

2. Without the aid of calculus, sketch graphs of the following quadratic polynomials, clearly indicating all intercepts with the axes:

(a) $P(x) = x^2$ (c) $P(x) = (x - 2)^2$ (e) $P(x) = 2x^2 + 5x - 3$
 (b) $P(x) = (x - 1)(x + 3)$ (d) $P(x) = 9 - x^2$ (f) $P(x) = 4 + 3x - x^2$

3. Without the aid of calculus, sketch graphs of the following cubic polynomials, clearly indicating all intercepts with the axes:

(a) $y = x^3$ (d) $y = (x - 1)(x + 2)(x - 3)$ (g) $y = (2x + 1)^2(x - 4)$
 (b) $y = x^3 + 2$ (e) $y = x(2x + 1)(x - 5)$ (h) $y = x^2(1 - x)$
 (c) $y = (x - 4)^3$ (f) $y = (1 - x)(1 + x)(2 + x)$ (i) $y = (2 - x)^2(5 - x)$

4. Without the aid of calculus, sketch graphs of the following quartic polynomials, clearly indicating all intercepts with the axes:

(a) $F(x) = x^4$ (b) $F(x) = (x + 2)^4$ (f) $F(x) = (x + 2)^3(x - 5)$
 (c) $F(x) = x(3x + 2)(x - 3)(x + 2)$ (g) $F(x) = (2x - 3)^2(x + 1)^2$
 (d) $F(x) = (1 - x)(x + 5)(x - 7)(x + 3)$ (h) $F(x) = (1 - x)^3(x - 3)$
 (e) $F(x) = x^2(x + 4)(x - 3)$ (i) $F(x) = (2 - x)^2(1 - x^2)$

DEVELOPMENT

5. These polynomials are not factored, but the positions of their zeroes can be found by trial and error. Copy and complete each tables of values, and sketch a graph, stating how many zeroes there are, and between which integers they lie.

(a) $y = x^2 - 3x + 1$

x	-1	0	1	2	3	4
y						

(b) $y = 1 + 3x - x^3$

x	-2	-1	0	1	2	3
y						

6. Without the aid of calculus, sketch graphs of the following polynomial functions, clearly indicating all intercepts with the axes.

(a) $P(x) = x(x - 2)^3(x + 1)^2$ (c) $P(x) = x(2x + 3)^3(1 - x)^4$
 (b) $P(x) = (x + 2)^2(3 - x)^3$ (d) $P(x) = (x + 1)(4 - x^2)(x^2 - 3x - 10)$

7. Use the graphs drawn in the previous question to solve the following inequalities.

(a) $x(x - 2)^3(x + 1)^2 > 0$ (c) $x(2x + 3)^3(1 - x)^4 \geq 0$
 (b) $(x + 2)^2(3 - x)^3 \geq 0$ (d) $(x + 1)(4 - x^2)(x^2 - 3x - 10) < 0$

8. (a) Without using calculus, sketch a graph of the function $P(x) = x(x-2)^2(x+1)$.
 (b) Hence by translating or reflecting this graph, sketch the following functions:
 (i) $R(x) = -x(x-2)^2(x+1)$ (iii) $U(x) = (x-2)(x-4)^2(x-1)$
 (ii) $Q(x) = -x(-x-2)^2(-x+1)$ (iv) $V(x) = (x+3)(x+1)^2(x+4)$
9. (a) Find the monic quadratic polynomial that crosses the y -axis at $(0, -6)$ and the x -axis at $(3, 0)$.
 (b) Find the quadratic polynomial that has a minimum value of -3 when $x = -2$, and passes through the point $(1, 6)$.
 (c) Find the cubic polynomial that has zeroes $0, 1$ and 2 , and in which the coefficient of x^3 is 2 .
10. Consider the polynomial $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.
 (a) What condition on the coefficients is satisfied if $P(x)$ is: (i) even, (ii) odd?
 (b) Find the monic, even quartic that has y -intercept 9 and a zero at $x = 3$.
 (c) Find the odd quintic with zeroes at $x = 1$ and $x = 2$ and leading coefficient -3 .
11. (a) Prove that every odd polynomial function is zero at $x = 0$.
 (b) Prove that every odd polynomial $P(x)$ is divisible by x .
 (c) Find the polynomial $P(x)$ that is known to be monic, of degree 3 , and an odd function, and has one zero at $x = 2$.
12. By making a suitable substitution, factor the following polynomials. Without using calculus, sketch graphs showing all intercepts with the axes.
 (a) $P(x) = x^4 - 13x^2 + 36$ (c) $P(x) = (x^2 - 5x)^2 - 2(x^2 - 5x) - 24$
 (b) $P(x) = 4x^4 - 13x^2 + 9$ (d) $P(x) = (x^2 - 3x + 1)^2 - 4(x^2 - 3x + 1) - 5$
13. (a) Sketch graphs of the following polynomials, clearly labelling all intercepts with the axes. Do not use calculus to find further turning points.
 (i) $F(x) = x(x-4)(x+1)$ (iii) $F(x) = x(x+3)^2(5-x)$
 (ii) $F(x) = (x-1)^2(x+3)$ (iv) $F(x) = x^2(x-3)^3(x-7)$
 (b) Without the aid of calculus, draw graphs of the derivatives of each of the polynomials in part (a). You will not be able to find the x -intercepts or y -intercepts accurately.
 (c) Suppose that $G(x)$ is a primitive of $F(x)$. For each of the polynomials in part (a), state for what values of x the function $G(x)$ is increasing and decreasing.
14. Sketch, over $-\pi < x < \pi$: (a) $y = \cos x$ (b) $y = \cos^2 x$ (c) $y = \cos^3 x$
15. [Every cubic has odd symmetry in its point of inflexion.]
 (a) Suppose that the origin is the point of inflexion of $f(x) = ax^3 + bx^2 + cx + d$.
 (i) Prove that $b = d = 0$, and hence that $f(x)$ is an odd function.
 (ii) Hence prove that if ℓ is a line through the origin crossing the curve again at A and B , then O is the midpoint of the interval AB .
 (b) Use part (a), and arguments based on translations, to prove that if ℓ is a line through the point of inflexion I crossing the curve again at A and B , then I is the midpoint of AB .
 (c) Prove that if a cubic has turning points, then the midpoint of the interval joining them is the point of inflexion.

EXTENSION

16. At what points do the graphs of the polynomials $f(x) = (x+1)^n$ and $g(x) = (x+1)^m$ intersect? [HINT: Consider the cases where m and n are odd and even.]

17. [The motivation for this question is the power series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.]

For each integer $n > 0$, let $E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$.

(a) Show that: (i) $E_n(x) = E_{n-1}(x) + \frac{x^n}{n!}$ (ii) $E_n'(x) = E_{n-1}(x)$

(b) Show that if $x = \alpha$ is a zero of $E_n'(x)$, then $E_n(\alpha) = \frac{\alpha^n}{n!}$.

(c) Suppose that n is even.

(i) Show that every stationary point of $E_n(x)$ lies above the x -axis,

(ii) Show that $E_n(x)$ is positive, for all x , and concave up, for all x .

(iii) Show that $E_n(x)$ has one stationary point, which is a minimum turning point.

(d) Suppose that n is odd.

(i) Show that $E_n(x)$ is increasing for all x , and has exactly one zero.

(ii) Show that $E_n(x)$ has exactly one point of inflexion.

(iii) By factoring in pairs, show that $E_n(-n) < 0$.

(iv) Show that the inflexion is above the x -axis.

4C Division of Polynomials

The previous exercise had examples of adding, subtracting and multiplying polynomials, operations which are quite straightforward. The division of one polynomial by another, however, requires some explanation.

Division of Polynomials: It can happen that the quotient of two polynomials is again a polynomial; for example,

$$\frac{6x^3 + 4x^2 - 9x}{3x} = 2x^2 + \frac{4}{3}x - 3 \quad \text{and} \quad \frac{x^2 + 4x - 5}{x + 5} = x - 1.$$

But usually, division results in rational functions, not polynomials:

$$\frac{x^4 + 4x^2 - 9}{x^2} = x^2 + 4 - \frac{9}{x^2} \quad \text{and} \quad \frac{x + 4}{x + 3} = 1 + \frac{1}{x + 3}.$$

In this respect, there is a very close analogy between the set $\mathbf{Z}$ of all integers and the set of all polynomials. In both cases, everything works nicely for addition, subtraction and multiplication, but the results of division do not usually lie within the set. For example, although $20 \div 5 = 4$ is an integer, the division of two integers usually results in a fraction rather than an integer, as in $23 \div 5 = 4\frac{3}{5}$.

The Division Algorithm for Integers: On the right is an example of the well-known long division algorithm for integers, applied here to $197 \div 12$. The number 12 is called the *divisor*, 197 is called the *dividend*, 16 is called the *quotient*, and 5 is called the *remainder*.

$$\begin{array}{r} 16 \text{ remainder } 5 \\ 12 \overline{) 197} \\ \underline{12} \\ 77 \\ \underline{72} \\ 5 \end{array}$$

The result of the division can be written as $\frac{197}{12} = 16\frac{5}{12}$, but we can avoid fractions completely by writing the result as:

$$197 = 12 \times 16 + 5.$$

The remainder 5 had to be less than 12, otherwise the division process could have been continued. Thus the general result for division of integers can be expressed as follows:

8 **DIVISION OF INTEGERS:** Suppose that p (the dividend) and d (the divisor) are integers, with $d > 0$. Then there are unique integers q (the quotient) and r (the remainder) such that

$$p = dq + r \quad \text{and} \quad 0 \leq r < d.$$

When the remainder r is zero, then d is a *divisor* of p , and the integer p *factors into the product* $p = d \times q$.

The Division Algorithm for Polynomials: The method of dividing one polynomial by another is similar to the method of dividing integers.

9 **THE METHOD OF LONG DIVISION OF POLYNOMIALS:**

- At each step, divide the leading term of the remainder by the leading term of the divisor. Continue the process for as long as possible.
- Unless otherwise specified, express the answer in the form

$$\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}.$$

WORKED EXERCISE: Divide $3x^4 - 4x^3 + 4x - 8$ by: (a) $x - 2$ (b) $x^2 - 2$
Give results first in the standard manner, then using rational functions.

SOLUTION: The steps have been annotated to explain the method.

(a)

$$\begin{array}{r}
 3x^3 + 2x^2 + 4x + 12 \\
 x - 2 \overline{) 3x^4 - 4x^3 + 4x - 8} \\
 \underline{3x^4 - 6x^3} \\
 2x^3 \\
 \underline{2x^3 - 4x^2} \\
 4x^2 + 4x - 8 \\
 \underline{4x^2 - 8x} \\
 12x - 8 \\
 \underline{12x - 24} \\
 16
 \end{array}$$

(leave a gap for the missing term in x^2)
(divide x into $3x^4$, giving the $3x^3$ above)
(multiply $x - 2$ by $3x^3$ and then subtract)
(divide x into $2x^3$, giving the $2x^2$ above)
(multiply $x - 2$ by $2x^2$ and then subtract)
(divide x into $4x^2$, giving the $4x$ above)
(multiply $x - 2$ by $4x$ and then subtract)
(divide x into $12x$, giving the 12 above)
(multiply $x - 2$ by 12 and then subtract)
(this is the final remainder)

Hence $3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16$,

or, writing the result using rational functions,

$$\frac{3x^4 - 4x^3 + 4x - 8}{x - 2} = 3x^3 + 2x^2 + 4x + 12 + \frac{16}{x - 2}.$$

(b)

$$\begin{array}{r}
 3x^2 - 4x + 6 \\
 x^2 - 2 \overline{) 3x^4 - 4x^3 + 4x - 8} \\
 \underline{3x^4 - 6x^2} \\
 -4x^3 + 6x^2 + 4x - 8 \\
 \underline{-4x^3 + 8x} \\
 6x^2 - 4x - 8 \\
 \underline{6x^2 - 12} \\
 -4x + 4
 \end{array}$$

(divide x^2 into $3x^4$, giving the $3x^2$ above)
(multiply $x^2 - 2$ by $3x^2$ and then subtract)
(divide x^2 into $-4x^3$, giving the $-4x$ above)
(multiply $x^2 - 2$ by $-4x$ and then subtract)
(divide x^2 into $6x^2$, giving the 6 above)
(multiply $x^2 - 2$ by 6 and then subtract)
(this is the final remainder)

$$\begin{aligned} \text{Hence } 3x^4 - 4x^3 + 4x - 8 &= (x^2 - 2)(3x^2 - 4x + 6) + (-4x + 4), \\ \text{or } \frac{3x^4 - 4x^3 + 4x - 8}{x^2 - 2} &= 3x^2 - 4x + 6 + \frac{-4x + 4}{x^2 - 2}. \end{aligned}$$

The Division Theorem:

The division process illustrated above can be continued until the remainder is zero or has degree less than the degree of the divisor. Thus the general result for polynomial division is:

DIVISION OF POLYNOMIALS: Suppose that $P(x)$ (the dividend) and $D(x)$ (the divisor) are polynomials with $D(x) \neq 0$. Then there are unique polynomials $Q(x)$ (the quotient) and $R(x)$ (the remainder) such that

10 1. $P(x) = D(x)Q(x) + R(x)$,
 2. either $\deg R(x) < \deg D(x)$, or $R(x) = 0$.

When the remainder $R(x)$ is zero, then $D(x)$ is called a *divisor* of $P(x)$, and the polynomial $P(x)$ factors into the product $P(x) = D(x) \times Q(x)$.

For example, in the two worked exercises above:

- the remainder after division by the degree 1 polynomial $x - 2$ was the constant polynomial 16,
- the remainder after division by the degree 2 polynomial $x^2 - 2$ was the linear polynomial $-4x + 4$.

Exercise 4C

1. Perform each of the following integer divisions, and write the result in the form $p = dq + r$, where $0 \leq r < d$. For example, $30 = 4 \times 7 + 2$.
 (a) $63 \div 5$ (b) $125 \div 8$ (c) $324 \div 11$ (d) $1857 \div 23$
2. Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.
 (a) $(x^2 - 4x + 1) \div (x + 1)$ (e) $(4x^3 - 4x^2 + 7x + 14) \div (2x + 1)$
 (b) $(x^2 - 6x + 5) \div (x - 5)$ (f) $(x^4 + x^3 - x^2 - 5x - 3) \div (x - 1)$
 (c) $(x^3 - x^2 - 17x + 24) \div (x - 4)$ (g) $(6x^4 - 5x^3 + 9x^2 - 8x + 2) \div (2x - 1)$
 (d) $(2x^3 - 10x^2 + 15x - 14) \div (x - 3)$ (h) $(10x^4 - x^3 + 3x^2 - 3x - 2) \div (5x + 2)$
3. Express the answers to parts (a)–(d) of the previous question in rational form, that is, as $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, and hence find the primitive of the quotient $\frac{P(x)}{D(x)}$.
4. Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.
 (a) $(x^3 + x^2 - 7x + 6) \div (x^2 + 3x - 1)$ (b) $(x^3 - 4x^2 - 2x + 3) \div (x^2 - 5x + 3)$
 (c) $(x^4 - 3x^3 + x^2 - 7x + 3) \div (x^2 - 4x + 2)$
 (d) $(2x^5 - 5x^4 + 12x^3 - 10x^2 + 7x + 9) \div (x^2 - x + 2)$
5. (a) If the divisor of a polynomial has degree 3, what are the possible degrees of the remainder?
 (b) On division by $D(x)$, a polynomial has remainder $R(x)$ of degree 2. What are the possible degrees of $D(x)$?

DEVELOPMENT

6. Use long division to perform each of the following divisions. Take care to ensure that the columns line up correctly. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.

$$\begin{array}{ll} \text{(a)} (x^3 - 5x + 3) \div (x - 2) & \text{(d)} (2x^4 - 5x^2 + x - 2) \div (x^2 + 3x - 1) \\ \text{(b)} (2x^3 + x^2 - 11) \div (x + 1) & \text{(e)} (2x^3 - 3) \div (2x - 4) \\ \text{(c)} (x^3 - 3x^2 + 5x - 4) \div (x^2 + 2) & \text{(f)} (x^5 + 3x^4 - 2x^2 - 3) \div (x^2 + 1) \end{array}$$

Write the answers to parts (c) and (f) above in rational form, that is, in the form $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, and hence find the primitive of the quotient $\frac{P(x)}{D(x)}$.

7. Find the quotient and remainder in each of the following divisions. Fractions will be needed throughout the calculations.

$$\begin{array}{ll} \text{(a)} (x^2 + 4x + 7) \div (2x + 1) & \text{(c)} (x^3 - x^2 + x + 1) \div (2x - 3) \\ \text{(b)} (6x^3 - x^2 + 4x - 2) \div (3x - 1) & \end{array}$$

8. (a) Use long division to show that $P(x) = x^3 + 2x^2 - 11x - 12$ is divisible by $x - 3$, and hence express $P(x)$ as the product of three linear factors.

(b) Find the values of x for which $P(x) > 0$.

9. (a) Use long division to show that $F(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$ is divisible by $x^2 - x - 2$, and hence express $F(x)$ as the product of four linear factors.

(b) Find the values of x for which $F(x) \leq 0$.

10. (a) Write down the division identity statement when $30 \div 4$ and $30 \div 7$.

(b) Division of the polynomial $P(x)$ by $D(x)$ results in the quotient $Q(x)$ and remainder $R(x)$. Show that if $P(x)$ is divided by $Q(x)$, the remainder will still be $R(x)$. What is the quotient?

11. (a) Find the quotient and remainder when $x^4 - 2x^3 + x^2 - 5x + 7$ is divided by $x^2 + x - 1$.

(b) Find a and b if $x^4 - 2x^3 + x^2 + ax + b$ is exactly divisible by $x^2 + x - 1$.

(c) Hence factor $x^4 - 2x^3 + x^2 + 8x - 5$.

12. (a) Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $d(x) = x^2 + 4$. Express your answer in the form $f(x) = d(x)q(x) + r(x)$.

(b) Hence find the values of a and b such that $x^4 - x^3 + x^2 + ax + b$ is divisible by $x^2 + 4$.

(c) Hence factor $x^4 - x^3 + x^2 - 4x - 12$.

13. If $x^4 - 2x^3 - 20x^2 + ax + b$ is exactly divisible by $x^2 - 5x + 2$, find a and b .

EXTENSION

14. Two integers are said to be *relatively prime* if their highest common factor is 1. If a and b are relatively prime it is possible to find integers x and y such that $ax + by = 1$. For example 51 and 44 are relatively prime.

Repeated use of the division identity leads to:

$$\begin{aligned} 51 &= 44 \times 1 + 7 \\ 44 &= 7 \times 6 + 2 \\ 7 &= 3 \times 2 + 1 \end{aligned}$$

Reversing these steps leads to:

$$\begin{aligned} 1 &= 7 - 3 \times 2 \\ &= 7 - 3(44 - 7 \times 6) \\ &= 19 \times 7 - 3 \times 44 \\ &= 19(51 - 44 \times 1) - 3 \times 44 \\ &= 19 \times 51 - 22 \times 44 \end{aligned}$$

- (a) Use this method to find integers a and b such that $87a + 19b = 1$
- (b) Find polynomials $A(x)$ and $B(x)$ such that $1 = A(x)(x^2 - x) + B(x)(x^4 + 4x^2 - 4x + 4)$.
15. [The uniqueness of integer division and polynomial division]
- (a) Suppose that $p = dq + r$ and $p = dq' + r'$, where p, d, q, q', r and r' are integers with $d \neq 0$, and where $0 \leq r < d$ and $0 \leq r' < d$. Prove that $q = q'$ and $r = r'$.
- (b) Suppose that $P(x) = D(x)Q(x) + R(x)$ and $P(x) = D(x)Q'(x) + R'(x)$, where $P(x), D(x), Q(x), Q'(x), R(x)$ and $R'(x)$ are polynomials with $D(x) \neq 0$, and where $R(x)$ and $R'(x)$ each has degree less than $D(x)$ or is the zero polynomial. Prove that $Q(x) = Q'(x)$ and $R(x) = R'(x)$.

4D The Remainder and Factor Theorems

Long division of polynomials is a cumbersome process. It is therefore very useful to have the remainder and factor theorems, which provide information about the results of that division without the division actually being carried out. In particular, the factor theorem gives a simple test as to whether a particular linear function is a factor or not.

The Remainder Theorem: The remainder theorem is a remarkable result which, in the case of linear divisors, allows the remainder to be calculated without the long division being performed.

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THE REMAINDER THEOREM: Suppose that $P(x)$ is a polynomial and α is a constant. Then the remainder after division of $P(x)$ by $x - \alpha$ is $P(\alpha)$.

PROOF: Since $x - \alpha$ is a polynomial of degree 1, the division theorem tells us that there are unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = (x - \alpha)Q(x) + R(x),$$

and either $R(x) = 0$ or $\deg R(x) = 0$.

Hence $R(x)$ is a zero or nonzero constant, which we can simply write as r ,

and so $P(x) = (x - \alpha)Q(x) + r$.

Substituting $x = \alpha$ gives $P(\alpha) = (\alpha - \alpha)Q(\alpha) + r$

and rearranging, $r = P(\alpha)$, as required.

WORKED EXERCISE: Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$:

(a) by long division, (b) by the remainder theorem.

SOLUTION: In the previous worked exercise, performing the division showed that

$$3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16,$$

that is, that the remainder is 16. Alternatively, substituting $x = 2$ into $P(x)$,

remainder = $P(2)$ (this is the remainder theorem)

$$= 48 - 32 + 8 - 8$$

$$= 16, \text{ as expected.}$$

WORKED EXERCISE: The polynomial $P(x) = x^4 - 2x^3 + ax + b$ has remainder 3 after division by $x - 1$, and has remainder -5 after division by $x + 1$. Find a and b .

SOLUTION: Applying the remainder theorem for each divisor,

$$\begin{aligned} P(1) &= 3 \\ 1 - 2 + a + b &= 3 \\ a + b &= 4. \end{aligned} \quad (1)$$

Also $P(-1) = -5$

$$\begin{aligned} 1 + 2 - a + b &= -5 \\ -a + b &= -8. \end{aligned} \quad (2)$$

Adding (1) and (2), $2b = -4$,

subtracting them, $2a = 12$.

Hence $a = 6$ and $b = -2$.

The Factor Theorem: The remainder theorem tells us that the number $P(\alpha)$ is just the remainder after division by $x - \alpha$. But $x - \alpha$ being a factor means that the remainder after division by $x - \alpha$ is zero, so:

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THE FACTOR THEOREM: Suppose that $P(x)$ is a polynomial and α is a constant. Then $x - \alpha$ is a factor if and only if $P(\alpha) = 0$.

This is a very quick and easy test as to whether $x - \alpha$ is a factor of $P(x)$ or not.

WORKED EXERCISE: Show that $x - 3$ is a factor of $P(x) = x^3 - 2x^2 + x - 12$, and $x + 1$ is not. Then use long division to factor the polynomial completely.

SOLUTION: $P(3) = 27 - 18 + 3 - 12 = 0$, so $x - 3$ is a factor.

$P(-1) = -1 - 2 - 1 - 12 = -16 \neq 0$, so $x + 1$ is not a factor.

Long division of $P(x) = x^3 - 2x^2 + x - 12$ by $x - 3$ (which we omit) gives

$$P(x) = (x - 3)(x^2 + x + 4),$$

and since $\Delta = 1 - 16 = -15 < 0$ for the quadratic, this factorisation is complete.

Factoring Polynomials — The Initial Approach: The factor theorem gives us the beginnings of an approach to factoring polynomials. This approach will be further refined in the next two sections.

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FACTORIZING POLYNOMIALS — THE INITIAL APPROACH:

- Use trial and error to find an integer zero $x = \alpha$ of $P(x)$.
- Then use long division to factor $P(x)$ in the form $P(x) = (x - \alpha)Q(x)$.

If the coefficients of $P(x)$ are all integers, then all the integer zeroes of $P(x)$ are divisors of the constant term.

PROOF: We must prove the claim that if the coefficients of $P(x)$ are integers, then every integer zero of $P(x)$ is a divisor of the constant term.

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$,

where the coefficients $a_n, a_{n-1}, \dots, a_1, a_0$ are all integers,

and let $x = \alpha$ be an integer zero of $P(x)$.

Substituting into $P(\alpha) = 0$ gives $a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0$

$$\begin{aligned} a_0 &= -a_n \alpha^n - a_{n-1} \alpha^{n-1} - \cdots - a_1 \alpha \\ &= \alpha(-a_n \alpha^{n-1} - a_{n-1} \alpha^{n-2} - \cdots - a_1), \end{aligned}$$

and so a_0 is an integer multiple of α .

WORKED EXERCISE: Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION: Since all the coefficients are integers, the only integer zeroes are the divisors of the constant term -4 , that is $1, 2, 4, -1, -2$ and -4 .

$$P(1) = 1 + 1 - 9 + 11 - 4$$

$$= 0, \text{ so } x - 1 \text{ is a factor.}$$

After long division (omitted), $P(x) = (x - 1)(x^3 + 2x^2 - 7x + 4)$.

Let $Q(x) = x^3 + 2x^2 - 7x + 4$, then $Q(1) = 1 + 2 - 7 + 4$

$$= 0, \text{ so } x - 1 \text{ is a factor.}$$

Again after long division (omitted), $P(x) = (x - 1)(x - 1)(x^2 + 3x - 4)$.

Factoring the quadratic, $P(x) = (x - 1)^3(x + 4)$.

NOTE: In the next two sections we will develop methods that will often allow long division to be avoided.

Exercise 4D

- Without division, find the remainder when $P(x) = x^3 - x^2 + 2x + 1$ is divided by:

(a) $x - 1$	(c) $x + 2$	(e) $x - 5$
(b) $x - 3$	(d) $x + 1$	(f) $x + 3$
- Without division, find which of the following are factors of $F(x) = x^3 + 4x^2 + x - 6$.

(a) $x - 1$	(c) $x - 2$	(e) $x - 3$
(b) $x + 1$	(d) $x + 2$	(f) $x + 3$
- Without division, find the remainder when $P(x) = x^3 + 2x^2 - 4x + 5$ is divided by:

(a) $2x - 1$	(b) $2x + 3$	(c) $3x - 2$
- Find k , if $x - 1$ is a factor of $P(x) = x^3 - 3x^2 + kx - 2$.
 - Find m , if -2 is a zero of the function $F(x) = x^3 + mx^2 - 3x + 4$.
 - When the polynomial $P(x) = 2x^3 - x^2 + px - 1$ is divided by $x - 3$, the remainder is 2. Find p .
 - For what value of a is $3x^4 + ax^2 - 2$ divisible by $x + 1$?

DEVELOPMENT

- Show that $P(x) = x^3 - 8x^2 + 9x + 18$ is divisible by $x - 3$ and $x + 1$.
 - By considering the leading term and constant term, express $P(x)$ in terms of three linear factors and hence solve $P(x) \geq 0$.
- Show that $P(x) = 2x^3 - x^2 - 13x - 6$ is divisible by $x - 3$ and $2x + 1$.
 - By considering the leading term and constant term, express $P(x)$ in terms of three linear factors and hence solve $P(x) \leq 0$.
- Factor each of the following polynomials and sketch a graph, indicating all intercepts with the axes. You do not need to find any other turning points.

(a) $P(x) = x^3 + 2x^2 - 5x - 6$	(d) $P(x) = x^4 - x^3 - 19x^2 - 11x + 30$
(b) $P(x) = x^3 + 3x^2 - 25x + 21$	(e) $P(x) = 2x^3 + 11x^2 + 10x - 8$
(c) $P(x) = -x^3 + x^2 + 5x + 3$	(f) $P(x) = 3x^4 + 4x^3 - 35x^2 - 12x$

8. Solve the equations by first factoring the LHS:
- (a) $x^3 + 3x^2 - 6x - 8 = 0$ (d) $x^3 - 2x^2 - 2x - 3 = 0$
 (b) $x^3 - 4x^2 - 3x + 18 = 0$ (e) $6x^3 - 5x^2 - 12x - 4 = 0$
 (c) $x^3 + x^2 - 7x + 2 = 0$ (f) $2x^4 + 11x^3 + 19x^2 + 8x - 4 = 0$
9. (a) If $P(x) = 2x^3 + x^2 - 13x + 6$, evaluate $P(\frac{1}{2})$. Use long division to express $P(x)$ in factored form.
 (b) If $P(x) = 6x^3 + x^2 - 5x - 2$, evaluate $P(-\frac{2}{3})$. Express $P(x)$ in factored form.
10. (a) Factor each of the polynomials $P(x) = x^3 - 3x^2 + 4$, $Q(x) = x^3 + 2x^2 - 5x - 6$ and $R(x) = x^3 - 3x - 2$.
 (b) Hence determine the highest monic common factor of $P(x)$, $Q(x)$ and $R(x)$.
 (c) What is the monic polynomial of least degree that is exactly divisible by $P(x)$, $Q(x)$ and $R(x)$? Write the answer in factored form.
11. At time t , the position of a particle moving along the x -axis is given by the equation $x = t^4 - \frac{17}{3}t^3 + 8t^2 - 3t + 5$. Find the times at which the particle is stationary.
12. (a) The polynomial $2x^3 - x^2 + ax + b$ has a remainder of 16 on division by $x - 1$ and a remainder of -17 on division by $x + 2$. Find a and b .
 (b) A polynomial is given by $P(x) = x^3 + ax^2 + bx - 18$. Find a and b , if $x + 2$ is a factor and -24 is the remainder when $P(x)$ is divided by $x - 1$.
 (c) $P(x)$ is an odd polynomial of degree 3. It has $x + 4$ as a factor, and when it is divided by $x - 3$ the remainder is 21. Find $P(x)$.
 (d) Find p such that $x - p$ is a factor of $4x^3 - (10p - 1)x^2 + (6p^2 - 5)x + 6$.
13. (a) The polynomial $P(x)$ is divided by $(x - 1)(x + 2)$. Find the remainder, given that $P(1) = 2$ and $P(-2) = 5$. [HINT: The remainder may have degree 1.]
 (b) The polynomial $U(x)$ is divided by $(x + 4)(x - 3)$. Find the remainder, given that $U(-4) = 11$ and $U(3) = -3$.
14. (a) The polynomial $P(x) = x^3 + bx^2 + cx + d$ has zeroes at 0, 3 and -3 . Find b , c and d .
 (b) Sketch a graph of $y = P(x)$. (c) Hence solve $\frac{x^2 - 9}{x} > 0$.
15. (a) Show that the equation of the normal to the curve $x^2 = 4y$ at the point $(2t, t^2)$ is $x + ty - 2t - t^3 = 0$.
 (b) If the normal passes through the point $(-2, 5)$, find the value of t .
16. (a) Is either $x + 1$ or $x - 1$ a factor of $x^n + 1$, where n is a positive integer?
 (b) Is either $x + a$ or $x - a$ a factor of $x^n + a^n$, where n is a positive integer?
17. When a polynomial is divided by $(x - 1)(x + 3)$, the remainder is $2x - 1$.
 (a) Express this in terms of a division identity statement.
 (b) Hence, by evaluating $P(1)$, find the remainder when the polynomial is divided by $x - 1$.
18. (a) When a polynomial is divided by $(2x + 1)(x - 3)$, the remainder is $3x - 1$. What is the remainder when the polynomial is divided by $2x + 1$?
 (b) When $x^5 + 3x^3 + ax + b$ is divided by $x^2 - 1$, the remainder is $2x - 7$. Find a and b .
 (c) When a polynomial $P(x)$ is divided by $x^2 - 5$, the remainder is $x + 4$. Find the remainder when $P(x) + P(-x)$ is divided by $x^2 - 5$. [HINT: Write down the division identity statement.]

19. Each term of an arithmetic sequence $a, a + d, a + 2d, \dots$ is added to the corresponding term of the geometric sequence $b, ba, ba^2, \dots$ to form a third sequence S , whose first three terms are $-1, -2$ and 6 . (Note that the common ratio of the geometric sequence is equal to the first term of the arithmetic sequence.)
- Show that $a^3 - a^2 - a + 10 = 0$.
 - Find a , given that a is real.
 - Hence show that the n th term of S is given by $T_n = 2n - 4 + (-2)^{n-1}$.

EXTENSION

20. When a polynomial is divided by $x - p$, the remainder is p^3 . When the polynomial is divided by $x - q$, the remainder is q^3 . Find the remainder when the polynomial is divided by $(x - p)(x - q)$.
21. [Finding the equation of a cubic, given its two stationary points] Let $y_1 = f(x)$ be a cubic polynomial with stationary points at $(6, 12)$ and $(12, 4)$. Let $y_2 = g(x) = f(x) - 4$.
- Write down the coordinates of the minimum turning point of $g(x)$.
 - Hence write down the general form of the equation of $g(x)$ in factored form.
 - Find the value of $g(6)$.
 - In Exercise 4B, you proved that a cubic has odd symmetry in its point of inflexion. Use this fact to show that $g(9) = 4$.
 - Hence use simultaneous equations to find a and k and the equation of $g(x)$.
 - Hence find the equation of the cubic through the stationary points $(6, 12)$ and $(12, 4)$.
 - In Chapter Ten of the Year 11 volume, you solved this type of question by letting $f(x) = ax^3 + bx^2 + cx + d$ and forming four equations in the four unknowns. Check your answer by this method.
22. (a) If all the coefficients of a monic polynomial are integers, prove that all the rational zeroes are integers. [HINT: Look carefully at the proof under Box 13.]
- (b) If all the coefficients of a polynomial are integers, prove that the denominators of all the rational zeroes (in lowest terms) are divisors of the leading coefficient.
23. (a) Use the remainder theorem to prove that $a + b + c$ is a factor of $a^3 + b^3 + c^3 - 3abc$. Then find the other factor. [HINT: Regard it as a polynomial in a .]
- (b) Factor $ab^3 - ac^3 + bc^3 - ba^3 + ca^3 - cb^3$.

4E Consequences of the Factor Theorem

The factor theorem has a number of fairly obvious but very useful consequences, which are presented here as six successive theorems.

A. Several Distinct Zeroes: Suppose that several distinct zeroes of a polynomial have been found, probably using test substitutions into the polynomial.

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DISTINCT ZEROES: Suppose that $\alpha_1, \alpha_2, \dots, \alpha_s$ are distinct zeroes of a polynomial $P(x)$. Then $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_s)$ is a factor of $P(x)$.

PROOF: Since α_1 is a zero, $x - \alpha_1$ is a factor, and $P(x) = (x - \alpha_1)p_1(x)$.

Since $P(\alpha_2) = 0$ but $\alpha_2 - \alpha_1 \neq 0$, $p_1(\alpha_2)$ must be zero.

Hence $x - \alpha_2$ is a factor of $p_1(x)$, and $p_1(x) = (x - \alpha_2)p_2(x)$,

and thus $P(x) = (x - \alpha_1)(x - \alpha_2)p_2(x)$.

Continuing similarly for s steps, $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_s)$ is a factor of $P(x)$.

B. All Distinct Zeroes: If n distinct zeroes of a polynomial of degree n can be found, then the factorisation is complete, and the polynomial is the product of distinct linear factors.

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ALL DISTINCT ZEROES: Suppose that $\alpha_1, \alpha_2, \dots, \alpha_n$ are n distinct zeroes of a polynomial $P(x)$ of degree n . Then

$$P(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n),$$

where a is the leading coefficient of $P(x)$.

PROOF: By the previous theorem, $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ is a factor of $P(x)$, so $P(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)Q(x)$, for some polynomial $Q(x)$.

But $P(x)$ and $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ both have degree n , so $Q(x)$ is a constant.

Equating coefficients of x^n , the constant $Q(x)$ must be the leading coefficient.

Factoring Polynomials — Finding Several Zeroes First: If we can find more than one zero of a polynomial, then we have found a quadratic or cubic factor, and the long divisions required can be reduced or even avoided completely.

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FACTORING POLYNOMIALS — FINDING SEVERAL ZEROES FIRST:

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Using long division, divide $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

When this procedure is applied to the polynomial factored in the previous section, one rather than two long divisions is required.

WORKED EXERCISE: Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION: As before, all the coefficients are integers, so the only integer zeroes are the divisors of the constant term -4 , that is $1, 2, 4, -1, -2$ and -4 .

$$P(1) = 1 + 1 - 9 + 11 - 4 = 0, \text{ so } x - 1 \text{ is a factor.}$$

$$P(-4) = 256 - 64 - 144 - 44 - 4 = 0, \text{ so } x + 4 \text{ is a factor.}$$

After long division by $(x - 1)(x + 4) = x^2 + 3x - 4$ (omitted),

$$P(x) = (x^2 + 3x - 4)(x^2 - 2x + 1).$$

Factoring the quadratic, $P(x) = (x - 1)(x - 4) \times (x - 1)^2$

$$= (x - 1)^3(x + 4).$$

NOTE: The methods of the next section will allow this particular factoring to be done with no long divisions. The following worked exercise involves a polynomial that factors into distinct linear factors, so that nothing more than the factor theorem is required to complete the task.

WORKED EXERCISE: Factor $P(x) = x^4 - x^3 - 7x^2 + x + 6$ completely.

SOLUTION: The divisors of the constant term 6 are 1, 2, 3, 6, -1, -2, -3 and -6.

$$P(1) = 1 - 1 - 7 + 1 + 6 = 0, \text{ so } x - 1 \text{ is a factor.}$$

$$P(-1) = 1 + 1 - 7 - 1 + 6 = 0, \text{ so } x + 1 \text{ is a factor.}$$

$$P(2) = 16 - 8 - 28 + 2 + 6 = -12 \neq 0, \text{ so } x - 2 \text{ is not a factor.}$$

$$P(-2) = 16 + 8 - 28 - 2 + 6 = 0, \text{ so } x + 2 \text{ is a factor.}$$

$$P(3) = 81 - 27 - 63 + 3 + 6 = 0, \text{ so } x - 3 \text{ is a factor.}$$

We now have four distinct zeroes of a polynomial of degree 4.

Hence $P(x) = (x - 1)(x + 1)(x + 2)(x - 3)$ (notice that $P(x)$ is monic).

C. The Maximum Number of Zeroes: If a polynomial of degree n were to have $n + 1$ zeroes, then by the first theorem above, it would be divisible by a polynomial of degree $n + 1$, which is impossible.

17 MAXIMUM NUMBER OF ZEROES: A polynomial of degree n has at most n zeroes.

D. A Vanishing Condition: The previous theorem translates easily into a condition for a polynomial to be the zero polynomial.

18 A VANISHING CONDITION: Suppose that $P(x)$ is a polynomial in which no terms have degree more than n , yet which is zero for at least $n + 1$ distinct values of x . Then $P(x)$ is the zero polynomial.

PROOF: Suppose that $P(x)$ had a degree. This degree must be at most n since there is no term of degree more than n . But the degree must also be at least $n + 1$ since there are $n + 1$ distinct zeroes. This is a contradiction, so $P(x)$ has no degree, and is therefore the zero polynomial.

NOTE: Once again, the zero polynomial $Z(x) = 0$ is seen to be quite different in nature from all other polynomials. It is the only polynomial with an infinite number of zeroes; in fact every real number is a zero of $Z(x)$. Associated with this is the fact that $x - \alpha$ is a factor of $Z(x)$ for all real values of α , since $Z(x) = (x - \alpha)Z(x)$ (which is trivially true, because both sides are zero for all x). It is no wonder then that the zero polynomial does not have a degree.

E. A Condition for Two Polynomials to be Identically Equal: A most important consequence of this last theorem is a condition for two polynomials $P(x)$ and $Q(x)$ to be identically equal — written as $P(x) \equiv Q(x)$, and meaning that $P(x) = Q(x)$ for all values of x .

19 AN IDENTICALLY EQUAL CONDITION: Suppose that $P(x)$ and $Q(x)$ are polynomials of degree n which have the same values for at least $n + 1$ values of x . Then the polynomials $P(x)$ and $Q(x)$ are identically equal, written as $P(x) \equiv Q(x)$, that is, they are equal for all values of x .

PROOF: Let $F(x) = P(x) - Q(x)$.

Since $F(x)$ is zero whenever $P(x)$ and $Q(x)$ have the same value,

it follows that $F(x)$ is zero for at least $n + 1$ values of x ,

so by the previous theorem, $F(x)$ is the zero polynomial, and $P(x) \equiv Q(x)$.

WORKED EXERCISE: Find a , b , c and d , if $x^3 - x = a(x-2)^3 + b(x-2)^2 + c(x-2) + d$ for at least four values of x .

SOLUTION: Since they are equal for four values of x , they are identically equal.

$$\text{Substituting } x = 2, \quad 6 = d.$$

$$\text{Equating coefficients of } x^3, \quad 1 = a.$$

$$\text{Substituting } x = 0, \quad 0 = -8 + 4b - 2c + 6$$

$$2b - c = 1.$$

$$\text{Substituting } x = 1, \quad 0 = -1 + b - c + 6$$

$$b - c = -5.$$

Hence $b = 6$ and $c = 11$.

F. Geometrical Implications of the Factor Theorem: Here are some of the geometrical versions of the factor theorem — they are translations of the consequences given above into the language of coordinate geometry. They are simply generalisations of the similar remarks about the graphs of quadratics in Box 25 of Section 8I of the Year 11 volume.

GEOMETRICAL IMPLICATIONS OF THE FACTOR THEOREM:

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1. The graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.
2. The graphs of two distinct polynomial functions cannot intersect in more points than the maximum of the two degrees.
3. A line cannot intersect the graph of a polynomial of degree n in more than n points.

In parts (2) and (3), points where the two curves are tangent to each other count according to their multiplicity.

WORKED EXERCISE: By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x) = x^4 + 4x^3 + 2$ and $Q(x) = x^4 + 3x^3 + 3x$, and find where $P(x)$ is above $Q(x)$.

SOLUTION: Subtracting, $F(x) = x^3 - 3x + 2$.

Substituting, $F(1) = 1 - 3 + 2 = 0$, so $x - 1$ is a factor.

$F(-2) = -8 + 6 + 2 = 0$, so $x + 2$ is a factor.

After long division by $(x - 1)(x + 2) = x^2 + x - 2$,

$$F(x) = (x - 1)^2(x + 2).$$

Hence $y = P(x)$ and $y = Q(x)$ are tangent at $x = 1$, but do not cross there, and intersect also at $x = -2$, where they cross at an angle.

Since $F(x)$ is positive for $-2 < x < 1$ or $x > 1$, and negative for $x < -2$,

$P(x)$ is above $Q(x)$ for $-2 < x < 1$ or $x > 1$, and below it for $x < -2$.

A NOTE FOR 4 UNIT STUDENTS: The *fundamental theorem of algebra* is stated, but cannot be proven, in the 4 Unit course. It tells us that the graph of a polynomial of degree n intersects every line in exactly n points, provided first that points where the curves are tangent are counted according to their multiplicity, and secondly that complex points of intersection are also counted. As its name implies, this most important theorem provides the fundamental link between the algebra of polynomials and the geometry of their graphs, and allows the

degree of a polynomial to be defined either algebraically, as the highest index, or geometrically, as the number of times every line crosses it.

G. Behaviour at Simple and Multiple Zeroes — A Proof: We can now give a satisfactory proof of the theorem stated in Box 7 of Section 4B:

‘Suppose that $x = \alpha$ is a zero of multiplicity $m \geq 1$ of a polynomial $P(x)$.

- If $x = \alpha$ has even multiplicity, the curve is tangent to the x -axis at $x = \alpha$, and does not cross the x -axis there.
- If $x = \alpha$ has odd multiplicity at least 3, the curve is tangent to the x -axis at $x = \alpha$, and crosses the x -axis there at a point of inflexion.
- If $x = \alpha$ is a simple zero, then the curve crosses the x -axis at $x = \alpha$ and is not tangent to the x -axis there.’

PROOF: [The proof here is more suited to those taking the 4 Unit course.]

A. Differentiation is required since tangents are involved.

Let $P(x) = (x - \alpha)^m Q(x)$, where $Q(\alpha) \neq 0$,

that is, where $(x - \alpha)$ is not a factor of $Q(x)$.

Using the product rule, $P'(x) = m(x - \alpha)^{m-1}Q(x) + (x - \alpha)^m Q'(x)$
 $= (x - \alpha)^{m-1}(mQ(x) + (x - \alpha)Q'(x)).$

When $m = 1$, $P'(\alpha) = Q(\alpha)$, which is not zero since $Q(\alpha) \neq 0$,

but when $m \geq 2$, $P'(\alpha) = 0$.

Hence $x = \alpha$ is a zero of $P'(x)$ if and only if $m \geq 2$,

That is, the curve is tangent to the x -axis at $x = \alpha$ if and only if $m \geq 2$.

B. Since $Q(\alpha) \neq 0$, $P(x) = (x - \alpha)^m Q(x)$ will change sign around $x = \alpha$ when m is odd, and will not change sign around $x = \alpha$ when m is even. This completes the proof.

Exercise 4E

1. Use the factor theorem to write down in factored form:
 - (a) a monic cubic polynomial with zeroes -1 , 3 and 4 .
 - (b) a monic quartic polynomial with zeroes 0 , -2 , 3 and 1 .
 - (c) a cubic polynomial with leading coefficient 6 and zeroes at $\frac{1}{3}$, $-\frac{1}{2}$ and 1 .
2. (a) Show that 2 and 5 are zeroes of $P(x) = x^4 - 3x^3 - 15x^2 + 19x + 30$.
 (b) Hence explain why $(x - 2)(x - 5)$ is a factor of $P(x)$.
 (c) Divide $P(x)$ by $(x - 2)(x - 5)$ and hence express $P(x)$ as the product of four linear factors.
3. Use trial and error to find as many integer zeroes of $P(x)$ as possible. Use long division to divide $P(x)$ by the product of the known factors and hence express $P(x)$ in factored form.

(a) $P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$	(c) $P(x) = 6x^4 - 25x^3 + 17x^2 + 28x - 20$
(b) $P(x) = 2x^4 - 5x^3 - 5x^2 + 20x - 12$	(d) $P(x) = 9x^4 - 51x^3 + 85x^2 - 41x + 6$
4. (a) The polynomial $(a - 2)x^2 + (1 - 3b)x + (5 - 2c)$ has three zeroes. What are the values of a , b and c ?
 (b) The polynomial $(a + 1)x^3 + (b - 3)x^2 + (2c - 1)x + (5 - 4d)$ has four zeroes. What are the values of a , b , c and d ?

DEVELOPMENT

5. (a) If $3x^2 - 4x + 7 \equiv a(x + 2)^2 + b(x + 2) + c$, find a , b and c .
 (b) If $2x^3 - 8x^2 + 3x - 4 \equiv a(x - 1)^3 + b(x - 1)^2 + c(x - 1) + d$, find a , b , c and d .
 (c) Use similar methods to express $x^3 + 2x^2 - 3x + 1$ as a polynomial in $(x + 1)$.
 (d) If the polynomials $2x^2 + 4x + 4$ and $a(x + 1)^2 + b(x + 2)^2 + c(x + 3)^2$ are equal for three values of x , find a , b and c .
6. (a) A polynomial of degree 3 has a double zero at 2. When $x = 1$ it takes the value 6 and when $x = 3$ it takes the value 8. Find the polynomial.
 (b) Two zeroes of a polynomial of degree 3 are 1 and -3 . When $x = 2$ it takes the value -15 and when $x = -1$ it takes the value 36. Find the polynomial.
7. Show that $x^2 - 3x + 2$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$, where m and n are positive integers.
8. If two polynomials have degrees m and n respectively, what is the maximum number of points that their graphs can have in common?
9. Explain why a cubic with three distinct zeroes must have two turning points.
10. The line $y = k$ meets the curve $y = ax^3 + bx^2 + cx + d$ four times. Find the values of a , b , c , d and k .
11. Find the turning points of the following polynomials and hence state how many zeroes they have.

(a) $12x - x^3 + 3$	(c) $4x^3 - x^4 - 1$
(b) $7 + 5x - x^2 - x^3$	(d) $3x^4 - 4x^3 + 2$
12. By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x)$ and $Q(x)$.

(a) $P(x) = 2x^3 - 4x^2 + 3x + 1$, $Q(x) = x^3 + x^2 - 8$	(b) $P(x) = x^4 + x^3 + 10x - 4$, $Q(x) = x^4 + 7x^2 - 6x + 8$
(c) $P(x) = -2x^3 + 3x^2 - 25$, $Q(x) = -3x^3 - x^2 + 11x + 5$	(d) $P(x) = x^4 - 3x^2 - 2$, $Q(x) = x^3 - 5x$
(e) $P(x) = x^4 + 4x^3 - x + 5$, $Q(x) = x^3 - 3x^2 - 2x + 5$	
13. Suppose that the polynomial equation $P(x) = 0$ has a double root at $x = \alpha$. That is, $P(x) = (x - \alpha)^2 Q(x)$, for some polynomial $Q(x)$, where $Q(\alpha) \neq 0$.
 - (a) Find $P'(x)$ and show that $P'(\alpha) = 0$.
 - (b) If $x = 1$ is a double root of $x^4 + ax^3 + bx^2 - 5x + 1 = 0$, find a and b .
 - (c) Given that $P(x) = 2x^4 - 20x^3 + 74x^2 - 120x + 72$, find $P'(x)$ and hence show that 2 and 3 are double roots of $P(x)$. Factor $P(x)$ completely.

EXTENSION

14. Show that if the polynomials $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - x + a$ have a common factor of degree 2, then $a + b = 0$.
15. [Wallis' product and sine as an infinite product] This extraordinary expression of $\sin \pi x$ as an infinite product result is not possible for us to prove rigorously:

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2}\right) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right) \left(1 - \frac{x^2}{4^2}\right) \cdots$$

- (a) Here is a wildly invalid, but still interesting, justification inspired by the factor theorem for polynomials. First, the function $\sin \pi x$ is zero at every integer value of x , so regarding $\sin \pi x$ as a sort of polynomial of infinite degree, $(x - n)$ must be a factor for all n . Writing this another way, x is a factor, and $\left(1 - \frac{x^2}{n^2}\right)$ is a factor for all n . Secondly, the constant multiple π can be justified (invalidly again) because $\sin \pi x \rightarrow \pi x$ as $x \rightarrow 0$, and each of the other factors on the RHS has limit 1 as $x \rightarrow 0$.
- (b) By substituting $x = \frac{1}{2}$ into the expression in part (a), derive from it the identity called *Wallis' product* (which is, in contrast, accessible by methods of the 4 Unit course):

$$\frac{\pi}{2} = \frac{2^2}{1 \times 3} \times \frac{4^2}{3 \times 5} \times \frac{6^2}{5 \times 7} \times \frac{8^2}{7 \times 9} \times \cdots = \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \cdots,$$

and use a calculator or computer to investigate the speed of convergence.

- (c) By other substitutions into part (a), and using part (b), prove that

$$\begin{aligned}\sqrt{2} &= \frac{2^2}{1 \times 3} \times \frac{6^2}{5 \times 7} \times \frac{10^2}{9 \times 11} \times \frac{14^2}{13 \times 15} \times \cdots = \frac{4}{3} \times \frac{36}{35} \times \frac{100}{99} \times \frac{196}{195} \times \cdots \\ \frac{3}{2} &= \frac{2^2}{1 \times 3} \times \frac{4^2}{3 \times 5} \times \frac{8^2}{7 \times 9} \times \frac{10^2}{9 \times 11} \times \frac{14^2}{13 \times 15} \times \cdots = \frac{4}{3} \times \frac{16}{15} \times \frac{64}{63} \times \frac{100}{99} \times \frac{196}{195} \times \cdots\end{aligned}$$

4F The Zeroes and the Coefficients

We have already shown in Chapter Eight of the Year 11 volume that if a quadratic $P(x) = ax^2 + bx + c$ has zeroes α and β , then their sum $\alpha + \beta$ and their product $\alpha\beta$ can easily be calculated from the coefficients without ever finding α or β themselves.

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SUM AND PRODUCT OF ZEROES OF A QUADRATIC:

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = +\frac{c}{a}.$$

This section will generalise these results to polynomials of arbitrary degree. The general result is a little messy to state, so we shall deal with quadratic, cubic and quartic polynomials first.

The Zeroes of a Quadratic: Reviewing the work in Chapter Eight of the Year 11 volume, suppose that α and β are the zeroes of a quadratic $P(x) = ax^2 + bx + c$. By the factor theorem (see Box 15), $P(x)$ is a multiple of the product $(x - \alpha)(x - \beta)$:

$$\begin{aligned}P(x) &= a(x - \alpha)(x - \beta) \\ &= ax^2 - a(\alpha + \beta)x + a\alpha\beta\end{aligned}$$

Now equating terms in x and constants gives the results obtained before:

$$\begin{aligned}-a(\alpha + \beta) &= b & a(\alpha\beta) &= c \\ \alpha + \beta &= -\frac{b}{a} & \text{and} & \alpha\beta = \frac{c}{a}\end{aligned}$$

The Zeroes of a Cubic: Suppose now that the cubic polynomial

$$P(x) = ax^3 + bx^2 + cx + d$$

has zeroes α , β and γ . Again by the factor theorem (see Box 15), $P(x)$ is a multiple of the product $(x - \alpha)(x - \beta)(x - \gamma)$:

$$\begin{aligned} P(x) &= a(x - \alpha)(x - \beta)(x - \gamma) \\ &= ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \gamma\alpha)x - a\alpha\beta\gamma \end{aligned}$$

Now equating coefficients of terms in x^2 , x and constants gives the new results:

22	ZEROES AND COEFFICIENTS OF A CUBIC: $\alpha + \beta + \gamma = -\frac{b}{a}$ $\alpha\beta + \beta\gamma + \gamma\alpha = +\frac{c}{a}$ $\alpha\beta\gamma = -\frac{d}{a}$

The middle formula is best read as ‘the sum of the products of pairs of zeroes’.

The Zeroes of a Quartic: Suppose that the four zeroes of the quartic polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are α , β , γ and δ . By the factor theorem (see Box 15), $P(x)$ is a multiple of the product $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$:

$$\begin{aligned} P(x) &= a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \\ &= ax^4 - a(\alpha + \beta + \gamma + \delta)x^3 + a(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 \\ &\quad - a(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)x + a\alpha\beta\gamma\delta. \end{aligned}$$

Equating coefficients of terms in x^3 , x^2 , x and constants now gives:

23	ZEROES AND COEFFICIENTS OF A QUARTIC: $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$ $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = +\frac{c}{a}$ $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a}$ $\alpha\beta\gamma\delta = +\frac{e}{a}$

The second formula gives ‘the sum of the products of pairs of zeroes’, and the third formula gives ‘the sum of the products of triples of zeroes’.

The General Case: Apart from the sum and product of zeroes, notation is a major difficulty here, and the results are better written in sigma notation. Suppose that the n zeroes of the degree n polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

are $\alpha_1, \alpha_2, \dots, \alpha_n$. Using similar methods gives:

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ZEROES AND COEFFICIENTS OF A POLYNOMIAL: $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$

$$\sum_{i < j} \alpha_i \alpha_j = +\frac{a_{n-2}}{a_n}$$

$$\sum_{i < j < k} \alpha_i \alpha_j \alpha_k = -\frac{a_{n-3}}{a_n}$$

.....

$$\alpha_1 \alpha_2 \cdots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

It is unlikely that anything apart from the first and last formulae would be required.

WORKED EXERCISE: Let α , β and γ be the roots of the cubic equation $x^3 - 3x + 2 = 0$. Use the formulae above to find:

- (a) $\alpha + \beta + \gamma$ (c) $\alpha\beta + \beta\gamma + \gamma\alpha$ (e) $\alpha^2 + \beta^2 + \gamma^2$
 (b) $\alpha\beta\gamma$ (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ (f) $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$

Check the result with the factorisation $x^3 - 3x + 2 = (x - 1)^2(x + 2)$ obtained in the last worked exercise of the previous section.

SOLUTION:

- (a) $\alpha + \beta + \gamma = \frac{-0}{1} = 0$ (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$
 (b) $\alpha\beta\gamma = \frac{-2}{1} = -2$ $= \frac{-3}{-2} = \frac{3}{2}$
 (c) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-3}{1} = -3$
 (e) $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$,
 so $0^2 = \alpha^2 + \beta^2 + \gamma^2 + 2 \times (-3)$
 $\alpha^2 + \beta^2 + \gamma^2 = 6$.
 (f) $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$
 $= \alpha\beta(\alpha + \beta + \gamma) + \beta\gamma(\beta + \gamma + \alpha) + \gamma\alpha(\gamma + \alpha + \beta) - 3\alpha\beta\gamma$
 $= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$
 $= (-3) \times 0 - 3 \times (-2)$
 $= 6$

Since $x^3 - 3x + 2 = (x - 1)^2(x + 2)$, the actual roots are 1, 1 and -2 , hence

- (a) $\alpha + \beta + \gamma = 1 + 1 - 2 = 0$ (e) $\alpha^2 + \beta^2 + \gamma^2 = 1 + 1 + 4 = 6$
 (b) $\alpha\beta\gamma = 1 \times 1 \times (-2) = -2$ (f) $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$
 $= 1 \times 1 + 1 \times 1 + 1 \times (-2)$
 $+ 1 \times 4 + 4 \times 1 + (-2) \times 1$
 $= 6$,
 (c) $\alpha\beta + \beta\gamma + \gamma\alpha = 1 - 2 - 2 = -3$
 (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1 + 1 - \frac{1}{2} = 1\frac{1}{2}$

all of which agree with the previous calculations.

Factoring Polynomials Using the Factor Theorem and the Sum and Product of Zeroes:

Long division can be avoided in many situations by applying the sum and product of zeroes formulae after one or more zeroes have been found. The full menu for the 3 Unit course now runs as follows:

FACTORING POLYNOMIALS — THE FULL 3 UNIT MENU:**25**

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Use sum and product of zeroes to find the other zeroes.
- Alternatively, use long division of $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

In the following worked exercise, we factor a polynomial factored twice already, but this time there is no need for any long division.

WORKED EXERCISE: Factor $F(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION: As before, $F(1) = 1 + 1 - 9 + 11 - 4 = 0$,
and $F(-4) = 256 - 64 - 144 - 44 - 4 = 0$.

Let the zeroes be 1, -4 , α and β .

$$\begin{aligned}\text{Then } \alpha + \beta + 1 - 4 &= -1 \\ \alpha + \beta &= 2.\end{aligned}\tag{1}$$

$$\begin{aligned}\text{Also } \alpha\beta \times 1 \times (-4) &= -4 \\ \alpha\beta &= 1.\end{aligned}\tag{2}$$

From (1) and (2), $\alpha = \beta = 1$, and so $F(x) = (x - 1)^3(x + 4)$.

WORKED EXERCISE: Factor completely the cubic $G(x) = x^3 - x^2 - 4$.

SOLUTION: First, $G(2) = 8 - 4 - 4 = 0$.

Let the zeroes be 2, α and β .

$$\begin{aligned}\text{Then } 2 + \alpha + \beta &= 1 \\ \alpha + \beta &= -1,\end{aligned}\tag{1}$$

$$\begin{aligned}\text{and } 2 \times \alpha \times \beta &= 4 \\ \alpha\beta &= 4.\end{aligned}\tag{2}$$

$$\begin{aligned}\text{Substituting (1) into (2), } \alpha(-1 - \alpha) &= 2 \\ \alpha^2 + \alpha + 2 &= 0\end{aligned}$$

This is an irreducible quadratic, because $\Delta = -7$,
so the complete factorisation is $G(x) = (x - 2)(x^2 + x + 2)$.

NOTE: This procedure — developing the irreducible quadratic factor from the sum and product of zeroes — is really little easier than the long division it avoids.

Forming Identities with the Coefficients: If some information can be gained about the roots of a polynomial equation, it may be possible to form an identity with the coefficients of the polynomial.

WORKED EXERCISE: If one root of the cubic $f(x) = ax^3 + bx^2 + cx + d$ is the opposite of another, prove that $ad = bc$.

SOLUTION: Let the zeroes be α , $-\alpha$ and β .

$$\begin{aligned}\text{First, } \alpha - \alpha + \beta &= -\frac{b}{a} \\ a\beta &= -b.\end{aligned}\tag{1}$$

$$\begin{aligned}\text{Secondly, } -\alpha^2 - \alpha\beta + \beta\alpha &= \frac{c}{a} \\ a\alpha^2 &= -c.\end{aligned}\tag{2}$$

Thirdly,
$$-\alpha^2\beta = -\frac{d}{a}$$

$$a\alpha^2\beta = d. \quad (3)$$

Taking $(1) \times (2) \div (3)$,
$$a = \frac{bc}{d}$$

$$ad = bc, \text{ as required.}$$

A NOTE FOR 4 UNIT STUDENTS: The 4 Unit course develops one further technique for factoring polynomials. It is proven that if α is a zero of $P(x)$, then α is a zero of $P'(x)$ if and only if it is at least a double zero of $P(x)$. Thus multiple zeroes can be uncovered by testing whether a known zero is also a zero of the first derivative. Question 14 in Exercise 4E presented this idea, and it was implicit in paragraph G of Section 4E, but it is not required at 3 Unit level.

Exercise 4F

- If α and β are the roots of the quadratic equation $x^2 - 4x + 2 = 0$, find:

(a) $\alpha + \beta$	(d) $\frac{1}{\alpha} + \frac{1}{\beta}$	(g) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$
(b) $\alpha\beta$	(e) $(\alpha + 2)(\beta + 2)$	(h) $\alpha\beta^3 + \alpha^3\beta$
(c) $\alpha^2\beta + \alpha\beta^2$	(f) $\alpha^2 + \beta^2$	(i) $\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$
- If α, β and γ are the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$, find:

(a) $\alpha + \beta + \gamma$	(d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$	(g) $(\alpha\beta)^2\gamma + (\alpha\gamma)^2\beta + (\beta\gamma)^2\alpha$
(b) $\alpha\beta + \alpha\gamma + \beta\gamma$	(e) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$	(h) $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha\beta\gamma$	(f) $(\alpha + 1)(\beta + 1)(\gamma + 1)$	(i) $(\alpha\beta)^{-2} + (\alpha\gamma)^{-2} + (\beta\gamma)^{-2}$

Now find the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$ by factoring the LHS. Hence check your answers for expressions (a)–(i).
- If a_1, a_2, a_3 and a_4 are the roots of the equation $x^4 - 5x^3 + 2x^2 - 4x - 3 = 0$, find:

(a) $\sum_i a_i$	(c) $\sum_{i < j < k} a_i a_j a_k$	(e) $\sum_i (a_i)^{-1}$	(g) $\sum_{i < j < k} (a_i a_j a_k)^{-1}$
(b) $\sum_{i < j} a_i a_j$	(d) $a_1 a_2 a_3 a_4$	(f) $\sum_{i < j} (a_i a_j)^{-1}$	(h) $\sum_i (a_i)^2$
- In each of the following questions, find each coefficient in turn by considering the sums and products of the roots.
 - Form a quadratic equation with roots -3 and 2 .
 - Form a cubic equation with roots $-3, 2$ and 1 .
 - Form a quartic equation with roots $-3, 2, 1$ and -1 .
- Show that $x = 1$ and $x = -2$ are zeroes of $P(x)$, and use the sum and product of zeroes to find the other one or two zeroes. Note any multiple zeroes.

(a) $P(x) = x^3 - 2x^2 - 5x + 6$	(c) $P(x) = x^4 + 3x^3 - 3x^2 - 7x + 6$
(b) $P(x) = 2x^3 + 3x^2 - 3x - 2$	(d) $P(x) = 3x^4 - 5x^3 - 10x^2 + 20x - 8$
- Use trial and error to find two integer zeroes of $F(x)$. Then use the sum and product of zeroes to find any other zeroes. Note any multiple zeroes.

- (a) $F(x) = x^4 - 6x^2 - 8x - 3$ (c) $F(x) = x^4 - 8x^3 + 6x^2 + 40x + 25$
 (b) $F(x) = x^4 - 15x^2 + 10x + 24$ (d) $F(x) = x^4 + x^3 - 3x^2 - 4x - 4$

DEVELOPMENT

7. If α and β are the roots of the equation $2x^2 + 5x - 4 = 0$, find:
 (a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$ (e) $|\alpha - \beta|$
8. Consider the polynomial $P(x) = x^3 - x^2 - x + 10$.
 (a) Show that -2 is a zero of $P(x)$.
 (b) Given that the zeroes of $P(x)$ are -2 , α and β , show that $\alpha + \beta = 3$ and $\alpha\beta = 5$.
 (c) Solve simultaneously the two equations in part (b) (you will need to form a quadratic in α), and hence show that there are no such real numbers α and β .
 (d) Hence state how many times the graph of the cubic crosses the x -axis.
9. (a) Suppose that $x - 3$ and $x + 1$ are factors of $x^3 - 6x^2 + ax + b$. Find a and b , and hence use sum and product of zeroes to factor the polynomial.
 (b) Suppose that $2x^3 + ax^2 - 14x + b$ has zeroes at -2 and 4 . Find a and b , and hence use sum and product of zeroes to find the other zero.
10. (a) Find values of a and b for which $x^3 + ax^2 - 10x + b$ is exactly divisible by $x^2 + x - 12$, and then factor the cubic.
 (b) Find values of a and b for which $x^2 - x - 20$ is a factor of $x^4 + ax^3 - 23x^2 + bx + 60$, and then find all the zeroes.
11. The polynomial $P(x) = x^3 - Lx^2 + Lx - M$ has zeroes α , $\frac{1}{\alpha}$ and β .
 (a) Show that: (i) $\alpha + \frac{1}{\alpha} + \beta = L$ (ii) $1 + \alpha\beta + \frac{\beta}{\alpha} = L$ (iii) $\beta = M$
 (b) Show that either $M = 1$ or $M = L - 1$.
12. The cubic equation $x^3 - Ax^2 + 3A = 0$, where $A > 0$, has roots α , β and $\alpha + \beta$.
 (a) Use the sum of the roots to show that $\alpha + \beta = \frac{1}{2}A$.
 (b) Use the sum of the products of pairs of roots to show that $\alpha\beta = -\frac{1}{4}A^2$.
 (c) Show that $A = 2\sqrt{6}$.
13. (a) Find the roots of the equation $4x^3 - 8x^2 - 3x + 9 = 0$, given that two of the roots are equal. [HINT: Let the roots be α , α and β .]
 (b) Find the roots of the equation $3x^3 - x^2 - 48x + 16 = 0$, given that the sum of two of the roots is zero. [HINT: Let the roots be α , $-\alpha$ and β .]
 (c) Find the roots of the equation $2x^3 - 5x^2 - 46x + 24 = 0$, given that the product of two of the roots is 3. [HINT: Let the roots be α , $\frac{3}{\alpha}$ and β .]
 (d) Find the zeroes of the polynomial $P(x) = 2x^3 - 13x^2 + 22x - 8$, given that one zero is the product of the other two. [HINT: Let the zeroes be α , β and $\alpha\beta$.]
14. (a) Find the roots of the equation $9x^3 - 27x^2 + 11x + 7 = 0$, if the roots form an arithmetic sequence. [HINT: Let the roots be $\alpha - d$, α and $\alpha + d$.] Then find the point of inflexion of $y = 9x^3 - 27x^2 + 11x + 7$, and show that its x -coordinate is one of the roots.
 (b) Find the zeroes of the polynomial $P(x) = 8x^3 - 14x^2 + 7x - 1$, if the zeroes form a geometric sequence. [HINT: Let the zeroes be $\frac{\alpha}{r}$, α and αr .]

- (c) Solve the equation $2x^3 - 3x^2 - 3x + 2 = 0$, given that the roots are in geometric progression.
15. (a) Two of the roots of the equation $x^3 + 3x^2 - 4x + a = 0$ are opposites. Find the value of a and the three roots.
(b) Two of the roots of the equation $4x^3 + ax^2 - 47x + 12 = 0$ are reciprocals. Find the value of a and the three roots.
(c) Find α and β if the zeroes of the polynomial $x^4 - 3x^3 - 8x^2 + 12x + 16 = 0$ are $\alpha, 2\alpha, \beta$ and 2β .
16. (a) If one root of the equation $x^3 - bx^2 + cx - d = 0$ is equal to the product of the other two, show that $(c + d)^2 = d(b + 1)^2$.
(b) If the roots of the equation $x^3 + ax^2 + bx + c = 0$ form an arithmetic sequence, show that $9ab = 2a^3 + 27c$, and that one of the roots is $-\frac{1}{3}a$.
(c) If the roots of the equation $x^3 + ax^2 + bx + c = 0$ form a geometric sequence, show that $b^3 = a^3c$, and that one of the roots is $-\sqrt[3]{c}$.
(d) The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has two zeroes which are opposites and two zeroes which are reciprocals. Show that:
(i) $b = 1 + d$ (ii) $c = ad$
(e) If the zeroes of the cubic $y = x^3 + ax^2 + bx + c = 0$ form an arithmetic sequence, show that the point of inflexion lies on the x -axis.
17. Consider the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$, which has zeroes α, β and γ such that $\gamma = \alpha + \beta$.
(a) Show that $\alpha + \beta = -\frac{b}{2a}$. (b) Show that $\alpha\beta = \frac{2d}{b}$.
(c) Hence show that α and β are also the roots of the equation $2abx^2 + b^2x + 4ad = 0$.
18. (a) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β and γ . Evaluate:
(i) $\alpha + \beta + \gamma$ (iii) $\alpha\beta\gamma$
(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ (iv) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$
(b) The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a, \cos b$ and $\cos c$. Using the results in (a), prove that $\sec a + \sec b + \sec c = 1$.
19. (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
(b) Show that the cubic equation $8x^3 - 6x + 1 = 0$ reduces to the form $\cos 3\theta = -\frac{1}{2}$ by substituting $x = \cos\theta$.
(c) Hence find the three solutions to the cubic equation.
(d) Use the sum and product of roots to evaluate:
(i) $\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} - \cos \frac{\pi}{9}$ (iii) $\sec \frac{2\pi}{9} + \sec \frac{4\pi}{9} - \sec \frac{\pi}{9}$
(ii) $\cos \frac{2\pi}{9} \cos \frac{4\pi}{9} \cos \frac{\pi}{9}$ (iv) $\cos^2 \frac{2\pi}{9} + \cos^2 \frac{4\pi}{9} + \cos^2 \frac{\pi}{9}$

EXTENSION

- 20.** If α , β and γ are the roots of the equation $x^3 + 5x - 4 = 0$, evaluate $\alpha^3 + \beta^3 + \gamma^3$.
- 21.** If $x^n - 1 = 0$ has n roots, $\alpha_1, \alpha_2, \dots, \alpha_n$, what is $(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_n)$?
- 22.** Suppose that the equation $x^3 + ax^2 + bx + c = 0$ has roots α, β and γ . If $\gamma = \alpha + \beta$, show that $a^3 - 4ab + 8c = 0$. [HINT: Use your work in question 17.]
- 23.** If $y = x^4 + bx^3 + cx^2 + dx + e$ has two double zeroes, express d and e in terms of b and c .

24. Suppose that $P(x) = x^3 + cx + d$ has zeroes α , β and γ . Show that:

(a) $\alpha\beta = c + \gamma^2$ (b) $(\alpha - \beta)^2 = -3\gamma^2 - 4c$ (c) $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -4c^3 - 27d^2$
 Check these results for the polynomial $P(x) = (x - 1)(x - 2)(x + 3)$.

NOTE: For any monic cubic, the expression $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2$ is called the *discriminant*. Students taking the 4 Unit course may like to prove that the cubic has three distinct real zeroes if and only if the discriminant is positive.

4G Geometry using Polynomial Techniques

This final section adds the methods of the preceding sections, particularly the sum and product of roots, to the available techniques for studying the geometry of various curves. The standard technique is to examine the roots of the equation formed in the process of solving two curves simultaneously.

Midpoints and Tangents: When two curves intersect, we can form the equation whose solutions are the x - or y -coordinates of points of intersection of the two curves. The midpoint of two points of intersection can then be found using the average of the roots. Tangents can be identified as corresponding to double roots.

The following worked exercise could also be done using quadratic equations, but it is a very clear example of the use of sum and product of roots.

WORKED EXERCISE: The line $y = 2x$ meets the parabola $y = x^2 - 2x - 8$ at the two points $A(\alpha, 2\alpha)$ and $B(\beta, 2\beta)$.

- Show that α and β are roots of $x^2 - 4x - 8 = 0$, and hence find the coordinates of the midpoint M of AB .
- Use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to find the horizontal distance $|\alpha - \beta|$ from A to B . Then use Pythagoras' theorem and the gradient of the line to find the length of AB .
- Find the value of b for which $y = 2x + b$ is a tangent to the parabola, and find the point T of contact.

SOLUTION:

- (a) Solving the line and the parabola simultaneously,

$$x^2 - 2x - 8 = 2x$$

$$x^2 - 4x - 8 = 0.$$

Hence $\alpha + \beta = 4,$

and $\alpha\beta = -8.$

Averaging the roots, M has x -coordinate $x = 2$,
 and substituting into the line, $M = (2, 4)$.

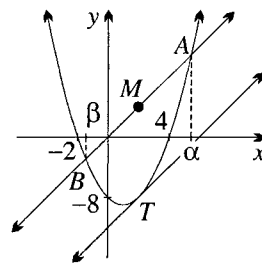
- (b) We know that
$$\begin{aligned} (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= 16 + 32 \\ &= 48 \end{aligned}$$

and so $|\alpha - \beta| = 4\sqrt{3}.$

Since the line has gradient 2, the vertical distance is $8\sqrt{3}$,

so using Pythagoras,
$$\begin{aligned} AB^2 &= (4\sqrt{3})^2 + (8\sqrt{3})^2 \\ &= 16 \times 15 \end{aligned}$$

$$AB = 4\sqrt{15}.$$



- (c) Solving $y = x^2 - 2x - 8$ and $y = 2x + b$ simultaneously,

$$x^2 - 4x - (8 + b) = 0$$

Since the line is a tangent, let the roots be θ and θ .

Then using the sum of roots, $\theta + \theta = 4$,

so $\theta = 2$,

Using the product of roots, $\theta^2 = -8 - b$

and since $\theta = 2$, $b = -12$.

So the line $y = 2x - 12$ is a tangent at $T(2, -8)$.

Locus Problems Using Sum and Product of Roots: The preceding theory can make some rather obscure-looking locus problems quite straightforward.

WORKED EXERCISE: A line through the point $P(-1, 0)$ crosses the cubic $y = x^3 - x$ at two further points A and B .

- (a) Sketch the situation, and find the locus of the midpoint M of AB .
 (b) Find the line through P tangent to the cubic at a point distinct from P .

SOLUTION:

- (a) Let $y = m(x + 1)$ be a general line through $P(-1, 0)$.

Solving the line simultaneously with the cubic,

$$x^3 - x = mx + m$$

$$x^3 - (m + 1)x - m = 0.$$

Let the x -coordinates of A and B be α and β respectively.

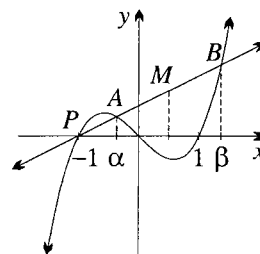
Then $\alpha + \beta + (-1) = 0$, using the sum of roots,

$$\alpha + \beta = 1.$$

Hence the midpoint M of AB has x -coordinate $\frac{1}{2}(\alpha + \beta) = \frac{1}{2}$,
 and the locus of M is therefore the line $x = \frac{1}{2}$.

But the line does not extend below the cubic, so $y \geq -\frac{3}{8}$.

- (b) For the line to be a tangent, $\alpha = \beta$,
 and since $\alpha + \beta = 1$, we must have $\alpha = \beta = \frac{1}{2}$.
 Using the product of roots, $\alpha \times \beta \times (-1) = m$,
 so $m = -\frac{1}{4}$, and the line is $y = -\frac{1}{4}(x + 1)$.



Exercise 4G

NOTE: These are geometrical questions, and sketches should be drawn every time — the algebraic result should look reasonable on the diagram. Questions 1–11 have been carefully structured to indicate the intended methods, and questions 12–15 should be done by similar methods.

- (a) Show that the x -coordinates of the points of intersection of the parabola $y = x^2 - 6x$ and the line $y = 2x - 16$ satisfy the equation $x^2 - 8x + 16 = 0$.
 (b) Solve this equation, and hence show that the line is a tangent to the parabola. Find the point T of contact.
- (a) Show that the x -coordinates of the points of intersection of the line $y = b - 2x$ and the parabola $y = x^2 - 6x$ satisfy the equation $x^2 - 4x - b = 0$.
 (b) Suppose now that the line is a tangent to the parabola, so that the roots are α and α .

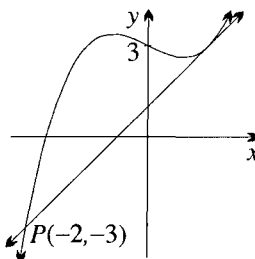
- (i) Using the sum of roots, show that $\alpha = 2$.
 - (ii) Using the product of roots, show that $\alpha^2 = -b$, and hence find b .
 - (iii) Find the equation of the tangent and its point T of contact.
3. The line $y = x + 1$ meets the parabola $y = x^2 - 3x$ at A and B .
- (a) Show that the x -coordinates α and β of A and B satisfy the equation $x^2 - 4x - 1 = 0$.
 - (b) Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB .
 - (c) Use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to show that the horizontal distance $|\alpha - \beta|$ between A and B is $2\sqrt{5}$.
 - (d) Use the gradient to explain why the vertical distance between A and B is also $2\sqrt{5}$, and hence use Pythagoras' theorem to find the length AB .

NOTE: Sketches in question 4–6 require the factorisation $x^3 - 5x^2 + 6x = x(x-2)(x-3)$.

4. (a) Show that the x -coordinates of the points of intersection of the line $y = 3 - x$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + 7x - 3 = 0$.
- (b) Show that $x = 1$ and $x = 3$ are roots, and use the sum of roots to find the third root.
- (c) Explain why the line is a tangent to the cubic, find the point of contact and the other point of intersection.
5. (a) Show that the x -coordinates of the points of intersection of the line $y = mx$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + (6 - m)x = 0$.
- (b) Suppose now that the line is a tangent to the cubic at a point other than the origin, so that the roots are 0 , α and α .
- (i) Using the sum of roots, show that $\alpha = 2\frac{1}{2}$.
 - (ii) Using the product of pairs of roots, show that $\alpha^2 = 6 - m$, and hence find m .
 - (iii) Find the equation of the tangent and its point T of contact.
6. The line $y = x - 2$ meets the cubic $y = x^3 - 5x^2 + 6x$ at $F(2, 0)$, and also at A and B .
- (a) Show that the x -coordinates α and β of A and B satisfy $x^3 - 5x^2 + 5x + 2 = 0$.
 - (b) Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB .
 - (c) Show that $\alpha\beta = -1$, then use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to show that the horizontal distance $|\alpha - \beta|$ between A and B is $\sqrt{13}$.
 - (d) Hence use Pythagoras' theorem to find the length AB .

DEVELOPMENT

7. Suppose that the cubic $F(x) = x^3 + ax^2 + bx + c$ has a relative minimum at $x = \alpha$ and a relative maximum at $x = \beta$.
- (a) By examining the zeroes of $F'(x)$, prove that $\alpha + \beta = -\frac{2}{3}a$.
 - (b) Deduce that the point of inflexion occurs at $x = \frac{1}{2}(\alpha + \beta)$.
8. A line is drawn from the point $A(-1, -7)$ on the curve $y = x^3 - 3x^2 + 4x + 1$ to touch the curve again at P .
- (a) Write down the equation of the line, given that it has gradient m .
 - (b) Find the cubic equation whose roots represent the x -coordinates of the points of intersection of the line and the curve.
 - (c) Explain why the roots of this equation are -1 , α and α , and hence find the point T of contact and the value of m .

9. The point $P(p, p^3)$ lies on the curve $y = x^3$. A straight line through P cuts the curve again at A and B .
- Find the equation of the straight line through P if it has gradient m .
 - Show that the x -coordinates of A and B satisfy the equation $x^3 - mx + mp - p^3 = 0$.
 - Hence find the x -coordinate of the midpoint M of AB , and show that for fixed p , M always lies on a line that is parallel to the y -axis.
10. (a) The cubic $x^3 - (m+1)x + (6-2m) = 0$ has a root at $x = -2$ and a double root at $x = \alpha$. Find m and α .
- (b) Write down the equation of the line ℓ passing through the point $P(-2, -3)$ with gradient m .
- (c) The diagram shows the curve $y = x^3 - x + 3$ and the point $P(-2, -3)$ on the curve. The line ℓ cuts the curve at P , and is tangent to the curve at another point A on the curve. Find the equation of the line ℓ .
- 
11. (a) Use the factor theorem to factor the polynomial $y = x^4 - 4x^3 - 9x^2 + 16x + 20$, given that there are four distinct zeroes, then sketch the curve.
- (b) The line $\ell: y = mx + b$ touches the quartic $y = x^4 - 4x^3 - 9x^2 + 16x + 20$ at two distinct points A and B . Explain why the x -coordinates α and β of A and B are double roots of $x^4 - 4x^3 - 9x^2 + (16-m)x + (20-b) = 0$.
- (c) Use the theory of the sum and product of roots to write down four equations involving α , β , m and b .
- (d) Hence find m and b , and write down the equation of ℓ .
12. (a) Find k and the points of contact if the parabola $y = x^2 - k$ touches the quartic $y = x^4$ at two points.
- (b) Find $c > 0$ and the points of contact if the hyperbola $xy = c^2$ touches the cubic $y = -x^3 + x$ at two points.
- (c) Find k and the point T of contact if the parabola $y = x^2 - k$ touches the cubic $y = x^3$.
- (d) Find α if the quadratic $y = ax(x-1)$ is tangent to the circle $x^2 + y^2 = 1$ at $x = \alpha$. [HINT: The curves always intersect when $x = 1$.]
13. (a) The variable line $y = 3x + b$ with gradient 3 meets the circle $x^2 + y^2 = 16$ at A and B . Find the locus of the midpoint M of AB .
- (b) The fixed point $F(0, 2)$ lies inside the circle $x^2 + y^2 = 16$. A variable line ℓ through F meets the circle at A and B . Find and describe the locus of the midpoint M of AB .
- (c) The parabola $y = x(x-a)$ meets the cubic $y = x^3 - 3x^2 + 2x$ at $O(0, 0)$, A and B . Find, including any restrictions, the locus of the midpoint M of AB as a varies.
- (d) The line $y = mx + b$ meets the hyperbola $xy = 1$ at A and B . Find the locus of the midpoint M of AB if: (i) m is constant, (ii) b is constant.
- (e) The parabola with vertex at $F(-1, -1)$ meets the hyperbola $xy = 1$ again at A and B . Find the locus of the midpoint M of AB .
14. (a) Find a and the points of contact if the parabola $y = x^2 - a$ touches the circle $x^2 + y^2 = 1$ at two distinct points.
- (b) Find a and any other points of intersection if the parabola $y = x^2 - a$ touches the circle $x^2 + y^2 = 1$ at exactly one point.
- (c) Find a if the circle $x^2 + (y-a)^2 = a^2$ intersects the parabola $y = x^2$ at a point which is a fourfold zero of the quartic formed when solving the two equations simultaneously.

- (d) By solving the line $y = mx + b$ simultaneously with the cubic $y = x^3 - 6x^2 - 2x + 1$ and insisting that there be a triple root, find the point of inflexion of the cubic without using calculus.
- (e) Find the line which touches the quartic $y = x^2(x - 2)(x - 6)$ at two distinct points A and B , and find the distance AB .
15. [A cubic has odd symmetry in its point of inflexion.] The line $y = mx + n$ meets the cubic $y = ax^3 + bx^2 + cx + d$ in three distinct points A , B and C . Show that if $AB = BC$, then B is the point of inflexion.

EXTENSION

16. A circle passing through the origin O is tangent to the hyperbola $xy = 1$ at A , and intersects the hyperbola again at two distinct points B and C . Prove that $OA \perp BC$.
17. The diagram to the right shows the circle $x^2 + y^2 = 1$ and the parabola $y = (\lambda x - 1)(x - 1)$, where λ is a constant. The circle and parabola meet in the four points

$$P(1, 0), \quad Q(0, 1), \quad A(\alpha, \phi), \quad B(\beta, \psi).$$

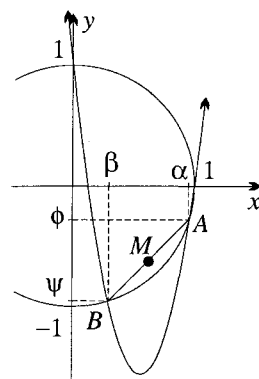
The point M is the midpoint of the chord AB .

- (a) Show that the x -coordinates of the points of intersection of the two curves satisfy the equation

$$\lambda^2 x^4 - 2\lambda(1 + \lambda)x^3 + (\lambda^2 + 4\lambda + 2)x^2 - 2(1 + \lambda)x = 0.$$

- (b) Use the formula for the sum of the roots to show that the x -coordinate of M is $\frac{\lambda + 2}{2\lambda}$.

- (c) Use a similar method to find the y -coordinate of M , and hence show that the locus of M is the line through the origin O parallel to PQ .
- (d) For what values of λ is the parabola tangent to the circle in the fourth quadrant?
- (e) For what values of λ are the four points P , Q , A and B distinct, with real numbers as coordinates.



18. [Harmonic conjugates] The line $\ell: y = mx - mb$ through the point $P(b, 0)$ outside the circle $x^2 + y^2 = 1$ meets the circle at the points A and B with x -coordinates α and β .
- (a) Show that α and β satisfy the equation $(m^2 + 1)x^2 - 2m^2bx + (m^2b^2 - 1) = 0$.
- (b) Show that if ℓ is a tangent to the circle, then $m^2(b^2 - 1) = 1$. Hence find the equation of the line ST joining the points S and T of the tangents to the circle from P .
- (c) The general line ℓ meets ST at Q . Prove that Q divides AB internally in the same ratio as P divides AB externally.

The Binomial Theorem

In the previous chapter we discussed the factoring of a polynomial into irreducible factors, so that it could be written in a form such as

$$P(x) = (x - 4)^2(x + 1)^3(x^2 + x + 1).$$

In this chapter we will now study in more detail the individual factors like $(x - 4)^2$ and $(x + 1)^3$ which appear in such a factorisation. For example, we know already that

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1.$$

The coefficients in the general expansion of $(x + a)^n$ will be investigated through the patterns they form when they are written down in the *Pascal triangle*. These patterns lead to a formula for the coefficients, called the *binomial theorem*, and this formula is the key to further study of the binomial expansion and its coefficients. We will be able to apply much of this work in Chapter Ten on probability, because the systematic counting required there turns out to be closely related to the binomial theorem.

STUDY NOTES: Sections 5A and 5B develop the Pascal triangle and apply it to numerical problems on binomial expansions, first of $(1 + x)^n$ and then of $(x + y)^n$. Section 5C introduces the notation $n!$ for factorials in preparation for the binomial theorem itself in Section 5D. Section 5E uses the binomial theorem to find the maximum coefficient and term in a binomial expansion, then Section 5F turns attention to some of the identities relating the binomial coefficients and to the resulting patterns in the Pascal triangle.

The notation ${}^nC_r$ is introduced in a preliminary manner in the notes of Section 5B, but Exercise 5B has been written so that use of the new notation can be delayed until Exercise 5D.

5A The Pascal Triangle

This section is restricted to the expansion of $(1 + x)^n$ and to the various techniques arising from such expansions. The techniques are based on the Pascal triangle and its basic properties, but the proofs of these properties will be left until Section 5B.

Some Expansions of $(1 + x)^n$: Here are the expansions of $(1 + x)^n$ for low values of n . The calculations have been carried out using two rows so that like terms can be written above each other in columns. In this way, the process by which the coefficients build up can be followed better.

$$\begin{aligned}
 (1+x)^0 &= 1 & (1+x)^3 &= 1(1+x)^2 + x(1+x)^2 \\
 & & &= 1 + 2x + x^2 \\
 (1+x)^1 &= 1 + x & &+ x + 2x^2 + x^3 \\
 & & &= 1 + 3x + 3x^2 + x^3 \\
 (1+x)^2 &= 1(1+x) + x(1+x) & (1+x)^4 &= 1(1+x)^3 + x(1+x)^3 \\
 &= 1 + x & &= 1 + 3x + 3x^2 + x^3 \\
 &+ x + x^2 & &+ x + 3x^2 + 3x^3 + x^4 \\
 &= 1 + 2x + x^2 & &= 1 + 4x + 6x^2 + 4x^3 + x^4
 \end{aligned}$$

Notice how the expansion of $(1+x)^2$ has 3 terms, that of $(1+x)^3$ has 4 terms, and so on. In general, the expansion of $(1+x)^n$ has $n+1$ terms, from the constant term in $x^0 = 1$ to the term in x^n . Be careful — this is inclusive counting — there are $n+1$ numbers from 0 to n inclusive.

The Pascal Triangle and the Addition Property: When the coefficients in the expansions of $(1+x)^n$ are arranged in a table, the result is known as the *Pascal triangle*. The table below contains the first five rows of the triangle, copied from the expansions above, plus the next four rows, obtained by continuing these calculations up to $(1+x)^8$.

Coefficient of:									
n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Four properties of this triangle should quickly become obvious. They will be used in this section, and proven formally in the next.

BASIC PROPERTIES OF THE PASCAL TRIANGLE:

1

1. Each row starts and ends with 1.
2. Each row is reversible.
3. The sum of each row is 2^n .
4. [The addition property] Every number in the triangle, apart from the 1s, is the sum of the number directly above, and the number above and to the left.

The first three properties should be reasonably obvious after looking at the expansions at the start of the section. The fourth property, called the *addition property*, however, needs attention. Three numbers in the Pascal triangle above have been boxed as an example of this — notice that

$$1 + 3 = 4.$$

The expansions on the first page of this chapter were written with the columns aligned to make this property stand out. For example, $1 + 3 = 4$ arises like this — in the expansion of $(1 + x)^4$, the coefficient of x^3 is the sum of the coefficients of x^3 and x^2 in the expansion of $(1 + x)^3$.

The whole Pascal triangle can be constructed using these rules, and the first question in the following exercise asks for the first thirteen rows to be calculated.

Using Pascal's Triangle: The following worked exercises illustrate various calculations involving the coefficients of $(1 + x)^n$ for low values of n .

WORKED EXERCISE: Use the Pascal triangle to write out the expansions of:

(a) $(1 - x)^4$ (b) $(1 + 2a)^6$ (c) $(1 - \frac{2}{3}x)^5$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad (1 - x)^4 &= 1 + 4(-x) + 6(-x)^2 + 4(-x)^3 + (-x)^4 \\ &= 1 - 4x + 6x^2 - 4x^3 + x^4 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (1 + 2a)^6 &= 1 + 6(2a) + 15(2a)^2 + 20(2a)^3 + 15(2a)^4 + 6(2a)^5 + (2a)^6 \\ &= 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad (1 - \frac{2}{3}x)^5 &= 1 + 5(-\frac{2}{3}x) + 10(-\frac{2}{3}x)^2 + 10(-\frac{2}{3}x)^3 + 5(-\frac{2}{3}x)^4 + (-\frac{2}{3}x)^5 \\ &= 1 - \frac{10}{3}x + \frac{40}{9}x^2 - \frac{80}{27}x^3 + \frac{80}{81}x^4 - \frac{32}{243}x^5 \end{aligned}$$

WORKED EXERCISE:

(a) Write out the expansion of $\left(1 + \frac{5}{x}\right)^2$, then write out the first four terms in the expansion of $(1 - x)^8$.

(b) Hence find, in the expansion of $\left(1 + \frac{5}{x}\right)^2 (1 - x)^8$:

(i) the term independent of x , (ii) the term in x .

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \left(1 + \frac{5}{x}\right)^2 &= 1 + 10x^{-1} + 25x^{-2} \\ (1 - x)^8 &= 1 - 8x + 28x^2 - 56x^3 + \dots \end{aligned}$$

(b) Hence in the expansion of $\left(1 + \frac{5}{x}\right)^2 (1 - x)^8$:

$$\begin{aligned} \text{(i) constant term} &= 1 \times 1 + (10x^{-1}) \times (-8x) + (25x^{-2}) \times (28x^2) \\ &= 1 - 80 + 700 \\ &= 621. \end{aligned}$$

$$\begin{aligned} \text{(ii) term in } x &= 1 \times (-8x) + (10x^{-1}) \times (28x^2) + (25x^{-2}) \times (-56x^3) \\ &= -8x + 280x - 1400x \\ &= -1128x. \end{aligned}$$

WORKED EXERCISE: By expanding the first few terms of $(1 + 0.02)^8$, find an approximation of 1.02^8 correct to five decimal places.

SOLUTION:

$$\begin{aligned} (1 + 0.02)^8 &= 1 + 8 \times 0.02 + 28 \times (0.02)^2 + 56 \times (0.02)^3 + 70 \times (0.02)^4 + \dots \\ &= 1 + 0.16 + 0.0112 + 0.000448 + 0.00001120 + \dots \\ &\doteq 1.17166 \end{aligned}$$

WORKED EXERCISE: Find the value of k if, in the expansion of $(1 + 2kx)^6$:

- (a) the terms in x^4 and x^3 have coefficients in the ratio $2 : 3$,
 (b) the terms in x^2 , x^3 and x^4 have coefficients in arithmetic progression.

$$\begin{aligned}\text{SOLUTION: } (1 + 2kx)^6 &= \dots + 15(2kx)^2 + 20(2kx)^3 + 15(2kx)^4 + \dots \\ &= \dots + 60k^2x^2 + 160k^3x^3 + 240k^4x^4 + \dots\end{aligned}$$

(a) Put $\frac{240k^4}{160k^3} = \frac{2}{3}$.

Then $\frac{3}{2}k = \frac{2}{3}$
 $k = \frac{4}{9}$.

(b) Put $240k^4 - 160k^3 = 160k^3 - 60k^2$.

Then $240k^4 - 320k^3 + 60k^2 = 0$

$$12k^4 - 16k^3 + 3k^2 = 0.$$

Either $k = 0$, or $12k^2 - 16k + 3 = 0$.

For the quadratic, $\Delta = 256 - 144 = 112 = 16 \times 7$,

so $k = 0$ or $\frac{16 + 4\sqrt{7}}{24}$ or $\frac{16 - 4\sqrt{7}}{24}$

$$= 0 \text{ or } \frac{1}{6}(4 + \sqrt{7}) \text{ or } \frac{1}{6}(4 - \sqrt{7}).$$

WORKED EXERCISE: [A harder example] Expand $(1 + x + x^2)^4$ using the Pascal triangle, by writing $1 + x + x^2 = 1 + (x + x^2)$, and writing $x + x^2 = x(1 + x)$.

$$\begin{aligned}\text{SOLUTION: } (1 + x + x^2)^4 &= \left(1 + (x(x + 1))\right)^4 \\ &= 1 + 4x(1 + x) + 6x^2(1 + x)^2 + 4x^3(1 + x)^3 + x^4(1 + x)^4 \\ &= 1 + 4x(1 + x) + 6x^2(1 + 2x + x^2) + 4x^3(1 + 3x + 3x^2 + x^3) \\ &\quad + x^4(1 + 4x + 6x^2 + 4x^3 + x^4) \\ &= 1 + (4x + 4x^2) + (6x^2 + 12x^3 + 6x^4) + (4x^3 + 12x^4 + 12x^5 + 4x^6) \\ &\quad + (x^4 + 4x^5 + 6x^6 + 4x^7 + x^8) \\ &= 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8\end{aligned}$$

Exercise 5A

- Complete all the rows of Pascal's triangle for $n = 0, 1, 2, 3, \dots, 12$. Keep this in a prominent place for use in the rest of this chapter.
- Using Pascal's triangle of binomial coefficients, give the expansions of each of the following:

(a) $(1 + x)^6$	(f) $(1 + 2y)^4$	(i) $\left(1 - \frac{1}{x}\right)^8$	(k) $\left(1 + \frac{y}{x}\right)^5$
(b) $(1 - x)^6$	(g) $\left(1 + \frac{x}{3}\right)^7$	(j) $\left(1 + \frac{2}{x}\right)^5$	(l) $\left(1 + \frac{3x}{y}\right)^4$
(c) $(1 + x)^9$	(h) $(1 - 3z)^3$		
(d) $(1 - x)^9$			
(e) $(1 + c)^5$			
- Continue the calculations of the expansions of $(1 + x)^n$ at the beginning of this section, expanding $(1 + x)^5$ and $(1 + x)^6$ in the same manner. Keep your work in columns, so that the addition property of the Pascal triangle is clear.
- Find the specified term in each of the following expansions.

(a) For $(1 + x)^{11}$:	(i) find the term in x^2 ,	(ii) find the term in x^8 .

- (b) For $(1-x)^7$: (i) find the term in x^3 , (ii) find the term in x^5 .
 (c) For $(1+2x)^6$: (i) find the term in x^4 , (ii) find the term in x^5 .
 (d) For $\left(1-\frac{3}{x}\right)^4$: (i) find the term in x^{-1} , (ii) find the term in x^{-2} .

5. Sketch on one set of axes:

(a) $y = (1-x)^0$, $y = (1-x)^2$, $y = (1-x)^4$, $y = (1-x)^6$.

(b) $y = (1-x)^1$, $y = (1-x)^3$, $y = (1-x)^5$.

6. Expand $(1+x)^9$ and $(1+x)^{10}$, and show that the sum of the coefficients of the second expansion is twice the sum of the coefficients in the first expansion.

DEVELOPMENT

7. Without expanding, simplify:

(a) $1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$

(b) $1 - 6(x+1) + 15(x+1)^2 - 20(x+1)^3 + 15(x+1)^4 - 6(x+1)^5 + (x+1)^6$

8. Find the coefficient of x^4 in the expansion of $(1-x)^4 + (1-x)^5 + (1-x)^6$.

9. Find integers a and b such that:

(a) $(1+\sqrt{3})^5 = a + b\sqrt{3}$

(c) $(1+3\sqrt{2})^4 = a + b\sqrt{2}$

(b) $(1-\sqrt{5})^3 = a + b\sqrt{5}$

(d) $(1-2\sqrt{3})^6 = a + b\sqrt{3}$

10. Expand and simplify:

(a) $(1+\sqrt{3})^5 + (1-\sqrt{3})^5$

(b) $(1+\sqrt{3})^5 - (1-\sqrt{3})^5$

11. Verify by direct expansion, and by taking out the common factor, that:

(a) $(1+x)^4 - (1+x)^3 = x(1+x)^3$

(b) $(1+x)^7 - (1+x)^6 = x(1+x)^6$

12. (a) Expand the first few terms of $(1+x)^6$, hence evaluate 1.003^6 to five decimal places.

(b) Similarly, expand $(1-4x)^5$, and hence evaluate 0.96^5 to five decimal places.

(c) Expand $(1+x)^8 - (1-x)^8$, and hence evaluate $1.002^8 - 0.998^8$ to five decimal places.

13. (a) (i) Expand $(1+x)^4$ as far as the term in x^2 .

(ii) Hence find the coefficient of x^2 in the expansion of $(1-5x)(1+x)^4$.

(b) (i) Expand $(1+2x)^5$ as far as the term in x^3 .

(ii) Hence find the coefficient of x^3 in the expansion of $(2-3x)(1+2x)^5$.

(c) (i) Expand $(1-3x)^4$ as far as the term in x^3 .

(ii) Hence find the coefficient of x^3 in the expansion of $(2+x)^2(1-3x)^4$.

14. Find the coefficient of:

(a) x^3 in $(3-4x)(1+x)^4$

(d) x^0 in $\left(1-\frac{x}{3}\right)^3 \left(1+\frac{2}{x}\right)^2$

(b) x in $(1+3x+x^2)(1-x)^3$

(e) x^5 in $(1+5x)^4(1-3x)^2$

(c) x^4 in $(5-2x^3)(1+2x)^5$

(f) x^3 in $(1+3x)^3(1-x)^7$

15. Determine the value of the term independent of x in the expansion of:

(a) $(1+2x)^4 \left(1-\frac{1}{x^2}\right)^6$

(b) $\left(1-\frac{x}{3}\right)^5 \left(1+\frac{2}{x}\right)^3$

16. (a) In the expansion of $(1+x)^6$:
- (i) find the term in x^2 ,
 - (ii) find the term in x^3 ,
 - (iii) find the ratio of the term in x^2 to the term in x^3 ,
 - (iv) find the values of (i), (ii) and (iii) when $x = 3$.
- (b) In the expansion of $\left(1 + \frac{2}{3x}\right)^7$:
- (i) find the term in x^{-5} ,
 - (ii) find the term in x^{-6} ,
 - (iii) find the ratio of the term in x^{-5} to the term in x^{-6} ,
 - (iv) find the values of (i), (ii) and (iii) when $x = 2$.
17. (a) When $(1+2x)^5$ is expanded in increasing powers of x , the third and fourth terms in the expansion are equal. Find the value of x .
- (b) When $(1+x)^5$, where $x \neq 0$, is expanded in increasing powers of x , the first, second and fourth terms in the expansion form a geometric sequence. Find the value of x .
- (c) When $(1+x)^7$ is expanded in increasing powers of x , the fifth, sixth and seventh terms in the expansion form an arithmetic sequence. Find the value of x .
18. (a) Find the coefficients of x^4 and x^5 in the expansion of $(1+kx)^8$. Hence find k if these coefficients are in the ratio $1:4$.
- (b) Find the coefficients of x^3 and x^4 in the expansion of $(1+kx)^6$. Hence find k if these coefficients are in the ratio $8:3$.
- (c) Find the coefficients of x^5 and x^6 in the expansion of $(1 - \frac{3}{4}kx)^9$. Hence find k if these coefficients are equal.
19. Use Pascal's triangle to help evaluate the integrals arising from the following questions.
- (a) Find the area bounded by the curve $y = x(1-x)^5$ and the x -axis, where $0 \leq x \leq 1$.
 - (b) Find the area bounded by the curve $y = x^4(1-x)^4$ and the x -axis, where $0 \leq x \leq 1$.
 - (c) Find the volume of the solid formed when the region between the x -axis and the curve $y = \sqrt{x(1-x)^3}$, for $0 \leq x \leq 1$, is revolved around the x -axis.
20. If $\$P$ is invested at the compound interest rate R per annum for n years, and interest is compounded annually, the accumulated amount is $\$A$, where $A = P(1+R)^n$.
- (a) Write down as decimals all terms in the expansion of $(1+0.04)^3$.
 - (b) Hence find the amount to which an investment of $\$1000$ will grow, if it is invested for 3 years at a rate of 4% per annum, and interest is compounded annually.
21. By writing $(1+x+3x^2)^6$ as $(1+A)^6$, where $A = x+3x^2$, expand $(1+x+3x^2)^6$ as far as the term in x^3 . Hence evaluate $(1.0103)^6$ to four decimal places.
22. [Patterns in Pascal's triangle] Check the following results using the triangle you constructed in question 1. (These will not be proven until later.)
- (a) The sum of the numbers in the row beginning 1, n , ... is equal to 2^n .
 - (b) If the second member of a row is a prime number, all the numbers in that row excluding the 1s are divisible by it.
 - (c) [The hockey stick pattern] Starting at any 1 on the left side of the triangle, go diagonally downwards any number of steps. Then the sum of these numbers is the number directly below the last number. For example, if you start at the 1 on the left hand side of the row 1, 3, 3, 1 and move down the diagonal 1, 4, 10, 20 the total of these numbers, namely 35, is found directly below 20.

- (d) [The powers of 11] If a row is made into a single number by using each element as a digit of the number, the number is a power of 11 (except that after the row 1, 4, 6, 4, 1, the pattern gets confused by carrying).
- (e) Find the diagonal and the column containing the triangular numbers, and show that adding adjacent pairs gives the square numbers.
23. [These geometrical results should be related to the numbers in the Pascal triangle.]
- (a) Place three points on the circumference of a circle. How many line segments and triangles can be formed using these three points?
- (b) Place four points on the circumference of a circle. How many segments, triangles and quadrilaterals can be formed using these four points?
- (c) What happens if five points are placed on the circle.
- (d) How many pentagons could you form if you placed seven points on the circumference of a circle?

EXTENSION

24. [The Pascal pyramid] By considering the expansion of $(1 + x + y)^n$, where $0 \leq n \leq 4$, calculate the first five layers of the Pascal pyramid.

5B Further Work with the Pascal Triangle

We pass now to the more general case of the expansion of $(x + y)^n$. Because x and y are both variables, the symmetries of the expansion will be more obvious, and this section offers proofs of the addition property and the other basic patterns in the Pascal triangle.

The Pattern of the Indices in the Expansion of $(x + y)^n$: Here are the expansions of $(x + y)^n$ for low values of n . Again, the calculations have been carried out with like terms written in the same column so that the addition property is clear.

$$\begin{array}{ll}
 (x + y)^0 = 1 & (x + y)^3 = x(x + y)^2 + y(x + y)^2 \\
 & = x^3 + 2x^2y + xy^2 \\
 (x + y)^1 = x + y & \quad + x^2y + 2xy^2 + y^3 \\
 & = x^3 + 3x^2y + 3xy^2 + y^3 \\
 (x + y)^2 = x(x + y) + y(x + y) & \\
 = x^2 + xy & (x + y)^4 = x(x + y)^3 + y(x + y)^3 \\
 \quad + xy + y^2 & = x^4 + 3x^3y + 3x^2y^2 + xy^3 \\
 = x^2 + 2xy + y^2 & \quad + x^3y + 3x^2y^2 + 3xy^3 + y^4 \\
 & = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
 \end{array}$$

The pattern for the indices of x and y is straightforward. The expansion of $(x + y)^3$, for example, has four terms, and in each term the indices of x and y are whole numbers adding to 3. Similarly the expansion of $(x + y)^4$ has five terms, and in each term the indices of x and y are whole numbers adding to 4. The useful phrase for this is that $(x + y)^n$ is *homogeneous of degree n in x and y together*.

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THE TERMS OF $(x + y)^n$: The expansion of $(x + y)^n$ has $n + 1$ terms, and in each term the indices of x and y are whole numbers adding to n . That is, the expression $(x + y)^n$ is *homogeneous of degree n in x and y together*, and so also is its expansion.

The reason for this is clear, and formal proof by induction should not be necessary. In each successive expansion, the terms of the previous expansion are multiplied first by x and then by y , so the sum of the indices goes up by 1. We now know that in general

$$(x + y)^n = *x^n + *x^{n-1}y + *x^{n-2}y^2 + \cdots + *x^2y^{n-2} + *xy^{n-1} + *y^n,$$

where $*$ denotes the different coefficients.

A Symbol for the Coefficients: The coefficients here are, of course, the same as in the previous section, as can be seen by replacing x and y by 1 and x in the expansion of $(x + y)^n$, and we will first deal with the special case of the expansion of $(1 + x)^n$. To investigate these coefficients further, we take the approach of giving names to the things we want to study.

3 THE DEFINITION OF ${}^nC_r$: Define the number ${}^nC_r$ to be the coefficient of x^r in the expansion of $(1 + x)^n$.
The symbol is usually read as ‘ n choose r ’, and the notations ${}^nC_r$ and $\binom{n}{r}$ are both used for these coefficients.

This definition will need some thought. Defining a number as a coefficient in an expansion is standard practice in mathematics, but it seems very strange the first time it is encountered. We can now write out the expansion of $(1 + x)^n$.

4 THE EXPANSION OF $(1 + x)^n$: Using the notation ${}^nC_r$ for the coefficients,
$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n.$$

There are $n + 1$ terms, and the general term of the expansion is
term in $x^r = {}^nC_r x^r$.
Alternatively, using sigma notation, the expansion can be written as
$$(1 + x)^n = \sum_{r=0}^n {}^nC_r x^r.$$

WORKED EXERCISE:

- (a) Write out the expansion of $(1 + x)^2$ and $(1 + x)^3$ using ${}^nC_r$ notation.
(b) Hence give the values of ${}^2C_0$, ${}^2C_1$, ${}^2C_2$ and of ${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$.

SOLUTION:

- (a) $(1 + x)^2 = {}^2C_0 + {}^2C_1 x + {}^2C_2 x^2$
and $(1 + x)^3 = {}^3C_0 + {}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3$
(b) But $(1 + x)^2 = 1 + 2x + x^2$,
so ${}^2C_0 = 1$, ${}^2C_1 = 2$ and ${}^2C_2 = 1$.
Also $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$,
so ${}^3C_0 = 1$, ${}^3C_1 = 3$, ${}^3C_2 = 3$ and ${}^3C_3 = 1$.

The Expansion of $(x + y)^n$: The coefficients in the expansion of $(x + y)^n$ are the same numbers ${}^nC_r$ as in the expansion of $(1 + x)^n$. Thus we can now write out the expansion of $(x + y)^n$ as well.

THE EXPANSION OF $(x + y)^n$: Using the ${}^nC_r$ notation,

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \cdots + {}^nC_n y^n.$$

There are $n + 1$ terms, and the general term of the expansion is

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$$\text{term in } x^{n-r}y^r = {}^nC_r x^{n-r}y^r.$$

Alternatively, using sigma notation, the expansion can be written as

$$(x + y)^n = \sum_{r=0}^n {}^nC_r x^{n-r}y^r.$$

The Pascal Triangle: The Pascal triangle of the previous section now becomes the table of values of the function ${}^nC_r$, with the rows indexed by n and the columns by r :

${}^nC_r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

The boxed numbers provide another example of the addition property of the Pascal triangle, and will be discussed further below.

Using the General Expansion: The general expansion of $(x + y)^n$ is applied in the same way as the expansion of $(1 + x)^n$.

WORKED EXERCISE: Use the Pascal triangle to write out the expansions of:

(a) $(2 - 3x)^4$

(b) $(5x + \frac{1}{5}a)^5$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad (2 - 3x)^4 &= 2^4 + 4 \times 2^3 \times (-3x) + 6 \times 2^2 \times (-3x)^2 \\ &\quad + 4 \times 2 \times (-3x)^3 + (-3x)^4 \\ &= 16 - 96x + 72x^2 - 216x^3 + 81x^4 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (5x + \frac{1}{5}a)^5 &= (5x)^5 + 5 \times (5x)^4 \times \frac{1}{5}a + 10 \times (5x)^3 \times (\frac{1}{5}a)^2 \\ &\quad + 10 \times (5x)^2 \times (\frac{1}{5}a)^3 + 5 \times (5x) \times (\frac{1}{5}a)^4 + (\frac{1}{5}a)^5 \\ &= 3125x^5 + 625ax^4 + 50a^2x^3 + 2a^3x^2 + \frac{1}{25}a^4x + \frac{1}{3125}a^5 \end{aligned}$$

WORKED EXERCISE: Use the Pascal triangle to write out the expansion of $(2x + x^{-2})^6$, leaving the terms unsimplified. Hence find:

(a) the term independent of x ,

(b) the term in x^{-3} .

$$\begin{aligned}\text{SOLUTION: } (2x + x^{-2})^6 &= (2x)^6 + 6 \times (2x)^5 \times (x^{-2}) + 15 \times (2x)^4 \times (x^{-2})^2 \\ &\quad + 20 \times (2x)^3 \times (x^{-2})^3 + 15 \times (2x)^2 \times (x^{-2})^4 + 6 \times (2x) \times (x^{-2})^5 + (x^{-2})^6\end{aligned}$$

(a) Constant term

$$\begin{aligned}&= 15 \times (2x)^4 \times (x^{-2})^2 \\ &= 15 \times 2^4 \times x^4 \times x^{-4} \\ &= 240.\end{aligned}$$

(b) Term in x^{-3}

$$\begin{aligned}&= 20 \times (2x)^3 \times (x^{-2})^3 \\ &= 20 \times 2^3 \times x^3 \times x^{-6} \\ &= 160.\end{aligned}$$

WORKED EXERCISE: Expand $(2 - 3x)^7$ as far as the term in x^2 , and hence find the term in x^2 in the expansion of $(5 + x)(2 - 3x)^7$.

$$\begin{aligned}\text{SOLUTION: } (2 - 3x)^7 &= 2^7 - 7 \times 2^6 \times (3x) + 21 \times 2^5 \times (3x)^2 - \dots \\ &= 128 - 1344x + 6048x^2 - \dots\end{aligned}$$

Hence the term in x^2 in the expansion of $(5 + x)(2 - 3x)^7$

$$\begin{aligned}&= 5 \times 6048x^2 - x \times 1344x \\ &= 28\,896x^2.\end{aligned}$$

Proofs of the First Three Basic Properties: The first three basic properties of the Pascal triangle can now be expressed in ${}^nC_r$ notation and proven straightforwardly.

BASIC PROPERTIES OF THE PASCAL TRIANGLE:

1. Each row starts and ends with 1, that is,
 ${}^nC_0 = {}^nC_n = 1$, for all cardinals n .
- 6 2. Each row is reversible, that is,
 ${}^nC_r = {}^nC_{n-r}$, for all cardinals n and r with $r \leq n$.
3. The sum of each row is 2^n , that is,
 ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$, for all cardinals n .

PROOF: Each proof begins with the general expansion

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \dots + {}^nC_n y^n.$$

Parts 1 and 3 then proceed by substitution, and part 2 by equating coefficients. These methods are both commonly required for solving problems, and they should be studied carefully.

1. Substituting $x = 1$ and $y = 0$, $(1 + 0)^n = {}^nC_0 + 0 + \dots + 0$,

and so as required, $1 = {}^nC_0$.

Substituting $x = 0$ and $y = 1$, $(0 + 1)^n = 0 + 0 + \dots + 0 + {}^nC_n$

and so as required, $1 = {}^nC_n$.

2. We know that $(x + y)^n = (y + x)^n$.

Now $(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + \dots + {}^nC_{n-1} xy^{n-1} + {}^nC_n y^n$,

and $(y + x)^n = {}^nC_0 y^n + {}^nC_1 y^{n-1}x + \dots + {}^nC_{n-1} yx^{n-1} + {}^nC_n x^n$.

Equating coefficients of like terms in the two expansions,

$${}^nC_0 = {}^nC_n, \quad {}^nC_1 = {}^nC_{n-1}, \quad {}^nC_2 = {}^nC_{n-2}, \quad \dots, \quad {}^nC_n = {}^nC_0,$$

and in general ${}^nC_{n-r} = {}^nC_r$, for $r = 0, 1, 2, \dots, n$.

3. Substituting $x = 1$ and $y = 1$, $(1 + 1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$,

and so as required, $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$.

Proof of the Addition Property of the Pascal Triangle: The addition property also needs to be restated in ${}^nC_r$ notation. In words, it says that every number in the triangle is the sum of the number directly above it, and the number above and to the left of it (apart from the first and the last numbers of each row). The boxed numbers in the Pascal triangle above provide an example of this — they show that

$${}^5C_2 = {}^4C_2 + {}^4C_1 \quad (\text{that is, } 10 = 6 + 4).$$

The general statement, in symbolic form, is therefore:

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THE ADDITION PROPERTY: If n and r are positive integers with $r \leq n$, then

$${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}, \text{ for } 1 \leq r \leq n.$$

PROOF: The expansions at the start of Sections 5A and 5B were written so that the columns aligned to make the addition property obvious. A formal proof will require examination of the coefficients in the expansion of $(1+x)^{n+1}$. We begin by noticing that

$$\begin{aligned} (1+x)^{n+1} &= (1+x)(1+x)^n \\ &= (1+x)^n + x(1+x)^n. \end{aligned}$$

On the LHS, the general term in x^r is

$${}^{n+1}C_r x^r.$$

On the RHS, the term in x^r in the first expression is

$${}^nC_r x^r,$$

and the term in x^r in the second expression is

$$x \times {}^nC_{r-1} x^{r-1} = {}^nC_{r-1} x^r,$$

so the general term in x^r on the RHS is the sum

$$({}^nC_r + {}^nC_{r-1})x^r.$$

Equating coefficients of these two terms proves the result.

Exercise 5B

NOTE: Questions 3 and 4 should be omitted by those wanting to delay the introduction of ${}^nC_r$ notation until Section 5D.

1. Use Pascal's triangle to expand each of the following:

(a) $(x+y)^4$	(d) $(p+q)^{10}$	(g) $(p-2q)^7$	(j) $(\frac{1}{2}r + \frac{1}{3}s)^5$
(b) $(x-y)^4$	(e) $(a-b)^9$	(h) $(3x+2y)^4$	(k) $\left(x + \frac{1}{x}\right)^6$
(c) $(r-s)^6$	(f) $(2x+y)^5$	(i) $(a - \frac{1}{2}b)^3$	

2. Use Pascal's triangle to expand each of the following:

(a) $(1+x^2)^4$	(c) $(x^2+2y^3)^6$	(e) $(\sqrt{x} + \sqrt{y})^7$
(b) $(1-3x^2)^3$	(d) $\left(x - \frac{1}{x}\right)^9$	(f) $\left(\frac{2}{x} + 3x^2\right)^5$

3. (a) Expand $(1+x)^4$, and hence write down the values of ${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$ and ${}^4C_4$.
[NOTE: ${}^nC_r$ is defined to be the coefficient of x^r in the expansion of $(1+x)^n$.]

(b) Hence find: (i) ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$ (ii) ${}^4C_0 - {}^4C_1 + {}^4C_2 - {}^4C_3 + {}^4C_4$

4. Use the values of ${}^nC_r$ from the Pascal triangle in the notes above to find:

(a) ${}^6C_0 + {}^6C_2 + {}^6C_4 + {}^6C_6$	(c) ${}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2$
(b) ${}^6C_1 + {}^6C_3 + {}^6C_5$	(d) $({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2$

5. Simplify the following without expanding the brackets:

(a) $y^5 + 5y^4(x-y) + 10y^3(x-y)^2 + 10y^2(x-y)^3 + 5y(x-y)^4 + (x-y)^5$
(b) $a^4 - 4a^3(a-b) + 6a^2(a-b)^2 - 4a(a-b)^3 + (a-b)^4$

- (c) $x^3 + 3x^2(2y - x) + 3x(2y - x)^2 + (2y - x)^3$
 (d) $(x + y)^6 - 6(x + y)^5(x - y) + 15(x + y)^4(x - y)^2 - 20(x + y)^3(x - y)^3$
 $+ 15(x + y)^2(x - y)^4 - 6(x + y)(x - y)^5 + (x - y)^6$

6. (a) (i) Expand $(4 + x)^5$ as far as the term in x^3 .
 (ii) Hence find the coefficient of x^3 in the expansion of $(3 - x)(4 + x)^5$.
 (b) (i) Expand $(1 - 2x)^6$ as far as the term in x^4 .
 (ii) Hence find the coefficient of x^4 in the expansion of $(1 - 3x)(1 - 2x)^6$.
 (c) (i) Expand $(3 - y)^7$ as far as the term in y^4 .
 (ii) Hence find the coefficient of y^4 in the expansion of $(1 - y)^2(3 - y)^7$.

DEVELOPMENT

7. (a) Expand and simplify $(x + y)^6 + (x - y)^6$.
 (b) Hence (and without a calculator) prove that $5^6 + 5^5 \times 3^3 + 5^3 \times 3^5 + 3^6 = 2^5(2^{12} + 1)$.
8. Find the coefficient of:
 (a) x^3 in $(2 - 5x)(x^2 - 3)^4$ (c) x^0 in $(3 - 2x)^2 \left(x + \frac{2}{x}\right)^5$
 (b) x^5 in $(x^2 - 3x + 11)(4 + x^3)^3$ (d) x^9 in $(x + 2)^3(x - 2)^7$
9. (a) (i) Use Pascal's triangle to expand $(x + h)^3$.
 (ii) If $f(x) = x^3$, simplify $f(x + h) - f(x)$.
 (iii) Hence use the definition $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$ to differentiate x^3 .
 (b) Similarly, differentiate x^5 from first principles.
10. (a) Show that $(3 + \sqrt{5})^6 + (3 - \sqrt{5})^6 = 20\,608$.
 (b) Show that $(2 + \sqrt{7})^4 + (2 - \sqrt{7})^4$ is rational.
 (c) Simplify $(5 + \sqrt{2})^5 - (5 - \sqrt{2})^5$.
 (d) If $(\sqrt{6} + \sqrt{3})^3 - (\sqrt{6} - \sqrt{3})^3 = a\sqrt{3}$, where a is an integer, find the value of a .
11. (a) Show that $\frac{1}{(\sqrt{3} - 1)^4} + \frac{1}{(\sqrt{3} + 1)^4} = \frac{(\sqrt{3} + 1)^4 + (\sqrt{3} - 1)^4}{(\sqrt{3} - 1)^4(\sqrt{3} + 1)^4}$ by putting the LHS over a common denominator. Then simplify the expression using Pascal's triangle.
 (b) Similarly, simplify $\frac{1}{(\sqrt{7} - \sqrt{5})^5} + \frac{1}{(\sqrt{7} + \sqrt{5})^5}$.
12. By starting with $((x + y) + z)^3$, expand $(x + y + z)^3$.
13. Expand $(x + 2y)^5$ and hence evaluate: (a) $(1.02)^5$ correct to five decimal places,
 (b) $(0.98)^5$ correct to five decimal places, (c) $(2.2)^5$ correct to four significant figures.
14. (a) Expand: (i) $\left(x + \frac{1}{x}\right)^3$ (ii) $\left(x + \frac{1}{x}\right)^5$ (iii) $\left(x + \frac{1}{x}\right)^7$
 (b) Hence, if $x + \frac{1}{x} = 2$, evaluate: (i) $x^3 + \frac{1}{x^3}$ (ii) $x^5 + \frac{1}{x^5}$ (iii) $x^7 + \frac{1}{x^7}$
15. Find the coefficients of x and x^{-3} in the expansion of $\left(3x - \frac{a}{x}\right)^5$. Hence find the values of a if these coefficients are in the ratio 2 : 1.

16. The coefficients of the terms in a^3 and a^{-3} in the expansion of $\left(ma + \frac{n}{a^2}\right)^6$ are equal, where m and n are nonzero real numbers. Prove that $m^2 : n^2 = 10 : 3$.
17. (a) Expand $\left(x + \frac{1}{x}\right)^6$. (b) If $U = x + \frac{1}{x}$, express $x^6 + \frac{1}{x^6}$ in the form $U^6 + AU^4 + BU^2 + C$. State the values of A , B and C .

EXTENSION

18. Find the term independent of x in the expansion of $(x + 1 + x^{-1})^4$.
19. [The Sierpinski triangle fractal]
- Draw an equilateral triangle of side length 1 unit on a piece of white paper. Join the midpoints of the sides of this triangle to form a smaller triangle. Colour it black. Repeat this process on all white triangles that remain. What do you notice?
 - Draw up Pascal's triangle in the shape of an equilateral triangle, then colour all the even numbers black and leave the odd numbers white. What do you notice? This pattern will be more evident if you take at least the first 16 rows — perhaps use a computer program to generate 100 rows of Pascal's triangle.

5C Factorial Notation

So far, we have been using the Pascal triangle to supply the binomial coefficients. There is a formula for ${}^nC_r$, but it involves taking products like

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$$

The notation $7!$, read as 'seven factorial', used here for this product is new. This section will develop familiarity with the notation and some algorithms for handling it, in preparation for the formula for ${}^nC_r$ in the next section.

The Definition of Factorials: The number 'n factorial', written as $n!$, is the product of all the positive integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

But it is better to define $n!$ recursively. This means first defining $0!$, and then saying exactly how to proceed from $(n-1)!$ to $n!$:

$$\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, \text{ for } n \geq 1. \end{cases}$$

This form of the definition gives more insight into how to manipulate factorial notation, and it also avoids the dots ... in the first definition.

TWO DEFINITIONS OF $n!$ (CALLED n FACTORIAL):

- For each cardinal n , define $n!$ to be the product of all positive integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

- Define the function $n!$ recursively by

$$\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, \text{ for } n \geq 1. \end{cases}$$

The fact that $0! = 1$ requires some thought. An empty product is 1, because if nothing has yet been multiplied, the register has been set back to 1. In a similar way, an empty sum is 0, because if nothing has yet been added, the register has been set back to 0. In any case, $0!$ is defined to be equal to 1. So using the recursive definition,

$$\begin{array}{lll} 0! = 1 & 4! = 4 \times 3! = 24 & 8! = 8 \times 7! = 40\,320 \\ 1! = 1 \times 0! = 1 & 5! = 5 \times 4! = 120 & 9! = 9 \times 8! = 362\,880 \\ 2! = 2 \times 1! = 2 & 6! = 6 \times 5! = 720 & 10! = 10 \times 9! = 3\,628\,800 \\ 3! = 3 \times 2! = 6 & 7! = 7 \times 6! = 5040 & 11! = 11 \times 10! = 39\,916\,800 \end{array}$$

and so on, increasing very quickly indeed. Calculators have a factorial button labelled $x!$ or $n!$. Use it straight away to convince yourself that at least the *calculator* believes that $0! = 1$. Notice also the error message if n is not a cardinal number — the domain of the function $n!$ is $\mathbf{N} = \{0, 1, 2, \dots\}$.

Unrolling Factorials: The recursive definition of $n!$ given above is very useful in calculations. Successive applications of the definition can be thought of as *unrolling* the factorial further and further:

$$\begin{aligned} 8! &= 8 \times 7! \quad (\text{unrolling once}) \\ &= 8 \times 7 \times 6! \quad (\text{unrolling twice}) \\ &= 8 \times 7 \times 6 \times 5! \quad (\text{unrolling three times}) \end{aligned}$$

and so on. This idea is vital when there are fractions involved.

WORKED EXERCISE: Simplify the following using unrolling techniques:

$$(a) \frac{10!}{7!} \qquad (b) \frac{(n+2)!}{(n-1)!} \qquad (c) \frac{n!}{(n-r)!}$$

SOLUTION:

$$\begin{aligned} (a) \frac{10!}{7!} &= \frac{10 \times 9 \times 8 \times 7!}{7!} & (b) \frac{(n+2)!}{(n-1)!} &= \frac{(n+2)(n+1)n(n-1)!}{(n-1)!} \\ &= 10 \times 9 \times 8 & &= (n+2)(n+1)n \\ &= 720 \end{aligned}$$

$$\begin{aligned} (c) \frac{n!}{(n-r)!} &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}} \end{aligned}$$

A Lemma to be Used Later: A *lemma* is a theorem, usually of a technical nature, whose principal purpose is to assist in the proof of a later theorem. The following lemma will be used in the proof of the binomial theorem in the next section. Its proof is not easy, but it is an excellent example of the unrolling technique.

A LEMMA ABOUT FACTORIALS: Let n and r be cardinal numbers, with $1 \leq r \leq n$.

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$$\text{Then } \frac{n!}{(n-r)! r!} + \frac{n!}{(n-r+1)! (r-1)!} = \frac{(n+1)!}{(n-r+1)! r!}.$$

PROOF: A common denominator for the two fractions is required:

$$\begin{aligned}
 \text{LHS} &= \frac{n!}{(n-r)! \times r!} + \frac{n!}{(n-r+1)! \times (r-1)!} \\
 &= \frac{n!}{(n-r)! \times r \times (r-1)!} + \frac{n!}{(n-r+1) \times (n-r)! \times (r-1)!} \\
 &= \frac{(n-r+1) \times n! + r \times n!}{r \times (r-1)! \times (n-r+1) \times (n-r)!} \\
 &= \frac{((n-r+1) + r) \times n!}{r! \times (n-r+1)!} \\
 &= \frac{(n+1) \times n!}{r! \times (n-r+1)!} \\
 &= \frac{(n+1)!}{r! \times (n-r+1)!} \\
 &= \text{RHS.}
 \end{aligned}$$

Exercise 5C

1. Evaluate the following:

(a) $3!$	(f) $\frac{9!}{4!}$	(i) $\frac{10!}{8! \times 2!}$	(l) $\frac{10!}{5! \times 3! \times 2!}$
(b) $7!$	(g) $\frac{15!}{14!}$	(j) $\frac{12!}{3! \times 9!}$	(m) $\frac{15!}{3! \times 5! \times 9!}$
(c) $10!$	(h) $\frac{8!}{3!}$	(k) $\frac{8!}{4! \times 4!}$	(n) $\frac{12!}{2! \times 3! \times 4! \times 5!}$
(d) $1!$			
(e) $0!$			

2. If $f(x) = x^6$, find:

- (a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$ (d) $f^{(4)}(x)$ (e) $f^{(5)}(x)$ (f) $f^{(6)}(x)$ (g) $f^{(7)}(x)$

3. Simplify by unrolling factorials appropriately:

(a) $\frac{n!}{(n-1)!}$	(c) $\frac{n(n-1)!}{n!}$	(e) $\frac{(n+2)!}{n!}$	(g) $\frac{(n-2)!(n-1)!}{n!(n-3)!}$
(b) $n \times (n-1)!$	(d) $\frac{(n+1)!}{(n-1)!}$	(f) $\frac{(n-2)!}{n!}$	(h) $\frac{n!(n-1)!}{(n+1)!}$

4. Simplify by taking out a common factor:

(a) $8! - 7!$	(c) $8! + 6!$	(e) $9! + 8! + 7!$
(b) $(n+1)! - n!$	(d) $(n+1)! + (n-1)!$	(f) $(n+1)! + n! + (n-1)!$

DEVELOPMENT

5. Write each expression as a single fraction:

(a) $\frac{1}{n!} + \frac{1}{(n-1)!}$	(b) $\frac{1}{n!} - \frac{1}{(n+1)!}$	(c) $\frac{1}{(n+1)!} - \frac{1}{(n-1)!}$

6. (a) If $f(x) = x^n$, find: (i) $f'(x)$ (ii) $f''(x)$ (iii) $f^{(n)}(x)$ (iv) $f^{(k)}(x)$, where $k \leq n$.

(b) If $f(x) = \frac{1}{x}$, find: (i) $f'(x)$ (ii) $f''(x)$ (iii) $f^{(5)}(x)$ (iv) $f^{(n)}(x)$.

7. (a) Show that $k \times k! = (k+1)! - k!$

(b) Hence by considering each individual term as a difference of two terms, sum the series $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$

8. Prove that $\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!}$.
9. (a) Find what power of: (i) 2, (ii) 10, is a divisor of $10!$
 (b) Find what power of: (i) 2, (ii) 5, (iii) 7, (iv) 13, is a divisor of $100!$
10. [A relationship between higher derivatives of polynomials and factorials]
 (a) If $f(x) = 11x^3 + 7x^2 + 5x + 3$, show that:
 (i) $f(0) = 3 \times 0!$ (ii) $f'(0) = 5 \times 1!$ (iii) $f''(0) = 7 \times 2!$
 (iv) $f'''(0) = 11 \times 3!$ (v) $f^{(k)}(0) = 0$, for all $k > 4$.

Hence explain why $f(x)$ can be written $f(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$.

- (b) Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is any polynomial, then

$$f^{(k)}(0) = \begin{cases} a_k k!, & \text{for } k = 0, 1, 2, \dots, n, \\ 0, & \text{for } k > n, \end{cases}$$

and hence explain why $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) \times x^k}{k!}$.

11. (a) Evaluate $\frac{k}{(k+1)!}$, for $k = 1, 2, 3, 4$ and 5 .
 (b) Evaluate $\sum_{k=1}^n \frac{k}{(k+1)!}$, for $n = 1, 2, 3, 4$ and 5 .
 (c) Make a reasonable guess about the value of $\sum_{k=1}^n \frac{k}{(k+1)!}$, and prove this result by mathematical induction. Hence find $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{(k+1)!}$.
 (d) Prove that $\frac{k}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$, and hence produce an alternative proof of part (c) using a collapsing sequence.

EXTENSION

12. Express using factorial notation:

(a) $30 \times 28 \times 26 \times \dots \times 2$ (b) $29 \times 27 \times 25 \times \dots \times 1$ (c) $\frac{30 \times 28 \times 26 \times \dots \times 2}{29 \times 27 \times 25 \times \dots \times 1}$

13. [Maclaurin series] The infinite power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0) \times x^k}{k!}$ at the end of question 10(b)

can also be generated by a function $f(x)$ whose higher derivatives do not eventually vanish. The resulting series is called the *Maclaurin series* of $f(x)$, and for most straightforward functions, if the Maclaurin series does converge, it converges to the original function $f(x)$.

- (a) (i) Find the Maclaurin series expansion of $f(x) = \frac{1}{1-x}$.
 (ii) For what values of x does this series converge, and what is its limit?
 (b) Find the Maclaurin series for $f(x) = \log(1-x)$. Refer to the last question in Exercise 12D of the Year 11 volume for a discussion of the convergence of this series.

- (c) (i) Find the Maclaurin series for $f(x) = \sin x$. Refer to the last question in Exercise 14I of the Year 11 volume to see why this series always converges to $\sin x$.
- (ii) Find the Maclaurin series for $f(x) = e^x$. Refer to the last two questions in Exercise 13C of the Year 11 volume to see why this series always converges to e^x .
- (iii) Hence find the first four nonzero terms of the Maclaurin series for $e^x \sin x$.
14. [Stirling's formula] The following formula is too difficult to prove at this stage (see question 24 of Exercise 6F for a preliminary lemma), but it is most important because it provides a continuous function that approximates $n!$ for integer values of n :

$$n! \doteq \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}, \text{ in the sense that the percentage error } \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Show that the formula has an error of approximately 2.73% for $3!$ and 0.83% for $10!$. Find the percentage error for $60!$

5D The Binomial Theorem

There is a rather straightforward formula for the coefficients ${}^nC_r$. It can be discovered by looking along a typical line of the Pascal triangle to see how each entry can be calculated from the entry to the left.

An Investigation for a Formula for ${}^7C_r$: Here is the line corresponding to $n = 7$:

${}^7C_r$	0	1	2	3	4	5	6	7
7	1	7	21	35	35	21	7	1

And here is how to work along the line:

The first entry in the line is 1.

For the entry 7, multiply by $7 = \frac{7}{1}$.

For the entry 21, multiply by $3 = \frac{6}{2}$.

For the entry 35, multiply by $\frac{5}{3}$.

For the entry 35, multiply by $1 = \frac{4}{4}$.

For the entry 21, multiply by $\frac{3}{5}$.

For the entry 7, multiply by $\frac{1}{3} = \frac{2}{6}$.

For the final entry 1, multiply by $\frac{1}{7}$.

Now we can write each entry ${}^7C_r$ as a product of fractions:

$${}^7C_0 = 1$$

$${}^7C_1 = \frac{7}{1} = 7$$

$${}^7C_2 = \frac{7 \times 6}{1 \times 2} = 21$$

$${}^7C_3 = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$$

$${}^7C_4 = \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} = 35$$

$${}^7C_5 = \frac{7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5} = 21$$

$${}^7C_6 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 7$$

$${}^7C_7 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 1$$

Statement of the Binomial Theorem: By now one should be reasonably convinced that the general formula for ${}^nC_r$ is

$${}^nC_r = \frac{\overbrace{n \times (n-1) \times (n-2) \times (n-3) \times \cdots}^{r \text{ factors}}}{\underbrace{1 \times 2 \times 3 \times 4 \times \cdots}_{r \text{ factors}}}.$$

This is not yet very elegant. The denominator is $1 \times 2 \times 3 \times 4 \times \cdots \times r = r!$. The numerator is $n \times (n-1) \times \cdots \times (n-r+1)$, which is $\frac{n!}{(n-r)!}$, by the unrolling procedures. This gives a very concise formulation of the binomial theorem.

THE BINOMIAL THEOREM: For all cardinal numbers n ,

$${}^nC_r = \frac{n!}{r!(n-r)!}, \text{ for } r = 0, 1, \dots, n.$$

Alternatively,
$${}^nC_r = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{1 \times 2 \times \cdots \times r}.$$

The difficult proof will be given at the end of this section. See the Extension section of Exercise 5E for an alternative and easier proof using differentiation. A further interesting proof by combinatoric methods will be given in Chapter Ten. Notice that the formula for ${}^nC_r$ remains unchanged when r is replaced by $n-r$:

$${}^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = {}^nC_r,$$

confirming the symmetry of each row in the Pascal triangle, as proven in Section 5B.

Scientific calculators have a button labelled $\boxed{{}^nC_r}$ which will find values of ${}^nC_r$. For low values of n and r , the answers are exact, but for high values they are only approximations.

Examples of the Binomial Theorem: Here are some worked examples using the formula to calculate ${}^nC_r$ for some values of n and r .

WORKED EXERCISE: Evaluate, using the binomial theorem: (a) ${}^8C_5$ (b) ${}^nC_3$

$$\begin{aligned} \text{SOLUTION: (a) } {}^8C_5 &= \frac{8!}{3! \times 5!} & \text{(b) } {}^nC_3 &= \frac{n(n-1)(n-2)(n-3)!}{(n-3)! \times 3!} \\ &= \frac{8 \times 7 \times 6 \times \cancel{5!}}{3 \times 2 \times 1 \times \cancel{5!}} & &= \frac{n(n-1)(n-2)}{6} \\ &= 56 \end{aligned}$$

WORKED EXERCISE: Find ${}^{16}C_5$, leaving your answer factored into primes.

$$\begin{aligned} \text{SOLUTION: } {}^{16}C_5 &= \frac{16 \times 15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4 \times 5} \\ &= 2 \times 14 \times 13 \times 12 \\ &= 2^4 \times 3 \times 7 \times 13 \quad (\text{Check this on the calculator.}) \end{aligned}$$

WORKED EXERCISE:

- (a) Find the general term in the expansion of $(2x^2 - x^{-1})^{20}$.
 (b) (i) Find the term in x^{34} . (ii) Find the term in x^{-5} .
 Give each coefficient as a numeral, and factored into primes.

SOLUTION:

$$\begin{aligned} \text{(a) General term} &= {}^{20}C_r \times (2x^2)^{20-r} \times (-x^{-1})^r \\ &= {}^{20}C_r \times 2^{20-r} \times x^{40-2r} \times (-1)^r \times x^{-r} \\ &= {}^{20}C_r \times 2^{20-r} \times (-1)^r \times x^{40-3r}. \end{aligned}$$

(b) (i) To obtain the term in x^{34} , $40 - 3r = 34$

$$r = 2.$$

$$\begin{aligned}\text{Hence the term in } x^{34} &= {}^{20}C_2 \times (2x^2)^{20-2} \times (-x^{-1})^2 \\ &= \frac{20 \times 19}{1 \times 2} \times 2^{18} \times x^{36} \times x^{-2} \\ &= 2^{19} \times 5 \times 19 \times x^{34} \\ &= 49\,807\,360 x^{34}.\end{aligned}$$

(Check this on the calculator using ${}^{20}C_2 \times 2^{18}$.)

(ii) To obtain the term in x^{-5} , $40 - 3r = -5$

$$r = 15.$$

$$\begin{aligned}\text{Hence the term in } x^{-5} &= {}^{20}C_{15} \times (2x^2)^{20-15} \times (-x^{-1})^{15} \\ &= -\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} \times 2^5 \times x^{10} \times x^{-15} \\ &= -19 \times 3 \times 17 \times 16 \times 2^5 \times x^{-5} \\ &= -2^9 \times 3 \times 17 \times 19 \times x^{-5} \\ &= -496\,128 x^{-5}.\end{aligned}$$

(Check this on the calculator using ${}^{20}C_{15} \times 2^5$.)

WORKED EXERCISE: In the expansion of $\left(x + \frac{1}{x}\right)^{40} \left(x - \frac{1}{x}\right)^{40}$, find the term independent of x . Give your answer in the form ${}^nC_r$, and also as a numeral.

SOLUTION: We can write $\left(x + \frac{1}{x}\right)^{40} \left(x - \frac{1}{x}\right)^{40} = (x^2 - x^{-2})^{40}$.

$$\begin{aligned}\text{Hence the term independent of } x &= {}^{40}C_{20} \times (x^2)^{20} \times (-x^{-2})^{20} \\ &= {}^{40}C_{20} \\ &\doteq 1.378 \times 10^{11} \quad (\text{using the calculator}).\end{aligned}$$

The Values of ${}^nC_r$ for $r = 0, 1$ and 2 : The particular formulae for ${}^nC_r$ for $n = 0, 1$ and 2 are important enough to be memorised:

$$\begin{aligned}{}^nC_0 &= \frac{n!}{0! n!} \\ &= 1\end{aligned}\quad \begin{aligned}{}^nC_1 &= \frac{n!}{1! (n-1)!} \\ &= n\end{aligned}\quad \begin{aligned}{}^nC_2 &= \frac{n!}{2! (n-2)!} \\ &= \frac{n \times (n-1)}{1 \times 2} \\ &= \frac{1}{2}n(n-1)\end{aligned}$$

By the symmetry of the rows, these are also the values of ${}^nC_n$, ${}^nC_{n-1}$ and ${}^nC_{n-2}$.

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SOME PARTICULAR VALUES OF ${}^nC_r$: For all cardinals n ,

$$\begin{aligned}{}^nC_0 &= {}^nC_n = 1, \\ {}^nC_1 &= {}^nC_{n-1} = n, \\ {}^nC_2 &= {}^nC_{n-2} = \frac{1}{2}n(n-1).\end{aligned}$$

WORKED EXERCISE: Find the value of n if:

(a) ${}^nC_2 = 55$

(b) ${}^nC_2 + {}^nC_1 + {}^nC_0 = 29$

SOLUTION: We know that ${}^nC_0 = 1$ and ${}^nC_1 = n$ and ${}^nC_2 = \frac{1}{2}n(n-1)$.

(a) $\frac{1}{2}n(n-1) = 55$

$$n^2 - n - 110 = 0$$

$$(n-11)(n+10) = 0$$

Since $n \geq 0$, $n = 11$.

(b) ${}^nC_2 + {}^nC_1 + {}^nC_0 = 29$

$$\frac{1}{2}n(n-1) + n + 1 = 29$$

$$n^2 - n + 2n + 2 = 58$$

$$n^2 + n - 56 = 0$$

$$(n-7)(n+8) = 0$$

Since $n \geq 0$, $n = 7$.

Proof of the Binomial Theorem: The demanding proof of the binomial theorem uses mathematical induction. The key to the proof is the addition property of the Pascal triangle, proven in Section 5B, because it allows ${}^{k+1}C_r$ to be expressed as the sum of ${}^kC_r$ and ${}^kC_{r-1}$. Towards the end of part B, the proof uses the technical lemma, proven in the previous section, about adding two fractions involving factorials.

PROOF: The proof is by mathematical induction on the degree n . The ‘result’ that the proof keeps referring to is the statement that

$${}^nC_r = \frac{n!}{(n-r)!r!}, \text{ for } r = 0, 1, 2, \dots, n,$$

which says that the formula holds for all values of r from 0 to n . In other words, we shall prove that if any one row of the triangle obeys the theorem, then the next row also obeys it.

A. We prove the result for $n = 0$. In this case, $r = 0$ is the only possible value of r , and so there is only a single formula to prove, namely ${}^0C_0 = \frac{0!}{0! \times 0!}$.

Here LHS = 1, because of the expansion $(x+y)^0 = 1$.

Also RHS = 1, since $0! = 1$ by definition.

B. Suppose that k is a value of n for which the result is true.

That is, ${}^kC_r = \frac{k!}{(k-r)!r!}$, for $r = 0, 1, 2, \dots, k$. (**)

We now prove the result for $n = k+1$.

That is, we prove ${}^{k+1}C_r = \frac{(k+1)!}{(k-r+1)!r!}$, for $r = 0, 1, 2, \dots, k+1$.

The proof of this will be in two parts. The first part confirms that the formula is true for the two ends of the row when $r = 0$ and $r = k+1$, and the second part proves it true for the other values $r = 1, 2, \dots, k$.

$$\begin{aligned} 1) \text{ When } r = 0, \quad \text{LHS} &= {}^{k+1}C_0 \\ &= 1, \text{ as proven in Section 5B,} \\ \text{RHS} &= \frac{(k+1)!}{(k+1)! \times 0!} \\ &= 1, \text{ since } 0! = 1. \end{aligned}$$

$$\begin{aligned}
 \text{When } r = k + 1, \text{ LHS} &= {}^{k+1}C_{k+1} \\
 &= 1, \text{ again as proven in Section 5B,} \\
 \text{RHS} &= \frac{(k+1)!}{0! \times (k+1)!} \\
 &= 1, \text{ since } 0! = 1.
 \end{aligned}$$

Hence the formula is true for $r = 0$ and $r = k + 1$.

- 2) Now suppose that $r = 1, 2, \dots, k$.

$$\begin{aligned}
 \text{LHS} &= {}^{k+1}C_r \\
 &= {}^kC_r + {}^kC_{r-1}, \text{ by the addition property with } n = k + 1, \\
 &= \frac{k!}{(k-r)!r!} + \frac{k!}{(k-r+1)!(r-1)!}, \text{ by the induction hypothesis (**),} \\
 &= \frac{(k+1)!}{(k-r+1)!r!}, \text{ by the lemma in Section 5C,} \\
 &= \text{RHS.}
 \end{aligned}$$

C. It follows now from A and B by mathematical induction that the result is true for all cardinals n .

Exercise 5D

- Use the result ${}^nC_r = \frac{n!}{r!(n-r)!}$ to evaluate the following. Check your answers for parts (a)–(i) against the Pascal triangle you developed in Exercise 5A.

(a) ${}^5C_2$	(d) ${}^{13}C_5$	(g) ${}^9C_1$	(j) $\frac{{}^{10}C_6}{{}^7C_2}$	(k) $\frac{{}^8C_5}{{}^4C_3}$	(l) $\frac{{}^{12}C_7}{{}^4C_2}$
(b) ${}^{10}C_4$	(e) ${}^{12}C_7$	(h) ${}^{11}C_{10}$			
(c) ${}^6C_3$	(f) ${}^8C_8$	(i) ${}^7C_3$			
- (a) Evaluate: (i) ${}^8C_3$ and ${}^8C_5$, (ii) ${}^7C_4$ and ${}^7C_3$.
 (b) If ${}^nC_3 = {}^nC_2$, find the value of n .
- (a) By evaluating the LHS and RHS, verify the following results for $n = 8$ and $r = 3$:
 (i) ${}^nC_r = {}^nC_{n-r}$ (ii) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
 (b) Use these two identities to solve the following equations for n :
 (i) ${}^5C_3 + {}^5C_4 = {}^nC_4$ (iii) ${}^nC_{10} = {}^nC_{20}$
 (ii) ${}^nC_7 + {}^nC_8 = {}^{11}C_8$ (iv) ${}^{12}C_4 = {}^{12}C_n$
- Find the specified terms in each of the following expansions.

(a) For $(2+x)^7$:	(i) find the term in x^2 ,	(ii) find the term in x^4 .
(b) For $(x+\frac{1}{2}y)^{14}$:	(i) find the term in x^9y^5 ,	(ii) find the term in x^5y^9 .
(c) For $(\frac{1}{2}x-3y^2)^{11}$:	(i) find the term in $x^{10}y^2$,	(ii) find the term in x^5y^{12} .
(d) For $(a-b^{\frac{1}{2}})^{20}$:	(i) find the term in $a^3b^{\frac{17}{2}}$,	(ii) find the term in a^2b^9 .
- (a) Use the binomial theorem to obtain formulae for:
 (i) ${}^nC_0$ (ii) ${}^nC_1$ (iii) ${}^nC_2$ (iv) ${}^nC_3$
 (b) Hence solve each of the following equations for n :
 (i) ${}^9C_2 - {}^nC_1 = {}^6C_3$ (iii) ${}^nC_2 + {}^6C_2 = {}^7C_2$ (v) ${}^nC_1 + {}^nC_2 = {}^5C_2$
 (ii) ${}^nC_2 = 36$ (iv) ${}^nC_2 + {}^nC_1 = 22 - {}^nC_0$ (vi) ${}^nC_3 + {}^nC_2 = 8{}^nC_1$
 (c) Use the formula for ${}^nC_2$ to show that ${}^nC_2 + {}^{n+1}C_2 = n^2$, and verify the result on the third column of the Pascal triangle.

DEVELOPMENT

6. (a) In the expansion of $(1+x)^{16}$, find the ratio of the term in x^{13} to the term in x^{11} .
 (b) Find the ratio of the coefficients of x^{14} and x^5 in the expansion of $(1+x)^{20}$.
 (c) In the expansion of $(2+x)^{18}$, find the ratio of the coefficients of x^{10} and x^{16} .
7. Consider the expansion $\left(x^2 + \frac{1}{x}\right)^9 = \sum_{i=0}^9 {}^9C_i (x^2)^{9-i} \left(\frac{1}{x}\right)^i$.
- (a) Show that each term in the expansion of $\left(x^2 + \frac{1}{x}\right)^9$ can be written as ${}^9C_i x^{18-3i}$.
 (b) Hence find the coefficients of: (i) x^3 (ii) x^{-3} (iii) x^0
8. In the expansion of $(2x^3 + 3x^{-2})^{10}$, the general term is ${}^{10}C_k (2x^3)^{10-k} (3x^{-2})^k$.
- (a) Show that this general term can be written as ${}^{10}C_k 2^{10-k} 3^k x^{30-5k}$.
 (b) Hence find the coefficients of the following terms, giving your answer factored into primes: (i) x^{10} (ii) x^{-5} (iii) x^0
9. (a) Show that the general term in the expansion of $\left(\frac{x}{2} - \frac{5}{x}\right)^{15}$ can be written as ${}^{15}C_j (-1)^j 5^j 2^{j-15} x^{15-2j}$.
 (b) Hence find, without simplifying, the coefficients of: (i) x^{11} (ii) x (iii) x^{-5}
10. Find the term independent of x in each expansion:
 (a) $\left(x + \frac{3}{x}\right)^8$ (b) $\left(2x^3 - \frac{1}{x}\right)^{12}$ (c) $(5x^4 - 2x^{-1})^{10}$ (d) $(ax^{-2} + \frac{1}{2}x)^6$
11. Find the coefficient of the power of x specified in each of the following expansions (leave the answer to part (f) unsimplified):
 (a) x^{15} in $\left(x^3 - \frac{2}{x}\right)^9$ (d) x^7 in $\left(5x^2 + \frac{1}{2x}\right)^8$
 (b) the constant term in $\left(\frac{1}{2x^3} + x\right)^{20}$ (e) x^{-1} in $(x^{-1} + \frac{2}{7}x)^5$
 (c) x^{-14} in $(x - 3x^{-4})^{11}$ (f) x^{11} in $\left(\frac{3x^2}{5} - \frac{1}{x}\right)^{19}$
12. Determine the coefficients of the specified terms in each of the following expansions:
 (a) For $(3+x)(1-x)^{15}$: (i) find the term in x^4 , (ii) find the term in x^{13} .
 (b) For $(2-5x+x^2)(1+x)^{11}$: (i) find the term in x^9 , (ii) find the term in x^3 .
 (c) For $(x-3)(x+2)^{15}$: (i) find the term in x^7 , (ii) find the term in x^{12} .
 (d) For $(1-2x-4x^2)\left(1-\frac{3}{x}\right)^9$: (i) find the term in x^0 , (ii) find the term in x^{-5} .
13. (a) Find the middle term when the terms in the following expansions are arranged in increasing powers of y .
 (i) $(2x-3y)^{10}$ (ii) $(x^{\frac{1}{2}} + y^{\frac{1}{3}})^{12}$ (iii) $(\frac{1}{5}x - y^2)^{18}$ (iv) $\left(\frac{2}{y} + \frac{x}{3}\right)^6$
 (b) Find the two middle terms when the terms in the following expansions are arranged in increasing powers of b .
 (i) $(a+3b)^5$ (ii) $(\frac{1}{2}a - \frac{1}{3}b)^{11}$ (iii) $(a^{\frac{1}{3}} + b^{\frac{1}{2}})^{15}$ (iv) $\left(\frac{1}{a} - \frac{b}{2}\right)^9$

14. (a) Find x if the terms in x^{10} and x^{11} in the expansion of $(5 + 2x)^{15}$ are equal.
 (b) Find x if the terms in x^{13} and x^{14} in the expansion of $(2 - 3x)^{17}$ are equal.
15. (a) Find the coefficient of x in the expansion of $\left(x + \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^4$.
 (b) Find the coefficient of x^2 in the expansion of $\left(x - \frac{1}{x}\right)^9 \left(x + \frac{1}{x}\right)^5$.
 (c) Find the coefficient of y^{-3} in the expansion of $\left(y + \frac{1}{y}\right)^{10} \left(y - \frac{1}{y}\right)^7$.
16. (a) In the expansion of $(2 + ax + bx^2)(1 + x)^{13}$, the coefficients of x^0 , x^1 and x^2 are all equal to 2. Find the values of a and b .
 (b) In the expansion of $(1 + x)^n$, the coefficient of x^5 is 1287. Find the value of n by trial and error, and hence find the coefficient of x^{10} .
17. The expression $(1 + ax)^n$ is expanded in increasing powers of x . Find the values of a and n if the first three terms are:
 (a) $1 + 28x + 364x^2 + \dots$ (b) $1 - \frac{10}{3}x + 5x^2 - \dots$
18. (a) In the expansion of $(2 + 3x)^n$, the coefficients of x^5 and x^6 are in the ratio 4 : 9. Find the value of n .
 (b) In the expansion of $(1 + 3x)^n$, the coefficients of x^8 and x^{10} are in the ratio 1 : 2. Find the value of n .
 (c) The expression $\left(3 + \frac{x}{5}\right)^n$ is expanded in increasing powers of x . When $x = 2$, the ratio of the 7th and 8th terms is 35 : 2. Find the value of n .
19. If n is a positive integer, use the binomial theorem to prove that $(5 + \sqrt{11})^n + (5 - \sqrt{11})^n$ is an integer.
20. Use binomial expansions and the binomial theorem to find the value of:
 (a) $(0.99)^{13}$ correct to five significant figures,
 (b) $(1.01)^{11}$ correct to four decimal places,
 (c) $(0.999)^{15}$ correct to five significant figures.
21. (a) When $(1 + x)^n$ is expanded in increasing powers of x , the ratios of three consecutive coefficients are 9 : 24 : 42. Find the value of n .
 (b) In the expansion of $(1 + x)^n$, the coefficients of x , x^2 and x^3 form an arithmetic progression. Find the value of n .
 (c) In the expansion of $(1 + x)^n$, the coefficients of x^4 , x^5 and x^6 form an AP.
 (i) Explain why $2 \times {}^nC_5 = {}^nC_4 + {}^nC_6$, and hence show that $n^2 - 21n + 98 = 0$.
 (ii) Hence find the two possible values of n .
22. By writing it as $((1 - x) + x^2)^4$, expand $(1 - x + x^2)^4$ in ascending powers of x as far as the term containing x^4 .
23. Show that there will always be a term independent of x in the expansion of $\left(x^p + \frac{1}{x^{2p}}\right)^{3n}$, where n is a positive integer, and find that term.

24. (a) Write down the term in x^r in the expansion of $(a - bx)^{12}$.
 (b) In the expansion of $(1 + x)(a - bx)^{12}$, the coefficient of x^8 is zero. Find the value of the ratio $\frac{a}{b}$ in simplest form.
25. [Divisibility problems]
 (a) Use the binomial theorem to show that $7^n + 2$ is divisible by 3, where n is a positive integer. [HINT: Write $7 = 6 + 1$.]
 (b) Use the binomial theorem to show that $5^n + 3$ is divisible by 4, where n is a positive integer.
 (c) Suppose that b, c and n are positive integers, and $a = b + c$. Use the binomial expansion of $(b + c)^n$ to show that $a^n - b^{n-1}(b + cn)$ is divisible by c^2 . Hence show that $5^{42} - 2^{48}$ is divisible by 9.
26. (a) Use the binomial theorem to expand $(x + h)^n$.
 (b) Hence use the definition $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ to differentiate x^n from first principles.
27. (a) Given that ${}^nC_r = \frac{n!}{r!(n-r)!}$, show that $\frac{r \times {}^nC_r}{{}^nC_{r-1}} = n - r + 1$.
 (b) Hence prove that
- $$\frac{{}^nC_1}{{}^nC_0} + \frac{2 \times {}^nC_2}{{}^nC_1} + \frac{3 \times {}^nC_3}{{}^nC_2} + \cdots + \frac{n \times {}^nC_n}{{}^nC_{n-1}} = \frac{n}{2}(n+1).$$
28. In the expansion of $(1 + 3x + ax^2)^n$, where n is a positive integer, the coefficient of x^2 is 0. Find, in terms of n , the value of: (a) a , (b) the coefficient of x^3 .
29. [APs in the Pascal triangle — see the last question in Exercise 5F for the general case.]
 (a) In the expansion of $(1 + x)^n$, the coefficients of x^{r-1} , x^r and x^{r+1} form an arithmetic sequence. Prove that $4r^2 - 4rn + n^2 - n - 2 = 0$.
 (b) Hence find three consecutive coefficients of the expansion of $(1 + x)^{14}$ which form an arithmetic sequence.

EXTENSION

30. (a) Show that $n \binom{n-1}{1} + n \binom{n-1}{2} + \cdots + n \binom{n-1}{n-2} = n(2^{n-1} - 2)$.
 (b) Hence use trial and error to find the smallest positive integer n such that
- $$n \binom{n-1}{1} + n \binom{n-1}{2} + \cdots + n \binom{n-1}{n-2} > 15\,000.$$
31. (a) If $r > p + 1$, show that ${}^rC_p = {}^{r+1}C_{p+1} - {}^rC_{p+1}$.
 (b) Hence deduce that for $n > p$, ${}^pC_p + {}^{p+1}C_p + {}^{p+2}C_p + \cdots + {}^nC_p = {}^{n+1}C_{p+1}$.
 (c) What is the significance of this result in the Pascal triangle?
32. (a) Find the value of $\frac{{}^nC_r}{{}^nC_{r-1}}$.
 (b) Evaluate $\frac{{}^nC_1}{{}^nC_0} + 2 \frac{{}^nC_2}{{}^nC_1} + 3 \frac{{}^nC_3}{{}^nC_2} + \cdots + n \frac{{}^nC_n}{{}^nC_{n-1}}$.
 (c) Prove the following identity, and verify it using the row indexed by $n = 4$:

$$({}^nC_0 + {}^nC_1)({}^nC_1 + {}^nC_2) \cdots ({}^nC_{n-1} + {}^nC_n) = {}^nC_0 {}^nC_1 {}^nC_2 \cdots {}^nC_n \times \frac{(n+1)^n}{n!}.$$

33. [A more general form of the binomial theorem] The binomial theorem can be written as a power series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

In this form, the theorem is true even when n is fractional or negative, provided that $|x| < 1$, in the sense that the infinite series on the RHS converges to $(1+x)^n$.

- (a) Prove, using the convergence of geometric series, that the result is true for $n = -1$.
 (b) Generate the binomial expansions of:

$$(i) \frac{1}{1+x} \quad (ii) \frac{1}{(1-x)^2} \quad (iii) \frac{1}{(1+x)^2} \quad (iv) \sqrt{1+x}$$

5E Greatest Coefficient and Greatest Term

In a typical binomial expansion like

$$(1+2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4,$$

the coefficients of successive terms rise and then fall. The greatest coefficient is the coefficient of x^3 , which is 32.

If we now make a substitution like $x = \frac{3}{4}$, the expansion becomes

$$(1 + \frac{3}{2})^4 = 1 + 6 + 13\frac{1}{2} + 13\frac{1}{2} + 5\frac{1}{16},$$

and again the terms rise and fall. There are two greatest terms, both $13\frac{1}{2}$.

The purpose of this section is to develop a systematic method, based on the binomial theorem, of finding these greatest terms and greatest coefficients. These methods, and the results, will have a particular role in some probability questions in Chapter Ten.

A Systematic Method: We will use the binomial theorem to find the ratio of successive coefficients or terms. This ratio will be greater than 1 when the coefficients or terms are increasing, and it will be less than 1 when the coefficients or terms are decreasing.

In the following worked exercise, the general method is applied to the two very simple expansions above — it is, of course, designed to be used with expansions of much higher degree, where writing out all the terms would be impossible. It is not necessary to use sigma notation — all that is needed is the general term — but we shall need the notation in the next section, and it does the job of clearly displaying the general term.

WORKED EXERCISE:

- (a) Write the expansion of $(1+2x)^4$ in the form $\sum_{k=0}^n t_k x^k$. Find the ratio $\frac{t_{k+1}}{t_k}$, and hence find the greatest coefficient.
- (b) Write the expansion of $(1+2x)^4$ in the form $\sum_{k=0}^n T_k$. Find the ratio $\frac{T_{k+1}}{T_k}$, and hence find the greatest term if $x = \frac{3}{4}$.

SOLUTION:

$$(a) \text{ Expanding, } (1+2x)^4 = \sum_{k=0}^4 {}^4C_k (2x)^k = \sum_{k=0}^4 {}^4C_k 2^k x^k,$$

$$\text{so } (1+2x)^4 = \sum_{k=0}^4 t_k x^k, \text{ where } t_k = {}^4C_k 2^k.$$

$$\begin{aligned} \text{Hence } \frac{t_{k+1}}{t_k} &= \frac{{}^4C_{k+1} 2^{k+1}}{{}^4C_k 2^k} \\ &= \frac{4!}{(k+1)!(3-k)!} \times \frac{k!(4-k)!}{4!} \times 2 \\ &= \frac{(4-k) \times 2}{k+1} \\ &= \frac{8-2k}{k+1}. \end{aligned}$$

To find where the coefficients are increasing, we solve $t_{k+1} > t_k$,

$$\text{that is, } \frac{t_{k+1}}{t_k} > 1 \quad (\text{notice that } t_k \text{ is positive}).$$

$$\text{From above, } \frac{8-2k}{k+1} > 1$$

$$8-2k > k+1$$

$$k < 2\frac{1}{3} \quad (\text{remember that } k \text{ is an integer}),$$

so $t_{k+1} > t_k$ for $k = 0, 1$ and 2 , and $t_{k+1} < t_k$ for $k = 3$.

$$\text{Hence } t_0 < t_1 < t_2 < t_3 > t_4,$$

and the greatest coefficient is $t_3 = {}^4C_3 \times 2^3 = 32$.

$$(b) \text{ From above, } (1+2x)^4 = \sum_{k=0}^4 T_k, \text{ where } T_k = {}^4C_k 2^k x^k.$$

$$\begin{aligned} \frac{T_{k+1}}{T_k} &= \frac{{}^4C_{k+1} 2^{k+1} x^{k+1}}{{}^4C_k 2^k x^k} \\ &= \frac{(8-2k)x}{k+1}, \text{ using the previous working,} \\ &= \frac{3(8-2k)}{4(k+1)}, \text{ substituting } x = \frac{3}{4}, \\ &= \frac{12-3k}{2k+2}, \text{ after cancelling the 2s.} \end{aligned}$$

To find where the terms are increasing, we solve $T_{k+1} > T_k$,

$$\text{that is, } \frac{T_{k+1}}{T_k} > 1 \quad (\text{again, } T_k \text{ is positive}).$$

$$\text{From above, } \frac{12-3k}{2k+2} > 1$$

$$12-3k > 2k+2$$

$$k < 2,$$

so $T_{k+1} > T_k$ for $k = 0$ and 1 , $T_{k+1} = T_k$ for $k = 2$, and $T_{k+1} < T_k$ for $k = 3$.

$$\text{Hence } T_0 < T_1 < T_2 = T_3 > T_4,$$

$$\text{and the greatest terms are } T_2 = {}^4C_2 \times \left(\frac{3}{2}\right)^2 = 13\frac{1}{2}$$

$$\text{and } T_3 = {}^4C_3 \times \left(\frac{3}{2}\right)^3 = 13\frac{1}{2}.$$

NOTE: Equality of successive terms or coefficients arises when the solution of the inequality is $k >$ some whole number, because then that whole number is the solution of the corresponding equality. There is no real need to make qualifications about this in the working, which is already complicated.

Exercise 5E

1. Let $(2 + 3x)^{12} = \sum_{k=0}^{12} t_k x^k$, where $t_k = {}^{12}C_k \times 2^{12-k} \times 3^k$ is the coefficient of x^k .

(a) Write down expressions for t_k and t_{k+1} , and show that $\frac{t_{k+1}}{t_k} = \frac{36 - 3k}{2k + 2}$.

(b) Hence show that t_7 is the greatest coefficient.

(c) Write down the greatest coefficient (and leave it factored).

2. Let $(7 + 3x)^{25} = \sum_{k=0}^{25} c_k x^k$.

(a) Write down expressions for c_k and c_{k+1} , and show that $\frac{c_{k+1}}{c_k} = \frac{75 - 3k}{7k + 7}$.

(b) Hence show that c_7 is the greatest coefficient.

(c) Write down the greatest coefficient (and leave it factored).

3. Let $(3 + 4x)^{13} = \sum_{k=0}^{13} T_k$, where $T_k = {}^{13}C_k \times 3^{13-k} \times (4x)^k$ is the term in x^k .

(a) Write down expressions for T_k and T_{k+1} , and show that $\frac{T_{k+1}}{T_k} = \frac{4x(13 - k)}{3(k + 1)}$.

(b) Hence show that when $x = \frac{1}{2}$, T_5 is the greatest term in the expansion.

(c) Write down the greatest term when $x = \frac{1}{2}$ (and leave it factored).

4. Let $(1 + 5x)^{21} = \sum_{k=0}^{21} T_k$, where T_k is the term in x^k .

(a) Write down expressions for T_k and T_{k+1} , and show that $\frac{T_{k+1}}{T_k} = \frac{5x(21 - k)}{k + 1}$.

(b) Hence show that when $x = \frac{3}{5}$, T_{16} is the greatest term in the expansion.

(c) Write down the greatest term when $x = \frac{3}{5}$ (and leave it factored).

5. (a) Let $(5 + 2x)^{15} = \sum_{k=0}^{15} t_k x^k$.

(i) Write down expressions for t_k and t_{k+1} , and show that $\frac{t_{k+1}}{t_k} = \frac{30 - 2k}{5k + 5}$.

(ii) Hence show that t_4 is the greatest coefficient.

(iii) Write down the greatest coefficient (and leave it factored).

(b) Let $(5 + 2x)^{15} = \sum_{k=0}^{15} T_k$, where T_k is the term in x^k .

(i) Write down T_k and T_{k+1} , and show that when $x = \frac{5}{3}$, $\frac{T_{k+1}}{T_k} = \frac{30 - 2k}{3k + 3}$.

(ii) Hence show that when $x = \frac{5}{3}$, T_6 is the greatest term in the expansion.

(iii) Write down the greatest term when $x = \frac{5}{3}$ (and leave it factored).

DEVELOPMENT

6. For each expansion, find: (i) the greatest coefficient, (ii) the greatest term if $x = \frac{2}{3}$.
 (a) $(1 + 4x)^{11}$ (b) $(2 + \frac{3}{4}x)^9$ (c) $(5x + 3)^{12}$ (d) $(5 + 6x)^{11}$
7. For each of the following expansions, find: (i) the coefficient with greatest absolute value, (ii) the term with greatest absolute value if $x = \frac{2}{3}$ and $y = 3$.
 (a) $(1 - 7x)^9$ (b) $(7 - 2x)^{14}$ (c) $(x - 2y)^{12}$ (d) $(2x - y)^{15}$
8. For each of the following expansions, find: (i) the greatest coefficient, (ii) the greatest term if $x = \frac{3}{2}$ and $y = \frac{4}{9}$.
 (a) $\left(2x^2 + \frac{3}{x}\right)^{10}$ (b) $(2x + 3y)^{12}$
9. Show that in the expansion of $(1 + x)^{14}$, where $x = \frac{2}{3}$, two consecutive terms are equal to each other and greater than any other term.
10. Let $(x + y)^n = \sum_{r=0}^n T_r$, where T_r is the term in $x^{n-r}y^r$.
 (a) Write down expressions for T_r and T_{r+1} and hence show that $\frac{T_{r+1}}{T_r} = \frac{(n-r)y}{(r+1)x}$.
 (b) Consider the expansion of $(a + 3b)^8$, where $a = 2$ and $b = \frac{3}{4}$. By substituting the appropriate values for x, y and n into the expression in (i), show that $\frac{T_{r+1}}{T_r} = \frac{72 - 9r}{8r + 8}$. Hence show that T_4 is the numerically greatest term in the expansion.
 (c) Find the greatest term in the expansion $(p + q)^{10}$, where $p = q = \frac{1}{2}$.
11. Let $(1 + x)^n = \sum_{r=0}^n t_r x^r$.
 (a) Find the values of n and r if $t_{r+1} = 5t_r$ and $t_{r+4} = 2t_{r+3}$.
 (b) Hence find the greatest coefficient in the expansion.
12. Let $(1 + 0.01)^{12} = \sum_{r=0}^{12} T_r$, where T_r is the term in $(0.01)^r$. Find the first value of r for which the ratio $\frac{T_{r+1}}{T_r} < 0.005$.
13. Let $(\sin \theta + \cos \theta)^{20} = \sum_{k=0}^{20} T_k$, where T_k is the term in $\sin^{20-k} \theta \cos^k \theta$. Find, to the nearest degree, the first positive angle for which $T_{14} > T_{15}$.
14. (a) Show that in the expansion of $(1 + x)^n$, where n is an even integer, the term with the greatest coefficient is the term in $x^{\frac{1}{2}n}$.
 (b) By considering the expansion of $(1 + x)^{2n}$, for any positive integer n , prove that the largest value of ${}^{2n}C_r$ is ${}^{2n}C_n$.

EXTENSION

15. [An alternative proof of the binomial theorem by differentiation] Let $f(x) = (1 + x)^n$.
 (a) Find the k th derivative of $f(x)$, and show that $f^{(k)}(0) = \frac{n!}{(n-k)!}$.

(b) Expand $f(x)$ using the binomial theorem, and show that $f^{(k)}(0) = k! {}^n C_k$.

(c) By equating these two expressions for $f^{(k)}(0)$, prove that ${}^n C_k = \frac{n!}{k!(n-k)!}$.

16. [The Poisson probability distribution] The probability that n car accidents occur at a given set of traffic lights during a year is

$$P_n = \frac{e^{-2.6} \times 2.6^n}{n!}, \text{ for } n = 0, 1, 2, \dots$$

By considering values of n for which $\frac{P_{n+1}}{P_n} \geq 1$, determine the most likely number of accidents at this intersection in a given one-year period.

5F Identities on the Binomial Coefficients

There are a great number of patterns in the Pascal triangle. Some are quite straightforward to recognise and to prove, others are more complicated. They can be very important in any application of the binomial theorem, and many of them will reappear in Chapter Ten on probability. Each pattern in the Pascal triangle is described by an identity on the binomial coefficients ${}^n C_k$ — these identities have a rather forbidding appearance, and it is important to take the time to interpret each identity as some sort of pattern in the Pascal triangle.

Methods of proof as well as the identities themselves are the subject of this section. Each proof begins with some form of the binomial expansion

$$(x + y)^n = \sum_{k=0}^n {}^n C_k x^{n-k} y^k.$$

Three approaches to generating identities from this expansion are developed in turn: substitutions, methods from calculus, and equating coefficients.

Here again is the first part of the Pascal triangle. Each identity that is obtained should be interpreted as a pattern in the triangle and verified there, either before or after the proof is completed.

$n \backslash r$	0	1	2	3	4	5	6	7	8		
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

The First Approach — Substitution: Substitutions into the basic binomial expansion or any subsequent development from it will yield identities.

WORKED EXERCISE: Obtain identities by substituting into the basic expansion:

- (a) $x = 1$ and $y = 1$ (b) $x = 1$ and $y = -1$ (c) $x = 1$ and $y = 2$

Then explain what pattern each identity describes in the Pascal triangle.

SOLUTION: We begin with the expansion $(x + y)^n = \sum_{k=0}^n {}^nC_k x^{n-k} y^k$.

- (a) Substituting $x = 1$ and $y = 1$ gives $2^n = \sum_{k=0}^n {}^nC_k$,

that is, ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

In the Pascal triangle, this means that the sum of every row is 2^n .

For example, $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$ (as proven in Section 5A).

- (b) Substituting $x = 1$ and $y = -1$ gives $0 = \sum_{k=0}^n {}^nC_k (-1)^k$,

that is, ${}^nC_0 - {}^nC_1 + {}^nC_2 - \cdots + (-1)^n {}^nC_n = 0$.

This means that the alternating sum of every row is zero.

For odd n , this is trivial: $1 - 5 + 10 - 10 + 5 - 1 = 0$,

but for even n , $1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$.

- (c) Substituting $x = 1$ and $y = 2$ gives $3^n = \sum_{k=0}^n {}^nC_k 2^k$,

that is, $1 \times {}^nC_0 + 2 \times {}^nC_1 + 2^2 \times {}^nC_2 + \cdots + 2^n \times {}^nC_n = 3^n$.

Taking as an example the row 1, 4, 6, 4, 1,

$$1 \times 1 + 2 \times 4 + 4 \times 6 + 8 \times 4 + 16 \times 1 = 81 = 3^4.$$

Second Approach — Differentiation and Integration: The basic expansion can be differentiated or integrated before substitutions are made. As always, integration involves finding an unknown constant.

WORKED EXERCISE: Consider the expansion $(1 + x)^n = \sum_{k=0}^n {}^nC_k x^k$.

- (a) Differentiate the expansion, then substitute $x = 1$ to obtain an identity.

- (b) Integrate the expansion, then substitute $x = -1$ to obtain an identity.

Then give an example of each identity on the Pascal triangle.

SOLUTION:

- (a) Differentiating, $n(1 + x)^{n-1} = \sum_{k=0}^n k \times {}^nC_k \times x^{k-1}$.

Substituting $x = 1$, $n \times 2^{n-1} = \sum_{k=0}^n k \times {}^nC_k$,

that is, $0 \times {}^nC_0 + {}^nC_1 + 2 \times {}^nC_2 + 3 \times {}^nC_3 + \cdots + n \times {}^nC_n = 0$.

Taking as an example the row 1, 4, 6, 4, 1,

$$0 \times 1 + 1 \times 4 + 2 \times 6 + 3 \times 4 + 4 \times 1 = 32 = 4 \times 2^3.$$

- (b) Integrating, $\frac{(1 + x)^{n+1}}{n + 1} = C + \sum_{k=0}^n \frac{{}^nC_k x^{k+1}}{k + 1}$, for some constant C .

To find the constant C of integration, substitute $x = 0$,

$$\text{then } \frac{1}{n+1} = C + \sum_{k=0}^n 0,$$

$$\text{so } C = \frac{1}{n+1}, \text{ and } \frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + \sum_{k=0}^n \frac{{}^nC_k x^{k+1}}{k+1}.$$

$$\text{Substituting } x = -1, \quad 0 = \frac{1}{n+1} + \sum_{k=0}^n \frac{{}^nC_k (-1)^{k+1}}{k+1},$$

$$\text{that is, } \frac{1}{n+1} - {}^nC_0 + \frac{{}^nC_1}{2} - \frac{{}^nC_2}{3} + \frac{{}^nC_3}{4} - \cdots + \frac{(-1)^{n+1} {}^nC_n}{n+1} = 0$$

$$\boxed{\times (-1)} \quad {}^nC_0 - \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} - \frac{{}^nC_3}{4} + \cdots + \frac{(-1)^n {}^nC_n}{n+1} = \frac{1}{n+1}.$$

Taking as an example the row 1, 4, 6, 4, 1,

$$1 \times 1 - \frac{1}{2} \times 4 + \frac{1}{3} \times 6 - \frac{1}{4} \times 4 + \frac{1}{5} \times 1 = 1 - 2 + 2 - 1 + \frac{1}{5} = \frac{1}{5}.$$

Third Approach — Equating Coefficients: The third method involves taking two equal expansions and equating coefficients.

WORKED EXERCISE: Taking $\left(x + \frac{1}{x}\right)^n \left(x + \frac{1}{x}\right)^n = \left(x + \frac{1}{x}\right)^{2n}$ and expanding and equating constants, prove that

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2 = {}^{2n}C_n.$$

Then interpret the identity on the Pascal triangle.

SOLUTION: The constant term on the RHS is ${}^{2n}C_n$.

The constant term on the LHS is the sum of the products

$${}^nC_0 \times {}^nC_n + {}^nC_1 \times {}^nC_{n-1} + {}^nC_2 \times {}^nC_{n-2} + \cdots + {}^nC_n \times {}^nC_0,$$

and because ${}^nC_{n-k} = {}^nC_k$, by the symmetry of the row, this constant term is

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2.$$

Equating the two constant terms,

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2 = {}^{2n}C_n, \text{ as required.}$$

This means that if the entries of any row are squared and added, the sum is the middle entry in the row twice as far down. For example, with the row 1, 3, 3, 1,

$$1^2 + 3^2 + 3^2 + 1^2 = 20 = {}^6C_3,$$

and with the row 1, 4, 6, 4, 1,

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = {}^8C_4.$$

Exercise 5F

1. Consider the identity $(1+x)^4 = {}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4$. Prove the following, and explain each result in terms of the row indexed by $n = 4$ in Pascal's triangle.
- (a) By substituting $x = 1$, show that ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4$.
- (b) (i) By substituting $x = -1$, show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = {}^4C_1 + {}^4C_3$.
(ii) Hence, by using the result of part (a), show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = 2^3$.
- (c) (i) Differentiate both sides of the identity.
(ii) By substituting $x = 1$, show that ${}^4C_1 + 2{}^4C_2 + 3{}^4C_3 + 4{}^4C_4 = 4 \times 2^3$.
(iii) By substituting $x = -1$, show that ${}^4C_1 - 2{}^4C_2 + 3{}^4C_3 - 4{}^4C_4 = 0$.
- (d) (i) By integrating both sides of the identity, show that for some constant K ,

$$\frac{1}{5}(1+x)^5 + K = {}^4C_0x + \frac{1}{2}{}^4C_1x^2 + \frac{1}{3}{}^4C_2x^3 + \frac{1}{4}{}^4C_3x^4 + \frac{1}{5}{}^4C_4x^5.$$

- (ii) By substituting $x = 0$, show that $K = -\frac{1}{5}$.
(iii) By substituting $x = -1$ in part (i), show that

$${}^4C_0 - \frac{1}{2}{}^4C_1 + \frac{1}{3}{}^4C_2 - \frac{1}{4}{}^4C_3 + \frac{1}{5}{}^4C_4 = \frac{1}{5}.$$

2. Consider the identity $(1+x)^n = \sum_{k=0}^n {}^nC_k x^k$. Prove the following, and explain each result in terms of the row indexed by $n = 5$ in Pascal's triangle.

- (a) By substituting $x = 1$, show that $\sum_{k=0}^n {}^nC_k = 2^n$.
- (b) (i) By substituting $x = -1$, show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots$.
(ii) Hence, by using the result of part (a), show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = 2^{n-1}$.
- (c) (i) Differentiate both sides of the identity.

- (ii) By substituting $x = 1$, show that $\sum_{k=1}^n k {}^nC_k = n 2^{n-1}$.

- (iii) By substituting $x = -1$, show that $\sum_{k=1}^n (-1)^{k-1} k {}^nC_k = 0$.

- (d) (i) By integrating both sides of the identity, show that for some constant K ,

$$\frac{1}{n+1}(1+x)^{n+1} + K = \sum_{k=0}^n {}^nC_k \frac{x^{k+1}}{k+1}.$$

- (ii) By substituting $x = 0$, show that $K = -\frac{1}{n+1}$.

- (iii) By substituting $x = -1$ in part (i), show that $\sum_{k=0}^n (-1)^k \frac{{}^nC_k}{k+1} = \frac{1}{n+1}$.

3. This question follows the same steps as question 2.

Consider the identity $(1+x)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r x^r$.

- (a) Show that $\sum_{r=0}^{2n} {}^{2n}C_r = 2^{2n}$. (b) Show that ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \dots + {}^{2n}C_{2n-1} = 2^{2n-1}$.

Check both results on the Pascal triangle, using $n = 3$ and $n = 4$.

(c) By differentiating both sides of the identity, show that:

$$(i) \sum_{r=1}^{2n} r {}^{2n}C_r = n 2^{2n} \qquad (ii) \sum_{r=1}^{2n} (-1)^{r-1} r {}^{2n}C_r = 0$$

Check both results on the Pascal triangle, using $n = 3$ and $n = 4$.

(d) By integrating both sides of the identity, show that:

$$(i) \sum_{r=0}^{2n} {}^{2n}C_r \frac{1}{r+1} = \frac{2^{2n+1} - 1}{2n+1} \quad [\text{HINT: The constant of integration is } -\frac{1}{2n+1}.]$$

$$(ii) \sum_{r=0}^{2n} (-1)^r \frac{{}^{2n}C_r}{r+1} = \frac{1}{2n+1}.$$

Check both results on the Pascal triangle, using $n = 3$ and $n = 4$.

4. (a) By equating the coefficients of x^3 on the RHS and LHS of the identity

$$(1+x)^3(1+x)^9 = (1+x)^{12},$$

show that ${}^3C_0 {}^9C_3 + {}^3C_1 {}^9C_2 + {}^3C_2 {}^9C_1 + {}^3C_3 {}^9C_0 = {}^{12}C_3$.

(b) By equating the coefficients of x^3 on the RHS and LHS of the identity

$$(1+x)^m(1+x)^n = (1+x)^{m+n},$$

show that ${}^mC_0 {}^nC_3 + {}^mC_1 {}^nC_2 + {}^mC_2 {}^nC_1 + {}^mC_3 {}^nC_0 = {}^{m+n}C_3$.

DEVELOPMENT

5. (a) Show that ${}^nC_k = {}^nC_{n-k}$.

(b) By comparing coefficients of x^{10} on both sides of $(1+x)^{10}(1+x)^{10} = (1+x)^{20}$, show that $\sum_{k=0}^{10} ({}^{10}C_k)^2 = {}^{20}C_{10}$.

(c) By comparing coefficients of x^n on both sides of the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that $\sum_{k=0}^n ({}^nC_k)^2 = {}^{2n}C_n$. Check this identity on the Pascal triangle by adding the squares of the rows indexed by $n = 1, 2, 3, 4, 5$ and 6 .

(d) By comparing coefficients of x^{n+1} on both sides of $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that

$${}^nC_0 \times {}^nC_1 + {}^nC_1 \times {}^nC_2 + {}^nC_2 \times {}^nC_3 + \cdots + {}^nC_{n-1} \times {}^nC_n = \frac{(2n)!}{(n-1)!(n+1)!}.$$

Check this identity on the rows indexed by $n = 3, 4, 5$ and 6 of the Pascal triangle.

(e) Prove that $(1+x)^n \left(1 + \frac{1}{x}\right)^n = \frac{1}{x^n} (1+x)^{2n}$. By equating coefficients of $\frac{1}{x}$, give an alternative proof of the result in part (d).

6. (a) By equating coefficients of $\frac{1}{x^4}$ in the expansion of $\left(1 + \frac{1}{x}\right)^4 \left(1 - \frac{1}{x}\right)^4 = \left(1 - \frac{1}{x^2}\right)^4$, prove that $({}^4C_0)^2 - ({}^4C_1)^2 + ({}^4C_2)^2 - ({}^4C_3)^2 + ({}^4C_4)^2 = {}^4C_2$.

(b) Generalise this result, and prove it, by considering the expansion of

$$\left(1 + \frac{1}{x}\right)^{2n} \left(1 - \frac{1}{x}\right)^{2n} = \left(1 - \frac{1}{x^2}\right)^{2n}.$$

Check your identity on the Pascal triangle, for $n = 4, 5$ and 6 .

7. (a) By expanding both sides of the identity $(1+x)^{n+4} = (1+x)^n(1+x)^4$, show that

$$\binom{n+4}{r} = \binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4},$$

and state the necessary restriction on r . Check this identity on the Pascal triangle, using $n = r = 5$ and using $n = 6$ and $r = 4$.

- (b) By expanding both sides of the identity $(1+x)^{p+q} = (1+x)^p(1+x)^q$, show that

$$\binom{p+q}{r} = \binom{p}{r} + \binom{p}{r-1}\binom{q}{1} + \binom{p}{r-2}\binom{q}{2} + \cdots + \binom{p}{1}\binom{q}{r-1} + \binom{q}{r},$$

and state the necessary restriction on r .

8. (a) By considering the expansion of $(1+x)^n$, show that:

$$(i) \sum_{k=0}^n {}^nC_k = 2^n \qquad (ii) \sum_{k=1}^n k {}^nC_k = n 2^{n-1}$$

- (b) Hence show that $\sum_{k=0}^n (k+1) {}^nC_k = 2^{n-1}(n+2)$. Check this identity on the Pascal triangle, using $n = 4, 5$ and 6 .

9. (a) Find the coefficient of x^{n+r} in the expansion of $(1+x)^{3n}$.

- (b) By writing $(1+x)^{3n}$ as $(1+x)^n(1+x)^{2n}$, prove that for $0 < r \leq n$,

$$\binom{n}{0}\binom{2n}{r} + \binom{n}{1}\binom{2n}{r+1} + \binom{n}{2}\binom{2n}{r+2} + \cdots + \binom{n}{n}\binom{2n}{r+n} = \binom{3n}{n+r}.$$

Check this identity on the Pascal triangle, using $n = 4$ and $r = 3$.

10. (a) Evaluate $\int_0^4 (1+x)^n dx$.

- (b) By expanding $(1+x)^n$ and then integrating, show that $\sum_{r=0}^n \frac{4^{r+1}}{r+1} {}^nC_r = \frac{5^{n+1} - 1}{n+1}$.

Check this identity on the Pascal triangle, for $n = 3$ and $n = 4$.

11. (a) Show that $x^n(1+x)^n \left(1 + \frac{1}{x}\right)^n = (1+x)^{2n}$.

- (b) Hence prove that $1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

12. (a) When the entries of the row 1, 5, 10, 10, 5, 1 indexed by $n = 5$ in Pascal's triangle are multiplied by 0, 1, 2, 3, 4, 5 respectively, the results are 0, 5, 20, 30, 20, 5. Ignoring the zero, this is five times the row 1, 4, 6, 4, 1. Formulate this result algebraically, for $n = 5$ and then for generally n , and prove it using the binomial theorem.

- (b) When the entries of the row 1, 5, 10, 10, 5, 1 are divided by 1, 2, 3, 4, 5 and 6 respectively, the result is $1, 2\frac{1}{2}, 3\frac{1}{2}, 2\frac{1}{2}, 1, \frac{1}{6}$. If you add $\frac{1}{6}$ at the start, this is $\frac{1}{6}$ th of the row 1, 6, 15, 20, 15, 6, 1. Formulate this result algebraically, for $n = 5$ and then for general n , and prove it using the binomial theorem.

13. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, show that

$$c_0c_2 + c_1c_3 + c_2c_4 + \cdots + c_{n-2}c_n = \frac{(2n)!}{(n+2)!(n-2)!}.$$

14. (a) Consider the row 1, 7, 21, 35, 35, 21, 7, 1 from Pascal's triangle. If a, b, c and d are any four consecutive terms from this row, show that $\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$.
- (b) Choose four consecutive terms from any other row and show that the identity holds.
- (c) Prove the identity by letting $a = {}^nC_{r-1}$, $b = {}^nC_r$, $c = {}^nC_{r+1}$ and $d = {}^nC_{r+2}$. You will need to use the addition property of Pascal's triangle.
15. (a) By using the substitution $u = \sin x$, prove that $\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx = \frac{1}{2k+1}$, where k is a positive integer.
- (b) By writing $\cos^{2n+1} x = \cos^{2n} x \cos x = (1 - \sin^2 x)^n \cos x$, show that

$$\cos^{2n+1} x = \sum_{k=0}^n {}^nC_k (-1)^k \sin^{2k} x \cos x.$$

(c) Hence, by using part (a), show that $\int_0^{\frac{\pi}{2}} \cos^{2n+1} x \, dx = \sum_{k=0}^n \frac{(-1)^k {}^nC_k}{2k+1}$.

(d) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$.

EXTENSION

16. [The hockey stick pattern] Use induction to prove the result from question 22(c) of Exercise 5A. That is, for any fixed value of n and p , where $n < p$, prove that

$${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{n+p}C_p = {}^{n+p+1}C_p.$$

You will need to use the addition property of Pascal's triangle.

17. By substituting $x = 1$ into the expansion of $(1+x)^{2n}$, prove that $\sum_{r=0}^n {}^{2n}C_r = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$.

18. (a) Show that $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n = \frac{(1+x)^{n+1} - 1}{x}$.

(b) By considering the coefficient of x^r on both sides of this identity, show that

$${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^rC_r = {}^{n+1}C_{r+1}.$$

19. By considering the identity $(1-x^2)^n = (1+x)^n(1-x)^n$ or otherwise, show that

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2$$

is zero, when n is odd, but that when n is even, its value is

$$\frac{(-1)^{\frac{n}{2}}(n+2)(n+4)\cdots(2n)}{2 \times 4 \times \cdots \times n} = \frac{(-1)^{\frac{n}{2}} n!}{\left(\left(\frac{n}{2}\right)!\right)^2}.$$

20. [APs in the Pascal triangle]

(a) Show that ${}^nC_{r-1} = \frac{r}{n-r+1} {}^nC_r$ and ${}^nC_{r+1} = \frac{n-r}{r+1} {}^nC_r$.

(b) Show that if ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ form an AP, then

- (i) $n+2 = (n-2r)^2$ is a perfect square, and
 (ii) $r = \frac{1}{2}(n - \sqrt{n+2})$ or $r = \frac{1}{2}(n + \sqrt{n+2})$.

(c) Hence find the first three rows of the Pascal triangle in which three consecutive terms form an AP, and identify those terms.

(d) Prove that four consecutive terms in a row of the Pascal triangle cannot form an AP.

Further Calculus

This chapter deals with four related topics which complete the treatment of calculus at 3 Unit level, apart from the extra work on rates of change in the next chapter. First, the systematic differentiation and integration of trigonometric functions is completed. Secondly, the reverse chain rule is extended to a more general method of integration called *integration by substitution*. Thirdly, the derivative is used to develop a very effective method, called *Newton's method*, of finding approximate solutions of equations. Fourthly, some of the earlier material on limits and inequalities is reviewed and summarised.

STUDY NOTES: The reciprocal trigonometric functions $\sec x$, $\operatorname{cosec} x$ and $\cot x$ are not central to the course, but Sections 6A and 6B are intended to teach sufficient familiarity with them, as well as reviewing and summarising the previous approaches to the calculus of trigonometric functions. Later questions in Exercises 6A and 6B become difficult, and many students may not want to pursue the exercises very far. Sections 6C and 6D develop integration by substitution, extending the reverse chain rule presented first in Chapter Ten of the Year 11 volume, and they should also provide a good summary of the previous methods of integration. It is suggested that 4 Unit students study Sections 6A–6D before or while they embark on the systematic integration of the 4 Unit course.

Section 6E develops two methods of finding approximate solutions of equations called *halving the interval* and *Newton's method*. The final Section 6F is very demanding. It is intended for 4 Unit students and for the more ambitious 3 Unit students, and could well be left until final revision. The section reviews and develops previous approaches to inequalities and limits, and involves arguments based on the derivative, on bounding a carefully chosen integral, on geometry, and on algebra.

6A Differentiation of the Six Trigonometric Functions

So far, the derivatives of $\sin x$, $\cos x$ and $\tan x$ have been established. While the derivatives of the other three trigonometric functions can be calculated when needed, the patterns become clearer when all six derivatives are listed, and it is recommended that they all be memorised.

Differentiating the Three Reciprocal Functions: The differentiation of the three functions $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$ depends on the formula for differentiating the reciprocal of a function:

$$\text{if } y = \frac{1}{u}, \text{ then by the chain rule, } \frac{dy}{dx} = -\frac{du/dx}{u^2}.$$

$$\begin{aligned} \text{A. Let } y &= \sec x \\ &= \frac{1}{\cos x}. \end{aligned}$$

$$\begin{aligned} \text{Then } y' &= -\frac{-\sin x}{\cos^2 x} \\ &= \sec x \tan x. \end{aligned}$$

$$\begin{aligned} \text{B. Let } y &= \operatorname{cosec} x \\ &= \frac{1}{\sin x}. \end{aligned}$$

$$\begin{aligned} \text{Then } y' &= -\frac{\cos x}{\sin^2 x} \\ &= -\operatorname{cosec} x \cot x. \end{aligned}$$

$$\begin{aligned} \text{C. Let } y &= \cot x \\ &= \frac{1}{\tan x}. \end{aligned}$$

$$\begin{aligned} \text{Then } y' &= -\frac{\sec^2 x}{\tan^2 x} \\ &= -\operatorname{cosec}^2 x. \end{aligned}$$

Here then is the list of all six derivatives.

THE DERIVATIVES OF THE SIX TRIGONOMETRIC FUNCTIONS:

1

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

The extensions of these standard forms to trigonometric functions of linear functions of x now follow easily. For example,

$$\frac{d}{dx} \sec(ax + b) = a \sec(ax + b) \tan(ax + b).$$

Remarks on these Derivatives: There are two patterns here that will help in memorising the results. These patterns should be studied in comparison with the graphs of all six trigonometric functions, reproduced again on the next full page.

First, the derivatives of the three co-functions — cosine, cotangent and cosecant — all begin with a negative sign. This is because the three co-functions all have negative gradient in the first quadrant, as can be seen from their graphs on the next page.

Secondly, the derivative of each co-function is obtained by adding the prefix ‘co-’, as well as adding the minus sign. For example,

$$\frac{d}{dx} \tan x = \sec^2 x \quad \text{and} \quad \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x.$$

WORKED EXERCISE:

(a) If $y = \tan x$, show that $y'' - 2y^3 - 2y = 0$.

(b) If $y = \sec x$, show that $y'' - 2y^3 + y = 0$.

SOLUTION:

(a) If $y = \tan x$,
then $y' = \sec^2 x$.

Using the chain rule,

$$\begin{aligned} y'' &= 2 \sec x \times \sec x \tan x \\ &= 2 \sec^2 x \tan x \\ &= 2(\tan^2 x + 1) \tan x. \end{aligned}$$

$$\text{Hence } y'' = 2y^3 + 2y.$$

(b) If $y = \sec x$,

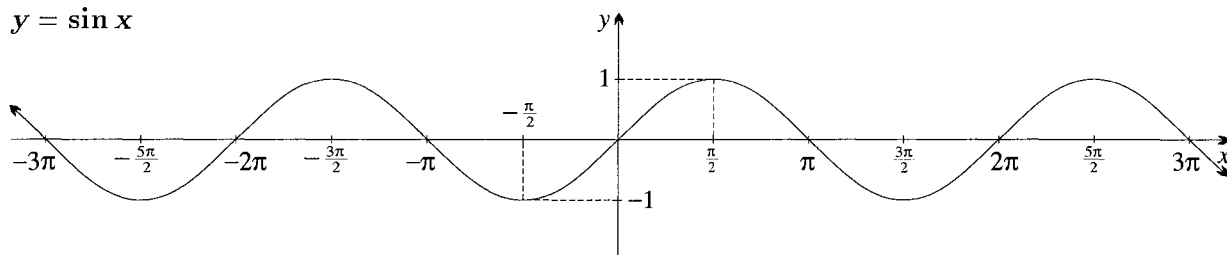
then $y' = \sec x \tan x$.

Using the product rule,

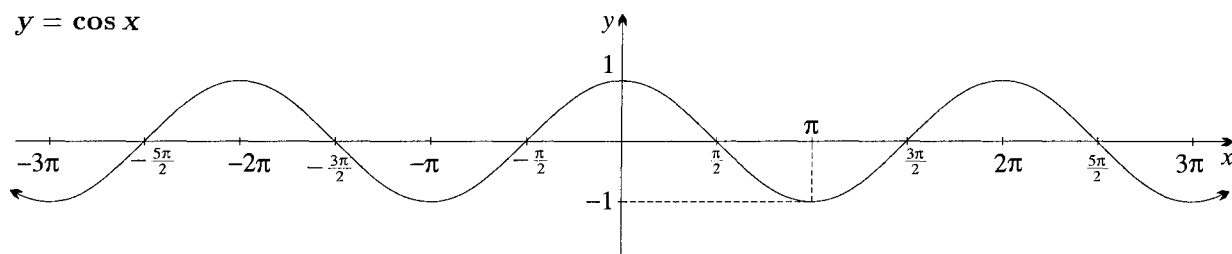
$$\begin{aligned} y'' &= (\sec x \tan x) \tan x + \sec x \sec^2 x \\ &= \sec x (\sec^2 x - 1) + \sec^3 x \\ &= 2 \sec^3 x - \sec x. \end{aligned}$$

$$\text{Hence } y'' = 2y^3 - y.$$

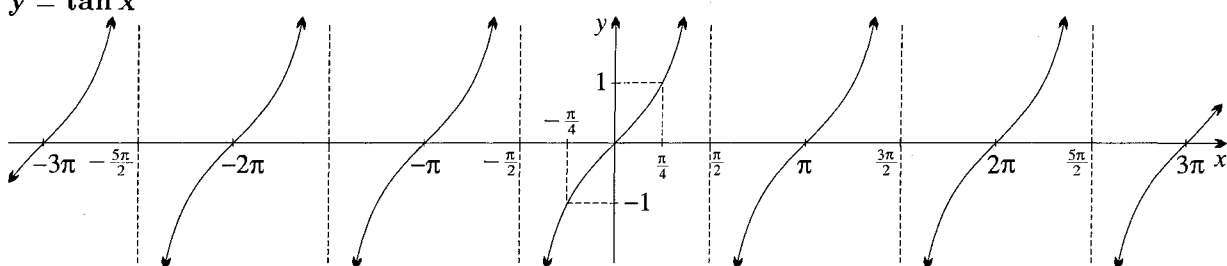
$y = \sin x$



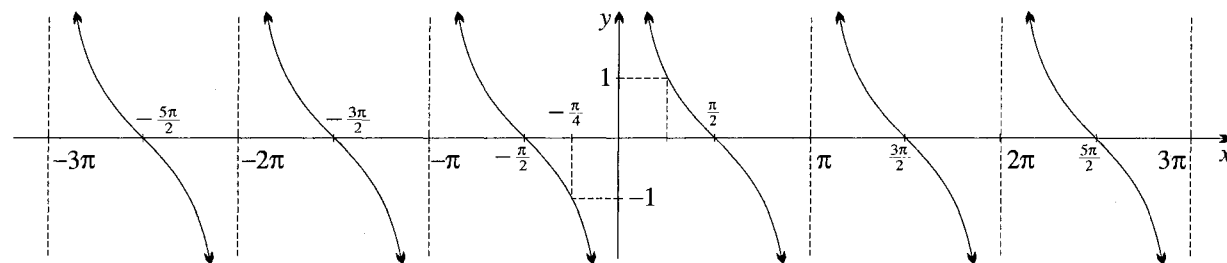
$y = \cos x$



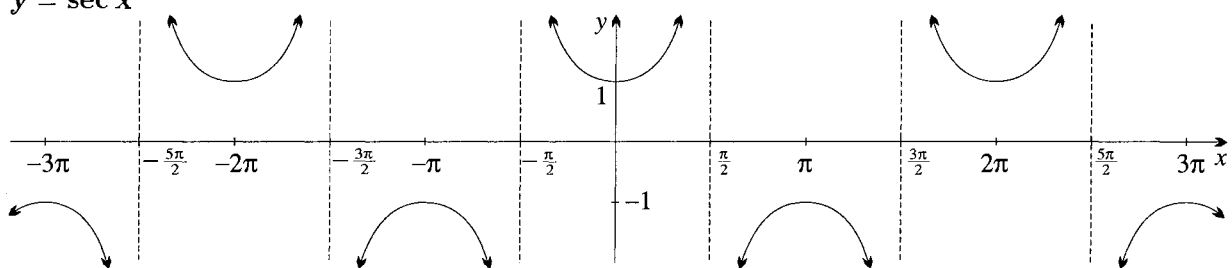
$y = \tan x$



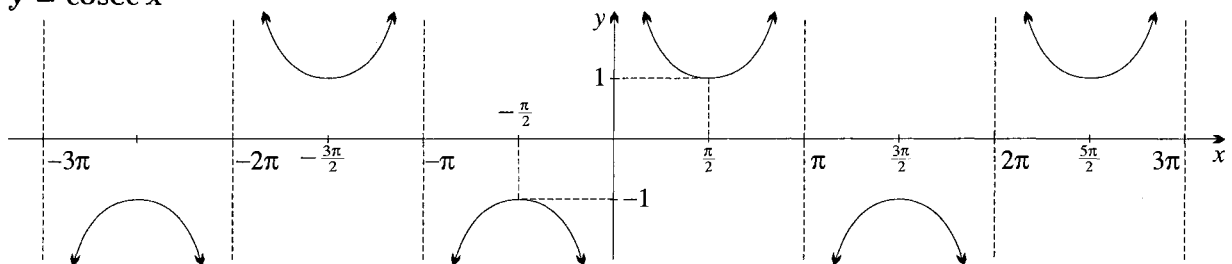
$y = \cot x$



$y = \sec x$



$y = \operatorname{cosec} x$



WORKED EXERCISE: Find any points on $y = \sec x$, for $0 \leq x \leq 2\pi$, where the tangent has gradient $\sqrt{2}$.

SOLUTION: Differentiating,

$$y' = \sec x \tan x.$$

Put $y' = \sqrt{2}$, then

$$\sec x \tan x = \sqrt{2}$$

$$\boxed{\times \cos^2 x}$$

$$\sin x = \sqrt{2} \cos^2 x.$$

$$\text{Since } \cos^2 x = 1 - \sin^2 x, \quad \sqrt{2} \sin^2 x + \sin x - \sqrt{2} = 0.$$

$$\text{Since } \Delta = 1 + 4 \times 2 = 9,$$

$$\begin{aligned} \sin x &= \frac{-1+3}{2\sqrt{2}} \text{ or } \frac{-1-3}{2\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \text{ or } -\sqrt{2}. \end{aligned}$$

The second value is less than -1 and so gives no solutions.

Hence $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, and the points are $(\frac{\pi}{4}, \sqrt{2})$ and $(\frac{3\pi}{4}, -\sqrt{2})$.

Exercise 6A

1. Differentiate with respect to x :

(a) $\sec x$

(c) $\cot x$

(e) $\cot(1-x)$

(b) $\operatorname{cosec} x$

(d) $\operatorname{cosec} 3x$

(f) $\sec(5x-2)$

2. Find the gradient of each curve at the point on it where $x = \frac{\pi}{6}$:

(a) $y = \sec 2x$

(b) $y = \cot 2x$

3. Find the equation of the tangent to each curve at the point indicated:

(a) $y = \cot 3x$ at $x = \frac{\pi}{12}$

(c) $y = \cos x + \sec x$ at $x = \frac{\pi}{3}$

(b) $y = \operatorname{cosec} x$ at $x = \frac{\pi}{4}$

(d) $y = \sec 5x$ at $x = \frac{\pi}{5}$

4. Differentiate with respect to x :

(a) $e^{\cot x}$

(c) $x \operatorname{cosec} x$

(e) $\sec^4 x$

(g) $e^{2x} \sec 2x$

(b) $\log_e(\sec x)$

(d) $\cot^2 x$

(f) $\log(\cot x)$

(h) $\frac{\operatorname{cosec}^2 x}{x^2}$

5. Consider the curve $y = \tan x + \cot x$, for $-\pi < x < \pi$.

(a) For which values of x in the given domain is y undefined?

(b) Is the function even or odd or neither?

(c) Show that the curve has no x -intercepts, and examine its sign in the four quadrants.

(d) Show that $y' = 0$ when $\tan^2 x = 1$.

(e) Find the stationary points in the given domain and determine their nature.

(f) Sketch the curve over the given domain.

(g) Show that the equation of the curve can be written as $y = 2 \operatorname{cosec} 2x$.

DEVELOPMENT

6. Show that:

(a) $\frac{d}{dx}(x \sec^2 x - \tan x) = 2x \sec^2 x \tan x$

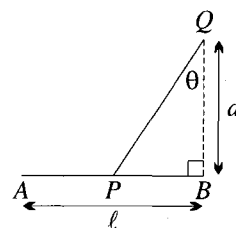
(c) $\frac{d}{dx} \left(\frac{1 + \tan x}{\sec x} \right) = \frac{1 - \tan x}{\sec x}$

(b) $\frac{d}{dx} \ln(\sec x + \tan x) = \sec x$

(d) $\frac{d}{dx} \tan^{-1}(\operatorname{cosec} x + \cot x) = -\frac{1}{2}$

7. If $y = \operatorname{cosec} x$, show that $y'' = 2y^3 - y$.

8. (a) Show that $\frac{d}{dx}(\sec x \tan x) = \sec x(2 \sec^2 x - 1)$.
 (b) Hence find the values of x for which the function $y = \sec x \tan x$ is decreasing in the interval $0 \leq x \leq 2\pi$.
9. Consider the function $f(x) = \tan x - \cot x - 4x$, defined for $0 < x < \pi$.
 (a) Show that $f'(x) = (\tan x - \cot x)^2$.
 (b) For what value of x in the domain $0 < x < \pi$ is $f(x)$ undefined?
 (c) Find any stationary points and determine their nature.
 (d) Sketch the graph of $f(x)$.
10. Consider the curve $y = 3\sqrt{3} \sec x - \operatorname{cosec} x$ over the domain $0 < x < 2\pi$.
 (a) For what values of x is y undefined? (b) Show that $y' = 0$ when $\tan x = -\frac{1}{\sqrt{3}}$.
 (c) Find the stationary points and determine their nature.
 (d) Use a calculator to examine the behaviour of y as $x \rightarrow 0^+$, as $x \rightarrow \frac{\pi}{2}^+$, as $x \rightarrow \pi^+$, and as $x \rightarrow \frac{3\pi}{2}^+$, and also as $x \rightarrow \frac{\pi}{2}^-$, as $x \rightarrow \pi^-$, as $x \rightarrow \frac{3\pi}{2}^-$, and as $x \rightarrow 2\pi^-$.
 (e) Hence sketch the curve.
11. Use a similar approach to the previous question to sketch $y = \operatorname{cosec} x + \sec x$ for $0 < x < 2\pi$.
12. (a) Show that $\frac{d}{dx} \tan^{-1}(\cot x) = -1$.
 (b) Show that $\frac{d}{dx} \cos^{-1}(\sin x) = -1$, provided that $\cos x > 0$.
 (c) Hence explain why each piece of $y = \cos^{-1}(\sin x) - \tan^{-1}(\cot x)$ is horizontal for $\cos x > 0$, and find the value of the constant when:
 (i) x is in the first quadrant, (ii) x is in the fourth quadrant.
13. Differentiate with respect to x :
 (a) $\cot \frac{1}{x}$ (b) $\log \log \sec x$ (c) $\frac{1}{\tan 3x - \sec 3x}$
14. A curve is defined parametrically by the equations $x = 2 \sec \theta$, $y = 3 \tan \theta$.
 (a) Show that $\frac{dy}{dx} = \frac{3 \sec \theta}{2 \tan \theta}$.
 (b) Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$.
15. (a) Using the t -formulae, or otherwise, show that:
 (i) $\frac{1 - \cos x}{\sin x} = \tan \frac{1}{2}x$ (ii) $\frac{1 + \sin x}{\cos x} = \tan(\frac{x}{2} + \frac{\pi}{4})$
 (b) Hence show that:
 (i) $\frac{d}{dx} (\ln \tan \frac{1}{2}x) = \operatorname{cosec} x$ (ii) $\frac{d}{dx} \log \tan(\frac{x}{2} + \frac{\pi}{4}) = \sec x$
16. In the diagram, AB is a major blood vessel and PQ is a minor blood vessel. Let $AB = \ell$ units, $BQ = d$ units and $\angle PQB = \theta$. It is known that the resistance to blood flow in a blood vessel is proportional to its length, and that the constant of proportionality varies from blood vessel to blood vessel. Let R be the sum of the resistances in AP and PQ .
 (a) Show that $R = c_1(\ell - d \tan \theta) + c_2 d \sec \theta$, where c_1 and c_2 are constants of proportionality.
 (b) If $c_2 = 2c_1$, find the value of θ that minimises R .

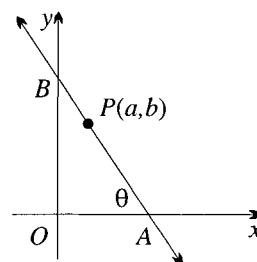


17. In the diagram, a line passes through the fixed point $P(a, b)$, where a and b are both positive, and meets the x -axis and y -axis at A and B respectively. Let $\angle OAB = \theta$.

(a) Show that $AB = a \sec \theta + b \operatorname{cosec} \theta$.

(b) Show that AB is minimum when $\tan \theta = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$.

(c) Show that the minimum distance is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.



EXTENSION

18. Differentiate implicitly to find $\frac{dy}{dx}$, given:

(a) $\cot y = \operatorname{cosec} x$

(b) $xy = \sec(x + y)$

19. Sketch the graph of the function $y = \frac{4}{\operatorname{cosec} x - \sec x}$, for $0 < x < 2\pi$.

[HINT: First find any x -intercepts, vertical asymptotes and stationary points.]

20. Use the result in question 6(d) to sketch $y = \tan^{-1}(\operatorname{cosec} x + \cot x)$.

6B Integration Using the Six Trigonometric Functions

Systematic integration of the trigonometric functions is not easy. The point of this section is learning the methods of integration — memorising results other than the six standard forms below is not required.

The Six Standard Forms: The first step is to reverse the six derivatives of the previous section to obtain the six standard forms for integration.

2

THE SIX STANDARD FORMS: Omitting constants of integration,

$$\begin{array}{lll} \int \cos x \, dx = \sin x & \int \sec^2 x \, dx = \tan x & \int \sec x \tan x \, dx = \sec x \\ \int \sin x \, dx = -\cos x & \int \operatorname{cosec}^2 x \, dx = -\cot x & \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x \end{array}$$

Again, linear extensions follow easily. For example,

$$\int \sec(ax + b) \tan(ax + b) \, dx = \frac{1}{a} \sec(ax + b) + C.$$

The Primitives of the Squares of the Trigonometric Functions: We have already integrated the squares of the trigonometric functions.

First, the primitives of $\sec^2 x$ and $\operatorname{cosec}^2 x$ are standard forms:

$$\int \sec^2 x \, dx = \tan x + C \quad \text{and} \quad \int \operatorname{cosec}^2 x \, dx = -\cot x + C.$$

Secondly, $\tan^2 x$ and $\cot^2 x$ can be integrated by writing them in terms of $\sec^2 x$ and $\operatorname{cosec}^2 x$ using the Pythagorean identities:

$$\tan^2 x = \sec^2 x - 1 \quad \text{and} \quad \cot^2 x = \operatorname{cosec}^2 x - 1.$$

Thirdly, $\sin^2 x$ and $\cos^2 x$ can be integrated by writing them in terms of $\cos 2x$:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \quad \text{and} \quad \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x.$$

The six results, and the methods of obtaining them, are listed below.

3

PRIMITIVES OF THE SQUARES OF THE SIX TRIGONOMETRIC FUNCTIONS:

$$\int \cos^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + C$$

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx = \frac{1}{2}x - \frac{1}{4} \sin 2x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) dx = -\cot x - x + C$$

The Primitives of the Six Trigonometric Functions: Surprisingly, it is harder to find primitives of the functions themselves than it is to find primitive of their squares. First, the primitives of $\sin x$ and $\cos x$ are standard forms:

$$\int \cos x \, dx = \sin x + C \quad \text{and} \quad \int \sin x \, dx = -\cos x + C.$$

Secondly, $\tan x$ and $\cot x$ can be integrated by the reverse chain rule:

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\log(\cos x) + C \end{aligned} \quad \left| \begin{array}{l} \text{Let } u = \cos x. \\ \text{Then } u' = -\sin x, \\ \text{and } \int \frac{1}{u} \frac{du}{dx} dx = \log u. \end{array} \right.$$

$$\begin{aligned} \int \cot x \, dx &= \int \frac{\cos x}{\sin x} \, dx \\ &= \log(\sin x) + C \end{aligned} \quad \left| \begin{array}{l} \text{Let } u = \sin x. \\ \text{Then } u' = \cos x, \\ \text{and } \int \frac{1}{u} \frac{du}{dx} dx = \log u. \end{array} \right.$$

Thirdly, the primitives of $\sec x$ and $\operatorname{cosec} x$ require some subtle tricks, whatever way they are found, and are beyond the 3 Unit course. One method is given here, but further details are left to the following exercise.

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \log(\sec x + \tan x) + C \end{aligned} \quad \left| \begin{array}{l} \text{Let } u = \sec x + \tan x. \\ \text{Then } u' = \sec x \tan x + \sec^2 x, \\ \text{and } \int \frac{1}{u} \frac{du}{dx} dx = \log u. \end{array} \right.$$

$$\begin{aligned} \int \operatorname{cosec} x \, dx &= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x} \, dx \\ &= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x} \, dx \\ &= -\log(\operatorname{cosec} x + \cot x) + C \end{aligned} \quad \left| \begin{array}{l} \text{Let } u = \operatorname{cosec} x + \cot x. \\ \text{Then } u' = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x, \\ \text{and } \int \frac{1}{u} \frac{du}{dx} dx = \log u. \end{array} \right.$$

Here are the six results and the methods of obtaining them.

PRIMITIVES OF THE SIX TRIGONOMETRIC FUNCTIONS:

4

$$\begin{aligned}\int \cos x \, dx &= \sin x + C \\ \int \sin x \, dx &= -\cos x + C \\ \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = -\log(\cos x) + C \\ \int \cot x \, dx &= \int \frac{\cos x}{\sin x} \, dx = \log(\sin x) + C \\ * \int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \log(\sec x + \tan x) + C \\ * \int \operatorname{cosec} x \, dx &= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x} \, dx = -\log(\operatorname{cosec} x + \cot x) + C\end{aligned}$$

*These forms are not required in the 3 Unit course.

A Special Case of the Reverse Chain Rule: The two functions $y = \cos x \sin^n x$ and $y = \sin x \cos^n x$ can be integrated easily using the reverse chain rule.

WORKED EXERCISE: Find primitives of: (a) $y = \sin x \cos^4 x$ (b) $y = \cos x \sin^n x$

SOLUTION:

$$\begin{aligned}\text{(a) } \int \sin x \cos^4 x \, dx &= -\int (-\sin x) \cos^4 x \, dx & \left| \begin{array}{l} \text{Let } u = \cos x. \\ \text{Then } \frac{du}{dx} = -\sin x, \\ \text{and } \int u^4 \frac{du}{dx} \, dx = \frac{1}{5} u^5. \end{array} \right. \\ &= -\frac{1}{5} \cos^5 x + C\end{aligned}$$

$$\begin{aligned}\text{(b) } \int \cos x \sin^n x \, dx &= \frac{\sin^{n+1} x}{n+1} + C & \left| \begin{array}{l} \text{Let } u = \sin x. \\ \text{Then } \frac{du}{dx} = \cos x, \\ \text{and } \int u^n \frac{du}{dx} \, dx = \frac{u^{n+1}}{n+1}. \end{array} \right.\end{aligned}$$

WORKED EXERCISE: [A harder question]

- (a) Find $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$ by writing $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$.
 (b) Find $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$ by writing $\cos^3 x = \cos x(1 - \sin^2 x)$.
 (c) Find $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$ by writing $\cos^4 x = (\frac{1}{2} + \frac{1}{2} \cos 2x)^2$.

SOLUTION:

$$\begin{aligned}\text{(a) } \int_0^{\frac{\pi}{2}} \cos^2 x \, dx &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\ &= \left[\frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_0^{\frac{\pi}{2}} \cos^3 x \, dx \\
 &= \int_0^{\frac{\pi}{2}} \cos x (1 - \sin^2 x) \, dx \\
 &= \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\frac{\pi}{2}} \\
 &\quad \text{(using the previous worked exercise)} \\
 &= \left(1 - \frac{1}{3}\right) - (0 - 0) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int_0^{\frac{\pi}{2}} \cos^4 x \, dx \\
 &= \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 \, dx \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right)^2 \, dx \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\right) \, dx \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x\right) \, dx \\
 &= \left[\frac{1}{4}x + \frac{1}{4} \sin 2x + \frac{1}{8}x + \frac{1}{32} \sin 4x\right]_0^{\frac{\pi}{2}} \\
 &= \left(\frac{\pi}{8} + 0 + \frac{\pi}{16} + 0\right) - (0 + 0 + 0 + 0) \\
 &= \frac{3\pi}{16}
 \end{aligned}$$

NOTE: Almost all the arguments above using primitives could have been replaced by arguments about symmetry. In particular, horizontal shifting and reflection in the x -axis will prove that

$$\int_0^{\frac{\pi}{2}} \cos 2x \, dx = \int_0^{\frac{\pi}{2}} \cos 4x \, dx = 0,$$

and arguments about reflection in $y = \frac{1}{2}$ will prove that

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 2x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 4x \, dx = \frac{\pi}{4}.$$

Students taking the 4 Unit course may like to investigate the symmetries involved.

Exercise 6B

1. Find:

$$\text{(a)} \quad \int \cos 2x \, dx$$

$$\text{(c)} \quad \int \sec^2 2x \, dx$$

$$\text{(e)} \quad \int \sec 2x \tan 2x \, dx$$

$$\text{(b)} \quad \int \sin 2x \, dx$$

$$\text{(d)} \quad \int \operatorname{cosec}^2 2x \, dx$$

$$\text{(f)} \quad \int \operatorname{cosec} 2x \cot 2x \, dx$$

2. Find:

$$\text{(a)} \quad \int \cos \frac{1}{3}x \, dx$$

$$\text{(d)} \quad \int \operatorname{cosec}^2 \frac{1}{5}(2x + 3) \, dx$$

$$\text{(b)} \quad \int \sin \frac{1}{2}(1 - x) \, dx$$

$$\text{(e)} \quad \int \sec(ax + b) \tan(ax + b) \, dx$$

$$\text{(c)} \quad \int \sec^2(4 - 3x) \, dx$$

$$\text{(f)} \quad \int \operatorname{cosec}(a - bx) \cot(a - bx) \, dx$$

3. Calculate the exact area bounded by each curve, the x -axis and the two vertical lines.

NOTE: In each case, the region lies completely above the x -axis.

$$\text{(a)} \quad y = \sec x \tan x, \quad x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{3},$$

$$\text{(c)} \quad y = \operatorname{cosec} \frac{1}{3}x \cot \frac{1}{3}x, \quad x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{4},$$

$$\text{(b)} \quad y = \operatorname{cosec}^2 2x, \quad x = \frac{\pi}{6} \text{ and } x = \frac{\pi}{4},$$

$$\text{(d)} \quad y = \tan x, \quad x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{3}.$$

4. (a) Show that $\frac{d}{dx}(\ln \sec x) = \tan x$, and hence find $\int_0^{\frac{\pi}{3}} \tan x \, dx$.
 (b) Show that $\frac{d}{dx}(\ln \sin 3x) = 3 \cot 3x$, and hence find $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \cot 3x \, dx$.
 (c) Show that $\frac{d}{dx}(\ln(\sec x + \tan x)) = \sec x$, and hence find $\int_0^{\frac{\pi}{4}} \sec x \, dx$.
 (d) Show that $\frac{d}{dx}(\frac{1}{2} \ln(\operatorname{cosec} 2x - \cot 2x)) = \operatorname{cosec} 2x$, and hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} 2x \, dx$.
5. Express $\sin^2 x$ in terms of $\cos 2x$, and hence find:
 (a) $\int \sin^2 x \, dx$ (b) $\int \sin^2 2x \, dx$ (c) $\int \sin^2 \frac{1}{4}x \, dx$ (d) $\int_0^{\frac{\pi}{3}} \sin^2 3x \, dx$
6. Express $\cos^2 x$ in terms of $\cos 2x$, and hence find:
 (a) $\int \cos^2 x \, dx$ (b) $\int \cos^2 6x \, dx$ (c) $\int \cos^2 \frac{1}{2}x \, dx$ (d) $\int_0^{\frac{\pi}{4}} \cos^2 2x \, dx$
7. (a) Find: (i) $\int \tan^2 2x \, dx$ (ii) $\int \cot^2 \frac{1}{2}x \, dx$
 (b) Evaluate: (i) $\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3 \tan^2 3x \, dx$ (ii) $\int_{\frac{\pi}{24}}^{\frac{\pi}{8}} \cot^2 4x \, dx$
8. (a) If $y' = \sin^2 x + \tan^2 x$ and $y = 1$ when $x = 0$, find y when $x = \frac{\pi}{4}$.
 (b) Given that $f'(x) = -\operatorname{cosec} 2x(\cot 2x + \operatorname{cosec} 2x)$ and $f(\frac{\pi}{4}) = 1$, find $f(\frac{\pi}{12})$.
9. Find the volume of the solid generated when the given curve is rotated about the x -axis.
 [HINT: In part (f), use the reverse chain rule.]
 (a) $y = \sec 2x$ between $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$, (d) $y = \sqrt{\cot x}$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$,
 (b) $y = \tan \frac{1}{2}x$ between $x = 0$ and $x = \frac{\pi}{2}$, (e) $y = \cot \frac{\pi}{2}x$ between $x = \frac{1}{2}$ and $x = 1$,
 (c) $y = \cos \pi x$ between $x = 0$ and $x = \frac{1}{2}$, (f) $y = \sec x \tan x$ between $x = 0$ and $x = \frac{\pi}{3}$.

DEVELOPMENT

10. Use the reverse chain rule to find:
 (a) $\int \sin^3 x \cos x \, dx$ [Let $u = \sin x$.]
 (b) $\int \cot^4 x \operatorname{cosec}^2 x \, dx$ [Let $u = \cot x$.]
 (c) $\int \sec^7 x \tan x \, dx$ [Let $u = \sec x$, and write $\sec^7 x \tan x = \sec^6 x \times \sec x \tan x$.]
 (d) $\int_0^{\pi} \cos^6 x \sin x \, dx$ (e) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan^3 x} \, dx$ (f) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^3 x \cot x \, dx$
11. Find:
 (a) $\int 2x \sec x^2 \tan x^2 \, dx$ (c) $\int e^x \cot e^x \, dx$
 (b) $\int \frac{\operatorname{cosec}^2 x}{1 + \cot x} \, dx$ (d) $\int \sec 2x \tan 2x e^{\sec 2x} \, dx$

12. Evaluate:

$$(a) \int_0^{\frac{\pi}{6}} \frac{1}{\sec 2x} dx$$

$$(c) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1 + \sin^3 x}{\sin^2 x} dx$$

$$(b) \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} dx$$

$$(d) \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec} x} dx$$

13. In each part, sketch the region defined by the given boundaries. Then find the area of the region, and the volume generated when the region is rotated about the x -axis.

$$(a) y = 1 + \sin x, x = 0, x = \pi, y = 0$$

$$(c) y = \sin x \cos x, x = \frac{\pi}{8}, x = \frac{3\pi}{8}, y = 0$$

$$(b) y = \sin x + \cos x, x = 0, x = \frac{\pi}{3}, y = 0$$

$$(d) y = \tan x + \cot x, x = \frac{\pi}{6}, x = \frac{\pi}{4}, y = 0$$

$$(e) y = 1 + \operatorname{cosec} x, x = \frac{\pi}{6}, x = \frac{5\pi}{6}, y = 0 \quad [\text{HINT: } \int \operatorname{cosec} x dx = -\ln(\operatorname{cosec} x + \cot x) + C]$$

14. (a) Show that $\frac{d}{dx}(\ln \sin x - x \cot x) = x \operatorname{cosec}^2 x$, and hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \operatorname{cosec}^2 x dx$.

(b) Show that $\frac{d}{dx}(\operatorname{cosec} x - \cot x) = \frac{1}{1 + \cos x}$, and hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx$.

(c) Show that $\frac{d}{dx}(\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x) = \sec^3 x \tan^3 x$, and hence find $\int_0^{\frac{\pi}{3}} \sec^3 x \tan^3 x dx$.

(d) Show that $\frac{d}{dx}(\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln(\sec x + \tan x)) = \sec^3 x$, hence find $\int_0^{\frac{\pi}{4}} \sec^3 x dx$.

(e) Show that $\frac{d}{dx}(\cot^3 x) = 3 \operatorname{cosec}^2 x - 3 \operatorname{cosec}^4 x$, and hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^4 x dx$.

(f) Show that $\frac{d}{dx}(\cos^3 x \sin x) = 4 \cos^4 x - 3 \cos^2 x$, and hence find $\int_0^{\frac{\pi}{2}} \cos^4 x dx$.

15. Find the value of $\lim_{R \rightarrow \infty} \left(\frac{1}{R} \int_0^R \sin^2 t dt \right)$, explaining your reasoning carefully.

16. Starting with $\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cot x) + C$, show that

$$\int \operatorname{cosec} x dx = \ln \left(\frac{1 - \cos x}{\sin x} \right) + C = \ln \left(\frac{\sin x}{1 + \cos x} \right) + C = \ln t + C, \text{ where } t = \tan \frac{1}{2} x.$$

EXTENSION

17. (a) Show that $\frac{d}{dx} \left(\frac{1}{n-1} \left(\tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x dx \right) \right) = \sec^n x$, for $n \geq 2$.

(b) Hence find the value of $\int_0^{\frac{\pi}{4}} \sec^7 x dx$.

6C Integration by Substitution

The reverse chain rule as we have been using it so far does not cover all the situations where the chain rule can be used in integration. This section and the next develop a more general method called *integration by substitution*. The first stage, covered in this section, begins by translating the reverse chain rule into a slightly more flexible notation. It involves substitutions of the form

‘Let u = some function of x .’

The Reverse Chain Rule — An Example: Here is an example of the reverse chain rule as we have been using it. The working is set out in full on the right.

WORKED EXERCISE: Find $\int x(1-x^2)^4 dx$.

$$\begin{array}{l} \text{SOLUTION: } \int x(1-x^2)^4 dx = -\frac{1}{2} \int (-2x)(1-x^2)^4 dx \\ \qquad \qquad \qquad = -\frac{1}{2} \times \frac{1}{5}(1-x^2)^5 + C \\ \qquad \qquad \qquad = -\frac{1}{10}(1-x^2)^5 + C \end{array} \quad \left| \begin{array}{l} \text{Let } u = 1-x^2. \\ \text{Then } \frac{du}{dx} = -2x, \\ \text{and } \int u^4 \frac{du}{dx} dx = \frac{1}{5}u^5. \end{array} \right.$$

Rewriting this Example as Integration by Substitution: We shall now rewrite this using a new notation. The key to this new notation is that the derivative $\frac{du}{dx}$ is treated as a fraction — the du and the dx are split apart, so that the statement

$$\frac{du}{dx} = -2x \quad \text{is written instead as} \quad du = -2x dx.$$

The new variable u no longer remains in the working column on the right, but is brought over into the main sequence of the solution on the left.

WORKED EXERCISE: Find $\int x(1-x^2)^4 dx$, using the substitution $u = 1-x^2$.

$$\begin{array}{l} \text{SOLUTION: } \int x(1-x^2)^4 dx = \int u^4 \left(-\frac{1}{2}\right) du \\ \qquad \qquad \qquad = -\frac{1}{2} \times \frac{1}{5}u^5 + C \\ \qquad \qquad \qquad = -\frac{1}{10}(1-x^2)^5 + C \end{array} \quad \left| \begin{array}{l} \text{Let } u = 1-x^2. \\ \text{Then } du = -2x dx, \\ \text{and } x dx = -\frac{1}{2} du. \end{array} \right.$$

WORKED EXERCISE: Find $\int \sin x \sqrt{1-\cos x} dx$, using the substitution $u = 1-\cos x$.

$$\begin{array}{l} \text{SOLUTION: } \int \sin x \sqrt{1-\cos x} dx = \int u^{\frac{1}{2}} du \\ \qquad \qquad \qquad = \frac{2}{3}u^{\frac{3}{2}} + C \\ \qquad \qquad \qquad = \frac{2}{3}(1-\cos x)^{\frac{3}{2}} + C \end{array} \quad \left| \begin{array}{l} \text{Let } u = 1-\cos x. \\ \text{Then } du = \sin x dx. \end{array} \right.$$

An Advance on the Reverse Chain Rule: Some integrals which can be done in this way could only be done by the reverse chain rule in a rather clumsy manner.

WORKED EXERCISE: Find $\int x\sqrt{1-x} dx$, using the substitution $u = 1-x$.

$$\begin{array}{l} \text{SOLUTION: } \int x\sqrt{1-x} dx = \int (1-u)\sqrt{u} du \\ \qquad \qquad \qquad = \int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du \\ \qquad \qquad \qquad = \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} + C \\ \qquad \qquad \qquad = \frac{2}{3}(1-x)^{\frac{3}{2}} - \frac{2}{5}(1-x)^{\frac{5}{2}} + C \end{array} \quad \left| \begin{array}{l} \text{Let } u = 1-x. \\ \text{Then } du = -dx, \\ \text{and } x = 1-u. \end{array} \right.$$

Substituting the Limits of Integration in a Definite Integral: A great advantage of this method is that the limits of integration can be changed from values of x to values of u . There is then no need ever to go back to x . The first worked exercise below repeats the previous integrand, but this time within a definite integral.

WORKED EXERCISE: Find $\int_0^1 x\sqrt{1-x} dx$, using the substitution $u = 1 - x$.

SOLUTION: $\begin{aligned} \int_0^1 x\sqrt{1-x} dx &= - \int_1^0 (1-u)\sqrt{u} du \\ &= - \int_1^0 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du \\ &= - \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_1^0 \\ &= -0 + \left(\frac{2}{3} - \frac{2}{5}\right) \\ &= \frac{4}{15} \end{aligned}$	Let $u = 1 - x$. Then $du = -dx$, and $x = 1 - u$. When $x = 0$, $u = 1$, when $x = 1$, $u = 0$.

WORKED EXERCISE: Find $\int_0^\pi \sin x \cos^6 x dx$, using the substitution $u = \cos x$.

SOLUTION: $\begin{aligned} \int_0^\pi \sin x \cos^6 x dx &= - \int_1^{-1} u^6 du \\ &= -\frac{1}{7} \left[u^7\right]_1^{-1} \\ &= -\frac{1}{7} \times (-1) + \frac{1}{7} \times 1 \\ &= \frac{2}{7} \end{aligned}$	Let $u = \cos x$. Then $du = -\sin x dx$. When $x = 0$, $u = 1$, when $x = \pi$, $u = -1$.

Exercise 6C

- Consider the integral $\int 2x(1+x^2)^3 dx$, and the substitution $u = 1 + x^2$.
 - Show that $du = 2x dx$.
 - Show that the integral can be written as $\int u^3 du$.
 - Hence find the primitive of $2x(1+x^2)^3$.
 - Check your answer by differentiating it.
- Repeat the previous question for each of the following indefinite integrals and substitutions.

(a) $\int 2(2x+3)^3 dx$ [Let $u = 2x+3$.]	(d) $\int \frac{3}{\sqrt{3x-5}} dx$ [Let $u = 3x-5$.]
(b) $\int 3x^2(1+x^3)^4 dx$ [Let $u = 1+x^3$.]	(e) $\int \sin^3 x \cos x dx$ [Let $u = \sin x$.]
(c) $\int \frac{2x}{(1+x^2)^2} dx$ [Let $u = 1+x^2$.]	(f) $\int \frac{4x^3}{1+x^4} dx$ [Let $u = 1+x^4$.]
- Consider the integral $\int \frac{x}{\sqrt{1-x^2}} dx$, and the substitution $u = 1 - x^2$.
 - Show that $x dx = -\frac{1}{2} du$.
 - Show that the integral can be written as $-\frac{1}{2} \int u^{-\frac{1}{2}} du$.
 - Hence find the primitive of $\frac{x}{\sqrt{1-x^2}}$.

4. Repeat the previous question for each of the following indefinite integrals and substitutions.

- (a) $\int x^3(x^4 + 1)^5 dx$ [Let $u = x^4 + 1$.] (d) $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^3} dx$ [Let $u = 1 + \sqrt{x}$.]
 (b) $\int x^2 \sqrt{x^3 - 1} dx$ [Let $u = x^3 - 1$.] (e) $\int \tan^2 2x \sec^2 2x dx$ [Let $u = \tan 2x$.]
 (c) $\int x^2 e^{x^3} dx$ [Let $u = x^3$.] (f) $\int \frac{e^{\frac{1}{x}}}{x^2} dx$ [Let $u = \frac{1}{x}$.]

5. Find the exact value of each definite integral, using the given substitution.

- (a) $\int_0^1 x^2(2 + x^3)^3 dx$ [Let $u = 2 + x^3$.] (f) $\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}} dx$ [Let $u = \sqrt{x}$.]
 (b) $\int_0^1 \frac{2x^3}{\sqrt{1 + x^4}} dx$ [Let $u = 1 + x^4$.] (g) $\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x dx$ [Let $u = \sin 2x$.]
 (c) $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x dx$ [Let $u = \cos x$.] (h) $\int_0^1 \frac{(\sin^{-1} x)^3}{\sqrt{1 - x^2}} dx$ [Let $u = \sin^{-1} x$.]
 (d) $\int_{\frac{1}{2}\sqrt{3}}^1 x\sqrt{1 - x^2} dx$ [Let $u = 1 - x^2$.] (i) $\int_0^2 \frac{x + 1}{\sqrt[3]{x^2 + 2x}} dx$ [Let $u = x^2 + 2x$.]
 (e) $\int_1^{e^2} \frac{\ln x}{x} dx$ [Let $u = \ln x$.] (j) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x} dx$ [Let $u = \tan x$.]

DEVELOPMENT

6. Use the substitution $u = x^3$ to find:

- (a) the exact area bounded by the curve $y = \frac{x^2}{1 + x^6}$, the x -axis and the line $x = 1$,
 (b) the exact volume generated when the region bounded by the curve $y = \frac{x}{(1 - x^6)^{\frac{1}{4}}}$, the x -axis and the line $x = 1$ is rotated about the x -axis.

7. Evaluate each of the following, using the substitution $u = \sin x$.

- (a) $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x} dx$ (c) $\int_0^{\frac{\pi}{2}} \cos^3 x dx$
 (b) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx$ (d) $\int_{\frac{\pi}{8}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x} dx$

8. Find each indefinite integral, using the given substitution.

- (a) $\int \frac{e^{2x}}{\sqrt{1 + e^{2x}}} dx$ [Let $u = e^{2x}$.] (c) $\int \frac{\tan x}{\ln \cos x} dx$ [Let $u = \ln \cos x$.]
 (b) $\int \frac{1}{x \ln x} dx$ [Let $u = \ln x$.] (d) $\int \tan^3 x \sec^4 x dx$ [Let $u = \tan x$.]

9. (a) A curve has gradient function $\frac{e^{2x}}{1 + e^{4x}}$ and passes through the point $(0, \frac{\pi}{8})$. Use the substitution $u = e^{2x}$ to find its equation.

(b) If $y'' = \frac{x}{(4 - x^2)^{\frac{3}{2}}}$, and when $x = 0$, $y' = 1$ and $y = \frac{1}{2}$, use the substitution $u = 4 - x^2$ to find y' and then find y as a function of x .

10. (a) Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.
 (b) Use the substitution $u = \sec x$ to find:
 (i) $\int_0^{\frac{\pi}{3}} 2^{\sec x} \sec x \tan x \, dx$ [HINT: $\int a^x \, dx = \frac{a^x}{\ln a}$] (ii) $\int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$
11. Evaluate each integral, using the given substitution.
 (a) $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \sin^2 x} \, dx$ [Let $u = \sin^2 x$.] (b) $\int_1^e \frac{\ln x + 1}{(x \ln x + 1)^2} \, dx$ [Let $u = x \ln x$.]
12. Use the substitution $u = \sqrt{x-1}$ to find $\int \frac{1}{2x\sqrt{x-1}} \, dx$.
13. The region R is bounded by the curve $y = \frac{x}{x+1}$, the x -axis and the vertical line $x = 3$.
 Use the substitution $u = x + 1$ to find:
 (a) the exact area of R ,
 (b) the exact volume generated when R is rotated about the x -axis.
14. (a) Use the substitution $u = \sqrt{x}$ to find $\int \frac{1}{\sqrt{x(1-x)}} \, dx$.
 (b) Evaluate the integral in part (a) again, using the substitution $u = x - \frac{1}{2}$.
 (c) Hence show that $\sin^{-1}(2x-1) = 2 \sin^{-1} \sqrt{x} - \frac{\pi}{2}$, for $0 < x < 1$.

EXTENSION

15. Use the substitution $u = x - \frac{1}{x}$ to show that $\int_1^{\frac{\sqrt{6}+\sqrt{2}}{2}} \frac{1+x^2}{1+x^4} \, dx = \frac{\pi}{4\sqrt{2}}$.

6D Further Integration by Substitution

The second stage of integration by substitution reverses the previous procedure and replaces x by a function of u . The substitutions are therefore of the form

‘Let $x = \text{some function of } u$.’

Substituting x by a Function of u : As a first example, here is a quite different substitution which solves the integral given in a worked example of the last section.

WORKED EXERCISE: Find $\int_0^1 x\sqrt{1-x} \, dx$, using the substitution $x = 1 - u^2$.

SOLUTION: $\int_0^1 x\sqrt{1-x} \, dx = \int_1^0 (1-u^2)u(-2u) \, du$ $= -2 \int_1^0 (u^2 - u^4) \, du$ $= -2 \left[\frac{1}{3}u^3 - \frac{1}{5}u^5 \right]_1^0$ $= -0 + 2\left(\frac{1}{3} - \frac{1}{5}\right)$ $= \frac{4}{15}$	<table border="0"> <tr> <td style="padding-right: 10px;">Let</td> <td>$x = 1 - u^2$.</td> </tr> <tr> <td>Then</td> <td>$dx = -2u \, du$,</td> </tr> <tr> <td>and</td> <td>$\sqrt{1-x} = u$.</td> </tr> <tr> <td>When</td> <td>$x = 0, \quad u = 1,$</td> </tr> <tr> <td>when</td> <td>$x = 1, \quad u = 0.$</td> </tr> </table>	Let	$x = 1 - u^2$.	Then	$dx = -2u \, du$,	and	$\sqrt{1-x} = u$.	When	$x = 0, \quad u = 1,$	when	$x = 1, \quad u = 0.$
Let	$x = 1 - u^2$.										
Then	$dx = -2u \, du$,										
and	$\sqrt{1-x} = u$.										
When	$x = 0, \quad u = 1,$										
when	$x = 1, \quad u = 0.$										

This question is a good example of the fact that an integral may be evaluated in a variety of ways. The following integral uses a trigonometric substitution, but can also be done through areas of segments.

WORKED EXERCISE: Find $\int_{3\sqrt{2}}^6 \sqrt{36 - x^2} dx$:

- (a) using the substitution $x = 6 \sin u$,
 (b) using the formula for the area of a segment.

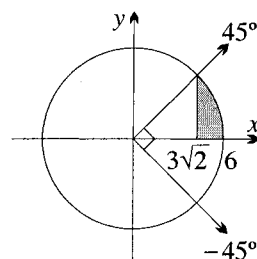
SOLUTION:

$$\begin{aligned} \text{(a)} \quad \int_{3\sqrt{2}}^6 \sqrt{36 - x^2} dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 6 \cos u \times 6 \cos u du \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 36 \left(\frac{1}{2} + \frac{1}{2} \cos 2u \right) du \\ &= \left[18u + 9 \sin 2u \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= (9\pi + 0) - \left(\frac{9\pi}{2} + 9 \right) \\ &= \frac{9}{2}(\pi - 2) \end{aligned}$$

Let $x = 6 \sin u$.
 Then $dx = 6 \cos u du$,
 and $\sqrt{36 - x^2} = 6 \cos u$.
 When $x = 3\sqrt{2}$, $u = \frac{\pi}{4}$,
 when $x = 6$, $u = \frac{\pi}{2}$.

- (b) The integral is sketched opposite. The shaded area is half the segment subtending an angle of 90° .

$$\begin{aligned} \text{Hence } \int_{3\sqrt{2}}^6 \sqrt{36 - x^2} dx &= \frac{1}{2} \times \frac{1}{2} \times 6^2 \left(\frac{\pi}{2} - \sin \frac{\pi}{2} \right) \\ &= 9 \left(\frac{\pi}{2} - 1 \right). \end{aligned}$$



NOTE: Careful readers may notice a problem here, in that given the value $x = 3\sqrt{2}$, u is determined by $\sin u = \frac{1}{2}\sqrt{2}$, so there are infinitely many possible values of u . A similar problem occurred in the previous worked exercise, where $0 = 1 - u^2$ had two solutions. These problems arise because the functions involved in the substitutions were $x = 1 - u^2$ and $x = 6 \sin u$, whose inverses were not functions. A full account of all this would require substitutions by restrictions of the functions given above so that they had inverse functions. In practice, however, this is rarely necessary, and it is certainly not a concern of this course.

As a rule of thumb, work with positive square roots, and with trigonometric functions, work in the same quadrants as were involved in the definitions of the inverse trigonometric functions in Chapter One.

Exercise 6D

1. Consider the integral $I = \int x(x-1)^5 dx$, and let $x = u + 1$.

- (a) Show that $dx = du$.
 (b) Show that $I = \int u^5(u+1) du$.
 (c) Hence find I .
 (d) Check your answer by differentiating it.

2. Using the same substitution as in the previous question, find:

(a) $\int \frac{x}{\sqrt{x-1}} dx$

(b) $\int \frac{x}{(x-1)^2} dx$

3. Consider the integral $J = \int x\sqrt{x+1} dx$, and let $x = u^2 - 1$.
- Show that $dx = 2u du$.
 - Show that $J = 2 \int (u^4 - u^2) du$.
 - Hence find J .
 - Check your answer by differentiating it.
4. Using the same substitution as in the previous question, find:
- $\int x^2\sqrt{x+1} dx$
 - $\int \frac{2x+3}{\sqrt{x+1}} dx$
5. Find each of the following indefinite integrals using the given substitution.
- $\int \frac{x-2}{x+2} dx$ [Let $x = u - 2$.]
 - $\int \frac{2x+1}{\sqrt{2x-1}} dx$ [Let $x = \frac{1}{2}(u+1)$.]
 - $\int 3x\sqrt{4x-5} dx$ [Let $x = \frac{1}{4}(u^2+5)$.]
 - $\int \frac{1}{1+\sqrt{x}} dx$ [Let $x = (u-1)^2$.]
6. Evaluate, using the given substitution:
- $\int_0^1 x(x+1)^3 dx$ [Let $x = u - 1$.]
 - $\int_0^{\frac{1}{2}} \frac{1+x}{1-x} dx$ [Let $x = 1 - u$.]
 - $\int_0^1 \frac{3x}{\sqrt{3x+1}} dx$ [Let $x = \frac{1}{3}(u-1)$.]
 - $\int_0^1 \frac{2-x}{(2+x)^3} dx$ [Let $x = u - 2$.]
 - $\int_0^4 x\sqrt{4-x} dx$ [Let $x = 4 - u^2$.]
 - $\int_1^5 \frac{x}{(2x-1)^{\frac{3}{2}}} dx$ [Let $x = \frac{1}{2}(u^2+1)$.]
 - $\int_0^4 \frac{1}{3+\sqrt{x}} dx$ [Let $x = (u-3)^2$.]
 - $\int_0^7 \frac{x^2}{\sqrt[3]{x+1}} dx$ [Let $x = u^3 - 1$.]

DEVELOPMENT

7. (a) Consider the integral $I = \int \frac{1}{\sqrt{5-4x-x^2}} dx$, and let $x = u - 2$.
Show that $I = \int \frac{1}{\sqrt{9-u^2}} du$, and hence find I .
- (b) Use a similar approach to find:
- $\int \frac{1}{x^2+2x+4} dx$ [Let $x = u - 1$.]
 - $\int \frac{1}{\sqrt{4-2x-x^2}} dx$ [Let $x = u - 1$.]
 - $\int_1^2 \frac{1}{\sqrt{3+2x-x^2}} dx$ [Let $x = u + 1$.]
 - $\int_3^7 \frac{1}{x^2-6x+25} dx$ [Let $x = u + 3$.]
8. (a) Consider the integral $J = \int \frac{1}{\sqrt{4-x^2}} dx$, and let $x = 2 \sin \theta$.
Show that $J = \int 1 d\theta$, and hence show that $J = \sin^{-1} \frac{x}{2} + C$.
- (b) Using a similar approach, find:
- $\int \frac{1}{9+x^2} dx$ [Let $x = 3 \tan \theta$.]
 - $\int \frac{-1}{\sqrt{3-x^2}} dx$ [Let $x = \sqrt{3} \cos \theta$.]
 - $\int \frac{1}{\sqrt{1-4x^2}} dx$ [Let $x = \frac{1}{2} \sin \theta$.]
 - $\int \frac{1}{1+16x^2} dx$ [Let $x = \frac{1}{4} \tan \theta$.]
 - $\int_0^3 \frac{1}{\sqrt{36-x^2}} dx$ [Let $x = 6 \sin \theta$.]
 - $\int_0^{\frac{2}{3}} \frac{1}{4+9x^2} dx$ [Let $x = \frac{2}{3} \tan \theta$.]

9. (a) Consider the integral $I = \int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$, and let $x = \sin \theta$.

Show that $I = \int \sec^2 \theta d\theta$, and hence show that $I = \frac{x}{\sqrt{1-x^2}} + C$.

- (b) Similarly, use the given substitution to find:

- (i) $\int \frac{1}{(4+x^2)^{\frac{3}{2}}} dx$ [Let $x = 2 \tan \theta$.] (iv) $\int \frac{1}{x^2 \sqrt{25-x^2}} dx$ [Let $x = 5 \cos \theta$.]
 (ii) $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$ [Let $x = \sin \theta$.] (v) $\int \frac{1}{x^2 \sqrt{9+x^2}} dx$ [Let $x = 3 \tan \theta$.]
 (iii) $\int_0^2 \sqrt{4-x^2} dx$ [Let $x = 2 \sin \theta$.] (vi) $\int_2^4 \frac{1}{x^2 \sqrt{x^2-4}} dx$ [Let $x = 2 \sec \theta$.]

10. (a) Sketch the region R bounded by $y = \frac{1}{x^2+1}$, the x - and y -axes, and the line $x = 1$.

- (b) Find the volume generated when R is rotated about the x -axis.

[HINT: Use the substitution $x = \tan \theta$.]

11. Find the equation of the curve $y = f(x)$ if $f'(x) = \frac{\sqrt{x^2-9}}{x}$ and $f(3) = 0$.

[HINT: Use the substitution $x = 3 \sec \theta$.]

12. Find the exact area of the region bounded by $y = \frac{x^3}{\sqrt{3-x^2}}$, the x -axis and the line $x = 1$.

[HINT: Use the substitution $x = \sqrt{3} \sin \theta$, followed by the substitution $u = \cos \theta$.]

13. [These are confirmations rather than proofs, since the calculus of trigonometric functions was developed on the basis of the formulae in parts (a) and (b).]

- (a) Use integration to confirm that the area of a circle is πr^2 .

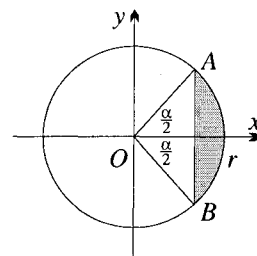
[HINT: Find the area bounded by the semicircle $y = \sqrt{r^2-x^2}$ and the x -axis and double it. Use the substitution $x = r \sin \theta$.]

- (b) The shaded area in the diagram to the right is the segment of a circle of radius r cut off by the chord AB subtending an angle α at the centre O .

- (i) Show that the area is $I = 2 \int_{r \cos \frac{1}{2}\alpha}^r \sqrt{r^2-x^2} dx$.

- (ii) Let $x = r \cos \theta$, and show that $I = -2r^2 \int_{\frac{1}{2}\alpha}^0 \sin^2 \theta d\theta$.

- (iii) Hence confirm that $I = \frac{1}{2}r^2(\alpha - \sin \alpha)$.



- (c) Use a similar approach to confirm that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab . Then justify the formula by regarding the ellipse as the unit circle stretched horizontally by a factor of a and vertically by a factor of b .

EXTENSION

14. (a) Multiply $\sec \theta$ by $\frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta}$, and hence find $\int \sec \theta d\theta$.

- (b) The region R is bounded by $y = \frac{x}{\sqrt{x^2+16}}$, the x -axis and line $x = 4$. Show that the volume generated by rotating R about the y -axis is $16\pi (\sqrt{2} - \ln(\sqrt{2}+1))$ units³.
 [HINT: Use the substitution $y = \sin \theta$ and the result in part (a).]

15. (a) Use the substitution $x = -u$ to show that $\int_{-2}^2 \frac{x^2}{e^x + 1} dx = \int_{-2}^2 \frac{x^2 e^x}{e^x + 1} dx$.

(b) Hence find $\int_{-2}^2 \frac{x^2}{e^x + 1} dx$.

6E Approximate Solutions and Newton's Method

Most equations cannot be solved exactly. This section deals with two methods of finding approximate solutions, called *halving the interval* and *Newton's method*. Each method produces a sequence of approximate solutions with increasingly greater accuracy, with Newton's method converging to the solution very fast indeed.

Approaching an Unknown Equation: Given an unknown equation, there are three successive questions to ask:

THREE QUESTIONS TO ASK ABOUT AN UNKNOWN EQUATION:

- 5
 1. Does the equation have a solution?
 2. How many solutions are there, and roughly where are they?
 3. How can approximations be found, correct to the required level of accuracy?

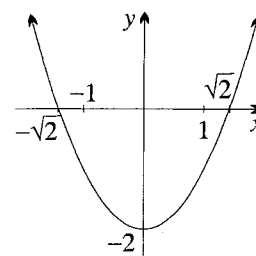
Any work on approximations should therefore be preceded by an exploratory table of values, and probably a graph, to give the rough locations of the solutions. These procedures were described in Section 3F of the Year 11 volume.

The easiest example of our methods is finding approximations to $\sqrt{2}$. This means finding the positive root of the equation $x^2 = 2$. We will write the equation as

$$x^2 - 2 = 0,$$

so that it has the form $f(x) = 0$, where $f(x) = x^2 - 2$. Then

x	-2	-1	0	1	2
$x^2 - 2$	2	-1	-2	-1	2



Hence there is solution between 1 and 2, and another between -2 and -1. We shall seek approximations to the solution $x = \sqrt{2}$ between 1 and 2.

Halving the Interval: This is simply a systematic approach to constructing a table of values near the solution. A function can only change sign at a zero or a discontinuity, hence we have trapped a solution between 1 and 2. If we keep halving the interval, the solution will be trapped successively within intervals of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$, until the desired order of accuracy is obtained.

APPROXIMATING SOLUTIONS BY HALVING THE INTERVAL: Given the equation $f(x) = 0$:

- 6
 1. Locate the solutions roughly by means of a table of values and/or a graph.
 2. To obtain a sequence approximating a particular solution, trap the solution within an interval, then keep halving the interval where the solution is trapped.

Each successive application of the method will halve the uncertainty of the approximation. Since $2^{10} = 1024 \div 1000$, it will take roughly ten further steps to obtain three further decimal places.

WORKED EXERCISE: Use the method of halving the interval to approximate $\sqrt{2}$ correct to three significant figures.

SOLUTION: We have already found that $\sqrt{2}$ lies between 1 and 2.

Let $f(x) = x^2 - 2$. Then by hand and by calculator,

x	1	2	$1\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{3}{8}$	$1\frac{7}{16}$	$1\frac{13}{32}$	$1\frac{27}{64}$	$1\frac{53}{128}$	$1\frac{107}{256}$	$1\frac{213}{512}$	$1\frac{425}{1024}$	$1\frac{849}{2048}$	$1\frac{1697}{4096}$
$f(x)$	-1	2	$\frac{1}{4}$	$-\frac{7}{16}$	$-\frac{7}{64}$	$\frac{17}{256}$	-	+	-	+	+	+	+	+

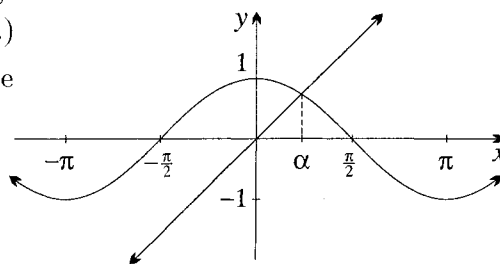
Hence $1\frac{53}{128} < \sqrt{2} < 1\frac{1697}{4096}$,

or in decimal form, $0.4140 < \sqrt{2} < 1.4144$, so that $\sqrt{2} \div 1.414$.

(Strictly speaking, one should round down the left-hand bound, and round up the right-hand bound.)

WORKED EXERCISE: Solve $\cos x = x$ correct to three decimal places, by halving the interval.

SOLUTION: The graph shows that there is exactly one solution, and that it lies between $x = 0$ and $x = 1$. Let the solution be $x = \alpha$, and consider the function $y = \cos x - x$.



x	0	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{23}{32}$	$\frac{47}{64}$	$\frac{95}{128}$	$\frac{189}{256}$	$\frac{379}{512}$	$\frac{757}{1024}$	$\frac{1513}{2048}$
y	1	-0.46	0.38	-0.02	0.19	0.09	0.03	+	-	+	-	-	+

Hence $\frac{1513}{2048} < \alpha < \frac{757}{1024}$,

or in decimal form, $0.7387 < \alpha < 0.7393$, so that $\alpha \div 0.739$.

Newton's Method: The function graphed below has an unknown root at $x = \alpha$, and $x = x_0$ is a known approximation to that root. Let $J = (x_0, 0)$.

Draw a tangent at $P(x_0, f(x_0))$,

and let it meet the x -axis at $K(x_1, 0)$ with angle of inclination θ .

Then x_1 will be a better approximation to α than x_0 .

Now $\tan \theta$ is the gradient of $y = f(x)$ at $x = x_0$,

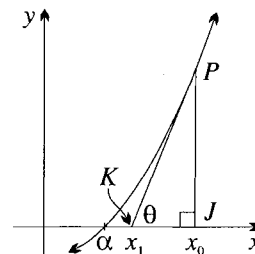
so $\tan \theta = f'(x_0)$.

In $\triangle JPK$, $JK = \frac{PJ}{\tan \theta}$,

that is, $x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$,

so $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

This formula is the basis of Newton's method.



NEWTON'S METHOD: Suppose that $x = x_0$ is an approximation to a root $x = \alpha$ of an equation $f(x) = 0$. Then, provided that the situation is favourable, a closer approximation is

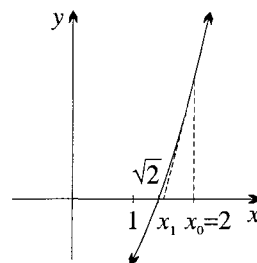
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The formula can be applied successively to produce a sequence of successively closer approximations to the root.

We will mention below some serious questions about what makes a ‘favourable situation’. For now, notice from the accompanying diagram that the function was carefully chosen so that it was increasing and concave up, with $x_0 > \alpha$.

WORKED EXERCISE:

- (a) Beginning with the approximation $x_0 = 2$ for $\sqrt{2}$, use one step of Newton’s method to obtain a better approximation x_1 .
- (b) Show that in general, $x_n = \frac{x_{n-1}^2 + 2}{2x_{n-1}}$.
- (c) Continue the process to obtain an approximation correct to eight decimal places.



SOLUTION:

- (a) Here $f(x) = x^2 - 2$
so $f'(x) = 2x$.

$$\begin{aligned}\text{Hence } x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{f(2)}{f'(2)} \\ &= 2 - \frac{2}{4} \\ &= 1\frac{1}{2}.\end{aligned}$$

$$\begin{aligned}\text{(b) In general, } x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{2x_{n-1}^2 - x_{n-1}^2 + 2}{2x_{n-1}} \\ &= \frac{x_{n-1}^2 + 2}{2x_{n-1}}.\end{aligned}$$

- (c) Continuing these calculations, $x_2 = \frac{x_1^2 + 2}{2x_1} = 1.416\ 666\ 666\ \dots$

$$x_3 = \frac{x_2^2 + 2}{2x_2} = 1.414\ 215\ 686\ \dots$$

$$x_4 = \frac{x_3^2 + 2}{2x_3} = 1.414\ 213\ 562\ \dots$$

$$x_5 = \frac{x_4^2 + 2}{2x_4} = 1.414\ 213\ 562\ \dots$$

Hence

$$\sqrt{2} \div 1.414\ 213\ 56.$$

A NOTE ON CALCULATORS: On many new calculators, the formula only needs to be entered once, after which each successive approximation can be obtained simply by pressing $\boxed{=}$. Enter the initial value x_0 and press $\boxed{=}$, then enter the formula using the key labelled $\boxed{\text{Ans}}$ whenever x_0 occurs in the formula.

A NOTE ON THE SPEED OF CONVERGENCE: It should be obvious from the diagram above that Newton’s method converges extremely rapidly once it gets going. As a rule of thumb, the number of correct decimal places doubles with each step. It would help intuition to continue these calculations using mathematical software capable of working with thirty or more decimal places.

Problem One — The Initial Approximation May Be on the Wrong Side: The original diagram above shows that Newton’s method works when the curve bulges towards the x -axis in the region between $x = \alpha$ and $x = x_0$. In other situations, the method can easily run into problems. The first problem is hopefully only a nuisance — in the example below, x_0 is chosen on the wrong side of the root, but the next approximation x_1 is on the favourable side, and the sequence then converges rapidly as before.

WORKED EXERCISE:

(a) Beginning with the approximate solution $x_0 = 0$ of $\cos x = x$, use one step of Newton's method to obtain x_1 .

(b) Show that in general, $x_n = \frac{x_{n-1} \sin x_{n-1} + \cos x_{n-1}}{1 + \sin x_{n-1}}$.

(c) Find an approximation correct to eight decimal places.

SOLUTION: The graph below shows $f(x) = \cos x - x$ in the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. With $x_0 = 0$, the next approximation is $x = 1$, as shown in part (a). Were x_0 chosen further to the left, more serious problems could occur.

(a) Let $f(x) = \cos x - x$.

Then $f'(x) = -\sin x - 1$.

$$\begin{aligned} \text{Hence } x_1 &= x_0 - \frac{\cos x_0 - x_0}{-\sin x_0 - 1} \\ &= 0 + \frac{\cos 0 - 0}{\sin 0 + 1} \\ &= 1. \end{aligned}$$

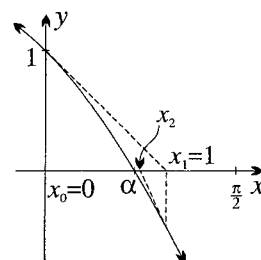
(b) Now that the approximation has crossed to the other side, convergence will be rapid.

$$\begin{aligned} \text{In general, } x_n &= x_{n-1} - \frac{\cos x_{n-1} - x_{n-1}}{-\sin x_{n-1} - 1} \\ &= \frac{x_{n-1}(\sin x_{n-1} + 1) + (\cos x_{n-1} - x_{n-1})}{\sin x_{n-1} + 1} \\ &= \frac{x_{n-1} \sin x_{n-1} + \cos x_{n-1}}{\sin x_{n-1} + 1}. \end{aligned}$$

(c) Continuing the process,

$$\begin{aligned} x_2 &= \frac{x_1 \sin x_1 + \cos x_1}{\sin x_1 + 1} = 0.750\,363\,867 \dots \\ x_3 &= \frac{x_2 \sin x_2 + \cos x_2}{\sin x_2 + 1} = 0.739\,112\,890 \dots \\ x_4 &= \frac{x_3 \sin x_3 + \cos x_3}{\sin x_3 + 1} = 0.739\,085\,133 \dots \\ x_5 &= \frac{x_4 \sin x_4 + \cos x_4}{\sin x_4 + 1} = 0.739\,085\,133 \dots \end{aligned}$$

Hence $\alpha \doteq 0.739\,085\,13$.



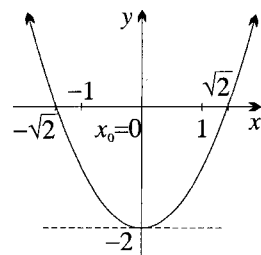
Problem Two — The Tangent May Be Horizontal: If the tangent at $x = x_0$ is horizontal, it will never meet the x -axis, hence there will be no approximation x_1 . The algebraic result is a zero denominator.

WORKED EXERCISE: Explain, algebraically and geometrically, why $x_0 = 0$ cannot be taken as a suitable first approximation when finding $\sqrt{2}$ by Newton's method.

SOLUTION: Here $f(x) = x^2 - 2$ and $f'(x) = 2x$.

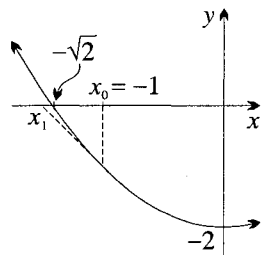
$$\begin{aligned} \text{Algebraically, } x_1 &= x_0 - \frac{x_0^2 - 2}{2x_0} \\ &= 0 - \frac{0^2 - 2}{2 \times 0}, \text{ which is undefined.} \end{aligned}$$

Geometrically, the tangent at $P(0, -2)$ is horizontal, so it never meets the x -axis, and x_1 cannot be found.



Problem Three — The Sequence May Converge to the Wrong Root:

In the previous example, if we were to choose $x_0 = -1$, beginning on the wrong side of the stationary point, then the sequence would converge to $-\sqrt{2}$ instead of to $\sqrt{2}$. The diagram shows this happening.



Problem Four — The Sequence May Oscillate, or even Move Away from the Root: The diagram below shows the curve $y = x^3 - 5x$, which has an inflexion at the origin. If we try to approximate the root $x = 0$ using Newton's method, then neither side is favourable, and the sequence will keep crossing sides. Worse still, if $x_0 = 1$, the sequence will simply oscillate between 1 and -1 , and if $x_0 > 1$, the sequence will move away from $x = 0$ instead of converging to it.

WORKED EXERCISE: Show that for $f(x) = x^3 - 5x$, one application of Newton's method will give $x_1 = \frac{2x_0^3}{3x_0^2 - 5}$.

(a) For $x_0 = 1$, show that the sequence of approximations oscillates.

(b) For $x_0 > 1$, show that the sequence will move away from $x = 0$.

SOLUTION: Since $f(x) = x^3 - 5x$, $f'(x) = 3x^2 - 5$.

$$\begin{aligned} \text{Hence } x_1 &= x_0 - \frac{x_0^3 - 5x_0}{3x_0^2 - 5} \\ &= \frac{3x_0^3 - 5x_0 - x_0^3 + 5x_0}{3x_0^2 - 5} \\ &= \frac{2x_0^3}{3x_0^2 - 5}. \end{aligned}$$

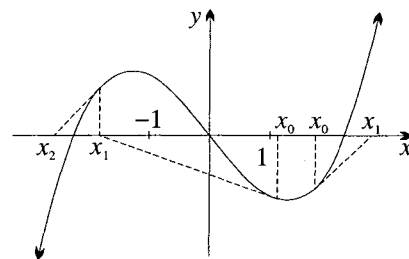
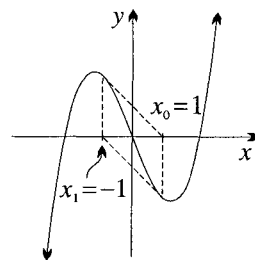
$$\begin{aligned} \text{(a) Substituting } x_0 = 1, x_1 &= \frac{2}{3 - 5} \\ &= -1. \end{aligned}$$

Then because $f(x)$ has odd symmetry, the sequence oscillates:

$$x_2 = 1, x_3 = -1, x_4 = 1, \dots$$

(b) When x_0 is to the right of the turning point, the tangent will slope upwards, and will meet the x -axis to the right of the positive zero — the sequence will then converge to that zero.

When x_0 is between $x = 1$ and the turning point, the tangent will be flatter than the tangent at $x = 1$, so x_1 will be to the left of -1 . Once the sequence moves outside the two turning points, it will converge to one of the other two zeroes. But if any of $x_0, x_1, x_2, \dots$ is ever at a turning point, the tangent will be horizontal and the method will terminate.



Problem Five — The Equation May Have No Solutions: The final Extension problem in the following exercise pursues the consequences when Newton's method is applied to the function $f(x) = 1 + x^2$, which has no zeroes at all. It is in such situations that Newton's method becomes a topic within modern chaos theory.

Exercise 6E

- If $P(x) = x^2 - 2x - 1$, show that $P(2) < 0$ and $P(3) > 0$, and therefore that there is a root of the equation $x^2 - 2x - 1 = 0$ between 2 and 3.
 - Evaluate $P(\frac{5}{2})$ and hence show that the root to the equation $P(x) = 0$ lies in the interval $2 < x < 2\frac{1}{2}$.
 - Which end of this interval is the root closer to? Justify your answer by using the halving the interval method a second time.

2. (a) (i) Show that the equation $x^3 + x^2 + 2x - 3 = 0$ has a root between $x = 0$ and $x = 1$.
 (ii) Use halving the interval twice to find an approximation to the root.
- (b) (i) Show that the equation $x^4 + 2x^2 - 5 = 0$ has a root between 0.5 and 1.5.
 (ii) Use halving the interval until you can approximate the root to one decimal place.
3. (a) (i) Show that the function $F(x) = x^3 - \log_e(x + 1)$ has a zero between 0.8 and 0.9.
 (ii) Use halving the interval once to approximate the root to one decimal place.
- (b) (i) Show that the equation $\log_e x = \sin x$ has a root between 2 and 3.
 (ii) Use halving the interval to approximate the root to one decimal place.
- (c) (i) Show that the equation $e^x - \log_e x = 3$ has a root between 1 and 2.
 (ii) Use halving the interval to approximate the root to one decimal place.
4. (a) Beginning with the approximate solution $x_0 = 2$ of $x^2 - 5 = 0$, use one step of Newton's method to obtain a better approximation x_1 . Give your answer to one decimal place.
- (b) Show that in general, $x_{n+1} = \frac{x_n^2 + 5}{2x_n}$.
- (c) Use part (b) to find x_2, x_3, x_4 and x_5 , which should confirm the accuracy of x_4 to at least eight decimal places.

NOTE: Your calculator may be able to obtain each successive approximation simply by pressing $\boxed{=}$. Try doing this — enter $x_0 = 2$ and press $\boxed{=}$, then enter the formula in part (b) using the key labelled $\boxed{\text{Ans}}$ whenever x_0 is needed, then press $\boxed{=}$ to get x_1 . Now pressing $\boxed{=}$ successively should yield $x_2, x_3, x_4 \dots$

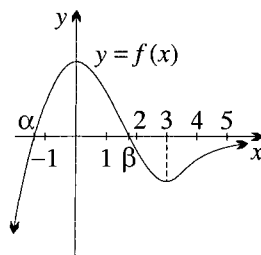
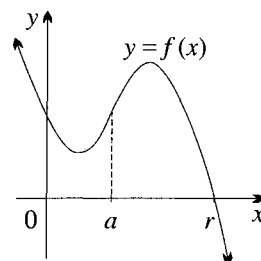
5. Repeat the steps of the previous question in each of the following cases.

- (a) $x^3 - 9x - 2 = 0, x_0 = 3$. Show that $x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 9}$.
 - (b) $e^x - 3x - 1 = 0, x_0 = 2$. Show that $x_{n+1} = \frac{e^{x_n}(x_n - 1) + 1}{e^{x_n} - 3}$.
 - (c) $2 \sin x - x = 0, x_0 = 2$. Show that $x_{n+1} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2 \cos x_n}$.
6. Use Newton's method twice to find the indicated root of each equation, giving your answer correct to two decimal places. Then continue the process to obtain an approximation correct to eight decimal places.
- (a) For $x^2 - 2x - 1 = 0$, approximate the root near $x = 2$.
 - (b) For $x^3 + x^2 + 2x - 3 = 0$, approximate the root near $x = 1$.
 - (c) For $x^4 + 2x^2 - 5 = 0$, approximate the root near $x = 1$.
 - (d) For $x^3 - \log_e(x + 1) = 0$, approximate the root near $x = 0.8$.
 - (e) For $\log_e x = \sin x$, approximate the root near $x = 2$.
 - (f) For $e^x - \log_e x = 3$, approximate the root near $x = 1$.

DEVELOPMENT

7. (a) Show that the equation $x^3 - 16 = 0$ has a root between 2 and 3.
- (b) Use halving the interval three times to find a better approximation to the root.
- (c) The actual answer to five decimal places is 2.51984. Was the final number you substituted the best approximation to the root?

8. Use Newton's method to find approximations correct to two decimal places. Then continue the process to obtain an approximation correct to eight decimal places.
- (a) $\sqrt{13}$ (b) $\sqrt[3]{35}$ (c) $\sqrt[5]{158}$
9. The closest integer to $\sqrt[4]{100}$ is 3. Use one application of Newton's method to show that $3\frac{19}{108}$ is a better approximation to $\sqrt[4]{100}$. Then obtain an approximation correct to eight decimal places.
10. Consider the polynomial $P(x) = 4x^3 + 2x^2 + 1$.
- (a) Show that $P(x)$ has a real zero α in the interval $-1 < x < 0$.
- (b) By sketching the graph of $P(x)$, show that α is the only real zero of $P(x)$.
- (c) Use Newton's method with initial value $\alpha \doteq -\frac{1}{4}$ to obtain a second approximation.
- (d) Explain from the graph of $P(x)$ why this second approximation is not a better approximation to α than $-\frac{1}{4}$ is.
11. Consider the graph of $y = f(x)$. The value a shown on the axis is taken as the first approximation to the solution r of $f(x) = 0$. Is the second approximation obtained by Newton's method a better approximation to r than a is? Give a reason for your answer.
12. The diagram shows the curve $y = f(x)$, which has turning points at $x = 0$ and $x = 3$ and a point of inflexion at $x = 4$. The equation $f(x) = 0$ has two real roots α and β . Determine which of the following cases applies when Newton's method is repeatedly applied with the given starting value x_0 :
- A. α is approximated. B. β is approximated.
- C. The sequence $x_1, x_2, x_3, \dots$ is moving away from both roots.
- D. The method breaks down at the first application.
- (a) $x_0 = -2$ (d) $x_0 = 0$ (g) $x_0 = 2$ (j) $x_0 = 3.1$
- (b) $x_0 = -1$ (e) $x_0 = 0.1$ (h) $x_0 = 2.9$ (k) $x_0 = 4$
- (c) $x_0 = -0.1$ (f) $x_0 = 1$ (i) $x_0 = 3$ (l) $x_0 = 5$
13. (a) On the same diagram, sketch the graphs of $y = e^{-\frac{1}{2}x}$ and $y = 5 - x^2$, showing all intercepts with the x and y axes.
- (b) On your diagram, indicate the negative root α of the equation $x^2 + e^{-\frac{1}{2}x} = 5$.
- (c) Show that $-2 < \alpha < -1$.
- (d) Use one iteration of Newton's method, with starting value $x_1 = -2$, to show that α is approximately $\frac{-18}{e+8}$.
14. (a) Suppose that we apply Newton's method with starting value $x_0 = 0$ repeatedly to the function $y = e^{-kx}$, where k is a positive constant.
- (i) Show that $x_{n+1} = x_n + \frac{1}{k}$.
- (ii) Describe the resulting sequence $x_1, x_2, x_3, \dots$.
- (b) Repeat part (a) with the function $y = x^{-k}$ (where once again $k > 0$) and starting value $x_0 = 1$.
- (c) What can we deduce from parts (a) and (b) about the rates at which e^{-kx} and x^{-k} approach zero as $x \rightarrow \infty$? Draw a diagram to illustrate this.



EXTENSION

15. Suppose that $a \geq 2$ is an integer which is not a perfect square. Our aim is to approximate $\sqrt{a}$ by applying Newton's method to the equation $x^2 - a = 0$. Let $x_0, x_1, x_2, \dots$ be the approximations obtained by successive applications of Newton's method, where the initial value x_0 is the smallest integer greater than $\sqrt{a}$.

- (a) Show that $x_{n+1} = \frac{x_n^2 + a}{2x_n}$, for $n \geq 0$.
 (b) Prove by induction that for all integers $n \geq 0$,

$$x_n - \sqrt{a} \leq 2\sqrt{a} \left(\frac{x_0 - \sqrt{a}}{2\sqrt{a}} \right)^{2^n}.$$

(Note that the index on the RHS is 2^n , not $2n$.)

- (c) Show that when Newton's method is applied to finding $\sqrt{3}$, using the initial value $x_0 = 2$, the twentieth approximation x_{20} is correct to at least one million decimal places.
16. Let $f(x) = 1 + x^2$ and let x_1 be a real number. For $n = 1, 2, 3, \dots$, define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(You may assume that $f'(x_n) \neq 0$.)

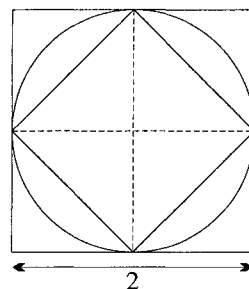
- (a) Show that $|x_{n+1} - x_n| \geq 1$, for $n = 1, 2, 3, \dots$.
 (b) Graph the function $y = \cot \theta$ for $0 < \theta < \pi$.
 (c) Use the graph to show that there exists a real number θ_n such that $x_n = \cot \theta_n$ and $0 < \theta_n < \pi$.
 (d) By using the formula for $\tan 2A$, deduce that $\cot \theta_{n+1} = \cot 2\theta_n$, for $n = 1, 2, 3, \dots$.
 (e) Find all points x_1 such that $x_1 = x_{n+1}$, for some value of n .

6F Inequalities and Limits Revisited

Arguments about inequalities and limits have occurred continually throughout our work. This demanding section is intended to revisit the subject and focus attention on some of the types of arguments being used. As mentioned in the Study Notes, it is intended for 4 Unit students — familiarity with arguments about inequalities and limits is required in that course — and for the more ambitious 3 Unit students, who may want to leave it until final revision.

A Geometrical Argument Proving an Inequality about π : The following worked exercise does nothing more than prove that π is between 2 and 4 — hardly a brilliant result — but it is a good illustration of the use of geometrical arguments.

WORKED EXERCISE: The outer square in the diagram to the right has side length 2. Find the areas of the circle and both squares, and hence prove that $2 < \pi < 4$.



SOLUTION: The circle has radius 1,
so area of circle $= \pi \times 1^2$
 $= \pi$.

The outer square has side length 2,
so area of outer square $= 2^2$
 $= 4$.

The inner square has diagonals of length 2,
so area of inner square $= \frac{1}{2} \times 2 \times 2$
 $= 2$.

But area of inner square $<$ area of circle $<$ area of outer square.

Hence $2 < \pi < 4$.

Arguments using Concavity and the Definite Integral: The following worked exercise applies two very commonly used principles to produce inequalities.

USING CONCAVITY AND THE DEFINITE INTEGRAL TO PRODUCE INEQUALITIES:

8

- If a curve is concave up in an interval, then the chord joining the endpoints of the curve lies above the curve.

- If $f(x) < g(x)$ in an interval $a < x < b$, then $\int_a^b f(x) dx < \int_a^b g(x) dx$.

WORKED EXERCISE:

- Using the second derivative, prove that the chord joining the points $A(0, 1)$ and $B(1, e)$ on the curve $y = e^x$ lies above the curve in the interval $0 < x < 1$.
- Find the equation of the chord, and hence prove that $\sqrt{e} < \frac{1}{2}(e + 1)$.
- By integrating over the interval $0 \leq x \leq 1$, prove that $e < 3$.

SOLUTION:

- Since $y = e^x$, $y' = e^x$ and $y'' = e^x$.

Since y'' is positive for all x , the curve is concave up everywhere.

In particular, the chord joining A and B lies above the curve.

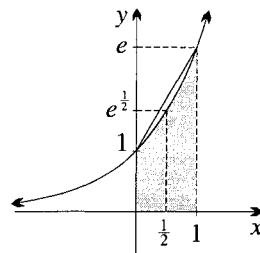
- The chord has gradient $= e - 1$ (the rise is $e - 1$, the run is 1),
so the chord is $y = (e - 1)x + 1$ (using $y = mx + b$).

When $x = \frac{1}{2}$, the line is above the curve $y = e^x$,

so substituting $x = \frac{1}{2}$, $e^{\frac{1}{2}} < \frac{1}{2}(e - 1) + 1$ (the chord is above the curve)
 $\sqrt{e} < \frac{1}{2}(e + 1)$, as required.

- Since $y = (e - 1)x + 1$ is above $y = e^x$ in the interval $0 < x < 1$,

$$\begin{aligned} \int_0^1 e^x dx &< \int_0^1 ((e - 1)x + 1) dx \\ \left[e^x \right]_0^1 &< \left[\frac{1}{2}(e - 1)x^2 + x \right]_0^1 \\ e - 1 &< \frac{1}{2}(e - 1) + 1 \\ 2e - 2 &< e - 1 + 2 \\ e &< 3, \text{ as required.} \end{aligned}$$



Extension — Algebraic Arguments about Inequalities: The result $\sqrt{e} < \frac{1}{2}(e+1)$ proven above is unremarkable, because it is true for any positive number x except 1. This is proven in the following worked exercise. The algebraic argument used there is normal in the 4 Unit course, but would seldom be required in the 3 Unit course.

WORKED EXERCISE: Show that $\sqrt{x} < \frac{1}{2}(x+1)$, for all $x \geq 0$ except $x = 1$.

SOLUTION: Suppose by way of contradiction that $\sqrt{x} \geq \frac{1}{2}(x+1)$.

Then

$$2\sqrt{x} \geq x+1.$$

Squaring,

$$4x \geq x^2 + 2x + 1$$

$$0 \geq x^2 - 2x + 1$$

$$0 \geq (x-1)^2.$$

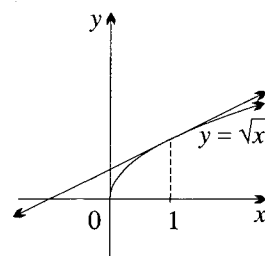
This is impossible except when $x = 1$, because a square can never be negative.

NOTE: Question 1 in the following exercise proves this result using arguments involving tangents and concavity.

Exercise 6F

1. The diagram shows the curve $y = \sqrt{x}$ and the tangent at $x = 1$.

- Show that the tangent has equation $y = \frac{1}{2}(x+1)$.
- Find y'' , and hence explain why the curve is concave down for $x > 0$.
- Hence prove graphically that $\sqrt{x} < \frac{1}{2}(x+1)$, for all $x \geq 0$ except $x = 1$. **NOTE:** This inequality was proven algebraically in the last worked exercise above.



2. (a) A regular hexagon is drawn inside a circle of radius 1 cm and centre O so that its vertices lie on the circumference, as shown in the first diagram.

- Show that $\triangle OAB$ is equilateral and hence find its area.

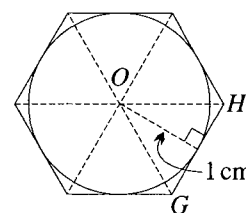
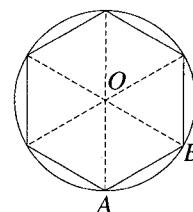
- Hence find the exact area of this hexagon.

- (b) Another regular hexagon is drawn outside the circle, as shown in the second diagram.

- Find the area of $\triangle OGH$.

- Hence find the exact area of this outer hexagon.

- (c) By considering the results in parts (a) and (b), show that $\frac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}$.



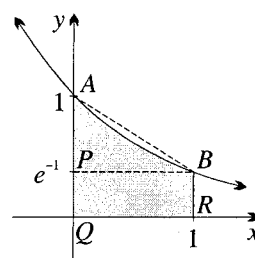
3. The diagram shows the points $A(0,1)$ and $B(1,e^{-1})$ on the curve $y = e^{-x}$.

- Show that the exact area of the region bounded by the curve, the x -axis and the vertical lines $x = 0$ and $x = 1$ is $(1 - e^{-1})$ square units.

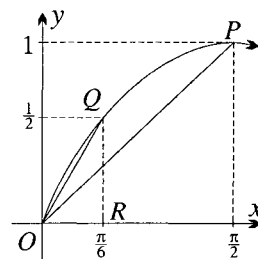
- (b) Find the area of:

- rectangle $PBRQ$, (ii) trapezium $ABRQ$.

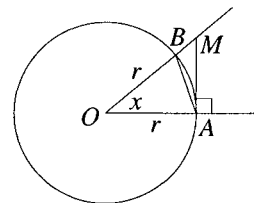
- (c) Use the areas found in the previous parts to show that $2 < e < 3$.



4. The diagram shows the curve $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$. The points $P(\frac{\pi}{2}, 1)$ and $Q(\frac{\pi}{6}, \frac{1}{2})$ lie on the curve.
- Find the equation of the tangent at O .
 - Find the equation of the chord OP , and hence show that $\frac{2x}{\pi} < \sin x < x$, for $0 < x < \frac{\pi}{2}$.
 - Find the equation of the chord OQ , and hence show that $\frac{3x}{\pi} < \sin x < x$, for $0 < x < \frac{\pi}{6}$.
 - By integrating $\sin x$ from 0 to $\frac{\pi}{6}$ and comparing this to the area of $\triangle ORQ$, show that $\pi < 12(2 - \sqrt{3}) \div 3.2$.



5. The diagram shows a circle with centre O and radius r , and a sector OAB subtending an angle of x radians at O . The tangent at A meets the radius OB produced at M .
- Find, in terms of r and x , the areas of:
 - $\triangle OAB$,
 - sector OAB ,
 - $\triangle OAM$.
 - Hence show that $\sin x < x < \tan x$, for $0 < x < \frac{\pi}{2}$.



6. (a) Prove, using mathematical induction, that for all positive integers n ,

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4) = \frac{1}{6}n(n+1)(2n+13).$$

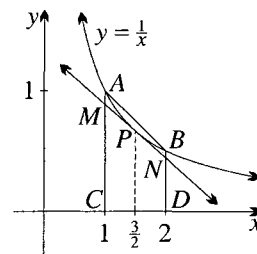
- (b) Hence find $\lim_{n \rightarrow \infty} \frac{1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4)}{n^3}$.

7. Suppose that $f(x) = \ln(1+x) - \ln(1-x)$.

- Find the domain of $f(x)$.
- Find $f'(x)$, and hence explain why $f(x)$ is an increasing function.

8. The points A , P and B on the curve $y = \frac{1}{x}$ have x -coordinates 1, $1\frac{1}{2}$ and 2 respectively. The points C and D are the feet of the perpendiculars drawn from A and B to the x -axis. The tangent to the curve at P cuts AC and BD at M and N respectively.

- Show that the tangent at P has equation $4x + 9y = 12$.
- Find the coordinates of M and N .
- Find the areas of the trapezia $ABDC$ and $MNDC$.
- Hence show that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.



DEVELOPMENT

9. Let $f(x) = \log_e x$.

- (a) Show that $f'(1) = 1$.

- (b) Use the definition of the derivative, that is, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, to show that

$$f'(1) = \lim_{h \rightarrow 0} \log_e(1+h)^{\frac{1}{h}}.$$

- (c) Combine parts (a) and (b) and replace h with $\frac{1}{n}$ to show that $\lim_{n \rightarrow \infty} \log_e(1 + \frac{1}{n})^n = 1$.

- (d) Hence show that $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$.

- (e) To how many decimal places is the RHS of the equation in part (d) accurate when $n = 10, 10^2, 10^3, 10^4, 10^5, 10^6$?

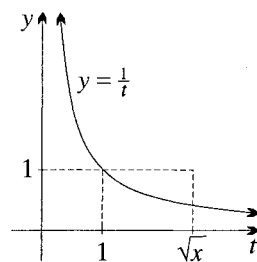
10. (a) Show, using calculus, that the graph of $y = \ln x$ is concave down throughout its domain.
 (b) Sketch the graph of $y = \ln x$, and mark two points $A(a, \ln a)$ and $B(b, \ln b)$ on the curve, where $0 < a < b$.
 (c) Find the coordinates of the point P that divides the interval AB in the ratio $2 : 1$.
 (d) Using parts (b) and (c), deduce that $\frac{1}{3} \ln a + \frac{2}{3} \ln b < \ln(\frac{1}{3}a + \frac{2}{3}b)$.
11. (a) Solve the equation $\sin 2x = 2 \sin^2 x$, for $0 < x < \pi$.
 (b) Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2 \sin^2 x$.
12. Evaluate $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$. [HINT: Multiply by $\frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x}$.]
13. (a) Suppose that $f(x) = \sqrt{1+x}$. Find $f'(8)$.
 (b) Sketch the curve $f(x) = \sqrt{1+x}$ and the tangent at $x = 8$. Hence show that $f'(x) < \frac{1}{6}$ for $x > 8$.
 (c) Deduce that $\sqrt{1+x} \leq 3 + \frac{1}{6}(x-8)$ when $x \geq 8$.
14. Let $f(x) = x^n e^{-x}$, where $n > 1$.
 (a) Show that $f'(x) = x^{n-1} e^{-x}(n-x)$.
 (b) Show that the graph of $f(x)$ has a maximum turning point at $(n, n^n e^{-n})$, and hence sketch the graph for $x \geq 0$. (Don't attempt to find points of inflexion.)
 (c) Explain, by considering the graph of $f(x)$ for $x > n$, why $x^n e^{-x} < n^n e^{-n}$ for $x > n$.
 (d) Deduce from part (c) that $(1 + \frac{1}{n})^n < e$. [HINT: Let $x = n+1$.]
15. (a) Show that $\frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$.
 (b) Hence find, as a fraction in lowest terms, the sum of the first 80 terms of the series $\frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} + \dots$.
 (c) Obtain an expression for $\sum_{r=2}^n \frac{1}{r^2-1}$, and hence find the limiting sum of the series.
16. A sequence is defined recursively by
- $$t_1 = \frac{1}{3} \quad \text{and} \quad t_{n+1} = t_n + t_n^2, \text{ for } n \geq 1.$$
- (a) Show that $\frac{1}{t_n} - \frac{1}{t_{n+1}} = \frac{1}{1+t_n}$.
 (b) Hence find the limiting sum of the series $\sum_{n=1}^{\infty} \frac{1}{1+t_n}$.
17. The function $f(x)$ is defined by $f(x) = x - \log_e(1+x^2)$.
 (a) Show that $f'(x)$ is never negative.
 (b) Explain why the graph of $y = f(x)$ lies completely above the x -axis for $x > 0$.
 (c) Hence prove that $e^x > 1+x^2$, for all positive values of x .
18. (a) Prove by induction that $2^n > n$, for all positive integers n .
 (b) Hence show that $1 < \sqrt[n]{n} < 2$, if n is a positive integer greater than 1.
 (c) Suppose that a and n are positive integers. It is known that if $\sqrt[n]{a}$ is a rational number, then it is an integer. What can we deduce about $\sqrt[n]{n}$, where n is a positive integer greater than 1?

19. Consider the function $y = e^x \left(1 - \frac{x}{10}\right)^{10}$.
- Show that $y' = -\frac{1}{10} x e^x \left(1 - \frac{x}{10}\right)^9$.
 - Find the two turning points of the graph of the function.
 - Discuss the behaviour of the function as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.
 - Sketch the graph of the function.
 - From your graph, deduce that $e^x \leq \left(1 - \frac{x}{10}\right)^{-10}$, for $x < 10$.
 - Hence show that $\left(\frac{11}{10}\right)^{10} \leq e \leq \left(\frac{10}{9}\right)^{10}$.
20. (a) (i) Prove by induction that $(1 + c)^n > 1 + cn$, for all integers $n \geq 2$, where c is a nonzero constant greater than -1 .
- Hence show that $\left(1 - \frac{1}{2^n}\right)^n > \frac{1}{2}$, for all integers $n \geq 2$.
- (b) (i) Solve the inequation $x^2 > 2x + 1$.
- Hence prove by induction that $2^n > n^2$, for all integers $n \geq 5$.
- (c) Suppose that $a > 0$, $b > 0$, and n is a positive integer.
- Divide the expression $a^{n+1} - a^n b + b^{n+1} - b^n a$ by $a - b$, and hence show that $a^{n+1} + b^{n+1} \geq a^n b + b^n a$.
 - Hence prove by induction that $\left(\frac{a+b}{2}\right)^n \leq \frac{a^n + b^n}{2}$.
21. Let $A(1, 1)$ and $B(k, \frac{1}{k})$, where $k > 1$, be points on the hyperbola $y = \frac{1}{x}$.
- Show that the tangents to the hyperbola at A and B intersect at $T\left(\frac{2k}{k+1}, \frac{2}{k+1}\right)$.
 - Suppose that A' , B' and T' are the feet of the perpendiculars drawn from A , B and T to the x -axis.
 - Show that the sum of the areas of the two trapezia $AA'T'T'$ and $TT'B'B$ is $\frac{2(k-1)}{k+1}$ square units.
 - Hence prove that $\frac{2u}{u+2} < \log(u+1) < u$, for all $u > 0$.

EXTENSION

22. The diagram shows the curve $y = \frac{1}{t}$, for $t > 0$.

- If $x > 1$, show that $\int_1^{\sqrt{x}} \frac{1}{t} dt = \frac{1}{2} \log x$.
- Explain why $0 < \frac{1}{2} \log x < \sqrt{x}$, for all $x > 1$.
- Hence show that $\lim_{x \rightarrow \infty} \left(\frac{\log x}{x}\right) = 0$.



23. (a) Given that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, show that:

(i) $e^{-\sin x} < e^{-\frac{2x}{\pi}}$ for $0 < x < \frac{\pi}{2}$,

(ii) $\int_0^{\frac{\pi}{2}} e^{-\sin x} dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}} dx$.

(b) Use the substitution $u = \pi - x$ to show that $\int_0^{\frac{\pi}{2}} e^{-\sin x} dx = \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x} dx$.

(c) Hence show that $\int_0^{\pi} e^{-\sin x} dx < \frac{\pi}{e}(e - 1)$.

24. (a) Show that $\frac{d}{dx}(x \ln x - x) = \ln x$.

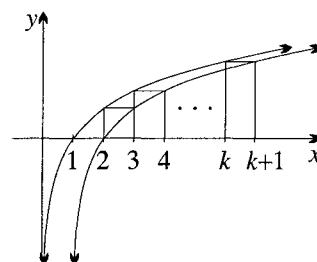
(b) Hence show that $\int_1^n \ln x dx = n \ln n - n + 1$.

- (c) Use the trapezoidal rule on the intervals with endpoints $1, 2, 3, \dots, n$ to show that

$$\int_1^n \ln x dx \div \frac{1}{2} \ln n + \ln(n-1)!$$

- (d) Hence show that $n! < e n^{n+\frac{1}{2}} e^{-n}$. NOTE: This is a preparatory lemma in the proof of Stirling's formula $n! \div \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$, which gives an approximation for $n!$ whose percentage error converges to 0 for large integers n .

25. The diagram shows the curves $y = \log x$ and $y = \log(x-1)$, and $k-1$ rectangles constructed between $x = 2$ and $x = k+1$, where $k \geq 2$.



- (a) Using the result in part (a) of the previous question, show that:

(i) $\int_2^{k+1} \log x dx = (k+1) \log(k+1) - \log 4 - k + 1$

(ii) $\int_2^{k+1} \log(x-1) dx = k \log k - k + 1$

- (b) Deduce that $k^k < k! e^{k-1} < \frac{1}{4}(k+1)^{k+1}$, for all $k \geq 2$.
26. (a) Show graphically that $\log_e x \leq x - 1$, for $x > 0$.
- (b) Suppose that $p_1, p_2, p_3, \dots, p_n$ are positive real numbers whose sum is 1. Show that

$$\sum_{r=1}^n \log_e(np_r) \leq 0.$$

- (c) Let $x_1, x_2, x_3, \dots, x_n$ be positive real numbers. Prove that

$$(x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

When does equality apply in this relationship?

[HINT: Let $s = x_1 + x_2 + x_3 + \cdots + x_n$, and then use part (b) with $p_1 = \frac{x_1}{s}, \dots$]

27. [The binomial theorem and differentiation by the product rule] Suppose that $y = uv$ is the product of two functions u and v of x .

- (a) Show that $y'' = u''v + 2u'v' + uv''$, and develop formulae for y''' , $y^{(4)}$ and $y^{(5)}$.
- (b) Find the fifth derivative of $y = (x^2 + x + 1)e^{-x}$.
- (c) Use sigma notation to write down a formula for the n th derivative $y^{(n)}$.

Rates and Finance

The various topics of this chapter are linked in three ways. First, exponential functions, to various bases, underlie the mathematics of natural growth, compound interest, geometric sequences and housing loans. Secondly, the rate of change in a quantity over time can be studied using the continuous functions presented towards the end of the chapter, or by means of the sequences that describe the changing values of salaries, loans and capital values. Thirdly, many of the applications in the chapter are financial. It is intended that by juxtaposing these topics, the close relationships amongst them in terms of content and method will be made clearer.

STUDY NOTES: Sections 7A and 7B review the earlier formulae of APs and GPs in the context of various practical applications, including salaries, simple interest and compound interest. Sections 7C and 7D concern the specific application of the sums of GPs to financial calculations that involve the payment of regular instalments while compound interest is being charged — superannuation and housing loans are typical examples. Sections 7E and 7F deal with the application of the derivative and the integral to general rates of change, Section 7E being a review of work on related rates of change in Chapter Seven of the Year 11 volume. Section 7G reviews natural growth and decay, in preparation for the treatment in Section 7H of modified equations of growth and decay.

For those who prefer to study the continuous rates of change first, it is quite possible to study Sections 7E–7G first and then return to the applications of APs and GPs in Sections 7A–7D. A handful of questions in Section 7G are designed to draw the essential links between exponential functions, GPs and compound interest, and these can easily be left until Sections 7A–7D have been completed.

Prepared spreadsheets may be useful here in providing experience of how superannuation funds and housing loans behave over time, and computer programs may be helpful in modelling rates of change of some quantities. The intention of the course, however, is to establish the relationships between these phenomena and the known theories of sequences, exponential functions and calculus.

7A Applications of APs and GPs

Arithmetic and geometric sequences were studied in Chapter Six of the Year 11 volume — this section will review the main results about APs and GPs and apply them to problems. Many of the applications will be financial, in preparation for the next three sections.

Formulae for Arithmetic Sequences: At this stage, it should be sufficient simply to list the essential definitions and formulae concerning arithmetic sequences.

ARITHMETIC SEQUENCES:

- A sequence T_n is called an *arithmetic sequence* if

$$T_n - T_{n-1} = d, \text{ for } n \geq 2,$$

where d is a constant, called the *common difference*.

- The n th term of an AP is given by

$$T_n = a + (n - 1)d,$$

1

where a is the first term T_1 .

- Three terms T_1 , T_2 and T_3 are in AP if $T_3 - T_2 = T_2 - T_1$.
- The *arithmetic mean* of a and b is $\frac{1}{2}(a + b)$.
- The sum S_n of the first n terms of an AP is

$$S_n = \frac{1}{2}n(a + \ell) \quad (\text{use when } \ell = T_n \text{ is known}),$$

$$\text{or } S_n = \frac{1}{2}n(2a + (n - 1)d) \quad (\text{use when } d \text{ is known}).$$

WORKED EXERCISE: [A simple AP] Gulgarindi Council is sheltering 100 couples taking refuge in the Town Hall from a flood. They are providing one chocolate per day per person. Every day after the first day, one couple is able to return home. How many chocolates will remain from an initial store of 12 000 when everyone has left?

SOLUTION: The chocolates eaten daily form a series $200 + 198 + \cdots + 2$, which is an AP with $a = 200$, $\ell = 2$ and $n = 100$,

$$\begin{aligned} \text{so number of chocolates eaten} &= \frac{1}{2}n(a + \ell) \\ &= \frac{1}{2} \times 100 \times (200 + 2) \\ &= 10\,100. \end{aligned}$$

Hence 1900 chocolates will remain.

WORKED EXERCISE: [Salaries and APs] Georgia earns \$25 000 in her first year, then her salary increases every year by a fixed amount $\$D$. If the total amount earned at the end of twelve years is \$600 000, find, correct to the nearest dollar:

- (a) the value of D , (b) her final salary.

SOLUTION: Her annual salaries form an AP with $a = 25\,000$ and $d = D$.

- (a) Put $S_{12} = 600\,000$.

$$\frac{1}{2}n(2a + (n - 1)d) = 600\,000$$

$$6(2a + 11d) = 600\,000$$

$$6(50\,000 + 11D) = 600\,000$$

$$50\,000 + 11D = 100\,000$$

$$D = 4545\frac{5}{11}$$

Hence the annual increment is about \$4545.

- (b) Final salary $= T_{12}$

$$= a + 11d$$

$$= 25\,000 + 11 \times 4545\frac{5}{11}$$

$$= \$75\,000.$$

OR $S_n = \frac{1}{2}n(a + \ell)$

$$600\,000 = \frac{1}{2} \times 12 \times (25\,000 + \ell)$$

$$100\,000 = 25\,000 + \ell$$

$$\ell = 75\,000,$$

so her final salary is \$75 000.

Formulae for Geometric Sequences: Geometric sequences involve the one further idea of the limiting sum.

GEOMETRIC SEQUENCES:

- A sequence T_n is called a *geometric sequence* if

$$\frac{T_n}{T_{n-1}} = r, \text{ for } n \geq 2,$$

where r is a constant, called the *common ratio*.

- The n th term of a GP is given by

$$T_n = ar^{n-1}.$$

- Three terms T_1, T_2 and T_3 are in GP if $\frac{T_3}{T_2} = \frac{T_2}{T_1}$.

- The *geometric mean* of a and b is $\sqrt{ab}$ or $-\sqrt{ab}$.

- The sum S_n of the first n terms of a GP is

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad (\text{easier when } r > 1),$$

$$\text{or } S_n = \frac{a(1 - r^n)}{1 - r} \quad (\text{easier when } r < 1).$$

- The limiting sum S_∞ exists if and only if $-1 < r < 1$, and then

$$S_\infty = \frac{a}{1 - r}.$$

The following worked example is a typical problem on GPs, involving both the n th term T_n and the n th partial sum S_n . Notice the use of the change-of-base formula to solve exponential equations by logarithms. For example,

$$\log_{1.05} 1.5 = \frac{\log_e 1.5}{\log_e 1.05}.$$

WORKED EXERCISE: [Inflation and GPs] The General Widget Company sells 2000 widgets per year, beginning in 1991, when the price was \$300 per widget. Each year, the price rises 5% due to cost increases.

- Find the total sales in 1996.
- Find the first year in which total sales will exceed \$900 000.
- Find the total sales from the foundation of the company to the end of 2010.
- During which year will the total sales of the company since its foundation first exceed \$20 000 000?

SOLUTION: The annual sales form a GP with $a = 600\,000$ and $r = 1.05$.

- The sales in any one year constitute the n th term T_n of the series,

$$\begin{aligned} \text{and } T_n &= ar^{n-1} \\ &= 600\,000 \times 1.05^{n-1}. \end{aligned}$$

$$\begin{aligned} \text{Hence sales in 1996} &= T_6 \\ &= 600\,000 \times 1.05^5 \\ &\div \$765\,769. \end{aligned}$$

- (b) Put $T_n > 900\,000$.
 Then $600\,000 \times 1.05^{n-1} > 900\,000$
 $1.05^{n-1} > 1.5$
 $n - 1 > \log_{1.05} 1.5$,
 and using the change-of-base formula, $n - 1 > \frac{\log_e 1.5}{\log_e 1.05} \doteq 8.31$
 $n > 9.31$.

Hence $n = 10$, and sales first exceed \$900 000 in 2000.

- (c) The total sales since foundation constitute the n th partial sum S_n of the series,

$$\begin{aligned} \text{and} \quad S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{600\,000 \times (1.05^n - 1)}{0.05} \\ &= 12\,000\,000 \times (1.05^n - 1). \end{aligned}$$

$$\begin{aligned} \text{Hence total sales to 2010} &= S_{20} \\ &= 12\,000\,000(1.05^{20} - 1) \\ &\doteq \$19\,839\,572. \end{aligned}$$

- (d) Put $S_n > 20\,000\,000$.
 Then $12\,000\,000 \times (1.05^n - 1) > 20\,000\,000$
 $1.05^n > 2\frac{2}{3}$
 $n > \log_{1.05} 2\frac{2}{3}$,
 and using the change-of-base formula, $n > \frac{\log 2\frac{2}{3}}{\log 1.05} \doteq 20.1$.

Hence $n = 21$, and cumulative sales will first exceed \$20 000 000 in 2011.

Taking Logarithms when the Base is Less than 1, and Limiting Sums: When the base is less than 1, passing from an index inequation to a log inequation reverses the inequality sign. For example,

$$\left(\frac{1}{2}\right)^n < \frac{1}{8} \quad \text{means} \quad n > 3.$$

The following worked exercise demonstrates this. Moreover, the GP in the exercise has a limiting sum because the ratio is positive and less than 1. This limiting sum is used to interpret the word ‘eventually’.

WORKED EXERCISE: Sales from the Gumnut Softdrinks Factory in Wadelbri were 50 000 bottles in 2001, but are declining by 6% every year. Nevertheless, the company will always continue to trade.

- (a) In what year will sales first fall below 20 000?
 (b) What will the total sales from 2001 onwards be eventually?
 (c) What proportion of those sales will occur by the end of 2020?

SOLUTION: The sales form a GP with $a = 50\,000$ and $r = 0.94$.

- (a) Put $T_n < 20\,000$.
 Then $ar^{n-1} < 20\,000$
 $50\,000 \times 0.94^{n-1} < 20\,000$
 $0.94^{n-1} < 0.4$

$$n - 1 > \log_{0.94} 0.4 \quad (\text{the inequality reverses})$$

$$n - 1 > \frac{\log_e 0.4}{\log_e 0.94} \doteq 14.8$$

$$n \geq 15.8.$$

Hence $n = 16$, and sales will first fall below 20 000 in 2016.

Since $-1 < r < 1$, the series has a limiting sum.

$$\begin{aligned} \text{(b) Eventual sales} &= S_{\infty} \\ &= \frac{a}{1-r} \\ &= \frac{50\,000}{0.06} \\ &\doteq 833\,333. \end{aligned} \quad \begin{aligned} \text{(c) } \frac{\text{Sales to 2020}}{\text{eventual sales}} &= \frac{a(1-r^{20})}{1-r} \div \frac{a}{1-r} \\ &= 1-r^{20} \\ &= 1-0.94^{20} \\ &\doteq 71\%. \end{aligned}$$

WORKED EXERCISE: [A harder trigonometric application]

(a) Consider the series $1 - \tan^2 x + \tan^4 x - \dots$, where $-90^\circ < x < 90^\circ$.

(i) For what values of x does the series converge?

(ii) What is the limit when it does converge?

(b) In the diagram, $\triangle OA_1B_1$ is right-angled at O , OA_1 has length 1, and $\angle OA_1B_1 = x$, where $x < 45^\circ$.

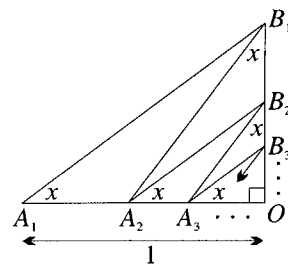
Construct $\angle OB_1A_2 = x$, and construct $A_2B_2 \parallel A_1B_1$.

Continue the construction of $A_3, B_3, A_4, \dots$

(i) Show that $A_1A_2 = 1 - \tan^2 x$ and $A_3A_4 = \tan^4 x - \tan^6 x$.

(ii) Find the limiting sum of $A_1A_2 + A_3A_4 + A_5A_6 + \dots$.

(iii) Find the limiting sum of $A_2A_3 + A_4A_5 + A_6A_7 + \dots$.



SOLUTION:

(a) The series is a GP with $a = 1$ and $r = -\tan^2 x$.

(i) Hence the series converges when $\tan^2 x < 1$,

that is, when $-1 < \tan x < 1$,

so from the graph of $\tan x$, $-45^\circ < x < 45^\circ$.

$$\begin{aligned} \text{(ii) When the series converges, } S_{\infty} &= \frac{a}{1-r} \\ &= \frac{1}{1+\tan^2 x} \\ &= \cos^2 x, \text{ since } 1+\tan^2 x = \sec^2 x. \end{aligned}$$

(b) (i) In $\triangle OA_1B_1$, $OB_1 = \tan x$.

$$\text{In } \triangle OB_1A_2, \frac{OA_2}{OB_1} = \tan x,$$

$$\text{so } OA_2 = \tan^2 x,$$

$$\text{hence } A_1A_2 = 1 - \tan^2 x.$$

(ii) Continuing the process, $OA_3 = OA_2 \times \tan^2 x = \tan^4 x$,

$$\text{and } OA_4 = OA_3 \times \tan^2 x = \tan^6 x,$$

$$\text{so } A_3A_4 = \tan^4 x - \tan^6 x.$$

$$\begin{aligned} \text{Hence } A_1A_2 + A_3A_4 + \dots &= 1 - \tan^2 x + \tan^4 x - \tan^6 x + \dots \\ &= \cos^2 x, \text{ by part (a).} \end{aligned}$$

- (iii) Every piece of OA_1 is on $A_1A_2 + A_3A_4 + \dots$ or on $A_2A_3 + A_4A_5 + \dots$,
 so $A_2A_3 + A_4A_5 + \dots = OA_1 - (A_1A_2 + A_3A_4 + \dots)$
 $= 1 - \cos^2 x$
 $= \sin^2 x$.

Exercise 7A

NOTE: The theory for this exercise was covered in Chapter Six of the Year 11 volume. This exercise is therefore a medley of problems on APs and GPs, with two introductory questions to revise the formulae for APs and GPs.

- Five hundred terms of the series $102 + 104 + 106 + \dots$ are added. What is the total?
 - In a particular arithmetic series, there are 48 terms between the first term 15 and the last term -10 . What is the sum of all the terms in the series?
 - Show that the series $100 + 97 + 94 + \dots$ is an AP, and find the common difference.
 - Show that the n th term is $T_n = 103 - 3n$, and find the first negative term.
 - Find an expression for the sum S_n of the first n terms, and show that 68 is the minimum number of terms for which S_n is negative.
- The first few terms of a particular series are $2000 + 3000 + 4500 + \dots$.
 - Show that it is a geometric series, and find the common ratio.
 - What is the sum of the first five terms?
 - Explain why the series does not converge.
 - Consider the series $18 + 6 + 2 + \dots$.
 - Show that it is a geometric series, and find the common ratio.
 - Explain why this geometric series has a limiting sum, and find its value.
 - Show that the limiting sum and the sum of the first ten terms are equal, correct to the first three decimal places.
- A secretary starts on an annual salary of \$30 000, with annual increments of \$2000.
 - Find his annual salary, and his total earnings, at the end of ten years.
 - In which year will his salary be \$42 000?
- An accountant receives an annual salary of \$40 000, with 5% increments each year.
 - Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.
 - In which year will her salary first exceed \$70 000?
- Lawrence and Julian start their first jobs on low wages. Lawrence starts at \$25 000 per annum, with annual increases of \$2500. Julian starts at the lower wage of \$20 000 per annum, with annual increases of 15%.
 - Find Lawrence's annual wages in each of the first three years, and explain why they form an arithmetic sequence.
 - Find Julian's annual wages in each of the first three years, and explain why they form a geometric sequence.
 - Show that the first year in which Julian's annual wage is the greater of the two will be the sixth year, and find the difference, correct to the nearest dollar.
- An initial salary of \$50 000 increases each year by \$3000. In which year will the salary first be at least twice the original salary?
 - An initial salary of \$50 000 increases by 4% each year. In which year will the salary first be at least twice the original salary?

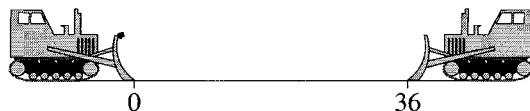
7. A certain company manufactures three types of shade cloth. The product with code SC50 cuts out 50% of harmful UV rays, SC75 cuts out 75% and SC90 cuts out 90% of UV rays. In the following questions, you will need to consider the amount of UV light let through.
- What percentage of UV light does each cloth let through?
 - Show that two layers of SC50 would be equivalent to one layer of SC75 shade cloth.
 - Use trial and error to find the minimum number of layers of SC50 that would be required to cut out at least as much UV light as one layer of SC90.
 - Similarly, find how many layers of SC50 would be required to cut out 99% of UV rays.
8. Olim, Pixi, Thi (pronounced 'tea'), Sid and Nee work in the sales division of a calculator company. Together they find that sales of scientific calculators are dropping by 150 per month, while sales of graphics calculators are increasing by 150 per month.
- Current sales of all calculators total 20 000 per month, and graphics calculators account for 10% of sales. How many graphics calculators are sold per month?
 - How many more graphics calculators will be sold per month by the sales team six months from now?
 - Assuming that current trends continue, how long will it be before all calculators sold by the company are graphics calculators?

DEVELOPMENT

9. One Sunday, 120 days before Christmas, Franksworth store publishes an advertisement saying '120 shopping days until Christmas'. Franksworth subsequently publishes similar advertisements every Sunday until Christmas.
- How many times does Franksworth advertise?
 - Find the sum of the numbers of days published in all the advertisements.
 - On which day of the week is Christmas?
10. A farmhand is filling a row of feed troughs with grain. The distance between adjacent troughs is 5 metres, and he has parked the truck with the grain 1 metre from the closest trough. He decides that he will fill the closest trough first and work his way to the far end. Each trough requires three bucketloads to fill it completely.
- How far will the farmhand walk to fill the 1st trough and return to the truck? How far for the 2nd trough? How far for the 3rd trough?
 - How far will the farmhand walk to fill the n th trough and return to the truck?
 - If he walks a total of 156 metres to fill the furthest trough, how many feed troughs are there?
 - What is the total distance he will walk to fill all the troughs?
11. Yesterday, a tennis ball used in a game of cricket in the playground was hit onto the science block roof. Luckily it rolled off the roof. After bouncing on the playground it reached a height of 3 metres. After the next bounce it reached 2 metres, then $1\frac{1}{3}$ metres and so on.
- What was the height reached after the n th bounce?
 - What was the height of the roof the ball fell from?
 - The last time the ball bounced, its height was below 1 cm for the first time. After that it rolled away across the playground.
 - Show that $(\frac{3}{2})^{n-1} > 300$.
 - How many times did the ball bounce?
12. A certain algebraic equation is being solved by the method of halving the interval, with the two starting values 4 units apart. The pen of a plotter begins at the left-hand value, and then moves left or right to the location of each successive midpoint. What total distance will the pen have travelled eventually?

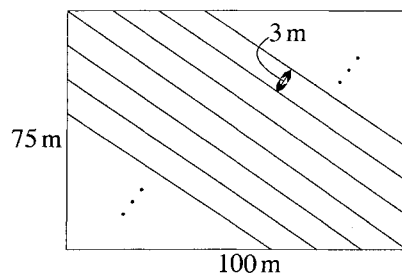
13. Theodor earns \$30 000 in his first year, and his salary increases each year by a fixed amount $\$D$.
- Find D if his salary in his tenth year is \$58 800.
 - Find D if his total earnings in the first ten years are \$471 000.
 - If $D = 2200$, in which year will his salary first exceed \$60 000?
 - If $D = 2000$, show that his total earnings first exceed \$600 000 during his 14th year.
14. Madeline opens a business selling computer stationery. In its first year, the business has sales of \$200 000, and each year sales are 20% more than the previous year's sales.
- In which year do annual sales first exceed \$1 000 000?
 - In which year do total sales since foundation first exceed \$2 000 000?
15. Madeline's sister opens a hardware store. Sales in successive years form a GP, and sales in the fifth year are half the sales in the first year. Let sales in the first year be $\$F$.
- Find, in exact form, the ratio of the GP.
 - Find the total sales of the company as time goes on, as a multiple of the first year's sales, correct to two decimal places.
16. [Limiting sums of trigonometric series]
- Find when each series has a limiting sum, and find that limiting sum:
 - $1 + \cos^2 x + \cos^4 x + \dots$
 - $1 + \sin^2 x + \sin^4 x + \dots$
 - Find, in terms of $t = \tan \frac{1}{2}x$, the limiting sums of these series when they converge:
 - $1 - \sin x + \sin^2 x - \dots$
 - $1 + \sin x + \sin^2 x + \dots$
 - Show that when these series converge:
 - $1 - \cos x + \cos^2 x - \dots = \frac{1}{2} \sec^2 \frac{1}{2}x$
 - $1 + \cos x + \cos^2 x + \dots = \frac{1}{2} \operatorname{cosec}^2 \frac{1}{2}x$

17.



Two bulldozers are sitting in a construction site facing each other. Bulldozer A is at $x = 0$, and bulldozer B is 36 metres away at $x = 36$. A bee is sitting on the scoop at the very front of bulldozer A. At 7:00 am the workers start up both bulldozers and start them moving towards each other at the same speed V m/s. The bee is disturbed by the commotion and flies at twice the speed of the bulldozers to land on the scoop of bulldozer B.

- Show that the bee reaches bulldozer B when it is at $x = 24$.
 - Immediately the bee lands, it takes off again and flies back to bulldozer A. Where is bulldozer A when the two meet?
 - Assume that the bulldozers keep moving towards each other and the bee keeps flying between the two, so that the bee will eventually be squashed.
 - Where will this happen?
 - How far will the bee have flown?
18. The area available for planting in a particular paddock of a vineyard measures 100 metres by 75 metres. In order to make best use of the sun, the grape vines are planted in rows diagonally across the paddock, as shown in the diagram, with a 3-metre gap between adjacent rows.

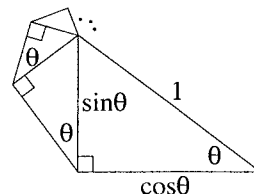


- What is the length of the diagonal of the field?
- What is the length of each row on either side of the diagonal?

- (c) Confirm that each row two away from the diagonal is 112.5 metres long.
 (d) Show that the lengths of these rows form an arithmetic sequence.
 (e) Hence find the total length of all the rows of vines in the paddock.

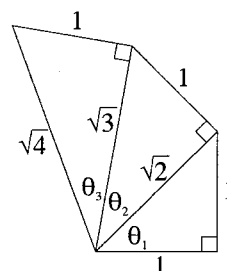
EXTENSION

19. The diagram shows the first few triangles in a spiral of similar right-angled triangles, each successive one built with its hypotenuse on a side of the previous one.



- (a) What is the area of the largest triangle?
 (b) Use the result for the ratio of areas of similar figures to show that the areas of successive triangles form a geometric sequence. What is the common ratio?
 (c) Hence show that the limiting sum of the areas of the triangles is $\frac{1}{2} \tan \theta$.

20. The diagram shows the beginning of a spiral created when each successive right-angled triangle is constructed on the hypotenuse of the previous triangle. The altitude of each triangle is 1, and it is easy to show by Pythagoras' theorem that the sequence of hypotenuse lengths is $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots$. Let the base angle of the n th triangle be θ_n . Clearly θ_n gets smaller, but does this mean that the spiral eventually stops turning? Answer the following questions to find out.



- (a) Write down the value of $\tan \theta_n$.

- (b) Show that $\sum_{n=1}^k \theta_n \geq \frac{1}{2} \sum_{n=1}^k \frac{1}{n}$. [HINT: $\theta \geq \frac{1}{2} \tan \theta$, for $0 \leq \theta \leq \frac{\pi}{4}$.]

- (c) By sketching $y = \frac{1}{x}$ and constructing the upper rectangle on each of the intervals

$$1 \leq x \leq 2, \quad 2 \leq x \leq 3, \quad 3 \leq x \leq 4, \quad \dots, \text{ show that } \sum_{n=1}^k \frac{1}{n} \geq \int_1^k \frac{1}{n} \, dn.$$

- (d) Does the total angle through which the spiral turns approach a limit?

7B Simple and Compound Interest

This section will review the formulae for simple and compound interest, but with greater attention to the language of functions and of sequences. Simple interest can be understood mathematically both as an arithmetic sequence and as a linear function. Compound interest or depreciation can be understood both as a geometric sequence and as an exponential function.

Simple Interest, Arithmetic Sequences and Linear Functions: The well-known formula for simple interest is $I = PRn$. But if we want the total amount A_n at the end of n units of time, we need to add the principal P — this gives $A_n = P + PRn$, which is a linear function of n . Substituting into this function the positive integers $n = 1, 2, 3, \dots$ gives the sequence

$$P + PR, \quad P + 2PR, \quad P + 3PR, \quad \dots$$

which is an AP with first term $P + PR$ and common difference PR .

SIMPLE INTEREST: Suppose that a principal $\$P$ earns simple interest at a rate R per unit time for n units of time. Then the simple interest $\$I$ earned is

$$I = PRn.$$

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The total amount $\$A_n$ after n units of time is a linear function of n ,

$$A_n = P + PRn.$$

This forms an AP with first term $P + PR$ and common difference PR .

Be careful that the interest rate here is a number, not a percentage. For example, if the interest rate is 7% pa, then $R = 0.07$. (The initials 'pa' stand for 'per annum', which is Latin for 'per year'.)

WORKED EXERCISE: Find the principal $\$P$, if investing $\$P$ at 6% pa simple interest yields a total of \$6500 at the end of five years.

SOLUTION: Put $P + PRn = 6500$.

Since $R = 0.06$ and $n = 5$, $P(1 + 0.30) = 6500$

$$\boxed{\div 1.3} \quad P = \$5000.$$

Compound Interest, Geometric Sequences and Exponential Functions: The well-known formula for compound interest is $A_n = P(1 + R)^n$. First, this is an exponential function of n , with base $1 + R$. Secondly, substituting $n = 1, 2, 3, \dots$ into this function gives the sequence

$$P(1 + R), P(1 + R)^2, P(1 + R)^3, \dots$$

which is a GP with first term $P(1 + R)$ and common ratio $1 + R$.

COMPOUND INTEREST: Suppose that a principal $\$P$ earns compound interest at a rate R per unit time for n units of time, compounded every unit of time. Then the total amount after n units of time is an exponential function of n ,

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$$A_n = P(1 + R)^n.$$

This forms a GP with first term $P(1 + R)$ and common ratio $1 + R$.

Note that the formula only works when compounding occurs after every unit of time. For example, if the interest rate is 18% per year with interest compounded monthly, then the units of time must be months, and the interest rate per month is $R = 0.18 \div 12 = 0.015$. Unless otherwise stated, compounding occurs over the unit of time mentioned when the interest rate is given.

PROOF: Although the formula was developed in earlier years, it is vital to understand how it arises, and how the process of compounding generates a GP.

The initial principal is P , and the interest is R per unit time.

Hence the amount A_1 at the end of one unit of time is

$$A_1 = \text{principal} + \text{interest} = P + PR = P(1 + R).$$

This means that adding the interest is effected by multiplying by $1 + R$.

Similarly, the amount A_2 is obtained by multiplying A_1 by $1 + R$:

$$A_2 = A_1(1 + R) = P(1 + R)^2.$$

Then, continuing the process,

$$A_3 = A_2(1 + R) = P(1 + R)^3,$$

$$A_4 = A_3(1 + R) = P(1 + R)^4,$$

so that when the money has been invested for n units of time,

$$A_n = A_{n-1}(1 + R) = P(1 + R)^n.$$

WORKED EXERCISE: Amelda takes out a loan of \$5000 at a rate of 12% pa, compounded monthly. She makes no repayments.

(a) Find the total amount owing at the end of five years.

(b) Find when, correct to the nearest month, the amount owing doubles.

SOLUTION: Because the interest is compounded every month, the units of time must be months. The interest rate is therefore 1% per month, and $R = 0.01$.

(a) $A_{60} = P \times 1.01^{60}$ (5 years is 60 months),
 $\div \$9083$.

(b) Put $A_n = 10\,000$.

$$\text{Then } 5000 \times 1.01^n = 10\,000$$

$$1.01^n = 2$$

$$n = \log_{1.01} 2$$

$$= \frac{\log 2}{\log 1.01}, \text{ using the change-of-base formula,}$$

$$\div 70 \text{ months.}$$

Depreciation: Depreciation is usually expressed as the loss per unit time of a percentage of the current price of an item. The formula for depreciation is therefore the same as the formula for compound interest, except that the rate is negative.

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DEPRECIATION: Suppose that goods originally costing $\$P$ depreciate at a rate R per unit time for n units of time. Then the total amount after n units of time is

$$A_n = P(1 - R)^n.$$

WORKED EXERCISE: An espresso machine bought on 1st January 2001 depreciates at $12\frac{1}{2}\%$ pa. In which year will the value drop below 10% of the original cost, and what will be the loss of value during that year, as a percentage of the original cost?

SOLUTION: In this case, $R = -0.125$ is negative, because the value is decreasing.

Let the initial value be P . Then $A_n = P \times 0.875^n$.

Put $A_n = 0.1 \times P$, to find when the value has dropped to 10%.

$$\text{Then } P \times 0.875^n = 0.1 \times P$$

$$n = \frac{\log 0.1}{\log 0.875}$$

$$\div 17.24.$$

Hence the depreciated value will drop below 10% during 2018.

$$\text{Loss during that year} = A_{17} - A_{18}$$

$$= (0.875^{17} - 0.875^{18})P,$$

$$\text{so percentage loss} = (0.875^{17} - 0.875^{18}) \times 100\% \\ \div 1.29\%.$$

8. Find the total value A_n when a principal P is invested at 12% pa compound interest for n years. Hence find the smallest number of years for the investment:
 - (a) to double, (b) to treble, (c) to quadruple, (d) to increase by a factor of 10.
9. A student was asked to find the original value, correct to the nearest dollar, of an investment earning 9% per annum, compounded annually for three years, given its current value of \$54 391.22.
 - (a) She incorrectly thought that since she was working in reverse, she should use the depreciation formula. What value did she get?
 - (b) What is the correct answer?
10. An amount of \$10 000 is invested for five years at 4% pa interest, compounded monthly.
 - (a) Find the final value of the investment.
 - (b) What rate of simple interest, correct to two significant figures, would be needed to yield the same final balance?
11. Xiao and Mai win a prize in the lottery and decide to put \$100 000 into a retirement fund offering 8.25% per annum interest, compounded monthly. How long will it be before their money has doubled? Give your answer correct to the nearest month.
12. The present value of a company asset is \$350 000. If it has been depreciating at $17\frac{1}{2}\%$ per annum for the last six years, what was the original value of the asset, correct to the nearest \$1000?
13. Thirwin, Neri, Sid and Nee each inherit \$10 000. Each invests the money for one year. Thirwin invests his money at 7.2% per annum simple interest. Neri invests hers at 7.2% per annum, compounded annually. Sid invests his at 7% per annum, compounded monthly. Nee invests in certain shares with a return of 8.1% per annum, but must pay stockbrokers' fees of \$50 to buy the shares initially and again to sell them at the end of the year. Who is furthest ahead at the end of the year?
14. (a) A principal P is invested at a compound interest rate of r per period.
 - (i) Write down A_n , the total value after n periods.
 - (ii) Hence find a formula for the number of periods required for the total value to reach twice the principal.
 (b) Suppose that a simple interest rate of R per period applied instead.
 - (i) Write down B_n , the total value after n periods.
 - (ii) Further suppose that for a particular value of n , $A_n = B_n$. Derive a formula for R in terms of r and n .

EXTENSION

15. (a) Write down the total value A_n of an investment P if a simple interest rate R is applied over n periods.
- (b) Show, by means of the binomial theorem, that the total value of the investment when compound interest is applied may be written as $A_n = P + PRn + P \sum_{k=2}^n {}^nC_k R^k$.
- (c) Explain what each of the three terms of the formula in part (b) represents.
16. (a) Write out the terms of $P(1 + R)^n$ as a binomial expansion.
- (b) Show that the term $P {}^nC_k R^k$ is the sum of interest earned for any, not necessarily consecutive, k years over the life of the investment.
- (c) What is the significance of the greatest term in the binomial expansion, in this context?

7C Investing Money by Regular Instalments

Many investment schemes, typically superannuation schemes, require money to be invested at regular intervals such as every month or every year. This makes things difficult, because each individual instalment earns compound interest for a different length of time. Hence calculating the value of these investments at some future time requires the theory of GPs.

This topic is intended to be an application of GPs, and learning formulae is not recommended.

Developing the GP and Summing It: The most straightforward way to solve these problems is to find what each instalment grows to as it accrues compound interest. These final amounts form a GP, which can then be summed.

WORKED EXERCISE: Robin and Robyn are investing \$10 000 in a superannuation scheme on 1st July each year, beginning in 2000. The money earns compound interest at 8% pa, compounded annually.

- How much will the fund amount to by 30th June 2020?
- Find the year in which the fund first exceeds \$700 000 on 30th June.
- What annual instalment would have produced \$1 000 000 by 2020?

SOLUTION: Because of the large numbers involved, it is usually easier to work with pronumerals, apart perhaps from the (fixed) interest rate.

Let M be the annual instalment, so $M = 10\,000$ in parts (a) and (b), and let A_n be the value of the fund at the end of n years.

After the first instalment is invested for n years, it amounts to $M \times 1.08^n$, after the second instalment is invested for $n - 1$ years, it amounts to $M \times 1.08^{n-1}$, after the n th instalment is invested for just 1 year, it amounts to $M \times 1.08$, so

$$A_n = 1.08M + 1.08^2M + \cdots + 1.08^nM.$$

This is a GP with first term $a = 1.08M$, ratio $r = 1.08$, and n terms.

$$\begin{aligned} \text{Hence } A_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{1.08M \times (1.08^n - 1)}{0.08} \\ A_n &= 13.5M \times (1.08^n - 1). \end{aligned}$$

- Substituting $n = 20$ and $M = 10\,000$,

$$A_n = 13.5 \times 10\,000 \times (1.08^{20} - 1)$$

$$\doteq \$494\,229.$$

- Substituting $M = 10\,000$ and $A_n = 700\,000$,

$$700\,000 = 13.5 \times 10\,000 \times (1.08^n - 1)$$

$$1.08^n - 1 = \frac{70}{13.5}$$

$$n = \frac{\log(\frac{70}{13.5} + 1)}{\log 1.08}$$

$$\doteq 23.68.$$

Hence the fund first exceeds \$700 000 on 30th June 2024.

- (c) Substituting
- $A_n = 1\,000\,000$
- and
- $n = 20$
- ,

$$1\,000\,000 = 13.5 \times M \times (1.08^{20} - 1)$$

$$M = \frac{1\,000\,000}{13.5 \times (1.08^{20} - 1)}$$

$$\doteq \$20\,234.$$

WORKED EXERCISE: Charmaine is offered the choice of two superannuation schemes, both of which will yield the same amount at the end of ten years.

- Pay \$600 per month, with interest of 7.8% pa, compounded monthly.
- Pay weekly, with interest of 7.8% pa, compounded weekly.

- (a) What is the final value of the first scheme?
 (b) What are the second scheme's weekly instalments?
 (c) Which scheme would cost her more per year?

SOLUTION: The following solution begins by generating the general formula for the amount A_n after n units of time, in terms of the instalment M and the rate R , and this formula is then applied in parts (a) and (b). An alternative approach would be to generate separately each of the formulae required in parts (a) and (b). Whichever approach is adopted, the formulae must be derived rather than just quoted from memory.

Let M be the instalment and R the rate per unit time,
 and let A_n be the value of the fund at the end of n units of time.

The first instalment is invested for n months, and so amounts to $M(1+R)^n$,
 the second instalment is invested for $n-1$ months, and so amounts to $M(1+R)^{n-1}$,
 and the last instalment is invested for 1 month, and so amounts to $M(1+R)$,
 so $A_n = M(1+R) + M(1+R)^2 + \cdots + M(1+R)^n$.

This is a GP with first term $a = M(1+R)$, ratio $r = (1+R)$, and n terms.

Hence $A_n = \frac{a(r^n - 1)}{r - 1}$

$$A_n = \frac{M(1+R) \times ((1+R)^n - 1)}{R}.$$

- (a) For the first scheme, the interest rate is $\frac{7.8}{12}\% = 0.65\%$ per month,
 so substitute $n = 120$, $M = 600$ and $R = 0.0065$.

$$A_n = \frac{600 \times 1.0065 \times (1.0065^{120} - 1)}{0.0065}$$

$$\doteq \$109\,257 \text{ (retain in the memory for part(b))}.$$

- (b) For the second scheme, the interest rate is $\frac{7.8}{52}\% = 0.15\%$ per week,
 so substituting $R = 0.0015$,

$$A_n = \frac{M \times 1.0015 \times (1.0015^n - 1)}{0.0015}.$$

Writing this formula with M as the subject,

$$M = \frac{A_n \times 0.0015}{1.0015 \times (1.0015^n - 1)},$$

and substituting $n = 520$ and $A_n = 109\,257$ (from memory),

$$M \doteq \$138.65 \text{ (retain in the memory for part(c))}.$$

- (c) This is about \$7210.04 per year, compared with \$7200 per year for the first.

An Alternative Approach Using Recursion: There is an alternative approach, using recursion, to developing the GPs involved in these calculations. Because the working is slightly longer, we have chosen not to display this method in the notes. It has, however, the advantage that its steps follow the progress of a banking statement. For those who are interested in the recursive method, it is developed in two structured questions at the end of the Development section in the following exercise.

Exercise 7C

- A company makes contributions of \$3000 on 1st July each year to the superannuation fund of one of its employees. The money earns compound interest at 6.5% per annum. In the following parts, round all currency amounts correct to the nearest dollar.
 - Let M be the annual contribution, and let A_n be the value of the fund at the end of n years.
 - How much does the first instalment amount to at the end of n years?
 - How much does the second instalment amount to at the end of $n - 1$ years?
 - What is the worth of the last contribution, invested for just one year?
 - Hence write down a series for A_n .
 - Hence show that $A_n = \frac{1.065 M(1.065^n - 1)}{0.065}$.
 - What will be the value of the fund after 25 years, and what will be the total amount of the contributions?
 - Suppose that the employee wanted to achieve a total investment of \$300 000 after 25 years, by topping up the contributions.
 - What annual contribution would have produced this amount?
 - By how much would the employee have to top up the contributions?
- A company increases the annual wage of an employee by 4% on 1st January each year.
 - Let M be the annual wage in the first year of employment, and let W_n be the wage in the n th year. Write down W_1 , W_2 and W_n in terms of M .
 - Hence show that the total amount paid to the employee is $A_n = \frac{M(1.04^n - 1)}{0.04}$.
 - If the employee starts on \$30 000 and stays with the company for 20 years, how much will the company have paid over that time? Give your answer correct to the nearest dollar.
- A person invests \$10 000 each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first payment is on 1st January 2001 and the last payment is on 1st January 2020.
 - How much did the person invest over the life of the fund?
 - Calculate, correct to the nearest dollar, the amount to which the 2001 payment has grown by the beginning of 2021.
 - Find the total value of the fund when it is paid out on 1st January 2021.

DEVELOPMENT

- Each year on her birthday, Jane's parents put \$20 into an investment account earning $9\frac{1}{2}\%$ per annum compound interest. The first deposit took place on the day of her birth. On her 18th birthday, Jane's parents gave her the account and \$20 cash in hand.

- (a) How much money had Jane's parents deposited in the account?
- (b) How much money did she receive from her parents on her 18th birthday?
5. A man about to turn 25 is getting married. He has decided to pay \$5000 each year on his birthday into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. If he dies before age 65, his wife will inherit the value of the insurance to that point. If he lives to age 65, the insurance company will pay out the value of the policy in full. Answer the following correct to the nearest dollar.
- (a) The man is in a dangerous job. What will be the payout if he dies just before he turns 30?
- (b) The man's father died of a heart attack just before age 50. Suppose that the man also dies of a heart attack just before age 50. How much will his wife inherit?
- (c) What will the insurance company pay the man if he survives to his 65th birthday?
6. In 2001, the school fees at a private girls' school are \$10 000 per year. Each year the fees rise by $4\frac{1}{2}\%$ due to inflation.
- (a) Susan is sent to the school, starting in Year 7 in 2001. If she continues through to her HSC year, how much will her parents have paid the school over the six years?
- (b) Susan's younger sister is starting in Year 1 in 2001. How much will they spend on her school fees over the next twelve years if she goes through to her HSC?
7. A woman has just retired with a payment of \$500 000, having contributed for 25 years to a superannuation fund that pays compound interest at the rate of $12\frac{1}{2}\%$ per annum. What was the size of her annual premium, correct to the nearest dollar?
8. John is given a \$10 000 bonus by his boss. He decides to start an investment account with a bank that pays $6\frac{1}{2}\%$ per annum compound interest.
- (a) If he makes no further deposits, what will be the balance of his account, correct to the nearest cent, 15 years from now?
- (b) If instead he also makes an annual deposit of \$1000 at the beginning of each year, what will be the balance at the end of 15 years?
9. At age 20, a woman takes out a life insurance policy in which she agrees to pay premiums of \$500 per year until she turns 65, when she is to be paid a lump sum. The insurance company invests the money and gives a return of 9% per annum, compounded annually. If she dies before age 65, the company pays out the current value of the fund plus 25% of the difference had she lived until 65.
- (a) What is the value of the payout, correct to the nearest dollar, at age 65?
- (b) Unfortunately she dies at age 53, just before her 35th premium is due.
- (i) What is the current value of the life insurance?
- (ii) How much does the life insurance company pay her family?
10. A finance company has agreed to pay a retired couple a pension of \$15 000 per year for the next twenty years, indexed to inflation which is $3\frac{1}{2}\%$ per annum.
- (a) How much will the company have paid the couple at the end of twenty years?
- (b) Immediately after the tenth annual pension payment is made, the finance company increases the indexed rate to 4% per annum to match the increased inflation rate. Given these new conditions, how much will the company have paid the couple at the end of twenty years?
11. A person pays \$2000 into an investment fund every six months, and it earns interest at a rate of 6% pa, compounded monthly. How much is the fund worth at the end of ten years?

NOTE: The following two questions illustrate an alternative approach to superannuation questions, using a recursive method to generate the appropriate GP. As mentioned in the notes above, the method has the disadvantage of requiring more steps in the working, but has the advantage that its steps follow the progress of a banking statement.

12. Cecilia deposits $\$M$ at the start of each month into a savings scheme that pays interest of 1% per month, compounded monthly. Let A_n be the amount in her account at the end of the n th month.
- Explain why $A_1 = 1.01M$.
 - Explain why $A_2 = 1.01(M + A_1)$, and why $A_{n+1} = 1.01(M + A_n)$, for $n \geq 2$.
 - Use the recursive formulae in part (b), together with the value of A_1 in part (a), to obtain expressions for $A_2, A_3, \dots, A_n$.
 - Use the formula for the n th partial sum of a GP to show that $A_n = 101M(1.01^n - 1)$.
 - If each deposit is $\$100$, how much will be in the fund after three years?
 - Hence find, correct to the nearest cent, how much each deposit M must be if Cecilia wants the fund to amount to $\$30\,000$ at the end of five years.
13. A couple saves $\$100$ at the start of each week in an account paying 10.4% pa interest, compounded weekly. Let A_n be the amount in the account at the end of the n th week.
- Explain why $A_1 = 1.002 \times 100$, and why $A_{n+1} = 1.002(100 + A_n)$, for $n \geq 2$.
 - Use these recursive formulae to obtain expressions for $A_2, A_3, \dots, A_n$.
 - Using GP formulae, show that $A_n = 50\,100(1.01^n - 1)$.
 - Hence find how many weeks it will be before the couple has $\$100\,000$.

EXTENSION

14. Let V be the value of an investment of $\$1000$ earning compound interest at the rate of 10% per annum for n years.
- Draw up a table of values for V of values of n between 0 and 7.
 - Plot these points and join them with a smooth curve. What type of curve is this?
 - On the same graph add upper rectangles of width 1, add the areas of these rectangles, and give your answer correct to the nearest dollar.
 - Compare your answer with the value of superannuation after seven years if $\$1000$ is deposited each year at the same rate of interest.
 - What do you notice?
 - What do you conclude?
15. (a) If you have access to a program like ExcelTM for Windows 98TM, try checking your answers to questions 1 to 10 using the built-in financial functions. In particular, the built-in ExcelTM function **FV(rate, nper, pmt, pv, type)** seems to produce an answer different from what might be expected. Investigate this and explain the difference.
- (b) If you have access to a program like MathematicaTM, try checking your answers to questions 1 to 10, using the following function definitions.
- Calculate the final value of a superannuation fund, invested for n years at a rate of r per annum with annual premiums of $\$m$, using

$$\text{Super}[n_, r_, m_] := m * (1 + r) * ((1 + r) ^ n - 1) / r.$$
 - Calculate the premiums if the final value of the fund is p , using

$$\text{SupContrib}[p_, n_, r_] := p * r / ((1 + r) * ((1 + r) ^ n - 1)).$$

7D Paying Off a Loan

Long-term loans such as housing loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time. The calculations associated with paying off a loan are therefore similar to the investment calculations of the previous section. The extra complication is that an investment fund is always in credit, whereas a loan account is always in debit because of the large initial loan that must be repaid.

Developing the GP and Summing It: As with superannuation, the most straightforward method is to calculate the final value of each payment as it accrues compound interest, and then add these final values up using the theory of GPs. We must also deal with the final value of the initial loan.

WORKED EXERCISE: Natasha and Richard take out a loan of \$200 000 on 1st January 2002 to buy a house. Interest is charged at 12% pa, compounded monthly, and they will repay the loan in monthly instalments of \$2200.

- Find the amount owing at the end of n months.
- Find how long it takes to repay: (i) the full loan, (ii) half the loan.
- How long would repayment take if they were able to pay \$2500 per month?
- Why would instalments of \$1900 per month never repay the loan?

NOTE: The first repayment is normally made at the end of the first repayment period. In this example, that means on the last day of each month.

SOLUTION: Let $P = 200\,000$ be the principal, let M be the instalment, and let A_n be the amount still owing at the end of n months.

To find a formula for A_n , we need to calculate the value of each instalment under the effect of compound interest of 1% per month, from the time that it is paid.

The first instalment is invested for $n - 1$ months, and so amounts to $M \times 1.01^{n-1}$, the second instalment is invested for $n - 2$ months, and so amounts to $M \times 1.01^{n-2}$, the n th instalment is invested for no time at all, and so amounts to M .

The initial loan, after n months, amounts to $P \times 1.01^n$.

Hence $A_n = P \times 1.01^n - (M + 1.01M + \cdots + 1.01^{n-1}M)$.

The bit in brackets is a GP with first term $a = M$, ratio $r = 1.01$, and n terms.

$$\begin{aligned} \text{Hence } A_n &= P \times 1.01^n - \frac{a(r^n - 1)}{r - 1} \\ &= P \times 1.01^n - \frac{M(1.01^n - 1)}{0.01} \\ &= P \times 1.01^n - 100M(1.01^n - 1) \end{aligned}$$

or, reorganising, $A_n = 100M - 1.01^n(100M - P)$.

- (a) Substituting $P = 200\,000$ and $M = 2200$ gives

$$\begin{aligned} A_n &= 100 \times 2200 - 1.01^n \times 20\,000 \\ &= 220\,000 - 1.01^n \times 20\,000. \end{aligned}$$

- (b) (i) To find when the loan is repaid, put $A_n = 0$:

$$\begin{aligned} 1.01^n \times 20\,000 &= 220\,000 \\ n &= \frac{\log 11}{\log 1.01} \\ &\doteq 20 \text{ years and 1 month.} \end{aligned}$$

(ii) To find when the loan is half repaid, put $A_n = 100\,000$:

$$\begin{aligned} 1.01^n \times 20\,000 &= 120\,000 \\ n &= \frac{\log 6}{\log 1.01} \\ &\doteq 15 \text{ years.} \end{aligned}$$

(c) Substituting instead $M = 2500$ gives $100M = 250\,000$,

so $A_n = 250\,000 - 1.01^n \times 50\,000$.

Put $A_n = 0$, for the loan to be repaid.

Then $1.01^n \times 50\,000 = 250\,000$

$$\begin{aligned} n &= \frac{\log 5}{\log 1.01} \\ &\doteq 13 \text{ years and 6 months.} \end{aligned}$$

(d) Substituting $M = 1900$ gives $100M = 190\,000$,

so $A_n = 190\,000 - 1.01^n \times (-10\,000)$, which is always positive.

This means that the debt would be increasing rather than decreasing.

Another way to understand this is to calculate

$$\begin{aligned} \text{initial interest per month} &= 200\,000 \times 0.01 \\ &= 2000, \end{aligned}$$

so initially, \$2000 of the instalment is required just to pay the interest.

The Alternative Approach Using Recursion: As with superannuation, the GP involved in loan-repayment calculations can be developed using an alternative recursive method, whose steps follow the progress of a banking statement. Again, this method is developed in two structured questions at the end of the Development section in the following exercise.

Exercise 7D

- I took out a personal loan of \$10 000 with a bank for five years at an interest rate of 18% per annum, compounded monthly.
 - Let P be the principal, let M be the size of each repayment to the bank, and let A_n be the amount owing on the loan after n months.
 - To what does the initial loan amount after n months?
 - Write down the amount to which the first instalment grows by the end of the n th month.
 - Do likewise for the second instalment and for the n th instalment.
 - Hence write down a series for A_n .
 - Hence show that $A_n = P \times 1.015^n - \frac{M(1.015^n - 1)}{0.015}$.
 - When the loan is paid off, what is the value of A_n ?
 - Hence find an expression for M in terms of P and n .
 - Given the values of P and n above, find M , correct to the nearest dollar.
- A couple takes out a \$250 000 mortgage on a house, and they agree to pay the bank \$2000 per month. The interest rate on the loan is 7.2% per annum, compounded monthly, and the contract requires that the loan be paid off within twenty years.

- (a) Again let A_n be the balance on the loan after n months, let P be the amount borrowed, and let M be the amount of each instalment. Find a series expression for A_n .
- (b) Hence show that $A_n = P \times 1.006^n - \frac{M(1.006^n - 1)}{0.006}$.
- (c) Find the amount owing on the loan at the end of the tenth year, and state whether this is more or less than half the amount borrowed.
- (d) Find A_{240} , and hence show that the loan is actually paid out in less than twenty years.
- (e) If it is paid out after n months, show that $1.006^n = 4$, and hence that $n = \frac{\log 4}{\log 1.006}$.
- (f) Find how many months early the loan is paid off.
3. As can be seen from the last two questions, the calculations involved with reducible loans are reasonably complex. For that reason, it is sometimes convenient to convert the reducible interest rate into a simple interest rate. Suppose that a mortgage is taken out on a \$180 000 house at 6.6% reducible interest per annum for a period of 25 years, with payments made monthly.
- (a) Using the usual pronumerals, explain why $A_{300} = 0$.
- (b) Find the size of each repayment to the bank.
- (c) Hence find the total paid to the bank, correct to the nearest dollar, over the life of the loan.
- (d) What amount is therefore paid in interest? Use this amount and the simple interest formula to calculate the simple interest rate per annum over the life of the loan, correct to two significant figures.

DEVELOPMENT

4. What is the monthly instalment necessary to pay back a personal loan of \$15 000 at a rate of $13\frac{1}{2}\%$ per annum over five years? Give your answer correct to the nearest dollar.
5. Most questions so far have asked you to round monetary amounts correct to the nearest dollar. This is not always wise, as this question demonstrates. A personal loan for \$30 000 is approved with the following conditions. The reducible interest rate is 13.3% per annum, with payments to be made at six-monthly intervals over five years.
- (a) Find the size of each instalment, correct to the nearest dollar.
- (b) Using this amount, show that $A_{10} \neq 0$, that is, the loan is not paid off in five years.
- (c) Explain why this has happened.
6. A couple have worked out that they can afford to pay \$19 200 each year in mortgage payments. If the current home loan rate is 7.5% per annum, with payments made monthly over a period of 25 years, what is the maximum amount that the couple can borrow and still pay off the loan?
7. A company borrows \$500 000 from the bank at an interest rate of 5% per annum, to be paid in monthly instalments. If the company repays the loan at the rate of \$10 000 per month, how long will it take? Give your answer in whole months with an appropriate qualification.
8. Some banks offer a 'honeymoon' period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed \$170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay \$1650 per month to the bank. The standard rate of interest is $8\frac{1}{2}\%$ pa, but the bank also offers a special rate of 6% pa for one year to people buying their first home.
- (a) Calculate the amount the couple would owe at the end of the first year, using the special rate of interest.

- (b) Use this value as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the loan contract?
9. A company buys machinery for \$500 000 and pays it off by 20 equal six-monthly instalments, the first payment being made six months after the loan is taken out. If the interest rate is 12% pa, compounded monthly, how much will each instalment be?
10. The current rate of interest on Bankerscard is 23% per annum, compounded monthly.
- (a) If a cardholder can afford to repay \$1500 per month on the card, what is the maximum value of purchases that can be made in one day if the debt is to be paid off in two months?
- (b) How much would be saved in interest payments if the card holder instead saved up the money for two months before making the purchase?
11. Over the course of years, a couple have saved up \$300 000 in a superannuation fund. Now that they have retired, they are going to draw on that fund in equal monthly pension payments for the next twenty years. The first payment is at the beginning of the first month. At the same time, any balance will be earning interest at $5\frac{1}{2}\%$ per annum, compounded monthly. Let B_n be the balance left immediately after the n th payment, and let M be the amount of the pension instalment. Also, let $P = 300\,000$ and R be the monthly interest rate.
- (a) Show that $B_n = P \times (1 + R)^{n-1} - \frac{M((1 + R)^n - 1)}{R}$.
- (b) Why is $B_{240} = 0$? (c) What is the value of M ?

NOTE: The following two questions illustrate the alternative approach to loan repayment questions, using a recursive method to generate the appropriate GP.

12. A couple buying a house borrow $\$P = \$150\,000$ at an interest rate of 6% pa, compounded monthly. They borrow the money at the beginning of January, and at the end of every month, they pay an instalment of $\$M$. Let A_n be the amount owing at the end of n months.
- (a) Explain why $A_1 = 1.005P - M$.
- (b) Explain why $A_2 = 1.005A_1 - M$, and why $A_{n+1} = 1.005A_n - M$, for $n \geq 2$.
- (c) Use the recursive formulae in part (b), together with the value of A_1 in part (a), to obtain expressions for $A_2, A_3, \dots, A_n$.
- (d) Using GP formulae, show that $A_n = 1.005^n P - 200M(1.005^n - 1)$.
- (e) Hence find, correct to the nearest cent, what each instalment should be if the loan is to be paid off in twenty years?
- (f) If each instalment is \$1000, how much is still owing after twenty years?
13. Eric and Enid borrow $\$P$ to buy a house at an interest rate of 9.6% pa, compounded monthly. They borrow the money on 15th September, and on the 14th day of every subsequent month, they pay an instalment of $\$M$. Let A_n be the amount owing after n months have passed.
- (a) Explain why $A_1 = 1.008P - M$, and why $A_{n+1} = 1.008A_n - M$, for $n \geq 2$.
- (b) Use these recursive formulae to obtain expressions for $A_2, A_3, \dots, A_n$.
- (c) Using GP formulae, show that $A_n = 1.008^n P - 125M(1.008^n - 1)$.
- (d) If the maximum instalment they can afford is \$1200, what is the maximum they can borrow, if the loan is to be paid off in 25 years? (Answer correct to the nearest dollar.)
- (e) Put $A_n = 0$ in part (c), and solve for n . Hence find how long will it take to pay off the loan of \$100 000 if each instalment is \$1000. (Round up to the next month.)

EXTENSION

14. A finance company has agreed to pay a retired couple a pension of \$19 200 per year for the next twenty years, indexed to inflation that is $3\frac{1}{2}\%$ per annum.
- How much will the company have paid the couple at the end of twenty years?
 - In return, the couple pay an up-front fee which the company invests at a compound interest rate of 7% per annum. The total value of the fee plus interest covers the pension payouts over the twenty-year period. How much did the couple pay the firm up front, correct to the nearest dollar?
15. [This question will be much simpler to solve using a computer for the calculations.] Suppose, using the usual notation, that a loan of \$ P at an interest rate of R per month is repaid over n monthly instalments of \$ M .
- Show that $M - (M + P)K^n + PK^{1+n} = 0$, where $K = 1 + R$.
 - Suppose that I can afford to repay \$650 per month on a \$20 000 loan to be paid back over three years. Use these figures in the equation above and apply Newton's method in order to find the highest rate of interest I can afford to meet. Give your answer correct to three significant figures.
 - Repeat the same problem using the bisection method, in order to check your answer.
16. A man aged 25 is getting married, and has decided to pay \$3000 each year into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. Once he reaches 65, the insurance company will pay out the value of the policy as a pension in equal monthly instalments over the next 25 years. During those 25 years, the balance will continue to earn interest at the same rate, but compounded monthly.
- What is the value of the policy when he reaches 65, correct to the nearest dollar?
 - What will be the size of pension payments, correct to the nearest dollar?

7E Rates of Change — Differentiating

A *rate of change* is the rate at which some quantity Q is changing. It is therefore the derivative $\frac{dQ}{dt}$ of Q with respect to time t , and is the gradient of the tangent to the graph of Q against time. A rate of change is always instantaneous unless otherwise stated, and should not be confused with an *average rate of change*, which is the gradient of a chord. This section will review the work on rates of change in Section 7H of the Year 11 volume, where the emphasis is on using the chain rule to calculate the rate of change of a given function. The next section will deal with the integration of rates.

Calculating Related Rates: As explained previously, the calculation of the relationship between two rates is simply an exercise in applying the chain rule.

6

RELATED RATES: Find a relation between the two quantities, then differentiate with respect to time, using the chain rule.

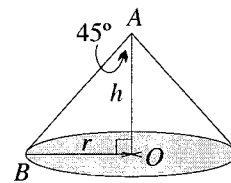
WORKED EXERCISE: Sand is being poured onto the top of a pile at the rate of $3\text{ m}^3/\text{min}$. The pile always remains in the shape of a cone with semi-vertical angle 45° . Find the rate at which:

- the height,
 - the base area,
- is changing when the height is 2 metres.

SOLUTION: Let the cone have volume V , height h and base radius r .

Since the semi-vertical angle is 45° , $r = h$ (isosceles $\triangle AOB$).

The rate of change of volume is known to be $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$.



- (a) We know that $V = \frac{1}{3}\pi r^2 h$,
and since $r = h$, $V = \frac{1}{3}\pi h^3$.

Differentiating with respect to time (using the chain rule with the RHS),

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \times \frac{dh}{dt} \\ &= \pi h^2 \frac{dh}{dt}.\end{aligned}$$

Substituting, $3 = \pi \times 2^2 \times \frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{3}{4\pi} \text{ m/min}.$$

- (b) The base area is $A = \pi h^2$ (since $r = h$).

$$\begin{aligned}\text{Differentiating, } \frac{dA}{dt} &= \frac{dA}{dh} \times \frac{dh}{dt} \\ &= 2\pi h \frac{dh}{dt}.\end{aligned}$$

$$\begin{aligned}\text{Substituting, } \frac{dA}{dt} &= 2 \times \pi \times 2 \times \frac{3}{4\pi} \\ &= 3 \text{ m}^2/\text{min}.\end{aligned}$$

WORKED EXERCISE: A 10 metre ladder is leaning against a wall, and the base is sliding away from the wall at 1 cm/s. Find the rate at which:

- (a) the height, (b) the angle of inclination,
is changing when the foot is already 6 metres from the wall.

SOLUTION: Let the height be y and the distance from the wall be x ,
and let the angle of inclination be θ . We know that $\frac{dx}{dt} = 0.01 \text{ m/s}$.

- (a) By Pythagoras' theorem, $x^2 + y^2 = 10^2$,

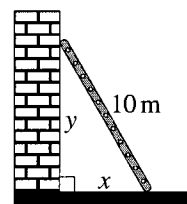
hence $y = \sqrt{100 - x^2}$.

$$\begin{aligned}\text{Differentiating, } \frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= \frac{-2x}{2\sqrt{100 - x^2}} \times \frac{dx}{dt} \\ &= -\frac{x}{\sqrt{100 - x^2}} \times \frac{dx}{dt}.\end{aligned}$$

Substituting $x = 6$ and $\frac{dx}{dt} = 0.01$,

$$\begin{aligned}\frac{dy}{dt} &= -\frac{6}{\sqrt{100 - 36}} \times 0.01 \\ &= -0.0075.\end{aligned}$$

Hence the height is decreasing at $\frac{3}{4} \text{ cm/s}$.



[Alternatively we can differentiate $x^2 + y^2 = 10^2$ implicitly.

This gives $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$.

When $x = 6$, $y = 8$ by Pythagoras' theorem, so substituting,

$$12 \times 0.01 + 16 \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -0.0075.$$

Hence the height is decreasing at $\frac{3}{4}$ cm/s.]

(b) By trigonometry, $x = 10 \cos \theta$.

$$\begin{aligned} \text{Differentiating, } \frac{dx}{dt} &= \frac{dx}{d\theta} \times \frac{d\theta}{dt} \\ &= -10 \sin \theta \times \frac{d\theta}{dt}. \end{aligned}$$

When $x = 6$, $\sin \theta = \frac{y}{10} = \frac{8}{10}$, so substituting,

$$\begin{aligned} 0.01 &= -10 \times \frac{8}{10} \times \frac{d\theta}{dt} \\ \frac{d\theta}{dt} &= -0.01 \times \frac{1}{8} \\ &= -\frac{1}{800}. \end{aligned}$$

Hence the angle of inclination is decreasing by $\frac{1}{800}$ radians per second, or, multiplying by $\frac{180}{\pi}$, by about 0.072° per second.

Exercise 7E

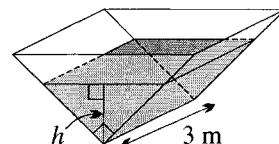
NOTE: This exercise reviews material already covered in Exercise 7H of the Year 11 volume.

- The sides of a square of side length x metres are increasing at a rate of 0.1 m/s.
 - Show that the rate of increase of the area is given by $\frac{dA}{dt} = 0.2x$ m²/s.
 - At what rate is the area of the square increasing when its sides are 5 metres long?
 - What is the side length when the area is increasing at 1.4 m²/s?
 - What is the area when the area is increasing at 0.6 m²/s?
- The diagonal of a square is decreasing at a rate of $\frac{1}{2}$ m/s.
 - Find the area A of a square with a diagonal of length ℓ .
 - Hence show that the rate of change of area is $\frac{dA}{dt} = -\frac{1}{2}\ell$ m²/s.
 - Find the rate at which the area is decreasing when:
 - the diagonal is 10 metres,
 - the area is 18 m².
 - What is the length of the diagonal when the area is decreasing at 17 m²/s?
- The radius r of a sphere is increasing at a rate of 0.3 m/s. In both parts, approximate π using a calculator and give your answer correct to three significant figures.

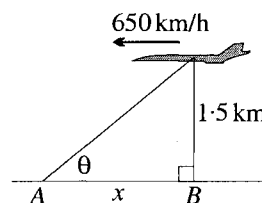
- (a) Show that the sphere's rate of change of volume is $\frac{dV}{dt} = 1.2\pi r^2$, and find the rate of increase of its volume when the radius is 2 metres.
- (b) Show that the sphere's rate of change of surface area is $\frac{dS}{dt} = 2.4\pi r$, and find the rate of increase of its surface area when the radius is 4 metres.
4. Jules is blowing up a spherical balloon at a constant rate of $200 \text{ cm}^3/\text{s}$.
- (a) Show that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.
- (b) Hence find the rate at which the radius is growing when the radius is 15 cm.
- (c) Find the radius and volume when the radius is growing at 0.5 cm/s .
5. A lathe is used to shave down the radius of a cylindrical piece of wood 500 mm long. The radius is decreasing at a rate of 3 mm/min .
- (a) Show that the rate of change of volume is $\frac{dV}{dt} = -3000\pi r$, and find how fast the volume is decreasing when the radius is 30 mm.
- (b) How fast is the circumference decreasing when the radius is: (i) 20 mm, (ii) 37 mm?

DEVELOPMENT

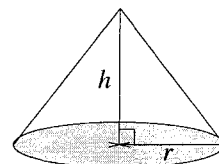
6. The water trough in the diagram is in the shape of an isosceles right triangular prism, 3 metres long. A jackaroo is filling the trough with a hose at the rate of 2 litres per second.



- (a) Show that the volume of water in the trough when the depth is $h \text{ cm}$ is $V = 300h^2 \text{ cm}^3$.
- (b) Given that 1 litre is 1000 cm^3 , find the rate at which the depth of the water is changing when $h = 20$.
7. An observer at A in the diagram is watching a plane at P fly overhead, and he tilts his head so that he is always looking directly at the plane. The aircraft is flying at 650 km/h at an altitude of 1.5 km . Let θ be the angle of elevation of the plane from the observer, and suppose that the distance from A to B , directly below the aircraft, is $x \text{ km}$.



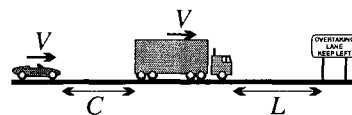
- (a) By writing $x = \frac{3}{2 \tan \theta}$, show that $\frac{dx}{d\theta} = -\frac{3}{2 \sin^2 \theta}$.
- (b) Hence find the rate at which the observer's head is tilting when the angle of inclination to the plane is $\frac{\pi}{3}$. Convert your answer from radians per hour to degrees per second, correct to the nearest degree.
8. Sand is poured at a rate of $0.5 \text{ m}^3/\text{s}$ onto the top of a pile in the shape of a cone, as shown in the diagram. Let the base have radius r , and let the height of the cone be h . The pile always remains in the same shape, with $r = 2h$.
- (a) Find the cone's volume, and show that it is the same as that of a sphere with radius equal to the cone's height.
- (b) Find the rate at which the height is increasing when the radius of the base is 4 metres.
9. A boat is observed from the top of a 100-metre-high cliff. The boat is travelling towards the cliff at a speed of 50 m/min . How fast is the angle of depression changing when the angle of depression is 15° ? Convert your answer from radians per minute to degrees per minute, correct to the nearest degree.



10. The volume of a sphere is increasing at a rate numerically equal to its surface area at that instant. Show that $\frac{dr}{dt} = 1$.
11. A point moves anticlockwise around the circle $x^2 + y^2 = 1$ at a uniform speed of 2 m/s.
- Find an expression for the rate of change of its x -coordinate in terms of x , when the point is above the x -axis. (The units on the axes are metres.)
 - Use your answer to part (a) to find the rate of change of the x -coordinate as it crosses the y -axis at $P(0, 1)$. Why should this answer have been obvious without this formula?

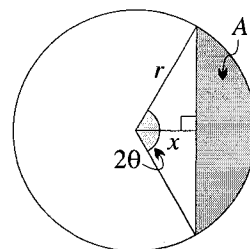
EXTENSION

12. A car is travelling C metres behind a truck, both travelling at a constant speed of V m/s. The road widens L metres ahead of the truck and there is an overtaking lane. The car accelerates at a uniform rate so that it is exactly alongside the truck at the beginning of the overtaking lane.

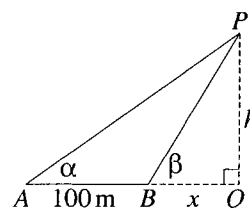


- What is the acceleration of the car?
- Show that the speed of the car as it passes the truck is $V \left(1 + \frac{2C}{L} \right)$.
- The objective of the driver of the car is to spend as little time alongside the truck as possible. What strategies could the driver employ?
- The speed limit is 100 km/h and the truck is travelling at 90 km/h, and is 50 metres ahead of the car. How far before the overtaking lane should the car begin to accelerate if applying the objective in part (c)?

13. The diagram shows a chord distant x from the centre of a circle. The radius of the circle is r , and the chord subtends an angle 2θ at the centre.



- Show that the area of the segment cut off by this chord is $A = r^2(\theta - \sin \theta \cos \theta)$.
 - Explain why $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dx} \times \frac{dx}{dt}$.
 - Show that $\frac{d\theta}{dx} = -\frac{1}{\sqrt{r^2 - x^2}}$.
 - Given that $r = 2$, find the rate of increase in the area if $\frac{dx}{dt} = -\sqrt{3}$ when $x = 1$.
14. The diagram shows two radars at A and B 100 metres apart. An aircraft at P is approaching and the radars are tracking it, hence the angles α and β are changing with time.
- Show that $x \tan \beta = (x + 100) \tan \alpha$.
 - Keeping in mind that x , α and β are all functions of time, use implicit differentiation to show that
- $$\frac{dx}{dt} = \frac{\dot{\alpha}(x + 100) \sec^2 \alpha - \dot{\beta}x \sec^2 \beta}{\tan \beta - \tan \alpha}.$$
- Use part (a) to find the value of x and the height of the plane when $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$.
 - At the angles given in part (c), it is found that $\frac{d\alpha}{dt} = \frac{5}{36}(\sqrt{3} - 1)$ radians per second and $\frac{d\beta}{dt} = \frac{5}{18}(\sqrt{3} - 1)$ radians per second. Find the speed of the plane.



7F Rates of Change — Integrating

In some situations, only the rate of change of a quantity as a function of time is known. The original function can then be obtained by integration, provided that the value of the function is known initially or at some other time.

WORKED EXERCISE: During a drought, the flow $\frac{dV}{dt}$ of water from Welcome Well gradually diminishes according to the formula $\frac{dV}{dt} = 3e^{-0.02t}$, where t is time in days after time zero, and V is the volume in megalitres of water that has flowed out.

- Show that $\frac{dV}{dt}$ is always positive, and explain this physically.
- Find an expression for the volume of water obtained after time zero.
- How much will flow from the well during the first 100 days?
- Describe the behaviour of V as $t \rightarrow \infty$, and find what percentage of the total flow comes in the first 100 days. Sketch the function.

SOLUTION:

- Since $e^x > 0$ for all x , $\frac{dV}{dt} = 3e^{-0.02t}$ is always positive.

V is always increasing, because V is the amount that has flowed out.

- We are given that $\frac{dV}{dt} = 3e^{-0.02t}$.

Integrating, $V = -150e^{-0.02t} + C$.

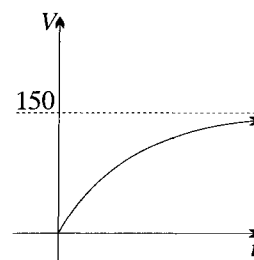
When $t = 0$, $V = 0$, so $0 = -150 + C$,

so $C = 150$, and $V = 150(1 - e^{-0.02t})$.

- When $t = 100$, $V = 150(1 - e^{-2})$
 $\doteq 129.7$ megalitres.

- As $t \rightarrow \infty$, $V \rightarrow 150$, since $e^{-0.02t} \rightarrow 0$.

$$\begin{aligned} \text{Hence proportion of flow in first 100 days} &= \frac{150(1 - e^{-2})}{150} \\ &= 1 - e^{-2} \\ &\doteq 86.5\%. \end{aligned}$$



WORKED EXERCISE: The rate at which ice on the side of Black Mountain is melting during spring changes with the time of day according to $\frac{dI}{dt} = -5 + 5 \cos \frac{\pi}{12}t$, where I is the mass in tonnes of ice remaining on the mountain, and t is the time in hours after midnight on the day measuring began.

- Initially, there were 2400 tonnes of ice. Find I as a function of t .
- Show that for all t , I is decreasing or stationary, and find when I is stationary.
- Show that the ice disappears at the end of the 20th day.

SOLUTION:

- We are given that $\frac{dI}{dt} = -5 + 5 \cos \frac{\pi}{12}t$.

Integrating, $I = -5t + \frac{60}{\pi} \sin \frac{\pi}{12}t + C$, for some constant C .

When $t = 0$, $I = 2400$, so $2400 = -0 - 0 + C$,
 so $C = 2400$, and $I = -5t + \frac{60}{\pi} \sin \frac{\pi}{12}t + 2400$.

- (b) Since $-5 \leq 5 \cos \frac{\pi}{12}t \leq 5$, $\frac{dI}{dt}$ can never be positive.

I is stationary when $\frac{dI}{dt} = 0$, that is, when $\cos \frac{\pi}{12}t = 1$.

The general solution for $t \geq 0$ is $\frac{\pi}{12}t = 0, 2\pi, 4\pi, 6\pi, \dots$
 $t = 0, 24, 48, 72, \dots$

That is, melting ceases at midnight on each successive day.

- (c) When $t = 480$, $I = -2400 + 0 + 2400 = 0$,
 so the ice disappears at the end of the 20th day.
 (Notice that I is never increasing, so there can only be one solution for t .)

Exercise 7F

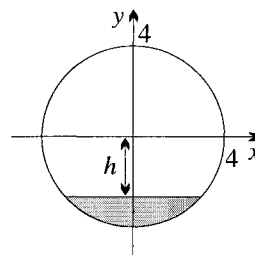
- Water is flowing out of a tank at the rate of $\frac{dV}{dt} = 5(2t - 50)$, where V is the volume in litres remaining in the tank at time t minutes after time zero.
 - When does the water stop flowing?
 - Given that the tank still has 20 litres left in it when the water flow stops, find V as a function of t .
 - How much water was initially in the tank?
- The rate at which a perfume ball loses its scent over time is $\frac{dP}{dt} = -\frac{2}{t+1}$, where t is measured in days.
 - Find P as a function of t if the initial perfume content is 6.8.
 - How long will it be before the perfume in the ball has run out and it needs to be replaced? (Answer correct to the nearest day.)
- A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by $\frac{dV}{dt} = -2 + \frac{1}{10}t \text{ m}^3/\text{s}$.
 - What is the initial flow rate, when the tap is fully on?
 - How long does it take to turn the tap off?
 - Given that when the tap has been turned off there are still 500 m^3 of water left in the tank, find V as a function of t .
 - Hence find how much water is released during the time it takes to turn the tap off.
 - Suppose that it is necessary to let out a total of 300 m^3 from the tank. How long should the tap be left fully on before gradually turning it off?
- The velocity of a particle is given by $\frac{dx}{dt} = e^{-0.4t}$.
 - Does the particle ever stop moving?
 - If the particle starts at the origin, find its displacement x as a function of time.
 - When does the particle reach $x = 1$? (Answer correct to two decimal places.)
 - Where does the particle move to eventually? (That is, find its limiting position.)

DEVELOPMENT

5. A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time t , is given by $\frac{dx}{dt} = 250(e^{-0.2t} - 1)$, where x is the height above the ground.
 - (a) What is its initial speed?
 - (b) What is its eventual speed?
 - (c) Find x as a function of t , if it is initially 200 metres above the ground.
6. Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\frac{dI}{dt} = -5 + 4 \cos \frac{\pi}{12}t$, where I is the tonnage of ice on the mountain at time t in hours since 2:00 am on 20th October.
 - (a) It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find I as a function of t .
 - (b) Explain, from the given rate, why the ice is always melting.
 - (c) The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.
7. As a particle moves around a circle, its angular velocity is given by $\frac{d\theta}{dt} = \frac{1}{1+t^2}$.
 - (a) Given that the particle starts at $\theta = \frac{\pi}{4}$, find θ as a function of t .
 - (b) Hence find t as a function of θ .
 - (c) Using the result of part (a), show that $\frac{\pi}{4} \leq \theta < \frac{3\pi}{4}$, and hence explain why the particle never moves through an angle of more than $\frac{\pi}{2}$.
8. The flow of water into a small dam over the course of a year varies with time and is approximated by $\frac{dW}{dt} = 1.2 - \cos^2 \frac{\pi}{12}t$, where W is the volume of water in the dam, measured in thousands of cubic metres, and t is the time measured in months from the beginning of January.
 - (a) What is the maximum flow rate into the dam and when does this happen?
 - (b) Given that the dam is initially empty, find W .
 - (c) The capacity of the dam is 25 200 m³. Show that it will be full in three years.
9. A certain brand of medicine tablet is in the shape of a sphere with diameter $\frac{1}{2}$ cm. The rate at which the pill dissolves is proportional to its surface area at that instant, that is, $\frac{dV}{dt} = kS$ for some constant k , and the pill lasts 12 hours before dissolving completely.
 - (a) Show that $\frac{dr}{dt} = k$, where r is the radius of the sphere at time t hours.
 - (b) Hence find r as a function of t . (c) Thus find k .
10. Sand is poured onto the top of a pile in the shape of a cone at a rate of 0.5 m³/s. The apex angle of the cone remains constant at 90°. Let the base have radius r and let the height of the cone be h .
 - (a) Find the volume of the cone, and show that it is one quarter of the volume of a sphere with the same radius.
 - (b) Find the rate of change of the radius of the cone as a function of r .
 - (c) By taking reciprocals and integrating, find t as a function of r , given that the initial radius of the pile was 10 metres.
 - (d) Hence find how long it takes, correct to the nearest second, for the pile to grow another 2 metres in height.

EXTENSION

11. (a) The diagram shows the spherical cap formed when the region between the lower half of the circle $x^2 + y^2 = 16$ and the horizontal line $y = -h$ is rotated about the y -axis. Find the volume V so formed.
- (b) The cap represents a shallow puddle of water left after some rain. When the sun comes out, the water evaporates at a rate proportional to its surface area (which is the circular area at the top of the cap).
- (i) Find this surface area A .
- (ii) We are told that $\frac{dV}{dt} = -kA$. Show that the rate at which the depth of the water changes is $-k$.
- (iii) The puddle is initially 2 cm deep and the evaporation constant is known to be $k = 0.025$ cm/min. Find how long it takes for the puddle to evaporate.



7G Natural Growth and Decay

This section will review the approaches to natural growth and decay developed in Section 13F of the Year 11 volume. The key idea here is that the exponential function $y = e^t$ is its own derivative, that is,

$$\text{if } y = e^t, \text{ then } \frac{dy}{dt} = e^t = y.$$

This means that at each point on the curve, the gradient is equal to the height. More generally,

$$\text{if } y = y_0 e^{kt}, \text{ then } \frac{dy}{dt} = ky_0 e^{kt} = ky.$$

This means that the rate of change of $y = Ae^{kt}$ is proportional to y .

The natural growth theorem says that, conversely, the only functions where the rate of growth is proportional to the value are functions of the form $y = Ae^{kt}$.

NATURAL GROWTH: Suppose that the rate of change of y is proportional to y :

$$\frac{dy}{dt} = ky, \text{ where } k \text{ is a constant of proportionality.}$$

Then $y = y_0 e^{kt}$, where y_0 is the value of y at time $t = 0$.

WORKED EXERCISE: The value V of some machinery is depreciating according to the law of natural decay $\frac{dV}{dt} = -kV$, for some positive constant k . Each year its value drops by 15%.

- (a) Show that $V = V_0 e^{-kt}$ satisfies this differential equation, where V_0 is the initial cost of the machinery.
- (b) Find the value of k , in exact form, and correct to four significant figures.
- (c) Find, correct to four significant figures, the percentage drop in value over five years.
- (d) Find, correct to the nearest 0.1 years, when the value has dropped by 90%.

SOLUTION:

(a) Substituting $V = V_0 e^{-kt}$ into $\frac{dV}{dt} = -kV$,

$$\begin{aligned} \text{LHS} &= \frac{d}{dt}(V_0 e^{-kt}) & \text{RHS} &= -k \times V_0 e^{-kt} \\ &= -kV_0 e^{-kt}, & &= \text{LHS.} \end{aligned}$$

Also, substituting $t = 0$ gives $V = V_0 e^0 = V_0$, as required.

(b) When $t = 1$, $V = 0.85 V_0$, so $0.85 V_0 = V_0 e^{-k}$

$$e^{-k} = 0.85$$

$$k = -\log_e 0.85$$

$$\doteq 0.1625.$$

(c) When $t = 5$, $V = V_0 e^{-5k}$

$$\doteq 0.4437 V_0,$$

so the value has dropped by

about 55.63% over the 5 years.

(d) Put $V = 0.1 V_0$.

$$\text{Then } V_0 e^{-kt} = 0.1 V_0$$

$$-kt = \log_e 0.1$$

$$t \doteq 14.2 \text{ years.}$$

Natural Growth and GPs: There are very close relationships between GPs and natural growth, as the following worked exercise shows.

WORKED EXERCISE: Continuing with the previous worked exercise:

- show that the values of the machinery after 0, 1, 2, ... years forms a GP, and find the ratio of the GP,
- find the loss of value during the 1st, 2nd, 3rd, ... years. Show that these losses form a GP, and find the ratio of the GP.

SOLUTION:

(a) The values after 0, 1, 2, ... years are $V_0, V_0 e^{-k}, V_0 e^{-2k}, \dots$

This sequence forms a GP with first term V_0 and ratio $e^{-k} = 0.85$.

(b) Loss of value during the first year = $V_0 - V_0 e^{-k}$

$$= V_0(1 - e^{-k}),$$

loss of value during the second year = $V_0 e^{-k} - V_0 e^{-2k}$

$$= V_0 e^{-k}(1 - e^{-k}),$$

loss of value during the third year = $V_0 e^{-2k} - V_0 e^{-3k}$

$$= V_0 e^{-2k}(1 - e^{-k}).$$

These losses form a GP with first term $V_0(1 - e^{-k})$ and ratio $e^{-k} = 0.85$.

A Confusing Term — The ‘Growth Rate’: Suppose that a population P is growing according to the equation $P = P_0 e^{0.08t}$. The constant $k = 0.08$ is sometimes called the ‘growth rate’, but this is a confusing term, because ‘growth rate’ normally refers to the instantaneous increase $\frac{dP}{dt}$ of the number of individuals per unit time. The constant k is better described as the *instantaneous proportional growth rate*, because the differential equation $\frac{dP}{dt} = kP$ shows that k is the proportionality constant relating the instantaneous rate of growth and the population.

It is important in this context not to confuse average rates of growth, represented by chords on the exponential graph, with instantaneous rates of growth, represented by tangents on the exponential graph. There are in fact four different rates — two instantaneous rates, one absolute and one proportional, and two average rates, one absolute and one proportional. The following worked exercise on inflation asks for all four of these rates.

WORKED EXERCISE: [Four different rates associated with natural growth]

The cost C of building an average house is rising according to the natural growth equation $C = 150\,000 e^{0.08t}$, where t is time in years since 1st January 2000.

- Show that $\frac{dC}{dt}$ is proportional to C , and find the constant of proportionality (this is the so-called ‘growth rate’, or, more correctly, the ‘instantaneous proportional growth rate’).
- Find the instantaneous rates at which the cost is increasing on 1st January 2000, 2001, 2002 and 2003, correct to the nearest dollar per year, and show that they form a GP.
- Find the value of C when $t = 1$, $t = 2$ and $t = 3$, and the average increases in cost over the first year, the second year and the third year, correct to the nearest dollar per year, and show that they form a GP.
- Show that the average increase in cost over the first year, the second year and the third year, expressed as a proportion of the cost at the start of that year, is constant.

SOLUTION:

- Differentiating, $\frac{dC}{dt} = 0.08 \times 150\,000 \times e^{0.08t} = 0.08 C$,

so $\frac{dC}{dt}$ is proportional to C , with constant of proportionality 0.08.

- Substituting into $\frac{dC}{dt} = 12\,000 e^{0.08t}$,

on 1st January 2000, $\frac{dC}{dt} = 12\,000 e^0 = \$12\,000$ per year,

on 1st January 2001, $\frac{dC}{dt} = 12\,000 e^{0.08} \doteq \$12\,999$ per year,

on 1st January 2002, $\frac{dC}{dt} = 12\,000 e^{0.16} \doteq \$14\,082$ per year,

on 1st January 2003, $\frac{dC}{dt} = 12\,000 e^{0.24} \doteq \$15\,255$ per year.

These form a GP with ratio $r = e^{0.08} \doteq 1.0833$.

- The values of C when $t = 0$, $t = 1$, $t = 2$ and $t = 3$ are respectively

$\$150\,000$, $150\,000 e^{0.08}$, $150\,000 e^{0.16}$ and $150\,000 e^{0.24}$,

so over the first year, increase $= 150\,000(e^{0.08} - 1) \doteq \$12\,493$,

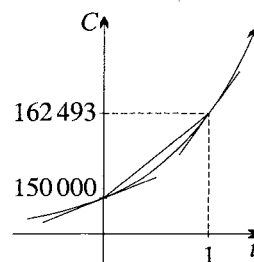
over the second year, increase $= 150\,000(e^{0.16} - e^{0.08})$

$= 150\,000 \times e^{0.08}(e^{0.08} - 1) \doteq \$13\,534$,

over the third year, increase $= 150\,000(e^{0.24} - e^{0.16})$

$= 150\,000 \times e^{0.16}(e^{0.08} - 1) \doteq \$14\,661$.

These increases form a GP with ratio $e^{0.08} \doteq 1.0833$.



(d) The three proportional increases are

$$\text{over the first year, } \frac{150\,000(e^{0.08} - 1)}{150\,000} = e^{0.08} - 1,$$

$$\text{over the second year, } \frac{150\,000 \times e^{0.08} \times (e^{0.08} - 1)}{150\,000 \times e^{0.08}} = e^{0.08} - 1,$$

$$\text{over the third year, } \frac{150\,000 \times e^{0.16} \times (e^{0.08} - 1)}{150\,000 \times e^{0.16}} = e^{0.08} - 1,$$

so the proportional increases are all equal to $e^{0.08} - 1 \doteq 8.33\%$.

Exercise 7G

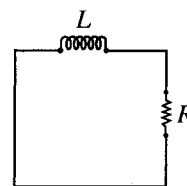
NOTE: This exercise is a review of the material covered in Section 13E of the Year 11 volume, with a little more stress laid on the rates.

- It is found that under certain conditions, the number of bacteria in a sample grows exponentially with time according to the equation $B = B_0 e^{0.1t}$, where t is measured in hours.
 - Show that B satisfies the differential equation $\frac{dB}{dt} = \frac{1}{10}B$.
 - Initially, the number of bacteria is estimated to be 1000. Find how many bacteria there are after three hours. Answer correct to the nearest bacterium.
 - Use parts (a) and (b) to find how fast the number of bacteria is growing after three hours.
 - By solving $1000 e^{0.1t} = 10\,000$, find, correct to the nearest hour, when there will be 10 000 bacteria.
- Twenty grams of salt is gradually dissolved in hot water. Assume that the amount S left undissolved after t minutes satisfies the law of natural decay, that is, $\frac{dS}{dt} = -kS$, for some positive constant k .
 - Show that $S = 20e^{-kt}$ satisfies the differential equation.
 - Given that only half the salt is left after three minutes, show that $k = \frac{1}{3} \log 2$.
 - Find how much salt is left after five minutes, and how fast the salt is dissolving then. (Answer correct to two decimal places.)
 - After how long, correct to the nearest second, will there be 4 grams of salt left undissolved?
 - Find the amounts of undissolved salt when $t = 0, 1, 2$ and 3 , correct to the nearest 0.01 g, show that these values form a GP, and find the common ratio.
- The population P of a rural town has been declining over the last few years. Five years ago the population was estimated at 30 000 and today it is estimated at 21 000.
 - Assume that the population obeys the law of natural decay $\frac{dP}{dt} = -kP$, for some positive constant k , where t is time in years from the first estimate, and show that $P = 30\,000e^{-kt}$ satisfies this differential equation.
 - Find the value of the positive constant k .
 - Estimate the population ten years from now.
 - The local bank has estimated that it will not be profitable to stay open once the population falls below 16 000. When will the bank close?

4. A chamber is divided into two identical parts by a porous membrane. The left part of the chamber is initially more full of a liquid than the right. The liquid is let through at a rate proportional to the difference in the levels x , measured in centimetres. Thus $\frac{dx}{dt} = -kx$.
- Show that $x = Ae^{-kt}$ is a solution of this equation.
 - Given that the initial difference in heights is 30 cm, find the value of A .
 - The level in the right compartment has risen 2 cm in five minutes, and the level in the left has fallen correspondingly by 2 cm.
 - What is the value of x at this time?
 - Hence find the value of k .
5. A radioactive substance decays with a half-life of 1 hour. The initial mass is 80 g.
- Write down the mass when $t = 0, 1, 2$ and 3 hours (no need for calculus here).
 - Write down the average loss of mass during the 1st, 2nd and 3rd hour, then show that the percentage loss of mass per hour during each of these hours is the same.
 - The mass M at any time satisfies the usual equation of natural decay $M = M_0 e^{-kt}$, where k is a constant. Find the values of M_0 and k .
 - Show that $\frac{dM}{dt} = -kM$, and find the instantaneous rate of mass loss when $t = 0, t = 1, t = 2$ and $t = 3$.
 - Sketch the M - t graph, for $0 \leq t \leq 1$, and add the relevant chords and tangents.

————— DEVELOPMENT —————

6. [The formulae for compound interest and for natural growth are essentially the same.] The cost C of an article is rising with inflation in such a way that at the start of every month, the cost is 1% more than it was a month before. Let C_0 be the cost at time zero.
- Use the compound interest formula of Section 7B to construct a formula for the cost C after t months. Hence find, in exact form and then correct to four significant figures:
 - the percentage increase in the cost over twelve months,
 - the time required for the cost to double.
 - The natural growth formula $C = C_0 e^{kt}$ also models the cost after t months. Use the fact that when $t = 1$, $C = 1.01 C_0$ to find the value of k . Hence find, in exact form and then correct to four significant figures:
 - the percentage increase in the cost over twelve months,
 - the time required for the cost to double.
7. A current i_0 is established in the circuit shown on the right. When the source of the current is removed, the current in the circuit decays according to the equation $L \frac{di}{dt} = -iR$.
- Show that $i = i_0 e^{-\frac{R}{L}t}$ is a solution of this equation.
 - Given that the resistance is $R = 2$ and that the current in the circuit decays to 37% of the initial current in a quarter of a second, find L . (NOTE: $37\% \doteq \frac{1}{e}$)



8. A tank in the shape of a vertical hexagonal prism with base area A is filled to a depth of 25 metres. The liquid inside is leaking through a small hole in the bottom of the tank, and it is found that the change in volume at any instant t hours after the tank starts leaking is proportional to the depth h metres, that is, $\frac{dV}{dt} = -kh$.

(a) Show that $\frac{dh}{dt} = -\frac{kh}{A}$.

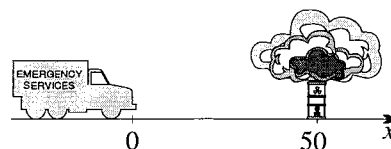
(b) Show that $h = h_0 e^{-\frac{k}{A}t}$ is a solution of this equation.

(c) What is the value of h_0 ?

(d) Given that the depth in the tank is 15 metres after 2 hours, find $\frac{k}{A}$.

(e) How long will it take to empty to a depth of just 5 metres? Answer correct to the nearest minute.

9. The emergency services are dealing with a toxic gas cloud around a leaking gas cylinder 50 metres away. The prevailing conditions mean that the concentration C in parts per million (ppm) of the gas increases proportionally to



the concentration as one moves towards the cylinder. That is, $\frac{dC}{dx} = kC$, where x is the distance in metres towards the cylinder from their current position.

(a) Show that $C = C_0 e^{kx}$ is a solution of the above equation.

(b) At the truck, where $x = 0$, the concentration is $C = 20\,000$ ppm. Five metres closer, the concentration is $C = 22\,500$ ppm. Use this information to find the values of the constants C_0 and k . (Give k exactly, then correct to three decimal places.)

(c) Find the gas concentration at the cylinder, correct to the nearest part per million.

(d) The accepted safe level for this gas is 30 parts per million. The emergency services calculate how far back from the cylinder they should keep the public, rounding their answer up to the nearest 10 metres.

(i) How far do they keep the public back?

(ii) Why do they round their answer up and not round it in the normal way?

10. Given that $y = A_0 e^{kt}$, it is found that at $t = 1$, $y = \frac{3}{4}A_0$.

(a) Show that it is not necessary to evaluate k in order to find y when $t = 3$.

(b) Find $y(3)$ in terms of A_0 .

11. (a) The price of shares in Bravo Company rose in one year from \$5.25 to \$6.10.

(i) Assuming the law of natural growth, show that the share price in cents is given by $B = 525e^{kt}$, where t is measured in months.

(ii) Find the value of k .

(b) A new information technology company, ComIT, enters the stock market at the same time with shares at \$1, and by the end of the year these are worth \$2.17.

(i) Again assuming natural growth, show that the share price in cents is given by $C = 100e^{\ell t}$.

(ii) Find the value of ℓ .

(c) During which month will the share prices in both companies be equal?

(d) What will be the (instantaneous) rate of increase in ComIT shares at the end of that month, correct to the nearest cent per month?

NOTE: The following two questions deal with finance, where rates are usually expressed not as instantaneous rates, but as average rates. It will usually take some work to relate the value k of the instantaneous rate to the average rate.

12. At any time t , the value V of a certain item is depreciating at an instantaneous rate of 15% of V per annum.
- Express $\frac{dV}{dt}$ in terms of V .
 - The cost of purchasing the item was \$12000. Write V as a function of time t years since it was purchased, and show that it is a solution of the equation in part (a).
 - Find V after one year, and find the decrease as a percentage of the initial value.
 - Find the instantaneous rate of decrease when $t = 1$.
 - How long, correct to the nearest 0.1 years, does it take for the value to decrease to 10% of its cost?
13. An investment of \$5000 is earning interest at the advertised rate of 7% per annum, compounded annually. (This is the average rate, not the instantaneous rate.)
- Use the compound interest formula to write down the value A of the investment after t years.
 - Use the result $\frac{d}{dt}(a^t) = a^t \log a$ to show that $\frac{dA}{dt} = A \log 1.07$.
 - Use the result $a = e^{\log a}$ to re-express the exponential term in A with base e .
 - Hence confirm that $\frac{dA}{dt} = A \log 1.07$.
 - Use your answer to either part (a) or part (c) to find the value of the investment after six years, correct to the nearest cent.
 - Hence find the instantaneous rate of growth after six years, again to the nearest cent.
14. (a) The population P_1 of one town is growing exponentially, with $P_1 = Ae^t$, and the population P_2 of another town is growing at a constant rate, with $P_2 = Bt + C$, where A , B and C are constants. When the first population reaches $P_1 = Ae$, it is found that $P_1 = P_2$, and also that both populations are increasing at the same rate.
- Show that the second population was initially zero (that is, that $C = 0$).
 - Draw a graph showing this information.
 - Show that the result in part (i) does not change if $P_1 = Aa^t$, for some $a > 1$.
[HINT: You may want to use the identity $a^t = e^{t \log a}$.]
- (b) Two graphs are drawn on the same axes, one being $y = \log x$ and the other $y = mx + b$. It is found that the straight line is tangent to the logarithmic graph at $x = e$.
- Show that $b = 0$, and draw a graph showing this information.
 - Show that the result in part (i) does not change if $y = \log_a x$, for some $a > 1$.
- (c) Explain the effect of the change of base in parts (a) and (b) in terms of stretching.
- (d) Explain in terms of a reflection why the questions in parts (a) and (b) are equivalent.

EXTENSION

15. The growing population of rabbits on Brair Island can initially be modelled by the law of natural growth, with $N = N_0 e^{\frac{1}{2}t}$. When the population reaches a critical value, $N = N_c$, the model changes to $N = \frac{B}{C + e^{-t}}$, with the constants B and C chosen so that both models predict the same rate of growth at that time.

- (a) Find the values of B and C in terms of N_c and N_0 .
 (b) Show that the population reaches a limit, and find that limit in terms of N_c .

7H Modified Natural Growth and Decay

In many situations, the rate of change of a quantity P is proportional not to P itself, but to the amount $P - B$ by which P exceeds some fixed value B . Mathematically, this means shifting the graph upwards by B , which is easily done using theory previously established.

The General Case: Here is the general statement of the situation.

8

MODIFIED NATURAL GROWTH: Suppose that the rate of change of a quantity P is proportional to the difference $P - B$, where B is some fixed value of P :

$$\frac{dP}{dt} = k(P - B), \text{ where } k \text{ is a constant of proportionality.}$$

Then $P = B + Ae^{kt}$, where A is the value of $P - B$ at time zero.

NOTE: Despite the following proof, memorisation of this general solution is not required. Questions will always give a solution in some form, and may then ask to verify by substitution that it is a solution of the differential equation.

PROOF: Let $y = P - B$ be the difference between P and B .

Then $\frac{dy}{dt} = \frac{dP}{dt} - 0$, since B is a constant,

$$= k(P - B), \text{ since we are given that } \frac{dP}{dt} = k(P - B),$$

so $\frac{dy}{dt} = ky$, since we defined y by $y = P - B$.

Hence, using the previous theory of natural growth,

$$y = y_0 e^{kt}, \text{ where } y_0 \text{ is the initial value of } y,$$

and substituting $y = P - B$,

$$P = B + Ae^{kt}, \text{ where } A \text{ is the initial value of } P - B.$$

WORKED EXERCISE: The large French tapestries that are hung in the permanently air-conditioned La Châtelle Hall have a normal water content W of 8 kg. When the tapestries were removed for repair, they dried out in the workroom atmosphere. When they were returned, the rate of increase of the water content was proportional to the difference from the normal 8 kg, that is,

$$\frac{dW}{dt} = k(8 - W), \text{ for some positive constant } k \text{ of proportionality.}$$

- (a) Prove that for any constant A , $W = 8 - Ae^{-kt}$ is a solution of the differential equation.
 (b) Weighing established that $W = 4$ initially, and $W = 6.4$ after 3 days.
 (i) Find the values of A and k .
 (ii) Find when the water content has risen to 7.9 kg.
 (iii) Find the rate of absorption of the water after 3 days.
 (iv) Sketch the graph of water content against time.

SOLUTION:

(a) Substituting $W = 8 - Ae^{-kt}$ into $\frac{dW}{dt} = k(8 - W)$,

$$\begin{aligned} \text{LHS} &= \frac{dW}{dt} & \text{RHS} &= k(8 - 8 + Ae^{-kt}) \\ &= kAe^{-kt}, & &= \text{LHS, as required.} \end{aligned}$$

(b) (i) When $t = 0$, $W = 4$, so $4 = 8 - A$
 $A = 4$.

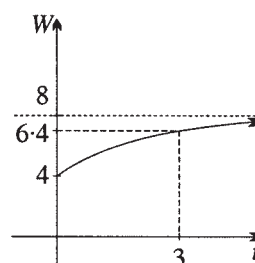
$$\begin{aligned} \text{When } t = 3, W = 6.4, \text{ so } 6.4 &= 8 - 4e^{-3k} \\ e^{-3k} &= 0.4 \\ k &= -\frac{1}{3} \log 0.4 \quad (\text{calculate and leave in memory}). \end{aligned}$$

(ii) Put $W = 7.9$, then $7.9 = 8 - 4e^{-kt}$
 $e^{-kt} = 0.025$

$$\begin{aligned} t &= -\frac{1}{k} \log 0.025 \\ &\div 12 \text{ days.} \end{aligned}$$

(iii) We know that $\frac{dW}{dt} = k(8 - W)$.

$$\begin{aligned} \text{When } t = 3, W = 6.4, \text{ so } \frac{dW}{dt} &= k \times 1.6 \\ &\div 0.49 \text{ kg/day.} \end{aligned}$$



Newton's Law of Cooling: Newton's law of cooling is a well-known example of natural decay. When a hot object is placed in a cool environment, the rate at which the temperature decreases is proportional to the difference between the temperature T of the object and the temperature E of the environment:

$$\frac{dT}{dt} = -k(T - E), \text{ where } k \text{ is a constant of proportionality.}$$

The same law applies to a cold body placed in a warmer environment.

WORKED EXERCISE: In a kitchen where the temperature is 20°C , Stanley takes a kettle of boiling water off the stove at time zero. Five minutes later, the temperature of the water is 70°C .

- (a) Show that $T = 20 + 80e^{-kt}$ satisfies the cooling equation $\frac{dT}{dt} = -k(T - 20)$, and gives the correct value of 100°C at $t = 0$. Then find k .
- (b) How long will it take for the water temperature to drop to 25°C ?
- (c) Graph the temperature-time function.

SOLUTION:

(a) Substituting $T = 20 + 80e^{-kt}$ into $\frac{dT}{dt} = -k(T - 20)$,

$$\begin{aligned} \text{LHS} &= \frac{dT}{dt} & \text{RHS} &= -k(20 + 80e^{-kt} - 20) \\ &= -80ke^{-kt}, & &= \text{LHS, as required.} \end{aligned}$$

Substituting $t = 0$, $T = 20 + 80 \times 1 = 100$, as required.

When $t = 5$, $T = 70$, so $70 = 20 + 80e^{-5k}$

$$e^{-5k} = \frac{5}{8}$$

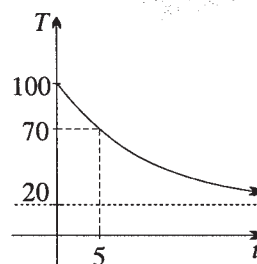
$$k = -\frac{1}{5} \log \frac{5}{8}.$$

(b) Substituting $T = 25$, $25 = 20 + 80e^{-kt}$

$$e^{-kt} = \frac{1}{16}$$

$$t = -\frac{1}{k} \log \frac{1}{16}$$

$$\doteq 29\frac{1}{2} \text{ minutes.}$$



Exercise 7H

- (a) Suppose that $P = 10\,000 + 2000e^{0.1t}$. (i) Show that $\frac{dP}{dt} = \frac{1}{10}(P - 10\,000)$.
 (ii) Find the value of P when $t = 0$, and state what happens as $t \rightarrow \infty$.

(b) Suppose that $P = 10\,000 + 2000e^{-0.1t}$. (i) Show that $\frac{dP}{dt} = -\frac{1}{10}(P - 10\,000)$.
 (ii) Find the value of P when $t = 0$, and state what happens as $t \rightarrow \infty$.

(c) Suppose that $P = 10\,000 - 2000e^{-0.1t}$. (i) Show that $\frac{dP}{dt} = -\frac{1}{10}(P - 10\,000)$.
 (ii) Find the value of P when $t = 0$, and state what happens as $t \rightarrow \infty$.
- The rate of increase of a population P of green and purple flying bugs is proportional to the excess of the population over 2000, that is, $\frac{dP}{dt} = k(P - 2000)$, for some constant k . Initially, the population is 3000, and three weeks later the population is 8000.

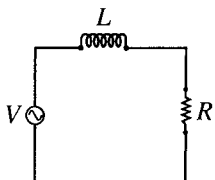
(a) Show that $P = 2000 + Ae^{kt}$ satisfies the differential equation, where A is constant.
 (b) By substituting $t = 0$ and $t = 3$, find the values of A and k .
 (c) Find the population after seven weeks, correct to the nearest ten bugs.
 (d) Find when the population reaches 500 000, correct to the nearest 0.1 weeks.
- During the autumn, the rate of decrease of the fly population F in Wanzenthal Valley is proportional to the excess over 30 000, that is, $\frac{dF}{dt} = -k(F - 30\,000)$, for some positive constant k . Initially, there are 1 000 000 flies in the valley, and ten days later the number has halved.

(a) Show that $F = 30\,000 + Be^{-kt}$ satisfies the differential equation, where B is constant.
 (b) Find the values of B and k .
 (c) Find the population after four weeks, correct to the nearest 1000 flies.
 (d) Find when the population reaches 35 000, correct to the nearest day.
- A hot cup of coffee loses heat in a colder environment according to Newton's law of cooling, $\frac{dT}{dt} = -k(T - T_e)$, where T is the temperature of the coffee in degrees Celsius at time t minutes, T_e is the temperature of the environment and k is a positive constant.

(a) Show that $T = T_e + Ae^{-kt}$ is a solution of this equation, for any constant A .
 (b) I make myself a cup of coffee and find that it has already cooled from boiling to 90°C . The temperature of the air in the office is 20°C . What are the values of T_e and A ?
 (c) The coffee cools from 90°C to 50°C after six minutes. Find k .
 (d) Find how long, correct to the nearest second, it will take for the coffee to reach 30°C .

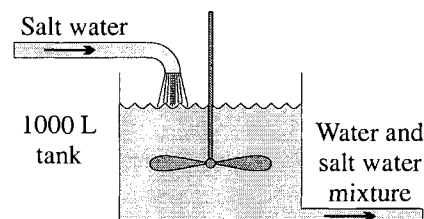
5. A tray of meat is taken out of the freezer at -9°C and allowed to thaw in the air at 25°C . The rate at which the meat warms follows Newton's law of cooling and so $\frac{dT}{dt} = -k(T-25)$, with time t measured in minutes.
- Show that $T = 25 - Ae^{-kt}$ is a solution of this equation, and find the value of A .
 - The meat reaches 8°C in 45 minutes. Find the value of k .
 - Find the temperature it reaches after another 45 minutes.
6. A 1 kilogram weight falls from rest through the air. When both gravity and air resistance are taken into account, it is found that its velocity is given by $v = 160(1 - e^{-\frac{1}{16}t})$. The velocity v is measured in metres per second, and downwards has been taken as positive.
- Confirm that the initial velocity is zero. Show that the velocity is always positive for $t > 0$, and explain this physically.
 - Show that $\frac{dv}{dt} = \frac{1}{16}(160 - v)$, and explain what this represents.
 - What velocity does the body approach?
 - How long does it take to reach one eighth of this speed?

DEVELOPMENT

7. A chamber is divided into two identical parts by a porous membrane. The left compartment is initially full and the right is empty. The liquid is let through at a rate proportional to the difference between the level x cm in the left compartment and the average level. Thus $\frac{dx}{dt} = k(15 - x)$.
- Show that $x = 15 + Ae^{-kt}$ is a solution of this equation.
 - What value does the level in the left compartment approach?
 - Hence explain why the initial height is 30 cm.
 - Thus find the value of A .
 - The level in the right compartment has risen 6 cm in 5 minutes. Find the value of k .
8. The diagram shows a simple circuit containing an inductor L and a resistor R with an applied voltage V . Circuit theory tells us that $V = RI + L \frac{dI}{dt}$, where I is the current at time t seconds.
- 
- Prove that $I = \frac{V}{R} + Ae^{-\frac{R}{L}t}$ is a solution of the differential equation, for any constant A .
 - Given that initially the current is zero, find A in terms of V and R .
 - Find the limiting value of the current in the circuit.
 - Given that $R = 12$ and $L = 8 \times 10^{-3}$, find how long it takes for the current to reach half its limiting value. Give your answer correct to three significant figures.
9. When a person takes a pill, the medicine is absorbed into the bloodstream at a rate given by $\frac{dM}{dt} = -k(M - a)$, where M is the concentration of the medicine in the blood t minutes after taking the pill, and a and k are constants.
- Show that $M = a(1 - e^{-kt})$ satisfies the given equation, and gives an initial concentration of zero.
 - What is the limiting value of the concentration?
 - Find k , if the concentration reaches 99% of the limiting value after 2 hours.

- (d) The patient starts to notice relief when the concentration reaches 10% of the limiting value. When will this occur, correct to the nearest second?

10. In the diagram, a tank initially contains 1000 litres of pure water. Salt water begins pouring into the tank from a pipe and a stirring blade ensures that it is completely mixed with the pure water. A second pipe draws the water and salt water mixture off at the same rate, so that there is always a total of 1000 litres in the tank.



- (a) If the salt water entering the tank contains 2 grams of salt per litre, and is flowing in at the constant rate of w litres/min, how much salt is entering the tank per minute?
- (b) If there are Q grams of salt in the tank at time t , how much salt is in 1 litre at time t ?
- (c) Hence write down the amount of salt leaving the tank per minute.
- (d) Use the previous parts to show that $\frac{dQ}{dt} = -\frac{w}{1000}(Q - 2000)$.
- (e) Show that $Q = 2000 + Ae^{-\frac{wt}{1000}}$ is a solution of this differential equation.
- (f) Determine the value of A .
- (g) What happens to Q as $t \rightarrow \infty$?
- (h) If there is 1 kg of salt in the tank after $5\frac{3}{4}$ hours, find w .

EXTENSION

11. [Alternative proof of the modified natural growth theorem] Suppose that a quantity P changes at a rate proportional to the difference between P and some fixed value B , that is, $\frac{dP}{dt} = k(P - B)$.
- (a) Take reciprocals, integrate, and hence show that $\log(P - B) = kt + C$.
- (b) Take exponentials and finally show that $P - B = Ae^{kt}$.
12. It is assumed that the population of a newly introduced species on an island will usually grow or decay in proportion to the difference between the current population P and the ideal population I , that is, $\frac{dP}{dt} = k(P - I)$, where k may be positive or negative.
- (a) Prove that $P = I + Ae^{kt}$ is a solution of this equation.
- (b) Initially 10 000 animals are released. A census is taken 7 weeks later and again at 14 weeks, and the population grows to 12 000 and then 18 000. Use these data to find the values of I , A and k .
- (c) Find the population after 21 weeks.
13. [The coffee drinkers' problem] Two coffee drinkers pour themselves a cup of coffee each just after the kettle has boiled. The woman adds milk from the fridge, stirs it in and then waits for it to cool. The man waits for the coffee to cool first, then just before drinking adds the milk and stirs. If they both begin drinking at the same time, whose coffee is cooler? Justify your answer mathematically. Assume that the air temperature is colder than the coffee and that the milk is colder still. Also assume that after the milk is added and stirred, the temperature drops by a fixed percentage.

Euclidean Geometry

The methods and structures of modern mathematics were established first by the ancient Greeks in their studies of geometry and arithmetic. It was they who realised that mathematics must proceed by rigorous proof and argument, that all definitions must be stated with absolute precision, and that any hidden assumptions, called *axioms*, must be brought out into the open and examined. Their work is extraordinary for their determination to prove details that may seem common sense to the layman, and for their ability to ask the most important questions about the subjects they investigated. Many Greeks, like the mathematician Pythagoras and the philosopher Plato, spoke of mathematics in mystical terms as the highest form of knowledge, and they called their results *theorems* — the Greek word *theorem* means ‘a thing to be gazed upon’ or ‘a thing contemplated by the mind’, from *θεωρέω* ‘behold’ (our word *theatre* comes from the same root).

Of all the Greek books, Euclid’s *Elements* has been the most influential, and was still used as a textbook in nineteenth-century schools. Euclid constructs a large body of theory in geometry and arithmetic beginning from almost nothing — he writes down a handful of initial assumptions and definitions that mostly seem trivial, such as ‘Things that are each equal to the same thing are equal to one another’. As is common in Greek mathematics, Euclid introduces geometry first, and then develops arithmetic ideas from it. For example, the product of two numbers is usually understood as the area of a rectangle. Such intertwining of arithmetic and geometry is still characteristic of the most modern mathematics, and has been evident in our treatment of the calculus, which has drawn its intuitions equally from algebraic formulae and from the geometry of curves, tangents and areas.

Geometry done using the methods established in Euclid’s book is called *Euclidean geometry*. We have assumed throughout this text that students were familiar from earlier years with the basic methods and results of Euclidean geometry, and we have used these geometric results freely in arguments. This chapter and the next will now review Euclidean geometry from its beginnings and develop it a little further. Our foundations can unfortunately be nothing like as rigorous as Euclid’s. For example, we shall assume the four standard congruence tests rather than proving them, and our second theorem is his thirty-second. Nevertheless, the arguments used here are close to those of Euclid, and are strikingly different from those we have used in calculus and algebra. The whole topic is intended to provide a quite different insight into the nature of mathematics.

Constructions with straight edge and compasses are central to Euclid’s arguments, and we have therefore included a number of construction problems in an unsystematic fashion. They need to be proven, and they need to be drawn. Their importance lies not in any practical use, but in their logic. For example, three

famous constructions unsolved by the Greeks — the trisection of a given angle, the squaring of a given circle (essentially the construction of π) and the doubling in volume of a given cube (essentially the construction of $\sqrt[3]{2}$) — were an inspiration to mathematicians of the nineteenth century grappling with the problem of defining the real numbers by non-geometric methods. All three constructions were eventually proven to be impossible.

STUDY NOTES: Most of this material will have been covered in Years 9 and 10, but perhaps not in the systematic fashion developed here. Attention should therefore be on careful exposition of the logic of the proofs, on the logical sequence established by the chain of theorems, and on the harder problems. The only entirely new work is in the final Section 8I on intercepts.

Many of the theorems are only stated in the notes, with their proofs left to structured questions in the following exercise. All such questions have been placed at the start of the Development section, even though they may be more difficult than succeeding problems, and are marked ‘COURSE THEOREM’ — working through these proofs is an essential part of the course.

There are many possible orders in which the theorems of this course could have been developed, but the order given here is that established by the Syllabus. All theorems marked as course theorems may be used in later questions, except where the intention of the question is to provide a proof of the theorem. Students should note carefully that *the large number of further theorems proven in the exercises cannot be used in subsequent questions*.

8A Points, Lines, Parallels and Angles

The elementary objects of geometry are points, lines and planes. Rigorous definitions of these things are possible, but very difficult. Our approach, therefore, will be the same as our approach to the real numbers — we shall describe some of their properties and list some of the assumptions we shall need to make about them.

Points, Lines and Planes: These simple descriptions should be sufficient.

POINTS: A *point* can be described as having a position but no size. The mark opposite has a definite width, and so is not a point, but it represents a point in our imagination.

P

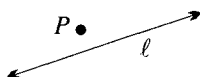
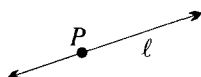


LINES: A *line* has no breadth, but extends infinitely in both directions. The drawing opposite has width and has ends, but it represents a line in our imagination.



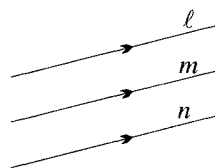
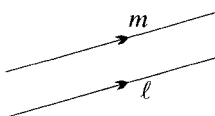
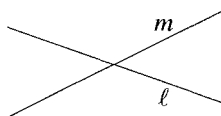
PLANES: A *plane* has no thickness, and it extends infinitely in all directions. Almost all our work is two-dimensional, and takes place entirely in a fixed plane.

Points and Lines in a Plane: Here are some of the assumptions that we shall be making about the relationships between points and lines in a plane.



POINT AND LINE: Given a point P and a line ℓ , the point P may or may not lie on the line ℓ .

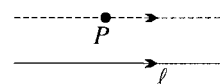
TWO POINTS: Two distinct points A and B lie on one and only one line, which can be named AB or BA .



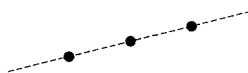
TWO LINES: Given any two distinct lines ℓ and m in a plane, either the lines intersect in a single point, or the lines have no point in common and are called *parallel lines*, written as $\ell \parallel m$.

THREE PARALLEL LINES: If two lines are each parallel to a third line, then they are parallel to each other.

THE PARALLEL LINE THROUGH A GIVEN POINT: Given a line ℓ and a point P not on ℓ , there is one and only one line through P parallel to ℓ .



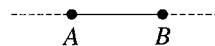
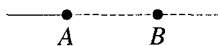
Collinear Points and Concurrent Lines: A third point may or may not lie on the line determined by two other points. Similarly, a third line may or may not pass through the point of intersection of two other lines.



COLLINEAR POINTS: Three or more distinct points are called *collinear* if they all lie on a single line.

CONCURRENT LINES: Three or more distinct lines are called *concurrent* if they all pass through a single point.

Intervals and Rays: These definitions rely on the idea that a point on a line divides the rest of the line into two parts. Suppose that A and B are any two distinct points on a line ℓ .



RAYs:

The ray AB consists of the *endpoint* A together with B and all the other points of ℓ on the same side of A as B is.

OPPOSITE RAY:

The ray that starts at this same endpoint A , but goes in the opposite direction, is called the *opposite ray*.

INTERVALs:

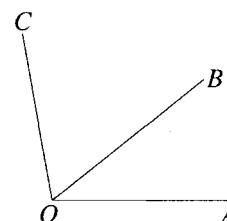
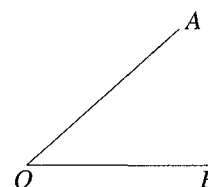
The *interval* AB consists of all the points lying on ℓ between A and B , including these two endpoints.

LENGTHS OF INTERVALs: We shall assume that intervals can be measured, and their lengths compared and added and subtracted with compasses.

Angles: We need to distinguish between an angle and the size of an angle.

ANGLES: An *angle* consists of two rays with a common endpoint. The two rays OA and OB in the diagram form an angle named either $\angle AOB$ or $\angle BOA$. The common endpoint O is called the *vertex* of the angle, and the rays OA and OB are called the *arms* of the angle.

ADJACENT ANGLES: Two angles are called *adjacent angles* if they have a common vertex and a common arm. In the diagram opposite, $\angle AOB$ and $\angle BOC$ are adjacent angles with common vertex O and common arm OB . Also, the overlapping angles $\angle AOC$ and $\angle AOB$ are adjacent angles, having common vertex O and common arm OA .

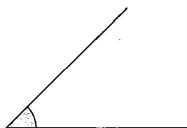
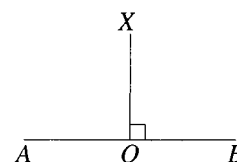


MEASURING ANGLES: The size of an angle is the amount of turning as one arm is rotated about the vertex onto the other arm. The units of degrees are based on the ancient Babylonian system of dividing the revolution into 360 equal parts — there are about 360 days in a year, and so the sun moves about 1° against the fixed stars every day. The measurement of angles is based on the obvious assumption that the sizes of adjacent angles can be added and subtracted.

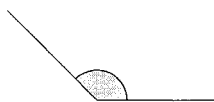
REVOLUTIONS: A *revolution* is the angle formed by rotating a ray about its endpoint once until it comes back onto itself. A revolution is defined to measure 360° .

STRAIGHT ANGLES: A *straight angle* is the angle formed by a ray and its opposite ray. A straight angle is half a revolution, and so measures 180° .

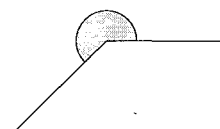
RIGHT ANGLES: Suppose that AOB is a line, and OX is a ray such that $\angle XOA$ is equal to $\angle XOB$. Then $\angle XOA$ is called a *right angle*. A right angle is half a straight angle, and so measures 90° .



ACUTE ANGLES:
An *acute angle* is an angle greater than 0° and less than a right angle.



OBTUSE ANGLES:
An *obtuse angle* is an angle greater than a right angle and less than a straight angle.



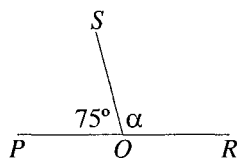
REFLEX ANGLES:
A *reflex angle* is an angle greater than a straight angle and less than a revolution.

Angles at a Point: Two angles are called *complementary* if they add to 90° . For example, 15° is the *complement* of 75° . Two angles are called *supplementary* if they add to 180° . For example, 105° is the *supplement* of 75° . Our first theorem relies on the assumption that adjacent angles can be added.

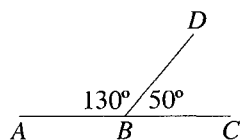
COURSE THEOREM — ANGLES IN A STRAIGHT LINE AND IN A REVOLUTION:

1

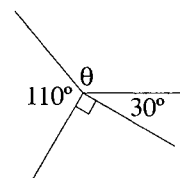
- Two adjacent angles in a straight line are supplementary.
- Conversely, if adjacent angles are supplementary, they form a straight line.
- Adjacent angles in a revolution add to 360° .



Given that PQR is a line,
 $\alpha = 105^\circ$ (angles
in a straight angle).

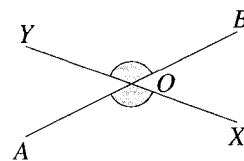


A, B and C are collinear
(adjacent angles are
supplementary).



$\theta + 110^\circ + 90^\circ + 30^\circ = 360^\circ$
(angles in a revolution),
 $\theta = 130^\circ$.

Vertically Opposite Angles: Each pair of opposite angles formed when two lines intersect are called *vertically opposite angles*. In the diagram to the right, AB and XY intersect at O . The marked angles $\angle AOX$ and $\angle BOY$ are vertically opposite. The unmarked angles $\angle AOY$ and $\angle BOX$ are also vertically opposite.

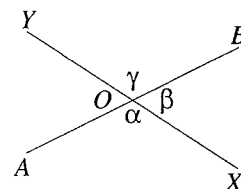


2 COURSE THEOREM: Vertically opposite angles are equal.

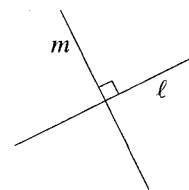
GIVEN: Let the lines AB and XY intersect at O .
Let $\alpha = \angle AOX$, let $\beta = \angle BOX$, and let $\gamma = \angle BOY$.

AIM: To prove that $\alpha = \gamma$.

PROOF: $\alpha + \beta = 180^\circ$ (straight angle $\angle AOB$),
and $\gamma + \beta = 180^\circ$ (straight angle $\angle XOY$),
so $\alpha = \gamma$.



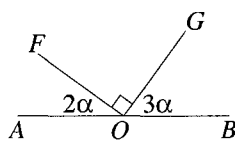
Perpendicular Lines: Two lines ℓ and m are called *perpendicular*, written as $\ell \perp m$, if they intersect so that one of the angles between them is a right angle. Because adjacent angles on a straight line are supplementary, all four angles must be right angles.



Using Reasons in Arguments: Geometrical arguments require reasons to be given for each statement — the whole topic is traditionally regarded as providing training in the writing of mathematical proofs. These reasons can be expressed in ordinary prose, or each reason can be given in brackets after the statement it justifies. All reasons should, wherever possible, give the names of the angles or lines or triangles referred to, otherwise there can be ambiguities about exactly what argument has been used. The authors of this book have boxed the theorems and assumptions that can be quoted as reasons.

WORKED EXERCISE: Find α or θ in each diagram below.

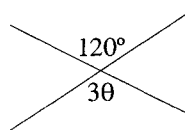
(a)



SOLUTION:

$$\begin{aligned} \text{(a)} \quad 2\alpha + 90^\circ + 3\alpha &= 180^\circ \\ &\text{(straight angle } \angle AOB), \\ 5\alpha &= 90^\circ \\ \alpha &= 18^\circ. \end{aligned}$$

(b)

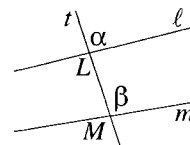


$$\begin{aligned} \text{(b)} \quad 3\theta &= 120^\circ \\ &\text{(vertically opposite angles),} \\ \theta &= 40^\circ. \end{aligned}$$

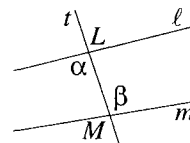
Angles and Parallel Lines: The standard results about alternate, corresponding and co-interior angles are taken as assumptions.

TRANSVERSALS: A *transversal* is a line that crosses two other lines (the two other lines may or may not be parallel). In each of the three diagrams below, t is a transversal to the lines ℓ and m , meeting them at L and M respectively.

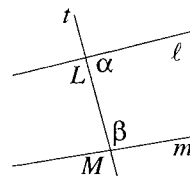
CORRESPONDING ANGLES: In the first diagram opposite, the two angles marked α and β are called *corresponding angles*, because they are in corresponding positions around the two vertices L and M .



ALTERNATE ANGLES: In the second diagram opposite, the two angles marked α and β are called *alternate angles*, because they are on alternate sides of the transversal t (they must also be inside the region between the lines ℓ and m).



CO-INTERIOR ANGLES: In the third diagram opposite, the two angles marked α and β are called *co-interior angles*, because they are inside the two lines ℓ and m , and on the same side of the transversal t .



Our assumptions about corresponding, alternate and co-interior angles fall into two groups. The first group are consequences arising when the lines are parallel.

- 3** **ASSUMPTION:** Suppose that a transversal crosses two lines.
- If the lines are parallel, then any two corresponding angles are equal.
 - If the lines are parallel, then any two alternate angles are equal.
 - If the lines are parallel, then any two co-interior angles are supplementary.

The second group are often neglected. They are the converses of the first group, and give conditions for the two lines to be parallel.

- 4** **ASSUMPTION:** Suppose that a transversal crosses two lines.
- If any pair of corresponding angles are equal, then the lines are parallel.
 - If any pair of alternate angles are equal, then the lines are parallel.
 - If any two co-interior angles are supplementary, then the lines are parallel.

WORKED EXERCISE: [A problem requiring a construction]

Find θ in the diagram opposite.

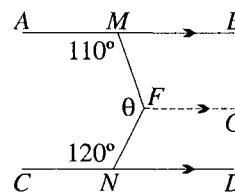
SOLUTION: Construct $FG \parallel AB$.

Then $\angle MFG = 110^\circ$ (alternate angles, $FG \parallel AB$),

and $\angle NFG = 120^\circ$ (alternate angles, $FG \parallel CD$),

so $\theta + 110^\circ + 120^\circ = 360^\circ$ (angles in a revolution at F),

$$\theta = 130^\circ.$$

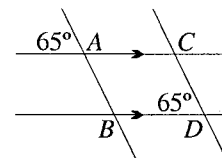


WORKED EXERCISE: Given that $AC \parallel BD$, prove that $AB \parallel CD$.

SOLUTION: $\angle CAB = 65^\circ$ (vertically opposite at A),

so $\angle ABD = 115^\circ$ (co-interior angles, $AC \parallel BD$),

so $AB \parallel CD$ (co-interior angles are supplementary).



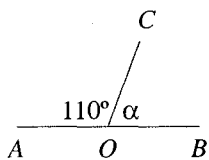
NOTE: A phrase like '(co-interior angles)' alone is never sufficient as a reason. If the two angles are being proven supplementary, the fact that the lines are parallel must also be stated. If the two lines are being proven parallel, the fact that the co-interior angles are supplementary must be stated.

Exercise 8A

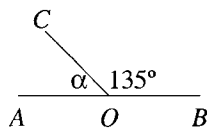
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find the angles α and β in the diagrams below, giving reasons.

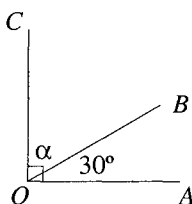
(a)



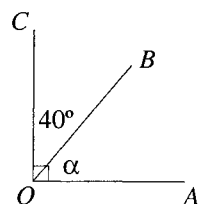
(b)



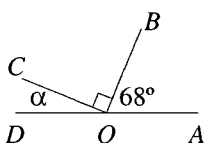
(c)



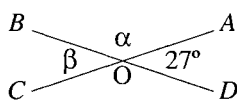
(d)



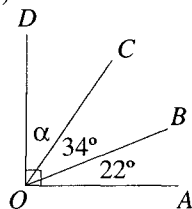
(e)



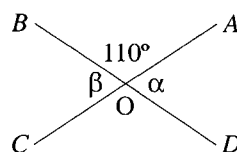
(f)



(g)

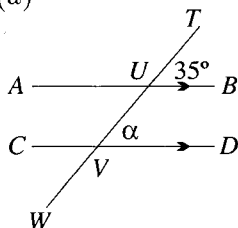


(h)

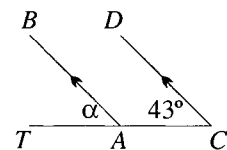


2. Find the angles α and β in each figure below, giving reasons.

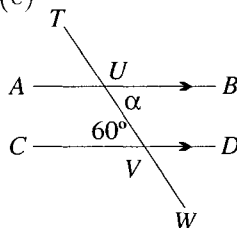
(a)



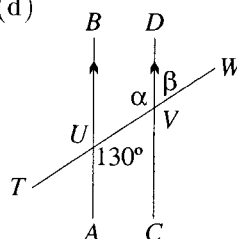
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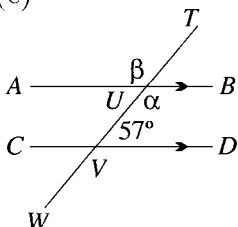
(c)



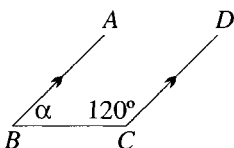
(d)



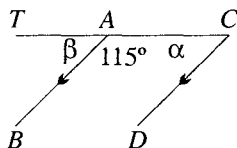
(e)



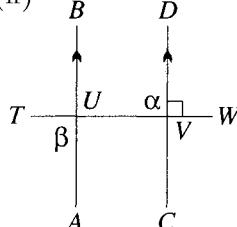
(f)



(g)

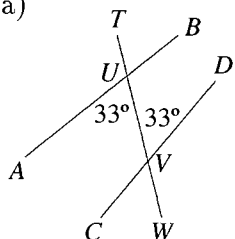


(h)

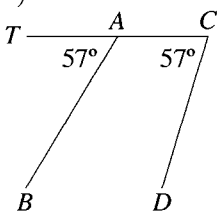


3. Show that $AB \parallel CD$ in the diagrams below, giving all reasons.

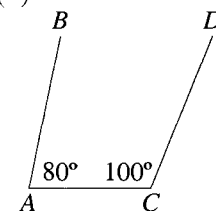
(a)



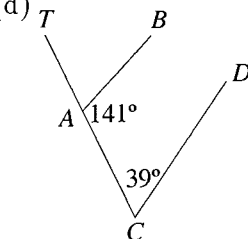
(b)



(c)

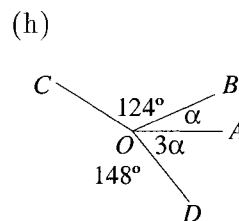
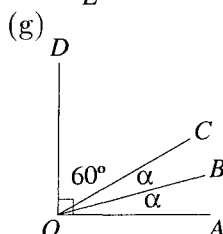
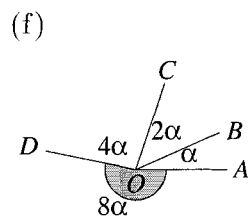
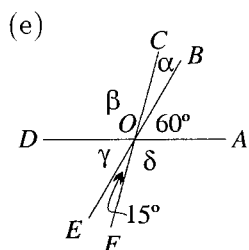
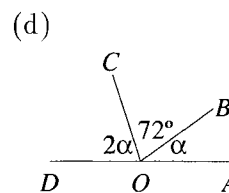
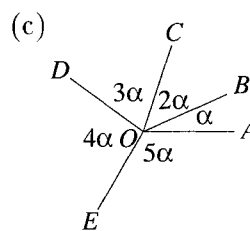
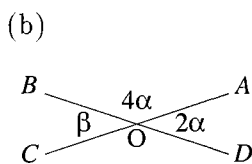
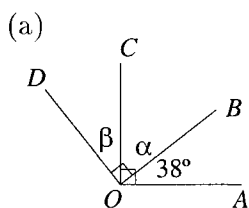


(d)

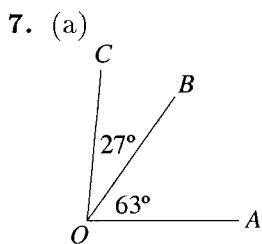
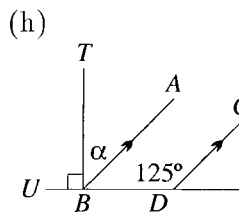
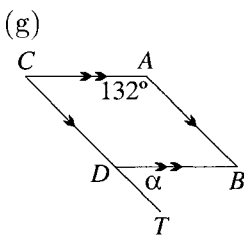
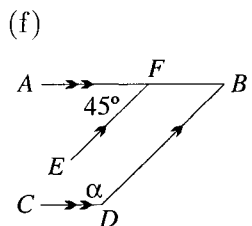
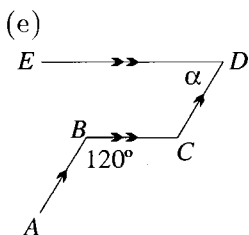
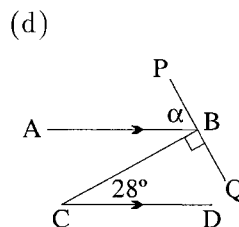
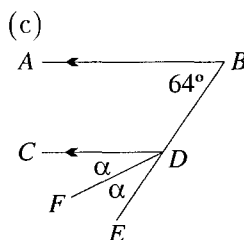
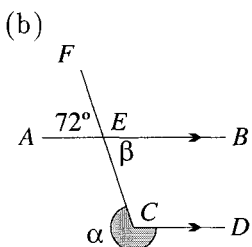
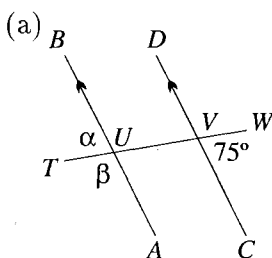


4. (a) Sketch a transversal crossing two non-parallel lines so that a pair of alternate angles formed by the transversal are about 45° and 65° .
 (b) Repeat part (a) so that a pair of corresponding angles are about 90° and 120° .
 (c) Repeat part (a) so that a pair of co-interior angles are both about 80° .

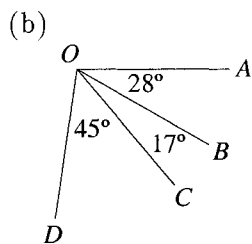
5. Find the angles α , β , γ and δ in the diagrams below, giving reasons.



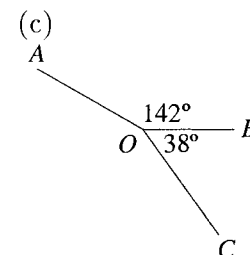
6. Find the angles α and β in each diagram below. Give all steps in your arguments.



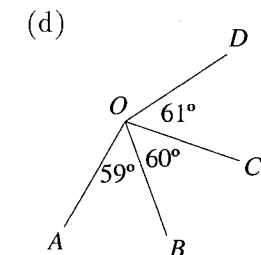
Show that $OC \perp OA$.



Show that $OD \perp OA$.



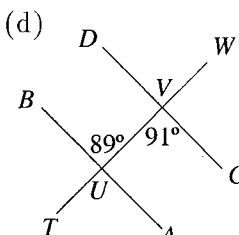
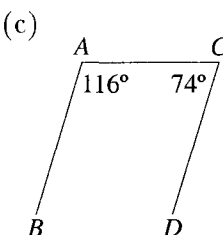
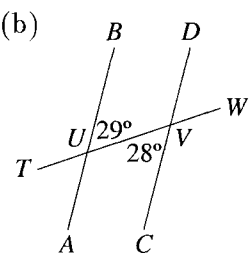
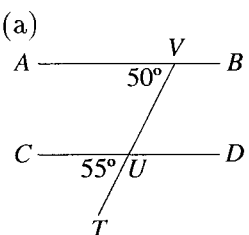
Show that A, O and C are collinear.



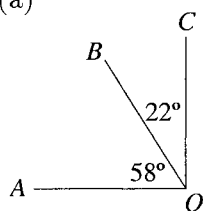
Show that A, O and D are collinear.

DEVELOPMENT

8. Show that AB is not parallel to CD in the diagrams below, giving all reasons.

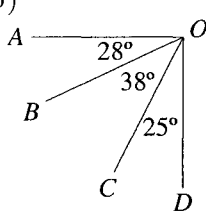


9. (a)



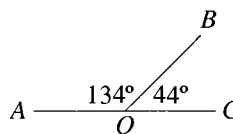
Show that OC is not perpendicular to OA .

(b)



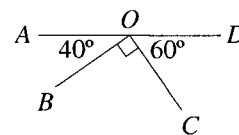
Show that OD is not perpendicular to OA .

(c)



Show that A, O and C are not collinear.

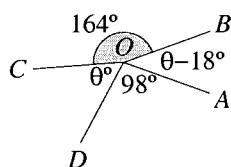
(d)



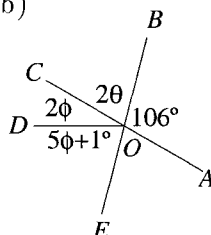
Show that A, O and D are not collinear.

10. Find θ and ϕ in the diagrams below, giving reasons.

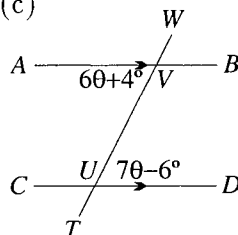
(a)



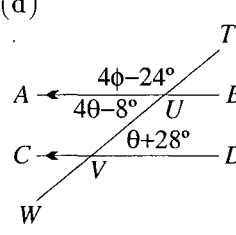
(b)



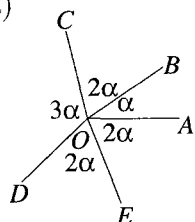
(c)



(d)

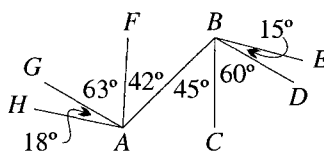


11. (a)



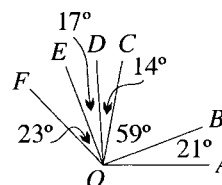
Name all straight angles and vertically opposite angles in the diagram.

(b)



Which two lines in the diagram above are parallel?

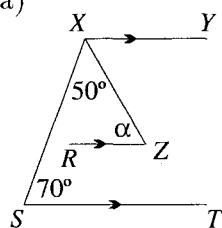
(c)



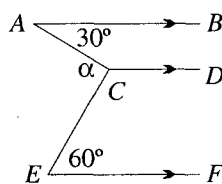
Which two lines in the diagram above form a right angle?

12. Find the angle α in each diagram below.

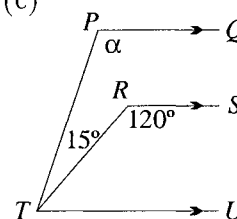
(a)



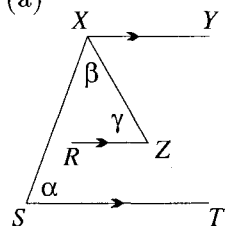
(b)



(c)

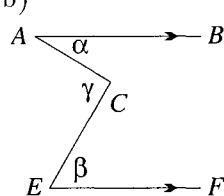


13. (a)



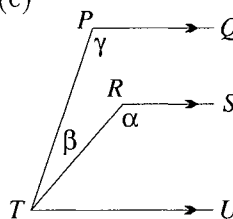
Show that $\gamma = 180^\circ - (\alpha + \beta)$.

(b)



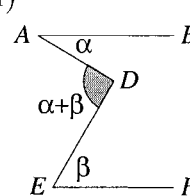
Show that $\gamma = \alpha + \beta$.

(c)



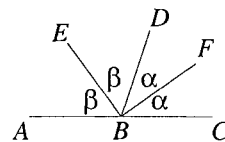
Show that $\gamma = \alpha - \beta$.

(d)

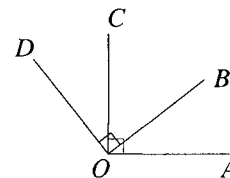


Show that $EF \parallel AB$.

14. THEOREM: The bisectors of adjacent supplementary angles form a right angle. In the diagram to the right, $\angle ABD$ and $\angle DBC$ are adjacent supplementary angles. Given that the line FB bisects $\angle DBC$ and the line EB bisects $\angle ABD$, prove that $\angle FBE = 90^\circ$.

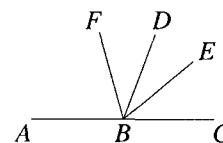


15. In the diagram to the right, the line CO is perpendicular to the line AO , and the line DO is perpendicular to the line BO . Show that the angles $\angle AOD$ and $\angle BOC$ are supplementary.

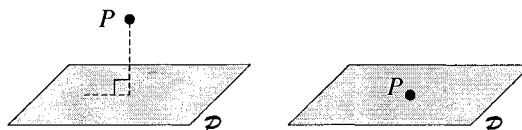


EXTENSION

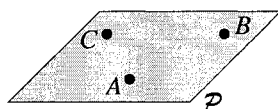
16. THEOREM: A generalisation of the result in question 14. In the diagram opposite, $\angle ABD$ and $\angle DBC$ are adjacent supplementary angles. Suppose that EB divides $\angle DBC$ in the ratio of $k : \ell$, and that FB also divides $\angle DBA$ in the ratio $k : \ell$. Find $\angle FBE$ in terms of k and ℓ .



17. Give concrete examples of the following:
- three distinct planes meeting at a point,
 - three distinct planes meeting at a line,
 - three distinct parallel planes,
 - three distinct planes intersecting in three distinct lines,
 - two distinct parallel planes intersecting with a third plane,
 - a line parallel to a plane,
 - a line intersecting a plane.
18. There are two possible configurations of a point and a plane. Either the point is in the plane or it is not, as shown in the diagram.
- What are the possible configurations of a line and a plane? Draw a diagram of each situation.
 - What are the possible configurations of two lines? Draw a diagram of each situation.
 - What are the possible configurations of two planes? Draw a diagram of each situation.

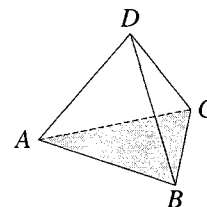


19. (a)



There is only one plane that passes through any three given non-collinear points. What are three other ways of determining a plane? Draw a diagram of each situation.

- (b)



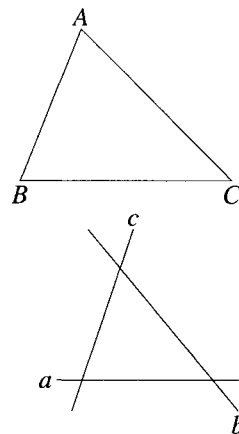
Two lines in space are called *skew* if they neither intersect nor are parallel. Given the tetrahedron $ABCD$ above, name all pairs of skew lines such that each passes through two of its vertices.

8B Angles in Triangles and Polygons

Having introduced angles and intervals, we can now begin to develop the relationships between the sizes of angles and the lengths of intervals. When three intervals are joined into a closed figure, they form a triangle, four such intervals form a quadrilateral, and more generally, an arbitrary number of such intervals form a polygon. Accordingly, this section is a study of angles in polygons. Sections 8C–8E then study the relationships between angles and lengths in triangles and quadrilaterals.

Triangles: A *triangle* is formed by taking any three non-collinear points A , B and C and constructing the intervals AB , BC and CA . The three intervals are called the *sides* of the triangle, and the three points are called its *vertices* (the singular is *vertex*).

Alternatively, a triangle can be formed by taking three non-concurrent lines a , b and c . Provided no two are parallel, the intersections of these lines form the vertices of the triangle.



Interior Angles of a Triangle: A triangle is a *closed* figure, meaning that it divides the plane into an inside and an outside. The three angles inside the triangle at the vertices are called the *interior angles*, and our first task is to prove that their sum is always 180° .

5 COURSE THEOREM: The sum of the interior angles of a triangle is a straight angle.

GIVEN: Let ABC be a triangle.

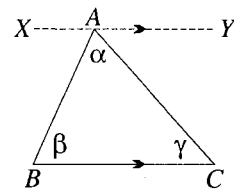
Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$.

AIM: To prove that $\alpha + \beta + \gamma = 180^\circ$.

CONSTRUCTION: Construct XAY through the vertex A parallel to BC .

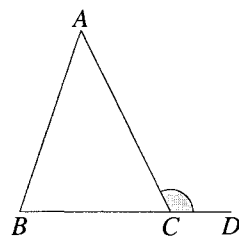
PROOF: $\angle XAB = \beta$ (alternate angles, $XAY \parallel BC$),
and $\angle YAC = \gamma$ (alternate angles, $XAY \parallel BC$).

Hence $\alpha + \beta + \gamma = 180^\circ$ (straight angle).



Exterior Angles of a Triangle: Suppose that ABC is a triangle, and suppose that the side BC is produced to D (the word ‘produced’ simply means ‘extended in the direction BC ’). Then the angle $\angle ACD$ between the side AC and the extended side CD is called an *exterior angle* of the triangle.

There are two exterior angles at each vertex, and because they are vertically opposite, they must be equal in size. Also, an exterior angle and the interior angle adjacent to it are adjacent angles on a straight line, so they must be supplementary. The exterior angles and interior angles are related as follows.



6 COURSE THEOREM: An exterior angle of a triangle equals the sum of the interior opposite angles.

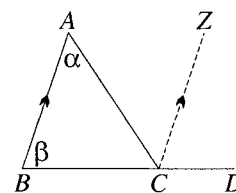
GIVEN: Let ABC be a triangle with BC produced to D . Let $\angle A = \alpha$ and $\angle B = \beta$.

AIM: To prove that $\angle ACD = \alpha + \beta$.

CONSTRUCTION: Construct the ray CZ through the vertex C parallel to BA .

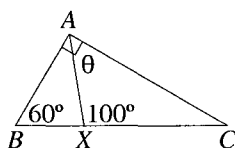
PROOF: $\angle ZCD = \beta$ (corresponding angles, $BA \parallel CZ$),
and $\angle ACZ = \alpha$ (alternate angles, $BA \parallel CZ$).

Hence $\angle ACD = \alpha + \beta$ (adjacent angles).

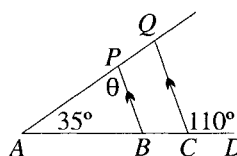


WORKED EXERCISE: Find θ in each diagram below.

(a)



(b)



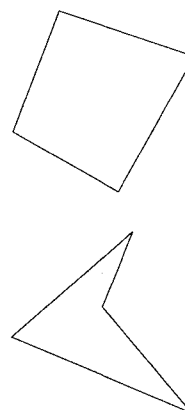
SOLUTION:

(a) $\angle C = 30^\circ$
(angle sum of $\triangle ABC$),
so $\theta = 50^\circ$
(angle sum of $\triangle ACX$).

(b) $\angle PBC = 110^\circ$
(corresponding angles, $BP \parallel CQ$),
so $\theta = 75^\circ$
(exterior angle of $\triangle ABP$).

Quadrilaterals: A *quadrilateral* is a closed plane figure bounded by four intervals. As with triangles, the intervals are called *sides*, and their four endpoints are called *vertices*. (The sides can't cross each other, and no vertex angle can be 180° .)

A quadrilateral may be *convex*, meaning that all its interior angles are less than 180° , or *non-convex*, meaning that one interior angle is greater than 180° . The intervals joining pairs of opposite vertices are called *diagonals* — notice that both diagonals of a convex quadrilateral lie inside the figure, but only one diagonal of a non-convex quadrilateral lies inside it. In both cases, we can prove that the sum of the interior angles is 360° .



7

COURSE THEOREM: The sum of the interior angles of a quadrilateral is two straight angles.

GIVEN: Let $ABCD$ be a quadrilateral, labelled so that the diagonal AC lies inside the figure.

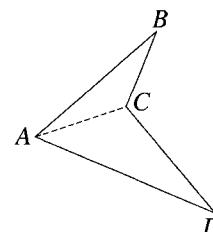
AIM: To prove that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360^\circ$.

CONSTRUCTION: Join the diagonal AC .

PROOF: The interior angles of $\triangle ABC$ have sum 180° ,
and the interior angles of $\triangle ADC$ have sum 180° .

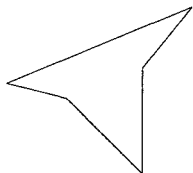
But the interior angles of quadrilateral $ABCD$
are the sums of the interior angles of $\triangle ABC$ and $\triangle ADC$.

Hence the sum of the interior angles of $ABCD$ is 360° .



Polygons: A *polygon* is a closed figure bounded by any number of straight sides (*polygon* is a Greek word meaning ‘many-angled’). A polygon is named according to the number of sides it has, and there must be at least three sides or else there would be no enclosed region. Here are some of the names:

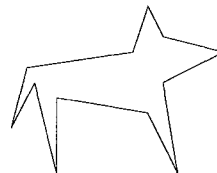
3 sides: triangle	6 sides: hexagon	9 sides: nonagon
4 sides: quadrilateral	7 sides: heptagon	10 sides: decagon
5 sides: pentagon	8 sides: octagon	12 sides: dodecagon



A pentagon



An octagon



A dodecagon

Like quadrilaterals, polygons can be *convex*, meaning that every interior angle is less than 180° , or *non-convex*, meaning that at least one interior angle is greater than 180° . A polygon is convex if and only if every one of its diagonals lies inside the figure. Notice that even a non-convex polygon must have at least one diagonal completely inside the figure.

The following theorem generalises the theorems about the interior angles of triangles and quadrilaterals to polygons with any number of sides.

8 COURSE THEOREM: The interior angles of an n -sided polygon have sum $180(n-2)^\circ$.

When the polygon is non-convex, the proof requires mathematical induction because we need to keep chopping off a triangle whose angle sum is 180° —this is carried through in question 23 of the following exercise. The situation is far easier when the polygon is convex, and the following proof is restricted to that case.

GIVEN: Let $A_1 A_2 \dots A_n$ be a convex polygon.

AIM: To prove that $\angle A_1 + \angle A_2 + \dots + \angle A_n = 180(n-2)^\circ$.

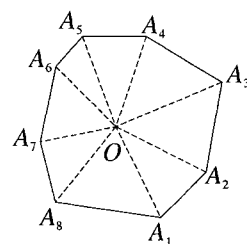
CONSTRUCTION: Choose any point O inside the polygon, and construct the intervals $OA_1, OA_2, \dots, OA_n$, giving n triangles $A_1OA_2, A_2OA_3, \dots, A_nOA_1$.

PROOF: The angle sum of the n triangles is $180n^\circ$.

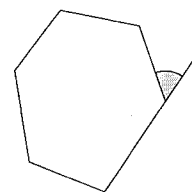
But the angles at O form a revolution, with size 360° .

Hence for the interior angles of the polygon,

$$\begin{aligned} \text{sum} &= 180n^\circ - 360^\circ \\ &= 180(n-2)^\circ. \end{aligned}$$



The Exterior Angles of a Polygon: An exterior angle of a convex polygon at any vertex is the angle between one side produced and the other side, just as in a triangle. We will ignore exterior angles of non-convex polygons, because they would have to involve negative angles. There is a surprisingly simple formula for the sum of the exterior angles.



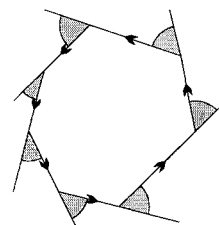
9 COURSE THEOREM: The sum of the exterior angles of a convex polygon is 360° .

PROOF: At each vertex, the interior and exterior angles add to 180° , so the sum of all interior and exterior angles is $180n^\circ$.

But the interior angles add to $180(n-2)^\circ$.

Hence the exterior angles must add to $2 \times 180^\circ = 360^\circ$.

Exterior Angles as the Amount of Turning: If one walks around a polygon, the exterior angle at each vertex is the angle one turns at that vertex. Thus the sum of all the exterior angles is the amount of turning when one walks right around the polygon. Clearly walking around a polygon involves a total turning of 360° , and the previous theorem can be interpreted as saying just that. In this way, the theorem can be generalised to say that when one walks around any closed curve, the amount of turning is always 360° (provided that the curve doesn't cross itself).



Regular Polygons: A *regular polygon* is a polygon in which all sides are equal and all interior angles are equal. Simple division gives:

10 COURSE THEOREM: In an n -sided regular polygon:

- each exterior angle is $\frac{360^\circ}{n}$,
- each interior angle is $\frac{180(n-2)^\circ}{n}$.

Substitution of $n = 3$ and $n = 4$ gives the familiar results that each angle of an equilateral triangle is 60° , and each angle of a square is 90° .

WORKED EXERCISE: Find the sizes of each exterior angle and each interior angle in a regular 12-sided polygon.

SOLUTION: The exterior angles have sum 360° , so each exterior angle is $360^\circ \div 12 = 30^\circ$. Hence each interior angle is 150° (angles in a straight angle).

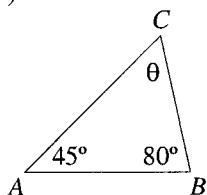
Alternatively, using the formula, each interior angle is $\frac{180 \times 12}{12} = 180^\circ$.

Exercise 8B

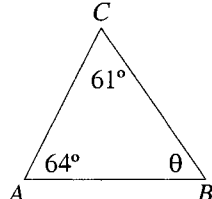
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Use the angle sum of a triangle to find θ in the diagrams below, giving reasons.

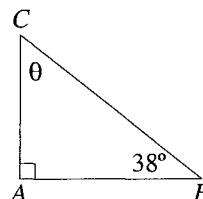
(a)



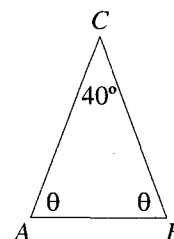
(b)

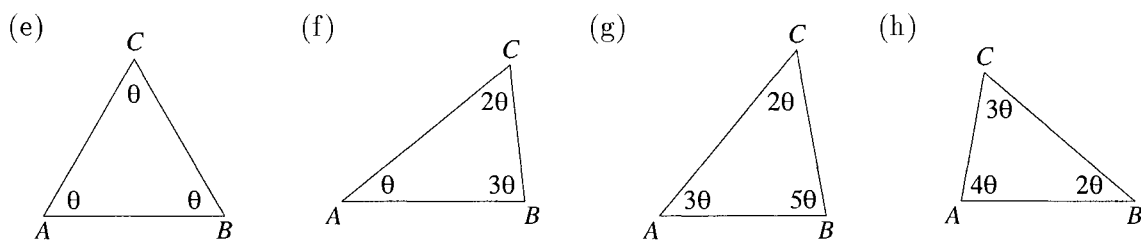


(c)

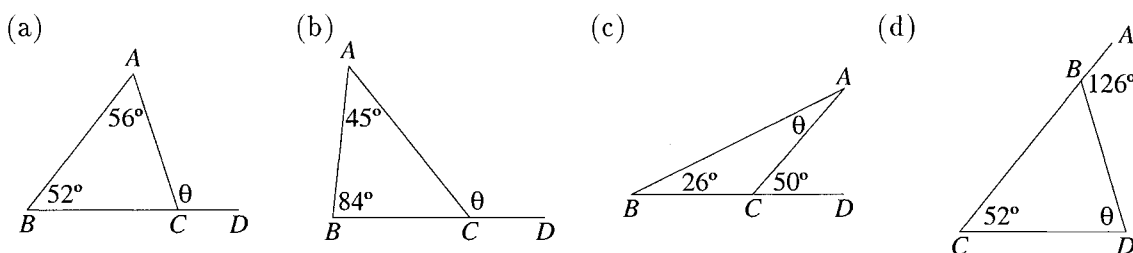


(d)

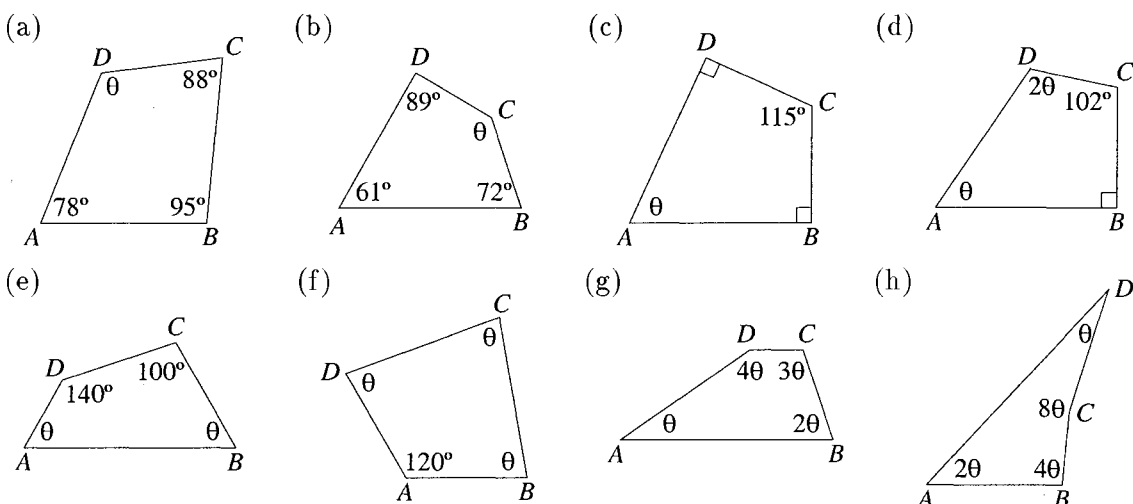




2. Use the exterior angle of a triangle theorem to find θ , giving reasons.



3. Use the angle sum of a quadrilateral to find θ in the diagrams below, giving reasons.



4. Demonstrate the formula $180(n-2)^\circ$ for the angle sum of a polygon by drawing examples of the following non-convex polygons and dissecting them into $n-2$ triangles:

- (a) a pentagon, (b) a hexagon, (c) an octagon, (d) a dodecagon.

5. Find the size of each (i) interior angle, (ii) exterior angle, of a regular polygon with:
(a) 5 sides, (b) 6 sides, (c) 8 sides, (d) 9 sides, (e) 10 sides, (f) 12 sides.

6. (a) Find the number of sides of a regular polygon if each interior angle is:

- (i) 135° (ii) 144° (iii) 172° (iv) 178°

(b) Find the number of sides of a regular polygon if its exterior angle is:

- (i) 72° (ii) 40° (iii) 18° (iv) $\frac{1}{2}^\circ$

(c) Why is it not possible for a regular polygon to have an interior angle equal to 123° ?

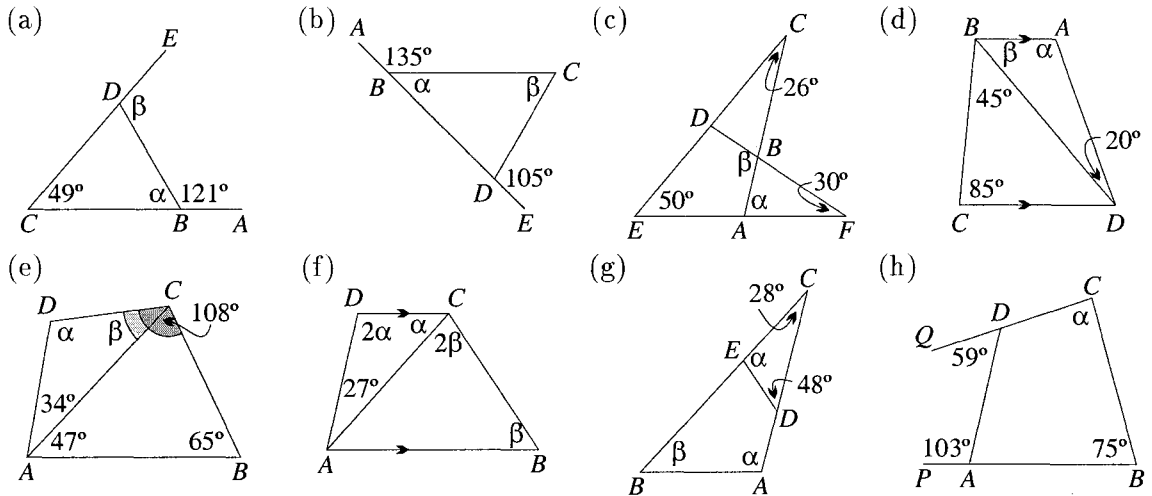
(d) Why is it not possible for a regular polygon to have an exterior angle equal to 71° ?

7. By drawing a diagram, find the number of diagonals of each polygon, and verify that the number of diagonals of a polygon with n sides is $\frac{1}{2}n(n-3)$:

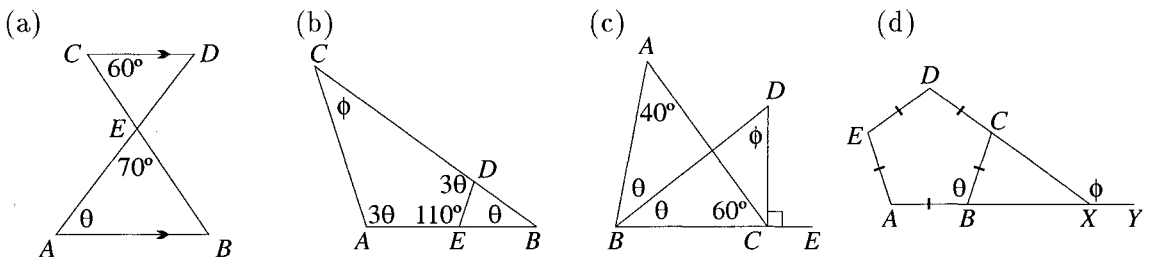
- (a) a convex pentagon, (b) a convex hexagon, (c) a convex octagon.

(This will be proven by mathematical induction in question 23.)

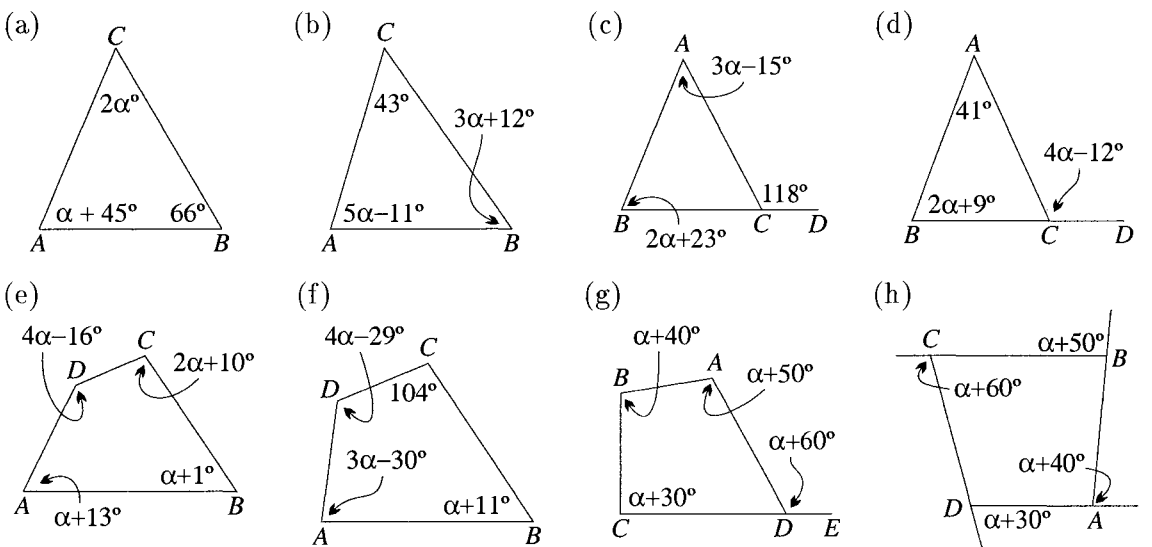
8. Find the angles α and β in the diagrams below. Give all steps in your argument.



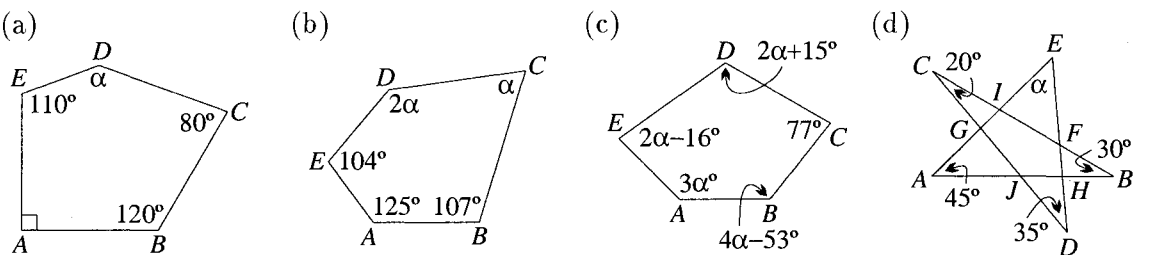
9. Find the angles θ and ϕ in the diagrams below, giving all reasons.



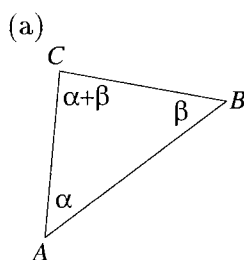
10. Find the value of α in the diagrams below, giving all reasons.



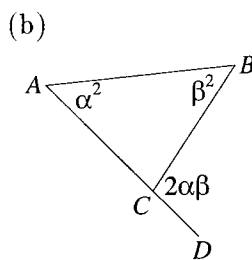
11. Find the values of α in the diagrams below, giving all reasons.



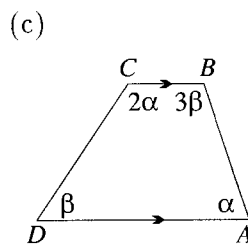
12. Prove the given relationships in the diagrams below.



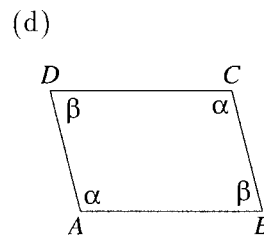
Show that $\alpha + \beta = 90^\circ$.



Show that $\alpha = \beta$.



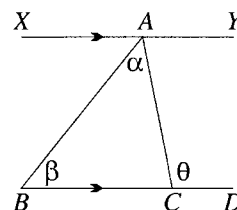
Show that $\alpha = 72^\circ$ and $\beta = 36^\circ$.



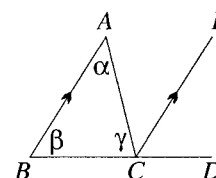
Show that $AB \parallel CD$ and $AD \parallel BC$.

DEVELOPMENT

13. COURSE THEOREM: An alternative proof of the exterior angle theorem. Given a triangle ABC with BC produced to D , construct the line XY through the vertex A parallel to BD . Let $\angle CAB = \alpha$ and $\angle ABC = \beta$. Use alternate angles twice to prove that $\angle ACD = \alpha + \beta$.

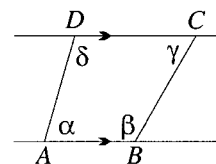


14. COURSE THEOREM: An alternative proof that the angle sum of a triangle is 180° . Let ABC be a triangle with BC produced to D . Construct the line CE through C parallel to BA . Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. Prove that $\alpha + \beta + \gamma = 180^\circ$.

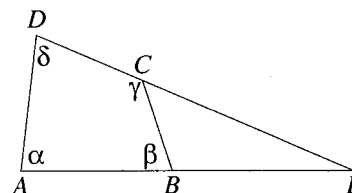


15. COURSE THEOREM: An alternative approach to proving that the angle sum of a quadrilateral is 360° .

(a) Suppose that a quadrilateral has a pair of parallel sides, and name them AB and CD as shown. Use the assumptions about parallel lines and transversals to prove that the interior angle sum of quadrilateral $ABCD$ is 360° .



(b) Suppose that in quadrilateral $ABCD$ there is no pair of parallel sides. Extend sides AB and DC to meet at E as shown. Use the theorems about angles in triangles to prove that the interior angle sum of quadrilateral $ABCD$ is 360° .



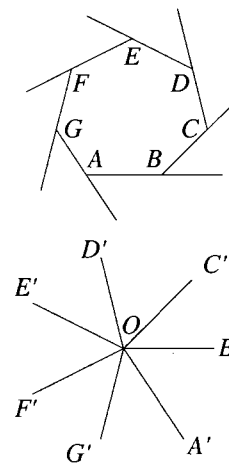
16. (a) Determine the ratio of the sum of the interior angles to the sum of the exterior angles in a polygon with n sides.

(b) Hence determine if it is possible to have these angles in the ratio: (i) $\frac{8}{3}$ (ii) $\frac{7}{2}$

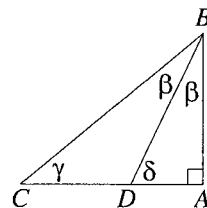
17. Convince yourself that the sum of the exterior angles of a polygon is 360° by carrying out the following constructions. Draw a polygon $ABCD \dots$ and pick a point O outside the polygon. From O draw OB' in the same direction as AB . Next draw OC' in the same direction as BC . Then do the same for CD and so on around the polygon. The diagrams show the result for the heptagon $ABCDEFG$.

(a) What is the sum of the angles at O ?

(b) How are the exterior angles of the polygon related to the angles at O ?



18. In the right-angled triangle ABC opposite, $\angle CAB = 90^\circ$, and the bisector of $\angle ABC$ meets AC at D . Let $\angle ABD = \beta$, $\angle ACB = \gamma$ and $\angle ADB = \delta$. Show that $\delta = 45^\circ + \frac{1}{2}\gamma$.



19. Three of the angles in a convex quadrilateral are equal. What is:

(a) the smallest possible size, (b) the largest possible size, of these three equal angles?

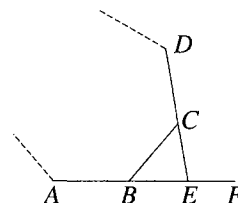
20. Let AB , BC and CD be three consecutive sides of a regular polygon with n sides. Produce AB to F , and produce DC to meet AF at E .

(a) Find the size of $\angle CEF$ as a function of n .

(b) Now suppose that $\angle CEF$ is the interior angle of another regular polygon with m sides. Find m in terms of n .

(c) Hence find all pairs of regular polygons that are related in this way.

(d) In each case, if the first polygon has sides of length 1, what is the length of the sides of the second polygon?



21. SEQUENCES AND GEOMETRY:

(a) The three angles of a triangle ABC form an arithmetic sequence. Show that the middle-sized angle is 60° .

(b) The three angles of a triangle PQR form a geometric sequence. Show that the smallest angle and the common ratio cannot both be integers.

22. (a) A quadrilateral in which all angles are equal need not have all sides equal (it is in fact a rectangle). Prove, nevertheless, that opposite sides are parallel.

(b) Prove that if all angles of a hexagon are equal, then opposite sides are parallel.

(c) Prove more generally that this holds for polygons with $2n$ sides.

23. MATHEMATICAL INDUCTION IN GEOMETRY:

(a) Use mathematical induction to prove that for $n \geq 3$, a polygon with n sides has $\frac{1}{2}n(n-3)$ diagonals. Begin with a triangle, which has no diagonals.

(b) Use mathematical induction to prove that the sum of the interior angles of any polygon with $n \geq 3$ sides, convex or non-convex, is $180(n-2)^\circ$. Begin the induction step by choosing three adjacent vertices P_k , P_{k+1} and P_1 of the $(k+1)$ -gon so that $\angle P_k P_{k+1} P_1$ is acute, and joining the diagonal $P_1 P_k$ to form a triangle and a polygon with k sides.

EXTENSION

24. TRIGONOMETRY IN GEOMETRY: Suppose that a regular polygon has n sides of length 1.

(a) What will be the length of the side of the regular polygon with $2n$ sides that is formed by cutting off the vertices of the given polygon?

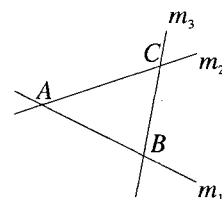
(b) Confirm your answer in the case of:

(i) cutting the corners off an equilateral triangle to form a regular hexagon,

(ii) cutting the corners off a square to form a regular octagon.

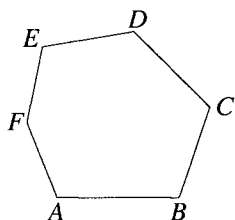
25. TRIGONOMETRY IN GEOMETRY: Three lines with nonzero gradients m_1 , m_2 and m_3 intersect at the points A , B and C . The acute angles α , β and γ , between each pair of lines, are

found using the usual formula $\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$.

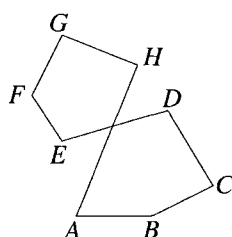


- (a) If one of the angles of $\triangle ABC$ is obtuse, explain why one of the acute angles found must be the sum of the other two.
- (b) If the signs of m_1 , m_2 and m_3 are all the same, what can be deduced about $\triangle ABC$?
- (c) If all angles of $\triangle ABC$ are acute, what can be deduced about the sign of $m_1 m_2 m_3$?
26. In a polygon with n sides, none of which are vertical and none horizontal, and all interior angles equal, determine the sign of the product of the gradients of all the sides.
27. Counting clockwise turns as negative and anticlockwise turns as positive, through how many revolutions would you turn if you followed the alphabet around the following figures?

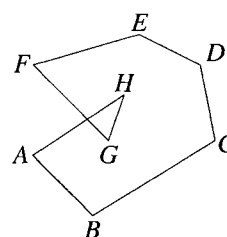
(a)



(b)



(c)



8C Congruence and Special Triangles

As in all branches of mathematics, symmetry is a vital part of geometry. In Euclidean geometry, symmetry is handled by means of congruence, and later through the more general idea of similarity. It is only by these methods that relationships between lengths and angles can be established.

Congruence: Two figures are called *congruent* if one figure can be picked up and placed so that it fits exactly on top of the other figure. More precisely, using the language of transformations:

11

CONGRUENCE: Two figures S and T are called *congruent*, written as $S \equiv T$, if one figure can be moved to coincide with the other figure by means of a sequence of rotations, reflections and translations.



The congruence sets up a correspondence between the elements of the two figures. In this correspondence, angles, lengths and areas are preserved.

12

PROPERTIES OF CONGRUENT FIGURES: If two figures are congruent.

- matching angles have the same size,
- matching intervals have the same length,
- matching regions have the same area.

Congruent Triangles: In practice, almost all of our congruence arguments concern congruent triangles. Euclid's geometry book proves four tests for the congruence of two triangles, but we shall take them as assumptions.

13

STANDARD CONGRUENCE TESTS FOR TRIANGLES: Two triangles are congruent if:

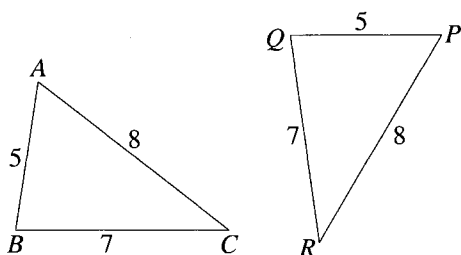
SSS the three sides of one triangle are respectively equal to the three sides of another triangle, or

SAS two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, or

AAS two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, or

RHS the hypotenuse and one side of one right triangle are respectively equal to the hypotenuse and one side of another right triangle.

These standard tests are known from earlier years, and have already been discussed in Sections 4H–4J of the Year 11 volume, where they were related to the sine and cosine rules. As mentioned in those sections, there is no ASS test — two sides and a non-included angle — and we constructed two non-congruent triangles with the same ASS specifications. Here are examples of the four tests.

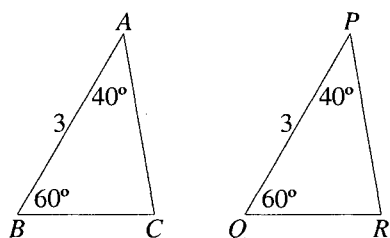


$\triangle ABC \equiv \triangle PQR$ (SSS).

Hence $\angle P = \angle A$, $\angle Q = \angle B$

and $\angle R = \angle C$

(matching angles of congruent triangles).

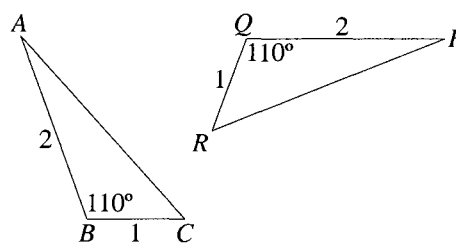


$\triangle ABC \equiv \triangle PQR$ (AAS).

Hence $QR = BC$ and $RP = CA$

(matching sides of congruent triangles),

and $\angle R = \angle C$ (angle sums of triangles).

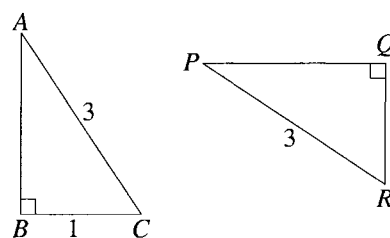


$\triangle ABC \equiv \triangle PQR$ (SAS).

Hence $\angle P = \angle A$, $\angle R = \angle C$

and $PR = AC$ (matching sides

and angles of congruent triangles).



$\triangle ABC \equiv \triangle PQR$ (RHS).

Hence $\angle P = \angle A$, $\angle R = \angle C$

and $PQ = AB$ (matching sides

and angles of congruent triangles).

Using the Congruence Tests: A fully set-out congruence proof has five lines — the first line introduces the triangles, the next three set out the three pairs of equal sides or angles, and the final line is the conclusion. Subsequent deductions from the congruence follow these five lines. Throughout the congruence proof, all vertices should be named in corresponding order. Each of the four standard congruence tests is used in one of the next four proofs.

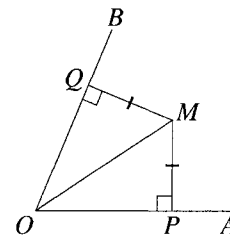
WORKED EXERCISE: The point M lies inside the arms of the acute angle $\angle AOB$. The perpendiculars MP and MQ to OA and OB respectively have equal lengths. Prove that $\triangle POM \equiv \triangle QOM$, and that OM bisects $\angle AOB$.

PROOF: In the triangles POM and QOM :

1. $OM = OM$ (common),
2. $PM = QM$ (given),
3. $\angle OPM = \angle OQM = 90^\circ$ (given),

so $\triangle POM \equiv \triangle QOM$ (RHS).

Hence $\angle POM = \angle QOM$ (matching angles).



WORKED EXERCISE: Prove that $\triangle ABC \equiv \triangle CDA$, and hence that $AD \parallel BC$.

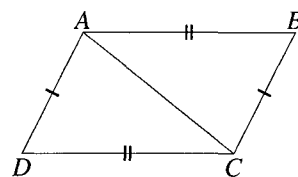
PROOF: In the triangles ABC and CDA :

1. $AC = CA$ (common),
2. $AB = CD$ (given),
3. $BC = DA$ (given),

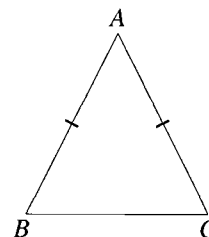
so $\triangle ABC \equiv \triangle CDA$ (SSS).

Hence $\angle BCA = \angle DAC$ (matching angles),

and so $AD \parallel BC$ (alternate angles are equal).



Isosceles Triangles: An *isosceles triangle* is a triangle in which two sides are equal. The two equal sides are called the *legs* of the triangle (the Greek word 'isosceles' literally means 'equal legs'), their intersection is called the *apex*, and the side opposite the apex is called the *base*. It is well known that the base angles of an isosceles triangle are equal.



14 COURSE THEOREM: If two sides of a triangle are equal, then the angles opposite those sides are equal.

GIVEN: Let ABC be an isosceles triangle with $AB = AC$.

AIM: To prove that $\angle B = \angle C$.

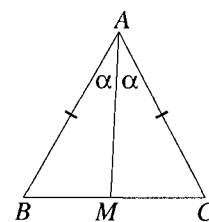
CONSTRUCTION: Let the bisector of $\angle A$ meet BC at M .

PROOF: In the triangles ABM and ACM :

1. $AM = AM$ (common),
2. $AB = AC$ (given),
3. $\angle BAM = \angle CAM$ (construction),

so $\triangle ABM \equiv \triangle ACM$ (SAS).

Hence $\angle ABM = \angle ACM$ (matching angles of congruent triangles).



A Test for a Triangle to be Isosceles: The converse of this result is also true, giving a test for a triangle to be isosceles.

15 COURSE THEOREM: Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.

GIVEN: Let ABC be a triangle in which $\angle B = \angle C = \beta$.

AIM: To prove that $AB = AC$.

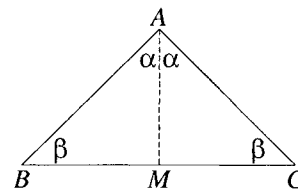
CONSTRUCTION: Let the bisector of $\angle A$ meet BC at M .

PROOF: In the triangles ABM and ACM :

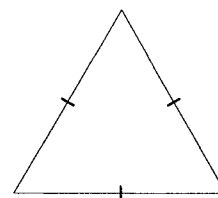
1. $AM = AM$ (common),
2. $\angle B = \angle C$ (given),
3. $\angle BAM = \angle CAM$ (construction),

so $\triangle ABM \equiv \triangle ACM$ (AAS).

Hence $AB = AC$ (matching sides of congruent triangles).



Equilateral Triangles: An equilateral triangle is a triangle in which all three sides are equal. It is therefore an isosceles triangle in three different ways, and the following property of and test for an equilateral triangle follow easily from the previous theorem and its converse.



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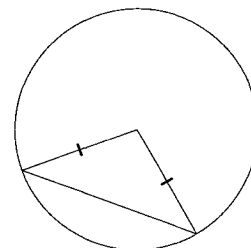
COURSE THEOREM: All angles of an equilateral triangle are equal to 60° .
Conversely, if all angles of a triangle are equal, then it is equilateral.

PROOF: Suppose that the triangle is equilateral, that is, all three sides are equal. Then all three angles are equal, and since their sum is 180° , they must each be 60° .

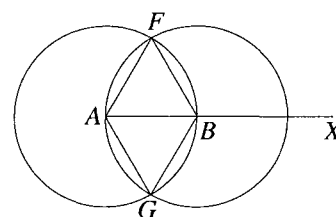
Conversely, suppose that all three angles are equal. Then all three sides are equal, meaning that the triangle is equilateral.

Circles and Isosceles Triangles: A circle is the set of all points that are a fixed distance (called the *radius*) from a fixed point (called the *centre*). Compasses are used for drawing circles, because the pencil is held at a fixed distance from the centre, where the compass-point is fixed in the paper.

If two points on the circumference are joined to the centre and to each other, then the equal radii mean that the triangle is isosceles. The following worked exercise shows how to construct an angle of 60° using straight edge and compasses.



WORKED EXERCISE: Construct a circle with centre on the end A of an interval AX , meeting the ray AX at B . With centre B and the same radius, construct a circle meeting the first circle at F and G . Prove that $\angle FAB = \angle GAB = 60^\circ$.



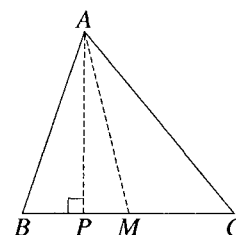
PROOF: Because they are all radii of congruent circles,

$$AF = AB = AG = BF = BG.$$

Hence $\triangle AFB$ and $\triangle AGB$ are both equilateral triangles, and so $\angle FAB = \angle GAB = 60^\circ$.

Medians and Altitudes: A *median* of a triangle joins a vertex to the midpoint of the opposite side. An *altitude* of a triangle is the perpendicular from a vertex to the opposite side. These two words are useful when talking about triangles.

In the diagram to the right, AP is one of the three altitudes in $\triangle ABC$. The point M is the midpoint of BC , and AM is one of the three medians in $\triangle ABC$.

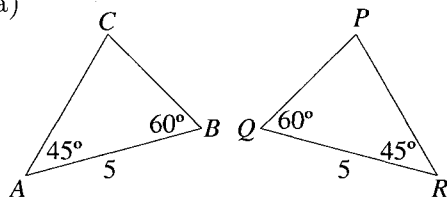


Exercise 8C

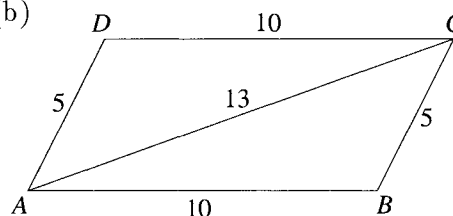
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. The two triangles in each pair below are congruent. Name the congruent triangles in the correct order and state which test justifies the congruence.

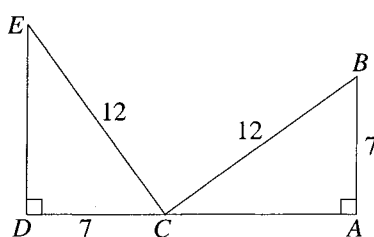
(a)



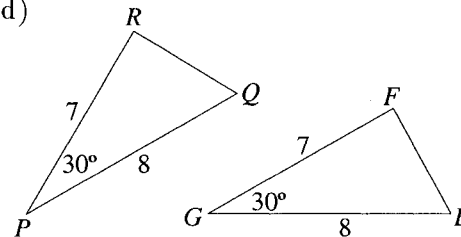
(b)



(c)

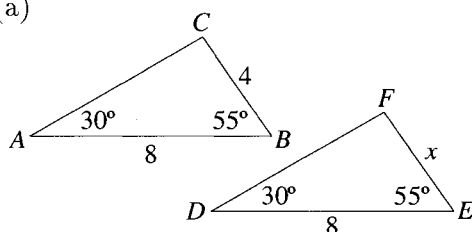


(d)

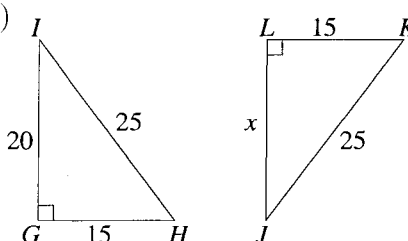


2. In each part, identify the congruent triangles, naming the vertices in matching order and giving a reason. Hence deduce the length of the side x .

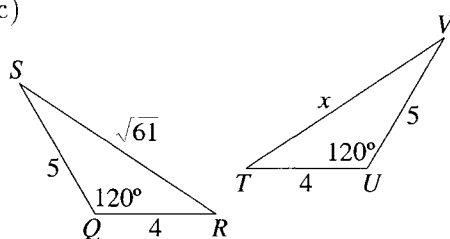
(a)



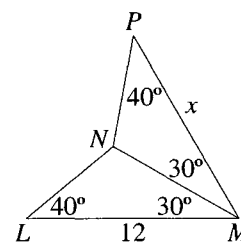
(b)



(c)

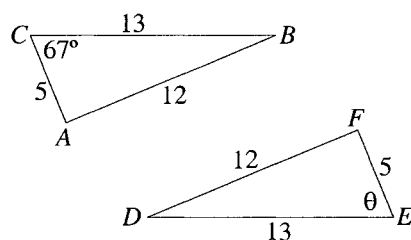


(d)

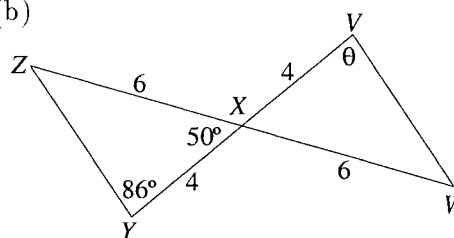


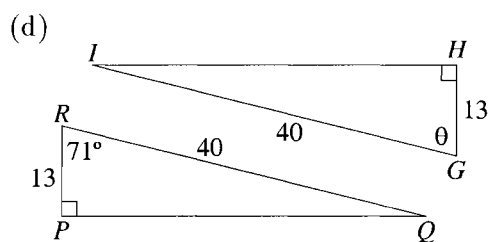
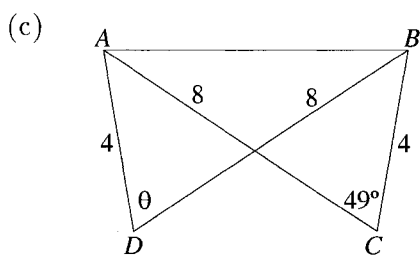
3. In each part, identify the congruent triangles, naming the vertices in matching order and giving a reason. Hence deduce the size of the angle θ .

(a)

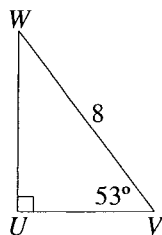
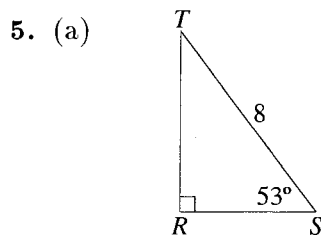
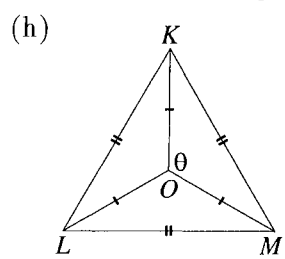
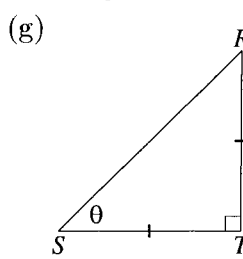
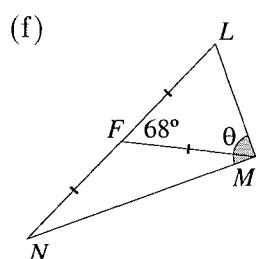
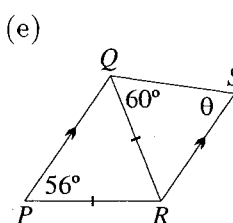
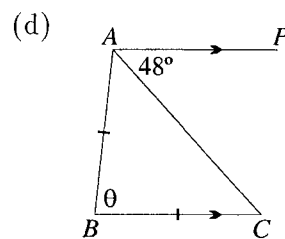
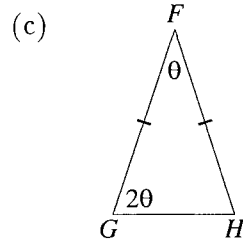
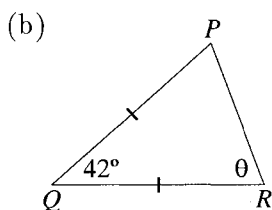
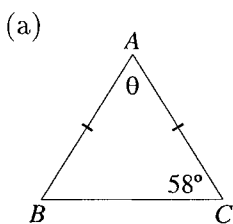


(b)





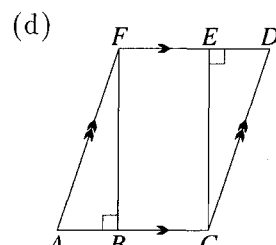
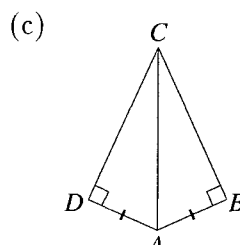
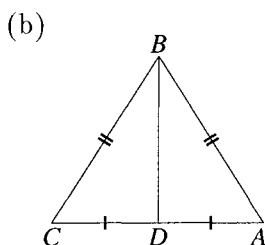
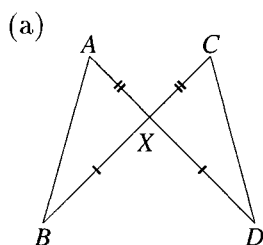
4. Find the size of angle θ in each diagram below, giving reasons.



When asked to show that the two triangles above were congruent, a student wrote $\triangle RST \equiv \triangle UVW$ (RHS). Although both triangles are indeed right-angled, explain why the reason given is incorrect. What is the correct reason?

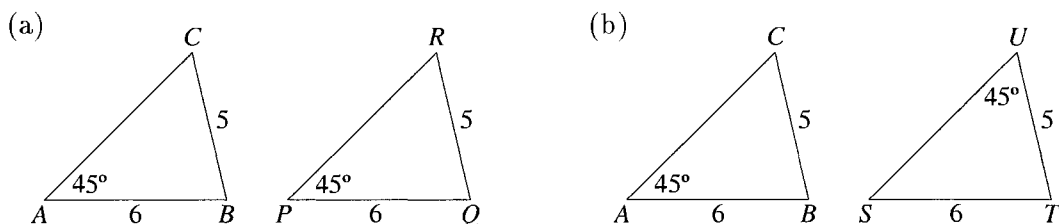
When asked to show that the two triangles above were congruent, another student wrote $\triangle GHI \equiv \triangle ABC$ (RHS). Again, although both triangles are right-angled, explain why the reason given is wrong. What is the correct reason?

6. In each part, prove that the two triangles in the diagram are congruent.



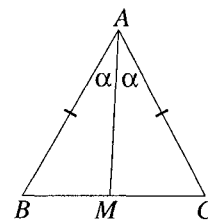
7. Let M be any point on the base BC of an isosceles triangle ABC . Using the facts that the legs AB and AC are equal, the base angles $\angle B$ and $\angle C$ are equal, and the side AM is common, is it possible to prove that the triangles ABM and ACM are congruent?

8. Explain why the given pairs of triangles cannot be proven to be congruent.

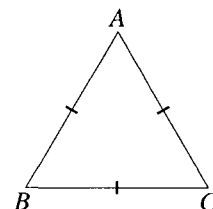


9. (a) What rotational and reflection symmetries does an isosceles triangle have?
 (b) What rotational and reflection symmetries does an equilateral triangle have?
10. INTERPRETING THE PROPERTIES OF ISOSCELES AND EQUILATERAL TRIANGLES USING TRANSFORMATIONS:

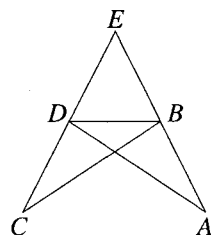
- (a) Sketched on the right is an isosceles triangle $\triangle BAC$ with $AB = AC$. The interval AM bisects $\angle BAC$.



- (i) Use the properties of reflections to explain why reflection in AM exchanges B and C , and hence explain why $\angle B = \angle C$, why M bisects BC , and why $AM \perp BC$.
 (ii) Name all the axes of symmetry of $\triangle ABC$.
 (b) The triangle $\triangle ABC$ on the right is equilateral.

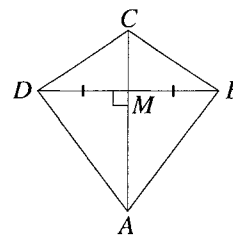


11. (a)



Given that $\triangle ABD \equiv \triangle CDB$ in the diagram above, prove that $\triangle BDE$ is isosceles.

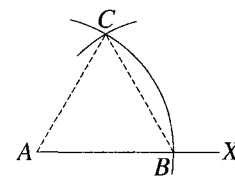
- (b)



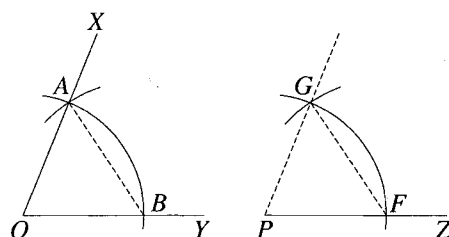
If $DM = MB$ and $AC \perp DB$, prove that $\triangle ABD$ and $\triangle CBD$ are isosceles.

DEVELOPMENT

12. CONSTRUCTION: *Constructing an angle of 60° .* Let AX be an interval. Construct an arc with centre A , meeting the line AX at B . With the same radius but with centre B , construct a second arc meeting the first one at C . Explain why $\triangle ABC$ is equilateral, and hence why $\angle BAC = 60^\circ$.

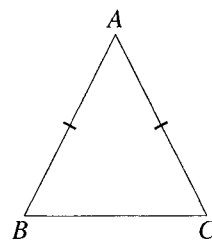
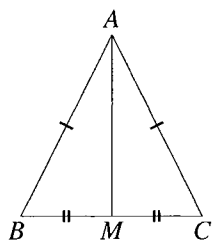


13. CONSTRUCTION: *Copying an angle.* Let $\angle XOY$ be an angle and PZ be an interval. Construct an arc with centre O meeting OX at A and OY at B . With the same radius, construct an arc with centre P , meeting PZ at F . With radius AB and centre F , construct an arc meeting the second arc at G .

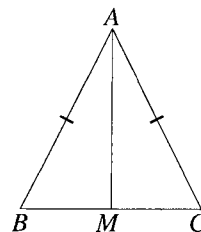


- (a) Prove that $\triangle AOB \equiv \triangle FPG$.
 (b) Hence prove that $\angle AOB = \angle FPG$.

14. COURSE THEOREM: *Three alternative proofs that the base angles of an isosceles triangle are equal.* Let ABC be an isosceles triangle with $AB = AC$.

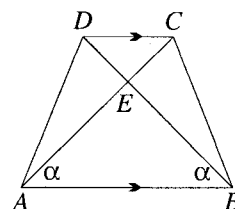


- (a) In the diagram above, the median AM has been constructed. Prove that the triangles AMB and AMC are congruent, and hence that $\angle B = \angle C$.
- (b) Draw your own triangle ABC , and on it construct the altitude AM . Prove that $\triangle AMB$ is congruent to $\triangle AMC$, and hence that $\angle B = \angle C$.
- (c) This is the most elegant proof, because it uses no construction at all. The two congruent triangles are the same triangle, but with the vertices in a different order.
- (i) Prove that $\triangle ABC \equiv \triangle ACB$.
- (ii) Hence prove that $\angle B = \angle C$.
15. THEOREM: *Properties of isosceles triangles.* In each part you will prove a property of an isosceles triangle. For each proof, use the same diagram, where $\triangle ABC$ is isosceles with $AB = AC$, and begin by proving that $\triangle AMB \equiv \triangle AMC$.



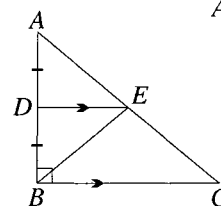
16. In the diagram, $AB \parallel DC$ and $\angle CAB = \angle ABD = \alpha$.

- (a) Show that $CE = DE$.
- (b) Prove that $\triangle ABC \equiv \triangle BAD$.
- (c) Hence show that $\angle DAC = \angle CBD$.



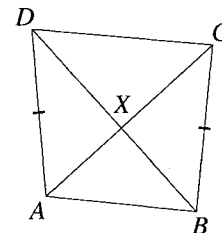
17. Triangle ABC has a right angle at B , D is the midpoint of AB , and DE is parallel to BC .

- (a) Prove that $\angle ADE$ is a right angle.
- (b) Prove that $\triangle AED \equiv \triangle BED$.
- (c) Prove that $BE = EC$.

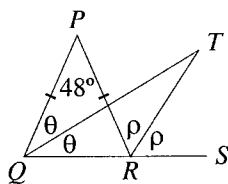


18. The diagonals AC and DB of quadrilateral $ABCD$ are equal and intersect at X . Also, $AD = BC$.

- (a) Show that $\triangle ABC \equiv \triangle BAD$.
- (b) Hence show that $\triangle ABX$ is isosceles.
- (c) Thus show that $\triangle CDX$ is also isosceles.
- (d) Show that $AB \parallel DC$.



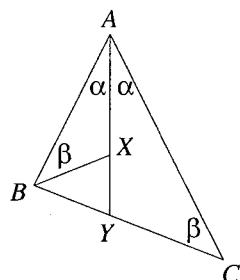
19. (a)



In the diagram, $\triangle PQR$ is isosceles with $PQ = PR$, and $\angle QPR = 48^\circ$. The interval QR is produced to S . The bisectors of $\angle PQR$ and $\angle PRS$ meet at the point T .

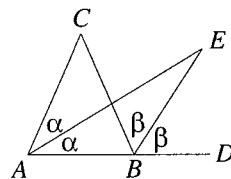
- Find $\angle PQR$.
- Find $\angle QTR$.

20. (a)



The bisector of $\angle BAC$ meets BC at Y . The point X is constructed on AY so that $\angle ABX = \angle ACB$. Prove that $\triangle BXY$ is isosceles.

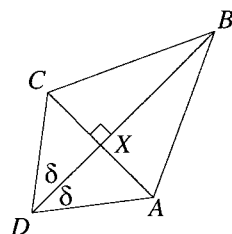
(b)



In $\triangle ABC$, AB is produced to D . AE bisects $\angle CAB$ and BE bisects $\angle CBD$.

- If $\triangle ABE$ is isosceles with $\angle A = \angle E$, show that $\triangle ABC$ is also isosceles.
- If $\triangle ABC$ is isosceles with $\angle A = \angle B$, under what circumstances will $\triangle ABE$ be isosceles?

(b)

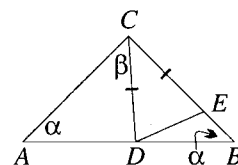


The diagonals AC and BD of quadrilateral $ABCD$ meet at right angles at X . Also, $\angle ADX = \angle CDX$.

- Prove that $AD = CD$.
- Hence prove that $AB = CB$.

21. In $\triangle ABC$, $\angle CAB = \angle CBA = \alpha$. Construct D on AB and E on CB so that $CD = CE$. Let $\angle ACD = \beta$.

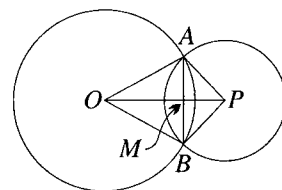
- Explain why $\angle CDB = \alpha + \beta$.
- Find $\angle DCB$ in terms of α and β .
- Hence find $\angle EDB$ in terms of β .



22. THEOREM: The line of centres of two intersecting circles is the perpendicular bisector of the common chord.

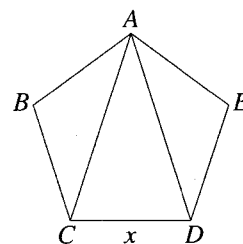
The diagram to the right shows two circles intersecting at A and B . The line of centres OP intersects AB at M .

- Explain why $\triangle ABO$ and $\triangle ABP$ are isosceles.
- Show that $\triangle AOP \equiv \triangle BOP$.
- Show that $\triangle AMO \equiv \triangle BMO$.
- Hence show that $AM = BM$ and $AB \perp OP$.

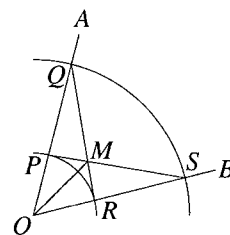


23. PENTAGONS AND TRIGONOMETRY: $ABCDE$ is a regular pentagon with side length x . Each interior angle is 108° .

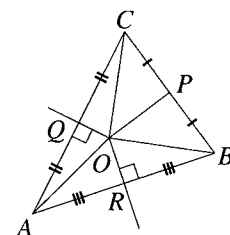
- State why $\triangle ABC$ is isosceles and find $\angle CAB$.
- Show that $\triangle ABC \equiv \triangle DEA$.
- Find $\angle CAD$.
- Find an expression for the area of the pentagon in terms of x and trigonometric ratios.



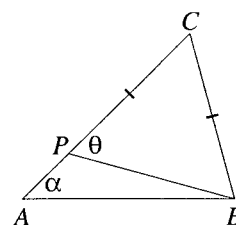
24. CONSTRUCTION: *Another construction to bisect an angle.* Given $\angle AOB$, draw two concentric circles with centre O , cutting OA at P and Q respectively, and OB at R and S respectively. Let PS and QR meet at M .



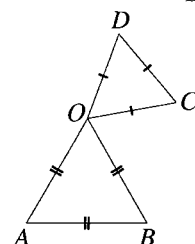
- (a) Prove that $\triangle POS \equiv \triangle ROQ$.
 (b) Hence prove that $\triangle PMQ \equiv \triangle RMS$.
 (c) Hence prove that OM bisects $\angle AOB$.
25. THE CIRCUMCENTRE THEOREM: *The perpendicular bisectors of the sides of a triangle are concurrent, and the resulting circumcentre is the centre of the circumcircle through all three vertices.* Let P, Q and R be the midpoints of the sides BC, CA and AB of $\triangle ABC$. Let the perpendiculars from Q and R meet at O , and join OA, OB, OC and OP .
- (a) Prove that $\triangle ORA \equiv \triangle ORB$.
 (b) Prove that $\triangle OQA \equiv \triangle OQC$.
 (c) Hence prove that $OA = OB = OC$, and $OP \perp BC$.



26. A GEOMETRIC INEQUALITY: *The angle opposite a longer side of a triangle is larger than the angle opposite a shorter side.* Suppose that $\triangle ABC$ is a triangle in which $CA > CB$. Construct P between C and A so that $CP = CB$, and let $\alpha = \angle A$ and $\theta = \angle CPB$.

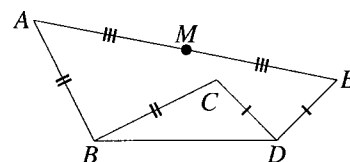


- (a) Explain why $\alpha < \theta$. (b) Explain why $\angle CBP = \theta$.
 (c) Hence prove that $\alpha < \angle CBA$.
27. A ROTATION THEOREM: The triangles OAB and OCD in the figure drawn to the right are both equilateral triangles, and they have a common vertex O . Prove that $AC = BD$.

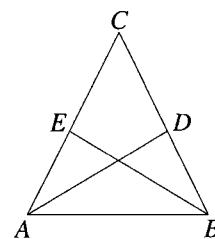


EXTENSION

28. In the diagram, B and D are fixed points on a horizontal line. A point C is chosen anywhere in the plane, and A is the image of C after a rotation of 90° (anticlockwise) about B . E is the image of C after a rotation of -90° (clockwise) about D . Find the location of M , the midpoint of AE , and show that this location is independent of the choice of C . [HINT: Let F be the foot of the altitude from C to BD . Add the points G and H , the two images of F under the two rotations, to the diagram.]



29. THREE TESTS FOR ISOSCELES TRIANGLES: Consider the triangle ABC , with D on the side BC and E on the side AC .
- (a) [Straightforward] Suppose that AD and BE are altitudes, and $AD = BE$. Show that $\triangle ABC$ is isosceles.
 (b) [More difficult] Suppose that AD and BE are medians, and $AD = BE$. Show that $\triangle ABC$ is isosceles.
 (c) [Extremely difficult] Suppose that AD and BE are angle bisectors, and $AD = BE$. Show that $\triangle ABC$ is isosceles.



8D Trapezia and Parallelograms

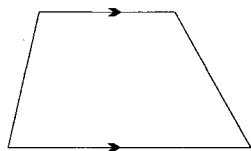
There are a series of important theorems concerning the sides and angles of quadrilaterals. If careful definitions are first given of five special quadrilaterals, these theorems can then be stated very elegantly as properties of these special quadrilaterals and tests for them. This section deals with trapezia and parallelograms, and the following section deals with rhombuses, rectangles and squares.

These theorems have been treated in earlier years, and most proofs have been left to structured questions in the following exercise. The proofs, however, are an essential part of the course, and should be carefully studied.

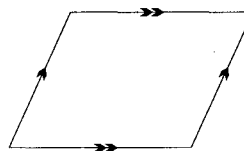
Definitions of Trapezia and Parallelograms: These figures are defined in terms of parallel sides. Notice that a parallelogram is a special sort of trapezium.

DEFINITIONS:

- 17
- A *trapezium* is a quadrilateral with at least one pair of opposite sides parallel.
 - A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.



A trapezium

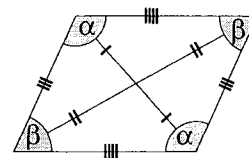


A parallelogram

Properties of and Tests for Parallelograms: The standard properties and tests concern the angles, the sides and the diagonals.

COURSE THEOREM: If a quadrilateral is a parallelogram, then:

- adjacent angles are supplementary, and
- opposite angles are equal, and
- opposite sides are equal, and
- the diagonals bisect each other.



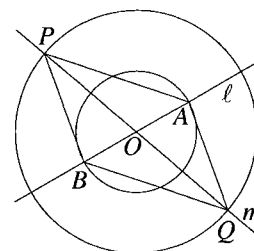
18

Conversely, a quadrilateral is a parallelogram if:

- the opposite angles are equal, or
- the opposite sides are equal, or
- one pair of opposite sides are equal and parallel, or
- the diagonals bisect each other.

WORKED EXERCISE: [A construction of a parallelogram] Two lines ℓ and m intersect at O , and concentric circles are constructed with centre O . Let ℓ meet the inner circle at A and B , and let m meet the outer circle at P and Q . Prove that $APBQ$ is a parallelogram.

PROOF: Since the point O is the midpoint of AB and of PQ , the diagonals of $APBQ$ bisect each other. Hence $APBQ$ is a parallelogram.

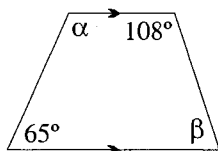


Exercise 8D

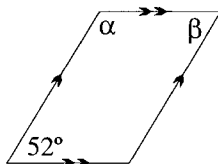
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find the angles α and β in the diagrams below, giving reasons.

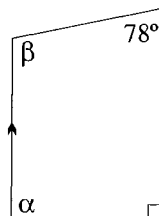
(a)



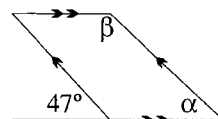
(b)



(c)

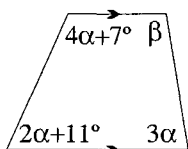


(d)

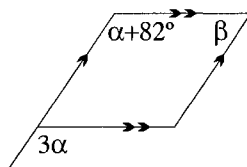


2. Write down an equation for α in each diagram below, giving reasons. Solve this equation to find the angles α and β , giving reasons.

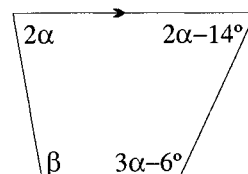
(a)



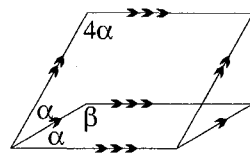
(b)



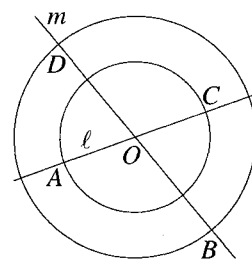
(c)



(d)



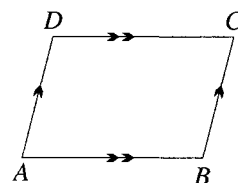
3. CONSTRUCTION: *Constructing a parallelogram from two equal parallel intervals.* Place a ruler with two parallel edges flat on the page, and draw 4 cm intervals AB and PQ on each side of the ruler. What theorem tells us that $ABQP$ is a parallelogram?
4. CONSTRUCTION: *Constructing a parallelogram from its diagonals.* Construct two lines ℓ and m meeting at O . Construct two circles C and D with the common centre O . Let ℓ meet C at A and C , and let m meet D at B and D . Use the tests for a parallelogram to explain why the quadrilateral $ABCD$ is a parallelogram.
5. Is it true that if one pair of opposite sides of a quadrilateral are parallel, and the other pair are equal, then the quadrilateral must be a parallelogram?
6. (a) What rotation and reflection symmetries does every parallelogram have?
(b) Can a trapezium that is not a parallelogram have any symmetries?
7. TRIGONOMETRY:
(a) If $ABCD$ is a parallelogram, show that $\sin A = \sin B = \sin C = \sin D$.
(b) Quadrilateral $ABCD$ is a trapezium with $AB \parallel DC$ and with $\angle A = \angle B$. Show that $\sin A = \sin B = \sin C = \sin D$.



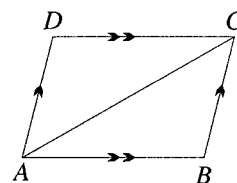
DEVELOPMENT

8. PROPERTIES OF A PARALLELOGRAM: In this question, you must use the definition of a parallelogram as a quadrilateral in which the opposite sides are parallel.

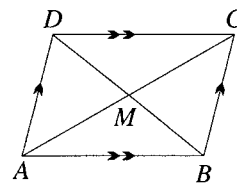
- (a) COURSE THEOREM: *Adjacent angles of a parallelogram are supplementary, and opposite angles are equal.* The diagram shows a parallelogram $ABCD$. Explain why $\angle A + \angle B = 180^\circ$ and $\angle A = \angle C$.



- (b) COURSE THEOREM: *Opposite sides of a parallelogram are equal.* The diagram shows a parallelogram $ABCD$ with diagonal AC .



- (i) Prove that $\triangle ACB \equiv \triangle CAD$.
 (ii) Hence show that $AB = DC$ and $BC = AD$.
- (c) COURSE THEOREM: *The diagonals of a parallelogram bisect each other.* The diagram shows a parallelogram $ABCD$ with diagonals meeting at M .

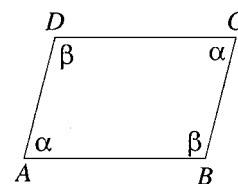


- (i) Prove that $\triangle ABM \equiv \triangle CDM$ (use part (b)).
 (ii) Hence show that $AM = MC$.

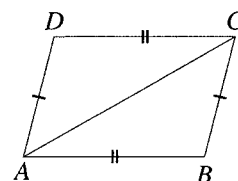
9. TESTS FOR A PARALLELOGRAM: These four theorems give the standard tests for a quadrilateral to be a parallelogram.

- (a) COURSE THEOREM: *If the opposite angles of a quadrilateral are equal, then it is a parallelogram.*

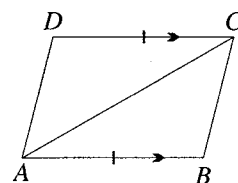
The diagram opposite shows a quadrilateral $ABCD$ in which $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$.



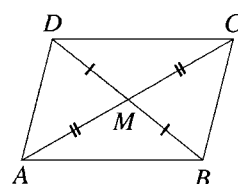
- (i) Prove that $\alpha + \beta = 180^\circ$.
 (ii) Hence show that $AB \parallel DC$ and $AD \parallel BC$.
- (b) COURSE THEOREM: *If the opposite sides of a quadrilateral are equal, then it is a parallelogram.*
 The diagram shows a quadrilateral $ABCD$ in which $AB = DC$ and $AD = BC$, with diagonal AC .



- (i) Prove that $\triangle ACB \equiv \triangle CAD$.
 (ii) Thus prove that $\angle CAB = \angle ACD$, and also that $\angle ACB = \angle CAD$.
 (iii) Hence show that $AB \parallel DC$ and $AD \parallel BC$.
- (c) COURSE THEOREM: *If one pair of opposite sides of a quadrilateral are equal and parallel, then it is a parallelogram.* The diagram shows a quadrilateral $ABCD$ in which $AB = DC$ and $AB \parallel DC$, with diagonal AC .

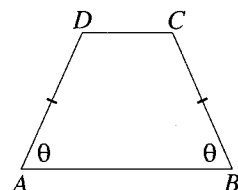


- (i) Prove that $\triangle ACB \equiv \triangle CAD$.
 (ii) Hence show that $AD \parallel BC$.
- (d) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*
 In the diagram, $ABCD$ is a quadrilateral in which the diagonals meet at M , with $AM = MC$ and $BM = MD$.



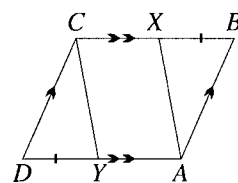
10. In quadrilateral $ABCD$, $\angle BAD = \angle ABC$ and $AD = BC$.

- (a) Prove that $\triangle BAD \equiv \triangle ABC$.
 (b) Why does $\angle ABD = \angle CAB$?
 (c) Show that $\angle DAC = \angle DBC$.
 (d) Prove that $ABCD$ is a trapezium.



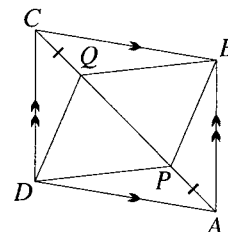
11. In the diagram, $ABCD$ is a parallelogram. The points X and Y lie on BC and AD respectively such that $BX = DY$.

- Explain why $\angle ABX = \angle CDY$.
- Explain why $AB = CD$.
- Show that $\triangle ABX \equiv \triangle CDY$.
- Hence prove that $AYCX$ is a parallelogram.



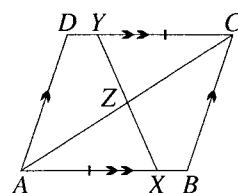
12. The diagram shows the parallelogram $ABCD$ with diagonal AC . The points P and Q lie on this diagonal in such a way that $AP = CQ$.

- Prove that $\triangle ABP \equiv \triangle CDQ$.
- Prove that $\triangle ADP \equiv \triangle CBQ$.
- Hence prove that $BQDP$ is a parallelogram.



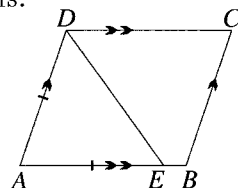
13. The diagram shows the parallelogram $ABCD$ and points X and Y on AB and CD respectively, with $AX = CY$. The diagonal AC intersects XY at Z .

- Prove that $\triangle AXZ \equiv \triangle CYZ$.
- Hence prove that XY is concurrent with the diagonals.



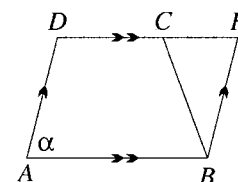
14. The previous two questions could have been solved more easily using the standard properties of and tests for a parallelogram. Explain these alternative proofs.

15. The diagram to the right shows a parallelogram $ABCD$. The point E is constructed on the side AB in such a way that $AD = AE$. Prove that the interval DE bisects the angle $\angle ADC$. [HINT: Begin by letting $\angle ADE = \theta$.]



16. THEOREM: The base angles of a trapezium are equal if and only if the non-parallel sides are equal. Let $ABCD$ be a trapezium with $AB \parallel DC$, but AD not parallel to BC . Construct $BF \parallel AD$ with F on DC , produced if necessary. Let $\angle DAB = \alpha$.

- Suppose first that $AD = BC$.
 - Prove that $BF = AD$.
 - Hence prove that $\angle ABC = \alpha$.
- Conversely, suppose that $\angle ABC = \alpha$.
 - Prove that $\angle BFC = \alpha$.
 - Hence prove that $BC = AD$.



17. The diagonals of quadrilateral $ABCD$ meet at M , and $\triangle ABM \equiv \triangle DCM$.

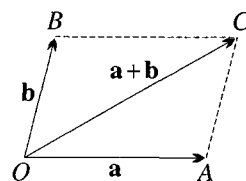
- Draw a diagram showing this information.
- Prove that $ABCD$ is a trapezium with equal base angles.

EXTENSION

18. Quadrilateral $ABCD$ is a parallelogram. A point X is chosen on AB and Y is constructed on DC so that $DX = BY$. Note that DX is not perpendicular to AB .

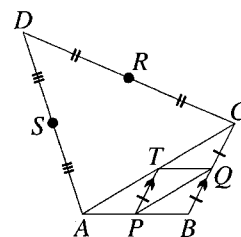
- Given that $DXBY$ is not a parallelogram, draw a picture of the situation.
- What type of quadrilateral is $DXBY$?
- What condition needs to be placed on DX in order to guarantee that $DXBY$ is a parallelogram?

19. **PARALLELOGRAMS AND VECTORS:** The diagram shows two vectors $\mathbf{a}$ and $\mathbf{b}$ starting from O . The parallelogram $OACB$ has been completed so that the diagonal OC represents the vector $\mathbf{a} + \mathbf{b}$. Draw three more parallelograms, each using O as one vertex, so that the diagonals from O represent the vectors: (a) $\mathbf{b} - \mathbf{a}$ (b) $-\mathbf{a} - \mathbf{b}$ (c) $\mathbf{a} - \mathbf{b}$



20. **THEOREM:** *The quadrilateral formed by joining the midpoints of the sides of a quadrilateral is a parallelogram.* In quadrilateral $ABCD$, the points Q , R and S are the midpoints of BC , CD and DA respectively. The two points P and T lie on AB and AC respectively such that $PT = BQ$ and $PT \parallel BQ$.

- (a) Explain why $PBQT$ is a parallelogram.
 (b) Show that the four triangles $\triangle APT$, $\triangle QPT$, $\triangle PBQ$ and $\triangle TQC$ are all congruent, and that P is the midpoint of AB .
 (c) Hence show that the line joining the midpoints of two adjacent sides of a quadrilateral is parallel to the diagonal joining those two sides.
 (d) Hence show that $PQRS$ is a parallelogram.

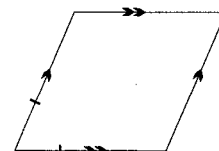


8E Rhombuses, Rectangles and Squares

Rhombuses, rectangles and squares are particular types of parallelograms, and their definitions in this course reflect that understanding. Again, most of the proofs have been encountered in earlier years, and are left to the exercises.

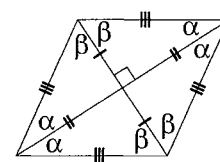
Rhombuses and their Properties and Tests: Intuitively, a rhombus is a 'pushed-over square', but its formal definition is:

- 19 **DEFINITION:** A *rhombus* is a parallelogram with a pair of adjacent sides equal.



As with the parallelogram, the standard properties and tests concern the sides, the vertex angles and the diagonals.

- 20 **COURSE THEOREM:** If a quadrilateral is a rhombus, then:
- all four sides are equal, and
 - the diagonals bisect each other at right angles, and
 - the diagonals bisect each vertex angle.
- Conversely, a quadrilateral is a rhombus if:
- all sides are equal, or
 - the diagonals bisect each other at right angles, or
 - the diagonals bisect each vertex angle.



PROOF: Since a rhombus is a parallelogram, its opposite sides are equal. Since also two adjacent sides are equal, all four sides must be equal.

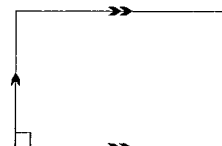
Conversely, suppose that all four sides of a quadrilateral are equal. Since opposite sides are equal, it must be a parallelogram, and since two adjacent sides are equal, it is therefore a rhombus.

This proves the first and fourth points. The remaining proofs are a little more complicated, and are left to the exercises.

Rectangles and their Properties and Tests: A rectangle is also defined as a special type of parallelogram.

21

DEFINITION: A *rectangle* is a parallelogram in which one angle is a right angle.



The standard properties and tests for a rectangle are:

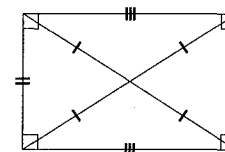
22

COURSE THEOREM: If a quadrilateral is a rectangle, then:

- all four angles are right angles, and
- the diagonals are equal and bisect each other.

Conversely, a quadrilateral is a rectangle if:

- all angles are equal, or
- the diagonals are equal and bisect each other.

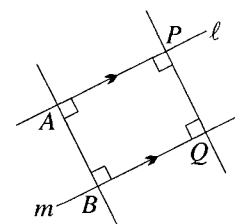


PROOF: Since a rectangle is a parallelogram, its opposite angles are equal and add to 360° . Since one angle is 90° , it follows that all angles are 90° .

Conversely, suppose that all angles of a quadrilateral are equal. Then since they add to 360° , they must each be 90° . Hence the opposite angles are equal, so the quadrilateral must be a parallelogram, and hence is a rectangle.

This proves the first and third points. The remaining proofs are left to structured exercises.

The Distance Between Parallel Lines: Suppose that AB and PQ are two transversals perpendicular to two parallel lines ℓ and m . Then $ABQP$ forms a rectangle, because all its vertex angles are right angles. Hence the opposite sides AB and PQ are equal. This allows a formal definition of the distance between two parallel lines.



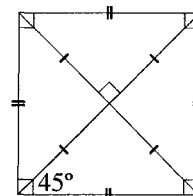
23

DEFINITION: The distance between two parallel lines is the length of a perpendicular transversal.

Squares: Rhombuses and rectangles are different special sorts of parallelograms. A square is simply a quadrilateral that is both a rhombus and a rectangle.

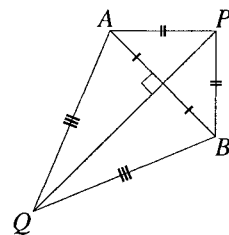
24

DEFINITION: A *square* is a quadrilateral that is both a rhombus and a rectangle.



It follows then from the previous theorems that all sides of a square are equal, all angles are right angles, and the diagonals bisect each other at right angles and meet each side at 45° .

A NOTE ON KITES: Kites are not part of the course, but they occur frequently in problems. A *kite* is usually defined as a quadrilateral in which two pairs of adjacent sides are equal, as in the diagram to the right, where $AP = BP$ and $AQ = BQ$. A question below develops the straightforward proof that the diagonal PQ is the perpendicular bisector of the diagonal AB , and bisects the vertex angles at P and Q . Another question deals with tests for kites.



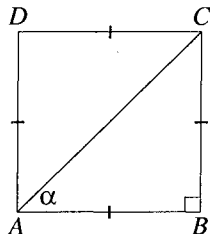
Theorems about kites, however, are not part of the course, and should not be quoted as reasons unless they have been developed earlier in the same question.

Exercise 8E

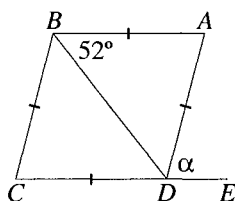
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find α in each of the figures below, giving reasons.

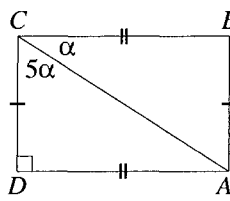
(a)



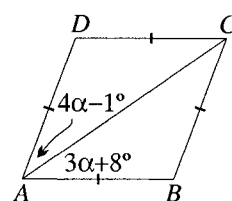
(b)



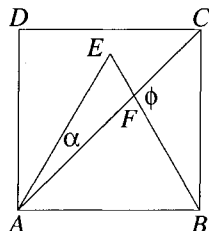
(c)



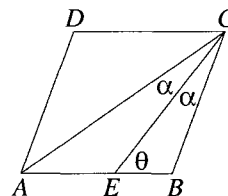
(d)



2. (a)



(b)



Inside the square $ABCD$ is an equilateral $\triangle ABE$. The diagonal AC intersects BE at F . Find the sizes of angles α and ϕ .

$ABCD$ is a rhombus with the diagonal AC shown. The line CE bisects $\angle ACB$. Show that $\theta = 3\alpha$.

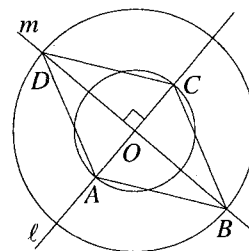
3. (a) What rotation and reflection symmetries does:

(i) every rectangle have, (ii) every rhombus have, (iii) every square have?

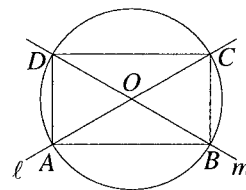
(b) What rotation and reflection symmetries does a circle have?

4. **CONSTRUCTIONS:** Constructing a rectangle, rhombus and square from their diagonals.

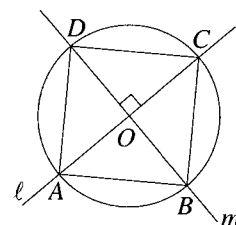
- (a) **RHOMBUS:** Construct any two perpendicular lines ℓ and m , and let them meet at O . Construct two circles C and D with the common centre O . Let ℓ meet C at A and C , and let m meet D at B and D . Use the standard tests for a rhombus to explain why the quadrilateral $ABCD$ is a rhombus.



- (b) **RECTANGLE:** Construct any two non-parallel lines ℓ and m , and let them meet at O . Construct a circle C with centre O and any radius. Let ℓ meet C at A and C , and let m meet C at B and D . Use the standard tests for a rectangle to explain why the quadrilateral $ABCD$ is a rectangle.



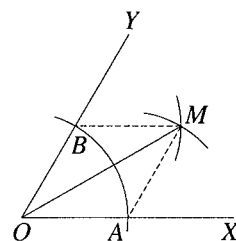
- (c) **SQUARE:** Construct any two perpendicular lines ℓ and m , and let them meet at O . Construct a circle C with centre O and any radius. Let ℓ meet C at A and C , and let m meet C at B and D . Use the standard tests for a square to explain why the quadrilateral $ABCD$ is a square.



5. CONSTRUCTION: *The bisector of a given angle.*

Given an angle $\angle XOY$, construct an arc with centre O and any radius meeting the arms OX and OY at A and B respectively. With the same radius and with centres A and B , construct two further arcs meeting at M .

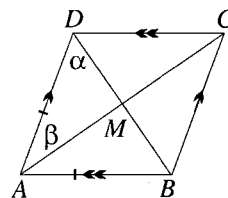
- (a) Why is the quadrilateral $OAMB$ a rhombus?
 (b) Hence prove that OM bisects $\angle XOY$.



DEVELOPMENT

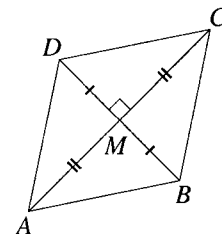
6. COURSE THEOREM: *The diagonals of a rhombus bisect each other at right angles, and bisect the vertex angles.* In the diagram, the diagonals of the rhombus $ABCD$ meet at M . Since a rhombus is a parallelogram, we already know that the diagonals bisect each other.

- (a) Let $\alpha = \angle ADB$. Explain why $\angle ABD = \alpha$.
 (b) Hence prove that $\angle CDB = \alpha$.
 (c) Let $\beta = \angle DAC$. Prove that $\angle BAC = \beta$.
 (d) Hence prove that $AC \perp BD$. (There is no need for congruence in this situation.)

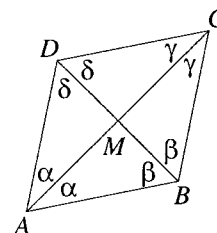


7. TESTS FOR A RHOMBUS: The following three parts are structured proofs of the standard tests for a rhombus listed in the notes above.

- (a) **COURSE THEOREM:** *If all sides of a quadrilateral are equal, then it is a rhombus.* Explain, using the previous theorems and the definition of a rhombus, why a quadrilateral with all sides equal must be a rhombus.
- (b) **COURSE THEOREM:** *If the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.* The diagram shows a quadrilateral $ABCD$ in which the diagonals bisect each other at right angles at M .
- (i) What previous theorem proves that the quadrilateral $ABCD$ is a parallelogram?
- (ii) Prove that $\triangle AMD \equiv \triangle AMB$, and hence that $AD = AB$. The quadrilateral $ABCD$ is then a rhombus by definition.
- (c) **COURSE THEOREM:** *If the diagonals of a quadrilateral bisect each vertex angle, then it is a rhombus.* The diagram shows a quadrilateral $ABCD$ in which the diagonals bisect each vertex angle. Let α, β, γ and δ be as shown.

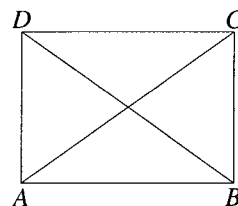


- (i) Prove that $\alpha + \beta + \gamma + \delta = 180^\circ$.
- (ii) By taking the sum of the angles in $\triangle ABC$ and $\triangle ADC$, prove that $\beta = \delta$.
- (iii) Similarly, prove that $\alpha = \gamma$, and state why $ABCD$ is a parallelogram.
- (iv) Finally, prove that $AB = AD$.



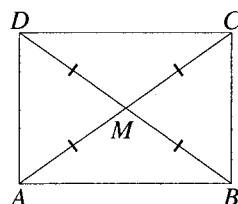
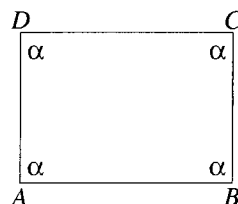
8. PROPERTIES OF RECTANGLES: The following two parts are structured proofs of the standard properties of a rectangle listed in the notes above.

- (a) COURSE THEOREM: *All the angles in a rectangle are right angles.* Use the definition of a rectangle — as a parallelogram with one angle a right angle — and the properties of a parallelogram to prove that all four angles of a rectangle are right angles.
- (b) COURSE THEOREM: *The diagonals of a rectangle are equal and bisect each other.* The diagram shows a rectangle $ABCD$, with diagonals drawn.

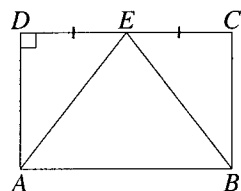


9. TESTS FOR A RECTANGLE: The following two parts are structured proofs of the standard tests for a rectangle listed in the notes above.

- (a) COURSE THEOREM: *If all angles of a quadrilateral are equal, then it is a rectangle.* The diagram shows a quadrilateral $ABCD$ in which all angles are equal.
- (i) Prove that all angles are right angles.
- (ii) Hence prove that $ABCD$ is a rectangle.
- (b) COURSE THEOREM: *If the diagonals of a quadrilateral are equal and bisect each other, then it is a rectangle.* The diagram shows a quadrilateral $ABCD$ in which the diagonals, meeting at M , are equal and bisect each other.
- (i) Explain why $ABCD$ is a parallelogram.
- (ii) Let $\alpha = \angle BAM$, and explain why $\angle ABM = \alpha$.
- (iii) Let $\beta = \angle MBC$, and explain why $\angle MCB = \beta$.
- (iv) Using the angle sum of the triangle ABC , prove that $\angle ABC = 90^\circ$.



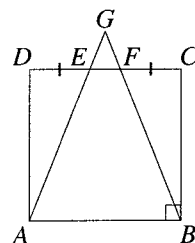
10. (a)



The point E is the midpoint of the side CD of the rectangle $ABCD$.

- (i) Prove that $\triangle BCE \equiv \triangle ADE$.
- (ii) Hence show that $\triangle ABE$ is isosceles.

(b)



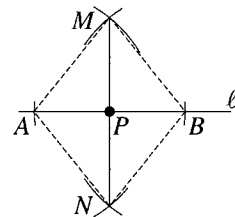
The points E and F are on the side CD in the square $ABCD$, with $CF = DE$. Produce AE and BF to meet at G .

- (i) Prove that $\triangle BCF \equiv \triangle ADE$.
- (ii) Hence show that $\triangle ABG$ is isosceles.

11. CONSTRUCTION: A right angle at a point on a line.

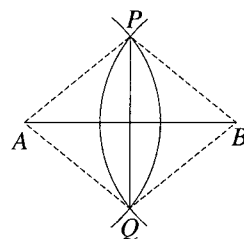
Given a point P on a line ℓ , construct an arc with centre P meeting ℓ at A and B . With increased radius, construct arcs with centres at A and B meeting at M and N .

- (a) Why is the quadrilateral $AMBN$ a rhombus?
 (b) Hence prove that P lies on MN and $MN \perp AB$.

**12. CONSTRUCTION:** The perpendicular bisector of an interval.

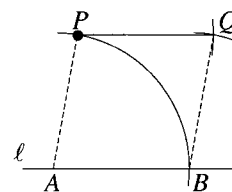
Given an interval AB , construct arcs of the same radius, greater than $\frac{1}{2}AB$, with centres at A and B . Let the arcs meet at P and Q .

- (a) Why is the quadrilateral $APBQ$ a rhombus?
 (b) Hence prove that PQ bisects AB and $PQ \perp AB$.

**13. CONSTRUCTION:** The line parallel to a given line through a given point.

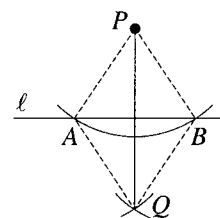
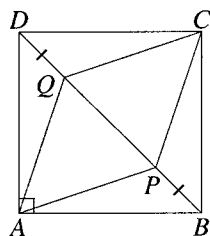
Given a line ℓ and a point P not on ℓ , choose a point A on ℓ . With centre A and radius AP , construct an arc meeting ℓ at B . With the same radius, draw arcs with centres at B and P meeting at Q .

- (a) Why is the quadrilateral $APQB$ a rhombus?
 (b) Hence prove that $PQ \parallel \ell$.

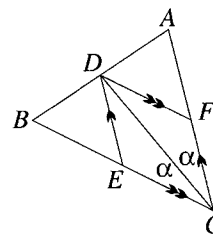
**14. CONSTRUCTION:** The line perpendicular to a given line through a given point.

Given a line ℓ and a point P not on ℓ , construct an arc with centre P meeting ℓ at A and B . With the same radius, draw arcs with centres at A and B , intersecting at Q .

- (a) Why is the quadrilateral $APBQ$ a rhombus?
 (b) Hence prove that $PQ \perp \ell$.

**15. (a)**

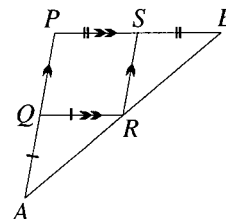
- P and Q lie on the diagonal BD of square $ABCD$, and $BP = DQ$. (i) Prove that $\triangle ABP \equiv \triangle CBP \equiv \triangle ADQ \equiv \triangle CDQ$.
 (ii) Hence show that $APCQ$ is a rhombus.

(b)

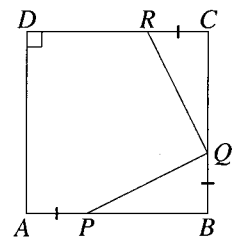
- In the triangle ABC , DC bisects $\angle BCA$, $DE \parallel AC$ and $DF \parallel BC$.
 (i) Explain why is $DECF$ a parallelogram.
 (ii) Show that $DECF$ is a rhombus.

16. The parallelogram $PQRS$ is inscribed in $\triangle PBA$ with R on AB . It is found that $QA = QR$ and $PS = SB$.

- (a) Prove that $\triangle BSR \equiv \triangle RQA$.
 (b) Hence prove that $PQRS$ is a rhombus.

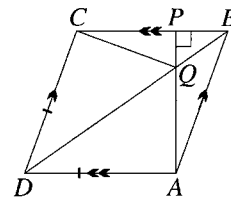
**17.** In the square $ABCD$, P is on AB , Q is on BC and R is on CD , with $AP = BQ = CR$.

- (a) Prove that $\triangle PBQ \equiv \triangle QCR$.
 (b) Prove that $\angle PQR$ is a right angle.



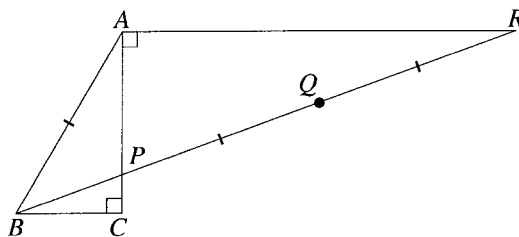
18. In the rhombus $ABCD$, AP is constructed perpendicular to BC and intersects the diagonal BD at Q .

- State why $\angle ADB = \angle CDB$.
- Prove that $\triangle AQD \equiv \triangle CQD$.
- Show that $\angle DAQ$ is a right angle.
- Hence find $\angle QCD$.



19. The triangles ABC and APR are both right-angled at the vertices marked in the diagram. The midpoint of PR is Q , and it is found that $PQ = QR = AB$.

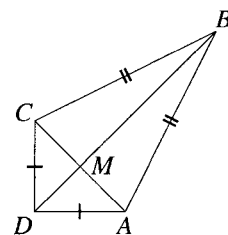
- Explain why $\angle PBC = \angle PRA$.
- Construct the point S that completes the rectangle $APSR$. Explain why Q is also the midpoint of AS and why $PQ = AQ$.
- Hence prove that $\angle PBA = 2 \times \angle PBC$.



20. TWO THEOREMS ABOUT KITES: A kite is defined to be a quadrilateral in which two pairs of adjacent sides are equal. [NOTE: This definition is not part of the course.]

- THEOREM: The diagonals of a kite are perpendicular and one bisects the other.
The diagram shows a kite $ABCD$ with $AB = BC$ and $AD = DC$. The diagonals AC and BD intersect at M .

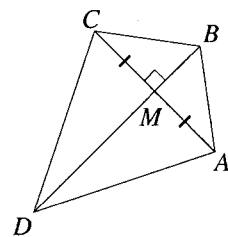
- Prove that $\triangle BAD \equiv \triangle BCD$.
- Use the properties of the isosceles triangle ABC to prove that DB bisects AC at right angles.



- THEOREM: If the diagonals of a quadrilateral are perpendicular, and one is bisected by the other, then the quadrilateral is a kite.

The diagram shows a quadrilateral with perpendicular diagonals meeting at M , and $AM = MC$.

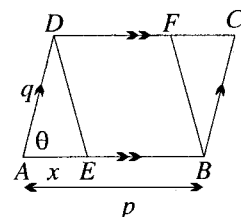
- Prove that $\triangle BAM \equiv \triangle BCM$.
- Hence prove that $BA = BC$.
- Similarly, prove that $DA = DC$.



EXTENSION

21. TRIGONOMETRY: The quadrilateral $ABCD$ is a parallelogram with $AB = p$, $AD = q$, $p > q$, and $\angle BAD = \theta$. The points E on AB and F on CD are chosen so that $EBFD$ is a rhombus. Let $AE = x$. Show that

$$x = \frac{p^2 - q^2}{2(p - q \cos \theta)}.$$



8F Areas of Plane Figures

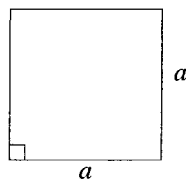
The standard area formulae are well known. Some of them were used in the development of the definite integral, which extended the idea of area to regions with curved boundaries. The formulae below apply to figures with straight edges, and their proofs by dissection are reviewed below.

Course Theorem — Area Formulae for Quadrilaterals and Triangles: The various area formulae are based on the definition of the area of a rectangle as length times breadth, and on the assumption that area remains constant when regions are dissected and rearranged. The first two formulae below are therefore definitions. The other four formulae can be proven using the diagrams below, which need to be studied until the logic of each dissection becomes clear.

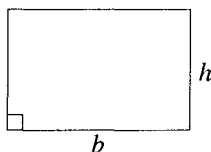
STANDARD AREA FORMULAE:

- 25**
- **SQUARE:** area = (side length)²
 - **RECTANGLE:** area = (length) × (breadth)
 - **PARALLELOGRAM:** area = (base) × (perpendicular height)
 - **TRIANGLE:** area = $\frac{1}{2} \times (\text{base}) \times (\text{perpendicular height})$
 - **RHOMBUS:** area = $\frac{1}{2} \times (\text{product of the diagonals})$
 - **TRAPEZIUM:** area = (average of parallel sides) × (perpendicular height)

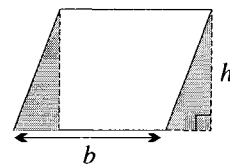
PROOF:



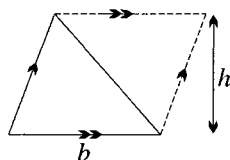
Square: area = a^2



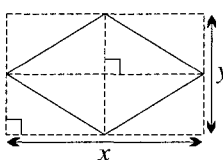
Rectangle: area = bh



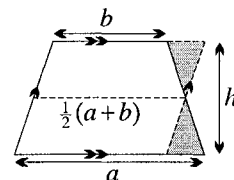
Parallelogram: area = bh



Triangle: area = $\frac{1}{2}bh$



Rhombus: area = $\frac{1}{2}xy$



Trapezium: area = $\frac{1}{2}h(a + b)$

NOTE: Because rhombuses are parallelograms, their areas can also be calculated using the formula area = (base) × (perpendicular height) associated with parallelograms. The formula area = $\frac{1}{2} \times (\text{product of the diagonals})$ gives another, and quite different, approach that is often forgotten in problems.

Because squares are rhombuses, their area can also be calculated using their diagonals. But the diagonals of a square are equal, so the formula becomes

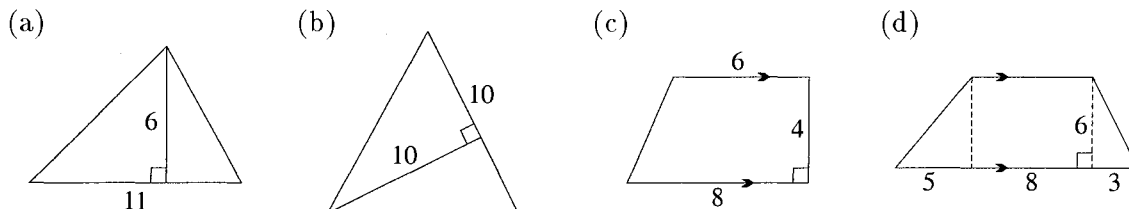
$$\text{area of square} = \frac{1}{2} \times (\text{square of the diagonal}).$$

The Area of the Circle: The area of a circle is πr^2 , where r is the radius. The proof of this result was discussed in Section 11B of the Year 11 volume as a preliminary to integration — because the boundary is curved, some sort of infinite dissection is necessary, and the proof therefore belongs to the theory of the definite integral.

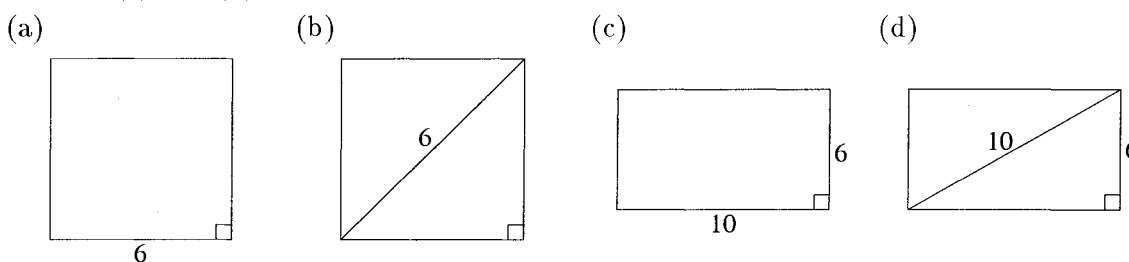
Exercise 8F

NOTE: The calculation of areas is so linked with Pythagoras' theorem that it is inconvenient to separate them in exercises. Pythagoras' theorem has therefore been used freely in the questions of this exercise, although its formal review is in the next section.

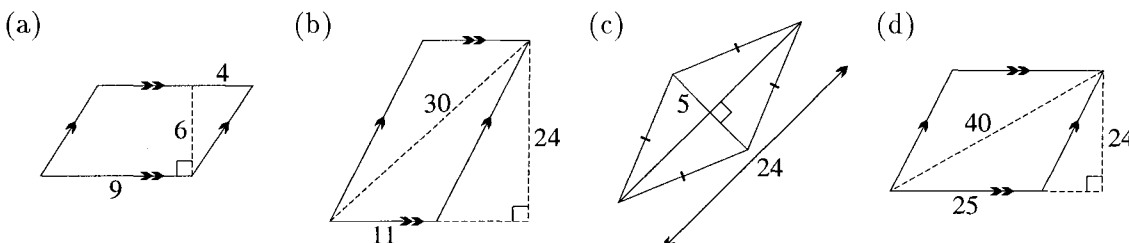
1. Find the areas of the following figures:



2. Find the area A and the perimeter P of the squares in parts (a) and (b) and the rectangles in parts (c) and (d). Use Pythagoras' theorem to find missing lengths where necessary.



3. Find the area A and the perimeter P of the following figures, using Pythagoras' theorem where necessary. Then find the lengths of any missing diagonals.



4. (a) Explain why the area of a square is half the square of the diagonal.
 (b) Show that the area of a rectangle with sides a and b is the same as the area of the square whose side length s is the geometric mean $\sqrt{ab}$ of the sides of the rectangle.
 (c) THE TWO AREA FORMULAE FOR TRIANGLES: Let $\triangle ABC$ be right-angled at C . Explain why the formula $A = \frac{1}{2} \times (\text{base}) \times (\text{height})$ for the area of the triangle is identical to the trigonometric area formulae $A = \frac{1}{2}ab \sin C$.

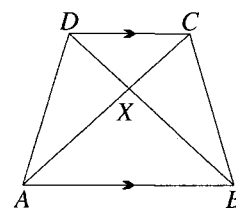
DEVELOPMENT

5. THEOREM: A median of a triangle divides the triangle into two triangles of equal area. Sketch a triangle ABC . Let M be the midpoint of BC , and join the median AM .

- (a) Explain why $\triangle ABM$ and $\triangle ACM$ have the same perpendicular height.
 (b) Hence explain why $\triangle ABM$ and $\triangle ACM$ have the same area.

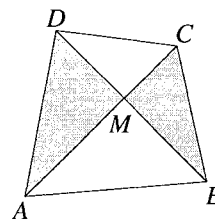
6. THEOREM: The two triangles formed by the diagonals and the non-parallel sides of a trapezium have the same area. In the trapezium $ABCD$, $AB \parallel DC$ and AC intersects BD at X .

- (a) Explain why $\text{area } \triangle ABC = \text{area } \triangle ABD$.
 (b) Hence explain why $\text{area } \triangle BCX = \text{area } \triangle ADX$.

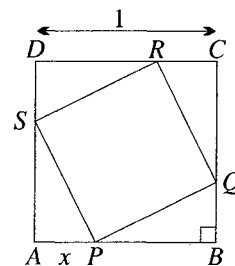


7. THEOREM: Conversely, a quadrilateral in which the diagonals form a pair of opposite triangles of equal area is a trapezium. The diagonals of the quadrilateral $ABCD$ meet at M , and $\triangle AMD$ and $\triangle BMC$ have equal areas.

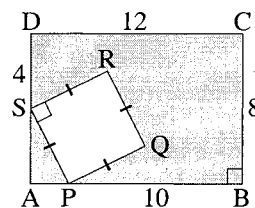
- (a) Prove that $\triangle ABD$ and $\triangle ABC$ have equal areas.
 (b) Hence prove that $AB \parallel DC$.



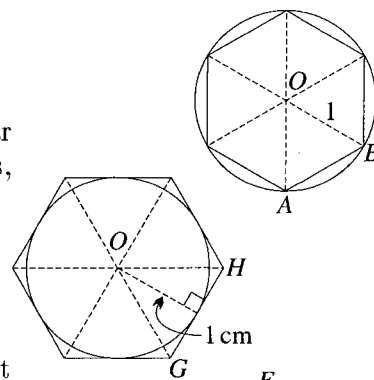
8. Prove that the four small triangles formed by the two diagonals of a parallelogram all have the same area. Under what circumstances are they all congruent?
9. The diagonals of a parallelogram form the diameters of two circles.
 (a) Why are they concentric?
 (b) If the diagonals are in the ratio $a : b$, what is the ratio of the areas of the circles?
 (c) Under what circumstances do the circles coincide?
10. In the diagram to the right, $ABCD$ and $PQRS$ are squares, and $AB = 1$ metre. Let $AP = x$.
 (a) Find an expression for the area of $PQRS$ in terms of x .
 (b) What is the minimum area of $PQRS$, and what value of x gives this minimum?
 (c) Explain why the result is the same if the total area of the four triangles is maximised.



11. The diagram shows a rectangle with a square offset in one corner. All dimensions shown are in metres.
 (a) Find the area of the square.
 (b) Hence find the shaded area outside the square.

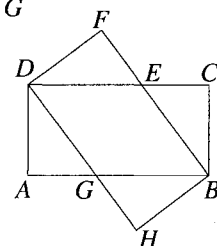


12. (a) The diagram shows a regular hexagon inscribed in a circle of radius 1 and centre O .
 (i) Find the area of $\triangle AOB$.
 (ii) Hence find the area of the hexagon.
 (b) The second diagram on the right shows another regular hexagon escribed around a circle of radius 1, that is, each side is tangent to the circle.
 (i) Find the area of $\triangle OGH$.
 (ii) Find the area of the hexagon.
 (c) Hence explain why $\frac{3}{2}\sqrt{3} < \pi < 2\sqrt{3}$.



13. In the diagram opposite, $ABCD$ and $BFDH$ are congruent rectangles with $AB = 8$ and $BC = 6$.

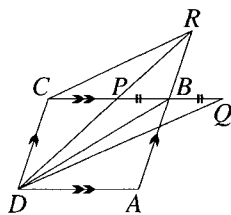
- (a) Explain why $\triangle ADG \equiv \triangle HBG$.
 (b) Show that $AG = \frac{AB^2 - AD^2}{2AB}$ by using Pythagoras' theorem, and hence find AG .
 (c) Hence find the area of $BEDG$.



14. A parallelogram has sides of length a and b , and one vertex angle has size θ .
 (a) Show that the area of the parallelogram is $A = ab \sin \theta$.
 (b) Use the cosine rule to find the squares on the diagonals in terms of a , b and θ .
 (c) Circles are drawn with the two diagonals as diameters. What is the area of the annulus between the two circles?

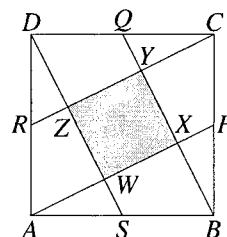
EXTENSION

15. (a)



The diagram above shows a parallelogram $ABCD$. The point P lies on the side BC , and the side CB is produced to Q so that $BQ = BP$. The intervals AB and DP are produced so that they intersect at R . Show that the areas of $\triangle DQB$ and $\triangle CPR$ are equal.

(b)

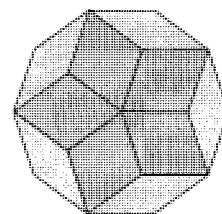


The diagram above shows a square $ABCD$ with the midpoints of each side being P , Q , R and S as shown. The intervals AP , BQ , CR and DS intersect at W , X , Y and Z as shown. Find the ratio of the areas of the small square $WXYZ$ and the large square $ABCD$.

16. The diagram shows the tessellation of a decagon by two types of rhombus, one fat and the other thin. The lengths of the sides of each rhombus and the decagon are all 1 cm.

- (a) Find both angles in each rhombus, and confirm that the interior angle at each vertex of the decagon is correct.
 (b) Hence show that the area of the decagon is

$$A = 5 \sin 36^\circ (2 \cos 36^\circ + 1) \text{ cm}^2.$$



17. A DIFFICULT EQUAL-AREA PROBLEM: The diagram shows the design of the clock-face on the stone towers at Martin Place and Central Railway Station in Sydney. The design within the inner circle seems to be based on dividing it into 24 regions of equal area. Let the inner circle have radius 1.

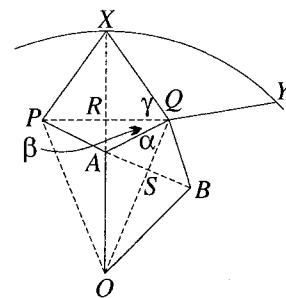
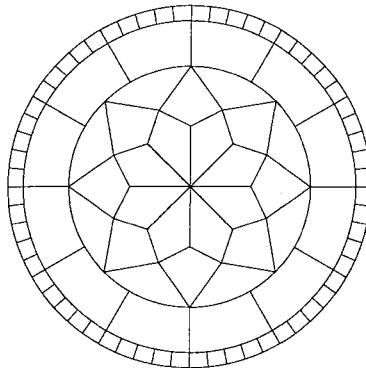
- (a) Show that the radius OA of the circle through the eight points inside the circle where three edges meet is $\frac{1}{2}$.
 (b) Use the area of the kite $APXQ$ to show that $RQ = \frac{\pi}{12}$.
 (c) Use the kite $OAQB$ to find the radius OQ of the circle through the eight points where four edges meet.
 (d) Show that $\cos \frac{\pi}{8} = \frac{1}{2} \sqrt{2 + \sqrt{2}}$, and hence find OR .
 (e) Find the lengths AR and RX , and hence show that

$$\tan \beta = \frac{\pi(\sqrt{2} + 1) - 6}{\pi} \quad \text{and} \quad \tan \gamma = \frac{12 - \pi(\sqrt{2} + 1)}{\pi}.$$

- (f) Use AB and the area of $\triangle ABQ$ to show that

$$\tan \alpha = \frac{6 - 3\sqrt{2}}{2\pi - 3\sqrt{2}}.$$

- (g) Hence find the angle between opposite edges at the eight points where four edges meet, correct to the nearest minute.



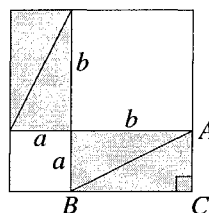
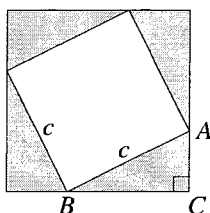
8G Pythagoras' Theorem and its Converse

Pythagoras' theorem hardly needs introduction, having been the basis of so much of the course. But its proof needs attention, and the converse theorem and its interesting proof by congruence will be new for many students.

Pythagoras' Theorem: The following proof by dissection of Pythagoras' theorem is very quick, and is one of hundreds of known proofs.

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PYTHAGORAS' THEOREM: In a right triangle, the square on the hypotenuse equals the sum of the squares on the other two sides.



GIVEN: Let $\triangle ABC$ be a right triangle with $\angle C = 90^\circ$.

AIM: To prove that $AC^2 + BC^2 = AB^2$.

CONSTRUCTION: As shown.

PROOF: Behold! (To quote an Indian text — is anything further required?)

Pythagorean Triads: A *Pythagorean triad* consists of three positive integers a , b and c such that $a^2 + b^2 = c^2$. For example,

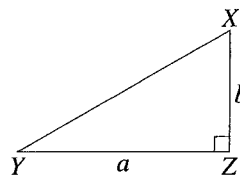
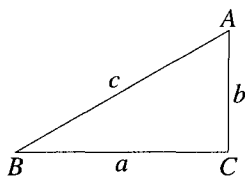
$$3^2 + 4^2 = 5^2 \quad \text{and} \quad 5^2 + 12^2 = 13^2,$$

so 3, 4, 5 and 5, 12, 13 are Pythagorean triads. Such triads are very convenient, because they can be the side lengths of a right triangle. An extension question below gives a complete list of Pythagorean triads.

Converse of Pythagoras' Theorem: The converse of Pythagoras' theorem is also true, and its proof is an application of congruence. The proof of the converse uses the forward theorem, and is consequently rather subtle.

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CONVERSE OF PYTHAGORAS' THEOREM: If the sum of the squares on two sides of a triangle equals the square on the third side, then the angle included by the two sides is a right angle.



GIVEN: Let ABC be a triangle whose sides satisfy the relation $a^2 + b^2 = c^2$.

AIM: To prove that $\angle C = 90^\circ$.

CONSTRUCTION: Construct $\triangle XYZ$ in which $\angle Z = 90^\circ$, $YZ = a$ and $XZ = b$.

PROOF: Using Pythagoras' theorem in $\triangle XYZ$,

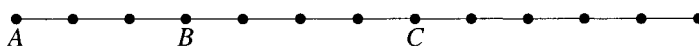
$$\begin{aligned} XY^2 &= a^2 + b^2 \quad (\text{because } XY \text{ is the hypotenuse}) \\ &= c^2 \quad (\text{given}), \end{aligned}$$

so $XY = c$.

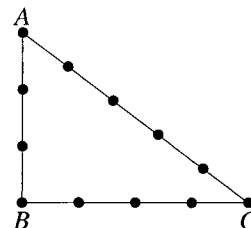
Hence the triangles ABC and XYZ are congruent by the SSS test,

and so $\angle C = \angle Z = 90^\circ$ (matching angles of congruent triangles).

WORKED EXERCISE: A long rope is divided into twelve equal sections by knots along its length. Explain how it can be used to construct a right angle.



SOLUTION: Let A be one end of the rope. Let B be the point 3 units along, and let C be the point a further 4 units along. Join the two ends of the rope, and stretch the rope into a triangle with vertices A , B and C . Then since 3, 4, 5 is a Pythagorean triad, the triangle will be right-angled at B .



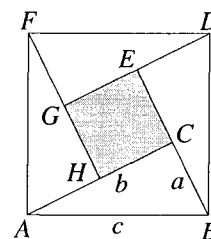
Exercise 8G

- Which of the following triplets are the sides of a right-angled triangle?
(a) 30, 24, 18 (b) 28, 24, 15 (c) 26, 24, 10 (d) 25, 24, 7 (e) 24, 20, 13
- Find the unknown side of each of the following right-angled triangles with base b , altitude a and hypotenuse c . Leave your answer in surd form where necessary.
(a) $a = 12$, $b = 5$ (b) $a = 4$, $b = 5$ (c) $b = 15$, $c = 20$ (d) $a = 3$, $c = 7$
- PYTHAGORAS' THEOREM AND THE COSINE RULE:** Let ABC be a triangle right-angled at C . Write down, with c^2 as subject, the cosine rule and Pythagoras' theorem, and explain why they are identical.
- A paddock on level ground is 2 km long and 1 km wide. Answer these questions, correct to the nearest second.
(a) If a farmer walks from one corner to the opposite corner along the fences in 40 minutes, how long will it take him if he walks across the diagonal?
(b) If his assistant jogs along the diagonal in 15 minutes, how long will it take him if he jogs along the fences?
- (a) Use Pythagoras' theorem to find an equation for the altitude a of an isosceles triangle with base $2b$ and equal legs s . Hence find the area of an isosceles triangle with:
(i) equal legs 15 cm and base 24 cm, (ii) equal legs 18 cm and base 20 cm.
(b) Write down the altitude in the special case where $s = 2b$. What type of triangle is this and what is its area?
- (a) The diagonals of a rhombus are 16 cm and 30 cm. (i) What are the lengths of the sides? (ii) Use trigonometry to find the vertex angles, correct to the nearest minute.
(b) A rhombus with 20 cm sides has a 12 cm diagonal. How long is the other diagonal?
(c) One diagonal of a rhombus is 20 cm, and its area is 100 cm^2 .
(i) How long is the other diagonal? (ii) How long are its sides?
- The sides of a rhombus are 5 cm, and its area is 24 cm^2 .
(a) Let the diagonals have lengths $2x$ and $2y$, and show that $xy = 12$ and $x^2 + y^2 = 25$.
(b) Solve for x and hence find the lengths of the diagonals.

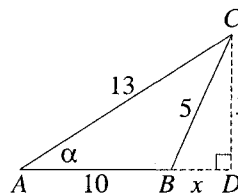
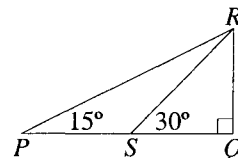
8. (a) THE t -FORMULAE: Two sides of a right-angled triangle are $2t$ and $t^2 - 1$.
 (i) Show that the hypotenuse is $t^2 + 1$. (ii) What are the two possible lengths of the hypotenuse if another side of the triangle is 8 cm?
 (b) Show that if a and b are integers with $b < a$, then $a^2 - b^2$, $2ab$, $a^2 + b^2$ is a Pythagorean triad. Then generate and check the Pythagorean triads given by:
 (i) $a = 2$, $b = 1$ (ii) $a = 3$, $b = 2$ (iii) $a = 4$, $b = 3$ (iv) $a = 7$, $b = 4$

DEVELOPMENT

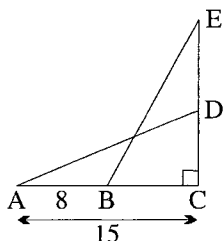
9. COURSE THEOREM: An alternative proof of Pythagoras' theorem. The triangle ABC is right-angled at C . Let the sides be $AB = c$, $BC = a$ and $CA = b$, with $b > a$. The triangles BDE , DFG and FAH are congruent to $\triangle ABC$.



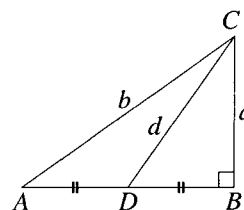
- (a) Explain why $HC = b - a$.
 (b) Find, in terms of the sides a , b and c , the areas of:
 (i) the square $ABDF$, (ii) the square $CEHG$,
 (iii) the four triangles.
 (c) Hence show that $a^2 + b^2 = c^2$.
10. Let AD be an altitude of $\triangle ABC$, and suppose that $BD = p^2$, $CD = q^2$ and $AD = pq$.
 (a) Find AB^2 and AC^2 . (b) Hence show that $\angle A = 90^\circ$.
11. In the diagram, $\triangle PQR$ and $\triangle QRS$ are both right-angled at Q , with $\angle RPQ = 15^\circ$ and $\angle RSQ = 30^\circ$.
 (a) Find $\angle PRS$ and hence show that $PS = RS$.
 (b) Given that $QR = 1$ unit, write down the lengths of QS and RS and deduce that $\tan 15^\circ = 2 - \sqrt{3}$.
12. (a) In triangle ABC , $AB = 10$, $BC = 5$ and $AC = 13$. The altitude is $CD = y$. Let $BD = x$ and $\angle A = \alpha$.
 (b) Use Pythagoras' theorem to write down a pair of equations for x and y .
 (c) Solve for x , and hence find $\cos \alpha$ without finding α .



13. (a)



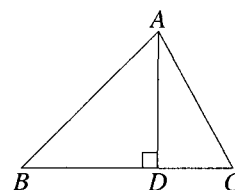
- (b)



In the diagram, $AD = BE = 25$. $AB = 8$ and $AC = 15$. Find the length of DE .

In the right-angled $\triangle ABC$, the point D bisects the base. Show that $4d^2 = b^2 + 3a^2$.

14. The altitude through A in $\triangle ABC$ meets the opposite side BC at D . Use Pythagoras' theorem in $\triangle ADB$ and $\triangle ADC$ to show that $AB^2 + DC^2 = AC^2 + BD^2$.



15. Triangle ABC is right-angled at A . Show that:
 (a) $(b + c)^2 - a^2 = a^2 - (b - c)^2$
 (b) $(a + b + c)(-a + b + c) = (a - b + c)(a + b - c)$

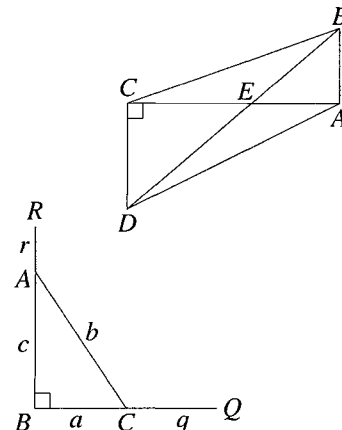
16. The quadrilateral $ABCD$ is a parallelogram with diagonal AC perpendicular to CD . The two diagonals intersect at E . Use Pythagoras' theorem to show that

$$DE^2 + 3EA^2 = AD^2.$$

[HINT: Begin by letting $CA = 2d$, $CD = a$ and $DA = c$.]

17. The triangle ABC has a right angle at B and the sides opposite the respective vertices are a , b and c . The side BC is produced a distance q to Q while BA is produced a distance r to R . Show that

$$QA^2 + RC^2 = QR^2 + AC^2.$$

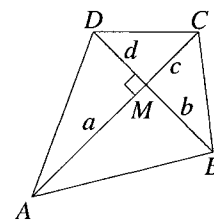


18. VARIANTS OF PYTHAGORAS' THEOREM:

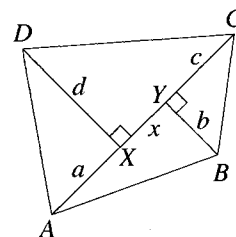
- Use Pythagoras' theorem to prove that the semicircle on the hypotenuse of a right-angled triangle equals the sum of the semicircles on the other two sides.
- Prove that the equilateral triangle on the hypotenuse of a right-angled triangle equals the sum of the equilateral triangles on the other two sides.

19. THEOREM: *If the diagonals of a quadrilateral are perpendicular, then the sums of squares on opposite sides are equal.* Let $ABCD$ be a quadrilateral, with diagonals meeting at right angles at M .

- Find expressions for AB^2 , BC^2 , CD^2 and AD^2 in terms of a , b , c and d .
- Hence show that $AB^2 + CD^2 = BC^2 + AD^2$.



20. THEOREM: *Conversely, if the sums of squares of opposite sides of a quadrilateral are equal, then the diagonals are perpendicular.* Let $ABCD$ be a quadrilateral in which $AB^2 + CD^2 = AD^2 + BC^2$. Let X and Y be the feet of the perpendiculars from B and D respectively to AC . Let $AX = a$, $BY = b$, $CY = c$, $DX = d$ and $XY = x$.

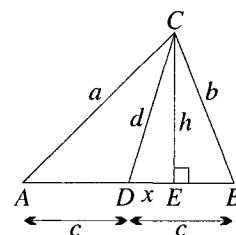


- Use Pythagoras' theorem to show that

$$a^2 + b^2 + c^2 + d^2 = (a + x)^2 + b^2 + (c + x)^2 + d^2.$$

- Hence show that $x = 0$ and $AC \perp BD$.

21. APOLLONIUS' THEOREM: *The sum of the squares on two sides of a triangle is equal to twice the sum of the square on half the third side and the square on the median to the third side.* The diagram shows $\triangle ABC$ with $AC = a$, $BC = b$ and $AB = 2c$. The median CD has length d . Let the altitude CE have length h , and let $DE = x$.

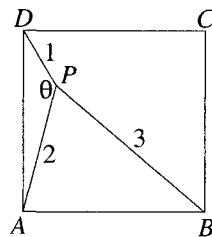


- Use Pythagoras' theorem to write down three equations.
- Eliminate h and x from these equations, and hence show that $a^2 + b^2 = 2(c^2 + d^2)$, as required.

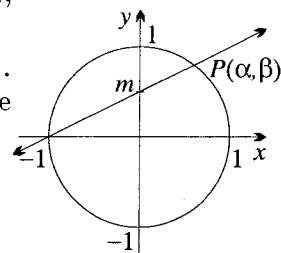
EXTENSION

22. (a) Show that $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$.
 (b) Hence show that the set of integers that are the sum of two squares is closed under multiplication. (That is, prove that if two integers are each the sum of two squares, then their product is also the sum of two squares.)

23. The diagram shows a square $ABCD$ with a point P inside it which is 1 unit from D , 2 units from A and 3 units from B . Let $\angle APD = \theta$.



- (a) Show that $\theta = \frac{3\pi}{4}$.
 (b) Show that if P is outside the square, then $\theta = \frac{\pi}{4}$.
 (c) Is the situation possible if P is 3 units from C instead of from B ?
24. A QUESTION MORE EASILY DONE BY COORDINATE GEOMETRY:
 (a) The points P and Q divide a given interval AB internally and externally respectively in the ratio $1 : 2$. The point X lies on the circle with diameter PQ . Prove that $AX : XB = 1 : 2$.
 [HINT: Drop the perpendicular from X to AB , and use Pythagoras' theorem.]
 (b) Now suppose that P and Q divide the given interval AB internally and externally respectively in the ratio $1 : \lambda$. Prove that $AX : XB = 1 : \lambda$.
 (c) Repeat part (b) using coordinate geometry with the origin at A .
25. PYTHAGOREAN TRIADS: Suppose that $a^2 + b^2 = c^2$, where a , b and c are integers.
 (a) Prove that one of a and b is even, and the other odd. [HINT: Find all possible remainders when the square of each number is divided by 4.]
 (b) Prove that one of the three integers is divisible by 5. [HINT: Find all the possible remainders when the square of a number is divided by 5.]
26. A LIST OF ALL PYTHAGOREAN TRIADS: A Pythagorean triad a , b , c is called *primitive* if there is no common factor of a , b and c .
 (a) Show that every Pythagorean triad is a multiple of a primitive Pythagorean triad.
 (b) Show that if a , b , c is a Pythagorean triad, then the point $P(\alpha, \beta)$, where $\alpha = a/c$ and $\beta = b/c$, lies on the unit circle $x^2 + y^2 = 1$.
 (c) Suppose that $m = p/q$ is any rational gradient between 0 and 1. Show that the line with gradient m through $M(-1, 0)$ meets the unit circle $x^2 + y^2 = 1$ again at $P(\alpha, \beta)$, where
- $$\alpha = \frac{1 - m^2}{1 + m^2} \text{ and } \beta = \frac{2m}{1 + m^2} \text{ are both rational,}$$
- and hence show that $q^2 - p^2$, $2pq$, $q^2 + p^2$ is a Pythagorean triad.
 (d) Show that if the integers p and q in part (c) are relatively prime and not both odd, then $q^2 - p^2$, $2pq$, $q^2 + p^2$ is a primitive Pythagorean triad.
 (e) Show that part (d) is a complete list of primitive Pythagorean triads.



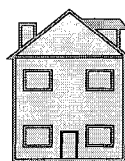
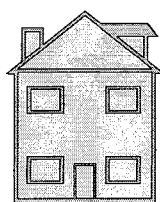
8H Similarity

Similarity generalises the study of congruence to figures that have the same shape but not necessarily the same size. Its formal definition requires the idea of an *enlargement*, which is a stretching in all directions by the same factor.

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SIMILARITY: Two figures $\mathcal{S}$ and $\mathcal{T}$ are called *similar*, written as $\mathcal{S} \parallel \mathcal{T}$, if one figure can be moved to coincide with the other figure by means of a sequence of rotations, reflections, translations, and enlargements.

The enlargement ratio involved in these transformations is called the *similarity ratio* of the two figures.



Like congruence, similarity sets up a correspondence between the elements of the two figures. In this correspondence, angles are preserved, and the ratio of two matching lengths equals the similarity ratio. Since an area is the product of two lengths, the ratio of the areas of matching regions is the square of the similarity ratio. Likewise, if the idea is extended into three-dimensional space, then the ratio of the volumes of matching solids is the cube of the similarity ratio.

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SIMILARITY RATIO: If two similar figures have similarity ratio $1 : k$, then

- matching angles have the same size,
- matching intervals have lengths in the ratio $1 : k$,
- matching regions have areas in the ratio $1 : k^2$,
- matching solids have volumes in the ratio $1 : k^3$.

Similar Triangles: As with congruence, most of our arguments concern triangles, and the four standard tests for similarity of triangles will be assumptions. These four tests correspond exactly with the four standard congruence tests, except that equal sides are replaced by proportional sides (the AAS congruence test thus corresponds to the AA similarity test). An example of each test is given below.

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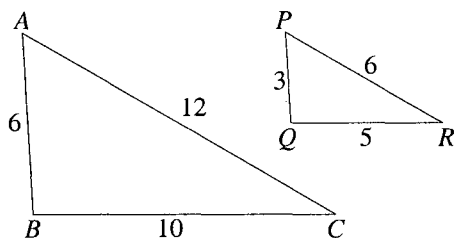
STANDARD SIMILARITY TESTS FOR TRIANGLES: Two triangles are similar if:

SSS the three sides of one triangle are respectively proportional to the three sides of another triangle, or

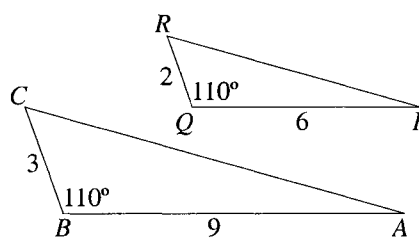
SAS two sides of one triangle are respectively proportional to two sides of another triangle, and the included angles are equal, or

AA two angles of one triangle are respectively equal to two angles of another triangle, or

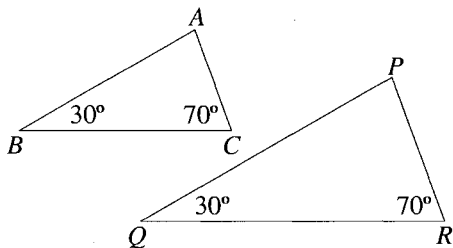
RHS the hypotenuse and one side of a right triangle are respectively proportional to the hypotenuse and one side of another right triangle.



$\triangle ABC \sim \triangle PQR$ (SSS),
with similarity ratio $2 : 1$.
Hence $\angle P = \angle A$, $\angle Q = \angle B$
and $\angle R = \angle C$
(matching angles of similar triangles).



$\triangle ABC \sim \triangle PQR$ (SAS),
with similarity ratio $3 : 2$.
Hence $\angle P = \angle A$, $\angle R = \angle C$
and $PR = \frac{2}{3}AC$ (matching sides
and angles of similar triangles).

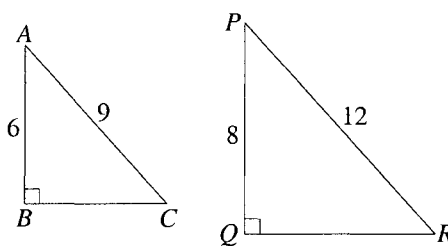


$$\triangle ABC \parallel \triangle PQR \quad (\text{AA}).$$

$$\text{Hence } \frac{PQ}{AB} = \frac{QR}{BC} = \frac{RP}{CA}$$

(matching sides of similar triangles),

and $\angle P = \angle A$ (angle sums of triangles).



$$\triangle ABC \parallel \triangle PQR \quad (\text{RHS}),$$

with similarity ratio 3 : 4.

Hence $\angle P = \angle A$, $\angle R = \angle C$

and $QR = \frac{4}{3}BC$ (matching sides and angles of similar triangles).

Using the Similarity Tests: Similarity tests should be set out in exactly the same way as congruence tests — the AA similarity test, however, will need only four lines. The similarity ratio should be mentioned if it is known. Keeping vertices in corresponding order is even more important with similarity, because the corresponding order is needed when writing down the proportionality of sides.

WORKED EXERCISE: A tower TC casts a 300-metre shadow CN , and a man RA 2 metres tall casts a 2.4-metre shadow AY . Show that $\triangle TCN \parallel \triangle RAY$, and find the height of the tower and the similarity ratio.

SOLUTION: In the triangles TCN and RAY :

1. $\angle TCN = \angle RAY = 90^\circ$ (given),
2. $\angle CNT = \angle AYR = \text{angle of elevation of the sun},$

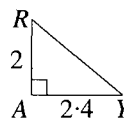
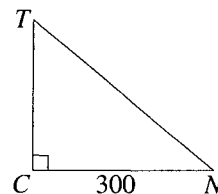
so $\triangle TCN \parallel \triangle RAY$ (AA).

$$\text{Hence } \frac{TC}{CN} = \frac{RA}{AY} \quad (\text{matching sides of similar triangles})$$

$$\frac{TC}{300} = \frac{2}{2.4}$$

$$TC = 250 \text{ metres.}$$

The similarity ratio is $300 : 2.4 = 125 : 1$.



WORKED EXERCISE: Prove that the interval PQ joining the midpoints of two adjacent sides AB and BC of a parallelogram $ABCD$ is parallel to the diagonal AC , and cuts off a triangle of area one eighth the area of the parallelogram.

PROOF: In the triangles BPQ and BAC :

1. $\angle PBQ = \angle ABC$ (common),
2. $PB = \frac{1}{2}AB$ (given),
3. $QB = \frac{1}{2}CB$ (given),

so $\triangle BPQ \parallel \triangle BAC$ (SAS), with similarity ratio 1 : 2.

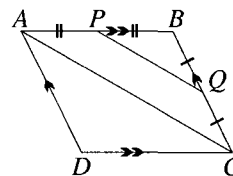
Hence $\angle BPQ = \angle BAC$ (matching angles of similar triangles),

so $PQ \parallel AC$ (corresponding angles are equal).

Also, $\text{area } \triangle BPQ = \frac{1}{4} \times \text{area } \triangle BAC$ (matching areas),

and $\text{area } \triangle ABC = \text{area } \triangle CDA$ (congruent triangles),

so $\text{area } \triangle BPQ = \frac{1}{8} \times \text{area of parallelogram } ABCD$.



Midpoints of Sides of Triangles: Similarity can be applied to configurations involving the midpoints of sides of triangles. The following theorem and its converse are standard results, and will be generalised in Section 8I.

31 COURSE THEOREM: The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

GIVEN: Let P and Q be the midpoints of the sides AB and AC of $\triangle ABC$.

AIM: To prove that $PQ \parallel BC$ and $PQ = \frac{1}{2}BC$.

PROOF: In the triangles APQ and ABC :

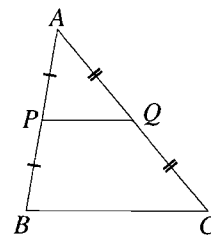
1. $AP = \frac{1}{2}AB$ (given),
2. $AQ = \frac{1}{2}AC$ (given),
3. $\angle A = \angle A$ (common),

so $\triangle APQ \parallel \triangle ABC$ (SAS), with similarity ratio 1 : 2.

Hence $\angle APQ = \angle ABC$ (matching angles of similar triangles),

so $PQ \parallel BC$ (corresponding angles are equal).

Also, $PQ = \frac{1}{2}BC$ (matching sides of similar triangles).



The Converse Theorem: Since there are two conclusions, there are several different theorems that could be regarded as the converse. The following theorem, however, is standard, and very useful.

32 COURSE THEOREM: Conversely, the interval through the midpoint of one side of a triangle and parallel to another side bisects the third side.

GIVEN: Let P be the midpoint of the side AB of $\triangle ABC$.

Let the line parallel to BC through P meet AC at Q .

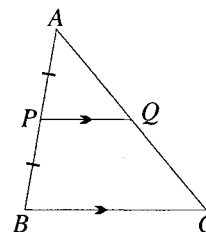
AIM: To prove that $AQ = \frac{1}{2}AC$.

PROOF: In the triangles APQ and ABC :

1. $\angle PAQ = \angle BAC$ (common),
2. $\angle APQ = \angle ABC$ (corresponding angles, $PQ \parallel BC$),

so $\triangle APQ \parallel \triangle ABC$ (AA), and the similarity ratio is $AP : AB = 1 : 2$.

Hence $AQ = \frac{1}{2}AC$ (matching sides of similar triangles).



Equal Ratios of Intervals and Equal Products of Intervals: The fact that the ratios of two pairs of intervals are equal can be just as well expressed by saying that the products of two pairs of intervals are equal:

$$\frac{AB}{BC} = \frac{XY}{YZ} \quad \text{is the same as} \quad AB \times YZ = BC \times XY.$$

The following worked exercise is one of the best known examples of this.

WORKED EXERCISE: Prove that the square on the altitude to the hypotenuse of a right triangle equals the product of the intercepts on the hypotenuse cut off by the altitude.

GIVEN: Let ABC be a triangle with $\angle A = 90^\circ$.

Let AP be the altitude to the hypotenuse.

AIM: To prove that $AP^2 = BP \times CP$.

PROOF: Let $\angle B = \beta$.

Then $\angle BAP = 90^\circ - \beta$ (angle sum of $\triangle BAP$),

so $\angle CAP = \beta$ (adjacent angles in the right angle $\angle BAC$).

In the triangles PAB and PCA :

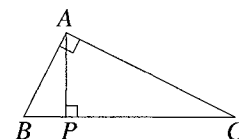
$$1. \quad \angle APB = \angle CPA = 90^\circ \quad (\text{given}),$$

$$2. \quad \angle ABP = \angle CAP \quad (\text{proven above}),$$

so $\triangle PAB \parallel \triangle PCA$ (AA).

$$\text{Hence} \quad \frac{BP}{AP} = \frac{AP}{CP} \quad (\text{matching sides of similar triangles}),$$

$$\text{so} \quad BP \times CP = AP^2.$$



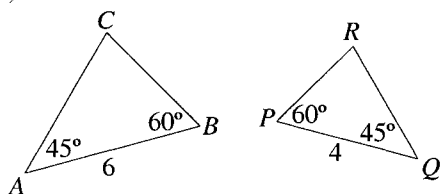
A NOTE ON THE GEOMETRIC MEAN: Recall from Chapter Six of the Year 11 volume that g is a *geometric mean* of a and b if $g^2 = ab$, because then the sequence a, g, b forms a GP with ratio $\frac{g}{a} = \frac{b}{g}$. Thus the previous result could be restated in the form of a theorem: *The altitude to the hypotenuse of a right-angled triangle is the geometric mean of the intercepts on the hypotenuse.*

Exercise 8H

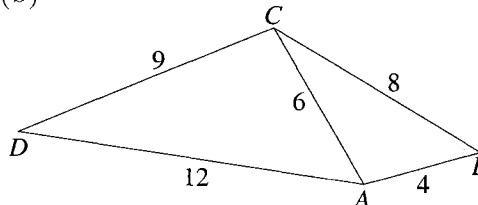
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Both triangles in each pair are similar. Name the similar triangles in the correct order and state which test is used.

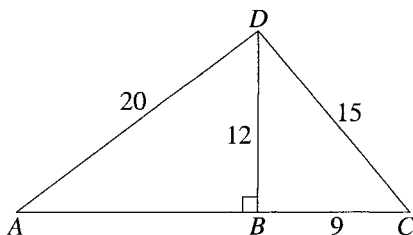
(a)



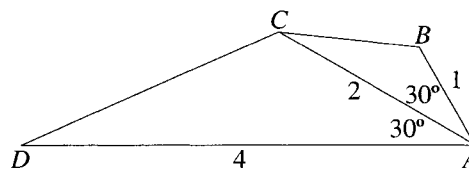
(b)



(c)

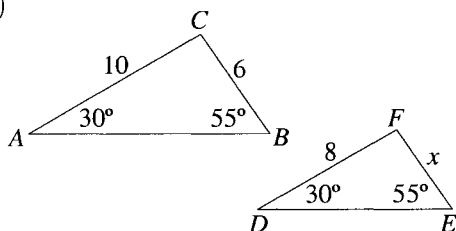


(d)

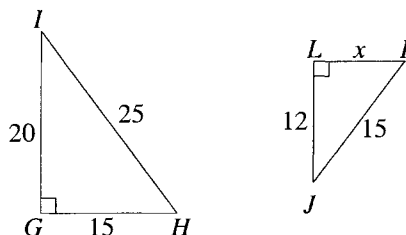


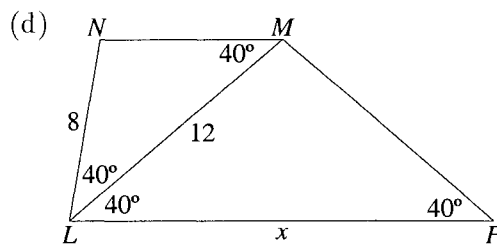
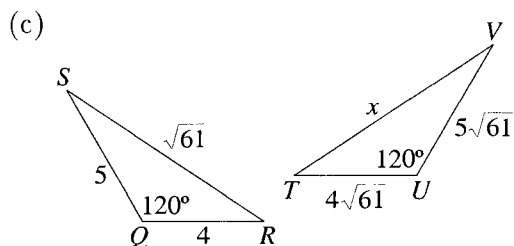
2. Identify the similar triangles, giving a reason, and hence deduce the length of the side x .

(a)

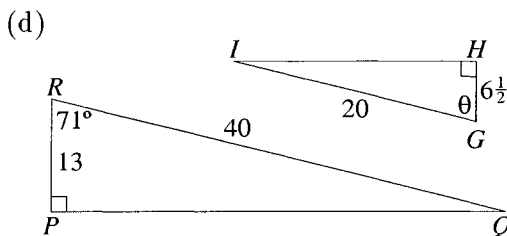
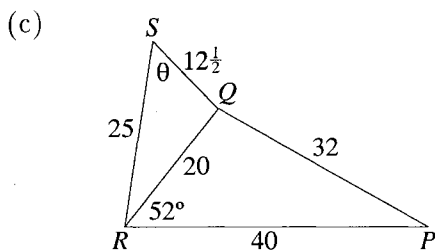
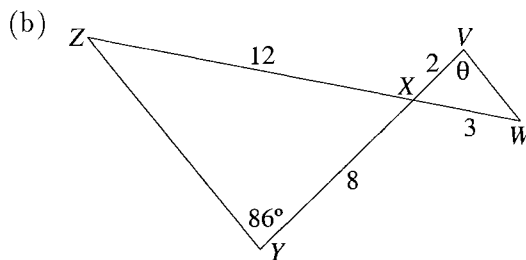
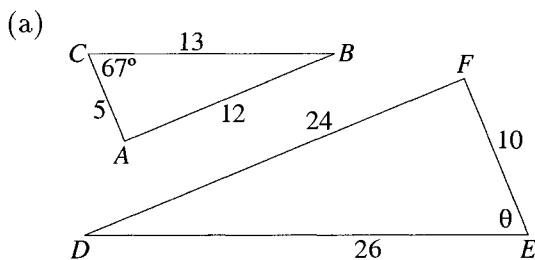


(b)

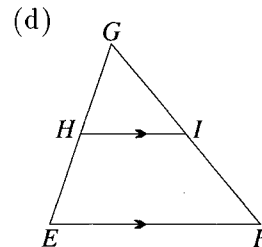
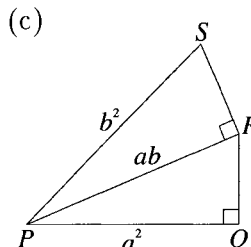
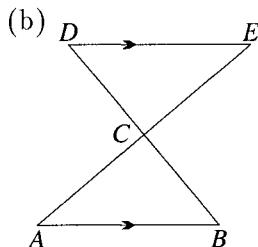
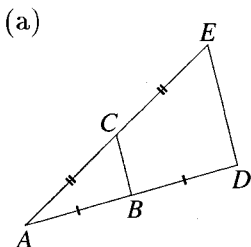




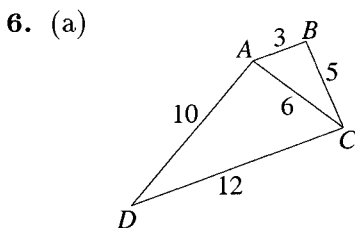
3. Identify the similar triangles, giving a reason, and hence deduce the size of the angle θ . In part (b), prove that $VW \parallel ZY$.



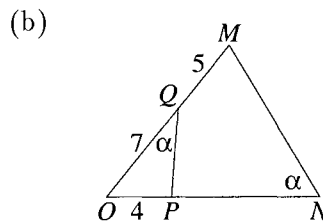
4. Prove that the triangles in each pair are similar.



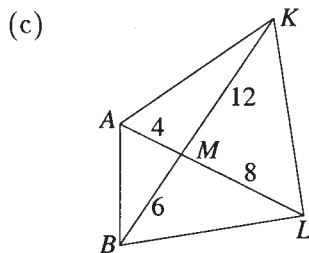
5. (a) A building casts a shadow 24 metres long, while a man 1.6 metres tall casts a 0.6-metre shadow. Draw a diagram, and use similarity to find the height of the building.
 (b) An architect builds a model of a house to a scale of 1 : 200. The house will have a swimming pool 10 metres long, with surface area 60 m^2 and volume 120 m^3 . What will the length and area of the model pool be, and how much water is needed to fill it?
 (c) Two coins of the same shape and material but different in size weigh 5 grams and 20 grams. If the larger coin has diameter 2 cm, what is the diameter of the smaller coin?



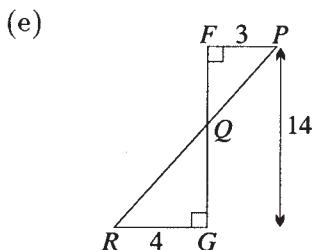
Show that $\triangle ADC \parallel \triangle BCA$,
and hence that $AB \parallel DC$.



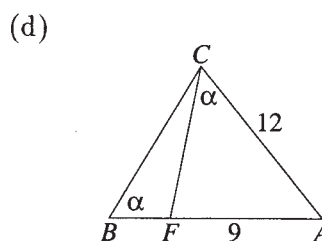
Show that $\triangle OPQ \parallel \triangle OMN$,
and hence find ON and PN .



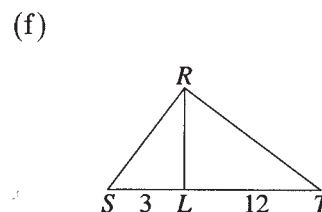
Show that $\triangle AMB \parallel \triangle LMK$.
What type of quadrilateral is $ABLK$?



Show that $\triangle FPQ \parallel \triangle GRQ$,
and hence find FQ , GQ , PQ and RQ .



Show that $\triangle ABC \parallel \triangle ACF$,
and hence find AB and FB .



Given that $RL \perp ST$ and $SR \perp TR$,
show that $\triangle LSR \parallel \triangle LRT$,
and hence find RL .

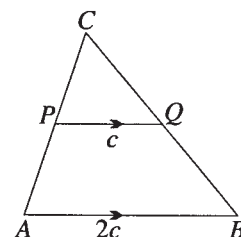
DEVELOPMENT

7. COURSE THEOREM: If two triangles are similar in the ratio $1 : k$, then their areas are in the ratio $1 : k^2$. Suppose that $\triangle ABC \parallel \triangle PQR$ (with vertices named in corresponding order) and let the ratio of corresponding sides be $1 : k$.

- Write down the area of $\triangle ABC$ in terms of a , b and $\angle C$.
- Do the same for $\triangle PQR$, and hence show that the areas are in the ratio $1 : k^2$.

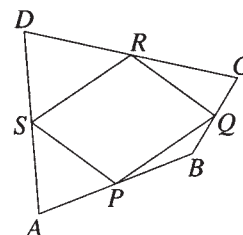
8. THEOREM: The interval parallel to one side of a triangle and half its length bisects the other two sides. In triangle ABC , suppose that PQ is parallel to AB and half its length.

- Prove that $\triangle ABC \parallel \triangle PQC$.
- Hence show that $CP = \frac{1}{2}CA$ and $CQ = \frac{1}{2}CB$.

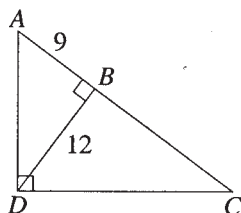


9. THEOREM: The quadrilateral formed by the midpoints of the sides of a quadrilateral is a parallelogram. Let $ABCD$ be a quadrilateral, and let P , Q , R and S be the midpoints of the sides AB , BC , CD and DA respectively.

- Prove that $\triangle PBQ \parallel \triangle ABC$, and hence that $PQ \parallel AC$.
- Similarly, prove that $PQ \parallel SR$ and $PS \parallel QR$.

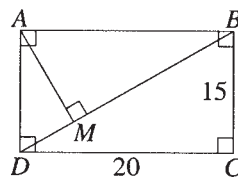


10. (a)



Show that $\triangle ABD \parallel \triangle ADC$,
and hence find AD , DC and BC .

(b)

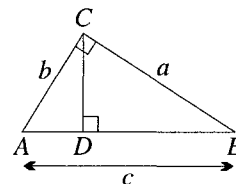


Use Pythagoras' theorem and similarity to
find AM , BM and DM .

11. THEOREM: Prove that the intervals joining the midpoints of the sides of any triangle dissect the triangle into four congruent triangles, each similar to the original triangle.

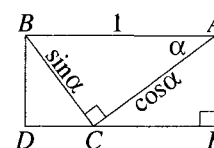
NOTE: Many of the hundreds of proofs of Pythagoras' theorem are based on similarity. The next three questions lead you through three of these proofs.

12. ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM: In the triangle ABC , there is a right angle at C , and the sides opposite the respective vertices are a , b and c . Let CD be the altitude from C to AB .



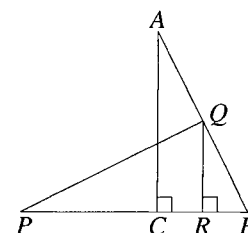
- (a) Prove that $\triangle CBD \parallel \triangle ABC$, and hence find BD in terms of the given sides.
 (b) Similarly, prove that $\triangle ACD \parallel \triangle ABC$, and find AD .
 (c) Hence prove that $a^2 + b^2 = c^2$.

13. ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM: In the rectangle $ABDE$, $\triangle ABC$ is right-angled at C , and $AB = 1$. Let $\angle BAC = \alpha$, then $BC = \sin \alpha$ and $AC = \cos \alpha$.



- (a) Prove that $\triangle ABC \parallel \triangle BCD$, and hence find DC .
 (b) Similarly, prove that $\triangle ABC \parallel \triangle CAE$, and find EC .
 (c) Hence prove that $\sin^2 \alpha + \cos^2 \alpha = 1$.

14. ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM: In the diagram, $\triangle ABC \equiv \triangle PQR$. The side PR is on the same line as BC , and the vertex Q is on AB . In $\triangle ABC$, there is a right angle at C . Let $AB = c$, $BC = a$ and $AC = b$.

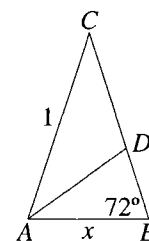


- (a) Prove that $\triangle PBQ \parallel \triangle ABC$, and hence find PB in terms of the given sides.
 (b) Similarly, prove that $\triangle QBR \parallel \triangle ABC$, and hence show that $PB = b + \frac{a^2}{b}$.
 (c) Hence prove that $a^2 + b^2 = c^2$.

15. Explain why the following pairs of figures are, or are not, similar:

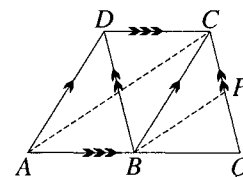
- (a) two squares, (b) two rectangles, (c) two rhombuses, (d) two equilateral triangles, (e) two isosceles triangles, (f) two circles, (g) two parabolas, (h) two regular hexagons.

16. In the diagram, $\triangle ABC$ is isosceles, with $\angle ABC = 72^\circ$, $CB = CA = 1$, and $AB = x$. The bisector of $\angle CAB$ meets BC at D .



- (a) Show that $\triangle ABC \parallel \triangle BDA$.
 (b) Use part (a) to find the exact value of x .
 (c) Explain why $\cos 72^\circ = \frac{x}{2}$, and hence write down the exact value of $\cos 72^\circ$.

17. In the figure, $ABCD$ is a parallelogram. The line PC , parallel to BD , meets AB produced at Q .

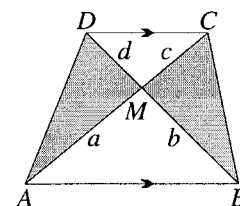


- (a) Prove that $AB = BQ$.
 (b) The midpoint of CQ is P . Prove that $\triangle PBQ \parallel \triangle CAQ$.
 (c) Hence prove that $\angle PBQ = \angle CAB$.

18. THEOREM: *The triangles formed by the diagonals and the parallel sides of a trapezium are similar, and the other two triangles have equal areas.*

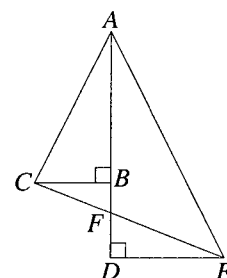
- (a) In the trapezium $ABCD$, the diagonals intersect at M .
Let $AM = a$, $BM = b$, $CM = c$ and $DM = d$, and let $\angle AMB = \theta$.

- (i) Prove that the unshaded triangles are similar.
(ii) Hence prove that $ad = bc$.
(iii) Prove that the shaded triangles have the same area.
(b) Now suppose that $a = 6$, $b = 4$, $c = 3$ and $d = 2$, with $AB = 8$ and $DC = 4$.
(i) Show that $\cos \theta = -\frac{1}{4}$ and $\sin \theta = \frac{\sqrt{15}}{4}$.
(ii) Hence find the area of the trapezium in exact form.



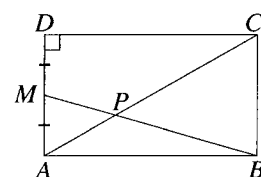
19. In the diagram, $\triangle ABC \parallel \triangle ADE$ and $\angle B$ is a right angle. The interval CE intersects BD at F . Let $AB = a$ and let the ratio of similarity be $AB : AD = k : \ell$.

- (a) Prove that $\triangle FBC \parallel \triangle FDE$.
(b) What is the ratio of the lengths $BF : FD$?
(c) Show that $FD = \frac{a\ell(\ell - k)}{k(\ell + k)}$.
(d) Now suppose that $AB : AD = 2 : 3$ and FD is an integer. What are the possible values of a ?



20. In the rectangle $ABCD$, $AB = 2 \times AD$, M is the midpoint of AD , and BM intersects AC at P .

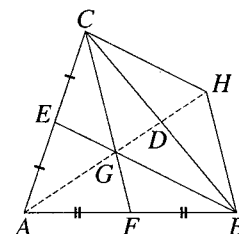
- (a) Show that $\triangle APM \parallel \triangle CPB$.
(b) Show that $3 \times CP = 2 \times CA$.
(c) Show that $9 \times CP^2 = 5 \times AB^2$.



21. THEOREM: *The medians of a triangle are concurrent.*

In the triangle ABC , E and F are the midpoints of AC and AB respectively, and BE and CF intersect at G . The interval AG is produced to H so that $AG = GH$, and AH intersects BC at D .

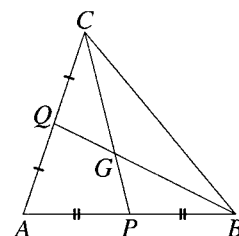
- (a) Prove that $\triangle AFG \parallel \triangle ABH$.
(b) Hence show that $GC \parallel BH$.
(c) Similarly, prove that $GB \parallel CH$, and hence that $GBHC$ is a parallelogram.
(d) Hence prove that $BD = DC$.



22. THEOREM: *The medians of a triangle are concurrent, and the resulting centroid trisects each median.*

Concurrency of the medians was proven in the previous question, so it remains to prove that they trisect each other. Let P and Q be the midpoints of the sides AB and AC of $\triangle ABC$. Let the medians PC and QB meet at G .

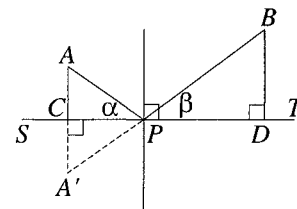
- (a) Prove that $\triangle ABC \parallel \triangle APQ$.
(b) Hence prove that $PQ = \frac{1}{2}BC$ and $PQ \parallel BC$.
(c) Prove that $\triangle PQG \parallel \triangle CBG$, with similarity ratio $1 : 2$.
(d) Hence deduce the given theorem.



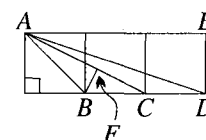
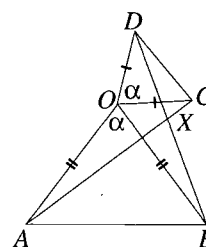
23. SEQUENCES AND GEOMETRY: Find the ratio of the sides in a right-angled triangle if:
- (a) the sides are in AP, (b) the sides are in GP.

EXTENSION

24. FOR REFLECTED LIGHT, THE ANGLE OF INCIDENCE EQUALS THE ANGLE OF REFLECTION: Suppose that a light source is at A above a reflective surface ST , and the reflected light is observed at B . Further suppose that at the point of reflection P , the angle of incidence is $(90^\circ - \alpha)$ and the angle of reflection is $(90^\circ - \beta)$. This means that $\angle APS = \alpha$ and $\angle BPT = \beta$. Let the image of A in the reflecting surface be at A' and let $A'A$ intersect ST at C . We will assume that light travels in a straight line and therefore that $A'PB$ is a straight line.



- (a) Explain why $AC = A'C$ and $AP = A'P$.
 (b) Prove that $\triangle APC \equiv \triangle A'PC$.
 (c) Thus prove that $\triangle APC \parallel \triangle BPD$.
 (d) Hence prove that the angle of incidence is equal to the angle of reflection.
25. Triangles ABO and CDO are similar isosceles triangles with a common vertex O . In both triangles, $\angle O = \alpha$ and $\triangle ABO$ is the larger of the two triangles. AC and DB are joined and meet (produced if necessary) at X .
- (a) Prove that $\triangle ODB \equiv \triangle OCA$.
 (b) Show that $\angle BXA = \alpha$.
 (c) Now suppose that $\triangle CDO$ is fixed and $\triangle ABO$ rotates about O . What is the locus of X ?
 (d) The kite $OAPB$ is completed so that P is on the circumcircle of $\triangle ABO$. Show that $\angle PXB = \frac{1}{2}\alpha$.
26. Three equal squares are placed side by side as shown in the diagram, and AB , AC and AD are drawn. Prove that $\angle BAC = \angle DAE$. [HINT: Construct $BF \perp AC$ as shown.]



8I Intercepts on Transversals

The previous theorem concerning the midpoints of the sides of a triangle can be generalised in two ways. First, the midpoint can be replaced by a point dividing the side in any given ratio. Secondly, the theorem can be applied to the intercepts cut off a transversal by three parallel lines. The word *intercept* needs clarification.

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INTERCEPTS: A point P on an interval AB divides the interval into two *intercepts* AP and PB .



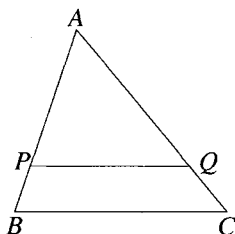
This section, unlike previous sections, will be entirely new for most students.

Points on the Sides of Triangles: The proofs of the following theorems are similar to the proofs of the previous two theorems, and are left to the exercise.

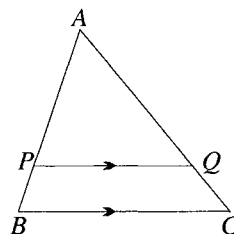
34

COURSE THEOREM: If two points P and Q divide two sides AB and AC respectively of a triangle in the same ratio $k : \ell$, then the interval PQ is parallel to the third side BC , and $PQ : BC = k : k + \ell$.

Conversely, a line parallel to one side of a triangle divides the other two sides in the same ratio.



Given that $AP : PB = AQ : QC = k : \ell$,
it follows that $PQ \parallel BC$
and $PQ : BC = k : k + \ell$ (intercepts).



Given that $PQ \parallel BC$,
it follows that
 $AP : PB = AQ : QC$ (intercepts).

Transversals to Three Parallel Lines: The previous theorems about points on the sides of a triangle can be applied to the intercepts cut off by three parallel lines.

35

COURSE THEOREM: If two transversals cross three parallel lines, then the ratio of the intercepts on one transversal is the same as the ratio of the intercepts on the other transversal.

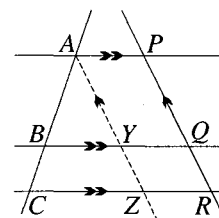
In particular, if three parallel lines cut off equal intercepts on one transversal, then they cut off equal intercepts on all transversals.

The second part follows from the first part with $k : \ell = 1 : 1$, so it will be sufficient to prove only the first part.

GIVEN: Let two transversals ABC and PQR cross three parallel lines, and let $AB : BC = k : \ell$.

AIM: To prove that $PQ : QR = k : \ell$.

CONSTRUCTION: Construct the line through A parallel to the line PQR , and let it meet the other two parallel lines at Y and Z respectively.



PROOF: The configuration in $\triangle ACZ$ is the converse part of the previous theorem, and so $AY : YZ = k : \ell$ (intercepts).

But the opposite sides of the parallelograms $APQY$ and $YQZR$ are equal,
so $AY = PQ$ and $YZ = QR$.

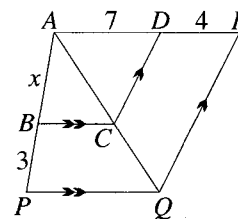
Hence $PQ : QR = k : \ell$.

WORKED EXERCISE: Find x in the diagram opposite.

SOLUTION: $\frac{x}{3} = \frac{AC}{CQ}$ (intercepts in $\triangle APQ$),

and $\frac{AC}{CQ} = \frac{7}{4}$ (intercepts in $\triangle AQR$).

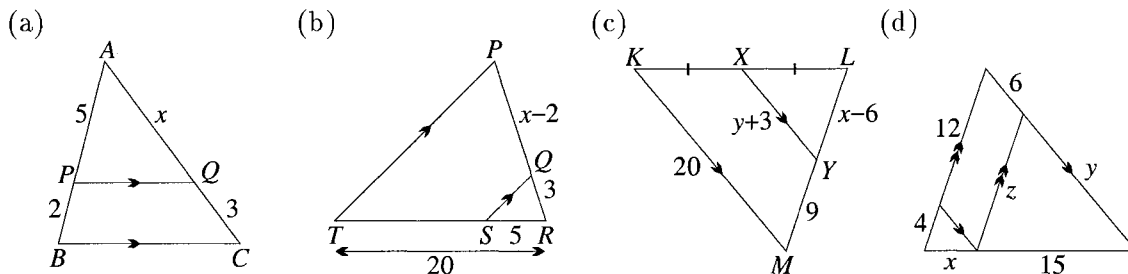
Hence $\frac{x}{3} = \frac{7}{4}$
 $x = 5\frac{1}{4}$.



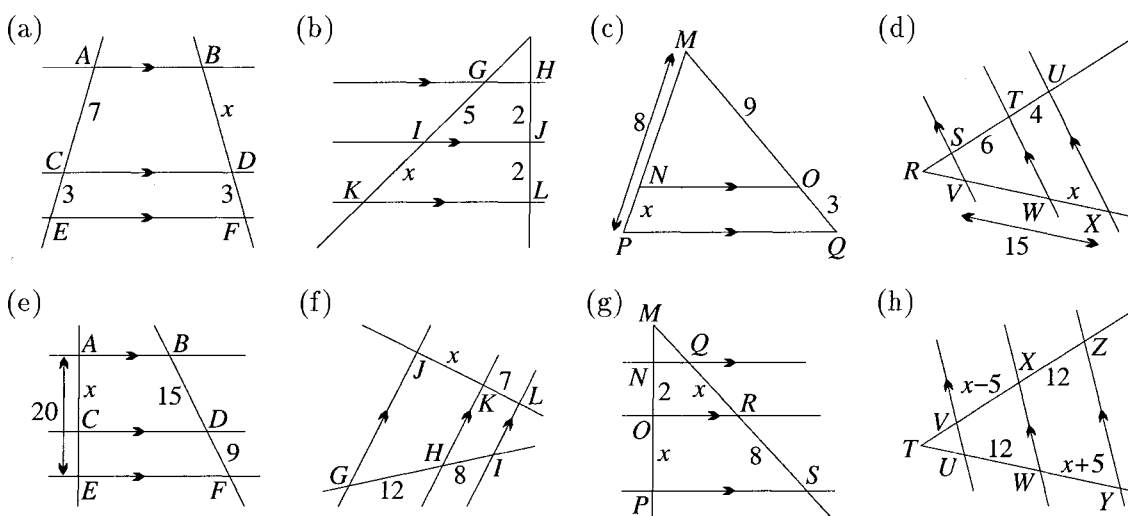
Exercise 8I

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

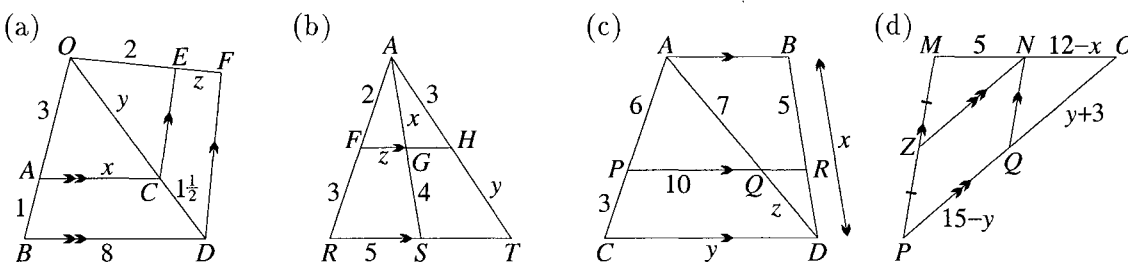
1. Find the values of x , y and z in the following diagrams.



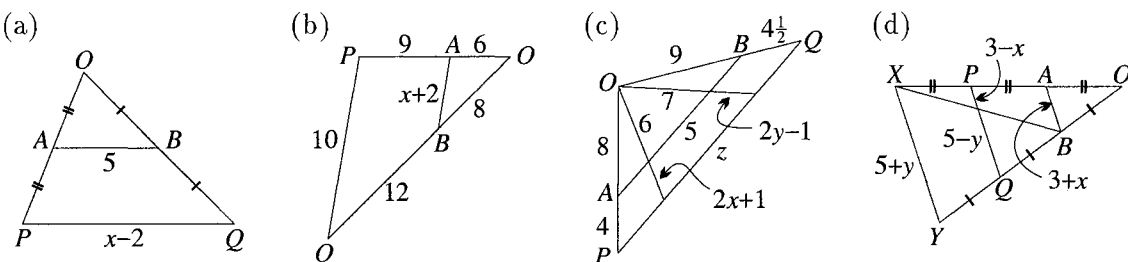
2. Find the value of x in each diagram below.



3. Find x , y and z in the diagrams below.

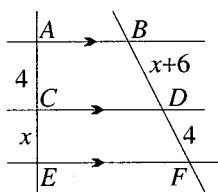


4. Give a reason why $AB \parallel PQ \parallel XY$ as appropriate, then find x and y .

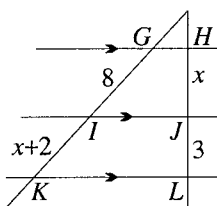


5. Write down a quadratic equation for x and hence find the value of x in each case.

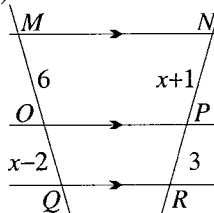
(a)



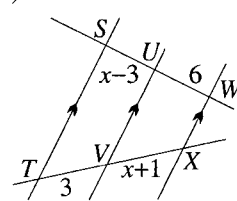
(b)



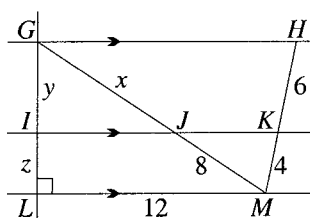
(c)



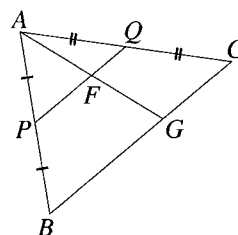
(d)



6. (a)



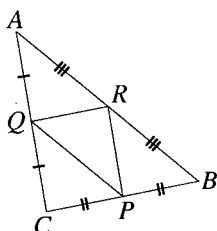
(b)



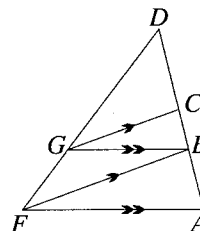
Use Pythagoras' theorem to find x , y and z .

Find $AF : AG$.

(c)



(d)



What sort of quadrilateral is $ARPQ$? Find the ratio of areas of $ARPQ$ and $\triangle ABC$.

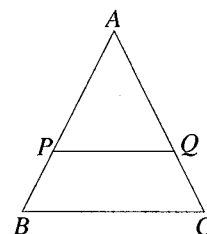
Show that $FG : GD = BC : CD$ and that $AF : BG = BF : CG$.

DEVELOPMENT

7. **COURSE THEOREM:** If two points P and Q divide two sides AB and AC respectively of a triangle in the same ratio $k : \ell$, then the interval PQ is parallel to the third side BC and $PQ : BC = k : k + \ell$. In $\triangle ABC$, the points P and Q divide the sides AB and AC respectively in the ratio $k : \ell$.

(a) Prove that $\triangle ABC \parallel \triangle APQ$.

(b) Hence prove that $PQ \parallel BC$ and $PQ : BC = k : k + \ell$.

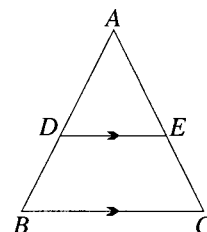


8. **COURSE THEOREM:** Conversely, a line parallel to one side of a triangle divides the other two sides in the same ratio. In $\triangle ABC$, the interval DE is parallel to BC .

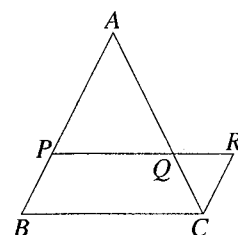
(a) Prove that $\triangle ABC \parallel \triangle ADE$.

(b) Let $DE : BC = k : (k + \ell)$. Show that

$$AD : DB = AE : EC = k : \ell.$$

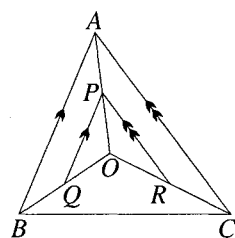


9. **ALTERNATIVE PROOF OF COURSE THEOREM:** If two points P and Q divide two sides AB and AC respectively of a triangle in the same ratio $k : \ell$, then the interval PQ is parallel to the third side BC and $PQ : BC = k : (k + \ell)$. In $\triangle ABC$, the points P and Q divide the sides AB and AC respectively in the ratio $k : \ell$. PQ is produced to R so that $PQ : QR = k : \ell$ and CR is joined.



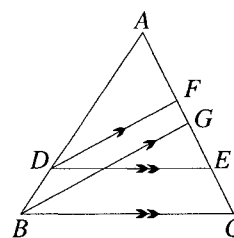
- (a) Show that $\triangle APQ \parallel \triangle CRQ$.
 (b) Hence show that $PBCR$ is a parallelogram.
 (c) Hence show that $PQ \parallel BC$ and $PQ : BC = k : (k + \ell)$.

10. (a)



Choose any point O inside $\triangle ABC$ and join O to each vertex. Choose any point P on OA and then construct $PQ \parallel AB$ and $PR \parallel AC$. Prove that $QR \parallel BC$.

(b)



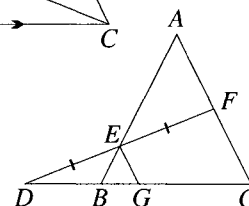
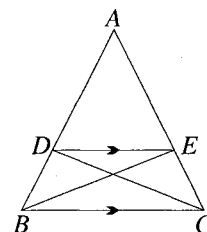
In $\triangle ABC$, the line DE is parallel to the base BC . A point G is chosen on AC and then DF is constructed parallel to BG . Prove that $AF : AG = DE : BC$.

11. The triangle ABC is isosceles, with $AB = AC$, and DE is parallel to BC .

- (a) Use the intercepts theorem to prove that $DB = EC$.
 (b) Show that $\triangle BCD \equiv \triangle CBE$.

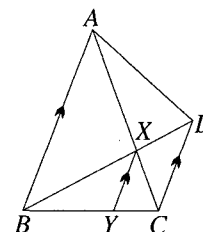
12. The triangle ABC is isosceles, with $AB = AC$. The base CB is produced to D . The points E on AB and F on AC are chosen so that E is the midpoint of the straight line DEF . G is the point on the base such that $CG = GD$.

- (a) Prove that $EG \parallel AC$.
 (b) Hence show that $FC = 2 \times EB$.



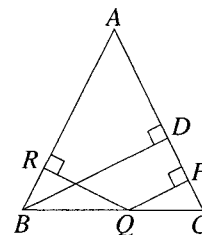
13. The diagram shows a trapezium $ABCD$ with $AB \parallel DC$. The diagonals AC and BD intersect at X , and XY is constructed parallel to AB , intersecting BC at Y .

- (a) Prove that $AB : CD = BY : YC$.
 (b) In a certain trapezium, the length of AB is 18 cm. Given that $BY : BC = 3 : 4$, what is the length of the shorter side?



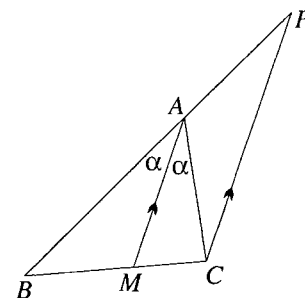
14. The triangle ABC is isosceles with $AB = AC$, and D is a point on AC such that $BD \perp AC$. Choose any point Q on the base, and construct the perpendiculars to the equal sides, with $QP \perp AC$ and $QR \perp AB$.

- (a) Reflect the triangle RBQ in the line BQ , and hence show that $RQ + PQ = BD$.
 (b) Construct CE perpendicular to AB at E and use the ratios of intercepts to prove the same result.



15. (a) Two vertical poles of height 10 metres and 15 metres are 8 metres apart. Wire stretches from the top of each pole to the foot of the other. Find how high above the ground the wires cross. How would this height change if the poles were 11 metres apart?
 (b) In a narrow laneway 2.4 metres wide between two buildings, a 4-metre ladder rests on one wall with its foot against the other wall, and a 3-metre ladder rests on the opposite wall. The ladders touch at their crossover point. How high is that crossover point? [HINT: You will need the height each ladder reaches up the wall, then use similarity.]

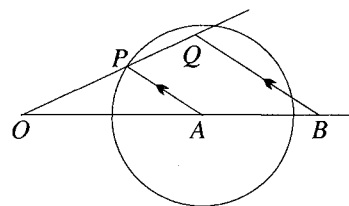
16. THEOREM: The bisector of the angle at a vertex of a triangle divides the opposite side in the ratio of the including sides. Suppose that M lies on the side BC of $\triangle ABC$, and $\angle BAM = \angle CAM = \alpha$. Let the line through C parallel to MA meet BA produced at P .



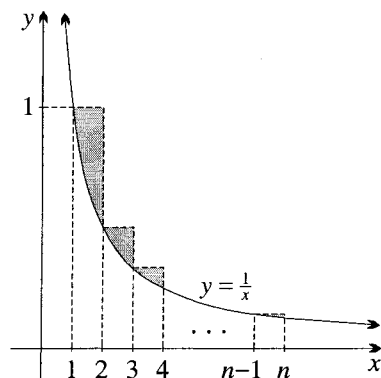
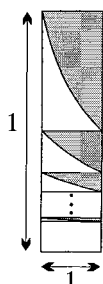
- (a) Prove that $\triangle APC$ is isosceles with $AP = AC$.
 (b) Hence show that $BM : MC = BA : AC$.
17. THEOREM: Conversely, if the interval joining a vertex of a triangle to a point on the opposite side divides that side in the ratio of the including sides, then the interval bisects the vertex angle. Prove this using a similar construction.

EXTENSION

18. In the diagram, A is the centre of a circle with radius R . O is a fixed point outside the circle and B is another fixed point on OA . For a given point P on the circle, the point Q on the line OP is chosen so that $AP \parallel BQ$. Describe the locus of Q as P moves around the circle.
 [HINT: Suppose that A divides OB in the ratio $k : \ell$.]



19. [The harmonic series and Euler's constant]



The right-hand diagram above shows the curve $y = 1/x$ (not to scale). Upper rectangles have been constructed on the intervals $1 \leq x \leq 2$, $2 \leq x \leq 3$, ... and $n-1 \leq x \leq n$. Let E_n be the total area of the shaded regions inside the rectangles and above the curve.

- (a) By considering the difference between the area of the rectangles and the area under the curve, show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n = E_n + \frac{1}{n}$.
- (b) The left-hand diagram above shows the shaded regions stacked on top of each other inside a unit square (be careful, the diagrams are not drawn to scale). By drawing appropriate diagonals, show that as $n \rightarrow \infty$, E_n converges to a limit γ between $\frac{1}{2}$ and 1.
- (c) Hence show that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = \gamma$. Then use your calculator, or a computer, to get some idea of the value of γ by substituting some values of n . [NOTE: The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called the *harmonic series* — the word comes from music, because if pipes are built of lengths $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ then the notes they sound will be the series of harmonics of the first pipe. The strange number $\gamma \doteq 0.577$ is called *Euler's constant*. It remains unknown even whether γ is rational or irrational.]

μηδεις ἀγεωμέτρητος εἰσίτω

'Let no-one enter who does not know geometry.'

(Inscribed over the doorway to Plato's Academy in Athens.)

Circle Geometry

Circles have already been studied using coordinate methods, and circles were essential in the development of the trigonometric functions. Many important properties of circles, however, remain to be developed, and the methods of Euclidean geometry are particularly suited to this task — first, the circle is easily defined geometrically in terms of centre and radius, compasses being designed to implement this definition, and secondly, angles are handled far more easily in Euclidean geometry than in coordinate geometry.

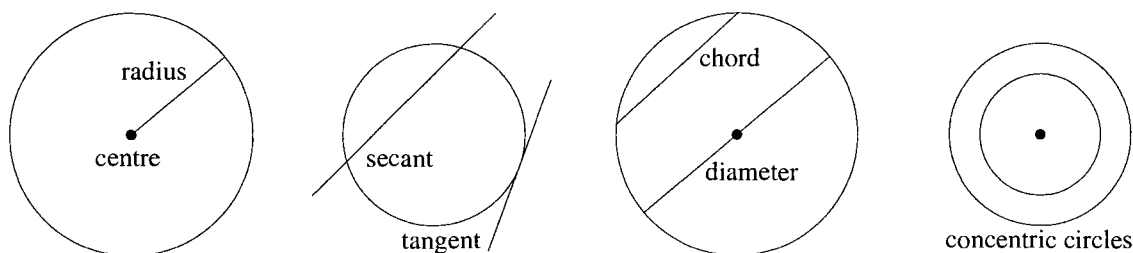
STUDY NOTES: Although this material may be familiar from earlier years, the emphasis now is less on numerical work and more on the logical development of the theory and on its applications to the proof of further results. Most students will therefore find the chapter rather demanding. Sections 9A–9D deal with angles at the centre and circumference of circles. Three difficult converse theorems here are quite new — these converses concern the circumcircle of a right triangle, and two tests for the concyclicity of four points. Sections 9E–9G then examine tangents to circles and the angles they form with diameters and chords.

As in the previous chapter, all the course theorems have been boxed. Some proofs are written out in the notes, and some are presented in structured questions placed at the start of the following development section. All these proofs are important — working through these proofs is an essential part of the course.

Some of the Extension sections of these exercises are longer than normal, but 3 Unit students should be reassured that these questions, as always, are beyond the standards of the 3 Unit HSC papers. The 4 Unit HSC papers usually contain a difficult geometry question, and many of the standard results associated with these questions have therefore been included in the Extension sections.

9A Circles, Chords and Arcs

The first group of theorems concern angles at the centre of a circle and their relationship with chords and arcs. The section ends with the crucial theorem that any set of three non-collinear points lie on a unique circle. First, some definitions:

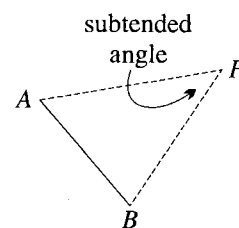


CIRCLE, CENTRE, RADIUS, TANGENT, SECANT, CHORD, DIAMETER:

- 1
- A *circle* is the set of all points that are a fixed distance (called the *radius*) from a given point (called the *centre*).
 - A *radius* is the interval joining the centre and any point on the circle.
 - A *tangent* is a line touching a circle in one point.
 - A *secant* is the line through two distinct points on a circle.
 - A *chord* is the interval joining two distinct points on a circle.
 - A *diameter* is a chord through the centre.
 - Two circles with a common centre are called *concentric*.

Subtended angles: We shall speak of *subtended angles* throughout this chapter, particularly angles subtended by chords of circles at the centre and at a point on the circumference.

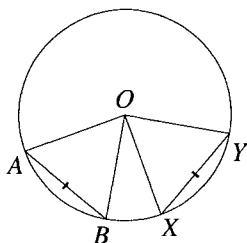
2 **ANGLES SUBTENDED BY AN INTERVAL:** The *angle subtended* at a point P by an interval AB is the angle $\angle APB$ formed at P by joining AP and BP .



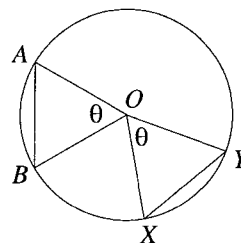
A Chord and the Angle Subtended at the Centre: The straightforward congruence proofs of this theorem and its converse have been left to the following exercise.

3 **COURSE THEOREM:** In the same circle or in circles of equal radius:

- Chords of equal length subtend equal angles at the centre.
- Conversely, chords subtending equal angles at the centre have equal lengths.



$\angle AOB = \angle XOY$
(equal chords AB and XY subtend equal angles at the centre O).



$AB = XY$
(chords subtending equal angles at the centre O are equal).

WORKED EXERCISE: In the diagram below, the chords AB , BC and CD have equal lengths. Prove that $AC = BD = 5$, then find AD .

SOLUTION: The three equal chords subtend equal angles at the centre O ,

so $\angle AOB = \angle BOC = \angle COD = 30^\circ$,

and $\angle AOC = 60^\circ$.

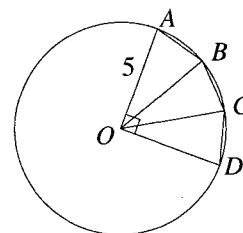
But $OA = OC$ (radii),

so $\triangle OAC$ is equilateral, and $AC = 5$.

Similarly, $\triangle OBD$ is equilateral, and $BD = 5$.

Secondly, $AD^2 = 5^2 + 5^2$ (Pythagoras),

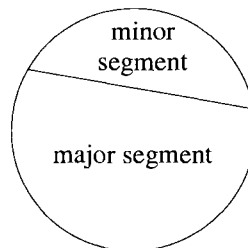
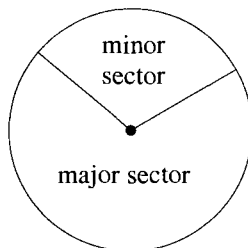
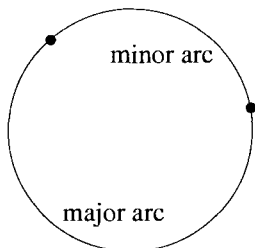
hence $AD = 5\sqrt{2}$.



Arcs, Sectors and Segments: Here again are the basic definitions.

ARCS, SECTORS AND SEGMENTS:

- 4
- Two points on a circle dissect the circle into a *major arc* and a *minor arc*, called *opposite arcs*.
 - Two radii of a circle dissect the region inside the circle into a *major sector* and a *minor sector*, called *opposite sectors*.
 - A chord of a circle dissects the region inside the circle into a *major segment* and a *minor segment*, called *opposite segments*.

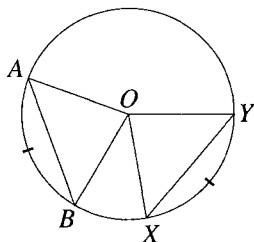


A fundamental assumption of the course is that arc length is proportional to the angle subtended at the centre. In particular, we shall assume that:

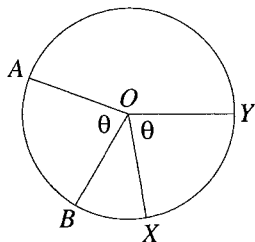
COURSE ASSUMPTION: In the same circle or in circles of equal radius:

- 5
- Equal arcs subtend equal angles at the centre.
 - Conversely, arcs subtending equal angles at the centre are equal.
 - Equal arcs cut off equal chords.
 - Conversely, equal chords cut off equal arcs.

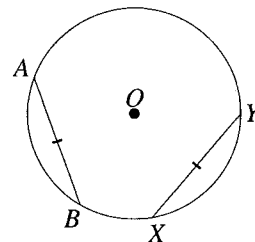
The first two statements can be proven informally by rotating one arc onto the other. The last two statements then follow from the first two, using the previous theorem. In the following diagrams, O is the centre of each circle.



$\angle AOB = \angle XOY$ and
 $AB = XY$ (arcs AB
 and XY are equal).



$\text{arc } AB = \text{arc } XY$ (arcs
 subtending equal angles
 at the centre are equal).



$\text{arc } AB = \text{arc } XY$
 (equal chords AB and XY
 cut off equal arcs).

WORKED EXERCISE: Two equal chords AB and XY of a circle intersect at E . Use equal arcs to prove that $AX = BY$ and that $\triangle EBX$ is isosceles.

SOLUTION:

First, $\text{arc } AB = \text{arc } XY$ (equal chords cut off equal arcs),

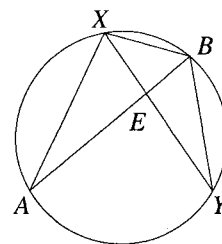
so $\text{arc } AX = \text{arc } BY$ (subtracting arc XB from each arc),

so $AX = BY$ (equal arcs cut off equal chords).

Secondly, $\triangle ABX \equiv \triangle YXB$ (SSS),

so $\angle ABX \equiv \angle YXB$ (matching angles of congruent triangles),

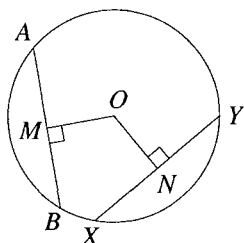
hence $EX = EB$ (opposite angles are equal).



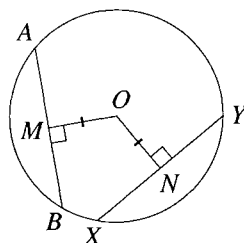
Chords and Distance from the Centre: The following theorem and its converse about the distance from a chord to the centre are often combined with Pythagoras' theorem in mensuration problems about circles. They are proven in the exercises.

COURSE THEOREM: In the same circle or in circles of equal radius:

- 6
- Equal chords are equidistant from the centre.
 - Conversely, chords that are equidistant from the centre are equal.



If $AB = XY$,
then $OM = ON$ (equal chords are equidistant from the centre O).



If $OM = ON$,
then $AB = XY$ (chords equidistant from the centre O are equal).

Chords, Perpendiculars and Bisectors: The radii from the endpoints of a chord are equal, and so the chord and the two radii form an isosceles triangle. The following important theorems are really restatements of theorems about isosceles triangles.

COURSE THEOREM:

- 7
- The perpendicular from the centre of a circle to a chord bisects the chord.
 - Conversely, the interval from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
 - The perpendicular bisector of a chord of a circle passes through the centre.

PROOF:

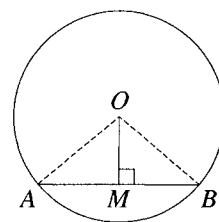
- A. To prove the first part, suppose that AB is a chord of a circle with centre O .
Let the perpendicular from O meet AB at M . We must prove that $AM = MB$.

In the triangles AMO and BMO :

1. $OM = OM$ (common),
2. $OA = OB$ (radii),
3. $\angle OMA = \angle OMB = 90^\circ$ (given),

so $\triangle AMO \equiv \triangle BMO$ (RHS).

Hence $AM = BM$ (matching sides of congruent triangles).



- B. To prove the second part, suppose that AB is a chord of a circle with centre O .
Let M be the midpoint of AB . We must prove that $OM \perp AB$.

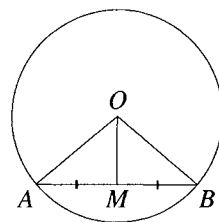
In the triangles AMO and BMO :

1. $OM = OM$ (common),
2. $OA = OB$ (radii),
3. $AM = BM$ (given),

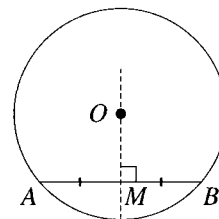
so $\triangle AMO \equiv \triangle BMO$ (SSS).

Hence $\angle AMO = \angle BMO$ (matching angles of congruent triangles).

But AMB is a straight line, and so $\angle AMO = 90^\circ$.



C. To prove the third part, suppose that AB is a chord of a circle with centre O . As proven in the first part, the perpendicular from O to AB bisects AB , and hence is the perpendicular bisector of AB . Hence the perpendicular bisector of AB passes through O , as required.

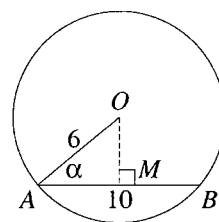


WORKED EXERCISE: In a circle of radius 6 units, a chord of length 10 units is drawn.

- How far is the chord from the centre?
- What is the sine of the angle between the chord and a radius at an endpoint of the chord?

SOLUTION: Let the centre be O and the chord be AB .

Construct the perpendicular OM from O to AB , and join the radius OA .



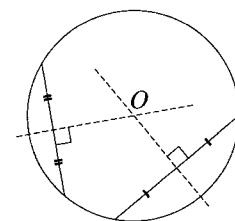
- Then $AM = MB$ (perpendicular from centre to chord),
so $OM^2 = 6^2 - 5^2$ (Pythagoras),
and $OM = \sqrt{11}$.

- Also, $\sin \alpha = \frac{1}{6}\sqrt{11}$.

Constructing the Centre of a Given Circle: The third part of the previous theorem gives a method of constructing the centre of a given circle.

8

COURSE CONSTRUCTION: Given a circle, construct any two non-parallel chords, and construct their perpendicular bisectors. The point of intersection of these bisectors is the centre of the circle.



PROOF: Since every perpendicular bisector passes through the centre, the centre must lie on every one of them, so the centre must be their single common point.

Constructing the Circle through Three Non-collinear Points: Any two distinct points determine a unique line. Three points may or may not be collinear, but if they are not, then they lie on a unique circle, constructed as described here.

9

COURSE THEOREM: Given any three non-collinear points, there is one and only one circle through the three points. Its centre is the intersection of any two perpendicular bisectors of the intervals joining the points.

The circle is called the *circumcircle* of the triangle formed by the three points, and its centre is called the *circumcentre*.

GIVEN: Let ABC be a triangle, and let O be the intersection of the perpendicular bisectors OP and OQ of BC and CA respectively.

AIM: To prove:

- The circle with centre O and radius OC passes through A and B .
- Every circle through A , B and C has centre O and radius OC .

CONSTRUCTION: Join AO , BO and CO .

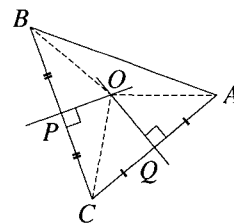
PROOF:

A. In the triangles BOP and COP :

1. $OP = OP$ (common),
2. $BP = CP$ (given),
3. $\angle BPO = \angle CPO = 90^\circ$ (given),

so $\triangle BOP \equiv \triangle COP$ (SAS).Hence $BO = CO$ (matching sides of congruent triangles).Similarly, $\triangle AOQ \equiv \triangle COQ$ and $AO = CO$.Hence $BO = CO = AO$,and the circle with centre O and radius OC passes through A and B .

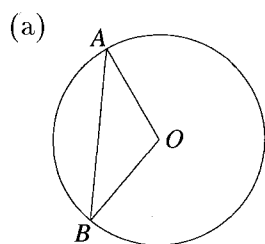
B. Now suppose that some circle with centre Z passes through A , B and C . We have already shown that the perpendicular bisector of a chord passes through the centre, and so Z lies on both OP and OQ . Hence O and Z coincide, and the radius is OC .



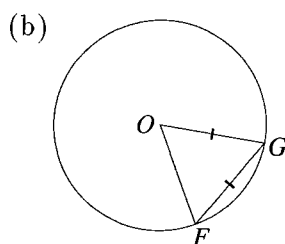
Exercise 9A

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, any point labelled O is the centre of the circle.

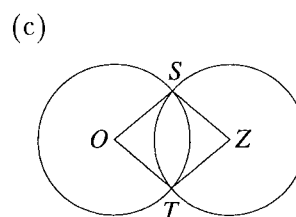
1. In part (c), O and Z are the centres of the two circles of equal radii.



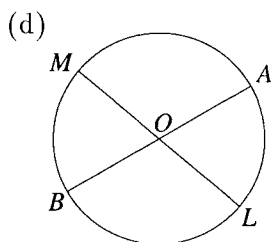
Prove that $\triangle OAB$ is isosceles.



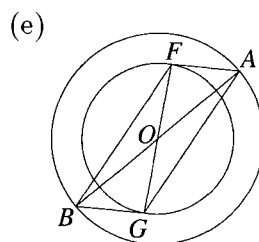
Prove that $\triangle OFG$ is equilateral.



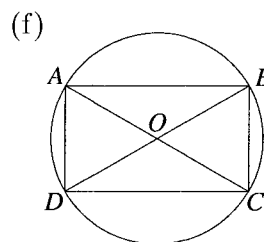
Prove that $OSZT$ is a rhombus.



Prove that arcs AL and MB have equal lengths.

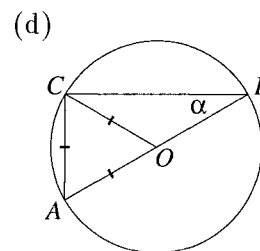
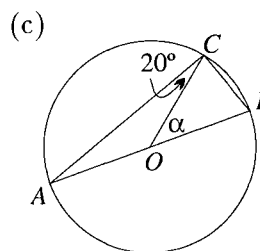
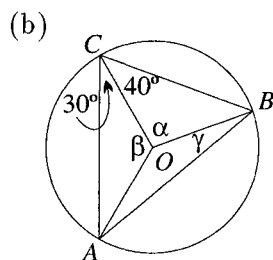
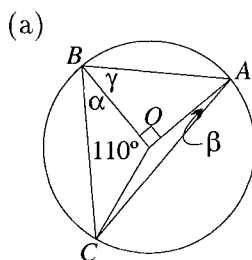


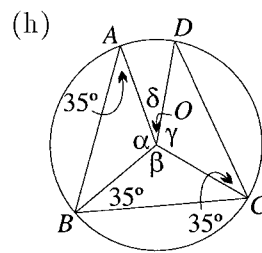
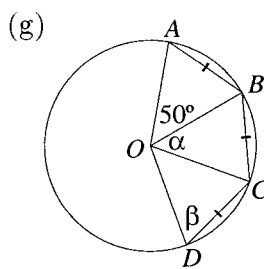
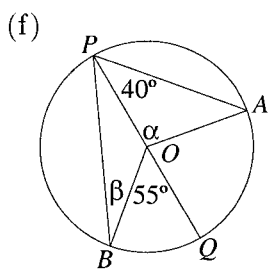
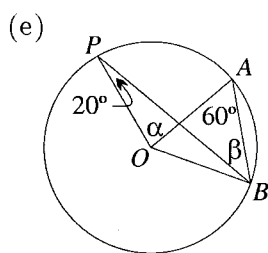
Prove that $AFBG$ is a parallelogram.



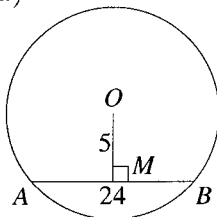
Prove that $ABCD$ is a rectangle.

2. Find α , β , γ and δ . In parts (g) and (h), prove that $\text{arc } ABC = \text{arc } BCD$ and $AC = BD$.



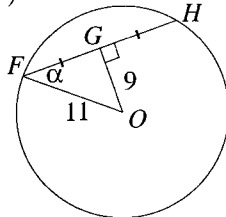


3. (a)



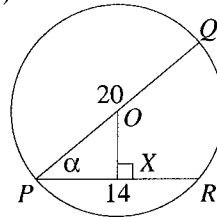
Find AO .

(b)



Find FH and $\cos \alpha$.

(c)

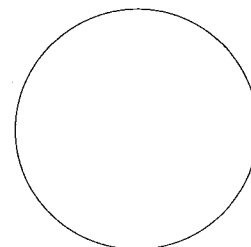


Find OX , QR and $\cos \alpha$.

4. CONSTRUCTION: Construct the centre of a given circle.

(a) Trace the circle drawn to the right, then use the construction given in Box 8 to find its centre.

(b) Trace it again, then use and explain this alternative construction. Construct any chord AB , and construct its perpendicular bisector — let the bisector meet the circle at P and Q , and construct the midpoint O of PQ .



5. CONSTRUCTION: Construct the circumcircle of a given triangle.

Place three non-collinear points towards the centre of a page, then use the construction given in Box 9 to construct the circle through these three points.

DEVELOPMENT

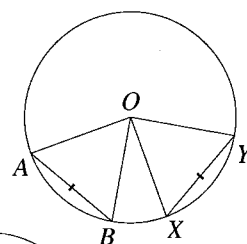
6. COURSE THEOREM: Equal chords subtend equal angles at the centre, and are equidistant from the centre.

In the diagram opposite, AB and XY are equal chords.

(a) Prove that $\triangle AOB \equiv \triangle XOY$.

(b) Prove that $\angle AOB = \angle XOY$.

(c) Prove that the chords are equidistant from the centre.

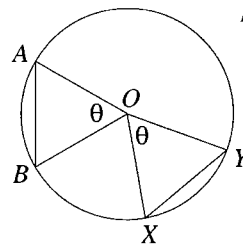


7. COURSE THEOREM: Two chords subtending equal angles at the centre have equal lengths.

In the diagram opposite, the angles $\angle AOB$ and $\angle XOY$ subtended by AB and XY are equal.

(a) Prove that $\triangle AOB \equiv \triangle XOY$.

(b) Hence prove that $AB = XY$.



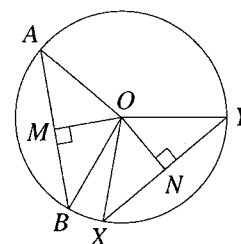
8. COURSE THEOREM: Two chords equidistant from the centre have equal lengths.

In the diagram opposite, $OM = ON$.

(a) Prove that $\triangle OAM \equiv \triangle OXN$.

(b) Prove that $\triangle OBM \equiv \triangle OYN$.

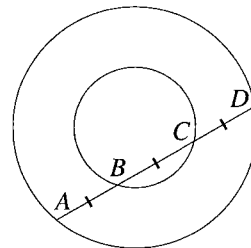
(c) Hence prove that $AB = XY$.



16. In the diagram to the right, the two concentric circles have radii 1 and 2 respectively.

- (a) What is the length of the chord AD ?
 (b) How far is the chord from the centre O ?

[HINT: Let $2x = AB = BC = CD$, and let h be the distance from the centre O . Then use Pythagoras' theorem.]



17. TRIGONOMETRY: A chord of length ℓ subtends an angle θ at the centre of a circle of radius r .
 (a) Prove that $\ell^2 = 2r^2(1 - \cos \theta)$.
 (b) Prove that $\ell = 2r \sin \frac{1}{2}\theta$.
 (c) Use trigonometric identities to reconcile the two results.
18. COORDINATE GEOMETRY: Using the result of Box 9, or otherwise, find the centre and radius of the circle passing through A , B and the origin $O(0,0)$ in each case:
 (a) $A = (4,0)$, $B = (4,8)$ (c) $A = (4,0)$, $\triangle ABO$ equilateral
 (b) $A = (4,0)$, $B = (2,12)$ (d) $A = (6,2)$, $B = (2,6)$

EXTENSION

19. The ratio of the length of a chord of a circle to the diameter is $\lambda : 1$. The chord moves around the circle so that its length is unchanged. Explain why the locus of the midpoint M of the chord is a circle, and find the ratio of the areas of the two circles.
20. An n -sided regular polygon is inscribed in a circle. Let the ratio of the perimeter of the polygon to the circumference of the circle be $\lambda : 1$, and let the ratio of the area of the polygon to the area of the circle be $\mu : 1$.
 (a) Find λ and μ for $n = 3, 4, 6$ and 8 .
 (b) Find expressions of λ and μ as functions of n , explain why they both have limit 1, and find the smallest value of n for which: (i) $\lambda > 0.999$ (ii) $\mu > 0.999$

9B Angles at the Centre and Circumference

This section studies the relationship between angles at the centre of a circle and angles at the circumference. The converse of the angle in a semicircle theorem is new work.

Angles in a Semicircle: An *angle in a semicircle* is an angle at the circumference subtended by a diameter of the circle. Traditionally, the following theorem is attributed to the early Greek mathematician Thales, and is said to be the first mathematical theorem ever formally proven.

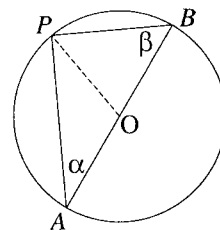
10 COURSE THEOREM: An angle in a semicircle is a right angle.

GIVEN: Let AOB be a diameter of a circle with centre O , and let P be a point on the circle distinct from A and B .

AIM: To prove that $\angle APB = 90^\circ$.

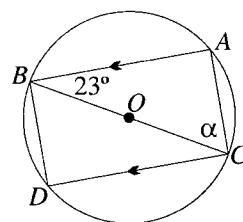
CONSTRUCTION: Join OP .

PROOF: Let $\angle A = \alpha$ and $\angle B = \beta$.
 Now $OA = OP = OB$ (radii of circle),
 forming two isosceles triangles $\triangle AOP$ and $\triangle BOP$,
 and so $\angle APO = \alpha$ and $\angle BPO = \beta$.
 But $(\alpha + \beta) + \alpha + \beta = 180^\circ$ (angle sum of $\triangle ABP$),
 so $\alpha + \beta = 90^\circ$, and $\angle APB = 90^\circ$.



WORKED EXERCISE: Find α , and prove that A , O and D are collinear.

SOLUTION: First, $\angle BAC = 90^\circ$ (angle in a semicircle),
 so $\alpha = 67^\circ$ (angle sum of $\triangle BAC$).
 Secondly, $\angle ACD = 90^\circ$ (co-interior angles, $AB \parallel CD$),
 and $\angle D = 90^\circ$, (angle in a semicircle),
 so $ABCD$ is a rectangle (all angles are right angles).
 Since the diagonals of a rectangle bisect each other,
 the diagonal AD passes through the midpoint O of BC .



Converse of the Angle in a Semicircle Theorem: The converse theorem essentially says 'every right angle is an angle in a semicircle', so its statement must assert the existence of the semicircle, given a right triangle.

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COURSE THEOREM: Conversely, the circle whose diameter is the hypotenuse of a right triangle passes through the third vertex of the triangle.

OR

The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle.

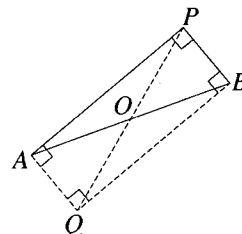
GIVEN: Let ABP be a triangle right-angled at P .

AIM: To prove that P lies on the circle with diameter AB .

CONSTRUCTION: Complete $\triangle APB$ to a rectangle $APBQ$, and let the diagonals AB and PQ intersect at O .

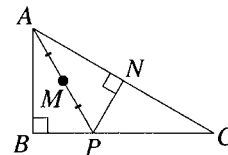
PROOF: The diagonals of the rectangle $APBQ$ are equal, and bisect each other.

Hence $OA = OB = OP = OQ$, as required.



WORKED EXERCISE: From any point P on the side BC of a triangle ABC right-angled at B , a perpendicular PN is drawn to the hypotenuse. Prove that the midpoint M of AP is equidistant from B and N .

SOLUTION: Since AP subtends right angles at N and B , the circle with diameter AP passes through B and N . Hence the centre M of the circle is equidistant from B and N .



Angles at the Centre and Circumference: A semicircle subtends a straight angle at the centre, which is twice the right angle it subtends at the circumference. This relationship can be generalised to a theorem about angles at the centre and circumference standing on any arc.

12

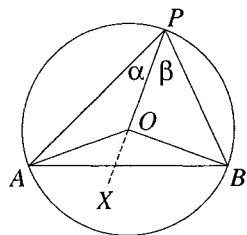
COURSE THEOREM: The angle subtended at the centre of a circle by an arc is twice any angle at the circumference standing on the same arc.

The angle 'standing on an arc' means the angle subtended by the chord joining its endpoints.

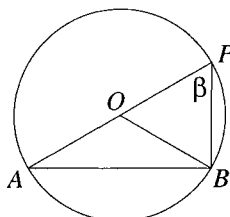
GIVEN: Let AB be an arc of a circle with centre O , and let P be a point on the opposite arc.

AIM: To prove that $\angle AOB = 2 \times \angle APB$.

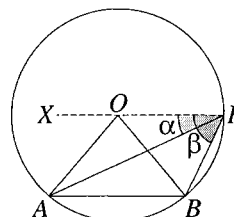
CONSTRUCTION: Join PO , and produce to X . Let $\angle APO = \alpha$ and $\angle BPO = \beta$.



Case 1



Case 2



Case 3

PROOF: There are three cases, depending on the position of P .

In each case, the equal radii $OA = OP = OB$ form isosceles triangles.

CASE 1: $\angle PAO = \alpha$ and $\angle PBO = \beta$ (base angles of isosceles triangles).

Hence $\angle AOX = 2\alpha$ and $\angle BOX = 2\beta$ (exterior angles),

and so $\angle AOB = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times \angle APB$, as required.

The other two cases are left to the exercises.

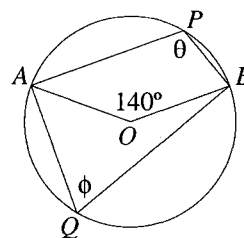
NOTE: The converse of this theorem is also true, but is not specifically in the course. It is set as an exercise in the Extension section following.

WORKED EXERCISE: Find θ and ϕ in the diagram opposite, where O is the centre of the circle.

SOLUTION: First, $\phi = 70^\circ$ (angles on the same arc APB).

Secondly, reflex $\angle AOB = 220^\circ$ (angles in a revolution),

so $\theta = 110^\circ$ (angles on the same arc AQB).

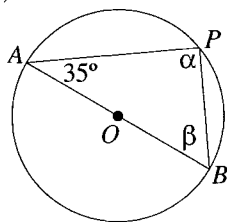


Exercise 9B

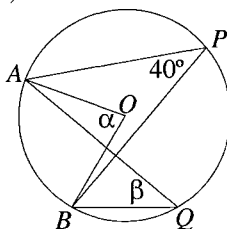
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles.

1. Find α , β , γ and δ in each diagram below.

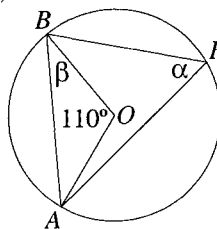
(a)



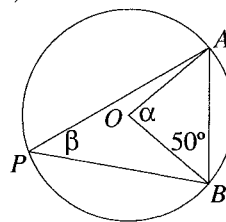
(b)

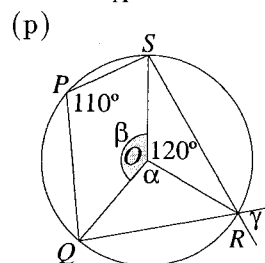
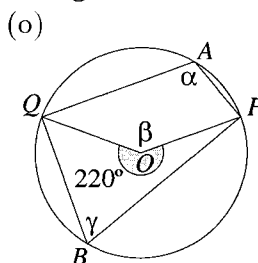
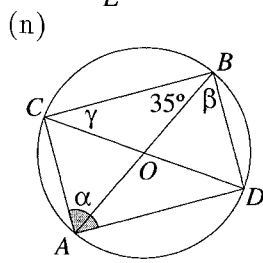
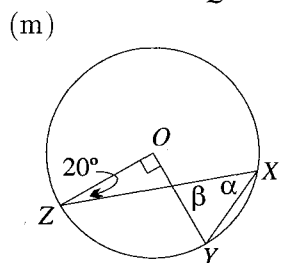
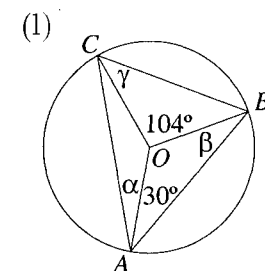
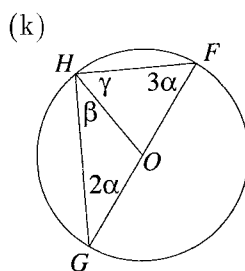
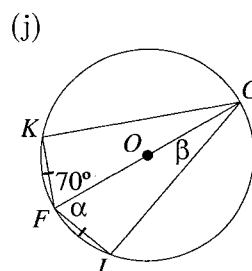
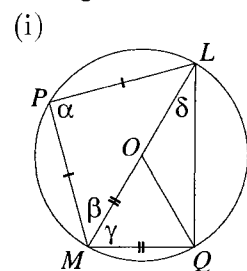
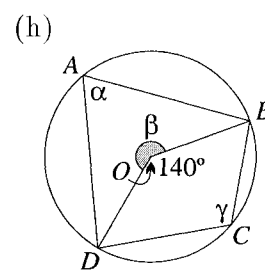
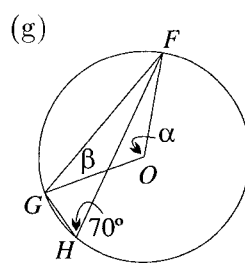
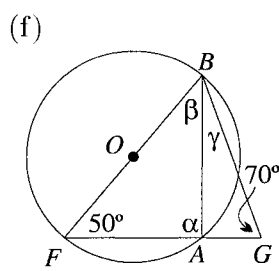
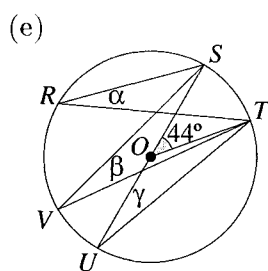


(c)

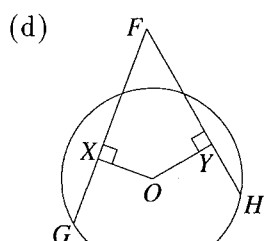
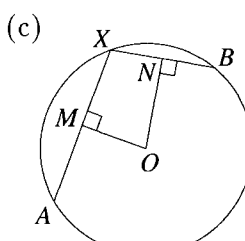
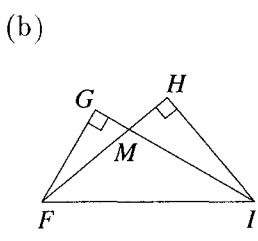
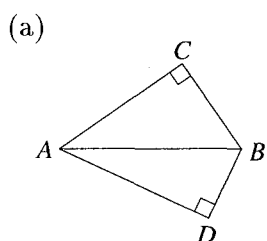


(d)

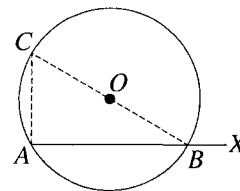




2. In each diagram, name a circle containing four points, and name a diameter of it. Give reasons for your answers.



3. A photographer is photographing the façade of a building. To do this effectively, he has to position himself so that the two ends of a building subtend a right angle at his camera. Describe the locus of his possible positions, and explain why he must be a constant distance from the midpoint of the building.
4. CONSTRUCTION: *Constructing a right angle at the end-point of an interval.* Let AX be an interval. With any centre O above or below the interval AX , construct a circle with radius OA . Let the circle pass through AX again at B . Construct the diameter through B , and let it meet the circle again at C . Prove that $AC \perp AX$.



DEVELOPMENT

5. COURSE THEOREM: Complete the other two cases of the proof that the angle at the centre subtended by an arc is twice the angle at the circumference subtended by that arc.

6. ALTERNATIVE PROOFS THAT AN ANGLE IN A SEMICIRCLE IS A RIGHT ANGLE: Let AB be a diameter of a circle with centre O , and let P be any other point on the circle.

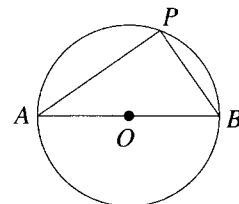
(a) Euclid's proof, Book 1, Proposition XX: Produce AP to Q , and join OP . Let $\angle A = \alpha$ and $\angle B = \beta$.

(i) Explain why $\angle QPB = \alpha + \beta$ and $\angle APB = \alpha + \beta$.

(ii) Hence prove that $\angle APB$ is a right angle.

(b) Proof using rectangles: Join PO and produce it to the diameter POR . Use the diagonal test to prove that $APBR$ is a rectangle, and hence that $\angle APB = 90^\circ$.

(c) Proof using intercepts: Let M be the midpoint of AP . Explain why $OM \perp AP$ and $OM \parallel BP$. Hence prove that $\angle P$ is a right angle.



7. ALTERNATIVE PROOFS OF THE CONVERSE: Let $\triangle ABP$ be right-angled at P .

(a) Proof using intercepts: Let O and M be the midpoints of AB and AP respectively.

(i) Prove that $OM \perp AP$.

(ii) Prove that $\triangle AOM \equiv \triangle POM$.

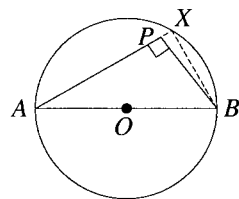
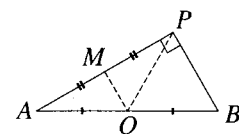
(iii) Explain why O is equidistant from A , B and P .

(b) A proof using the forward theorem: Construct the circle with diameter AB . Let AP (produced if necessary) meet the circle again at X . We must prove that the points P and X coincide.

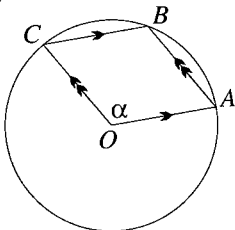
(i) Explain why $\angle AXB = 90^\circ$.

(ii) Explain why $PB \parallel XB$.

(iii) Explain why the points P and X coincide.

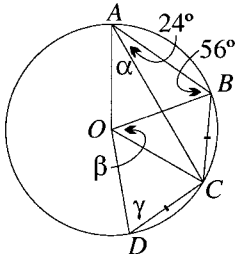


8. (a)



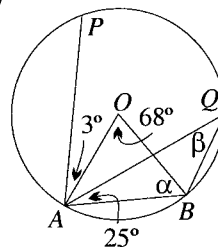
Explain why $\angle B = \alpha$, and find reflex $\angle O$. Then prove that $\alpha = 120^\circ$.

(b)



Find α , β and γ .

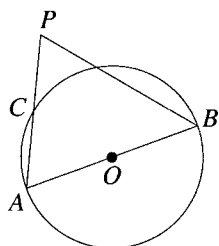
(c)



Find α and β . Then prove that $AP \parallel BQ$.

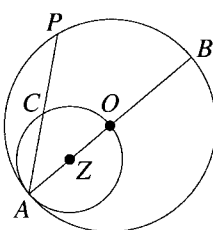
9. In each case, prove that C is the midpoint of AP . In part (a), $AB = PB$.

(a)



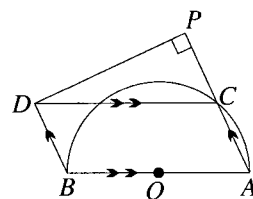
[HINT: Join BC .]

(b)



[HINT: Join OC and PB .]

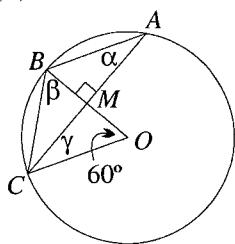
(c)



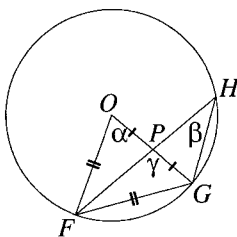
[HINT: Join BC .]

10. Give careful arguments to find α , β and γ in each diagram. In part (a), prove also that $OM = MB$. [HINT: Parts (b) and (c) will need congruence.]

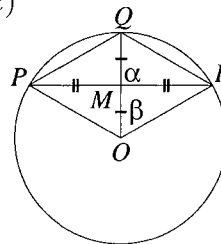
(a)



(b)

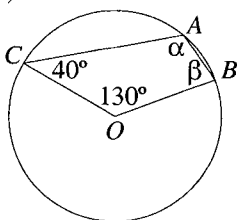


(c)

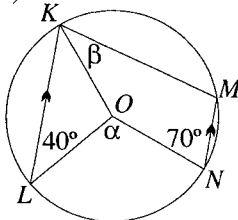


11. Find α , β , γ and δ in each diagram. Begin part (c) by proving that $\alpha = 120^\circ$.

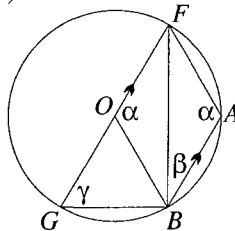
(a)



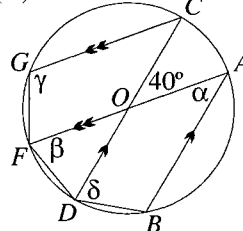
(b)



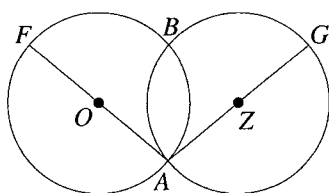
(c)



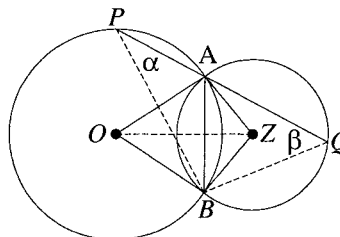
(d)



12. (a)



(b)



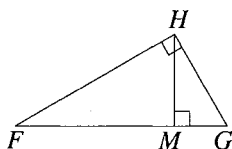
AOF and AZG are both diameters.

- Join AB , and hence prove that $\angle ABF = \angle ABG = 90^\circ$.
- Show that the points F , B and G are collinear.
- If the radii are equal, prove that $FB = BG$.

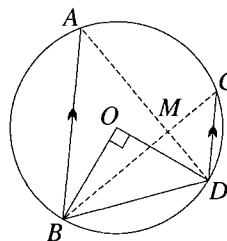
A line through A meets the two circles again at P and Q . Let $\angle P = \alpha$ and $\angle Q = \beta$.

- Prove that $\triangle AOZ \equiv \triangle BOZ$.
- Prove that OZ bisects $\angle AOB$ and $\angle AZB$.
- Prove that $\angle BOZ = \alpha$ and $\angle BZO = \beta$.
- Prove that $\angle PBQ = \angle OBZ$.

13. (a)



(b)



- Prove that the circles FMH , HMG and GHF have diameters FH , HG and GF respectively.
- Prove that the sum of the areas of the circles FMH and GMH equals the area of the circle FHG .

- Prove that $\angle A = \angle C = 45^\circ$.
- Prove that $AD \perp BC$.
- Prove that M lies on the circle BDO .

EXTENSION

14. MINIMISATION: In a rectangle inscribed in a circle, let length : breadth = $\lambda : 1$.
- Show that the ratio of the areas of the circle and the rectangle is $\frac{\pi}{4} \left(\lambda + \frac{1}{\lambda} \right)$.
 - Prove that the ratio of the areas has its minimum when the rectangle is a square, and find this minimum ratio.
 - Find λ when the ratio of the areas is twice its minimum value.
15. THEOREM: *The converse of the angle at the centre and circumference theorem.*
Use the method of question 7(c) to prove that if $\triangle AOB$ is isosceles with apex O , and a point P lies on the same side of AB as O such that $\angle AOB = 2\angle APB$, then the circle with centre O and radius $OA = OB$ also passes through C .
16. CIRCULAR MOTION: A horse is travelling around a circular track at a constant rate, and a punter standing at the edge of the track is following him with binoculars. Use circle geometry to prove that the punter's binoculars are rotating at a constant rate.

9C Angles on the Same and Opposite Arcs

The previous theorem relating angles at the centre and circumference has two important consequences. First, any two angles on the same arc are equal. Secondly, two angles in opposite arcs are supplementary, or alternatively, the opposite angles of a cyclic quadrilateral are supplementary,

Angles at the Circumference Standing on the Same Arc: An angle subtended by an arc at the circumference of a circle is also called 'an angle in a segment', just as an angle in a semicircle is called 'an angle in a semicircle'. This accounts for the alternative statement of the theorem:

- 13 COURSE THEOREM: Two angles in the same or equal segments are equal.
OR
Two angles at the circumference standing on the same or equal arcs are equal.

The proof of this theorem relates the two angles at the circumference back to the single angle at the centre (the case of 'equal arcs' is left to the reader):

GIVEN: Let AB be an arc of a circle with centre O , and let P and Q be points on the opposite arc.

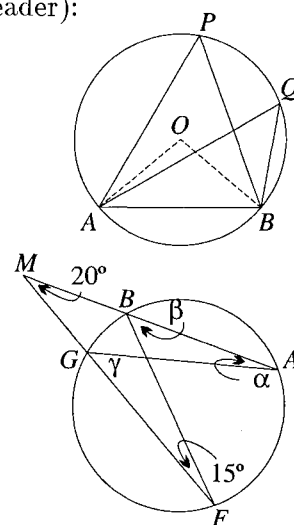
AIM: To prove that $\angle APB = \angle AQB$.

CONSTRUCTION: Join AO and BO .

PROOF: $\angle AOB = 2 \times \angle APB$ (angles on the same arc AB),
and $\angle AOB = 2 \times \angle AQB$ (angles on the same arc AB).
Hence $\angle APB = \angle AQB$.

WORKED EXERCISE: Find α , β and γ in the diagram opposite.

SOLUTION: $\alpha = 15^\circ$ (angles on the same arc BG),
 $\beta = 35^\circ$ (exterior angle of $\triangle BFM$),
 $\gamma = 35^\circ$ (angles on the same arc AF).



Cyclic Quadrilaterals: A *cyclic quadrilateral* is a quadrilateral whose vertices lie on a circle (we say that the quadrilateral is *inscribed* in the circle). A cyclic quadrilateral is therefore formed by taking two angles standing on opposite arcs, which is why its study is relevant here.

COURSE THEOREM:

14

- Opposite angles of a cyclic quadrilateral are supplementary.
- An exterior angle of a cyclic quadrilateral equals the opposite interior angle.

GIVEN: Let $ABCD$ be a cyclic quadrilateral, with side BC produced to T , and let O be the centre of the circle $ABCD$. Let $\angle A = \alpha$ and $\angle C = \gamma$.

AIM: To prove: (a) $\alpha + \gamma = 180^\circ$ (b) $\angle DCT = \alpha$

CONSTRUCTION: Join BO and DO .

PROOF: There are two angles at O , one reflex, one non-reflex.

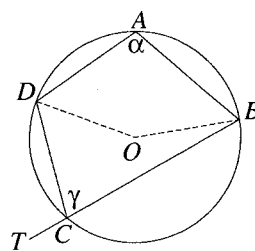
(a) Taking angles on the arc BCD , $\angle BOD = 2\alpha$ (facing C),

Taking angles on the arc BAD , $\angle BOD = 2\gamma$ (facing A).

Hence $2\alpha + 2\gamma = 360^\circ$ (angles in a revolution),

so $\alpha + \gamma = 180^\circ$, as required.

(b) Also, $\angle DCT = 180^\circ - \gamma$ (straight angle),
 $= \alpha$, by part (a).



WORKED EXERCISE: In the diagram below, prove that X , A and Y are collinear.

SOLUTION: Join AB , AX and AY , and let $\angle P = \theta$.

$$\angle XAB = 180^\circ - \theta$$

(opposite angles of cyclic quadrilateral $ABPX$).

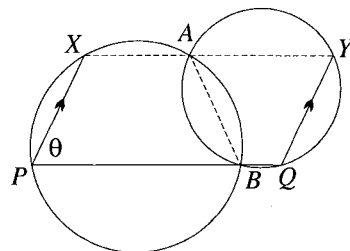
Also $\angle Q = 180^\circ - \theta$

(co-interior angles, $PX \parallel QY$),

so $\angle YAB = \theta$

(opposite angles of cyclic quadrilateral $ABQY$).

Hence $\angle XAY = 180^\circ$, and so XAB is a straight line.

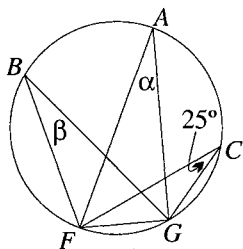


Exercise 9C

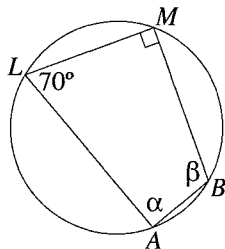
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles.

1. Find α , β and γ as appropriate in each diagram below.

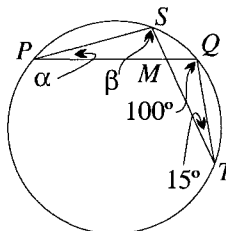
(a)



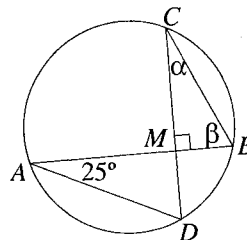
(b)

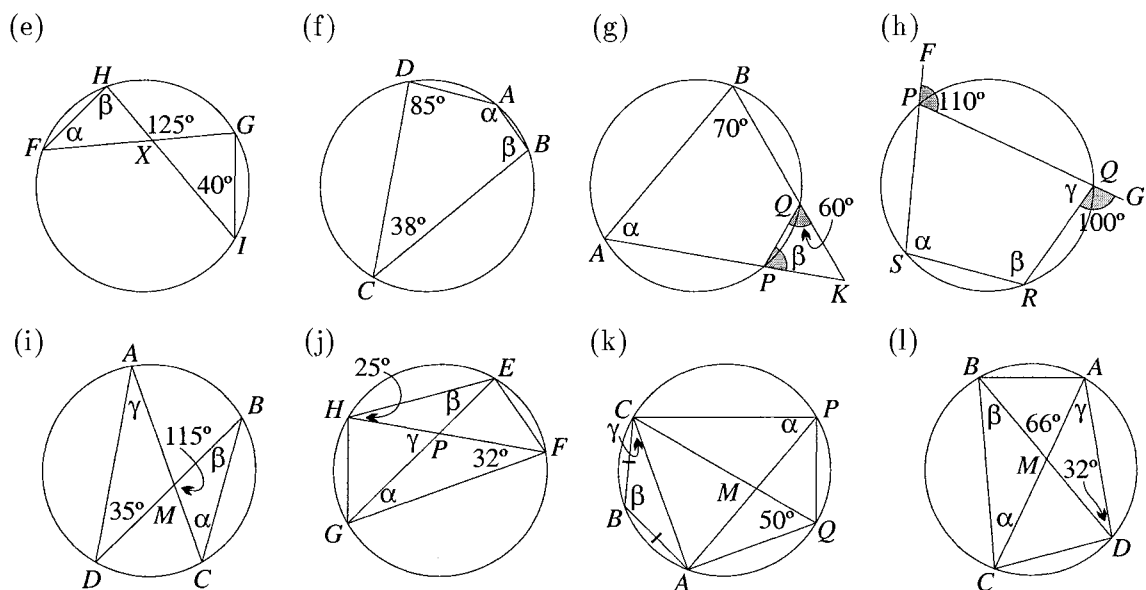


(c)

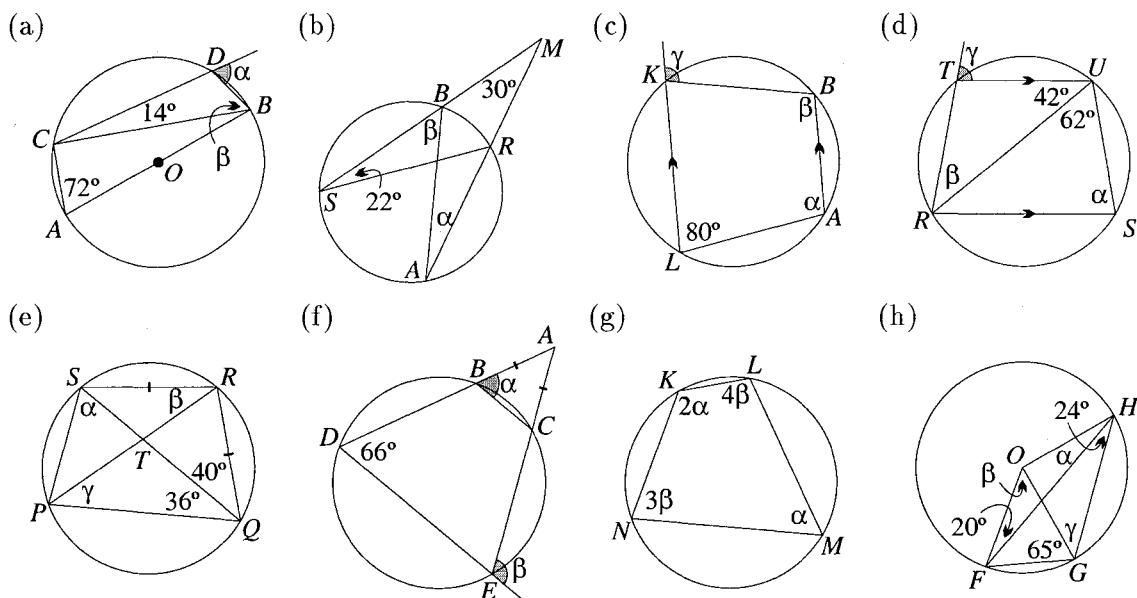


(d)



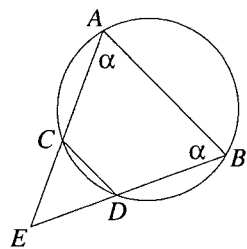


2. Find α , β and γ as appropriate in each diagram.



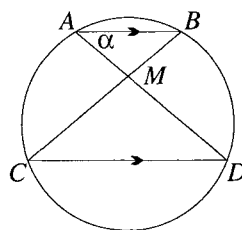
3. Suppose that $ABCD$ is a cyclic quadrilateral. Draw a diagram of $ABCD$, and then explain why $\sin A = \sin C$ and $\sin B = \sin D$.

4. (a)



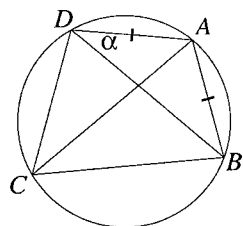
Prove that $CD \parallel AB$.
Prove that $EC = ED$.

(b)



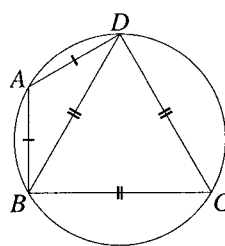
Prove that $\angle A = \angle B = \angle C = \angle D$.
Prove that $AD = BC$.

(c)



Prove that $\angle ACB = \alpha$.
Prove that AC bisects $\angle DCB$.

(d)



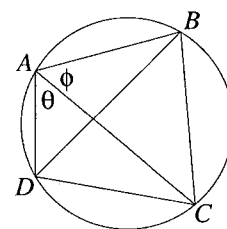
Prove that $AB \perp BC$.

DEVELOPMENT

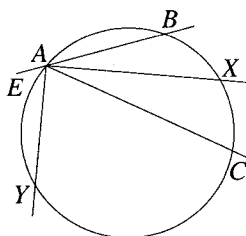
5. ALTERNATIVE PROOF THAT THE OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL ARE SUPPLEMENTARY:

In the diagram opposite:

- (a) Prove that $\angle DBC = \theta$ and $\angle BDC = \phi$.
(b) Hence prove that $\angle DAB$ and $\angle DCB$ are supplementary.

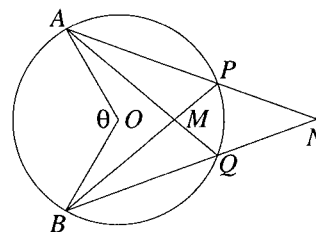


6. (a)



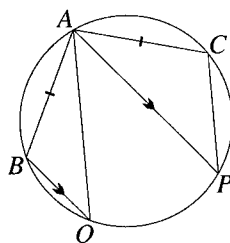
AX bisects $\angle CAB$, AY bisects $\angle CAE$.
Prove that $\angle YAX = 90^\circ$.
Prove that $\angle YCX = 90^\circ$.

(b)



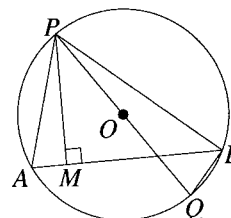
Give a reason why $\angle APB = \frac{1}{2}\theta$.
Show that $\angle BPN = \angle AQN = 180^\circ - \frac{1}{2}\theta$.
Show that $\angle AMB + \angle ANB = \theta$.

(c)



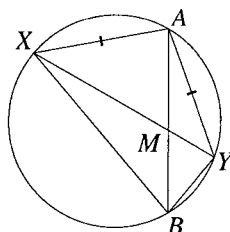
Give a reason why $\angle Q = \angle P$.
Prove that $AQ \parallel CP$.

(d)



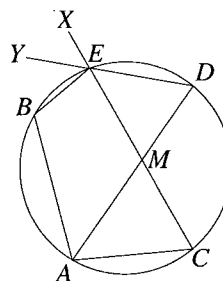
Give a reason why $\angle A = \angle Q$.
Prove that $\angle APM = \angle QPB$.

(e)



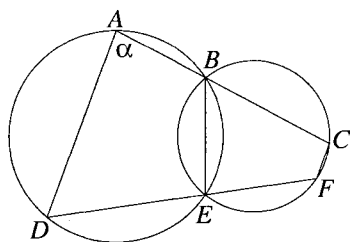
Give a reason why $\angle BXY = \angle BAY$.
Prove that AB bisects $\angle XBY$.
Prove that $\angle XMB = \angle AYB$.

(f)



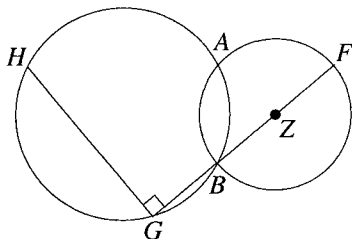
Give a reason why $\angle BAD = \angle BEY$.
Given that DA bisects $\angle BAC$,
prove that YE bisects $\angle XEB$.

7. (a)



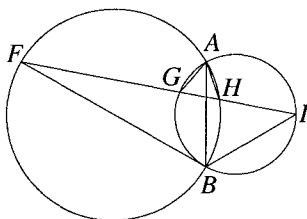
Prove that $\angle BEF = \alpha$ and find $\angle C$.
Prove that $AD \parallel CF$.

(c)



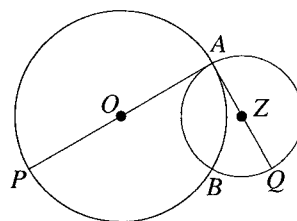
Find $\angle BAF$ and $\angle BAH$.
Prove that H, A and F are collinear.

(e)



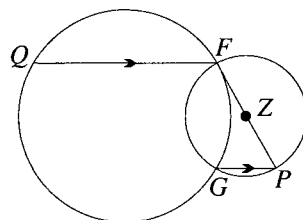
Give a reason why $\angle FBA = \angle FHA$.
Given that AB bisects $\angle FBI$,
prove that $AG = AH$.

(b)



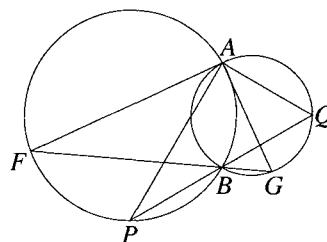
Find $\angle ABP$ and $\angle ABQ$.
Prove that P, B and Q are collinear.

(d)



Prove that QG is a diameter. If the radii are equal, prove that $QG \parallel FP$.

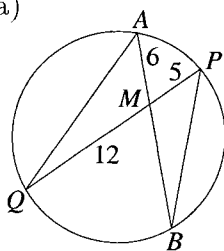
(f)



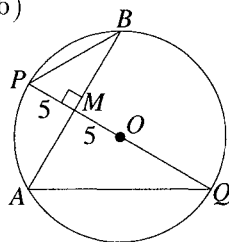
Give a reason why $\angle Q = \angle G$.
Given that FBG and PBQ are straight lines, prove that $\angle FAP = \angle GAQ$.

8. In each diagram, prove that $\triangle AMQ \parallel \triangle PMB$. Then find MB .

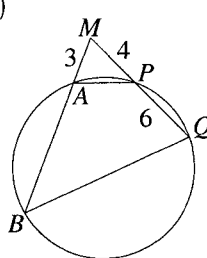
(a)



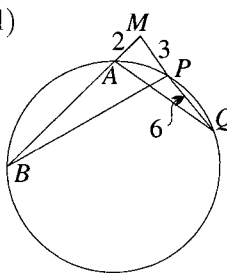
(b)



(c)



(d)

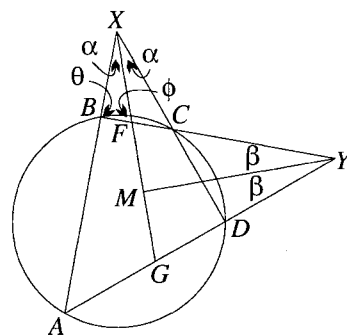


9. THEOREM: Let the two pairs of opposite sides of a cyclic quadrilateral meet, when produced, at X and Y respectively. Then the angle bisectors of $\angle X$ and $\angle Y$ are perpendicular.

In the diagram opposite:

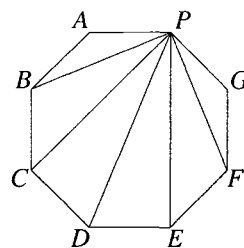
(a) Explain why $\angle XDA = \theta$.(b) Using $\triangle XGD$ and $\triangle XFB$, prove that $\angle XGD = \phi$.(c) Using $\triangle MYF$ and $\triangle MYG$, prove that $YM \perp XM$.

(d) How should this theorem be restated when a pair of opposite sides is parallel?



10. THEOREM: Diagonals in a regular polygon.

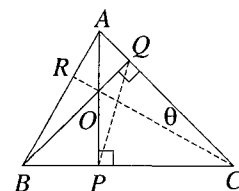
- (a) In the regular octagon opposite, use the circumcircle to prove that the six angles between adjacent diagonals at P are all equal. Hence find the value of $\alpha = \angle APB$.
- (b) More generally, prove that the angles between adjacent diagonals at any vertex of an n -sided regular polygon are all equal, and have the value $\frac{180^\circ}{n}$.



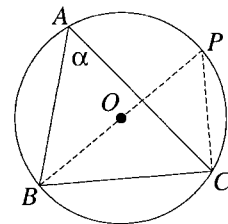
- 11.** (a) Prove that a cyclic parallelogram is a rectangle.
 (b) Prove that a cyclic rhombus is a square.
 (c) Prove that the non-parallel opposite sides of a cyclic trapezium are equal.
- 12.** Let A, B, C, D , and E be five points in order around a circle with centre O , and let AOE be a diameter. Prove that $\angle ABC + \angle CDE = 270^\circ$.
- 13.** (a) Prove that if two chords of a circle bisect each other, then they are both diameters.
 (b) Prove that if the chords AB and PQ intersect at M and $MA = MP$, then $MB = MQ$, $BP = AQ$ and $AP \parallel QB$.

14. THE ORTHOCENTRE THEOREM: *The three altitudes of a triangle are concurrent (their intersection is called the orthocentre of the triangle).*

In the diagram opposite, the two altitudes AP and BQ meet at O . Join CO and produce it to R , and join PQ .

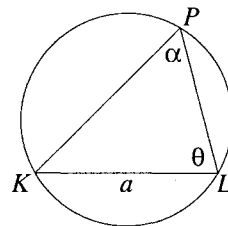


- (a) Explain why $OPCQ$ and $AQPB$ are cyclic.
- (b) Let $\angle ACR = \theta$, and explain why $\angle APQ = \angle ABQ = \theta$.
- (c) Use $\triangle OQC$ and $\triangle ORB$ to prove that $CR \perp AB$.
- 15. THE SINE RULE AND THE CIRCUMCIRCLE:** *The ratio of any side of a triangle to the sine of the opposite angle is the diameter of the circumcircle.*
 Let $\angle A$ in $\triangle ABC$ be acute, and let O be the centre of the circumcircle of $\triangle ABC$. Join BO and produce it to a diameter BOP , then join PC .
- (a) Let $\angle A = \alpha$, and explain why $\angle P = \alpha$.
- (b) Explain why $\triangle BPC$ is a right triangle.
- (c) Hence prove that $\frac{BC}{\sin \alpha} = BOP$.
- (d) Repeat the construction and proof when $\angle A$ is obtuse.



EXTENSION

- 16. MAXIMISATION:** In the diagram below, KL is a fixed chord of length a , and the point P varies on the major arc KL . Let y be the sum of the lengths of PK and PL .
- (a) Explain why α is constant as P varies.
- (b) Use the sine rule to prove that $y = \frac{a}{\sin \alpha} (\sin \theta + \sin(\theta + \alpha))$.
- (c) Find $\frac{dy}{d\theta}$, and show that $\frac{d^2y}{d\theta^2} = -y$.
- (d) Prove that y is maximum when $\theta = \frac{1}{2}(\pi - \alpha)$, then find and simplify the maximum value.



17. MATHEMATICAL INDUCTION: *The alternating sums of the angles of a cyclic polygon.*

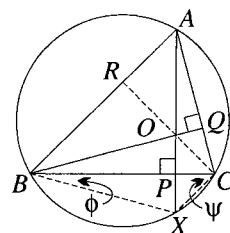
(a) Prove that if $ABCD$ is a cyclic quadrilateral, then $\angle A - \angle B + \angle C - \angle D = 0$.

(b) Prove that if $A_1A_2A_3A_4A_5A_6$ is a cyclic hexagon, then $\sum_{k=1}^6 (-1)^k \angle A_k = 0$.

[HINT: Use the major diagonal A_1A_4 to divide the hexagon into two cyclic quadrilaterals, then apply part (a) to each quadrilateral.]

(c) Use mathematical induction, and the same method as in part (b), to prove that for any cyclic polygon $A_1A_2 \dots A_{2n}$ with an even number of vertices, $\sum_{k=1}^{2n} (-1)^k \angle A_k = 0$.

18. THE ORTHOCENTRE THEOREM: *A proof using the circum-circle.* In the diagram, the two altitudes AP and BQ meet at O . Join CO and produce it to R . Produce AP to meet the circumcircle of $\triangle ABC$ at X , and join BX and CX . Let $\angle CBX = \phi$ and $\angle BCX = \psi$.



(a) Explain why $\angle CAX = \phi$.

(b) Using $\triangle QOA$ and $\triangle POB$, prove that $\angle PBO = \phi$.

(c) Prove that $\triangle POB \equiv \triangle PXB$, and hence that $PO = PX$.

(d) Prove that $\triangle POC \equiv \triangle PXC$, and hence that $\angle OCP = \psi$.

(e) By comparing $\triangle POC$ and $\triangle ROA$, prove that $CR \perp AB$.

19. The last two questions of Exercise 4J in the Year 11 volume contain a variety of algebraic results about cyclic quadrilaterals and their circumcircles, established using trigonometry. Those results and their proofs could be examined in the present context of Euclidean geometry. See also the related questions about the circumcircle and incircle of a triangle at the end of Exercises 4H and 4I in the Year 11 volume.

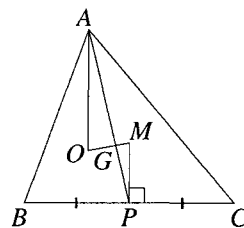
20. THE EULER LINE THEOREM: *The orthocentre, centroid and circumcentre of a triangle are collinear (the line is called the Euler line), with the centroid trisecting the interval joining the other two centres.*

Let M and G be the circumcentre and centroid respectively of $\triangle ABC$. Join MG , and produce it to a point O so that $OG : GM = 2 : 1$. We must prove that O is the orthocentre of $\triangle ABC$.

(a) Let P be the midpoint of BC . Use the fact that $AG : GP = 2 : 1$ to prove that $\triangle GMP \parallel \triangle GAO$.

(b) Hence prove that O lies on the altitude from A .

(c) Complete the proof.



9D Concyclic Points

A set of points is called *concyclic* if they all lie on a circle. The converses of the two theorems of the previous section provide two general tests for four points to be concyclic. There is an important logical structure here to keep in mind. First, any two distinct points lie on a unique line, but three points may or may not be collinear. Secondly, any three non-collinear points are concyclic, as proven in Section 9A, but four points may or may not be concyclic.

Concyclicity Test — Two Points on the Same Side of an Interval: We have proven that angles at the circumference standing on the same arc of a circle are equal. The converse of this is:

15 COURSE THEOREM: If two points lie on the same side of an interval, and the angles subtended at these points by the interval are equal, then the two points and the endpoints of the interval are concyclic.

The most satisfactory proof makes use of the forward theorem.

GIVEN: Let P and Q be points on the same side of an interval AB such that $\angle APB = \angle AQB = \alpha$.

AIM: To prove that the points A, B, P and Q are concyclic.

CONSTRUCTION: Construct the circle through A, B and P , and let the circle meet AQ (produced if necessary) at X . Join XB .

PROOF: Using the forward theorem,

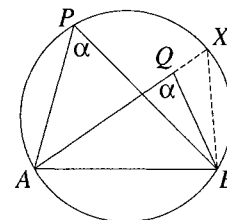
$$\angle AXB = \angle APB = \alpha \quad (\text{angles on the same arc } AB).$$

Hence $\angle AXB = \angle AQB$,

so $QB \parallel XB$ (corresponding angles are equal).

But QB and XB intersect at B , and are therefore the same line.

Hence Q and X coincide, and so Q lies on the circle.



WORKED EXERCISE: In the diagram opposite, $AB = AG$.

Prove that $ACGD$ is cyclic, and that $\angle ACD = \angle AGD$.

SOLUTION: Let $\angle B = \beta$.

Then $\angle AGB = \beta$ (base angles of isosceles $\triangle BAG$)

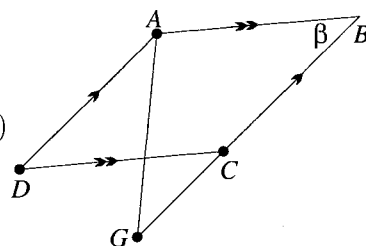
and $\angle ADC = \beta$

(opposite angles of parallelogram $ABCD$),

so the quadrilateral $ACGD$ is cyclic,

because AC subtends equal angles at D and G .

Hence $\angle ACD = \angle AGD$ (angles on the same arc AD).



Concyclicity Test — Cyclic Quadrilaterals: The converses of the two forms of the cyclic quadrilateral theorem are:

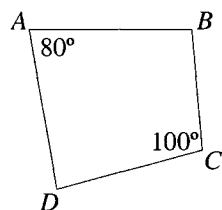
16 COURSE THEOREM:

- If one pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.
- If one exterior angle of a quadrilateral is equal to the opposite interior angle, then the quadrilateral is cyclic.

Since the exterior angle and the adjacent interior angle are supplementary, being angles in a straight angle, we need only prove the first test, and the second will follow immediately. The proof of the first test is similar to the previous proof, and is left to the exercises.

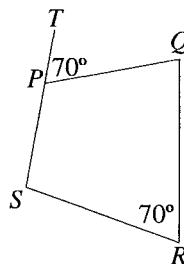
WORKED EXERCISE: Give reasons why each quadrilateral below is cyclic.

(a)



$ABCD$ is a cyclic quadrilateral
(opposite angles are supplementary).

(b)



$PQRS$ is a cyclic quadrilateral (exterior angle equals opposite interior angle).

NOTE: When the angles subtended by the interval are right angles, the four points are concyclic by the earlier theorem that a right angle was an angle in a semicircle, moreover the interval is then a diameter of the circle. These two tests for the concyclicity of four points should therefore be seen as generalisations of that theorem.

WORKED EXERCISE: Prove that if $FGCB$ is cyclic, then $FB = GC$.
Prove that if $FB = GC$, then $FGCB$ is cyclic.

SOLUTION: Let $\angle AFG = \alpha$.

Then $\angle AGF = \alpha$ (base angles of isosceles $\triangle AFG$).

Suppose first that $FGCB$ is cyclic.

Then $\angle C = \alpha$ (exterior angle of cyclic quadrilateral $FGCB$)

and $\angle B = \alpha$ (exterior angle of cyclic quadrilateral $FGCB$),

so $AB = AC$ (opposite angles of $\triangle ABC$ are equal),

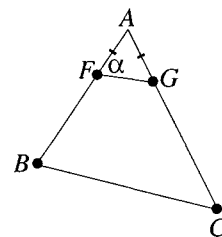
hence $FB = GC$ (subtracting the equal intervals AF and AG).

Suppose secondly that $FB = GC$.

Then $FG \parallel BC$ (intercepts on AB and AC),

so $\angle B = \alpha$ (corresponding angles, $FG \parallel BC$),

hence $FGCB$ is cyclic (exterior angle $\angle AGF$ equals interior opposite angle $\angle B$).

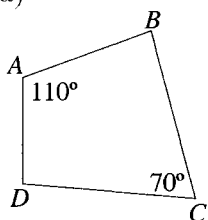


Exercise 9D

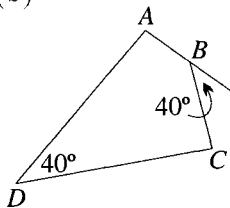
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O are centres of the appropriate circles.

1. In each diagram, give a reason why $ABCD$ is a cyclic quadrilateral.

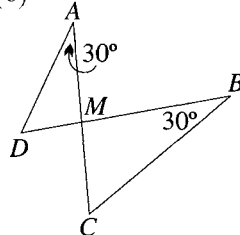
(a)



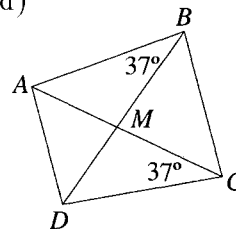
(b)



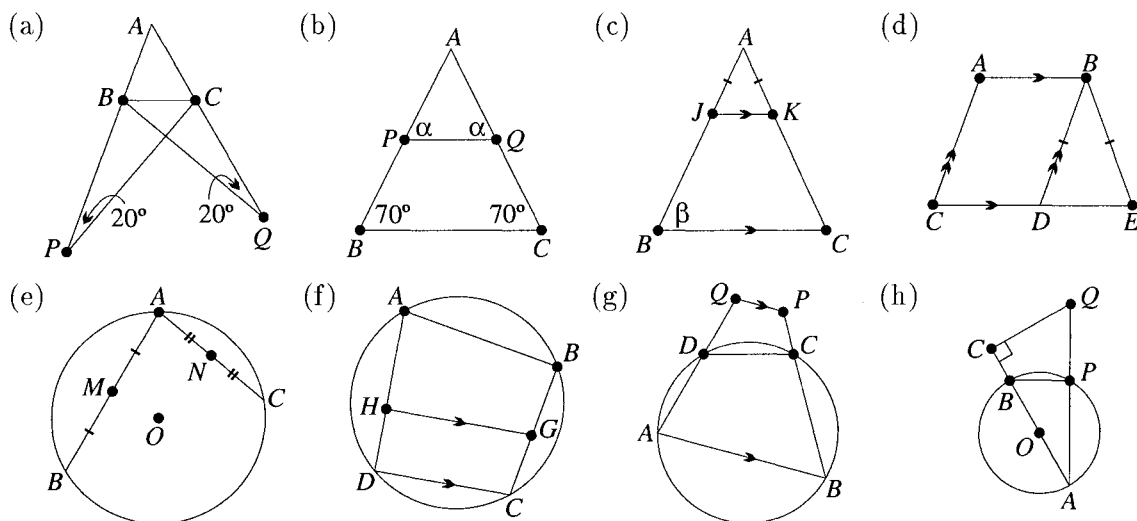
(c)



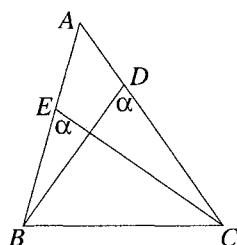
(d)



2. In each diagram, prove that the four darkened points are concyclic.

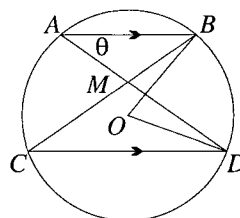


3. (a)



Prove that $BEDC$ is cyclic.
Hence prove that $\angle EBD = \angle ECD$,
and that $\angle ADE = \angle ABC$.

(b)



Prove that $\angle BMD = 2\theta$, and
hence prove that $BMOD$ is cyclic.
Hence prove that $\angle MBO = \angle MDO$.

4. (a) Prove that every rectangle is cyclic.

(b) Prove that any quadrilateral $ABCD$ in which $\angle A - \angle B + \angle C - \angle D = 0^\circ$ is cyclic.

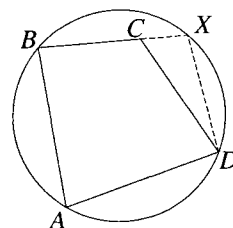
DEVELOPMENT

5. COURSE THEOREM: If one pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

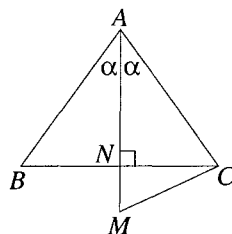
Let $ABCD$ be a quadrilateral in which $\angle A + \angle BCD = 180^\circ$.
Construct the circle through A, B and D , and let it meet BC
(produced if necessary) at X . Join DX .

(a) Prove that $\angle BXD + \angle A = 180^\circ$.

(b) Prove that $CD \parallel XD$, and that C and X coincide.



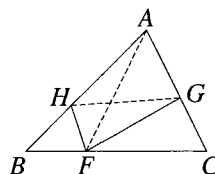
6. (a)



(i) Prove that if $ABMC$ is cyclic,
then $MC \perp AC$.

(ii) Prove that if $MC \perp AC$,
then $ABMC$ is cyclic.

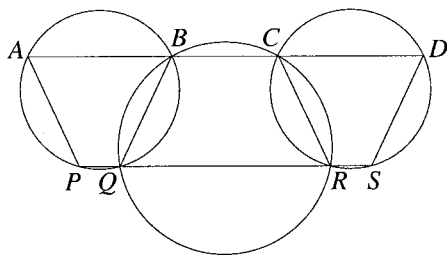
(b)



(i) Prove that if $\angle BHF = \angle AGF$, then
 $FGAH$ is cyclic and $\angle AHG = \angle AFG$.

(ii) Prove that if $\angle AHG = \angle AFG$, then
 $FGAH$ is cyclic and $\angle BHF = \angle AGF$.

7. (a)



In the diagram above, $ABCD$ and $PQRS$ are straight lines, not necessarily parallel.

- Show that $AP \parallel CR$.
- Show that $APSD$ is cyclic.

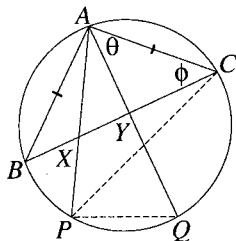
8. Let AB and XY be parallel intervals, with AY and BX meeting at M .

- Prove that if $AXYB$ is cyclic, then $MA = MB$.
- Prove that if $MA = MB$, then $AXYB$ is cyclic.

9. Let P, Q and R be the midpoints of three chords MA, MB and MC of a circle.

- Prove that $PQ \parallel AB$ and $QR \parallel BC$.
- Prove that M, P, Q and R are concyclic.

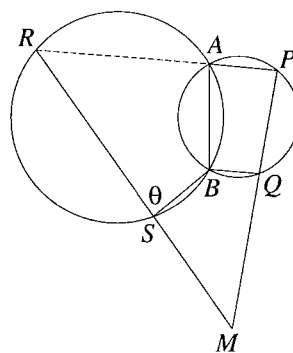
10. (a)



In the diagram above, $AB = AC$.

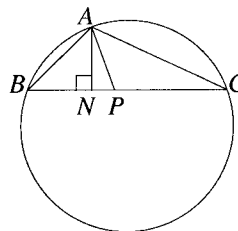
- Prove that $\angle CPQ = \theta$.
- Prove that $\angle CPA = \phi$.
- Hence prove that $PQYX$ is cyclic.

(b)



- Prove that $\angle PAB = \theta$.
- Prove that if S, B, Q and M are concyclic, then R, A , and P are collinear.
- Prove that if R, A and P are collinear, then $SBQM$ is cyclic.

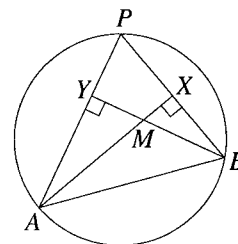
(b)



- Prove that if AP produced is a diameter of circle ABC , then $\angle BAN = \angle CAP$.
- Prove that if $\angle BAN = \angle CAP$, then AP produced is a diameter of circle ABC .

11. The chord AB of the circle opposite is fixed, and the point P varies on the major arc of the circle. The altitudes AX and BY of $\triangle ABP$ meet at M .

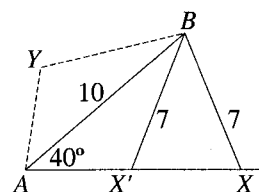
- Let $\angle P = \alpha$. Explain why α is constant.
- Explain why $PXMY$ is cyclic.
- Show that $\angle AMB = 180^\circ - \alpha$, and find the locus of M .



EXTENSION

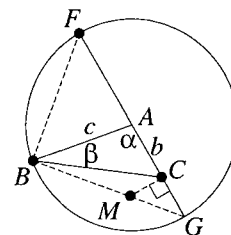
12. TRIGONOMETRY: The spurious ASS congruence test can be related to cyclic quadrilaterals.

The unbroken lines represent a construction of the two possible triangles ABX and ABX' in which $\angle BAX = 40^\circ$, $AB = 10$ and $BX = 7$. The broken lines represent $\triangle ABX'$ reflected about AB to $\triangle ABY$. Prove that the two triangles together form a cyclic quadrilateral $AXBY$.

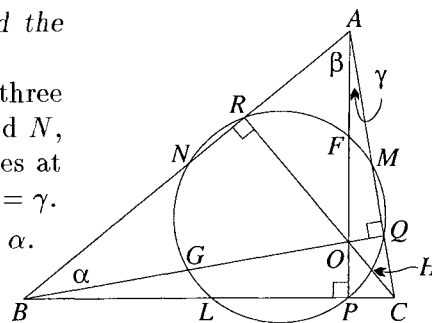
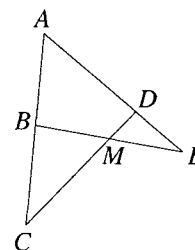


13. A TRIGONOMETRIC THEOREM: $\tan(B + \frac{1}{2}A) = \frac{c+b}{c-b} \tan \frac{1}{2}A$ in any triangle ABC .

Let ABC be a triangle in which $c > b$. Let $\angle ABC = \beta$ and $\angle CAB = \alpha$. Construct the circle with centre A passing through B , and construct the diameter $FACG$. Let the perpendicular to $FACG$ through C meet BG at M .



- Explain why $AF = c$ and $CG = c - b$.
 - Prove that C, M, B and F are concyclic.
 - Prove that $\angle BFC = \frac{1}{2}\alpha = \angle CMG$.
 - Prove that $\angle FBC = \beta + \frac{1}{2}\alpha = \angle FMC$.
 - Prove that $CM = (c - b) \cot \frac{1}{2}\alpha = (c + b) \cot(\beta + \frac{1}{2}\alpha)$.
 - Adapt the construction to prove the theorem when $c < b$.
14. ABC and ADE are any two intervals meeting at A . Let BE and DC meet at M , and let the circles CMB and EMD meet again at N . Prove that $ADNC$ and $ABNE$ are cyclic. [HINT: Join NM, NC and NE .]
15. Referring to the diagram in question 11, where the chord AB is constant and P varies:
- Explain why $AYXB$ is cyclic, and locate the centre of this circle.
 - Prove that $\angle YAX$ is constant, and that the interval XY has constant length.
 - What is the locus of the midpoint of XY ?
16. THE NINE-POINT CIRCLE THEOREM: *The circle through the feet of the three altitudes of a triangle passes through the three midpoints of the sides, and bisects the three intervals joining the orthocentre to the vertices. Its centre is the midpoint of the interval joining the circumcentre and the orthocentre.*
- In $\triangle ABC$ opposite, P, Q and R are the feet of the three altitudes. The circle PQR meets the sides at L, M and N , and the intervals joining the orthocentre to the vertices at F, G and H . Let $\angle ABO = \alpha, \angle BAO = \beta$ and $\angle CAO = \gamma$.
- Prove that $\angle RBO = \angle RPO = \angle QPO = \angle QCO = \alpha$.
 - Proceed similarly with β and γ .
 - Prove that $\alpha + \beta + \gamma = 180^\circ$.
 - Prove that $\angle RLQ = 2\alpha$, and hence that $BL = LC$.
 - Prove that $\angle LHC = \angle RPL$, and hence that $OH = HC$.
17. Suppose that $ABCD$ is a square, and a point P is placed so that $AP : BP : CP = 1 : 2 : 3$.
- Find the size of $\angle APB$.
 - Give a straight-edge-and-compasses construction of the point P .

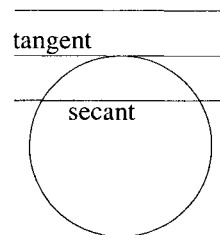


9E Tangents and Radii

Tangents were the object of intensive study in calculus, because the derivative was defined as the gradient of the tangent. Circles, however, were their original context, and the results in the remainder of this chapter are developed without reference to the derivative.

Tangent and Radius: We shall assume that given a circle, any line is a secant crossing a circle at two points, or is a tangent touching it at one point, or misses the circle entirely.

- 17 **DEFINITION:** A *tangent* is a line that meets a circle in one point, called the *point of contact*.

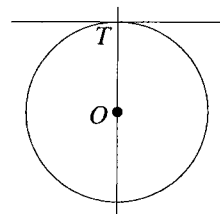


We shall also make the following assumption about the relationship between a tangent and the radius at the point of contact.

- 18 **COURSE ASSUMPTION:** At every point on a circle, there is one and only one tangent to the circle at that point. This tangent is the line through the point perpendicular to the radius at the point.

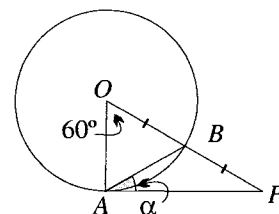
This result can easily be seen informally in two ways. First, a diameter is an axis of symmetry of a circle — this symmetry reflects the perpendicular line at the endpoint T onto itself, and so the perpendicular line cannot meet the circle again, and is therefore a tangent.

Alternatively, if a line ever comes closer to the centre than the radius to the centre, then it will cross the circle twice and be a secant, so a tangent at a point T on a circle must be a line whose point of closest approach to the centre is T — but the closest distance to the centre is the perpendicular distance, therefore the tangent is the line perpendicular to the radius.



WORKED EXERCISE: Find α in the diagram below, where O is the centre, and prove that PA is a tangent to the circle.

SOLUTION: $OA = OB$ (radii),
 so $\angle OAB = \angle OBA = 60^\circ$ (angle sum of isosceles $\triangle OAB$),
 so $BA = OB = PB$ ($\triangle OBA$ is equilateral).
 Hence $\alpha = \angle P = 30^\circ$ (exterior angle of isosceles $\triangle BAP$),
 so $\angle OAP = 90^\circ$ (adjacent angles).
 Hence PA is a tangent to the circle.



Tangents from an External Point: The first formal theorem about tangents concerns the two tangents to a circle from a point outside the circle.

- 19 **COURSE THEOREM:** The two tangents from an external point have equal lengths.

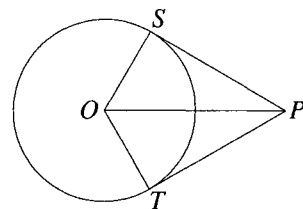
GIVEN: Let PS and PT be two tangents to a circle with centre O from an external point P .

AIM: To prove that $PS = PT$.

CONSTRUCTION: Join OP , OS and OT .

PROOF: In the triangles SOP and TOP :

1. $OS = OT$ (radii),
 2. $OP = OP$ (common),
 3. $\angle OSP = \angle OTP = 90^\circ$ (radius and tangent),
- so $\triangle SOP \equiv \triangle TOP$ (RHS).
 Hence $PS = PT$ (matching sides of congruent triangles).



WORKED EXERCISE: Use the construction established above to prove:

- The tangents from an external point subtend equal angles at the centre.
- The interval joining the centre and the external point bisects the angle between the tangents.

PROOF: Using the congruence $\triangle SOP \equiv \triangle TOP$ established above:

- $\angle SOP = \angle TOP$ (matching angles of congruent triangles),
- $\angle SPO = \angle TPO$ (matching angles of congruent triangles).

Touching Circles: Two circles are said to *touch* if they have a common tangent at the point of contact. They can touch *externally* or *internally*, as the two diagrams below illustrate.

20

COURSE THEOREM: When two circles touch (internally or externally), the two centres and the point of contact are collinear.

GIVEN: Let two circles with centres O and Z touch at T .

AIM: To prove that O , T and Z are collinear.

CONSTRUCTION: Join OT and ZT , and construct the common tangent XTY at T .

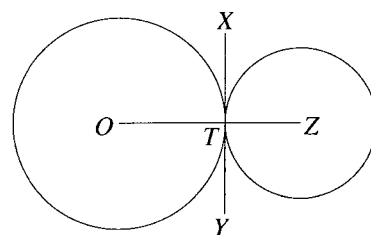
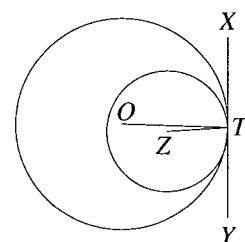
PROOF: There are two possible cases, because the circles can touch internally or externally, but the argument is practically the same in both. Since XY is a tangent and OT and ZT are radii,

$$\angle OTX = 90^\circ \text{ and } \angle ZTX = 90^\circ.$$

Hence $\angle OTZ = 180^\circ$ (when the circles touch externally),

or $\angle OTZ = 0^\circ$ (when the circles touch internally).

In both cases, O , T and Z are collinear.



Direct and Indirect Common Tangents: There are two types of *common tangents* to a given pair of circles:

21

DIRECT AND INDIRECT COMMON TANGENTS: A common tangent to a pair of circles:

- is called *direct*, if both circles are on the same side of the tangent,
- is called *indirect*, if the circles are on opposite sides of the tangent.

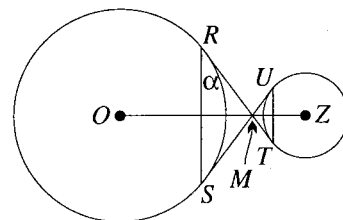
The two types are illustrated in the worked exercise below. Notice that according to this definition, the common tangent at the point of contact of two touching circles is a type of indirect common tangent if they touch externally, and a type of direct common tangent if they touch internally.

WORKED EXERCISE: Given two unequal circles and a pair of direct or indirect common tangents (notice that there are two cases):

- Prove that the two tangents have equal lengths.
- Prove that the four points of contact form a trapezium.
- Prove that their point of intersection is collinear with the two centres.

GIVEN: Let the two circles have centres O and Z .

Let the tangents RT and US meet at M .



CONSTRUCTION: Join OM and ZM .

AIM: To prove:

- (a) $RT = SU$, (b) $RS \parallel UT$,
 (c) OMZ is a straight line.

PROOF:

- (a) $RM = SM$ (tangents from an external point),
 and $TM = UM$ (tangents from an external point),
 so $RT = RM - MT = SM - MU = SU$ (direct case),
 or $RT = RM + MT = SM + MU = SU$ (indirect case).

- (b) Let $\alpha = \angle SRM$.

Then $\angle RSM = \alpha$ (base angles of isosceles $\triangle RMS$),

so $\angle RMS = 180^\circ - 2\alpha$ (angle sum of $\triangle RMS$),

so $\angle TMU = 180^\circ - 2\alpha$ (vertically opposite, or common, angle),

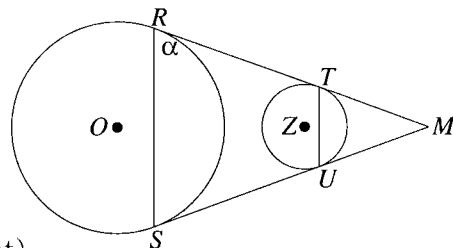
so $\angle UTM = \alpha$ (base angles of isosceles $\triangle TMU$).

Hence $RS \parallel TU$ (alternate or corresponding angles are equal).

- (c) From the previous worked exercise, both OM and ZM bisect the angle between the two tangents, and hence

$$\angle RMO = \angle TMZ = 90^\circ - \alpha.$$

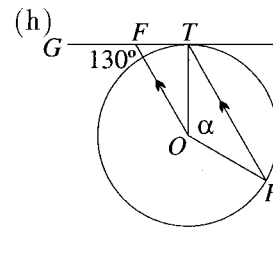
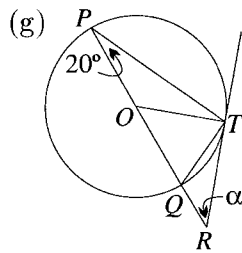
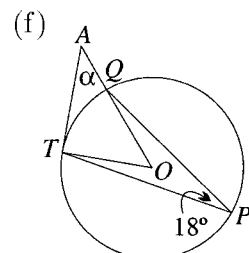
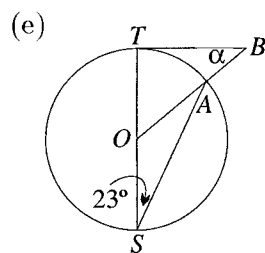
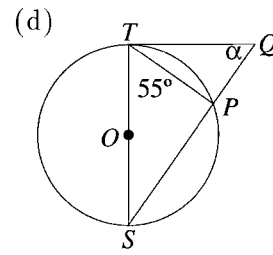
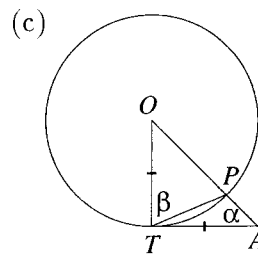
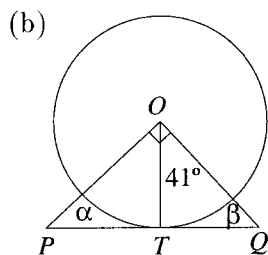
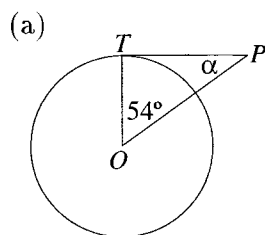
In the direct case, OM and ZM must be the same arm of the angle with vertex M . In the indirect case, OMZ is a straight line by the converse of the vertically opposite angles result.



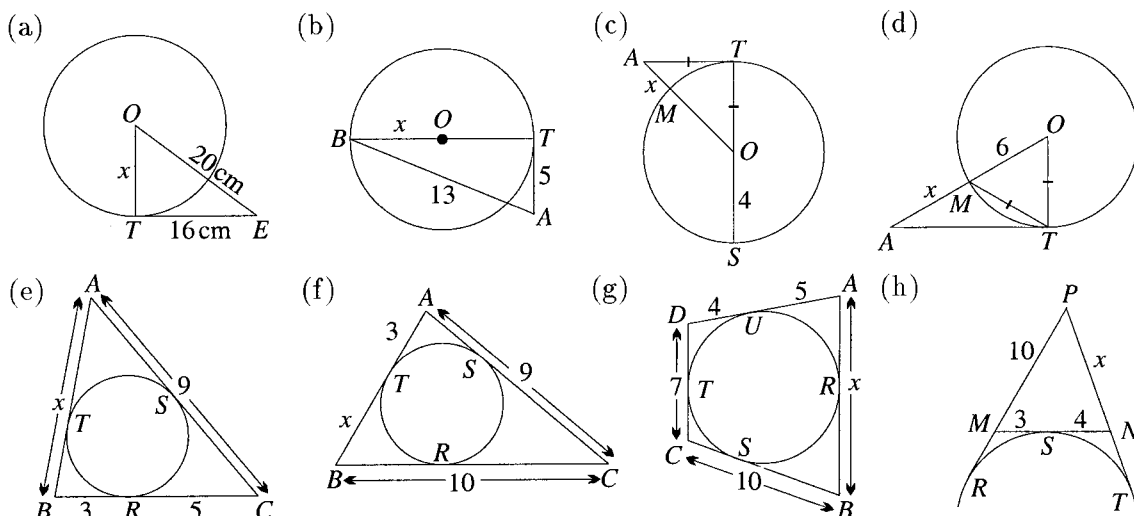
Exercise 9E

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles, and the obvious lines at points labelled R, S, T and U are tangents.

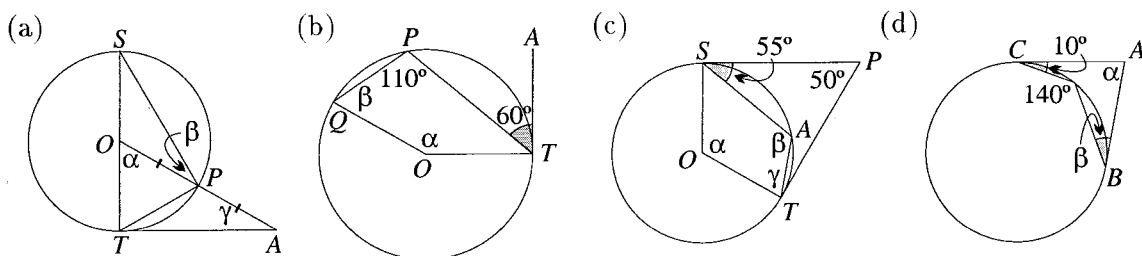
1. Find α and β in each diagram below.



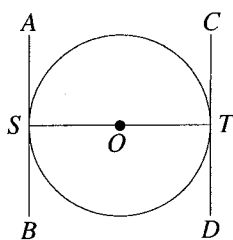
2. Find x in each diagram.



3. Find α , β and γ in each diagram below.

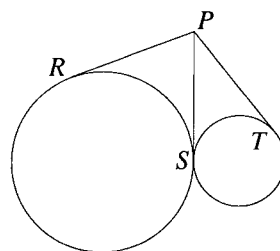


4. (a)



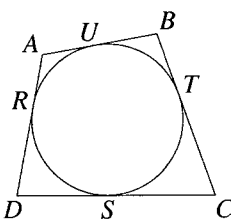
Prove that the tangents at S and at T are parallel.

(b)



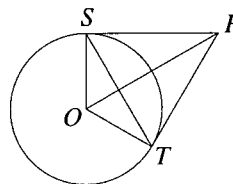
Prove that the three tangents PR , PS and PT from the point P on the common tangent at the point S of contact are equal.

(c)



Prove that $AB + DC = AD + BC$.

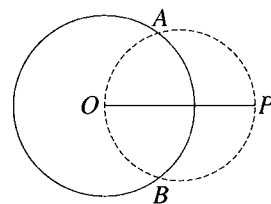
(d)



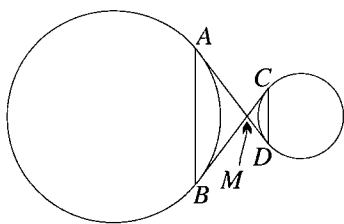
Prove that $OSPT$ is cyclic, and hence that $\angle OST = \angle OPT$ and $\angle TOP = \angle TSP$.

5. CONSTRUCTION: Construct the tangents to a given circle from a given external point.

Given a circle with centre O and an external point P , construct the circle with diameter OP , and let the two circles intersect at A and B . Prove that PA and PB are tangents.

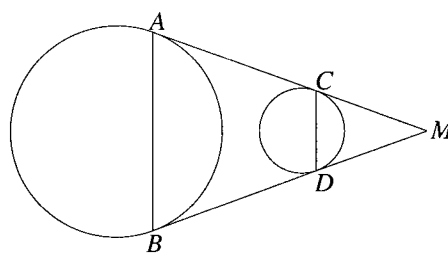


6. (a)



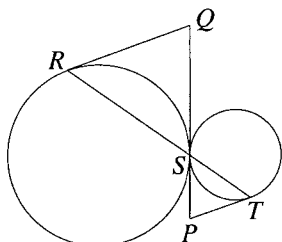
Prove: (i) the indirect common tangents AD and BC are equal, (ii) $AB \parallel CD$.

(b)



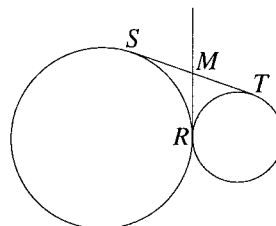
Prove: (i) the direct common tangents AC and BD are equal, (ii) $AB \parallel CD$.

(c)



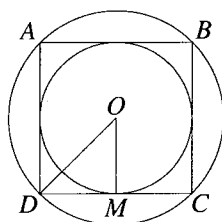
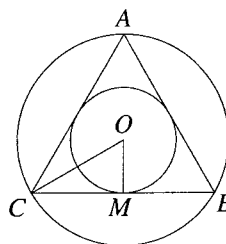
Prove that $PT \parallel RQ$.

(d)



Prove that the common tangent MR at the point of contact bisects the direct common tangent ST , and that $SR \perp TR$.

7. In each diagram, both circles have centre O and the inner circle has radius r . Find the radius of the outer circle if:

(a) $ABCD$ is a square,(b) ABC is an equilateral triangle,

DEVELOPMENT

8. (a) THEOREM: The line joining the centre of a circle to an external point is the perpendicular bisector of the chord joining the points of contact.

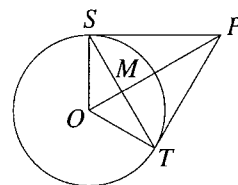
It was proven that $\angle TOM = \angle SOM$ in a worked exercise. Using this, prove by congruence that $TS \perp OP$.

(b) THEOREM: In the same notation, the semichord TM is the geometric mean of the intercepts PM and OM .

(i) Prove that $\triangle MPT \parallel \triangle MTO$.

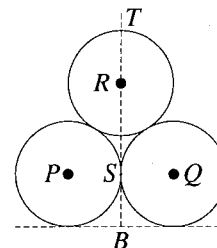
(ii) Hence prove that $TM^2 = PM \times OM$.

(c) Given that $OM = 7$ and $MP = 28$, find ST .

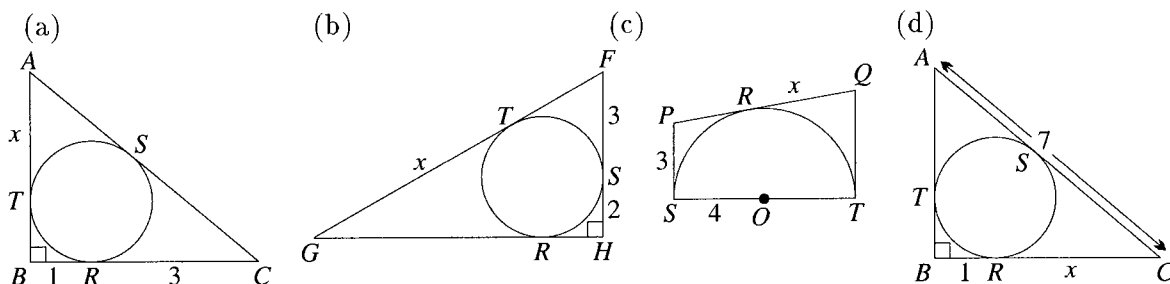


9. (a) Show that an equilateral triangle of side length $2r$ has altitude of length $r\sqrt{3}$.

(b) Hence find the height of the pile of three circles of equal radius r drawn to the right.



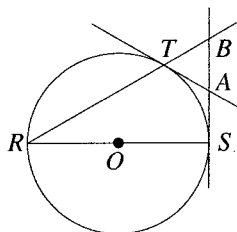
10. In each diagram, use Pythagoras' theorem to form an equation in x , and then solve it. [HINT: In part (c), drop a perpendicular from P to QT .]



11. In each diagram below, prove: (i) $\triangle PAT \parallel \triangle QBT$, (ii) $AP \parallel QB$.

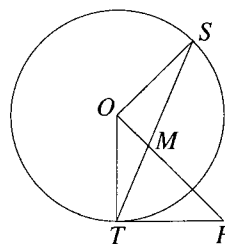


12. (a)



Prove that $BA = AS$.
[HINT: Join TO and TS , then let $\angle R = \theta$.]

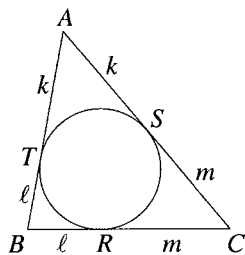
- (b)



Given that $PT = PM$, prove that PO is perpendicular to SO .
[HINT: Let $\angle TMP = \theta$.]

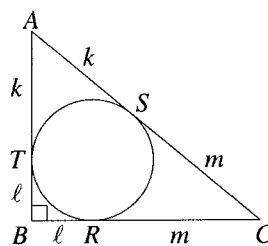
13. CONSTRUCTION: Construct the circle with a given point as centre and tangential to a given line not passing through the point. Use the fact that a tangent is perpendicular to the radius at the point of contact to find a ruler-and-compasses construction of the circle.

14. (a)



Suppose that the circle RST is inscribed in $\triangle ABC$. Prove that $k = \frac{1}{2}(-a + b + c)$, $\ell = \frac{1}{2}(a - b + c)$ and $m = \frac{1}{2}(a + b - c)$.

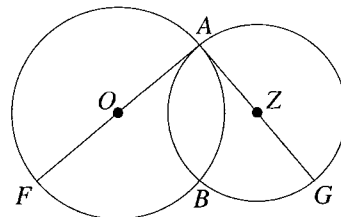
- (b)



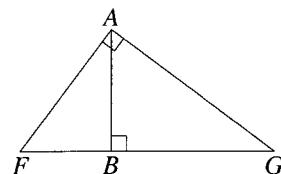
Suppose further that $\angle ABC = 90^\circ$. Prove that $k = \frac{\ell(m + \ell)}{m - \ell}$, and find a , b and c in terms of ℓ and m .

15. THEOREM: Suppose that two circles touch externally, and fit inside a larger circle which they touch internally. Then the triangle formed by the three centres has perimeter equal to the diameter of the larger circle. Prove this theorem using a suitable diagram.

16. (a) Two circles with centres O and Z intersect at A and B so that the diameters AOF and AZG are each tangent to the other circle.

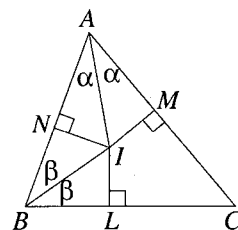


- (i) Prove that F , B and G are collinear.
- (ii) Prove that $\triangle AGB \parallel \triangle FAB$, and hence prove that AB is the geometric mean of FB and GB (meaning that $AB^2 = FB \times GB$).
- (b) Conversely, suppose that AB is the altitude to the hypotenuse of the right triangle AFG . Prove that the circles with diameters AF and AG intersect again at B , and are tangent to AG and AF respectively.



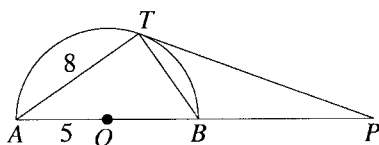
17. (a) Two circles of radii 5 cm and 3 cm touch externally. Find the length of the direct common tangent.
- (b) Two circles of radii 17 cm and 10 cm intersect, with a common chord of length 16 cm. Find the length of the direct common tangent.
- (c) Two circles of radii 5 cm and 4 cm are 3 cm apart at their closest point. Find the lengths of the direct and indirect common tangents.
18. Prove the following general cases of the previous question.
- (a) THEOREM: The direct common tangent of two circles touching externally is the geometric mean of their diameters (meaning that the square of the tangent is the product of the diameters).
- (b) THEOREM: The difference of the squares of the direct and indirect common tangents of two non-overlapping circles is the product of the two diameters.

19. THE INCENTRE THEOREM: The angle bisectors of the vertices of a triangle are concurrent, and their point of intersection (called the incentre) is the centre of a circle (called the incircle) tangent to all three sides. In the diagram opposite, the angle bisectors of $\angle A$ and $\angle B$ of $\triangle ABC$ meet at I . The intervals IL , IM and IN are perpendiculars to the sides.



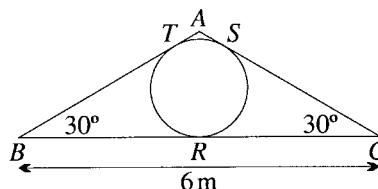
- (a) Prove that $\triangle AIN \equiv \triangle AIM$ and $\triangle BIL \equiv \triangle BIN$.
- (b) Prove that $IL = IM = IN$.
- (c) Prove that $\triangle CIL \equiv \triangle CIM$.
- (d) Complete the proof.

20. (a)



TRIGONOMETRY: The figure ATB in the diagram above is a semicircle. Find the exact values of the lengths TP and BP .

- (b)

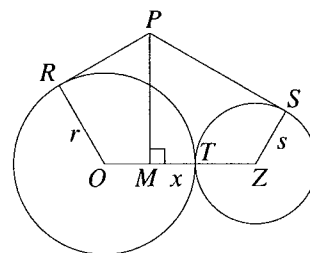


MENSURATION: This window is made in four pieces. Find the area of the small piece AST exactly and approximately.

EXTENSION

21. (a) **THEOREM:** *If two circles touch, the tangents to the two circles from a point outside both of them are equal if and only if the point lies on the common tangent at the point of contact.*

In the diagram to the right, use Pythagoras' theorem to prove that $PR = PS$ if and only if $x = 0$.



- (b) **THEOREM:** *Given three circles such that each pair of circles touches externally, the common tangents at the three points of contacts are equal and concurrent. They meet at the incentre of the triangle formed by the three centres, and the incircle passes through the three points of contact. Use the result of part (a) to prove this theorem.*
22. (a) Three circles of equal radius r are placed so that each is tangent to the other two. Find the area of the region contained between them, and the radius of the largest circle that can be constructed in this region.
- (b) Four spheres of equal radius r are placed in a stack so that each touches the other three. Find the height of the stack.
23. (a) **CONSTRUCTION:** Given two intersecting lines, construct the four circles of a given radius that are tangential to both lines.
- (b) **CONSTRUCTION:** Given two non-intersecting circles, construct their direct and indirect common tangents.
24. (a) **THEOREM:** Suppose that there are three circles of three different radii such that no circle lies within any other circle. Prove that the three points of intersection of the direct common tangents to each pair of circles are collinear. [HINT: Replace the three circles by three spheres lying on a table, then the direct common tangents to each pair of circles form a cone.]
- (b) **THEOREM:** Prove that the orthocentre of a triangle is the incentre of the triangle formed by the feet of the three altitudes.

9F The Alternate Segment Theorem

The word 'alternate' means 'the other one'. In the diagram below, the chord AB divides the circle into two segments — the angle $\alpha = \angle BAT$ lies in one of the segments, and the angle APB lies in the other segment. The alternate segment theorem claims that the two angles are equal.

The Alternate Segment Theorem: Stating the theorem verbally:

22

COURSE THEOREM: The angle between a tangent to a circle and a chord at the point of contact is equal to any angle in the alternate segment.

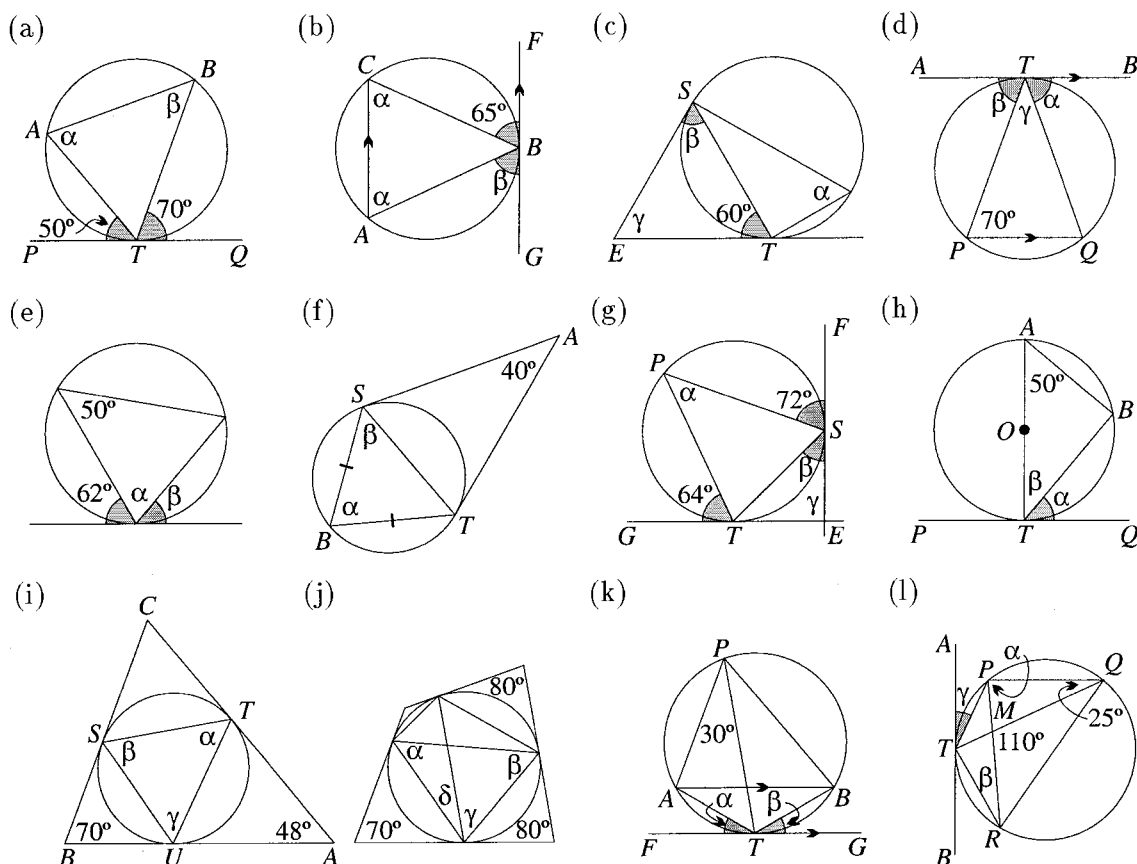
GIVEN: Let AB be a chord of a circle with centre O , and let SAT be the tangent at A . Let $\angle APB$ be an angle in the alternate (other) segment to $\angle BAT$.

AIM: To prove that $\angle APB = \angle BAT$.

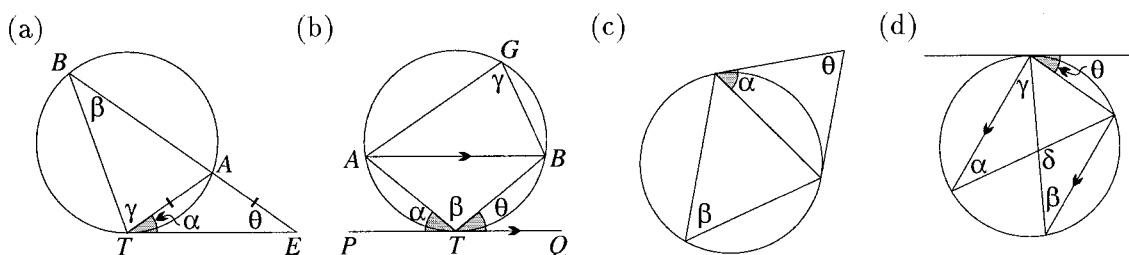
CONSTRUCTION: Construct the diameter AOQ from A , and join BQ .

1. State the alternate segment theorem, and draw several diagrams, with tangents and chords in different orientations, to illustrate it.

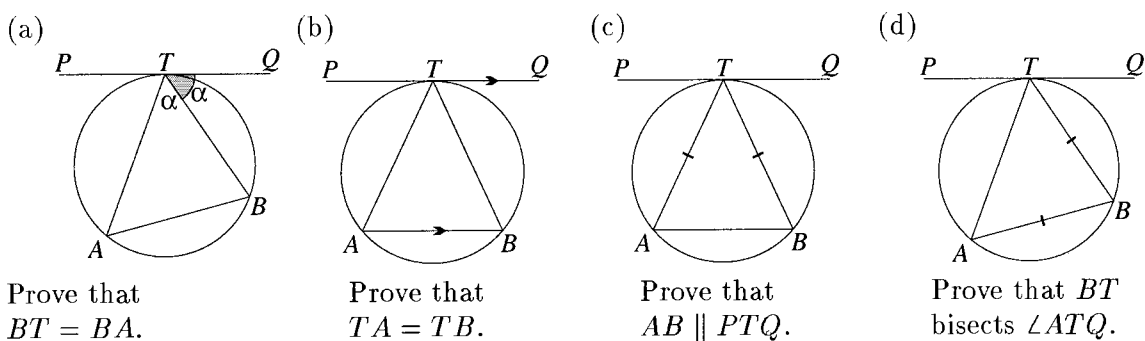
2. Find α , β , γ and δ in each diagram below.



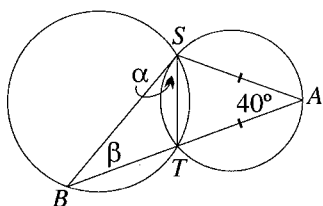
3. In each diagram below, express α , β and γ in terms of θ .



4. In each diagram below, PTQ is a tangent to the circle.

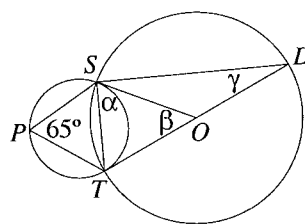


5. (a)



The line SB is a tangent, and $AS = AT$. Find α and β .

(b)



The tangents at S and T meet at the centre O . Find α , β and γ .

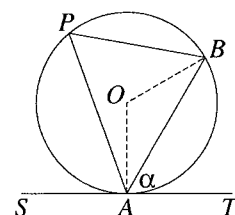
DEVELOPMENT

6. ANOTHER PROOF OF THE ALTERNATE SEGMENT THEOREM:

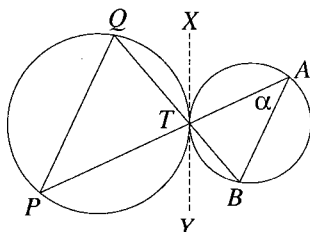
Let AB be a chord of a circle, and let SAT be the tangent at A . Let $\angle APB$ be an angle in the alternate segment to $\angle BAT$.

(a) Let $\alpha = \angle BAT$, and find $\angle OAB$.

(b) Find $\angle OBA$ and $\angle AOB$. (c) Hence show that $\angle P = \alpha$.

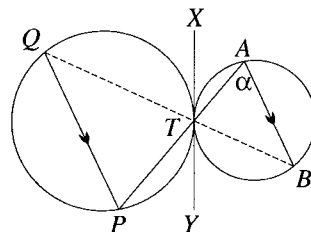


7. (a)



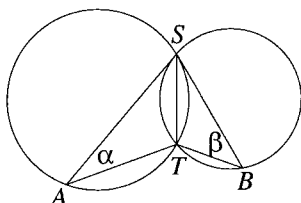
The two circles touch externally at T , and XTY is the common tangent at T . Prove that $AB \parallel QP$.

(b)



The two circles touch externally at T , and XTY is the common tangent there. Prove that the points Q , T and B are collinear.

8. (a)



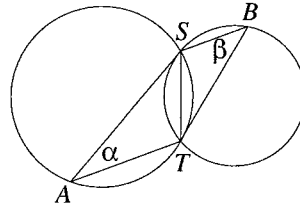
The lines SA and TB are tangents.

(i) Prove that $\angle SAT = \angle BST$.

(ii) Prove that $\triangle SAT \sim \triangle BST$.

(iii) Prove that $AT \times BT = ST^2$.

(b)



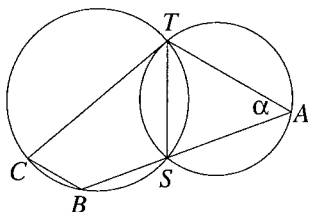
The lines SA and TB are tangents.

(i) Prove that $AT \parallel SB$.

(ii) Prove that $\triangle SAT \sim \triangle BTS$.

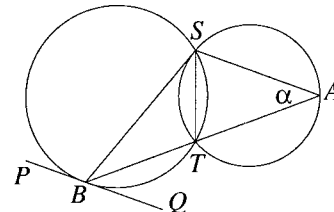
(iii) Prove that $AT \times BS = ST^2$.

9. (a)



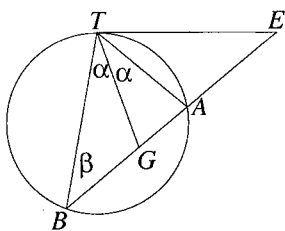
The line TC is a tangent. Prove that $TA \parallel CB$.

(b)



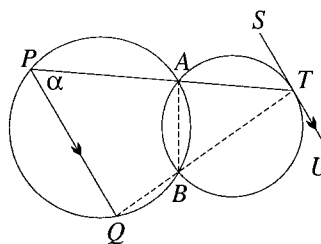
The lines SB and PBQ are tangents. Prove that $SA \parallel PBQ$.

10. (a)



The line TG bisects $\angle BTA$, and ET is a tangent. Prove that $ET = EG$.

(b)



The line STU is a tangent parallel to PQ . Prove that Q, B and T are collinear.

11. Investigate what happens in question 6, parts (a) and (b), when the two circles touch internally. Draw the appropriate diagrams and prove the corresponding results.

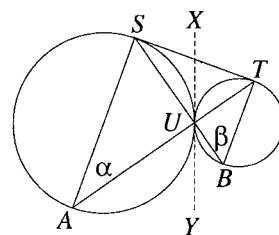
12. ST is a direct common tangent to two circles touching externally at U , and XUY is the common tangent at U .

(a) Prove that $AT \perp BS$.

(b) Prove that AS and BT are parallel diameters.

(c) Explain why if the two circles have different diameters, then AB is not a tangent to either circle.

(d) Prove that the circle through S, U and T has centre X and is tangent to AS and BT .



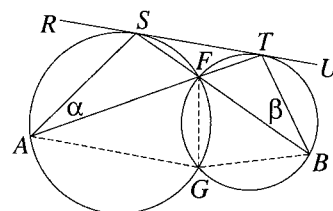
13. $RSTU$ is a direct common tangent to the two circles.

(a) Prove that $\angle RSA = \angle UTB$.

(b) Prove that $\triangle AST \parallel \triangle STB$.

(c) Prove that $ST^2 = AS \times BT$.

(d) Prove that if the points A, G and B are collinear, then $\angle SFA = 60^\circ$.



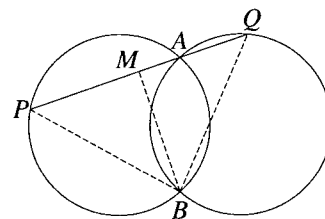
14. A LOCUS PROBLEM: Two circles of equal radii intersect at A and B . A variable line through A meets the two circles again at P and Q .

(a) Prove that $\angle QPB = \angle PQB$.

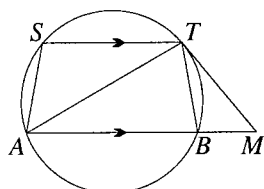
(b) Prove that $BM \perp PQ$, where M is the midpoint of PQ .

(c) What is the locus of M , as the line PAQ varies?

(d) What happens when Q lies on the minor arc AB ?

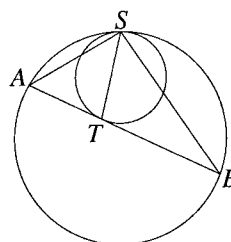


15. (a)



If $ST \parallel AB$ and TM is a tangent, prove that $\triangle TMB \parallel \triangle TAS$.

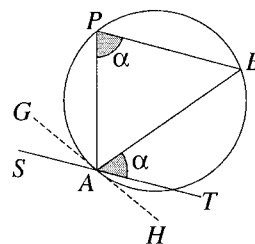
(b)



If the circles are tangent at S , and ATB is a tangent, prove that TS bisects $\angle ASB$.

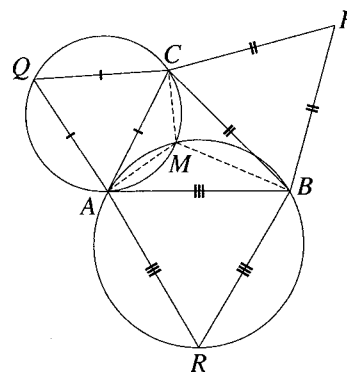
EXTENSION

16. THE CONVERSE OF THE ALTERNATE SEGMENT THEOREM: Suppose that the line SAT passes through the vertex A of $\triangle ABP$ and otherwise lies outside the triangle. Suppose also that $\angle BAT = \angle APB = \alpha$. Then the circle through A , B and P is tangent to SAT . Construct the circle through A , B and P , and let GAH be the tangent to the circle at A .



- (a) Prove that $\angle BAH = \alpha$.
 (b) Hence explain why the lines SAT and GAH coincide.

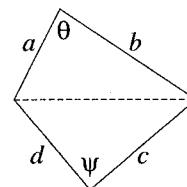
17. THEOREM: Let equilateral triangles ABR , BCP and CAQ be built on the sides of an acute-angled triangle ABC . Then the three circumcircles of the equilateral triangles intersect in a common point, and this point is the point of intersection of the three concurrent lines AP , BQ and CR . Construct the circles through A , C and Q and through A , B and R , and let the two circles meet again at M .



- (a) Prove that $\angle AMC = \angle AMB = 120^\circ$.
 (b) Prove that P , C , M and B are concyclic.
 (c) Prove that $\angle AMQ = 60^\circ$.
 (d) Hence prove the theorem.

18. The alternate segment theorem has an interesting relationship with the earlier theorem that two angles at the circumference subtended by the same arc are equal. Go back to that theorem (see Box 13), and ask what happens to the diagram as Q moves closer and closer to A . The alternate segment theorem describes what happens when Q is in the limiting position at A .

19. A MAXIMISATION THEOREM: A cyclic quadrilateral has the maximum area of all quadrilaterals with the same side lengths in the same order. Let the quadrilateral have fixed side lengths a , b , c and d , and variable opposite angles θ and ψ as shown. Let A be its area.



- (a) Explain why $A = \frac{1}{2}ab \sin \theta + \frac{1}{2}cd \sin \psi$.
 (b) By equating two expressions for the diagonal, and differentiating implicitly with respect to θ , prove that

$$\frac{d\psi}{d\theta} = \frac{ab \sin \theta}{cd \sin \psi}.$$

- (c) Hence prove that $\frac{dA}{d\theta} = \frac{ab \sin(\theta + \psi)}{2 \sin \psi}$, and thus prove the theorem.

9G Similarity and Circles

The theorems of the previous sections have concerned the equality of angles at the circumference of a circle. In this final section, we shall use these equal angles to prove similarity. The similarity will then allow us to work with intersecting chords, and with secants and tangents from an external point.

Intercepts on Intersecting Chords: When two chords intersect, each is broken into two intervals called *intercepts*. The first theorem tells us that the product of the intercepts on one chord equals the product of the intercepts on the other chord.

23 COURSE THEOREM: If two chords of a circle intersect, the product of the intercepts on the one chord is equal to the product of the intercepts on the other chord.

GIVEN: Let AB and PQ be chords of a circle intersecting at M .

AIM: To prove that $AM \times MB = PM \times MQ$.

CONSTRUCTION: Join AP and BQ .

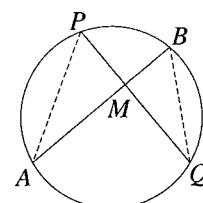
PROOF: In the triangles APM and QBM :

1. $\angle A = \angle Q$ (angles on the same arc PB),
2. $\angle AMP = \angle QMB$ (vertically opposite angles),

so $\triangle APM \parallel \triangle QBM$ (AA).

Hence $\frac{AM}{QM} = \frac{PM}{BM}$ (matching sides of similar triangles),

that is, $AM \times MB = PM \times MQ$.



Intercepts on Secants: When two chords need to be produced outside the circle, before they intersect, the same theorem applies, provided that we reinterpret the theorem as a theorem about secants from an external point. The *intercepts* are now the two intervals on the secant from the external point.

24 COURSE THEOREM: Given a circle and two secants from an external point, the product of the two intervals from the point to the circle on the one secant is equal to the product of these two intervals on the other secant.

GIVEN: Let M be a point outside a circle, and let MAB and MPQ be secants to the circle.

AIM: To prove that $AM \times MB = PM \times MQ$.

CONSTRUCTION: Join AP and BQ .

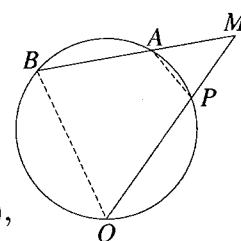
PROOF: In the triangles APM and QBM :

1. $\angle MAP = \angle MQB$ (external angle of cyclic quadrilateral),
2. $\angle AMP = \angle QMB$ (common),

so $\triangle APM \parallel \triangle QBM$ (AA).

Hence $\frac{AM}{QM} = \frac{PM}{BM}$ (matching sides of similar triangles),

that is, $AM \times MB = PM \times MQ$.



Intercepts on Secants and Tangents: A tangent from an external point can be regarded as a secant meeting the circle in two identical points. With this interpretation, the previous theorem still applies.

25 COURSE THEOREM: Given a circle, and a secant and a tangent from an external point, the product of the two intervals from the point to the circle on the secant is equal to the square of the tangent. In other words, the tangent is the geometric mean of the intercepts on the secant.

Recall the definitions of arithmetic and geometric means of two numbers a and b :

- The *arithmetic mean* is the number m such that $b - m = m - a$.

That is, $2m = a + b$ or $m = \frac{a+b}{2}$.

- The *geometric mean* is the number g such that $\frac{b}{g} = \frac{g}{a}$.

That is, $g^2 = ab$, or (if g is positive) $g = \sqrt{ab}$.

GIVEN: Let M be a point outside a circle. Let MAB be a secant to the circle, and let MT be a tangent to the circle.

AIM: To prove that $AM \times MB = TM^2$.

CONSTRUCTION: Join AT and BT .

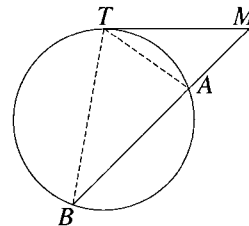
PROOF: In the triangles ATM and TBM :

1. $\angle ATM = \angle TBM$ (alternate segment theorem),
2. $\angle AMT = \angle TMB$ (common),

so $\triangle ATM \parallel \triangle TBM$ (AA).

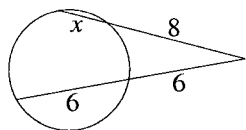
Hence $\frac{AM}{TM} = \frac{TM}{BM}$ (matching sides of similar triangles),

that is, $AM \times MB = TM^2$.

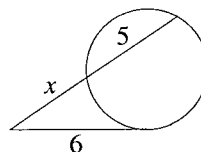


WORKED EXERCISE: Find x in the two diagrams below.

(a)



(b)



SOLUTION:

$$\begin{aligned} \text{(a)} \quad 8(x+8) &= 6 \times 12 \\ &\text{(intercepts on intersecting chords)} \\ x+8 &= 9 \\ x &= 1. \end{aligned}$$

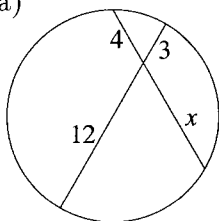
$$\begin{aligned} \text{(b)} \quad x(x+5) &= 6^2 \quad \text{(tangent and secant)} \\ x^2 + 5x - 36 &= 0 \\ (x+9)(x-4) &= 0 \\ x &= 4 \quad (x \text{ must be positive}). \end{aligned}$$

Exercise 9G

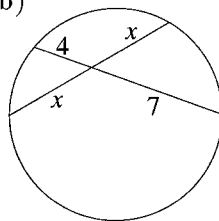
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles, and the obvious lines at points labelled R , S , T and U are tangents.

1. Find x in each diagram below.

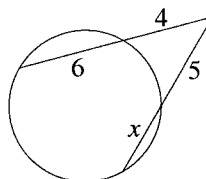
(a)



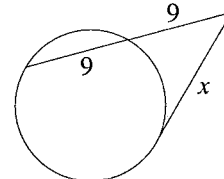
(b)

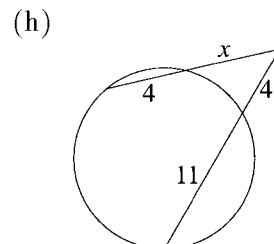
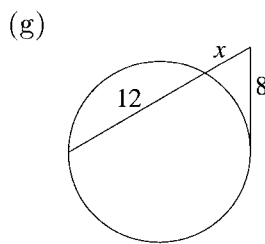
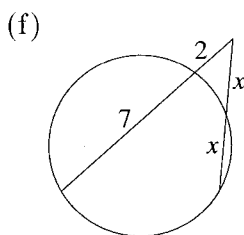
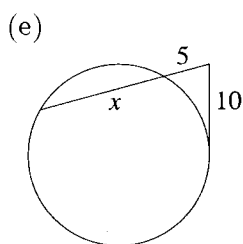


(c)

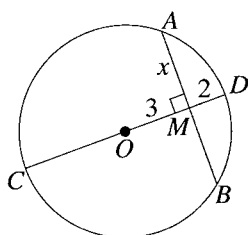


(d)



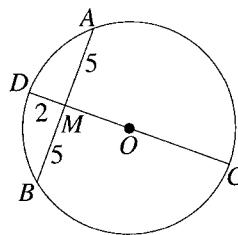


2. (a)



- (i) Explain why $MB = x$.
- (ii) Find x .
- (iii) Find the area of $CADB$.

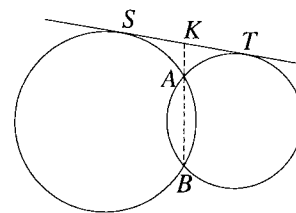
(b)



- (i) Explain why $CD \perp AB$.
- (ii) Find the radius OC .
- (iii) Find the area of $CADB$.

DEVELOPMENT

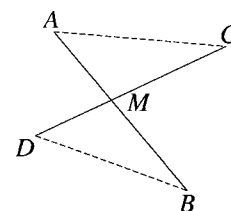
3. **THEOREM:** When two circles intersect, the common chord of the two circles bisects each direct common tangent.
- In the diagram, ST is a direct common tangent.



- (a) Give a reason why $SK^2 = KA \times KB$.
- (b) Hence prove that $SK = TK$.

4. **CONVERSE OF THE INTERSECTING CHORDS THEOREM:** If the products of the intercepts on two intersecting intervals are equal, then the four endpoints of the two intervals are concyclic.

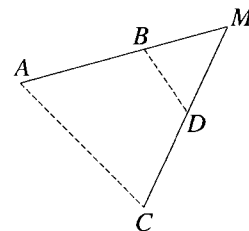
In the diagram opposite, $AM \times BM = CM \times DM$.



- (a) Prove that $\frac{AM}{CM} = \frac{DM}{BM}$.
 - (b) Prove that $\triangle AMC \parallel \triangle DMB$.
 - (c) Prove that $\angle CAM = \angle BDM$.
 - (d) Prove that $ACBD$ is cyclic.
5. **CONVERSE OF THE SECANTS FROM AN EXTERNAL POINT THEOREM:** Let two intervals ABM and CDM meet at their common endpoint M , and suppose that

$$AM \times BM = CM \times DM.$$

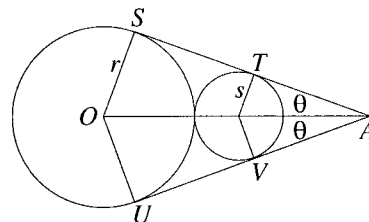
Then $ABDC$ is cyclic.



- (a) Prove that $\triangle AMC \parallel \triangle DMB$.
- (b) Prove that $\angle CAM = \angle BDM$.
- (c) Prove that $ACBD$ is cyclic.

- (b) **TRIGONOMETRY WITH TOUCHING CIRCLES:** Suppose that two circles touch externally, and that their radii are r and s , with $r > s$. Let their direct common tangents meet at an angle 2θ . Show that

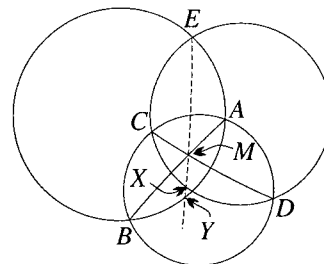
$$\frac{r}{s} = \frac{1 + \sin \theta}{1 - \sin \theta}.$$



EXTENSION

11. **THEOREM:** Given three circles such that each pair of circles overlap, then the three common chords are concurrent.

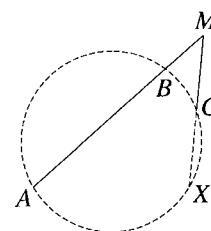
In the diagram opposite, the common chords AB and CD meet at M , and the line EM meets the two circles again at X and Y .



- (a) By applying the intersecting chord theorem three times, prove that $EM \times MX = EM \times MY$.
 (b) Explain why EM must be the third common chord.
 (c) Repeat the construction and proof when the common chords AB and CD meet outside the two circles.
12. **THEOREM:** Given three circles such that each pair of circles touch externally, then the three common tangents at the points of contact are concurrent.
 Prove this theorem by making suitable adaptations to the previous proof.

13. **CONVERSE OF THE SECANT AND TANGENT THEOREM:**
 Let the intervals ABM and CM meet at their common end-point M , and suppose that $MC^2 = MA \times MB$. Then MC is tangent to the circle ABC .

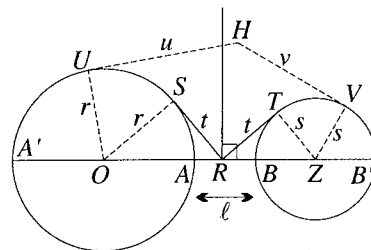
Construct the circle ABC , and suppose by way of contradiction that it meets MC again at X .



- (a) Prove that $MA \times MB = MC \times MX$.
 (b) Hence prove that C and X coincide.
14. **HARMONIC CONJUGATES:** In the configuration of question 9:
 (a) Prove that M divides $F'F$ internally in the same ratio that P divides $F'F$ externally. (M is called the *harmonic conjugate* of P with respect to F' and F).
 (b) Prove that $F'F$ is the *harmonic mean* of $F'M$ and $F'P$ (meaning that $\frac{1}{F'F}$ is the arithmetic mean of $\frac{1}{F'M}$ and $\frac{1}{F'P}$).

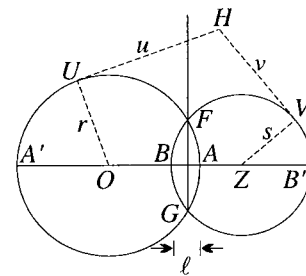
15. **THE RADICAL AXIS THEOREM:**

- (a) Suppose that two circles with centres O and Z and radii r and s do not overlap. Let the line OZ meet the circles at A' , A , B and B' as shown, and let $AB = \ell$. Choose R on AB so that the tangents RS and RT to the two circles have equal length t .



- (i) Prove that a point H outside both circles lies on the perpendicular to OZ through R if and only if the tangents from H to the two circles are equal.
 (ii) Prove that $AR : RB = AB' : A'B$.

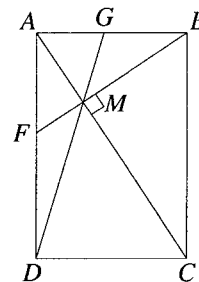
- (b) Suppose that two circles with centres O and Z and radii r and s overlap, meeting at F and G . Let the line OZ meet the circles at A' , A , B and B' as shown, with $AB = \ell$. Let OZ meet FG at R .



- (i) Prove that if H is any point outside both circles, then H lies on FG produced if and only if the tangents from H to the two circles are equal.
- (ii) Prove that $AR : RB = AB' : A'B$.

16. CONSTRUCTIONS TO SQUARE A RECTANGLE, A TRIANGLE AND A POLYGON:

- (a) Use the configuration in question 6 to construct a square whose area is equal to the area of a given rectangle.
- (b) Construct a square whose area is equal to the area of a given triangle.
- (c) Construct a square whose area is equal to the area of a given polygon.
- 17. CONSTRUCTION:** Construct the circle(s) tangent to a given line and passing through two given points not both on the line.
- 18. GEOMETRIC SEQUENCES IN GEOMETRY:** In the diagram below, $ABCD$ is a rectangle with $AB : BC = 1 : r$. The line through B perpendicular to the diagonal AC meets AC at M and meets the side AD at F . The line DM meets the side AB at G .



- (a) Write down five other triangles similar to $\triangle AMF$.
- (b) Show that the lengths FA , AB and BC form a GP.
- (c) Find the ratio $AG : GB$ in terms of r , and find r if AG and GB have equal lengths.
- (d) Is it possible to choose the ratio r so that DG is a common tangent to the circles with diameters AF and BC respectively?
- (e) Is it possible to choose the ratio r so that the points D , F , G and B are concyclic and distinct?
- 19. A DIFFICULT THEOREM:** Prove that the tangents at opposite vertices of a cyclic quadrilateral intersect on the secant through the other two vertices if and only if the two products of opposite sides of the cyclic quadrilateral are equal.

Probability and Counting

Probability arises when one performs an experiment that has various possible outcomes, but for which there is insufficient information to predict precisely which of these outcomes will occur. The classic examples of this are tossing a coin, throwing a die, or drawing a card from a pack. Probability, however, is involved in almost every experiment done in science, and is fundamental to understanding statistics. This chapter will first review some of the basic ideas of probability, using the language of sets, and then establish various systematic counting procedures that will allow more complicated probability questions to be solved. These counting procedures open up the link between probability theory and the binomial expansion, leading to the development of binomial probability.

STUDY NOTES: Sections 10A–10C review in a more systematic manner the basic ideas of probability from earlier years. The language of sets is used here, particularly in Section 10B to deal with ‘and’, ‘or’ and ‘not’ — it may help to review Section 1J in the Year 11 volume, which presented a straightforward account of the elementary ideas about sets. Section 10D deals with probability tree diagrams, which will be new to most students. Further work in probability requires some systematic counting procedures, which are developed in Sections 10E–10G, and applied to probability in Sections 10H and 10I. Section 10J on binomial probability combines the binomial theorem from Chapter Five with the counting procedures.

Throughout the chapter, attention should be given to the fallacies and confusions that inevitably arise in any discussion of probability.

10A Probability and Sample Spaces

Our first task is to develop a workable formula for probability that can serve as the foundation for the topic. That formula will rest on the idea of dividing the results of an experiment into equally likely possible outcomes. We will also need to make reference to probabilities that are experimentally determined.

Equally Likely Possible Outcomes: The idea of equally likely possible outcomes is well illustrated by the experiment of throwing a die. (A *die*, plural *dice*, is a cube with its corners and edges rounded so that it rolls easily, and with the numbers 1–6 printed on its six sides.) The outcome of this experiment is the number on the top face when the die stops rolling, giving six *possible outcomes* — 1, 2, 3, 4, 5, 6. This is a complete list of the possible outcomes, because each time the die is rolled, one and only one of these outcomes can occur.

Provided that the die is completely symmetric, that is, it is not *biased* in any way, we have no reason to expect that any one outcome is more likely to occur than any of the other five, and we call these six possible outcomes *equally likely possible outcomes*. With the results of the experiment thus divided into six equally likely possible outcomes, we now assign the probability $\frac{1}{6}$ to each of these six outcomes. Notice that the six probabilities are equal and they all add up to 1. The general case is as follows.

1

EQUALLY LIKELY POSSIBLE OUTCOMES: Suppose that the possible results of an experiment can be divided into n *equally likely possible outcomes* — meaning that one and only one of these n outcomes will occur, and there is no reason to expect one outcome to be more likely than another.

Then the probability $\frac{1}{n}$ is assigned to each of these equally likely possible outcomes.

Randomness: Notice that it has been assumed that the terms ‘more likely’ and ‘equally likely’ already have a meaning in the mind of the reader. There are many ways of interpreting these words. In the case of a thrown die, one could interpret the phrase ‘equally likely’ as meaning that the die is perfectly symmetric. Alternatively, one could interpret it as saying that we lack entirely the knowledge to make any statement of preference for one outcome over another.

The word *random* can be used here. In the context of equally likely possible outcomes, saying that a die is thrown ‘randomly’ means that we are justified in assigning the same probability to each of the six possible outcomes. In a similar way, we will speak about drawing a card ‘at random’ from a pack, or forming a queue of people in a ‘random order’.

The Fundamental Formula for Probability: Suppose that we need a throw of at least 3 on a die to win a game. Then getting at least 3 is called the particular *event* under discussion, the outcomes 3, 4, 5 and 6 are called *favourable* for this event, and the other two possible outcomes 1 and 2 are called *unfavourable*. The probability assigned to getting a score of at least 3 is then

$$\begin{aligned} P(\text{scoring at least 3}) &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \\ &= \frac{4}{6} \\ &= \frac{2}{3}. \end{aligned}$$

In general, if the results of an experiment can be divided into a number of equally likely possible outcomes, some of which are favourable for a particular event and the others unfavourable, then:

2

THE FUNDAMENTAL FORMULA FOR PROBABILITY:

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

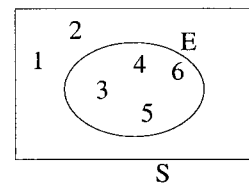
The Sample Space and the Event Space: The language of sets makes some of the theory of probability easier to explain — some review of Section 1J of the Year 11 volume may be helpful at this stage. The Venn diagram on the right shows the six possible outcomes when a die is thrown. The set of all these outcomes is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

This set is called the *sample space* and is represented by the outer rectangular box. The event ‘scoring at least 3’ is the set

$$E = \{3, 4, 5, 6\},$$

which is called the *event space* and is represented by the ellipse. In general, the set of all equally likely possible outcomes is called the *sample space*, and the set of all favourable outcomes is called the *event space*. The basic probability formula can then be restated in set language.



THE SAMPLE SPACE AND THE EVENT SPACE: Suppose that an event E has sample space S . Then, using the symbol $|A|$ for the number of members of a set A ,

3

$$\mathcal{P}(E) = \frac{|E|}{|S|}.$$

Probabilities Involving Playing Cards: So many questions in probability involve a pack of playing cards that any student of probability needs to be familiar with them — the reader is encouraged to acquire some cards and play some simple games with them. A pack of cards consists of 52 cards organised into four *suits*, each containing 13 cards. The four suits are

two black suits: ♣ clubs, ♠ spades,
two red suits: ♦ diamonds, ♥ hearts.

Each of the four suits contains 13 cards:

A (Ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (Jack), Q (Queen), K (King).

An ace can also be regarded as a 1. It is assumed that when a pack of cards is shuffled, the order is totally *random*, meaning that there is no reason to expect any one ordering of the cards to be more likely to occur than any other.

WORKED EXERCISE: A card is drawn at random from a pack of playing cards. Find the probability that the card is:

- (a) the 7 of hearts, (b) a heart, (c) a seven, (d) a red card,
(e) a picture card (Jack, Queen, King), (f) a green card, (g) red or black.

SOLUTION: In each case, there are 52 equally likely possible outcomes.

- (a) Since there is 1 seven of hearts, $\mathcal{P}(7\heartsuit) = \frac{1}{52}$.
(b) Since there are 13 hearts, $\mathcal{P}(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.
(c) Since there are 4 sevens, $\mathcal{P}(\text{seven}) = \frac{4}{52} = \frac{1}{13}$.
(d) Since there are 26 red cards, $\mathcal{P}(\text{red card}) = \frac{26}{52} = \frac{1}{2}$.
(e) Since there are 12 picture cards, $\mathcal{P}(\text{picture card}) = \frac{12}{52} = \frac{3}{13}$.
(f) Since no card is green, $\mathcal{P}(\text{green card}) = \frac{0}{52} = 0$.
(g) Since all 52 cards are red or black, $\mathcal{P}(\text{red or black card}) = \frac{52}{52} = 1$.

Impossible and Certain Events: Parts (f) and (g) of the previous worked exercise were intended to illustrate the probabilities of events that are impossible or certain. Since getting a green card is impossible, there are no favourable outcomes, so the probability is 0. Since all the cards are either red or black, and getting a red or black card is certain to happen, all possible outcomes are favourable outcomes, so the probability is 1. Notice that for the other five events, the probability lies between 0 and 1.

IMPOSSIBLE AND CERTAIN EVENTS:

4

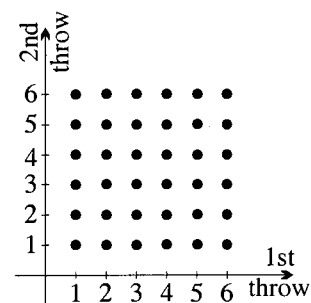
- An event has probability 0 if and only if it cannot happen.
- An event has probability 1 if and only if it is certain to happen.
- For any other event, $0 < P(\text{event}) < 1$.

Graphing the Sample Space: Many experiments consist of several *stages*. For example, when a die is thrown twice, the two throws can be regarded as two separate stages of the one experiment. The reason for using the word ‘sample space’ rather than ‘sample set’ is that the sample space of a multi-stage experiment takes on some of the characteristics of a space. In particular, the sample space of a two-stage experiment can be displayed on a two-dimensional graph, and the sample space of a three-stage experiment can be displayed in a three-dimensional graph. The following worked example shows how a two-dimensional dot diagram can be used for calculations with the sample space of a die thrown twice.

WORKED EXERCISE: A die is thrown twice. Find the probability that:

- (a) the pair is a double,
- (b) at least one number is four,
- (c) both numbers are greater than four,
- (d) both numbers are even,
- (e) the sum of the two numbers is six,
- (f) the sum is at most four.

SOLUTION: The horizontal axis in the diagram to the right represents the six possible outcomes of the first throw, and the vertical axis represents the six possible outcomes of the second throw. The 36 dots therefore represent the 36 different possible outcomes of the two-stage experiment, all equally likely, which is the full sample space. The various parts can now be answered by counting the dots representing the various event spaces.



- (a) Since there are 6 doubles, $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$.
- (b) Since 11 pairs contain a 4, $P(\text{at least one is a 4}) = \frac{11}{36}$.
- (c) Since 4 pairs consist only of 5 or 6, $P(\text{both greater than 4}) = \frac{4}{36} = \frac{1}{9}$.
- (d) Since 9 pairs have two even members, $P(\text{both even}) = \frac{9}{36} = \frac{1}{4}$.
- (e) Since 5 pairs have sum 6, $P(\text{sum is 6}) = \frac{5}{36}$.
- (f) Since 6 pairs have sum 2, 3 or 4, $P(\text{sum at most 4}) = \frac{6}{36} = \frac{1}{6}$.

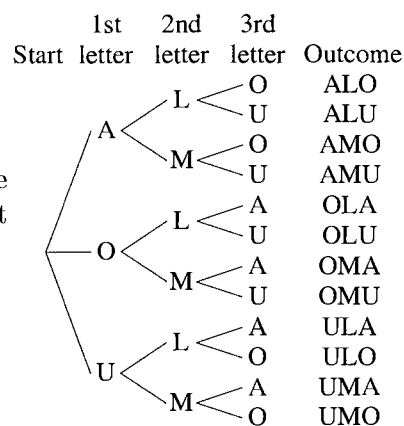
Tree Diagrams: Listing the sample space of a multi-stage experiment can be difficult, and the dot diagrams of the previous paragraph are hard to draw in more than two dimensions. Tree diagrams provide a very useful alternative way to display the sample space. Such diagrams have a column for each stage, plus an initial column labelled ‘Start’ and a final column listing the possible outcomes.

WORKED EXERCISE: A three-letter word is chosen in the following way. The first and last letters are chosen from the three vowels 'A', 'O' and 'U', with repetition not allowed, and the middle letter is chosen from 'L' and 'M'. List the sample space, then find the probability that:

- (a) the word is 'ALO',
- (b) the letter 'O' does not occur,
- (c) 'M' and 'U' do not both occur,
- (d) the letters are in alphabetical order.

SOLUTION: The tree diagram to the right lists all twelve equally likely possible outcomes. The two vowels must be different, because repetition was not allowed.

- (a) $P(\text{'ALO'}) = \frac{1}{12}$.
- (b) $P(\text{no 'O'}) = \frac{4}{12} = \frac{1}{3}$.
- (c) $P(\text{not both 'M' and 'U'}) = \frac{8}{12} = \frac{2}{3}$.
- (d) $P(\text{alphabetical order}) = \frac{4}{12} = \frac{1}{3}$.



The Meaning of 'Word': In the worked exercise above, and throughout this chapter, 'word' simply means an arrangement of letters — the arrangement doesn't have to have any meaning or be a word in the dictionary. Thus 'word' simply becomes a convenient device for discussing arrangements of things in particular orders.

Invalid Arguments: Arguments offered in probability theory can be invalid for all sorts of subtle reasons, and it is common for a question to ask for comment on a given argument. It is most important in such a situation that any fallacy in the given argument be explained — it is not sufficient only to offer an alternative argument with a different conclusion.

WORKED EXERCISE: Comment on the validity of these arguments.

- (a) 'When two coins are tossed together, there are three outcomes: two heads, two tails, and one of each. Hence the probability of getting one of each is $\frac{1}{3}$.'
- (b) 'Brisbane is one of fourteen teams in the Rugby League, so the probability that Brisbane wins the premiership is $\frac{1}{14}$.'

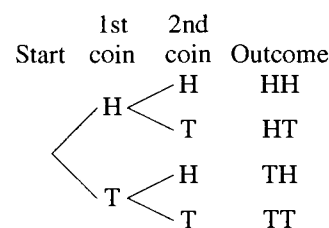
SOLUTION:

- (a) [Identifying the fallacy] The division of the results into the three given outcomes is correct, but no reason is offered as to why these outcomes are equally likely.

[Supplying the correct argument] The diagram on the right divides the results of the experiment into four *equally likely* possible outcomes; since two of these outcomes, HT and TH, are favourable to the event 'one of each', it follows that $P(\text{one of each}) = \frac{2}{4} = \frac{1}{2}$.

- (b) [Identifying the fallacy] The division into fourteen possible outcomes is correct provided that one assumes that a tie for first place is impossible, but no reason has been offered as to why each team is equally likely to win, so the argument is invalid.

[Offering a replacement question] What can be said with confidence is that if a team is selected at random from the fourteen teams, then the probability that it is the premiership-winning team is $\frac{1}{14}$.



NOTE: It is difficult to give a complete account of part (b). It is not clear that an exact probability can be assigned to the event ‘Brisbane wins’, although we can safely assume that those with knowledge of the game would have some idea of ranking the fourteen teams in order from most likely to win to least likely to win. If there is an organised system of betting, one might, or might not, agree to take this as an indication of the community’s collective wisdom on the fourteen probabilities.

Experimental Probability: When a drawing pin is thrown, there are two possible outcomes, point-up and point-down. But these two outcomes are not equally likely, and there seems to be no way to analyse the results of the experiment into equally likely possible outcomes. In the absence of any fancy arguments from physics about rotating pins falling on a smooth surface, however, we can gain some estimate of the two probabilities by performing the experiment a number of times.

The questions in the following worked example could raise difficult issues beyond the scope of this course, but the intention here is only that they be answered briefly in a common-sense manner.

WORKED EXERCISE: A drawing pin is thrown 400 times, and falls point-up 362 times.

- What probability does this experiment suggest for the result ‘point-up’?
- Discuss whether the results are inconsistent with a probability of:
 - $\frac{9}{10}$
 - $\frac{11}{12}$
 - $\frac{1}{2}$
 - $\frac{4}{5}$
- A machine repeats the experiment 1 000 000 times and the pin falls point-up 916 203 times. What answers would you now give for part (b)?

SOLUTION:

- These results suggest $\mathcal{P}(\text{point-up}) \doteq 0.905$, but with only 400 trials, there would be little confidence in this result past the second, or even the first, decimal place, since we would expect different runs of the same experiment to differ by small numbers. So we conclude that $\mathcal{P}(\text{point-up}) \doteq 0.9$.
- Since a probability of $\frac{9}{10}$ would predict about 360 point-up results and $\frac{11}{12}$ would predict about 367 point-up results, both these fractions seem consistent with the experiment. If the probability were $\frac{1}{2}$, however, we would have expected about 200 point-up results, so this can be excluded. Similarly, a probability of $\frac{4}{5}$ would predict about 320 point-up results, and can reasonably be excluded.
- A probability of $\frac{9}{10}$ would predict about 900 000 point-up results, and $\frac{11}{12}$ would predict about 916 667 point-up results. Hence the estimate of $\frac{11}{12}$ seems reasonable, but the estimate of $\frac{9}{10}$ can now reasonably be rejected.

Interpreting Probability as an Experimental Limit: The experimental meaning of probability is a little elusive. The probability of getting 5 or 6 on a die is $\frac{1}{3}$, but what does this mean when we repeatedly throw a die? If we throw the die 60 times, we would expect to get 5 or 6 about $\frac{1}{3} \times 60 = 20$ times, although if it happened 17–23 times, it would probably not disturb our confidence. If we threw the die 6000 times, we would expect to get 5 or 6 about $\frac{1}{3} \times 6000 = 2000$ times (but interestingly enough we would be quite surprised if we got 5 or 6 exactly 2000 times). Thus one interpretation of the statement that getting 5 or 6 has probability $\frac{1}{3}$ is:

$$\text{As (number of trials)} \rightarrow \infty, \quad \frac{\text{number of 5s and 6s}}{\text{number of trials}} \rightarrow \frac{1}{3}.$$

Some experiments, however, are inherently unrepeatable, and yet we seem to understand what it means to assign probabilities to them. For example, many people bet money on Australia beating England in a particular test series, and their bets seem to rely on some estimate of the probability that Australia will win that particular series. (Most would agree that this probability is much greater than $\frac{1}{2}$!)

Exercise 10A

1. A coin is tossed. Find the probability that it shows:
(a) a head, (b) a tail, (c) either a head or a tail, (d) neither a head nor a tail.
2. If a die is rolled, find the probability that the uppermost face is:
(a) three, (b) an even number, (c) a number greater than four, (d) a multiple of three.
3. A bag contains eight red balls, seven yellow balls and three green balls. A ball is selected at random. Find the probability it is: (a) red, (b) yellow or green, (c) not yellow.
4. In a bag there are four red capsicums, three green capsicums, six red apples and five green apples. One item is chosen at random. Find the probability that it is:
(a) green, (c) an apple, (e) a red apple,
(b) red, (d) a capsicum, (f) a green capsicum.
5. A letter is randomly selected from the 26 letters in the English alphabet. Find the probability that the letter is:
(a) the letter S, (c) a consonant, (e) either C, D or E,
(b) a vowel, (d) the letter γ , (f) one of the letters of the word MATHS.
[NOTE: The letter Y is normally classified as a consonant.]
6. A number is selected at random from the integers 1, 2, 3, ..., 19, 20. Find the probability of choosing:
(a) the number 4, (d) an odd number, (g) a multiple of 4,
(b) a number greater than 15, (e) a prime number, (h) the number e ,
(c) an even number, (f) a square number, (i) a rational number.
7. From a regular pack of 52 cards, one card is drawn at random. Find the probability that:
(a) it is black, (d) it is the jack of hearts, (g) it is a heart or a spade,
(b) it is red, (e) it is a club, (h) it is a red five or a black seven,
(c) it is a king, (f) it is a picture card, (i) it is less than a four.
8. A book has 150 pages. The book is randomly opened at a page. Find the probability that the page number is:
(a) greater than 140, (c) an odd number, (e) either 72 or 111,
(b) a multiple of 20, (d) a number less than 25, (f) a three-digit number.
9. An integer x , where $1 \leq x \leq 200$, is chosen at random. Determine the probability that it:
(a) is divisible by 5, (c) has two digits, (e) is greater than 172,
(b) is a multiple of 13, (d) is a square number, (f) has three equal digits.
10. A bag contains three times as many yellow marbles as blue marbles. If a marble is chosen at random, find the probability that it is: (a) yellow, (b) blue.

11. From a group of four students, Anna, Bill, Charlie and David, two are chosen at random to be on the Student Representative Council. List the sample space, and hence find the probability that:
- (a) Bill and David are chosen,
 - (b) Anna is chosen,
 - (c) Charlie is chosen but Bill is not,
 - (d) neither Anna nor David is selected.
12. A fair coin is tossed twice. Use a tree diagram to list the possible outcomes. Hence find the probability that the two tosses result in:
- (a) two heads, (b) a head and a tail, (c) a head on the first toss and a tail on the second.
13. A die is rolled and a coin is tossed. Use a tree diagram to list all the possible outcomes. Hence find the probability of obtaining:
- (a) a head and an even number,
 - (b) a tail and a number greater than four,
 - (c) a tail and a number less than four,
 - (d) a head and a prime number.
14. From the integers 2, 3, 8 and 9, two-digit numbers are formed in which no digit can be repeated in the same number.
- (a) Draw a tree diagram to illustrate the possible outcomes.
 - (b) If one of the two-digit numbers is chosen at random, find the probability that it is:
 - (i) the number 82,
 - (ii) a number greater than 39,
 - (iii) an even number,
 - (iv) a multiple of 3,
 - (v) a number ending in 2,
 - (vi) a perfect square.
15. A captain and vice-captain of a cricket team are to be chosen from Amanda, Belinda, Carol, Dianne and Emma.
- (a) Use a tree diagram to list the possible pairings, noting that order is important.
 - (b) Find the probability that:
 - (i) Carol is captain and Emma is vice-captain,
 - (ii) Belinda is either captain or vice-captain,
 - (iii) Amanda is not selected for either position,
 - (iv) Emma is vice-captain.
16. A hand of five cards contains a ten, jack, queen, king and ace. From the hand, two cards are drawn in succession, the first card not being replaced before the second card is drawn.
- (a) Find the probability that:
 - (i) the ace is selected,
 - (ii) the king is not selected,
 - (iii) the queen is the second card chosen.
 - (b) Repeat part (a) if the first card is replaced before the second card is drawn.
17. Two dice are thrown simultaneously. List the set of 36 possible outcomes on a two-dimensional graph, and hence find the probability of:
- (a) obtaining a three on the first throw,
 - (b) obtaining a four on the second throw,
 - (c) a double five,
 - (d) a total score of seven,
 - (e) a total score greater than nine,
 - (f) an even number on both dice,
 - (g) at least one two,
 - (h) neither a one nor a four appearing,
 - (i) a five and a number greater than three,
 - (j) the same number on both dice.
18. Fifty tagged fish were released into a dam known to contain fish. Later a sample of thirty fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just prior to the sample of thirty being removed.
19. A biased coin is tossed 300 times, and lands on heads 227 times.
- (a) What probability does this experiment suggest for the result 'heads'?
 - (b) Discuss whether the results are inconsistent with a probability of:
 - (i) $\frac{1}{2}$
 - (ii) $\frac{3}{4}$
 - (iii) $\frac{7}{9}$
 - (iv) $\frac{5}{8}$

DEVELOPMENT

20. A coin is tossed three times. Draw a tree diagram to illustrate the possible outcomes. Then find the probability of obtaining:
- (a) three heads, (c) at least two tails, (e) more heads than tails,
(b) a head and two tails, (d) at most one head, (f) a head on the second toss.
21. If the births of boys and girls are equally likely, determine the probability that:
- (a) in a family of two children there are:
(i) two girls, (ii) no girls, (iii) one boy and one girl.
(b) in a family of three children there are:
(i) three boys, (ii) two girls and one boy, (iii) more boys than girls.
22. An unbiased coin is tossed four times. Find the probability of obtaining:
- (a) four heads, (c) at least two heads, (e) two heads and two tails,
(b) exactly three tails, (d) at most one head, (f) more tails than heads.
23. A rectangular field is 60 metres long and 30 metres wide. A cow wanders randomly around the field. Find the probability that the cow is:
- (a) more than 10 metres from the edge of the field,
(b) not more than 10 metres from a corner of the field.
24. Comment on the following arguments. Identify precisely any fallacies in the arguments, and if possible, give some indication of how to correct them.
- (a) 'On every day of the year it either rains or it doesn't. Therefore the chance that it will rain tomorrow is $\frac{1}{2}$.'
- (b) 'When the Sydney Swans play Hawthorn, either Hawthorn wins, the Swans win or the game is a draw. Therefore the probability that the next game between these two teams results in a draw is $\frac{1}{3}$.'
- (c) 'When answering a multiple-choice test in which there are four possible answers given to each question, the chance that Peter answers a question correctly is $\frac{1}{4}$.'
- (d) 'A bag contains a number of red, white and black beads. If you choose one bead at random from the bag, the probability that it is black is $\frac{1}{3}$.'
- (e) 'Four players play in a knockout tennis tournament resulting in a single winner. A man with no knowledge of the game or the players declares that one particular player will win his semi-final, but lose the final. The probability that he is correct is $\frac{1}{4}$.'
25. If a coin is tossed n times, where $n > 1$, find the probability of obtaining:
- (a) n heads, (b) at least one head and at least one tail.

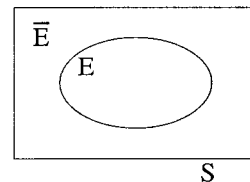
EXTENSION

26. [Buffon's needle problem — an example of continuous probability]
- (a) A needle, whose length is equal to the width of the floorboards, is thrown at random onto the floor. Show that if its inclination to the cracks is θ , then the probability that it lies across a crack is $\sin \theta$. Then, by integrating across all possible angles, show that the probability that it lies across a crack is $\frac{2}{\pi}$.
- (b) Hence devise a probabilistic experiment to compute π .
- (c) Find the probability if the needle has length half the width of the floorboards.
- (d) Find the probability if the needle has length twice the width of the floorboards, but take care — some angles are a problem because probabilities cannot ever exceed 1.

10B Probability and Venn Diagrams

The language of sets was introduced in the previous section when speaking about the sample space and the event space. One outcome of this is that we can use Venn diagrams to visualise the possible outcomes, and calculations become easier.

Complementary Events and the Word ‘Not’: It is often easier to find the probability that an event does not occur than the probability that it does occur. The *complementary event* of an event E is the event ‘ E does not occur’, and is written as $\bar{E}$. Using a Venn diagram, the complementary event $\bar{E}$ is represented by the region outside the circle in the diagram to the right.



Since $|\bar{E}| = |S| - |E|$, it follows that $P(\bar{E}) = 1 - P(E)$.

5 COMPLEMENTARY EVENTS: Suppose that E is an event with sample space S . Define the *complementary event* $\bar{E}$ to be the event ‘ E does not occur’. Then

$$P(\bar{E}) = 1 - P(E).$$

In Section 1J of the Year 11 volume, we defined the *complement* $\bar{E}$ of a set E to be the set of things in S but not in E , and commented that the complement of a set is closely linked to the word ‘not’. The new notation $\bar{E}$ for complementary event is quite deliberately the same notation as that for the complement of a set.

WORKED EXERCISE: What is the probability of failing to throw a double six when throwing a pair of dice?

SOLUTION: As discussed in the previous section, the double six is but one outcome amongst 36 possible outcomes, and so has probability $\frac{1}{36}$.

$$\begin{aligned} \text{Hence } P(\text{not throwing a double six}) &= 1 - P(\text{double six}) \\ &= \frac{35}{36}. \end{aligned}$$

WORKED EXERCISE: A card is drawn at random from a pack. Find the probability that it is an even number, or a picture card, or red.

NOTE: Remember that the word ‘or’ always means ‘and/or’ in logic and mathematics. Thus in this worked example, the words ‘or any two of these, or all three of these’ are understood, and need not be added.

SOLUTION: The complementary event $\bar{E}$ is drawing a card that is a black odd number less than ten. This complementary event has ten members:

$$\bar{E} = \{A\clubsuit, 3\clubsuit, 5\clubsuit, 7\clubsuit, 9\clubsuit, A\spadesuit, 3\spadesuit, 5\spadesuit, 7\spadesuit, 9\spadesuit\}.$$

There are 52 possible cards to choose, so using the complementary event formula above:

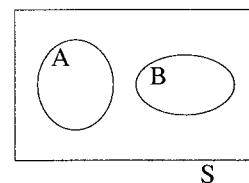
$$P(E) = 1 - \frac{10}{52} = \frac{42}{52} = \frac{21}{26}.$$

Mutually Exclusive Events and Disjoint Sets: Two events A and B with the same sample space S are called *mutually exclusive* if they cannot both occur. In the Venn diagram of such a situation, the two events A and B are represented as disjoint sets (*disjoint* means that their intersection is empty).

In this situation, the event ‘ A and B ’ is impossible, and has probability zero. On the other hand, the event ‘ A or B ’ is represented on the Venn diagram by the union $A \cup B$ of the two sets.

Since $|A \cup B| = |A| + |B|$ for disjoint sets, it follows that

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B).$$



6

MUTUALLY EXCLUSIVE EVENTS: Suppose that A and B are mutually exclusive events with sample space S . Then the event ‘ A or B ’ is represented by $A \cup B$, and

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B).$$

The event ‘ A and B ’ cannot occur, and has probability zero.

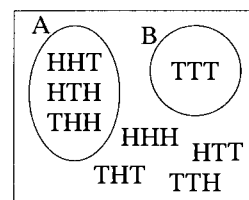
WORKED EXERCISE: If three coins are tossed, find the probability of tossing an odd number of tails.

SOLUTION: Let A be the event ‘one tail’ and B the event ‘three tails’. Then A and B are mutually exclusive, and

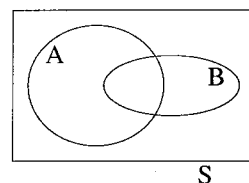
$$A = \{ \text{HHT, HTH, THH} \} \quad \text{and} \quad B = \{ \text{TTT} \}.$$

The full sample space has eight members altogether (question 20 in the previous exercise lists them all), so

$$\mathcal{P}(A \text{ or } B) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$



The Events ‘ A and B ’ and ‘ A or B ’ — The Addition Rule: More generally, suppose that A and B are any two events with the same sample space S , not necessarily mutually exclusive. The Venn diagram of the situation will now represent the two events A and B as overlapping sets within the same universal set S . The event ‘ A and B ’ will then be represented by the intersection $A \cap B$ of the two sets, and the event ‘ A or B ’ will be represented by the union $A \cup B$.



The general counting rule for sets is $|A \cup B| = |A| + |B| - |A \cap B|$, because the members of the intersection $A \cap B$ are counted in A and again in B , and so have to be subtracted. It follows then that $\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B)$ — this rule is often called the *addition rule* of probability.

7

THE EVENTS ‘ A OR B ’ AND ‘ A AND B ’: Suppose that A and B are two events with sample space S . Then the event ‘ A and B ’ is represented by the intersection $A \cap B$ and the event ‘ A or B ’ is represented by the union $A \cup B$, and

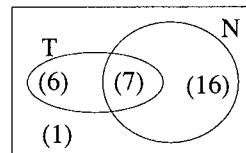
$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B).$$

It was explained in Section 1J of the Year 11 volume that the word ‘or’ is closely linked with the union of sets, and the word ‘and’ is closely linked with the intersection of sets. For this reason, the event ‘ A or B ’ is often written as ‘ $A \cup B$ ’, and the event ‘ A and B ’ is often written as ‘ $A \cap B$ ’ or just ‘ AB ’.

WORKED EXERCISE: In a class of 30 girls, 13 play tennis and 23 play netball. If 7 girls play both sports, what is the probability that a girl chosen at random from the class plays neither sport?

SOLUTION: Let T be the event 'she plays tennis',
and let N be the event 'she plays netball'.

$$\begin{aligned}\text{Then } \mathcal{P}(T) &= \frac{13}{30} \\ \mathcal{P}(N) &= \frac{23}{30} \\ \text{and } \mathcal{P}(N \text{ and } T) &= \frac{7}{30} \\ \text{Hence } \mathcal{P}(N \text{ or } T) &= \frac{13}{30} + \frac{23}{30} - \frac{7}{30} \\ &= \frac{29}{30}, \\ \text{and } \mathcal{P}(\text{neither sport}) &= 1 - \mathcal{P}(N \text{ or } T) \\ &= \frac{1}{30}.\end{aligned}$$



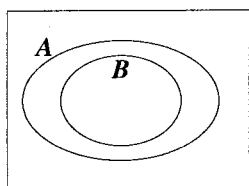
NOTE: An alternative approach is shown in the diagram. Starting with the 7 girls in the intersection, the numbers 6 and 16 can then be written into the respective regions 'tennis but not netball' and 'netball but not tennis'. Since these numbers add to 29, this leaves only one girl playing neither tennis nor netball.

Exercise 10B

- [A brief review of set notation. Students should refer to Exercise 1J of the Year 11 volume for a more substantial list of questions.]
 - Find $A \cup B$ and $A \cap B$ for each pair of sets:
 - $A = \{1, 3, 5\}$, $B = \{3, 5, 7\}$
 - $A = \{1, 3, 4, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$
 - $A = \{h, o, b, a, r, t\}$, $B = \{b, i, c, h, e, n, o\}$
 - $A = \{\text{prime numbers less than } 10\}$, $B = \{\text{odd numbers less than } 10\}$
 - Let $A = \{1, 3, 7, 10\}$ and $B = \{4, 6, 7, 9\}$, and take the universal set to be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List the members of:
 - $\overline{A}$
 - $\overline{B}$
 - $A \cap B$
 - $\overline{A \cap B}$
 - $A \cup B$
 - $\overline{A \cup B}$
- A student has a 22% chance of being chosen as a prefect. What is the chance that he will not be chosen as a prefect?
- When breeding labradors, the probability of breeding a black dog is $\frac{3}{7}$.
 - What is the probability of breeding a dog that is not black?
 - If you breed 56 dogs, how many would you expect to be not black?
- The chance of a new light globe being defective is $\frac{1}{15}$.
 - What is the probability that a new light globe will not be defective?
 - If 120 new light globes were checked, how many would you expect to be defective?
- A die is rolled. If n denotes the number on the uppermost face, find:
 - $\mathcal{P}(n = 5)$
 - $\mathcal{P}(n \neq 5)$
 - $\mathcal{P}(n = 4 \text{ or } n = 5)$
 - $\mathcal{P}(n = 4 \text{ and } n = 5)$
 - $\mathcal{P}(n \text{ is even or odd})$
 - $\mathcal{P}(n \text{ is neither even nor odd})$
 - $\mathcal{P}(n \text{ is even and divisible by three})$
 - $\mathcal{P}(n \text{ is even or divisible by three})$

6. A card is selected from a regular pack of 52 cards. Find the probability that the card:
- (a) is a jack,
 - (b) is a ten,
 - (c) is a jack or a ten,
 - (d) is a jack and a ten,
 - (e) is neither a jack nor a ten,
 - (f) is black,
 - (g) is a picture card,
 - (h) is a black picture card,
 - (i) is black or a picture card,
 - (j) is neither black nor a picture card.
7. A die is thrown. Let A be the event that an even number appears. Let B be the event that a number greater than two appears.
- (a) Are A and B mutually exclusive?
 - (b) Find: (i) $\mathcal{P}(A)$ (ii) $\mathcal{P}(B)$ (iii) $\mathcal{P}(A \text{ and } B)$ (iv) $\mathcal{P}(A \text{ or } B)$
8. Two dice are thrown. Let a and b denote the numbers rolled. Find:
- (a) $\mathcal{P}(a \text{ is odd})$
 - (b) $\mathcal{P}(b \text{ is odd})$
 - (c) $\mathcal{P}(a \text{ and } b \text{ are odd})$
 - (d) $\mathcal{P}(a \text{ or } b \text{ is odd})$
 - (e) $\mathcal{P}(\text{neither } a \text{ nor } b \text{ is odd})$
 - (f) $\mathcal{P}(a = 1)$
 - (g) $\mathcal{P}(b = a)$
 - (h) $\mathcal{P}(a = 1 \text{ and } b = a)$
 - (i) $\mathcal{P}(a = 1 \text{ or } b = a)$
 - (j) $\mathcal{P}(a \neq 1 \text{ and } a \neq b)$

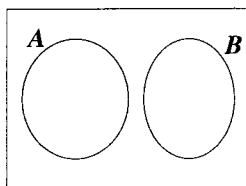
9. (a)



If $\mathcal{P}(A) = \frac{1}{2}$ and $\mathcal{P}(B) = \frac{1}{3}$, find:

- (i) $\mathcal{P}(\bar{A})$ (ii) $\mathcal{P}(\bar{B})$
- (iii) $\mathcal{P}(A \text{ and } B)$
- (iv) $\mathcal{P}(A \text{ or } B)$
- (v) $\mathcal{P}(\text{neither } A \text{ nor } B)$

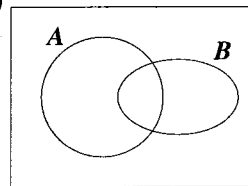
(b)



If $\mathcal{P}(A) = \frac{2}{5}$ and $\mathcal{P}(B) = \frac{1}{5}$, find:

- (i) $\mathcal{P}(\bar{A})$ (ii) $\mathcal{P}(\bar{B})$
- (iii) $\mathcal{P}(A \text{ or } B)$
- (iv) $\mathcal{P}(A \text{ and } B)$
- (v) $\mathcal{P}(\text{not both } A \text{ and } B)$

(c)



If $\mathcal{P}(A) = \frac{1}{2}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ and } B) = \frac{1}{6}$, find:

- (i) $\mathcal{P}(\bar{A})$ (ii) $\mathcal{P}(\bar{B})$
- (iii) $\mathcal{P}(A \text{ or } B)$
- (iv) $\mathcal{P}(\text{neither } A \text{ nor } B)$
- (v) $\mathcal{P}(\text{not both } A \text{ and } B)$

10. Use the addition rule $\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B)$ to answer the following questions:

- (a) If $\mathcal{P}(A) = \frac{1}{5}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ and } B) = \frac{1}{15}$, find $\mathcal{P}(A \text{ or } B)$.
- (b) If $\mathcal{P}(A) = \frac{1}{2}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ or } B) = \frac{5}{6}$, find $\mathcal{P}(A \text{ and } B)$.
- (c) If $\mathcal{P}(A \text{ or } B) = \frac{9}{10}$, $\mathcal{P}(A \text{ and } B) = \frac{1}{5}$ and $\mathcal{P}(A) = \frac{1}{2}$, find $\mathcal{P}(B)$.
- (d) If A and B are mutually exclusive and $\mathcal{P}(A) = \frac{1}{7}$ and $\mathcal{P}(B) = \frac{4}{7}$, find $\mathcal{P}(A \text{ or } B)$.

DEVELOPMENT

11. An integer n is picked at random, where $1 \leq n \leq 20$. The events A , B , C and D are:

A : an even number is chosen, B : a number greater than 15 is chosen,
 C : a multiple of 3 is chosen, D : a one-digit number is chosen.

- (a) (i) Are the events A and B mutually exclusive?
- (ii) Find $\mathcal{P}(A)$, $\mathcal{P}(B)$, $\mathcal{P}(A \text{ and } B)$ and hence evaluate $\mathcal{P}(A \text{ or } B)$.
- (b) (i) Are the events A and C mutually exclusive?
- (ii) Find $\mathcal{P}(A)$, $\mathcal{P}(C)$, $\mathcal{P}(A \text{ and } C)$ and hence evaluate $\mathcal{P}(A \text{ or } C)$.
- (c) (i) Are the events B and D mutually exclusive?
- (ii) Find $\mathcal{P}(B)$, $\mathcal{P}(D)$, $\mathcal{P}(B \text{ and } D)$ and hence evaluate $\mathcal{P}(B \text{ or } D)$.

12. List the twenty-five primes less than 100. A number is drawn at random from the integers from 1 to 100. Find the probability that:
- (a) it is prime, (b) it has remainder 1 after division by 4,
 - (c) it is prime and it has remainder 1 after division by 4,
 - (d) it is either prime or it has remainder 1 after division by 4.
13. In a group of 50 students, there are 26 who study Latin and 15 who study Greek and 8 who take both languages. Draw a Venn diagram and find the probability that a student chosen at random:
- (a) studies only Latin, (b) studies only Greek, (c) does not study either language.
14. During a game, all 21 members of an Australian Rules football team consume liquid. Some players drink only water, some players drink only GatoradeTM and some players drink both. If there are 14 players who drink water and 17 players who drink GatoradeTM:
- (a) How many drink both water and GatoradeTM?
 - (b) If one team member is selected at random, find the probability that:
 - (i) he drinks water but not GatoradeTM,
 - (ii) he drinks GatoradeTM but not water.
15. Each student in a music class of 28 studies either the piano or the violin or both. It is known that 20 study the piano and 15 study the violin. Find the probability that a student selected at random studies both instruments.
16. A group of 60 students was invited to try out for three sports: rugby, soccer and cross country — 32 tried out for rugby, 29 tried out for soccer, 15 tried out for cross country, 11 tried out for rugby and soccer, 9 tried out for soccer and cross country, 8 tried out for rugby and cross country, and 5 tried out for all three sports. Draw a Venn diagram and find the probability that a student chosen at random:
- (a) tried out for only one sport, (c) tried out for at least two sports,
 - (b) tried out for exactly two sports, (d) did not try out for a sport.
17. 43 people were surveyed and asked whether they drank CokeTM, SpriteTM or FantaTM. Three people drank all of these, while four people did not drink any of them. 19 drank CokeTM, 21 drank SpriteTM and 17 drank FantaTM. One person drank CokeTM and FantaTM but not SpriteTM. Find the probability that a person selected at random from the group drank SpriteTM only.

EXTENSION

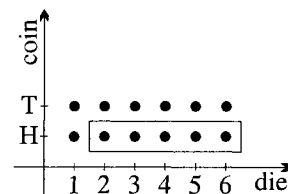
18. [The inclusion–exclusion principle] This rule allows you to count the number of elements contained in the union of the sets without counting any element more than once.
- (a) Given three finite sets A , B and C , find a rule for calculating $n(A \cup B \cup C)$. [HINT: Use a Venn diagram and pay careful attention to the elements in $A \cap B \cap C$.]
 - (b) A new car dealer offers three options to his customers: power steering, air conditioning and a CD player. He sold 72 cars without any options, 12 with all three options, 38 included power steering and air conditioning, 25 included power steering and a CD player, 22 included air conditioning and a CD player, 83 included power steering, 55 included air conditioning and 70 included a CD player. Using the formula established in (a), how many cars did he sell?
 - (c) Extend the formula to the unions of four sets, each with finitely many elements. Is it possible to draw a sensible Venn diagram of four sets?

10C Multi-Stage Experiments

This section deals with experiments that have a number of stages. The full sample space of such an experiment can quickly become too large to be conveniently listed, and instead we shall develop a rule for multiplying together the probabilities associated with each stage.

Two-Stage Experiments — The Product Rule: Here is a simple question about a two-stage experiment:

‘Throw a die, then toss a coin. What is the probability of obtaining at least two on the die followed by a head?’



Graphed on the right are the twelve possible outcomes of the experiment, all equally likely, with a box drawn around the five favourable outcomes. Thus

$$\mathcal{P}(\text{at least two and a head}) = \frac{5}{12}.$$

Now let us consider the two stages separately. The first stage is throwing a die, and we want the outcome A = ‘getting at least two’ — here there are six possible outcomes and five favourable outcomes, giving probability $\frac{5}{6}$. The second stage is tossing a coin, and we want the outcome B = ‘tossing a head’ — here there are two possible outcomes and one favourable outcome, giving probability $\frac{1}{2}$.

The full experiment then has $6 \times 2 = 12$ possible outcomes, and there are $5 \times 1 = 5$ favourable outcomes. Hence

$$\mathcal{P}(AB) = \frac{5 \times 1}{6 \times 2} = \frac{5}{6} \times \frac{1}{2} = \mathcal{P}(A) \times \mathcal{P}(B).$$

Thus the probability of the compound event ‘getting at least two and a head’ can be found by multiplying together the probabilities of the two stages. The argument here can easily be generalised to any two-stage experiment.

TWO-STAGE EXPERIMENTS: If A and B are independent events in successive stages of a two-stage experiment, then

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$$\mathcal{P}(AB) = \mathcal{P}(A) \times \mathcal{P}(B),$$

where the word ‘independent’ means that the outcome of one stage does not affect the outcome of the other stage.

Independent Events: The word ‘independent’ needs further discussion. In our example above, the throwing of the die clearly does not affect the tossing of the coin, so the two events are independent.

Here is a very common and important type of experiment where the two stages are not independent:

‘Choose an elector at random from the NSW population. First note the elector’s gender. Then ask if (s)he voted Labor or non-Labor in the last State election.’

In this example, we would suspect that the gender and the political opinion of a person may not be independent, and that there is *correlation* between them. This is in fact the case, as almost every opinion poll has shown over the years. Correlation is beyond this course, but is one of the most common things statisticians study in their routine work.

WORKED EXERCISE: A pair of dice are thrown twice. What is the probability that the first throw is a double and the second throw gives a sum of at least four?

SOLUTION: We saw in Section 10A that when two dice are thrown, there are 36 possible outcomes, graphed in the diagram to the right.

There are six doubles amongst the 36 possible outcomes,

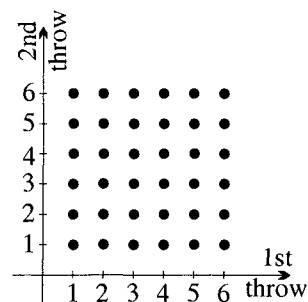
so $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$.

All but the pairs (1,1), (2,1) and (1,2) give a sum at least four,

so $P(\text{sum is at least four}) = \frac{33}{36} = \frac{11}{12}$.

Since the two stages are independent,

$$P(\text{double, sum at least four}) = \frac{1}{6} \times \frac{11}{12} = \frac{11}{72}.$$



Multi-Stage Experiments — The Product Rule: The same arguments clearly apply to an experiment with any number of stages.

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MULTI-STAGE EXPERIMENTS: If $A_1, A_2, \dots, A_n$ are independent events, then

$$P(A_1 A_2 \dots A_n) = P(A_1) \times P(A_2) \times \dots \times P(A_n).$$

WORKED EXERCISE: A coin is tossed four times. Find the probability that:

(a) every toss is a head,

(b) there is at least one head.

SOLUTION:

$$\begin{aligned} \text{(a) } P(\text{HHHH}) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{16}. \end{aligned}$$

$$\begin{aligned} \text{(b) } P(\text{at least one head}) &= 1 - P(\text{TTTT}) \\ &= 1 - \frac{1}{16} = \frac{15}{16}. \end{aligned}$$

Listing the Favourable Outcomes: The product rule is often combined with a listing of the favourable outcomes. A tree diagram may help in producing that listing, although this is hardly necessary in the straightforward worked exercise below, which continues the previous example.

WORKED EXERCISE: A coin is tossed four times. Find the probability that:

(a) the first three coins are heads,

(c) there are at least three heads,

(b) the middle two coins are tails,

(d) there are exactly two heads.

SOLUTION:

$$\text{(a) } P(\text{the first three coins are heads}) = P(\text{HHHH}) + P(\text{HHHT})$$

(notice that the two events HHHH and HHHT are mutually exclusive)

$$= \frac{1}{16} + \frac{1}{16}$$

(since each of these two probabilities is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$)

$$= \frac{1}{8}.$$

$$\text{(b) } P(\text{middle two are tails}) = P(\text{HTTH}) + P(\text{HTTT}) + P(\text{TTHH}) + P(\text{TTTT})$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}.$$

$$\text{(c) } P(\text{at least 3 heads}) = P(\text{HHHH}) + P(\text{HHHT}) + P(\text{HHTH}) + P(\text{HTHH}) + P(\text{THHH})$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{5}{16}.$$

$$\text{(d) } P(\text{exactly 2 heads}) = P(\text{HHTT}) + P(\text{HTHT}) + P(\text{THHT})$$

$$+ P(\text{HTTH}) + P(\text{THTH}) + P(\text{TTHH})$$

(since these are all the six possible orderings of H, H, T and T)

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{8}.$$

Sampling Without Replacement — An Extension of the Product Rule: The product rule can be extended to the following question, where the two stages of the experiment are not independent.

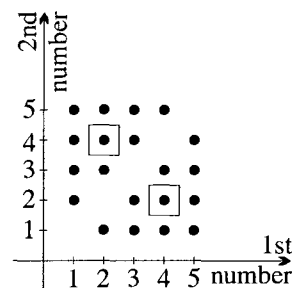
WORKED EXERCISE: A box contains five discs numbered 1, 2, 3, 4 and 5. Two numbers are drawn in succession, without replacement. What is the probability that both are even?

SOLUTION: The probability that the first number is even is $\frac{2}{5}$.

When this even number is removed, one even and three odd numbers remain, so the probability that the second number is also even is $\frac{1}{4}$.

$$\begin{aligned}\text{Hence } P(\text{both even}) &= \frac{2}{5} \times \frac{1}{4} \\ &= \frac{1}{10}.\end{aligned}$$

The graph on the right allows the calculation to be checked by examining its full sample space. Because doubles are not allowed, there are only 20 possible outcomes. The two boxed outcomes are the only outcomes that consist of two even numbers, giving the same probability of $\frac{2}{20} = \frac{1}{10}$.



Retelling the Experiment: Sometimes, the manner in which an experiment is told makes calculation difficult, but the experiment can be retold in a different manner so that the probabilities are the same, but the calculations are much simpler.

WORKED EXERCISE: Wes is sending Christmas cards to ten friends. He has two cards with Christmas trees, two with angels, two with snow, two with reindeer, and two with Santa Claus. What is the probability that Harry and Helmut get matching cards?

SOLUTION: Retell the process as follows. 'Wes decides to pair up the friends who will receive matching cards. First he writes down Harry's name, then he chooses a pair for Harry. He then proceeds similarly with the remaining eight names.'

All that now matters is the person he chooses to pair with Harry. Since there are nine names remaining, the probability that it is Helmut is $\frac{1}{9}$.

WORKED EXERCISE: From a room of ten people, five are chosen at random to be seated on the balcony for dinner. What is the probability that Sandra and Dinesh both sit on the balcony?

SOLUTION: Retell the method of choosing the random seating as 'Choose where Sandra sits, then choose where Dinesh sits'.

Sandra then has five chances out of ten of sitting on the balcony.

If she does, then Dinesh has four chances out of nine of sitting on the balcony.

$$\begin{aligned}\text{Hence } P(\text{both on balcony}) &= \frac{5}{10} \times \frac{4}{9} \\ &= \frac{2}{9}.\end{aligned}$$

Alternatively, retell the experiment as 'Choose, in order, the five people to sit inside, and find the probability that neither Sandra nor Dinesh sits inside'.

$$\begin{aligned}\text{Then } P(\text{neither inside}) &= \frac{8}{10} \times \frac{7}{9} \times \frac{6}{8} \times \frac{5}{7} \times \frac{4}{6} \\ &= \frac{2}{9}.\end{aligned}$$

Exercise 10C

- Suppose that A , B , C and D are independent events, with $P(A) = \frac{1}{8}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$ and $P(D) = \frac{2}{7}$. Use the product rule to find:
 - $P(AB)$
 - $P(AD)$
 - $P(BC)$
 - $P(ABC)$
 - $P(BCD)$
 - $P(ABCD)$
- A coin and a die are tossed. Use the product rule to find the probability of obtaining:
 - a three and a head,
 - a six and a tail,
 - an even number and a tail,
 - a number less than five and a head.
- One set of cards contains the numbers 1, 2, 3, 4 and 5, and another set contains the letters A, B, C, D and E. One card is drawn at random from each set. Use the product rule to find the probability of drawing:
 - 4 and B,
 - 2 or 5, then D,
 - 1, then A or B or C,
 - an odd number and C,
 - an even number and a vowel,
 - a number less than 3, and E,
 - the number 4, followed by a letter from the word MATHS.
- Two marbles are picked at random, one from a bag containing three red and four blue marbles, and the other from a bag containing five red and two blue marbles. Find the probability of drawing:
 - two red marbles,
 - two blue marbles,
 - a red marble from the first bag and a blue marble from the second.
- A box contains five light globes, two of which are faulty. Two globes are selected, one at a time without replacement. Find the probability that:
 - both globes are faulty,
 - neither globe is faulty,
 - the first globe is faulty and the second one is not,
 - the second globe is faulty and the first one is not.
- A box contains twelve red and ten green discs. Three discs are selected, one at a time without replacement.
 - What is the probability that the discs selected are red, green, red in that order?
 - What is the probability of this event if the disc is replaced after each draw?
- From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:
 - a spade then a heart,
 - two clubs,
 - a jack then a queen,
 - the king of diamonds then the ace of clubs.
 - Repeat the question if the first card is replaced before the second card is drawn.
- A coin is weighted so that it is twice as likely to fall heads as it is tails.
 - Write down the probabilities that the coin falls: (i) heads, (ii) tails.
 - If you toss the coin three times, find the probability of:
 - three heads,
 - three tails,
 - head, tail, head in that order.
- [Valid and invalid arguments] Identify any fallacies in the following arguments. If possible, give some indication of how to correct them.
 - 'The probability that a Year 12 student chosen at random likes classical music is 50%, and the probability that a student plays a classical instrument is 20%. Therefore the probability that a student chosen at random likes classical music and plays a classical instrument is 10%.'

- (b) 'The probability of a die showing a prime is $\frac{1}{2}$, and the probability that it shows an odd number is $\frac{1}{2}$. Hence the probability that it shows an odd prime number is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.'
- (c) 'I choose a team at random from an eight-team competition. The probability that it wins any game is $\frac{1}{2}$, so the probability that it defeats all of the other seven teams is $(\frac{1}{2})^7 = \frac{1}{128}$.'
- (d) 'A normal coin is tossed and shows heads eight times. Nevertheless, the probability that it shows heads the next time is still $\frac{1}{2}$.'

DEVELOPMENT

10. An archer fires three shots at a bulls-eye. He has a 90% chance of hitting the bulls-eye. Using H for hit and M for miss, list all eight possible outcomes. Then, assuming that successive shots are independent, use the product rule to find the probability that he will:
- (a) hit the bulls-eye three times, (d) hit the bulls-eye exactly once,
 (b) miss the bulls-eye three times, (e) miss the bulls-eye on the first shot only,
 (c) hit the bulls-eye on the first shot only, (f) miss the bulls-eye exactly once.
 [HINT: Part (d) requires adding the probabilities of HMM, MHM and MMH, and part (f) requires a similar calculation.]
11. A die is rolled twice. Using the product rule, find the probability of rolling:
- (a) a double two, (e) a four and then a one,
 (b) any double, (f) a one and a four in any order,
 (c) a number greater than three, then an odd number, (g) an even number, then a five,
 (d) a one and then a four, (h) a five and then an even number,
 (i) an even number and a five in any order.
12. There is a one-in-five chance that you will guess the correct answer to a multiple-choice question. The test contains five such questions — label the various possible results of the test as CCCCC, CCCCI, CCCIC, ... What is the chance that you will answer:
- (a) all five correctly, (b) all five incorrectly,
 (c) the first, third and fifth correctly, and the second and fourth incorrectly,
 (d) the first correctly and the remainder incorrectly,
 (e) exactly one correctly, [HINT: Add the probabilities of CHII, ICHI, IICH, IIIC, IIIC.]
 (f) exactly four correctly. [HINT: List the possible outcomes first.]
13. A die is thrown six times.
- (a) What is the probability that the n th throw is n on each occasion?
 (b) What is the probability that the n th throw is n on exactly five occasions?
14. From a bag containing two red and two green marbles, marbles are drawn one at a time without replacement until two green marbles have been drawn. Find the probability that:
- (a) exactly two draws are required, (c) exactly four draws are required,
 (b) at least three draws are required, (d) exactly three draws are required.
 [HINT: In part (c), consider the colour of the fourth marble drawn.]
15. Sophia, Gabriel and Elizabeth take their driving test. The chances that they pass are $\frac{1}{2}$, $\frac{5}{8}$ and $\frac{3}{4}$ respectively.
- (a) Find the probability that Sophia passes and the other two fail.
 (b) By listing the possible outcomes for one of the girls passing and the other two failing, find the probability that exactly one of the three passes.
 (c) If only one of them passes, find the probability that it is Gabriel.

16. (a) If a coin is tossed repeatedly, find the probability of obtaining at least one head in:
(i) two tosses, (ii) five tosses, (iii) ten tosses.
(b) Write down the probability of obtaining at least one head in n tosses.
(c) How many times would you need to toss a coin so that the probability of tossing at least one head is greater than 0.9999?
17. (a) When rolling a die n times, what is the probability of not rolling a six?
(b) Show that the probabilities of not rolling a six on 1, 2, 3, ... tosses of the coin form a GP, and write down the first term and common ratio.
(c) How many times would you need to roll a die so that the probability of rolling at least one six was greater than $\frac{9}{10}$?
18. One layer of tinting material on a window cuts out $\frac{1}{5}$ of the sun's UV rays.
(a) What fraction would be cut out by using two layers?
(b) How many layers would be required to cut out at least $\frac{9}{10}$ of the sun's UV rays?
19. In a lottery, the probability of the jackpot being won in any draw is $\frac{1}{60}$.
(a) What is the probability that the jackpot prize will be won in each of four consecutive draws?
(b) How many consecutive draws need to be made for there to be a greater than 98% chance that at least one jackpot prize will have been won?
20. [This question and the next are best done by retelling the story of the experiment, as explained in the notes above.] Nick has five different pairs of socks to last the week, and they are scattered loose in his drawer. Each morning, he gets up before light and chooses two socks at random. Find the probability that he wears a matching pair:
(a) on the first morning, (c) on the third morning, (e) every morning,
(b) on the last morning, (d) the first two mornings, (f) every morning but one.
21. Kia and Abhishek are two of twelve guests at a tennis party, where people are playing doubles on three courts. The twelve have been divided randomly into three groups of four. Find the probability that:
(a) Kia and Abhishek play on the same court,
(b) Kia and Abhishek both play on River court,
(c) Kia is on River court and Abhishek is on Rose court.

EXTENSION

22. [A notoriously confusing problem] In a television game show, the host shows the contestant three doors, only one of which conceals the prize, and the game proceeds as follows. First, the contestant chooses a door. Secondly, the host opens one of the other two doors, showing the contestant that it is not the prize door. Thirdly, the host invites the contestant to change her choice, if she wishes. Analyse the game, and advise the contestant what to do.
23. In a game, a player draws a card from a pack of 52. If he draws a two he wins. If he draws a three, four or five he loses. If he draws a heart that is not a two, three, four or five then he must roll a die. He wins only if he rolls a one. If he draws one of the other three suits and the card is not a two, three, four or five, then he must toss a coin. He wins only if he tosses a tail. Given that the player wins the game, what is the probability that he drew a black card?

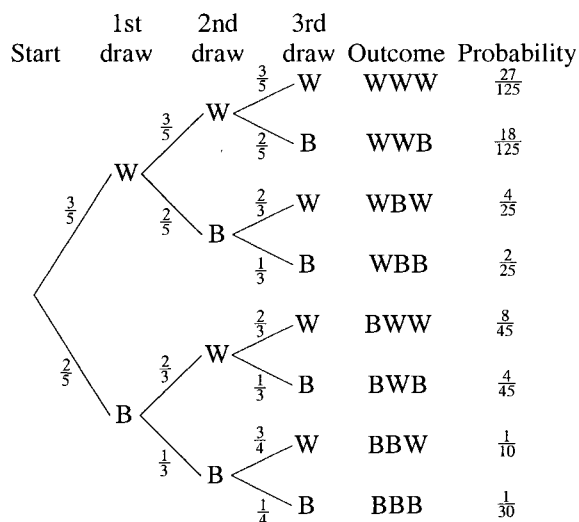
10D Probability Tree Diagrams

In more complicated problems, and particularly in unsymmetric situations, a *probability tree diagram* can be very useful in organising the various cases, in preparation for the application of the product rule.

Constructing a Probability Tree Diagram: A probability tree diagram differs from the tree diagrams used in Section 10A for counting possible outcomes, in that the relevant probabilities are written on the branches and then multiplied together in accordance with the product rule. An example should demonstrate the method. Notice that, as before, these diagrams have one column labelled 'Start', a column for each stage, and a column listing the outcomes, but there is now an extra column labelled 'Probability' at the end.

WORKED EXERCISE: A bag contains six white marbles and four blue marbles. Three marbles are drawn in succession. At each draw, if the marble is white, it is replaced, and if it is blue, it is not replaced. Find the probabilities of drawing zero, one, two and three blue marbles.

SOLUTION: With the ten marbles in the bag, the probabilities are $\frac{3}{5}$ and $\frac{2}{5}$.
If one blue marble is removed, the probabilities become $\frac{2}{3}$ and $\frac{1}{3}$.
If two blue marbles are removed, the probabilities become $\frac{3}{4}$ and $\frac{1}{4}$.



Multiplying probabilities along each arm, and then adding the cases,

$$\mathcal{P}(\text{no blue marbles}) = \frac{27}{125},$$

$$\mathcal{P}(\text{one blue marble}) = \frac{18}{125} + \frac{4}{25} + \frac{8}{45} = \frac{542}{1125},$$

$$\mathcal{P}(\text{two blue marbles}) = \frac{2}{25} + \frac{4}{45} + \frac{1}{10} = \frac{121}{450},$$

$$\mathcal{P}(\text{three blue marbles}) = \frac{1}{30}.$$

NOTE: Your calculator will show that the eight probabilities in the last column of the diagram add exactly to 1, and that the four answers above also add to 1. These are useful checks on the working.

An Infinite Probability Tree Diagram: Some situations generate an infinite probability tree diagram. In the following, more difficult worked example, the limiting sum of a GP is used to evaluate the resulting sum.

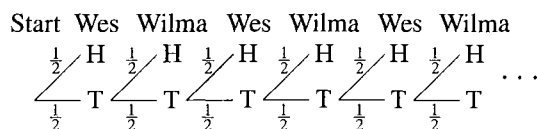
WORKED EXERCISE: Wes and Wilma toss a coin alternately, beginning with Wes. The first to toss heads wins. What probability of winning does Wes have?

SOLUTION: The branches on the tree diagram below terminate when a head is tossed, and the person who tosses that head wins the game. Space precludes writing in either the final outcome or the product of the probabilities!

From the diagram, $P(\text{Wes wins}) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$

This is the limiting sum of the GP with $a = \frac{1}{2}$ and $r = \frac{1}{4}$,

$$\begin{aligned} \text{so } P(\text{Wes wins}) &= \frac{a}{1-r} \\ &= \frac{\frac{1}{2}}{1-\frac{1}{4}} \\ &= \frac{2}{3}. \end{aligned}$$



Exercise 10D

- A bag contains three black and four white discs. A disc is selected from the bag, its colour is noted, and it is then returned to the bag before a second disc is drawn.
 - Draw a probability tree diagram and hence find the probability that:
 - both discs drawn are white,
 - the discs have different colours.
 - Repeat the question if the first ball is not replaced before the second one is drawn.
- Two light bulbs are selected at random from a large batch of bulbs in which 5% are defective. Draw a probability tree diagram and find the probability that:
 - both bulbs are defective,
 - at least one bulb works.
- One bag contains three red and two blue balls and another bag contains two red and three blue balls. A ball is drawn at random from each bag. Draw a probability tree diagram and hence find the probability that:
 - the balls have the same colour,
 - the balls have different colours.
- In group *A* there are three girls and seven boys, and in group *B* there are six girls and four boys. One person is chosen at random from each group. Draw a probability tree diagram and hence find the probability that:
 - both people chosen are of the same sex,
 - one boy and one girl are chosen.
- There is an 80% chance that Gary will pass his driving test and a 90% chance that Emma will pass hers. Draw a probability tree diagram, and find the chance that:
 - Gary passes and Emma fails,
 - Gary fails and Emma passes,
 - only one of Gary and Emma passes,
 - at least one fails.
- In an aviary there are four canaries, five cockatoos and three budgerigars. If two birds are selected at random, find the probability that:
 - both are canaries,
 - neither is a canary,
 - one is a canary and one is a cockatoo,
 - at least one is a canary.
- In a large co-educational school, the population is 47% female and 53% male. Two students are selected at random. Find, correct to two decimal places, the probability that:
 - both are male,
 - they are of different sexes.
- The probability of a woman being alive at 80 years of age is 0.2, and the probability of her husband being alive at 80 years of age is 0.05. What is the probability that:
 - they will both live to be 80 years of age,
 - only one of them will live to be 80?

9. The numbers 1, 2, 3, 4 and 5 are each written on a card. The cards are shuffled and one card is drawn at random. The number is noted and the card is then returned to the pack. A second card is selected, and in this way a two-digit number is recorded. For example, a 2 on the first draw and a 3 on the second results in the number 23.
- (a) What is the probability of:
- (i) the number 35 being recorded, (ii) an odd number being recorded?
- (b) Repeat the question if the first card is not returned to the pack before the second one is drawn.
10. A factory assembles calculators. Each calculator requires a chip and a battery. It is known that 1% of chips and 4% of batteries are defective. Find the probability that a calculator selected at random will have at least one defective component.
11. Alex and Julia are playing in a tennis tournament. They will play each other twice and each has an equal chance of winning the first game. If Alex wins the first game his confidence increases, and his probability of winning the second game is increased to 0.55. If he loses the first game he loses heart so that his probability of winning the second game is reduced to 0.25. Find the probability that Alex wins exactly one game.
12. In a raffle in which there are 200 tickets, the first prize is drawn and then the second prize is drawn without replacing the winning ticket. If you buy 15 tickets, find the probability that you win:
- (a) both prizes, (b) at least one prize.
13. The probability that a set of traffic lights will be green when you arrive at them is $\frac{3}{5}$. A motorist drives through two sets of lights. Assuming that the two sets of traffic lights are not synchronised, find the probability that:
- (a) both sets of lights will be green, (b) at least one set of lights will be green.

DEVELOPMENT

14. A box contains ten chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Hugo randomly selects and eats three chocolates from the box. Find the probability that:
- (a) the first chocolate Hugo eats is caramel, (b) Hugo eats three mint chocolates,
(c) Hugo eats exactly one caramel chocolate,
(d) Hugo eats at least two caramel chocolates.
15. In a bag there are four green, three blue and five red discs.
- (a) Two discs are drawn at random, and the first disc is not replaced before the second disc is drawn. Find the probability of drawing:
- (i) two red discs, (iv) a blue disc on the first draw,
(ii) one red and one blue disc, (v) two discs of the same colour,
(iii) at least one green disc, (vi) two differently coloured discs.
- (b) Repeat the above questions if the first disc is replaced before the second disc is drawn.
16. Max and Jack each throw a die. Find the probability that:
- (a) they do not throw the same number,
(b) the number thrown by Max is greater than the number thrown by Jack,
(c) the numbers they throw differ by three,
(d) the product of the numbers is even.

17. In basketball, the chance of a girl making a basket from the free-throw line is 0.7 and the chance of a boy making the basket is 0.65. Therefore if a boy and a girl are selected at random, the chance that at least one of them will shoot a basket is 1.35. Explain the problem with this argument.
18. A game is played in which two coloured dice are rolled once. The six faces of the black die are numbered 5, 7, 8, 10, 11, 14. The six faces of the white die are numbered 3, 6, 9, 12, 13, 15. The player wins if the number on the black die is bigger than the number on the white die.
- (a) Calculate the probability of a player winning the game.
 - (b) Calculate the probability that a player will lose at least once in two consecutive games.
 - (c) How many games must be played before you have a 90% chance of winning at least one game?
19. Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.
20. In a game, two dice are rolled and the score given is the maximum of the two numbers on the uppermost faces. For example, if the dice show a three and a five, the score is a five.
- (a) Find the probability that you score a one in a single throw of the two dice.
 - (b) What is the probability of scoring three consecutive ones in three rolls of the dice?
 - (c) Find the probability of a six in a single roll of the dice.
 - (d) Given that one of the dice shows a three, what is the probability of getting a score greater than five?
21. A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:
- (a) both Bob's cards are jacks, (b) at least one of Bob's cards is a jack,
 - (c) given that one of Bob's cards is a jack, the other one is also.
22. A twenty-sided die has the numbers from 1 to 20 on its faces.
- (a) If it is rolled twice, what is the probability that the same number appears on the uppermost face each time?
 - (b) If it is rolled three times, what is the probability that the number 15 appears on the uppermost face (i) exactly twice, (ii) at most once?
23. In each game of Sic Bo, three regular six-sided dice are thrown once.
- (a) In a single game, what is the probability that all three dice show six?
 - (b) What is the probability that exactly two of the dice show six?
 - (c) What is the probability that exactly two of the dice show the same number?
 - (d) What is the probability of rolling three different numbers on the dice?
24. Shadia has invented a game for one person. She throws two dice repeatedly until the sum of the two numbers shown is either six or eight. If the sum is six, she wins. If the sum is eight, she loses. If the sum is any other number, she continues to throw until the sum is six or eight.
- (a) What is the probability that she wins on the first throw?
 - (b) What is the probability that a second throw is needed?
 - (c) Find an expression for the probability that Shadia wins on her first, second or third throw.
 - (d) Calculate the probability that Shadia wins the game.

25. In a game, Anna and Ingrid take turns at drawing, and immediately replacing, a ball from an urn containing three blue and four green balls. The game is won when Anna draws a blue ball, or when Ingrid draws a green ball. Anna goes first. Find the probability that:
- (a) Anna wins on her first draw, (c) Anna wins in less than four of her turns,
(b) Ingrid wins on her first draw, (d) Anna wins the game.
26. A bag contains two green and two blue marbles. Marbles are drawn at random, one by one, without replacement, until two green marbles have been drawn.
- (a) What is the probability that exactly three draws will be required?
(b) If the marbles are replaced after each draw, and the first one drawn is green, find the probability that three draws will be required.
27. Prasad and Wilson are going to enlist in the Australian Army. The recruiting officer will be in town for seven consecutive days, starting on Monday and finishing the following Sunday. The boys must nominate three consecutive days on which to attend the recruitment office. They do this randomly and independently of one another.
- (a) By listing all the ways in which to choose the three consecutive days, find the probability that they both go on Monday.
(b) What is the probability that they meet at the recruitment office on Tuesday for the first time?
(c) Find the probability that Prasad and Wilson will not meet at the recruitment office.
(d) Hence find the probability that they will meet on at least one day at the recruitment office.
28. There are two white and three black discs in a bag. Two players A and B are playing a game in which they draw a disc alternately from the bag and then replace it. Player A must draw a white disc to win and player B must draw a black disc. Player A goes first. Find the probability that:
- (a) player A wins on the first draw, (c) player A wins in less than four of her draws,
(b) player B wins on her first draw, (d) player A wins the game.
29. A coin is tossed continuously until, for the first time, the same result appears twice in succession. That is, you continue tossing until you toss two heads or two tails in a row.
- (a) Find the probability that the game ends before the sixth toss of the coin.
(b) Find the probability that an even number of tosses will be required.

EXTENSION

30. A bag contains g green and b blue marbles. Three marbles are randomly selected from the bag.
- (a) Write down an expression for the probability that the three marbles chosen were green.
(b) If the bag had initially contained one additional green marble, the probability that the three marbles chosen were green would have been double that found in part (a).
- (i) Show that $b = \frac{g^2 - g - 2}{5 - g}$. (ii) Show that $b = -g - 4 + \frac{18}{5 - g}$.
- (iii) Sketch a graph of b against g , indicating the g -coordinates of any stationary points.
(iv) Hence determine all possible numbers of green and blue marbles.

10E Counting Ordered Selections

It should be clear by now that probability problems would become easier if we could develop greater sophistication in counting methods. This section and the next two take a break from probability questions to develop a more systematic approach to counting. Two questions dominate these sections:

1. Are the selections we are counting ordered or unordered?
2. If they are ordered, is repetition allowed or not?

Sections 10E and 10F develop the theory of counting ordered selections, with and without repetition, then Section 10G will deal with unordered selections.

The Multiplication Principle for Ordered Selections: An ordered selection can usually be regarded as a sequence of choices made one after the other. An efficient setting-out here is to use a box diagram to keep track of these successive choices.

WORKED EXERCISE: How many five-letter words can be formed in which the second and fourth letters are vowels?

NOTE: Unless otherwise indicated, always take 'y' as a consonant.

SOLUTION: We can select each letter in order:

1st letter	2nd letter	3rd letter	4th letter	5th letter
21	5	21	5	21

$$\begin{aligned}\text{Number of words} &= 21 \times 5 \times 21 \times 5 \times 21 \\ &= 231\,525.\end{aligned}$$

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MULTIPLICATION PRINCIPLE: Suppose that a selection is to be made in k stages.
 Suppose that the first stage can be chosen in n_1 ways, the second in n_2 ways,

 Then the number of ways of choosing the complete selection is $n_1 \times n_2 \times \cdots \times n_k$.

Ordered Selections With Repetition: A general formula can now be found for the number of ordered selections with repetition. Suppose that k -letter words are to be formed from n distinct letters, where any letter can be used any number of times. Then each successive letter in the word can be chosen in n ways:

1st letter	2nd letter	3rd letter	...	k th letter
n	n	n	...	n

giving n^k distinct words altogether.

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ORDERED SELECTIONS WITH REPETITION: The number of k -letter words formed from n distinct letters, allowing repetition, is n^k .

WORKED EXERCISE:

- (a) How many six-digit numbers can be formed entirely from odd digits?
- (b) How many of these numbers contain at least one 7?

SOLUTION:

- (a) There are five odd digits, so the number of such numbers is 5^6 .
 (b) We first count the number of these six-digit numbers not containing 7.

Such numbers are formed from the digits 1, 3, 5 and 9, so there are 4^6 of them.

Subtracting this from the answer to part (a),

$$\text{number of numbers} = 5^6 - 4^6 = 11\,529.$$

Ordered Selections Without Repetition: Counting ordered selections without repetition typically involves factorials, because as each stage is completed, the number of objects to choose from diminishes by 1.

WORKED EXERCISE: In how many ways can a class of 16 select a committee consisting of a president, a vice-president, a treasurer and a secretary?

SOLUTION: Select in order the president, the vice-president, the treasurer and the secretary (we assume that the same person cannot fill two roles).

president	vice-president	treasurer	secretary
16	15	14	13

Hence there are $16 \times 15 \times 14 \times 13 = \frac{16!}{12!}$ possible committees.

The General Case — Permutations: A *permutation* or *ordered set* is an arrangement of objects chosen from a certain set. For example, the words ABC, CED, EAB and DBC are all permutations of three letters taken from the 5-member set $\{A, B, C, D, E\}$.

The symbol ${}^n\text{P}_k$ is used to denote the number of permutations of k letters chosen without repetition from a set of n distinct letters. The previous worked exercise is easily generalised to show that there are $\frac{n!}{(n-k)!}$ such permutations, so this becomes the formula for ${}^n\text{P}_k$:

1st letter	2nd letter	3rd letter	4th letter	...	k th letter
n	$n-1$	$n-2$	$n-3$	...	$n-k+1$

$$\begin{aligned} \text{Hence } {}^n\text{P}_k &= n(n-1)(n-2)(n-3)\cdots(n-k+1) \\ &= \frac{n(n-1)(n-2)(n-3)\cdots 2 \times 1}{(n-k)(n-k-1)\cdots 2 \times 1} \\ &= \frac{n!}{(n-k)!}. \end{aligned}$$

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ORDERED SELECTIONS WITHOUT REPETITION (PERMUTATIONS): The number of k -letter words that can be formed without repetition from a set of n distinct letters is

$${}^n\text{P}_k = \frac{n!}{(n-k)!}.$$

For example, the number of permutations of three distinct letters taken from the 5-member set $\{A, B, C, D, E\}$ is ${}^5\text{P}_3 = \frac{5!}{2!} = 60$.

WORKED EXERCISE: How many eight-digit numbers, and how many nine-digit numbers, can be formed from the nine nonzero digits if no repetition is allowed?

SOLUTION:

$$\begin{aligned} \text{Number of 8-digit numbers} &= {}^9P_8 & \text{Number of 9-digit numbers} &= {}^9P_9 \\ &= \frac{9!}{1!} & &= \frac{9!}{0!} \\ &= 9! & &= 9! \end{aligned}$$

NOTE: It may seem surprising that these two results are equal. Notice, however, that every eight-digit number with distinct nonzero digits can be extended to a nine-digit number of this type simply by tacking the missing digit onto the left-hand end. This establishes a one-to-one correspondence between the two sets of numbers, for example,

$$96\,281\,745 \longleftrightarrow 396\,281\,745 \quad (\text{tacking the missing 3 onto the left}),$$

which explains why the two answers are the same. In general, ${}^nP_{n-1}$ and ${}^nP_n$ are both equal to $n!$

The Permutations of a Set: A *permutation of a set* is an arrangement of all the members of the set in a particular order. The number of such permutations is a special case of the previous paragraph — if the set has n members, then the number of permutations is ${}^nP_n = \frac{n!}{0!} = n!$

13 PERMUTATIONS OF A SET: The number of permutations of an n -member set, that is, the number of distinct orderings of the set, is ${}^nP_n = n!$

This result is so important that it should also be proven directly using a box diagram. In the boxes below, members are selected in turn to go into the first position, the second position, and so on.

1st position	2nd position	3rd position	...	n th position
n	$n-1$	$n-2$	...	1

Hence the number of orderings is $n(n-1)(n-2)\dots 1 = n!$ as expected.

WORKED EXERCISE: In how many ways can 20 people form a queue? Will the number of ways double with 40 people?

SOLUTION: With 20 people, number of ways = $20!$ [$\doteq 2.4 \times 10^{18}$].

With 40 people, number of ways = $40!$ [$\doteq 8.2 \times 10^{47}$].

Hence 40 people can form a queue in about 3×10^{29} times more ways than 20 people.

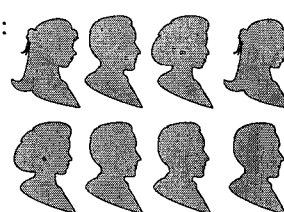
A General Counting Principle — Deal with the Restrictions First: Many problems have some restrictions in the way things can be arranged. These restrictions should be dealt with first. It is also important to keep in mind that the order of the boxes represents the order in which the choices are made, not the final ordering of the objects, and that they can be used in surprisingly flexible ways.

14 DEAL WITH THE RESTRICTIONS FIRST: When using boxes for counting ordered selections, deal with any restrictions first.
Remember that the boxes are in the order in which the selections are made.

WORKED EXERCISE: Eight people form two queues, each with four people. Albert will only stand in the left-hand queue, Beth will only stand in the right-hand queue, and Charles and Diana insist on standing in the same queue. How many ways can the two queues be formed?

SOLUTION: Place Albert in any of the 4 possible positions in the left-hand queue, Then place Beth in any of the 4 positions in the right-hand queue. Place Charles in any of the remaining 6 positions. Place Diana in one of the 2 remaining positions in the same queue. There remain 4 unfilled positions, which can be filled in $4!$ ways:

Albert	Beth	Charles	Diana	last four positions
4	4	6	2	$4!$



Hence number of ways $= 4 \times 4 \times 6 \times 2 \times 4!$
 $= 4608.$

A General Counting Principle — Compound Orderings: In some ordering problems, particular members must be grouped together. This produces a *compound ordering*, in which the various groups must first be ordered, and then the individuals ordered within each group.

15 COMPOUND ORDERINGS: First order the groups, then order the individuals within each group. (In this context, a group may sometimes consist of a single individual.)

WORKED EXERCISE: Four boys and four girls form a queue at the bus stop. There is one couple who want to stand together, the other three girls want to stand together, but the other three boys don't care where they stand. How many acceptable ways are there of forming the queue?

SOLUTION: There are five groups — the couple, the group of three girls, and the three groups each consisting of one individual boy. These five groups can be ordered in $5!$ ways. Then the individuals within each group must be ordered.

order the 5 groups	order the couple	order the 3 girls
$5!$	$2!$	$3!$

Hence number of ways $= 5! \times 2! \times 3! = 1440.$

Exercise 10E

1. Evaluate, using the formula ${}^n P_r = \frac{n!}{(n-r)!}$:

(a) ${}^6 P_2$

(c) ${}^3 P_3$

(e) ${}^5 P_5$

(g) ${}^9 P_5$

(b) ${}^{10} P_2$

(d) ${}^3 P_2$

(f) ${}^4 P_3$

(h) ${}^{10} P_8$

2. List all the permutations of the letters of the word DOG. How many are there?

3. List all the permutations of the letters EFGHI, beginning with F, taken three at a time.

4. Find how many arrangements of the letters of the word FRIEND are possible if the letters are taken: (a) four at a time, (b) six at a time.

5. Find how many four-digit numbers can be formed using the digits 5, 6, 7, 8 and 9 if:
(a) no digit is to be repeated, (b) any of the digits can occur more than once.
6. How many three-digit numbers can be formed using the digits 2, 3, 4, 5 and 6 if no digit can be repeated? How many of these are greater than 400?
7. In how many ways can seven people be seated in a row of seven different chairs?
8. Eight runners are participating in a 400-metre race.
(a) In how many ways can they finish?
(b) In how many ways can the gold, silver and bronze medals be awarded?
9. (a) If you toss a coin and roll a die, how many outcomes are possible?
(b) If you toss two coins and roll three dice, how many outcomes are possible?
10. A woman has four hats, three blouses, five skirts, two handbags and six pairs of shoes. In how many ways can she be attired, assuming that she wears one of each item?
11. Jack has six different football cards and Meg has another eight different football cards. In how many ways can one of Jack's cards be exchanged for one of Meg's cards?

DEVELOPMENT

12. In Sydney, phone numbers at present consist of eight digits, starting with the digit 9.
(a) How many phone numbers are possible?
(b) How many of these end in an odd number?
(c) How many consist of odd digits only?
(d) How many are there that do not contain a zero, and in which the consecutive digits alternate between odd and even?
13. Users of automatic teller machines are required to enter a four-digit pin number. Find how many pin numbers:
(a) are possible, (c) consist of odd digits only,
(b) consist of four distinct digits, (d) start and end with the same digit.
14. (a) If repetitions are not allowed, how many four-digit numbers can be formed from the digits 1, 2, ..., 8, 9?
(b) How many of these end in 1? (d) How many are divisible by 5?
(c) How many of these are even? (e) How many are greater than 7000?
15. Repeat the previous question if repetitions are allowed.
16. In Tasmania, a car licence plate consists of two letters followed by four digits. Find how many of these are possible:
(a) if there are no restrictions, (b) if there is no repetition of letters or digits,
(c) if the second letter is X and the third digit is 3,
(d) if the letters are D and Q and the digits are 3, 6, 7 and 9.
17. (a) In how many ways can the letters of the word NUMBER be arranged?
(b) How many begin with N? (c) How many begin with N and end with U?
(d) In how many is the N somewhere to the left of the U?

18. Find how many arrangements of the letters of the word **UNIFORM** are possible:
- (a) if the vowels must occupy the first, middle and last positions,
 - (b) if the word must start with U and end with M,
 - (c) if all the consonants must be in a group at the end of the word,
 - (d) if the M is somewhere to the right of the U.
19. Find how many arrangements of the letters of the word **BEHAVING**:
- (a) end in NG, (b) begin with three vowels, (c) have three vowels occurring together.
20. In how many ways can three Maths books, six Science books and four English books be placed on a shelf, if the books relating to each subject are to be kept together?
21. A Maths test is to consist of six questions. In how many ways can it be arranged so that:
- (a) the shortest question is first and the longest question is last?
 - (b) the shortest and longest questions are next to one another?
22. In how many ways can a boat crew of eight women be arranged if three of the women can only row on the bow side and two others can only row on the stroke side?
23. A motor bike can carry three people: the driver, one passenger behind the driver and one in the sidecar. If among five people, only two can drive, in how many ways can the bike be filled?
24. In Morse code, letters are formed by a sequence of dashes and dots. How many different letters is it possible to represent if a maximum of ten symbols are used?
25. Four boys and four girls are to sit in a row. Find how many ways this can be done if:
- (a) the boys and girls alternate, (b) the boys and girls sit in distinct groups.
26. Five-letter words are formed without repetition from the letters of **PHYSICAL**.
- (a) How many consist only of consonants? (c) How many begin with a vowel?
 - (b) How many begin with P and end with S? (d) How many contain the letter Y?
 - (e) How many have the two vowels occurring next to one another?
 - (f) How many have the letter A immediately following the letter L?
27. (a) How many seven-letter words can be formed without repetition from the letters of the word **INCLUDE**?
- (b) How many of these do not begin with I? (c) How many end in L?
 - (d) How many have the vowels and consonants alternating?
 - (e) How many have the C immediately following the D?
 - (f) How many have the letters N and D separated by exactly two letters?
 - (g) How many have the letters N and D separated by more than two letters?
28. Repeat parts (a)–(d) of the previous question if repetition is allowed.
29. (a) Find in how many ways ten people can be arranged in a line:
- (i) without restriction,
 - (ii) if one particular person must sit at either end,
 - (iii) if two particular people must sit next to one another,
 - (iv) if neither of two particular people can sit on either end of the row.
- (b) Find in how many ways can n people be placed in a row of n chairs:
- (i) if one particular person must be on either end of the row,
 - (ii) if two particular people must sit next to one another,
 - (iii) if two of them are not permitted to sit at either end.

30. Five boys and four girls form a queue at the cinema. There are two brothers who want to stand together, the remaining three boys wish to stand together, and the four girls don't mind where they stand. In how many ways can the queue be formed?
31. Eight people are to form two queues of four. In how many ways can this be done if:
- there are no restrictions?
 - Jim will only stand in the left hand queue?
 - Sean and Liam must stand in the same queue?
32. (a) How many five-figure numbers can be formed from the digits 2, 3, 4, 5 and 6?
 (b) How many of these numbers are greater than 56 432?
 (c) How many of these numbers are less than 56 432?
33. There are eight swimmers in a race. Find in how many ways can they finish if there are no dead heats and the swimmer in Lane 2 finishes:
- immediately after the swimmer in Lane 5,
 - after the swimmer in Lane 5.
34. (a) Integers are formed from the digits 2, 3, 4 and 5, with repetitions not allowed.
 (i) How many such numbers are there? (ii) How many of them are even?
 (b) Repeat the two parts to this question if repetitions are allowed.
35. (a) How many five-digit numbers can be formed from the digits 0, 1, 2, 3 and 4 if repetitions are not allowed?
 (b) How many of these are odd? (c) How many are divisible by 5?
36. Five backpackers arrive in a city where there are five youth hostels.
- How many different accommodation arrangements are there if there are no restrictions on where the backpackers stay?
 - How many different accommodation arrangements are there if each backpacker stays at a different youth hostel?
 - Suppose that two of the backpackers are brother and sister and wish to stay in the same youth hostel. How many different accommodation arrangements are there if the other three can go to any of the other youth hostels?
37. Numbers less than 4000 are formed from the digits 1, 3, 5, 8 and 9, without repetition.
- How many such numbers are there?
 - How many of them are odd?
 - How many of them are divisible by 5?
 - How many of them are divisible by 3?
38. (a) If ${}^8P_r = 336$, find the value of r . (b) If $7^{2n}P_n = 4^{2n+1}P_n$, find n .
 (c) Using the result ${}^nP_r = \frac{n!}{(n-r)!}$, prove that:
- $$(i) \quad {}^{n+1}P_r = {}^nP_r + r^n P_{r-1} \qquad (ii) \quad {}^nP_r = {}^{n-2}P_r + 2r {}^{n-2}P_{r-1} + r(r-1) {}^{n-2}P_{r-2}$$

EXTENSION

39. [Derangements] A *derangement* of n distinct objects is an arrangement of them so that none of the n objects appear in their original positions. For example, 4123 is a derangement of 1234, but 4132 is not. We will denote the number of derangements of n objects by $D(n)$.
- Find the values of $D(1)$, $D(2)$ and $D(3)$ by listing.
 - One way of determining the value of $D(n)$ for a particular value of n is by the inclusion/exclusion principle. Consider the case where $n = 4$.
 - How many ways are there of arranging 4 objects in a row?
 - How many of these would leave one specific object unmoved from its original position? This number would need to be subtracted from (a).

- (iii) However, in doing so, we have subtracted some of the arrangements twice. For example, some of them would leave both the first and second elements unchanged. Thus you need to add back the number of arrangements that leave any two of the objects unmoved. How many of these are there?
- (iv) Now you will need to subtract the number of arrangements that leave three objects unmoved and so on. Hence find an expression for $D(4)$.
- (c) Find an expression for $D(5)$. (d) Hence find an expression for $D(n)$.
40. [An alternative formula for derangements] As above, denote the number of derangements of n objects by $D(n)$. Suppose that $n > 2$.
- (a) Suppose that in a derangement of the n objects, the object in the n th place changes places with the object in the r th place. Explain why the number of ways this can happen is $(n-1)D(n-2)$.
- (b) The second alternative is that the object in the r th place moves to the n th place, but the object in the n th place does not move to the r th place. How many derangements are there of this type?
- (c) Hence find a recurrence relation for $D(n)$, stating the initial conditions.

10F Counting with Identical Elements, and Cases

This section covers two ideas — counting permutations in which there are identical elements, and counting in which the box methods of the previous section break down and cases have to be considered.

Counting with Identical Elements: Finding the number of different words formed by using all the letters of the word ‘PRESSES’ is complicated by the fact that there are three Ss and two Es. If the seven letters were all different, we would conclude:

$$\text{number of ways} = 7!$$

But we have *overcounted by a factor of* $2! = 2$, because the Es can be interchanged without changing the word. We have also *overcounted by a factor of* $3! = 6$, because the three Ss can be permuted amongst themselves in $3!$ ways without changing the word. Taking account of both overcountings:

$$\text{number of ways} = \frac{7!}{2! \times 3!} = 420.$$

This method is easily generalised. In the language of ‘words’:

COUNTING WITH IDENTICAL ELEMENTS: Suppose that a word of n letters has ℓ_1 alike of one type, ℓ_2 alike of another type, ..., ℓ_k alike of a final type.

16 Then the number of distinct words that can be formed from the letters is

$$\text{number of words} = \frac{n!}{\ell_1! \times \ell_2! \times \cdots \times \ell_k!}$$

WORKED EXERCISE: Four identical wine glasses and four identical tumblers are to be arranged in a line across the front of a cupboard.

- (a) How many ways can this be done?
- (b) How does this change if one of the wine glasses now is chipped, and two tumblers have now been replaced by two identical tumblers slightly different from the other two?

SOLUTION:

$$\begin{aligned} \text{(a) Number of ways} &= \frac{8!}{4! \times 4!} \\ &= 70. \end{aligned}$$

$$\begin{aligned} \text{(b) Number of ways} &= \frac{8!}{3! \times 1! \times 2! \times 2!} \\ &= 1680. \end{aligned}$$

Words Containing Only Two Different Letters: The theory about counting with identical letters assumes particular importance when there are only two types of letters, as in part (a) of the previous worked exercise. The answer to part (a) was $\frac{8!}{4! \times 4!}$, which is none other than ${}^8C_4$. This is true in general, and provides the essential link between the binomial theorem and probability.

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PERMUTATIONS OF WORDS CONTAINING ONLY TWO DIFFERENT LETTERS: Suppose that a word with n letters has k alike of one type and the remaining $n - k$ alike of another type. Then the number of distinct words that can be formed from the letters is

$$\text{number of words} = \frac{n!}{k! \times (n - k)!} = {}^nC_k.$$

WORKED EXERCISE: In a ten-question test, one mark is awarded for each question. In how many ways can a pupil score 7/10? For what other mark is there the same number of ways of achieving it?

SOLUTION: The mark of 7/10 means there were 7 correct and 3 incorrect,

$$\begin{aligned} \text{so number of ways} &= \frac{10!}{7! \times 3!} \\ &= 120. \end{aligned}$$

The other mark that would give 120 ways is 3/10, because for this mark there would be 3 correct and 7 incorrect. Notice that ${}^{10}C_7 = {}^{10}C_3 = 120$.

The Connection with the Binomial Theorem: The fact that the number of words with k As and $n - k$ Bs is ${}^nC_k$ can be seen directly from the binomial expansion. Below is the expansion of $(A + B)^5$ into 32 terms, initially without any simplification or collection of terms. There are $2^5 = 32$ terms, because each term involves choosing either A or B from each of the five brackets in turn.

$$\begin{aligned} (A + B)^5 &= (A + B)(A + B)(A + B)(A + B)(A + B) \\ &= AAAAA \\ &\quad + AAAAB + AAABA + AABAA + ABAAA + BAAAA \\ &\quad + AAABB + AABAB + ABAAB + BAAAB + AABBA \\ &\quad + ABABA + BAABA + ABBAA + BABAA + BBAAA \\ &\quad + AABBB + ABABB + BAABB + ABBAB + BABAB \\ &\quad + BBAAB + ABBBA + BABBA + BBABA + BBBAA \\ &\quad + ABBBB + BBBBB + BBABB + BBBAB + BBBBA \\ &\quad + BBBBB \\ &= A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5 \end{aligned}$$

This should make it clear that, for example, the coefficient ${}^5C_3 = 10$ of A^3B^2 is simply the number of all possible words formed from 3 As and 2 Bs. More generally, the coefficient ${}^nC_k$ of $A^k B^{n-k}$ in the expansion of $(A + B)^n$ is the number of all possible words formed from k As and $n - k$ Bs.

Using Cases: Many counting problems are too complicated to be analysed completely by a single box diagram. In such situations, the use of cases is unavoidable. Attention should be given, however, to minimising the number of different cases that need to be considered.

WORKED EXERCISE: How many six-letter words can be formed by using the letters of the word 'PRESSES'?

SOLUTION: We omit in turn each of the four letters 'P', 'R', 'E' and 'S'.

This leaves six letters which we must then arrange in order.

1. If an S is omitted, there are then 2 Es and 2 Ss,

$$\text{so number of words} = \frac{6!}{2! \times 2!} = 180.$$

2. If an E is omitted, there are then 3 Ss,

$$\text{so number of words} = \frac{6!}{3!} = 120.$$

3. If P or R is omitted (2 cases), there are then 2 Es and 3 Ss,

$$\begin{aligned} \text{so number of words} &= \frac{6!}{3! \times 2!} \times 2 \quad (\text{doubling for the two cases}) \\ &= 120. \end{aligned}$$

Hence the total number of words is $180 + 120 + 120 = 420$.

Exercise 10F

1. Find the number of permutations of the following words if all the letters are used:

(a) BOB	(d) TASMANIA	(g) EQUILATERAL
(b) ALAN	(e) BEGINNER	(h) COMMITTEE
(c) SNEEZE	(f) FOOTBALLS	(i) WOOLLOOMOOLOO

2. The six digits 1, 1, 1, 2, 2, 3 are used to form a six-digit number. How many numbers can be formed?
3. Six coins are lined up on a table. Find how many patterns are possible if there are:
 (a) five tails and one head, (b) four heads and two tails, (c) three tails and three heads.
4. Eight balls, identical except for colour, are arranged in a line. Find how many different arrangements are possible if:
 (a) all balls are of a different colour, (b) there are seven red balls and one white ball,
 (c) there are six red balls, one white ball and one black ball,
 (d) there are three red balls, three white balls and two black balls.
5. Five identical green chairs and three identical red chairs are arranged in a row. Find how many arrangements are possible:
 (a) if there are no restrictions, (b) if there must be a green chair on either end.

DEVELOPMENT

6. A motorist travels through eight sets of traffic lights, each of which is red or green. He is forced to stop at three sets of lights.
 - (a) In how many ways could this happen?
 - (b) What other number of red lights would give an identical answer to part (a)?
7. Find how many ways the letters of the word SOCKS can be arranged in a line:
 - (a) without restriction,
 - (b) so that the two Ss are together,
 - (c) so that the two Ss are separated by at least one other letter,
 - (d) so that the K is somewhere to the left of the C.
8. (a) Find the number of arrangements of the letters in SLOOPS if:
 - (i) there are no restrictions,
 - (iii) the two Os are to be separated,
 - (ii) the two Os are together,
 - (iv) the Os are together and the Ss are together.
 (b) How many arrangements of the letters in TATTOO are there, if the two Os are to be separated?
9. Find how many ways the letters of the word DECISIONS can be arranged:
 - (a) without restriction,
 - (b) so that the vowels and consonants alternate,
 - (c) so that the vowels come first followed by the consonants,
 - (d) so that the N is somewhere to the right of the D.
10. In how many ways can the letters of the word PROPORTIONALITY be arranged so that the vowels and consonants still occupy the same places?
11. A form has ten questions in order, each of which requires the answer 'Yes' or 'No'. Find the number of ways the form can be filled in:
 - (a) without restriction,
 - (f) if an odd number of answers are 'Yes',
 - (b) if the first and last answers are 'Yes',
 - (g) if exactly three answers are 'Yes', and they occur together,
 - (c) if two are 'Yes' and eight are 'No',
 - (h) if the first and last answers are 'Yes' and exactly four more are 'Yes'.
 - (d) if five are 'Yes' and five are 'No',
 - (e) if more than seven answers are 'Yes',
12. Containers are identified by a row of coloured dots on their lids. The colours used are yellow, green and purple. In any arrangement, there are to be no more than three yellow dots, no more than two green dots and no more than one purple dot.
 - (a) If six dots are used, what is the number of possible codes?
 - (b) What is the number of different codes possible if only five dots are used?
13. (a) How many five-letter words can be formed by using the letters of the word STRESS?
 (b) How many five-letter words can be formed by using the letters of the word BANANA?
14. Six-letter words are formed using two As and four Bs. By referring to the relevant section of the notes, show how all the different possible words can be derived from the expansion of $(A + B)^6$.
15. Find how many arrangements of the letters of the word TRANSITION are possible if:
 - (a) there are no restrictions,
 - (d) the Ns occupy the end positions,
 - (b) the Is are together,
 - (e) an N occupies the first but not the last position,
 - (c) the Is, Ns and Ts are together,
 - (f) the letter N is not at either end.

16. Ten coloured marbles are placed in a row.
- If they are all of different colours, how many arrangements are possible?
 - What is the minimum number of colours needed to guarantee at least 10 000 different patterns? [HINT: This will need a guess-and-check approach.]

EXTENSION

17. Bob is about to hang his eight shirts in the wardrobe. He has four different styles of shirt, two identical ones of each particular style. How many different arrangements are possible if no two identical shirts are next to one another?
18. Find how many arrangements there are of the letters of the word GUMTREE if:
- there are no restrictions,
 - the Es are together,
 - the Es are separated by:
 - one letter, (ii) two letters, (iii) three letters, (iv) four letters, (v) five letters,
 - the G is somewhere between the two Es,
 - the M is somewhere to the left of both Es and the U is somewhere between them,
 - the G is somewhere to the left of the U and the M is somewhere to the right of the U.
19. If the letters of the word GUMTREE and the letters of the word KOALA are combined and arranged into a single twelve-letter word, in how many of these arrangements do the letters of KOALA appear in their correct order, but not necessarily together?
20. Binary numbers consist of a sequence of 0s and 1s. One such number contains exactly a 0s and at most b 1s.
- If the number contains $a + b$ digits, show that there are ${}^{a+b}C_b$ possible sequences. You will need to recall that ${}^nC_r = \frac{n!}{r!(n-r)!}$.
 - Prove that: ${}^aC_0 + {}^{a+1}C_1 + {}^{a+2}C_2 + \cdots + {}^{a+b}C_b = {}^{a+b+1}C_b$.
 - Hence show that the total number of possible sequences is ${}^{a+b+1}C_b$.

10G Counting Unordered Selections

This section now turns from the ordered selections of the previous two sections to the counting of unordered selections. An unordered selection of distinct objects chosen from a certain set can be regarded simply as a subset of the set. For example, the four-member set $\{A, B, C, D\}$ has six two-member subsets:

$$\{A, B\} \quad \{A, C\} \quad \{A, D\} \quad \{B, C\} \quad \{B, D\} \quad \{C, D\}$$

The central result of the section is the following.

18	UNORDERED SELECTIONS: If a set S has n members, then the number of unordered selections of k members from the set is $\text{number of } k\text{-member subsets} = \frac{n!}{k! \times (n-k)!} = {}^nC_k.$ The total number of subsets of S is 2^n.

For example, the number of unordered selections, or subsets, of two distinct letters taken from the four-member set $\{A, B, C, D\}$ is ${}^4C_2 = \frac{4!}{2! \times 2!} = 6$. These are the six subsets listed above.

This result brings the whole theory of the binomial expansion into probability and counting, and it will dominate the remainder of this chapter. Two different proofs of the result will be offered.

An Example of the Result: Before giving any proof, however, here is an example to illustrate the result. Let S be the five-member set

$$S = \{A, B, C, D, E\}.$$

The total number of subsets of S is $2^5 = 32$, because choosing a subset or unordered selection from S requires looking at every member of S and deciding whether to include it in the subset or not — using the box notation:

Is A in or out?	Is B in or out?	Is C in or out?	Is D in?	Is E in?
2	2	2	2	2

Here is a list of those 32 subsets, arranged according to the number of their members:

- 1 0-member subset : $\emptyset$
- 5 1-member subsets: $\{A\}, \{B\}, \{C\}, \{D\}, \{E\}$
- 10 2-member subsets: $\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\},$
 $\{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}$
- 10 3-member subsets: $\{A, B, C\}, \{A, B, D\}, \{A, B, E\}, \{A, C, D\}, \{A, C, E\},$
 $\{A, D, E\}, \{B, C, D\}, \{B, C, E\}, \{B, D, E\}, \{C, D, E\}$
- 5 4-member subsets: $\{A, B, C, D\}, \{A, B, C, E\}, \{A, B, D, E\},$
 $\{A, C, D, E\}, \{B, C, D, E\}$
- 1 5-member subset : $\{A, B, C, D, E\}$

The numbers 1, 5, 10, 10, 5, 1 form the row of the Pascal triangle indexed by $n = 5$ and add to $2^5 = 32$. Thus at least for $n = 5$, the number of k -member subsets of an n -member set is indeed ${}^nC_k$.

A Proof Using Words Containing only Two Distinct Letters: Continuing with the example above, every subset of S can be described by visiting each member A, B, C, D and E of S in turn, answering ‘yes’ or ‘no’ as to whether that member is included in the subset. The result is that every subset of S corresponds to a five-letter word consisting entirely of Ys and Ns. For example, each of the ten three-member subsets of S is paired below with the corresponding word consisting of three Ys and two Ns:

$$\begin{array}{ll}
 \{A, B, C\} \longleftrightarrow \text{YYYNN} & \{A, D, E\} \longleftrightarrow \text{YNNYY} \\
 \{A, B, D\} \longleftrightarrow \text{YYNYN} & \{B, C, D\} \longleftrightarrow \text{NYYYN} \\
 \{A, B, E\} \longleftrightarrow \text{YYNNY} & \{B, C, E\} \longleftrightarrow \text{NYYNY} \\
 \{A, C, D\} \longleftrightarrow \text{YNNYY} & \{B, D, E\} \longleftrightarrow \text{NYYNY} \\
 \{A, C, E\} \longleftrightarrow \text{YNNYY} & \{C, D, E\} \longleftrightarrow \text{NNYYY}
 \end{array}$$

From the previous section, we know that the number of five-letter words consisting of three Ys and two Ns is $\frac{5!}{3! \times 2!} = {}^5C_3 = 10$, so this must be the number of three-member subsets. Moreover, the total number of five-letter words consisting entirely of Ys and Ns is $2^5 = 32$, because there are two choices for each letter in turn, so this must be the total number of subsets.

In general, let the members of the n -member set S be listed in some order:

$$S = \{ S_1, S_2, S_3, S_4, S_5, \dots, S_n \}.$$

Again, choosing a subset T of S means visiting every member of S and saying 'yes' or 'no' as to whether that member is to be included in the subset T . Thus there is a one-to-one correspondence between the subsets of S and the set of 2^n words of n letters consisting entirely of Ys and Ns. In particular, there is a one-to-one correspondence between the k -member subsets of S and the ${}^nC_k$ words consisting of k Ys and $n - k$ Ns.

A Proof Moving from Ordered Selections to Unordered Selections: We saw in Section 10E that the number of three-letter words formed without repetition from the five letters A, B, C, D and E is ${}^5P_3 = 5 \times 4 \times 3 = 60$. When we turn to unordered selections, however, there are six distinct words which all correspond, for example, to the three-member subset $\{B, C, E\}$:

$$BCE, BEC, CBE, CEB, EBC, ECB \longleftrightarrow \{B, C, E\}$$

The reason for this is that there are ${}^3P_3 = 3 \times 2 \times 1 = 6$ ways of ordering the subset $\{B, C, E\}$. Thus the correspondence between three-letter words and three-member subsets is many-to-one, with a six-fold overcounting. Hence the number of three-member subsets is $60 \div 6 = 10$, as required.

In general, ${}^nP_k = \frac{n!}{(n-k)!}$ words of k letters can be formed without repetition from the members of S . But every k -member subset can be ordered in ${}^kP_k = k!$ ways, so the correspondence between the ordered selections and the unordered selections is many-to-one with overcounting by a factor of $k!$. Hence the number of (unordered) k -member subsets of S is

$${}^nP_k \div {}^kP_k = \frac{n!}{(n-k)!} \div k! = \frac{n!}{k! \times (n-k)!} = {}^nC_k.$$

The Meaning of the Notation ${}^nC_k$: The notation ${}^nC_k$ does not originate with the binomial theorem, but with this piece of counting theory. Unordered selections are also called *combinations*, and the letter C in ${}^nC_k$ stands for 'combination', just as the letter P in ${}^nP_k$ stands for 'permutation'.

By a convenient, but false, etymology, the letter C also stands for 'choose'. This is the origin of the more recent convention of calling ${}^nC_k$ ' n choose k '.

WORKED EXERCISE: Ten people meet to play doubles tennis.

- How many ways can four people be selected from this group to play the first game? (Ignore the subsequent organisation into pairs.)
- How many of these ways will include Maria and exclude Alex?
- If there are four women and six men, how many ways can two men and two women be chosen for this game?
- Again with four women and six men, in how many ways will women be in the majority?

SOLUTION:

- (a) Number of ways = $^{10}C_4 = 210$.
- (b) Since Maria is included, three further people must be chosen, and since Alex is excluded, there are now eight people to choose these three from.
Hence number of ways = $^8C_3 = 56$.
- (c) Number of ways of choosing the women = $^4C_2 = 6$,
number of ways of choosing the men = $^6C_2 = 15$,
so number of ways of choosing all four = $15 \times 6 = 90$.
- (d) Number of ways with one man and three women = $^6C_1 \times ^4C_3 = 24$,
number of ways with four women = 1,
so number of ways with a majority of women = $24 + 1 = 25$.

A Natural (or Canonical) Correspondence — $^nC_k = ^nC_{n-k}$: Suppose that two people are to be chosen from five to make afternoon tea. This task can be done equally well by choosing the three people who will *not* be making tea. Thus for every choice of two people out of five, the remaining three people is a corresponding choice of three people out of five. Not only does this confirm that $^5C_2 = ^5C_3$, but it gives a one-to-one correspondence between the two-member and three-member subsets of a five-member set:

$$\begin{array}{ll}
 \{A, B\} \longleftrightarrow \{C, D, E\} & \{B, D\} \longleftrightarrow \{A, C, E\} \\
 \{A, C\} \longleftrightarrow \{B, D, E\} & \{B, E\} \longleftrightarrow \{A, C, D\} \\
 \{A, D\} \longleftrightarrow \{B, C, E\} & \{C, D\} \longleftrightarrow \{A, B, E\} \\
 \{A, E\} \longleftrightarrow \{B, C, D\} & \{C, E\} \longleftrightarrow \{A, B, D\} \\
 \{B, C\} \longleftrightarrow \{A, D, E\} & \{D, E\} \longleftrightarrow \{A, B, C\}
 \end{array}$$

In this correspondence, every subset T corresponds to its complement $\bar{T}$:

$$T \longleftrightarrow \bar{T}$$

This correspondence is a natural or *canonical* correspondence, but the correspondence is by no means restricted to mathematics. In normal language, a situation can often be described just as well by saying what it is not as by saying what it is, and we are all familiar with ‘invisible enemies’ or ‘unforgivable actions’ or ‘days when no rain fell’.

Words such as ‘at least’, ‘at most’, ‘not’ and ‘excluding’ should always be regarded as warnings that the problem may best be solved by considering the complementary situation.

WORKED EXERCISE: Let $S = \{2, 4, 6, 8, 10, 12\}$ be the set consisting of the first six positive even numbers.

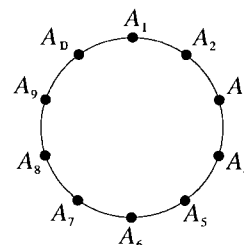
- (a) How many subsets of S contain at least two numbers?
- (b) How many subsets with at least two numbers do not contain 8?
- (c) How many subsets with at least two numbers do not contain 8 but do contain 10?

SOLUTION:

- (a) Number of 1-member and 0-member subsets $= {}^6C_1 + {}^6C_0 = 7$,
so number with at least 2 members $= 2^6 - 7 = 57$.
- (b) We consider now the 5-member set $T = \{2, 4, 6, 10, 12\}$.
Number of 1-member and 0-member subsets $= {}^5C_1 + {}^5C_0 = 6$,
so number with at least 2 members $= 2^5 - 6 = 26$.
- (c) Since 10 has already been chosen, we need to choose subsets with at least one member from the four-member set $U = \{2, 4, 6, 12\}$.
Number of 0-member subsets $= 1$ (the empty set),
so number of such subsets $= 2^4 - 1 = 15$.

WORKED EXERCISE: [A harder question] Ten points $A_1, A_2, \dots, A_{10}$ are arranged in order around a circle.

- (a) How many triangles can be drawn with these points as vertices?
- (b) How many pairs of such triangles can be drawn, if the vertices of the two triangles are distinct?
- (c) In how many such pairs will the triangles:
(i) not overlap, (ii) overlap?



SOLUTION:

- (a) To form a triangle, we must choose 3 points out of 10,
so number of triangles $= {}^{10}C_3 = 120$.
- (b) To form two triangles, first choose 6 points out of 10,
which can be done in ${}^{10}C_6 = 210$ ways.
Take any one of those 6 points, and choose the other 2 points in its triangle;
this can be done in ${}^5C_2 = 10$ ways.
Hence number of pairs of triangles $= 210 \times 10 = 2100$.
- (c) To form two non-overlapping triangles, we first choose 6 points out of 10,
which again can be done in ${}^{10}C_6 = 210$ ways.
These 6 points can be made into two non-overlapping triangles in 3 ways,
by arranging the 6 points in cyclic order, and choosing 3 adjacent points.
(i) Hence number of non-overlapping pairs $= 210 \times 3 = 630$.
(ii) By subtraction, number of overlapping pairs $= 2100 - 630 = 1470$.

Each Row of the Pascal Triangle Adds to 2^n : One identity about the Pascal triangle needs review here. The binomial expansion is

$$(x + y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_n x^0 y^n.$$

Substituting $x = y = 1$ gives $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$, which means that the row indexed by n in the Pascal triangle adds to 2^n .

We now have another interpretation of this identity. An n -member set has ${}^nC_k$ k -member subsets, and the total number of subsets is

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n.$$

Exercise 10G

- Two people are chosen from a group of five people called P, Q, R, S and T. List all possible combinations, and find how many there are.
- Find how many ways you can form a group of:
 - two people from a choice of seven,
 - three people from a choice of seven,
 - two people from a choice of six,
 - five people from a choice of nine.
- Find how many possible combinations there are if, from a group of ten people:
 - two people are chosen,
 - eight people are chosen.
 - Why are the answers identical?
- From a party of twelve men and eight women, find how many groups there are of:
 - five men and three women,
 - four women and four men.
- Four numbers are to be selected from the set of the first eight positive integers. Find how many possible combinations there are if:
 - there are no restrictions,
 - there are two odd numbers and two even numbers,
 - there is exactly one odd number,
 - all the numbers must be even,
 - there is at least one odd number.
- Four balls are simultaneously drawn from a bag containing three green and six blue marbles. Find how many drawings are possible if:
 - the balls may be of any colour,
 - there are exactly two green balls,
 - there are at least two green balls,
 - there are more blue balls than green balls.
- A committee of five is to be chosen from six men and eight women. Find how many committees are possible if:
 - there are no restrictions,
 - all members are to be female,
 - all members are to be male,
 - there are exactly two men,
 - there are four women and one man,
 - there is a majority of women,
 - a particular man must be included,
 - a particular man must not be included.

DEVELOPMENT

- What is the number of combinations of the letters of the word EQUATION taken four at a time (without repetition)?
 - How many contain four vowels?
 - How many contain the letter Q?
- A team of seven netballers is to be chosen from a squad of twelve players A, B, C, D, E, F, G, H, I, J, K and L. Find how many ways can they be chosen:
 - with no restrictions,
 - if the captain C is to be included,
 - if J and K are both to be excluded,
 - if A is included but H is not,
 - if one of F and L is to be included and the other excluded.
- Consider the digits 9, 8, 7, 6, 5, 4, 3, 2, 1 and 0. Find how many five-digit numbers are possible if the digits are to be in:
 - ascending order,
 - descending order.
 - Why do these two questions involve unordered selections?
- Twelve people arrive at a restaurant. There is one table for six, one table for four and one table for two. In how many ways can they be assigned to a table?

12. Twenty students, ten male and ten female, are to travel from school to the sports ground. Eight of them go in a minibus, six of them in cars, four of them on bikes and two walk.
- (a) In how many ways can they be distributed for the trip?
 - (b) In how many ways can they be distributed if none of the boys walk?
13. Ten points are chosen in a plane, no three of which are collinear.
- (a) How many lines can be drawn through pairs of the points?
 - (b) How many triangles can be drawn if each of the vertices is at one of the given points?
 - (c) How many of the triangles have a particular point as one of the vertices?
 - (d) How many of the triangles have two particular points making up one of the sides?
14. There are ten points in a plane, five of which are collinear. No other set of three of these points is collinear.
- (a) How many sets of three points can be selected from those five that are collinear?
 - (b) How many triangles can be formed using the ten points as vertices?
15. In how many ways can a group of six people be divided into two unequal groups?
16. From a standard deck of 52 playing cards, find how many five-card hands can be dealt:
- (a) consisting of black cards only,
 - (b) consisting of diamonds only,
 - (c) containing all four kings,
 - (d) consisting of three diamonds and two clubs,
 - (e) consisting of three twos and another pair,
 - (f) consisting of one pair and three of a kind.
17. (a) How many diagonals are there in a quadrilateral?
- (b) How many diagonals are there in a pentagon?
 - (c) How many diagonals are there in a decagon?
 - (d) How many diagonals are there in a polygon with n sides?
18. Twelve points are arranged in order around a circle.
- (a) How many triangles can be drawn with these points as vertices?
 - (b) How many pairs of such triangles can be drawn if the vertices of the two triangles are distinct?
 - (c) In how many such pairs will the triangles: (i) not overlap, (ii) overlap?
19. Let $S = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ be the set of the first ten positive odd integers.
- (a) How many subsets does S have?
 - (b) How many subsets of S contain at least three numbers?
 - (c) How many subsets with at least three numbers do not contain 7?
 - (d) How many subsets with at least three numbers do not contain 7 but do contain 19?
20. Find how many ways two numbers can be selected from the numbers 1, 2, ..., 8, 9 so that their sum is:
- (a) even, (b) odd, (c) divisible by 3, (d) divisible by 5, (e) divisible by 6.
21. There are ten basketballers in a team. Find how many ways:
- (a) the starting five can be chosen, (b) they can be split into two teams of five.
22. Nine players are to be divided into two teams of four and one umpire.
- (a) In how many ways can the teams be formed?
 - (b) If two particular people cannot be on the same team, how many different combinations are possible?

23. By considering its prime factorisation, find the number of positive divisors of 315 000.
24. Use the fact that ${}^nC_r$ is the number of unordered selections of r objects from n objects to provide combinatoric proofs of each of the following.
- (a) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ (d) ${}^{n+3}C_r = {}^nC_r + 3{}^nC_{r-1} + 3{}^nC_{r-2} + {}^nC_{r-3}$
 (b) ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ (e) ${}^{m+n}C_2 = {}^mC_2 + {}^mC_1 {}^nC_1 + {}^nC_2$
 (c) ${}^{n+2}C_r = {}^nC_r + 2{}^nC_{r-1} + {}^nC_{r-2}$ (f) ${}^{m+n}C_3 = {}^mC_3 + {}^mC_2 {}^nC_1 + {}^mC_1 {}^nC_2 + {}^nC_3$
25. The diagram shows a grid measuring 4 cm by 6 cm. The aim is to get from point A in the top left-hand corner to point B in the bottom right-hand corner by moving along the black lines either downwards or to the right. A single move is defined as shifting along one side of a single square, thus it takes you ten moves to get from A to B .
-
- (a) How many different routes are possible?
 (b) How many different routes are possible if you must pass through C ?
 (c) How many different routes are possible if you cannot move along the top line of the grid?
 (d) How many different routes are possible if you cannot move along the second row from the top of the grid?
26. (a) The six faces of a number of identical cubes are painted in six distinct colours. How many different cubes can be formed?
 (b) A die fits perfectly into a cubical box. How many ways are there of putting the die into the box?

EXTENSION

27. A piece of art receives a mark out of 100 for each of the categories design, technique and originality. In how many ways is it possible to score a mark of 200/300?
28. How many different combinations are there of three different integers between one and thirty inclusive such that their sum is divisible by three?
29. (a) How many doubles tennis games are possible, given a group of four players?
 (b) In how many ways can two games of doubles tennis be arranged, given a group of eight players?
 (c) Six married couples are to play in three games of doubles tennis. Find how many ways the pairings can be arranged if:
 (i) there are no restrictions, (ii) each game is to be a game of mixed doubles.

10H Using Counting in Probability

The purpose of this section is to apply the counting procedures of the last three sections to questions about probability. In these more complicated questions, counting procedures are required for counting both the sample space and the event space.

Counting the Sample Space and the Event Space: It is usually easier to use unordered selections, when this is possible, but as always, the two questions that need to be asked are: 'Is the selection ordered?' and if so, 'Is repetition allowed?'

WORKED EXERCISE: Three cards are dealt from a pack of 52.

- (a) Find the probability that one club and two hearts are dealt, in any order.
 (b) Find the probability that one club and two hearts are dealt in that order.

SOLUTION:

- (a) Let the sample space be the set of all unordered selections of 3 cards from 52,

$$\text{so number of unordered hands} = {}^{52}C_3 = \frac{52 \times 51 \times 50}{3 \times 2 \times 1}.$$

We can now choose the hand by choosing 1 club from 13 in ${}^{13}C_1 = 13$ ways,
 and choosing the 2 hearts from 13 in ${}^{13}C_2 = 78$ ways,
 so the hand can be chosen in 13×78 ways.

$$\text{Hence } \mathcal{P}(\text{1 club and 2 hearts}) = 13 \times 78 \times \frac{3 \times 2 \times 1}{52 \times 51 \times 50} = \frac{39}{850}.$$

- (b) Let the sample space be the set of all ordered selections of 3 cards from 52,

$$\text{so number of ordered hands} = {}^{52}P_3 = 52 \times 51 \times 50,$$

$$\text{and number of such hands in the order } \clubsuit\heartsuit\heartsuit = 13 \times 13 \times 12.$$

$$\text{Hence } \mathcal{P}(\clubsuit\heartsuit\heartsuit) = \frac{13 \times 13 \times 12}{52 \times 51 \times 50} = \frac{13}{850}.$$

NOTE: The answer to part (b) must be $\frac{1}{3}$ of the answer to part (a), because in a hand with one club and two hearts, the club can be any one of three positions. This indicates that it would be quite reasonable to do part (a) using ordered selections, and to do part (b) using unordered selections, although the methods chosen above are more natural to the way in which each question was worded.

Problems Requiring a Variety of Methods: The sample spaces in the two worked examples following are easily found, but a variety of methods is needed to establish the sizes of the various event spaces.

WORKED EXERCISE: A five-digit number is chosen at random. Find the probability:

- (a) that it is at least 60 000, (b) that it consists only of even digits,
 (c) that the digits are distinct,
 (d) that the digits are distinct and in increasing order.

SOLUTION: The first digit of a five-digit number cannot be zero, giving nine choices, but the other digits can be any one of the ten digits.

$$\text{Hence the number of five-digit numbers} = 9 \times 10 \times 10 \times 10 \times 10 = 90\,000.$$

- (a) To be at least 60 000, the first digit can be 6, 7, 8 or 9,

$$\text{so the number of favourable numbers is } 4 \times 10 \times 10 \times 10 \times 10 = 40\,000.$$

$$\text{Hence } \mathcal{P}(\text{at least 60 000}) = \frac{4}{9}.$$

- (b) If all the digits are even, there are four choices for the first digit (it cannot be zero) and five choices for each of the other four.

$$\text{Hence number of such numbers} = 4 \times 5 \times 5 \times 5 \times 5 = 2500,$$

$$\text{and } \mathcal{P}(\text{all digits are even}) = \frac{2500}{90\,000} = \frac{1}{36}.$$

- (c) This is counting without replacement:

1st digit	2nd digit	3rd digit	4th digit	5th digit
9	9	8	7	6

so number of such numbers = $9 \times 9 \times 8 \times 7 \times 6$

$$\text{and } \mathcal{P}(\text{digits are distinct}) = \frac{9 \times 9 \times 8 \times 7 \times 6}{90\,000} = \frac{189}{625}.$$

- (d) Every
- unordered*
- five-member subset of the set of nine nonzero digits can be arranged in exactly one way into a five-digit number with the digits in increasing order. (Note that the digit zero cannot be used, since a number can't begin with the digit zero.)

Hence number of such numbers

$$\begin{aligned} &= \text{number of unordered subsets of } \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ &= {}^9C_5 = 126, \end{aligned}$$

$$\text{so } \mathcal{P}(\text{digits are distinct and in increasing order}) = \frac{126}{90\,000} = \frac{7}{5000}.$$

WORKED EXERCISE: [A harder example] Continuing with the previous worked exercise, find the probability that:

- (a) the number contains at least one four,
- (b) the number contains at least one four and at least one five,
- (c) the number contains exactly three sevens,
- (d) the number contains at least three sevens.

SOLUTION:

- (a) This can be approached using the complementary event:

$$\text{number of five-digit numbers without a 4} = 8 \times 9 \times 9 \times 9 \times 9 = 52\,488,$$

$$\text{so number of five-digit numbers with a 4} = 90\,000 - 52\,488 = 37\,512.$$

$$\text{Hence } \mathcal{P}(\text{at least one 4}) = \frac{37\,512}{90\,000} = \frac{521}{1250}.$$

- (b) This can be approached using the addition theorem:

$$\text{number without a 4} = 52\,488,$$

$$\text{similarly number without a 5} = 52\,488,$$

$$\text{and number with no 5 and no 4} = 7 \times 8 \times 8 \times 8 \times 8 = 28\,672.$$

$$\text{Hence number with no 5 or no 4} = 52\,488 + 52\,488 - 28\,672 = 76\,304,$$

$$\text{and number with at least one 5 and at least one 4} = 90\,000 - 76\,304 = 13\,696.$$

$$\text{Hence } \mathcal{P}(\text{at least one 4 and at least one 5}) = \frac{13\,696}{90\,000} = \frac{856}{5625}.$$

- (c) Counting the number of five-digit numbers with exactly three 7s requires cases. First we count the five-digit strings with exactly three 7s, by first placing the three 7s and then choosing the first and second non-7 digits:

position of the three 7s	choose first non-7	choose second non-7
${}^5C_3 = 10$	9	9

giving 810 such strings. Secondly, we must subtract the number of five-digit strings with exactly three 7s and beginning with zero:

position of the three 7s ${}^4C_3 = 4$	choose the other non-7 9

giving 36 such strings. Hence there are $810 - 36 = 774$ such numbers, and

$$P(\text{number has exactly three 7s}) = \frac{774}{90\,000} = \frac{43}{5000}.$$

- (d) The number 77777 is the only five-digit number with five 7s.
Any five-digit number with exactly four 7s has one of the five forms

$$*7\,777, \quad 7*777, \quad 77*77, \quad 777*7, \quad 7777*,$$

where the * in $*7\,777$ is a nonzero digit. There are eight numbers of the first form, and nine of the other four forms, giving 44 numbers altogether.

Hence the number with at least three 7s = $774 + 44 + 1 = 819$,

$$\text{and} \quad P(\text{at least three 7s}) = \frac{819}{90\,000} = \frac{91}{10\,000}.$$

Exercise 10H

- A committee of three is to be selected from the nine members in a club.
 - How many different committees can be formed?
 - If there are five men in the club, what is the probability that the selected committee consists entirely of males?
- The integers from 1 to 10 inclusive are written on ten separate pieces of paper. Four pieces of paper are drawn at random. Find the probability that:
 - the four numbers drawn are 1, 2, 3 and 6,
 - the number 9 is one of the numbers drawn,
 - the number 8 is not drawn,
 - the number 7 is drawn but the number 1 is not.
- A bag contains three red, seven yellow and five blue balls. If three balls are drawn from the bag simultaneously, find the probability that:
 - all three balls are yellow,
 - all the balls are of the same colour,
 - there are two red balls and one blue ball,
 - all the balls are of different colours.
- A sports committee of five members is to be chosen from eight AFL footballers and seven soccer players. Find the probability that the committee will contain:
 - only AFL footballers,
 - only soccer players,
 - three soccer players and two AFL footballers,
 - at least one soccer player,
 - at most one soccer player,
 - Ian, a particular soccer player.
- From a standard pack of 52 cards, three are selected at random. Find the probability that:
 - they are the jack of spades, the two of clubs and the seven of diamonds,
 - all three are aces,
 - they are all diamonds,
 - they are all of the same suit,
 - they are all picture cards,
 - two are red and one is black,
 - one is a seven, one is an eight and one is a nine,
 - two are 7s and one is a 6,
 - exactly one is a diamond,
 - at least two of them are diamonds.

6. Repeat the previous question if the cards are selected from the pack one at a time, and each card is replaced before the next one is drawn.
7. Three boys and three girls are to sit in a row. Find the probability that:
 - (a) the boys and girls alternate,
 - (b) the boys and girls sit together,
 - (c) two specific girls sit next to one another.
8. A family of five are seated in a row at the cinema. Find the probability that:
 - (a) the parents sit on the end and the three children are in the middle,
 - (b) the parents sit next to one another.
9. Six people, of whom Patrick and Jessica are two, arrange themselves in a row. Find the probability that:
 - (a) Patrick and Jessica occupy the end positions,
 - (b) Patrick and Jessica are not next to each other.

DEVELOPMENT

10. The letters of PROMISE are arranged randomly in a row. Find the probability that:
 - (a) the word starts with R and ends with S,
 - (b) the letters P and R are next to one another,
 - (c) the letters P and R are separated by at least three letters,
 - (d) the vowels and the consonants alternate,
 - (e) the vowels are together.
11. The digits 3, 3, 4, 4, 4 and 5 are placed in a row to form a six-digit number. If one of these numbers is selected at random, find the probability that:
 - (a) it is even,
 - (b) it ends in 5,
 - (c) the 4s occur together,
 - (d) the number starts with 5 and then the 4s and 3s alternate,
 - (e) the 3s are separated by at least one other number.
12. The letters of the word PRINTER are arranged in a row. Find the probability that:
 - (a) the word starts with the letter E,
 - (b) the letters I and P are next to one another,
 - (c) there are three letters between N and T,
 - (d) there are at least three letters between N and T.
13. The letters of KETTLE are arranged randomly in a row. Find the probability that:
 - (a) the two letters E are together,
 - (b) the two letters E are not together,
 - (c) the two letters E are together and the two letters T are together,
 - (d) the Es and Ts are together in one group.
14. The letters of ENTERTAINMENT are arranged in a row. Find the probability that:
 - (a) the letters E are together,
 - (b) two Es are together and one is apart,
 - (c) all the letters E are apart,
 - (d) the word starts and ends with E.
15. A tank contains 20 tagged fish and 80 untagged fish. On each day, four fish are selected at random, and after noting whether they are tagged or untagged, they are returned to the tank. Answer the following questions, correct to three significant figures.
 - (a) What is the probability of selecting no tagged fish on a given day?
 - (b) What is the probability of selecting at least one tagged fish on a given day?
 - (c) Calculate the probability of selecting no tagged fish on every day for a week.
 - (d) What is the probability of selecting no tagged fish on exactly three of the seven days during the week?

16. A bag contains seven white and five black discs. Three discs are chosen from the bag. Find the probability that all three discs are black, if the discs are chosen:
- (a) without replacement,
 - (b) with replacement,
 - (c) so that after each draw the disc is replaced with one of the opposite colour.
17. Six people are to be divided into two groups, each with at least one person. Find the probability that:
- (a) there will be three in each group,
 - (b) there will be two in one group and four in the other,
 - (c) there will be one group of five and an individual.
18. A three-digit number is formed from the digits 3, 4, 5, 6 and 7 (no repetitions allowed). Find the probability that:
- (a) the number is 473,
 - (b) the number is odd,
 - (c) the number is divisible by 5,
 - (d) the number is divisible by 3,
 - (e) the number starts with 4 and ends with 7,
 - (f) the number contains the digit 3,
 - (g) the number contains the digits 3 and 5,
 - (h) the number contains the digits 3 or 5,
 - (i) all digits in the number are odd,
 - (j) the number is greater than 500.
19. The digits 1, 2, 3 and 4 are used to form numbers that may have 1, 2, 3 or 4 digits in them. If one of the numbers is selected at random, find the probability that:
- (a) it has three digits,
 - (b) it is even,
 - (c) it is greater than 200,
 - (d) it is odd and greater than 3000,
 - (e) it is divisible by 3.
20. (a) A senate committee of five members is to be selected from six Labor and five Liberal senators. What is the probability that Labor will have a majority on the committee?
- (b) The senate committee is to be selected from N Labor and five Liberal senators. Use trial and error to find the minimum value of N , given that the probability of Labor having a majority on the committee is greater than $\frac{3}{4}$.
21. Four basketball teams A, B, C and D each consist of ten players, and in each team, the players are numbered 1, 2, ..., 9, 10. Five players are to be selected at random from the four teams. Find the probability that of the five players selected:
- (a) three are numbered 4 and two are numbered 9,
 - (b) at least four are from the same team.
22. A poker hand of five cards is dealt from a standard pack of 52. Find the probability of obtaining:
- (a) one pair,
 - (b) two pairs,
 - (c) three of a kind,
 - (d) four of a kind,
 - (e) a full house (one pair and three of a kind),
 - (f) a straight (five cards in sequence regardless of suit),
 - (g) a flush, (five cards of the same suit),
 - (h) a royal flush (ten, jack, queen, king and ace in a single suit).
23. Four adults are standing in a room that has five exits. Each adult is equally likely to leave the room through any one of the five exits.
- (a) What is the probability that all four adults leave the room via the same exit?
 - (b) What is the probability that three particular adults use the same exit and the fourth adult uses a different exit?

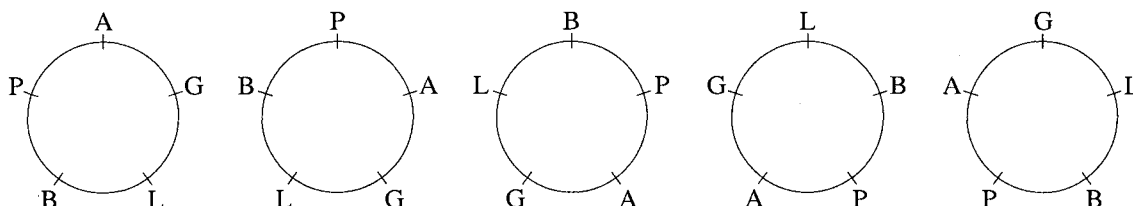
- (c) What is the probability that any three of the four adults come out the same exit, and the remaining adult comes out a different exit?
- (d) What is the probability that no more than two adults come out the same exit?
24. (a) Five diners in a restaurant choose randomly from a menu featuring five main courses. Find the probability that exactly one of the main courses is not chosen by any of the diners.
- (b) Repeat the question if there are n diners and a choice of n main courses.
25. [The birthday problem]
- (a) Assuming a 365-day year, find the probability that in a group of three people there will be at least one birthday in common. Answer correct to two significant figures.
- (b) If there are n people in the group, find an expression for the probability of at least one common birthday.
- (c) By choosing a number of values of n , plot a graph of the probability of at least one common birthday against n for $n \leq 50$.
- (d) How many people need to be in the group before the probability exceeds 0.5?
- (e) How many people need to be in the group before the probability exceeds 90%?

EXTENSION

26. During the seven games of the football season, Max and Bert must each miss three consecutive games. The games to be missed by each player are randomly and independently selected.
- (a) What is the probability that they both have the first game off together?
- (b) What is the probability that the second game is the first one missed by both players?
- (c) What is the probability that Max and Bert miss at least one of the same games?
27. Eight players make the quarter-finals at Wimbledon. The winner of each of the quarter-finals plays a semi-final to see who enters the final.
- (a) Assuming that all eight players are equally likely to win a match, show that the probability that any two particular players will play each other is $\frac{1}{4}$.
- (b) What is the probability that two particular people will play each other if the tournament starts with 16 players?
- (c) What is the probability that two particular players will meet in a similar knockout tournament if 2^n players enter?

10I Arrangements in a Circle

Arrangements in a circle or around a round table are complicated by the fact that two arrangements are regarded as equivalent if one can be rotated to produce the other. For example, all the five round-table seatings below of King Arthur, Queen Guinevere, Sir Lancelot, Sir Bors and Sir Percival are to be regarded as the same:

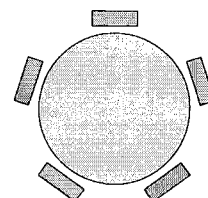


The Basic Algorithm: The most straightforward way of counting arrangements in a circle is to seat the people in order, dealing with the restrictions first as always, but reckoning that there is essentially only one way to seat the first person who sits down, because until that time, all the seats are identical.

- 19** **COUNTING ARRANGEMENTS IN A CIRCLE:** There is essentially only one way to seat the first person, because until then, all the seats are equivalent.

WORKED EXERCISE: Arthur, Guinevere, Lancelot, Bors and Percival sit around a round table. Find how many ways can this be done:

- without restriction,
- if Guinevere sits at Arthur's right hand,
- if Guinevere sits between Lancelot and Bors.
- if Arthur and Lancelot do not sit together.



SOLUTION:

(a)	Seat Arthur 1	Seat Guinevere 4	Seat Lancelot 3	Seat Bors 2	Seat Percival 1

Number of ways = 24.

(b)	Seat Arthur 1	Seat Guinevere 1	Seat Lancelot 3	Seat Bors 2	Seat Percival 1

Number of ways = 6.

(c)	Seat Guinevere 1	Seat Lancelot 2	Seat Bors 1	Seat Arthur 2	Seat Percival 1

Number of ways = 4.

(d)	Seat Arthur 1	Seat Lancelot 2	Seat Guinevere 3	Seat Bors 2	Seat Percival 1

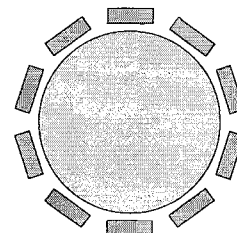
Number of ways = 12.

Arranging Groups Around a Circle: When arranging groups around a circle, the principle is the same as the principle for compound orderings established in Section 10E.

- 20** **ARRANGING GROUPS AROUND A CIRCLE:** First choose an order for each group. Then arrange the groups around the circle, reckoning that there is essentially only one way to place the first group.

WORKED EXERCISE: Five boys and five girls are to sit around a table. Find how many ways can this be done:

- without restriction,
- if the boys and girls alternate,
- if there are five couples, all of whom sit together,
- if the boys sit together and the girls sit together,
- if four couples sit together, but Walter and Maude do not.



SOLUTION:

(a)

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	9	8	7	6	5	4	3	2	1

Number of ways = $9! = 362\,880$.

(b)

1st boy	2nd boy	3rd boy	4th boy	5th boy	1st girl	2nd girl	3rd girl	4th girl	5th girl
1	4	3	2	1	5	4	3	2	1

Number of ways = $5! \times 4! = 2880$.

- (c) Each couple can be ordered in 2 ways, giving 2^5 orderings of the five couples. Then seat the five couples around the table:

1st couple	2nd couple	3rd couple	4th couple	5th couple
1	4	3	2	1

Number of ways = $2^5 \times 4! = 768$.

- (d) The boys can be ordered in $5!$ ways, and the girls in $5!$ ways also. Then seat the two groups around the table:

group of boys	group of girls
1	1

Number of ways = $5! \times 5! \times 1 = 14\,400$.

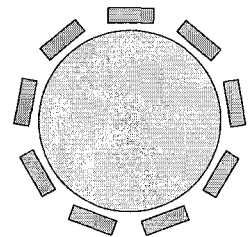
- (e) Order each of the four couples in 2 ways, giving 16 orderings of the couples. There are now four couples and two individuals to seat around the table, with the restriction that Maude does not sit next to Walter:

Walter	Maude	1st couple	2nd couple	3rd couple	4th couple
1	3	4	3	2	1

Number of ways = $2^4 \times 3 \times 4! = 1152$.

Probability in Arrangements Around a Circle: As always, counting allows probability problems to be solved by counting the sample space and the event space.

WORKED EXERCISE: Three Tasmanians, three New Zealanders and three people from NSW are seated at random around a round table. What is the probability that the three groups are seated together?



SOLUTION: Using the same boxes as before, there are $1 \times 8!$ possible orderings.

To find the number of favourable orderings,

first order each group in $3! = 6$ ways,

then order the three groups around the table in $1 \times 2 \times 1 = 2$ ways,

so the total number of favourable orderings is $6 \times 6 \times 6 \times 2$.

$$\begin{aligned} \text{Hence } \mathcal{P}(\text{groups are together}) &= \frac{6 \times 6 \times 6 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{3}{280}. \end{aligned}$$

Exercise 10!

1. (a) Find how many ways five people can be arranged in: (i) a line, (ii) a circle.
(b) Find how many ways ten people can be arranged in: (i) a line, (ii) a circle.
2. Eight people are arranged in: (a) a straight line, (b) a circle.
In how many ways can they be arranged so that two particular people sit together?
3. Bob, Betty, Ben, Brad and Belinda are to be seated at a round table. Find how many ways this can be done:
(a) if there are no restrictions, (d) if Brad is to sit between Bob and Ben,
(b) if Bob is to sit in his favourite chair, (e) if Belinda and Betty are not to sit next to one another.
(c) if Betty sits on Bob's right-hand side,
4. Four boys and four girls are arranged in a circle. Find how many ways this can be done:
(a) if there are no restrictions, (b) if the boys and the girls alternate,
(c) if the boys and girls are in distinct groups,
(d) if a particular boy and girl wish to sit next to one another,
(e) if two particular boys do not wish to sit next to one another,
(f) if one particular boy wants to sit between two particular girls.

DEVELOPMENT

5. The letters A, E, I, P, Q and R are arranged in a circle. Find the probability that:
(a) the vowels are together, (c) the vowels and consonants alternate,
(b) A is opposite R, (d) at least two vowels are next to one another.
6. Find how many ways the integers 1, 2, 3, 4, 5, 6, 7, 8 can be placed in a circle if:
(a) there are no restrictions, (d) at least three odd numbers are together,
(b) all the even numbers are together, (e) the two numbers 1 and 7 are next to one another.
(c) the odd and even numbers alternate,
7. A committee of three women and seven men is to be seated randomly at a round table.
(a) What is the probability that the three females will sit together?
(b) The committee elects a president and a vice-president. What is the probability that they are sitting opposite one another?
8. Find how many arrangements of n people around a circle are possible if:
(a) there are no restrictions, (c) two particular people sit apart,
(b) two particular people must sit together, (d) three particular people sit together.
9. Twelve marbles are to be placed in a circle. Find how many ways can this be done if:
(a) all the marbles are of different colours,
(b) there are eight red, three blue and one green marble.
10. There are two distinct round tables, each with five seats. In how many ways may a group of ten people be seated?
11. A sports committee consisting of four rowers, three basketballers and two cricketers sits at a circular table.
(a) How many different arrangements of the committee are possible if the rowers and basketballers both sit together in groups, but no rower sits next to a basketballer?
(b) One rower and one cricketer are related. If the conditions in (a) apply, what is the probability that these two members of the committee will sit next to one another?

EXTENSION

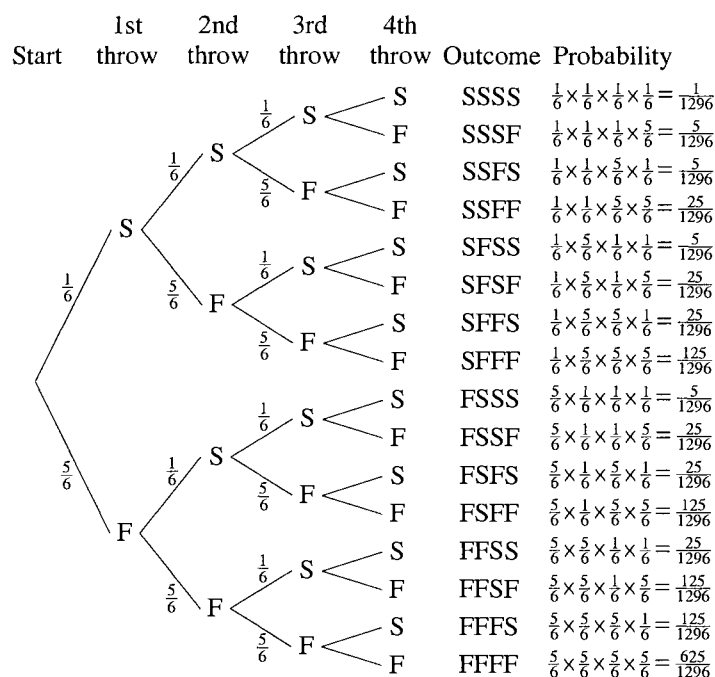
12. A group of n men and $n + 1$ women sit around a circular table. How many arrangements are possible if no two men are to sit next to one another?
13. (a) Consider a necklace of six differently coloured beads. Because the necklace can be turned over, clockwise and anticlockwise arrangements of the beads do not yield different orders. Hence find how many different arrangements there are of the six beads on the necklace.
- (b) In how many ways can ten different keys be placed on a key ring?
- (c) In how many ways can one yellow, two red and four green beads be placed on a bracelet if the beads are identical apart from colour? [HINT: This will require a listing of patterns to see if they are identical when turned over.]

10J Binomial Probability

This final section combines probability with the theory of the expansion of the binomial $(x + y)^n$, and provides a beautiful example of the relationship between probability and the theory of polynomials. We shall be concerned with multi-stage experiments in which:

- all the stages are identical, and
- each stage has only two possible outcomes, conventionally called ‘success’ and ‘failure’, and not necessarily equally likely.

Example — Repeatedly Attempting to Throw a Six on a Die: If a die is thrown four times, what is the probability of getting exactly two sixes? Let S (success) be the outcome ‘throwing a six’ and F (failure) be the outcome ‘not throwing a six’. Here is the probability tree diagram showing the sixteen possible outcomes, and their respective probabilities, when a die is thrown four times.



The outcome 'two sixes' can be obtained in ${}^4C_2 = 6$ different ways:

SSFF, SFSF, SFSS, FSSF, FSFS, FFSS,

because there are $\frac{4!}{2! \times 2!} = {}^4C_2$ ways of arranging two Ss and two Fs.

Each of these six outcomes has the same probability $(\frac{1}{6})^2 \times (\frac{5}{6})^2$, so

$$\mathcal{P}(\text{two sixes}) = {}^4C_2 \times (\frac{1}{6})^2 \times (\frac{5}{6})^2.$$

Similar arguments apply to the probabilities of getting 0, 1, 2, 3 and 4 sixes:

Result	Probability	Approximation
4 sixes	${}^4C_4 \times (\frac{1}{6})^4 \times (\frac{5}{6})^0$	0.000 77
3 sixes	${}^4C_3 \times (\frac{1}{6})^3 \times (\frac{5}{6})^1$	0.015 43
2 sixes	${}^4C_2 \times (\frac{1}{6})^2 \times (\frac{5}{6})^2$	0.115 74
1 six	${}^4C_1 \times (\frac{1}{6})^1 \times (\frac{5}{6})^3$	0.385 80
0 sixes	${}^4C_0 \times (\frac{1}{6})^0 \times (\frac{5}{6})^4$	0.482 25

The five probabilities of course add up to 1. Moreover, the five probabilities are the successive terms in the binomial expansion of

$$\begin{aligned} (\frac{1}{6} + \frac{5}{6})^4 &= {}^4C_0 \times (\frac{1}{6})^4 \times (\frac{5}{6})^0 + {}^4C_1 \times (\frac{1}{6})^3 \times (\frac{5}{6})^1 + {}^4C_2 \times (\frac{1}{6})^2 \times (\frac{5}{6})^2 \\ &\quad + {}^4C_3 \times (\frac{1}{6})^1 \times (\frac{5}{6})^3 + {}^4C_4 \times (\frac{1}{6})^0 \times (\frac{5}{6})^4. \end{aligned}$$

Binomial Probability — The General Case: Suppose that a multi-stage experiment consists of n identical stages, and at each stage the probability of 'success' is p and of 'failure' is q , where $p + q = 1$. Then

$$\mathcal{P}(k \text{ successes and } n - k \text{ failures in that order}) = p^k q^{n-k}.$$

But there are ${}^nC_k$ ways of ordering k successes and $(n - k)$ failures, so

$$\mathcal{P}(k \text{ successes and } n - k \text{ failures in any order}) = {}^nC_k p^k q^{n-k},$$

which is the term in $p^k q^{n-k}$ in the expansion of the binomial $(p + q)^n$.

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BINOMIAL PROBABILITY: Suppose that the probabilities of 'success' and 'failure' in any stage of an n -stage experiment are p and q respectively. Then

$$\mathcal{P}(k \text{ successes and } n - k \text{ failures in any order}) = {}^nC_k p^k q^{n-k}.$$

This is the term in $p^k q^{n-k}$ in the expansion of $(p + q)^n$.

WORKED EXERCISE: [This example shows also how to use complementary events and cases to answer questions.] Six cards are drawn at random from a pack of 52 playing cards, each being replaced before the next is drawn. Find, as fractions with denominator 4^6 , the probability that:

- (a) two are clubs, (c) at least one is a club,
- (b) one is a club, (d) at least four are clubs.

SOLUTION: There are 13 clubs in the pack, so at each stage the probability of drawing a club is $\frac{1}{4}$. Applying the formula with $p = \frac{1}{4}$ and $q = \frac{3}{4}$:

$$(a) \mathcal{P}(\text{two are clubs}) = {}^6C_2 \times (\frac{1}{4})^2 \times (\frac{3}{4})^4 = \frac{15 \times 3^4}{4^6} = \frac{1215}{4^6},$$

$$(b) \mathcal{P}(\text{one is a club}) = {}^6C_1 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{3}{4}\right)^5 = \frac{6 \times 3^5}{4^6} = \frac{1458}{4^6},$$

$$\begin{aligned}(c) \mathcal{P}(\text{at least one is a club}) &= 1 - \mathcal{P}(\text{all are non-clubs}) \\ &= 1 - \left(\frac{3}{4}\right)^6 \quad (\text{or } 1 - {}^6C_0 \times \left(\frac{1}{4}\right)^0 \times \left(\frac{3}{4}\right)^6) \\ &= \frac{3367}{4^6},\end{aligned}$$

$$\begin{aligned}(d) \mathcal{P}(\text{at least four are clubs}) &= \mathcal{P}(\text{four are clubs}) + \mathcal{P}(\text{five are clubs}) + \mathcal{P}(\text{six are clubs}) \\ &= {}^6C_4 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{3}{4}\right)^2 + {}^6C_5 \times \left(\frac{1}{4}\right)^5 \times \left(\frac{3}{4}\right)^1 + {}^6C_6 \times \left(\frac{1}{4}\right)^6 \times \left(\frac{3}{4}\right)^0 \\ &= \frac{15 \times 3^2 + 6 \times 3 + 1}{4^6} \\ &= \frac{154}{4^6}.\end{aligned}$$

An Example where $p = q = \frac{1}{2}$: A particular case of binomial probability is when the probabilities p and q of ‘success’ and ‘failure’ are both $\frac{1}{2}$.

WORKED EXERCISE: If a coin is tossed 100 times, what is the probability that it comes up heads exactly 50 times (correct to four significant figures)?

SOLUTION: Taking $p = q = \frac{1}{2}$,

$$\mathcal{P}(50 \text{ heads}) = {}^{100}C_{50} \times \left(\frac{1}{2}\right)^{50} \times \left(\frac{1}{2}\right)^{50} \doteq 0.0796.$$

NOTE: This is a fairly low probability. We might, without careful thought, have expected a higher probability than this, but of course any result from about 45 to 55 heads would be unlikely to surprise us. It should now be clear that in general

$$\mathcal{P}(k \text{ heads in } n \text{ tosses of a coin}) = {}^nC_k \times \left(\frac{1}{2}\right)^n.$$

Experimental Probabilities and Binomial Probability: Some of the most straightforward and important applications of binomial theory arise in situations where the probabilities of ‘success’ and ‘failure’ are determined experimentally.

WORKED EXERCISE: A light bulb is classed as ‘defective’ if it burns out in under 1000 hours. A company making light bulbs finds, after careful testing, that 1% of its bulbs are defective. If it packs its bulbs in boxes of 50, find, correct to three significant figures:

- (a) the probability that a box will contain no defective bulbs,
- (b) the probability that at least two bulbs in a box are defective.

SOLUTION: In this case, $p = 0.01$ and $q = 0.99$. Let X be the number of defective bulbs in the box. Then:

$$(a) \mathcal{P}(X = 0) = 0.99^{50} \doteq 0.605,$$

$$\begin{aligned}(b) \mathcal{P}(X \geq 2) &= 1 - (\mathcal{P}(X = 0) + \mathcal{P}(X = 1)) \\ &= 1 - (0.99^{50} + {}^{50}C_1 \times 0.01^1 \times 0.99^{49}) \\ &\doteq 0.089.\end{aligned}$$

Binomial Probability and the Maximum Term in a Binomial Expansion: The earlier theory in Chapter Seven on the maximum term in a binomial expansion has an important application in binomial probability, because it allows us to find the event with the greatest probability.

WORKED EXERCISE: A die is thrown 100 times.

- (a) What is the most likely number of sixes that will be thrown?
 (b) What is the probability that that particular number of sixes is thrown (answer correct to four significant figures)?

SOLUTION: Here $p = \frac{1}{6}$ and $q = \frac{5}{6}$.

$$\begin{aligned}
 \text{(a) Let } P_k &= \mathcal{P}(k \text{ sixes}). \\
 \text{Then } P_k &= {}^{100}C_k p^k q^{100-k} \\
 \text{and } P_{k+1} &= {}^{100}C_{k+1} p^{k+1} q^{99-k}, \\
 \text{so } \frac{P_{k+1}}{P_k} &= \frac{{}^{100}C_{k+1} p^{k+1} q^{99-k}}{{}^{100}C_k p^k q^{100-k}} \times \frac{k! \times (100-k)!}{100! \times p^k q^{100-k}} \\
 &= \frac{(100-k)p}{(k+1)q} \\
 &= \frac{100-k}{5k+5}, \text{ substituting } p = \frac{1}{6} \text{ and } q = \frac{5}{6}.
 \end{aligned}$$

To find the greatest value of P_k , we proceed as in Section 5E,

$$\text{solving } P_{k+1} > P_k, \text{ that is, } \frac{P_{k+1}}{P_k} > 1 \quad (1)$$

$$\begin{aligned}
 \frac{100-k}{5k+5} &> 1 \\
 5k+5 &< 100-k
 \end{aligned}$$

$$6k < 95$$

$$k < 15\frac{5}{6},$$

so the inequation (1) is true for $k = 0, 1, 2, \dots, 15$ and false otherwise.

Hence $P_0 < P_1 < \dots < P_{15} < P_{16} > P_{17} \dots > P_{100}$.

Thus the most likely number of sixes is 16.

$$\begin{aligned}
 \text{(b) Also, } P_{16} &= {}^{100}C_{16} \times \left(\frac{1}{6}\right)^{16} \times \left(\frac{5}{6}\right)^{84} \\
 &\doteq 0.1065.
 \end{aligned}$$

An Example where Each Stage is a Compound Event: Sometimes, when each stage of the experiment is itself a compound event, it may take some work to find the probability of success at each stage.

WORKED EXERCISE: Joe King and his sister Fay make shirts for a living. Joe works more slowly but more accurately, making 20 shirts a day, of which 2% are defective. Fay works faster, making 30 shirts a day, of which 4% are defective. If they send out their shirts in randomly mixed parcels of 30 shirts, what is the probability (to three significant figures) that no more than two shirts in a box are defective?

SOLUTION: If a shirt is chosen at random from one parcel, then using the product rule and the addition rule, the probability p that the shirt is defective is

$$\begin{aligned} p &= \mathcal{P}(\text{Joe made it, and it is defective}) \\ &\quad + \mathcal{P}(\text{Fay made it, and it is defective}) \\ &= \frac{20}{50} \times \frac{2}{100} + \frac{30}{50} \times \frac{4}{100} \\ &= \frac{4}{125}, \end{aligned}$$

so $p = \frac{4}{125}$ and $q = \frac{121}{125}$.

Let X be the number of defective shirts.

$$\begin{aligned} \text{Then } \mathcal{P}(X \leq 2) &= \mathcal{P}(X = 0) + \mathcal{P}(X = 1) + \mathcal{P}(X = 2) \\ &= \left(\frac{121}{125}\right)^{30} + {}^{30}C_1 \times \left(\frac{121}{125}\right)^{29} \times \frac{4}{125} + {}^{30}C_2 \times \left(\frac{121}{125}\right)^{28} \times \left(\frac{4}{125}\right)^2 \\ &\doteq 0.930. \end{aligned}$$

Exercise 10J

NOTE: Unless otherwise specified, leave your answers in unsimplified form.

- Assume that the probability that a child is female is $\frac{1}{2}$, and that sex is independent from child to child. Giving your answers as fractions in simplest form, find the probability that in a family of five children:
 - all are boys,
 - there are two girls and three boys,
 - there are four boys and one girl,
 - at least one will be a girl.
- In a one-day cricket game, a batsman has a chance of $\frac{1}{5}$ of hitting a boundary every time he faces a ball. If he faces all six balls in an over, what is the probability that he will hit exactly two boundaries, assuming that successive strikes are independent?
- During the wet season, it rains on average every two days out of three. Let W denote the number of wet days in a week. Assuming that the types of weather on successive days are independent events, find:
 - $\mathcal{P}(W = 3)$
 - $\mathcal{P}(\overline{W} = 2)$
 - $\mathcal{P}(W = 0)$
 - $\mathcal{P}(W \geq 1)$
- A marksman finds that on average he hits the target five times out of six. Assuming that successive shots are independent events, find the probability that in four shots:
 - he has exactly three hits,
 - he has exactly two misses.
- A jury roll contains 200 names, 70 of females and 130 of males. If twelve jurors are randomly selected, what is the probability of ending up with an all-male jury?
- A die is rolled twelve times. Find the probability that 5 appears on the uppermost face:
 - exactly three times,
 - exactly eight times,
 - ten or more times (that is ten, eleven or twelve times).
- A die is rolled six times. Let N denote the number of times that the number 3 is shown on the uppermost face. Find, correct to four decimal places:
 - $\mathcal{P}(N = 2)$
 - $\mathcal{P}(N < 2)$
 - $\mathcal{P}(N \geq 2)$
- Five out of six people surveyed think that Tasmania is the most beautiful state in Australia. What is the probability that in a group of 15 randomly selected people, at least 13 of them think that Tasmania is the most beautiful state in Australia?

9. An archer finds that on average he hits the bulls-eye nine times out of ten. Assuming that successive attempts are independent, find the probability that in twenty attempts:
- (a) he scores at least eighteen hits, (b) he misses at least once.
10. A torch manufacturer finds that on average 9% of the bulbs are defective. What is the probability that in a randomly selected batch of ten one-bulb torches:
- (a) there will be no more than two with defective bulbs,
(b) there will be at least two with defective bulbs.

————— DEVELOPMENT —————

11. A poll indicates that 55% of people support Labor party policy. If five people are selected at random, what is the probability that a majority of them will support Labor party policy? Give your answer correct to three decimal places.
12. In a multiple-choice test, there are ten questions, and each question has five possible answers, only one of which is correct. What is the probability of answering exactly seven questions correctly by chance alone? Give your answer correct to six significant figures.
13. The probability that a small earthquake occurs somewhere in the world on any one day is 0.95. Assuming that earthquake frequencies on successive days are independent, what is the probability that a small earthquake occurs on exactly 28 of January's 31 days? Leave your answer in index form.
14. The probability that a jackpot prize will be won in a given lottery is 0.012.
- (a) Find, correct to five decimal places, the probability that the jackpot prize will be won:
- (i) exactly once in ten independent lottery draws,
(ii) at least once in ten independent lottery draws.
- (b) The jackpot prize is initially \$10 000 and increases by \$10 000 each time the prize is not won. Find, correct to five decimal places, the probability that the jackpot prize will exceed \$200 000 when it is finally won.
15. (a) How many times must a die be rolled so that the probability of rolling at least one six is greater than 95%?
(b) How many times must a coin be tossed so that the probability of tossing at least one tail is greater than 99%?
16. Five families have three children each.
- (a) Find, correct to three decimal places, the probability that:
- (i) at least one of these families has three boys,
(ii) each family has more boys than girls.
- (b) What assumptions have been made in arriving at your answer?
17. Comment on the validity of the following arguments:
- (a) 'In the McLaughlin Library, 10% of the books are mathematics books. Hence if I go to a shelf and choose five books from that shelf, then the probability that all five books are mathematics books is 10^{-5} .'
- (b) During an election, 45% of voters voted for party A. Hence if I select a street at random, and then select a voter from each of four houses in the street, the probability that exactly two of those voters voted for party A is ${}^4C_2 \times (0.45)^2 \times (0.55)^2$.'

18. During winter it rains on average 18 out of 30 days. Five winter days are selected at random. Find, correct to four decimal places, the probability that:
- (a) the first two days chosen will be fine and the remainder wet,
 - (b) more rainy days than fine days have been chosen.
19. A tennis player finds that on average he gets his serve in eight out of every ten attempts and serves an ace once every fifteen serves. He serves four times. Assuming that successive serves are independent events, find, correct to six decimal places, the probability that:
- (a) all four serves are in,
 - (b) he hits at least three aces,
 - (c) he hits exactly three aces and the other serve lands in.
20. A man is restoring ten old cars, six of them manufactured in 1955 and four of them manufactured in 1962. When he tries to start them, on average the 1955 models will start 65% of the time and the 1962 models will start 80% of the time. Find, correct to four decimal places, the probability that at any time:
- (a) exactly three of the 1955 models and one of the 1962 models will start,
 - (b) exactly four of the cars will start. [HINT: You will need to consider five cases.]
21. An apple exporter deals in two types of apples, Red Delicious and Golden Delicious. The ratio of Red Delicious to Golden Delicious is 4 : 1. The apples are randomly mixed together before they are boxed. One in every fifty Golden Delicious and one in every one hundred Red Delicious apples will need to be discarded because they are undersized.
- (a) What is the probability that an apple selected from a box will need to be discarded?
 - (b) If ten apples are randomly selected from a box, find the probability that:
 - (i) all the apples will have to be discarded,
 - (ii) half of the apples will have to be discarded,
 - (iii) less than two apples will be discarded.
22. One bag contains three red and five white balls, and another bag contains four red and four white balls. One bag is chosen at random, a ball is selected from that bag, its colour is noted and then it is replaced.
- (a) Find the probability that the ball chosen is red.
 - (b) If the operation is carried out eight times, find the probability that:
 - (i) exactly three red balls are drawn,
 - (ii) at least three red balls are drawn.
23. (a) If six dice are rolled one hundred times, how many times would you expect the number of even numbers showing to exceed the number of odd numbers showing?
- (b) If eight coins are tossed sixty times, how many times would you expect the number of heads to exceed the number of tails?
24. [Probability and the greatest term in a binomial series] In each part, you will need to find the greatest term in the expansion of $(p + q)^n$, where p and q are the respective probabilities of 'success' and 'failure'. Give each probability unsimplified, and then correct to four significant figures.
- (a) A die is rolled 200 times. What is the most likely number of twos that will be thrown and what is the probability that that particular number of twos is thrown?
 - (b) A coin is tossed 41 times. What is the most likely number of heads that will be tossed, and what is the probability that that particular number of heads is tossed?
 - (c) From a 52-card pack, a card is drawn 35 times, and is replaced after each draw. What is the most likely number of aces that will be drawn, and what is the probability that that particular number of aces is drawn?

- (d) Repeat part (iii) for the number of spades that will be drawn.
25. If a fair coin is tossed $2n$ times, the probability of observing k heads and $2n - k$ tails is given by $P_k = \binom{2n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{2n-k}$.
- (a) Show that the most likely outcome is $k = n$.
- (b) Show that $P_n = \frac{(2n)!}{2^{2n}(n!)^2}$.

EXTENSION

26. A game is played using a barrel containing twenty similar balls numbered 1 to 20. The game consists of drawing four balls, without replacement, from the twenty balls in the barrel. The probability that any particular number is drawn in any game is 0.2.
- (a) Find, correct to four decimal places, the probability that the number 19 is drawn in exactly two of the next five games played.
- (b) Find, correct to five decimal places, the probability that the number 19 is drawn in at least two of the next five games played.
- (c) Let n be an integer where $4 \leq n \leq 20$.
- (i) What is the probability that, in any one game, all four selected numbers are less than or equal to n ?
- (ii) Show that the probability that, in any one game, n is the largest of the four numbers drawn is $\frac{{}^{n-1}C_3}{{}^{20}C_4}$.
27. (a) Expand $(a + b + c)^3$.
- (b) In a survey of football supporters, 65% supported Hawthorn, 24% followed Collingwood and 11% followed Sydney. Use the expansion in (a) to find, correct to five decimal places, the probability that if three people are randomly selected:
- (i) one supports Hawthorn, one supports Collingwood and one supports Sydney,
- (ii) exactly two of them support Collingwood,
- (iii) at least two of them support the same team.

Answers to Exercises

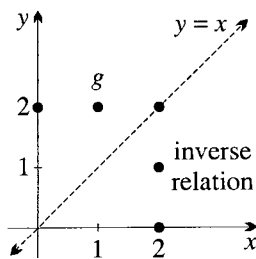
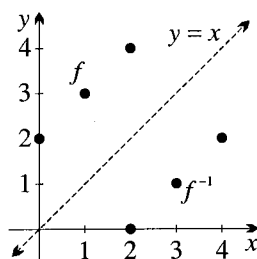
Chapter One

Exercise 1A (Page 5)

1(a) The inverse of f is $\{(2, 0), (3, 1), (4, 2)\}$.

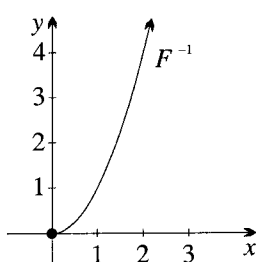
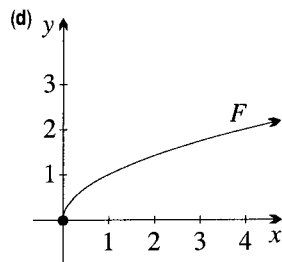
The inverse of g is $\{(2, 0), (2, 1), (2, 2)\}$.

(c) For f it is, for g it isn't.



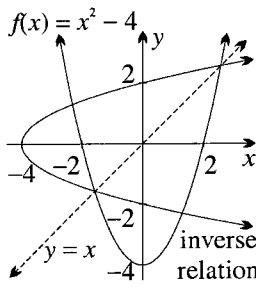
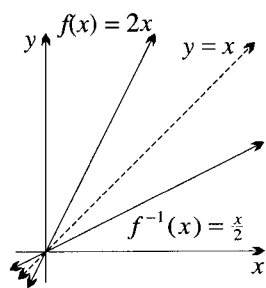
2(a) $3 \leq y \leq 5$ (b) domain: $3 \leq x \leq 5$, range: $0 \leq y \leq 2$ (c) $f^{-1}(x) = x - 3$

3(a) $0 \leq y \leq 2$ (b) domain: $0 \leq x \leq 2$, range: $0 \leq y \leq 4$ (c) $F^{-1}(x) = x^2$

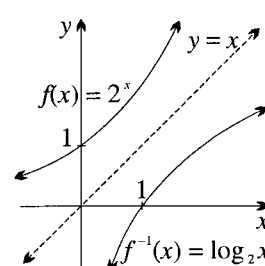
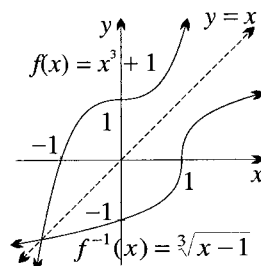


4(a) $f^{-1}(x) = \frac{1}{2}x$, both increasing.

(b) $f^{-1}(x) = \sqrt[3]{x-1}$, both increasing.

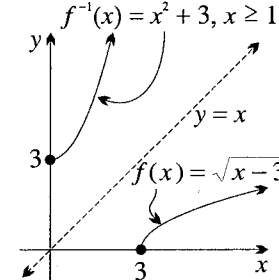
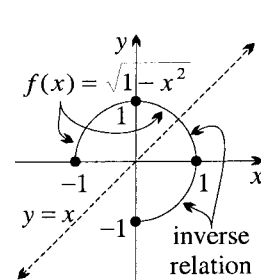


(c) The inverse is not a function, f is neither increasing nor decreasing. (d) The inverse is not a function, f is neither increasing nor decreasing.



(e) $f^{-1}(x) = \log_2 x$, both increasing.

(f) $f^{-1}(x) = x^2 + 3, x \geq 0$, both increasing.



5(a) Both x . (b) They are inverse functions.

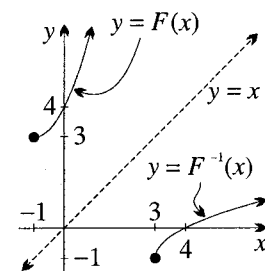
6(a) $g^{-1}(x) = \sqrt{x}$, domain: $x \geq 0$, range: $y \geq 0$

(b) $g^{-1}(x) = -\sqrt{x-2}$, domain: $x \geq 2$, range: $y \leq 0$ (c) $g^{-1}(x) = \sqrt{4-x^2}$, $-2 \leq x \leq 0$, domain: $-2 \leq x \leq 0$, range: $0 \leq y \leq 2$

7(a) $3x^2$ (b) $\frac{1}{3}(y+1)^{-\frac{2}{3}}$

8 $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \frac{dx}{dy} = 2y$

9(b) $F^{-1}(x) = -1 + \sqrt{x-3}$, domain: $x \geq 3$, range: $y \geq -1$



10(a) $x = e$

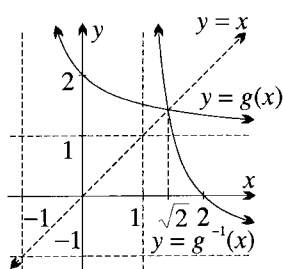
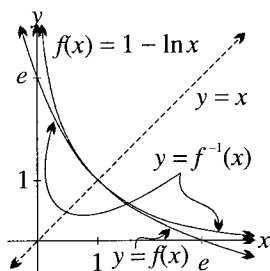
(b) Reflect $y = \ln x$ in the x -axis, then shift it one unit up.

(d) $f^{-1}(x) = e^{1-x}$, domain: all real x , range: $y > 0$

(e) Both are decreasing.

11(b) $g^{-1}(x) = \frac{2-x}{x-1}$, for $x > 1$, decreasing (c) $x = \sqrt{2}$.

It works because the graphs meet on the line of symmetry $y = x$.



12(a) $y = \sqrt[3]{-x}$ (b) $(-1, 1)$, $(0, 0)$ and $(1, -1)$

13(a) Shift two units to the left and four units down.

(b) x -intercepts: $-4, 0$, vertex: $(-2, -4)$.

(c) $x \geq -2$ (d) $x \geq -4$, increasing (e) $g^{-1}(x) = -2 + \sqrt{x+4}$

(f) $g(x)$ is neither, $g^{-1}(x)$ is increasing.

14(b) x -intercepts: $0, \sqrt{3}, -\sqrt{3}$, stationary points: $(-1, 2)$, $(1, -2)$, neither increasing nor decreasing (c) $-1 \leq x \leq 1$ (d) $-2 \leq x \leq 2$

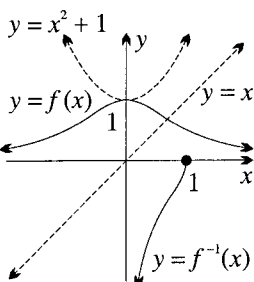
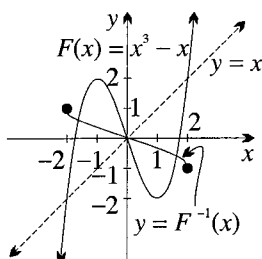
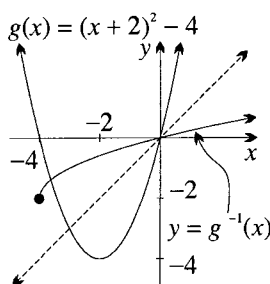
15(a) all real x (c) $f'(x) > 0$ for all real x . (d) For each value of y , there is only one value of x . That is, the graph of $f(x)$ passes the horizontal line test. $f^{-1}(x) = \log\left(\frac{x}{1-x}\right)$

16(a) neither (b) $x \leq 0$

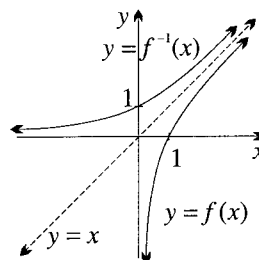
(c) $0 < x \leq 1$

(d) $f^{-1}(x) = -\sqrt{\frac{1-x}{x}}$

(e) increasing



17(b) No. The graph of the inverse is a vertical line, which is not a function.

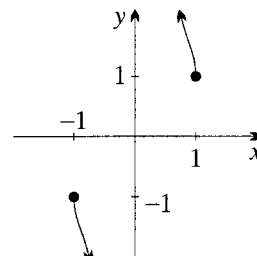
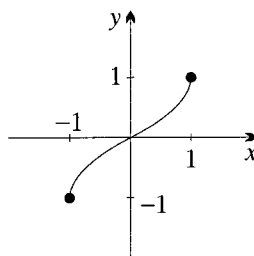


19(b) From part (a) we see, for example, that $g(\frac{1}{2}) = g(2)$, so the inverse is not a function.

(c)(i) $-1 \leq x \leq 1$ (iii) $g^{-1}(x) = \frac{1-\sqrt{1-x^2}}{x}$

(d) domain: $x \leq -1$ or $x \geq 1$, $g^{-1}(x) = \frac{1+\sqrt{1-x^2}}{x}$

(e) Because of the result in part (a).



20(a) vertex: $(2, \frac{10}{3})$, y -intercept: 4

(b) $x \geq 2$ (c) $x \geq \frac{10}{3}$

(d) The easy way is to solve $y = f(x)$ simultaneously with $y = x$. They intersect at $(4, 4)$ and $(6, 6)$. (e) $4 - N$

21(b) functions whose domain is $x = 0$ alone

22(a) all real x (b) 0

(d) $\frac{1}{2}(e^x + e^{-x})$, which is positive for all real x .

(e) $y = \frac{1}{2}e^x$ (f) $\sinh x$ is a one-to-one function.

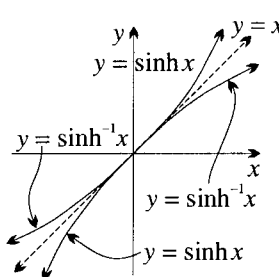
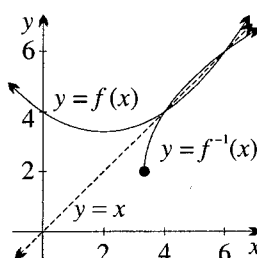
(i) $\frac{1}{\sqrt{1+x^2}}, \sinh^{-1} x + C$

24(b)(ii) $\frac{1}{6}$

Exercise 1B (Page 12)

1(a) 1.16 (b) 0.64 (c) 1.32 (d) 1.67 (e) 1.98 (f) 2.42

2(a) 0 (b) $\frac{\pi}{6}$ (c) 0 (d) $\frac{\pi}{4}$ (e) $-\frac{\pi}{2}$ (f) $\frac{\pi}{2}$ (g) 0 (h) $-\frac{\pi}{4}$ (i) $-\frac{\pi}{3}$ (j) $\frac{3\pi}{4}$ (k) $-\frac{\pi}{6}$ (l) π



3(a) 1.447 (b) 1.694 (c) 0.730 (d) -0.730 (e) 1.373
(f) -1.373

4(a) $\frac{\pi}{2}$ (b) 1 (c) 1 (d) $\frac{\pi}{6}$ (e) $\frac{1}{2}$ (f) $\frac{3\pi}{4}$ (g) $-\frac{\pi}{6}$
(h) 0 (i) $\frac{\pi}{3}$

5(a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$ (e) $-\frac{\pi}{2}$ (f) $\frac{\pi}{3}$

6(a)(i) $\frac{4}{5}$ (ii) $\frac{5}{12}$ (iii) $\frac{1}{3}\sqrt{5}$ (iv) $\frac{8}{17}$ (v) $\frac{3}{10}\sqrt{10}$
(vi) $-\frac{1}{3}\sqrt{7}$

7(a) $\frac{56}{65}$ (b) $\frac{10\sqrt{3}-\sqrt{5}}{20}$ (c) 1 (d) $\frac{56}{33}$

8(a) $-\frac{7}{9}$ (b) $\frac{12}{49}\sqrt{13}$ (c) $\frac{4}{3}$

9(b) $\frac{\pi}{4}$

13(a) 2 is outside the range of the inverse sine function, which is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. (b) It is because the sine curve is symmetrical about $x = \frac{\pi}{2}$. (c) $\pi - 2$

14(a) $\cot \theta$ (d) $-\frac{\pi}{2}$

15(a) $\frac{3x}{1-2x^2}$ (b) $x = \frac{1}{2}$ (note that $x \neq -1$)

16(a) $x = \frac{1}{3}$ (b) $x = \frac{1}{3}$ or 1

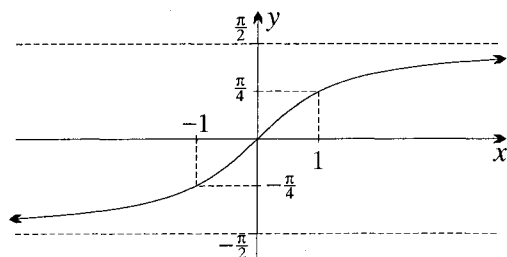
20 $x = -\frac{3}{2}$ or $\frac{1}{3}$

23(b) $0 < \tan^{-1} \left(\frac{1}{1+x^2} \right) \leq \frac{\pi}{4}$,

$0 \leq \tan^{-1} \left(\frac{x^2}{1+x^2} \right) < \frac{\pi}{4}$ (d) $\frac{\pi}{4} \leq y \leq \tan^{-1} \frac{4}{3}$

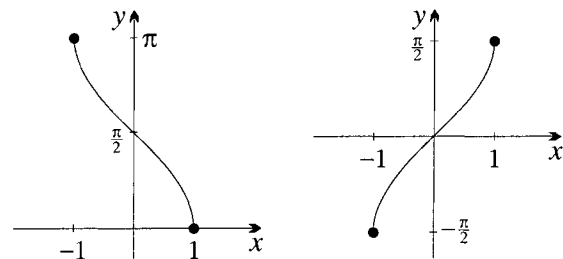
Exercise 1C (Page 17)

1(a) domain: all real x , range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$, odd

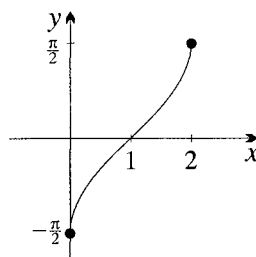


(b) domain: $-1 \leq x \leq 1$, range: $0 \leq y \leq \pi$, neither

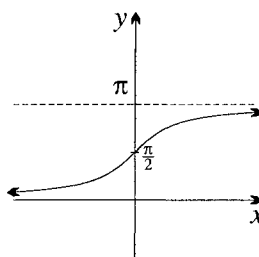
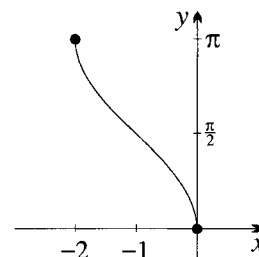
(c) domain: $-1 \leq x \leq 1$, range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, odd



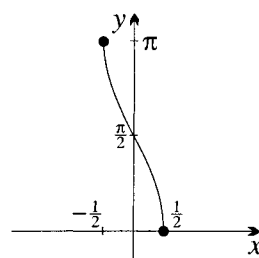
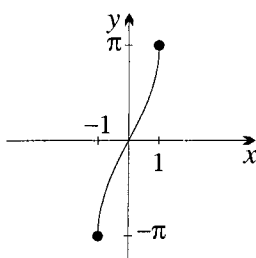
2(a) domain: $0 \leq x \leq 2$, range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, neither (b) domain: $-2 \leq x \leq 0$, range: $0 \leq y \leq \pi$, neither



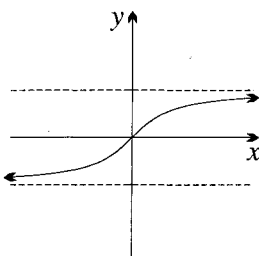
(c) domain: all real x , range: $0 < y < \pi$, neither



3(a) domain: $-1 \leq x \leq 1$, range: $-\pi \leq y \leq \pi$, odd (b) domain: $-\frac{1}{2} \leq x \leq \frac{1}{2}$, range: $0 \leq y \leq \pi$, neither

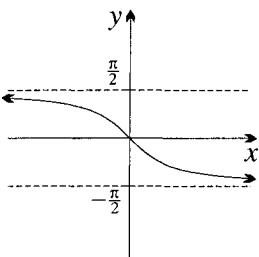
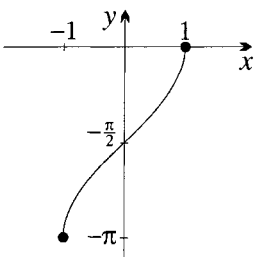


(c) domain: all real x , range: $-\frac{\pi}{4} < y < \frac{\pi}{4}$, odd

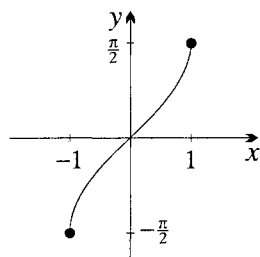


4(a) domain: $-1 \leq x \leq 1$, range: $-\pi \leq y \leq 0$, neither

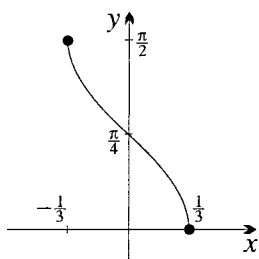
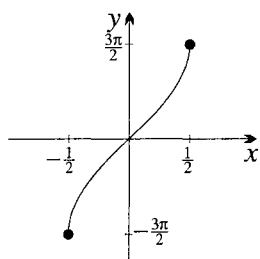
(b) domain: all real x , range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$, odd



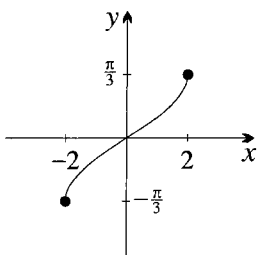
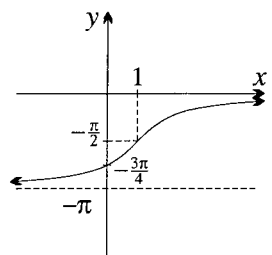
(c) domain: $-1 \leq x \leq 1$, range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, odd



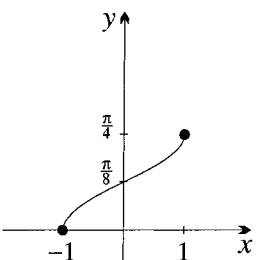
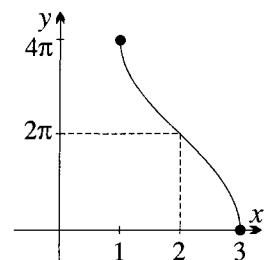
5(a) domain: $-\frac{1}{2} \leq x \leq \frac{1}{2}$, range: $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$, odd
 (b) domain: $-\frac{1}{3} \leq x \leq \frac{1}{3}$, range: $0 \leq y \leq \frac{\pi}{2}$, neither



(c) domain: all real x , range: $-\pi < y < 0$, neither
 (d) domain: $-2 \leq x \leq 2$, range: $-\frac{\pi}{3} \leq y \leq \frac{\pi}{3}$, odd

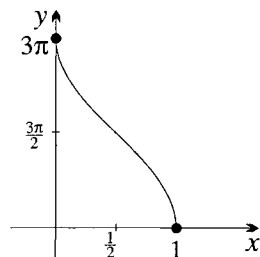
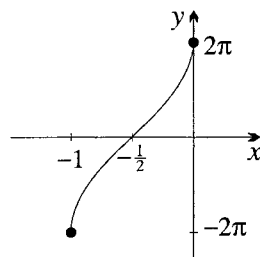


(e) domain: $1 \leq x \leq 3$, range: $0 \leq y \leq 4\pi$, neither
 (f) domain: $-1 \leq x \leq 1$, range: $0 \leq y \leq \frac{\pi}{4}$, neither

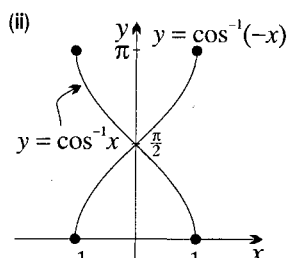
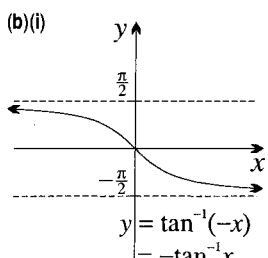
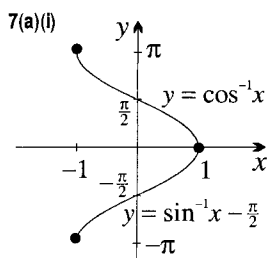
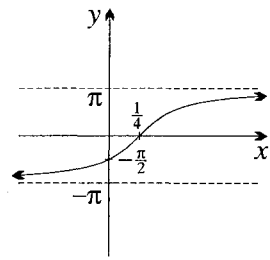
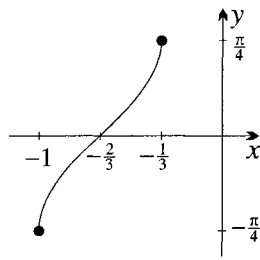


6(a)(i) $-1 \leq x \leq 0$ (ii) $-2\pi \leq y \leq 2\pi$

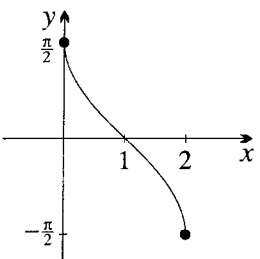
(b)(i) domain: $0 \leq x \leq 1$, range: $0 \leq y \leq 3\pi$



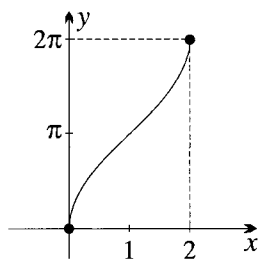
(ii) domain: $-1 \leq x \leq -\frac{1}{3}$, range: $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$
 (iii) domain: all real x , range: $-\pi < y < \pi$



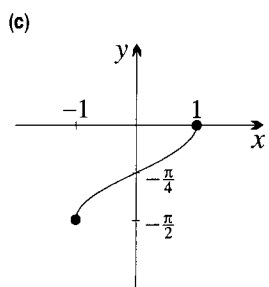
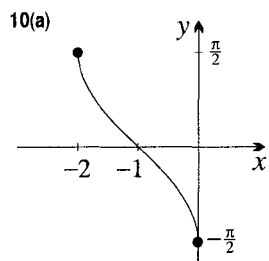
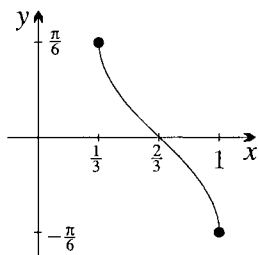
8(a) domain: $0 \leq x \leq 2$, range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
 (b) $y = \frac{\pi}{2}, 0, -\frac{\pi}{2}$ (c) $x = \frac{1}{2}$



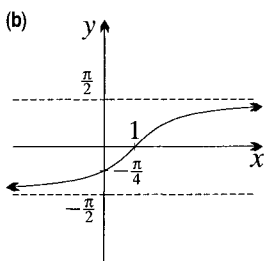
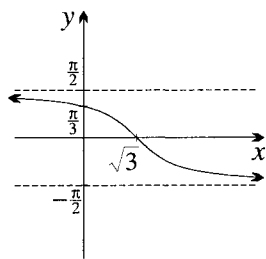
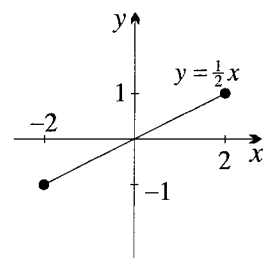
9(a) domain: $0 \leq x \leq 2$, range: $0 \leq y \leq 2\pi$
 (b) domain: all real x , range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



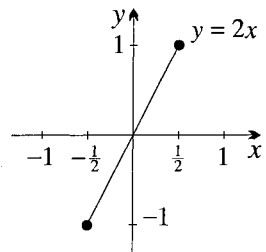
(c) domain: $\frac{1}{3} \leq x \leq 1$, range: $-\frac{\pi}{6} \leq y \leq \frac{\pi}{6}$



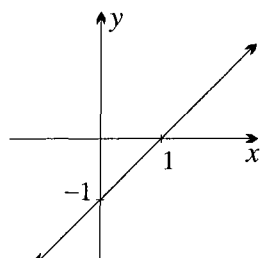
(b) odd



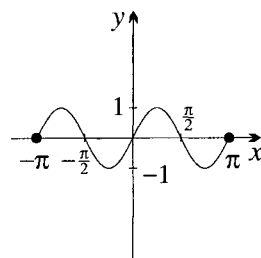
11(a) odd



(c) neither



12(a)



(b) $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

13(a) $-1 \leq x \leq 1$,

even

(b) $0 \leq \cos^{-1} x \leq \pi$,
so $\sin \cos^{-1} x \geq 0$.

14(a) domain: all real x ,
 $x \neq \frac{(2n+1)\pi}{2}$, where n is
an integer,

range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$,
odd

(b) x (c) π

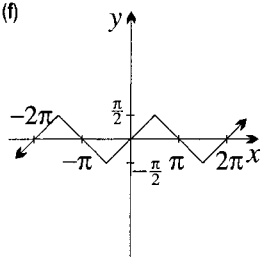
15 $\sin^{-1} \cos x$
 $= \frac{\pi}{2} - \cos^{-1} \cos x$,
so we reflect in the
 x -axis and then shift
 $\frac{\pi}{2}$ units up.
It's even.

16(a) domain: all real x , range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, pe-
riod: 2π , odd

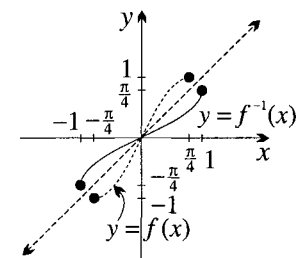
(b) x

(e) $\cos^{-1} \sin x = \frac{\pi}{2} - \sin^{-1} \sin x$, so we reflect in
the x -axis and then shift $\frac{\pi}{2}$ units up.

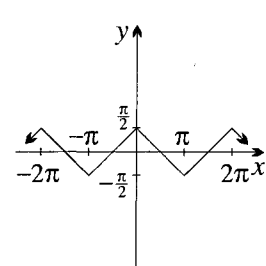
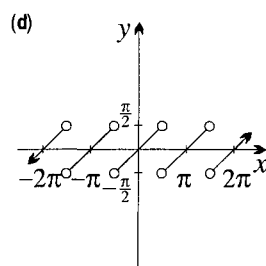
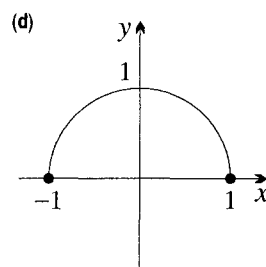
(f)



(c)



(d) $f^{-1}(x) = \frac{1}{2} \sin^{-1} x$,
odd

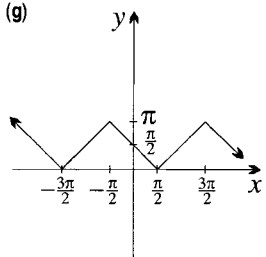


16(a) domain: all real x , range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, pe-
riod: 2π , odd

(b) x

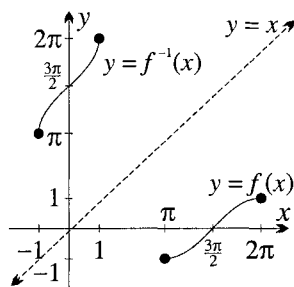
(e) $\cos^{-1} \sin x = \frac{\pi}{2} - \sin^{-1} \sin x$, so we reflect in
the x -axis and then shift $\frac{\pi}{2}$ units up.

(f)

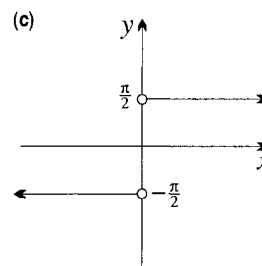


17(b) $\pi \leq x \leq 2\pi$

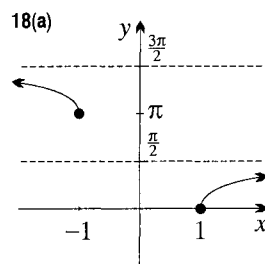
(e) $f^{-1}(x)$
 $= 2\pi - \cos^{-1} x$



14(a) $x \neq 0$, odd



18(a)



(b)(i) $\frac{\pi}{3}$ (ii) $\frac{4\pi}{3}$

15(a) $-\sqrt{t}$ (b) $\frac{t}{2}$

16(a) $x \geq 1$ or $x \leq -1$ (g)

(c) They are undefined.

(d) When $x > 1$,
 $f'(x) = \frac{1}{x\sqrt{x^2-1}}$,

and when $x < -1$,
 $f'(x) = \frac{-1}{x\sqrt{x^2-1}}$.

(e) $f'(x) > 0$ for $x > 1$

and for $x < -1$. (f)(i) $\frac{\pi}{2}$

(ii) $\frac{\pi}{2}$

17(a) domain: all real x , range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, odd

 (c) No, since $\frac{0}{0}$ is undefined. (d) $f'(x) = 1$ when $\cos x > 0$, and $f'(x) = -1$ when $\cos x < 0$.

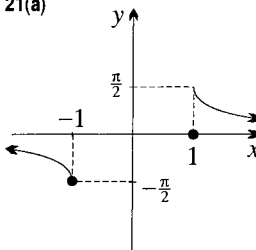
18(a) $-1 \leq x \leq 1$ (c) $g(x) = \frac{\pi}{2}$ for $0 \leq x \leq 1$.

19 $\tan^{-1} \frac{x+2}{1-2x}$ is $\tan^{-1} x + \tan^{-1} 2$ for $x < \frac{1}{2}$,

and is $\tan^{-1} x + \tan^{-1} 2 - \pi$ for $x > \frac{1}{2}$.

20(a) -1 (b) $-2\sqrt{1-x^2}y^2 - \frac{y}{x}$ (c) $\frac{x+y}{x-y}$

21(a)



domain:

$x \geq 1$ or $x \leq -1$,

range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$,

odd

Exercise 1D (Page 22)

2(a) $\frac{-1}{\sqrt{1-x^2}}$ (b) $\frac{1}{1+x^2}$ (c) $\frac{2}{\sqrt{1-4x^2}}$ (d) $\frac{3}{1+9x^2}$

(e) $\frac{-5}{\sqrt{1-25x^2}}$ (f) $\frac{-1}{\sqrt{1-x^2}}$ (g) $\frac{2x}{\sqrt{1-x^4}}$ (h) $\frac{3x^2}{1+x^6}$

(i) $\frac{1}{x^2+4x+5}$ (j) $\frac{1}{\sqrt{2x-x^2}}$ (k) $\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(l) $2x \tan^{-1} x + 1$ (m) $\frac{1}{\sqrt{25-x^2}}$

(n) $\frac{4}{16+x^2}$ (o) $\frac{-1}{2\sqrt{x-x^2}}$ (p) $\frac{1}{2\sqrt{x(1+x)}}$ (q) $\frac{-1}{1+x^2}$

3(a) 2 (b) 2 (c) 1 (d) -1

4(a) Tangent is $y = -6x + \pi$, normal is $y = \frac{1}{6}x + \pi$.

(b) Tangent is $y = \frac{1}{\sqrt{2}}x + \frac{\pi}{4} - 1$,

normal is $y = -\sqrt{2}x + \frac{\pi}{4} + 2$.

5(b) $\frac{\pi}{2}$

6(a) π (b) $\frac{\pi}{2}$

7(b) concave up

9(a) $\cos^{-1} x$ (b) $\frac{3e^{3x}}{\sqrt{1-e^{6x}}}$ (c) $\frac{2}{\sqrt{7+12x-4x^2}}$

(d) $\frac{1}{x^2-2x+2}$ (e) $\frac{e^x}{\sqrt{1-e^{2x}}}$ (f) $\frac{1}{2\sqrt{1-x^2}\sin^{-1} x}$

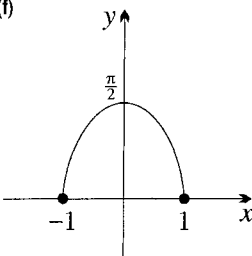
(g) $\frac{1}{2x\sqrt{\log x(1-\log x)}}$ (h) $\frac{\sin^{-1} \sqrt{1-x}}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}}$

(i) $\frac{1}{1+x^2}$

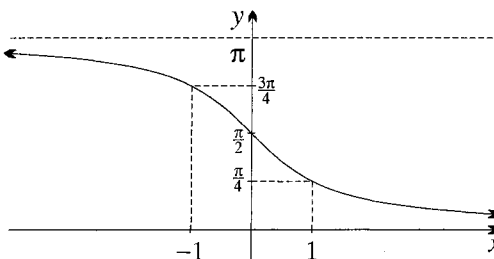
11(a) $-1 \leq x \leq 1$, even (f)

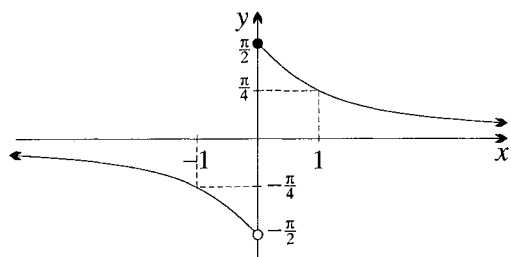
 (b) The y -axis, since the function is even.

(c) $\frac{-2x}{\sqrt{1-x^4}}$

 (e) The tangents at $x = 1$ and $x = -1$ are vertical.


13(c) $\frac{1}{45}$ rad/s

 22(a)(i) domain: all real x , range: $0 < y < \pi$, point symmetry about $(0, \frac{\pi}{2})$

 (ii) domain: all real x , range: $-\frac{\pi}{2} < y \leq \frac{\pi}{2}$, odd, except for the value at $x = 0$



(c) It is only true for the second function. (d) The first produces a continuous function with a natural symmetry. The second has the same range (apart from $y = \frac{\pi}{2}$ and $y = 0$) as $\tan^{-1} x$, but the value at $x = 0$ disturbs the symmetry.

23(c) $\frac{x}{(x^2-1)\sqrt{x^2-2}}$

(d) domain:

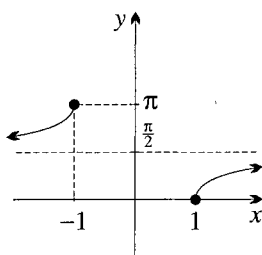
$x \geq 1$ or $x \leq -1$,

range: $0 \leq y \leq \pi$

excluding $y = \frac{\pi}{2}$,

point symmetry

about $(0, \frac{\pi}{2})$



Exercise 1E (Page 28)

2(a) $\cos^{-1} x + C$ (b) $\sin^{-1} \frac{x}{2} + C$ (c) $\frac{1}{3} \tan^{-1} \frac{x}{3} + C$

(d) $\sin^{-1} \frac{3x}{2} + C$ (e) $\frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C$

(f) $\cos^{-1} \frac{x}{\sqrt{5}} + C$

3(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{12}$ (e) $\frac{\pi}{6}$ (f) $\frac{5\pi}{12}$

4(a) $y = \sin^{-1} x + \pi$ (b) $y = \tan^{-1} \frac{x}{4} + \frac{\pi}{4}$

5(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

6(a) $\frac{1}{2} \sin^{-1} 2x + C$ (b) $\frac{1}{4} \tan^{-1} 4x + C$

(c) $\frac{1}{\sqrt{2}} \cos^{-1} \sqrt{2}x + C$ (d) $\frac{1}{3} \sin^{-1} \frac{3x}{2} + C$

(e) $\frac{1}{15} \tan^{-1} \frac{3x}{5} + C$ (f) $\frac{1}{2} \cos^{-1} \frac{2x}{\sqrt{3}} + C$

7(a) $\frac{\pi}{18}$ (b) $\frac{\pi}{12}$ (c) $\frac{2\pi}{9}\sqrt{3}$ (d) $\frac{5\pi}{24}$ (e) $\frac{\pi}{12}\sqrt{3}$

(f) $\frac{\pi}{120}\sqrt{10}$

9(c) $(\frac{\pi}{12} + \frac{1}{2}\sqrt{3} - 1) \text{ unit}^2$

10(b) $(1 - \frac{1}{2}\sqrt{3}) \text{ unit}^2$

11(b) $\frac{\pi}{2}$

12(a) $\frac{6x^2}{4+x^6}$ (b) $\frac{1}{6} \tan^{-1} \frac{x^3}{2} + C$

13(a) $\frac{\pi^2}{4\sqrt{7}} \text{ unit}^3$ (b) $\frac{\pi^2}{8} \text{ unit}^3$

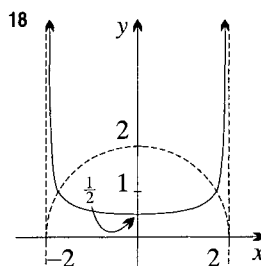
14(b) $\tan^{-1}(x+3) + C$

15(a) $\tan^{-1} x + \frac{x}{1+x^2}$ (b) $\frac{\pi}{4} - \frac{1}{2} \ln 2$

16(a) 0 (b) 0 (c) $\frac{3\pi}{4}$ (d) 0 (e) 0 (f) 18π

17(a)(i) 0 (b)(i) $f(0) = 0$ and $f'(x) < 0$ for $x > 0$.

(ii) $\frac{\pi-2}{8}$



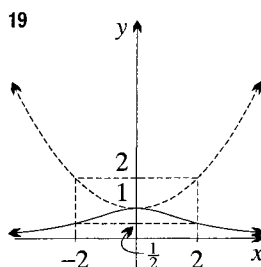
(c) domain:

$-2 \leq x \leq 2$,

range: $y \geq \frac{1}{2}$, even

(d) $\frac{\pi}{3} \text{ unit}^2$

(e) $\pi \text{ unit}^2$



(a) The y -axis, since it's an even function.

(b) domain: all real x , range: $0 < y \leq 1$

(d) 0 (e) $\pi \text{ unit}^2$

(f) $4 \tan^{-1} \frac{a}{2} \text{ unit}^2$

(g) $2\pi \text{ unit}^2$

21(b) 0.153 unit^2

22(a) $\frac{8011}{10200}$ (b) $I = \frac{\pi}{4}$, four decimal places

24(a) $2 \tan^{-1} \sqrt{x} + C$ (b) $\tan^{-1} e - \frac{\pi}{4}$

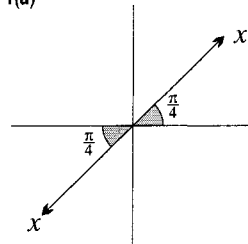
25(g) $\pi \div 3.092$, error $\div 0.050$

26(a) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 + \dots + \tan^{-1} n$

(b) $x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)$

Exercise 1F (Page 35)

1(a)



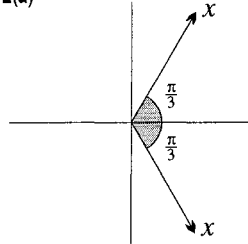
(b) $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}$

or $\frac{21\pi}{4}$

(c) $-\frac{3\pi}{4}, -\frac{7\pi}{4}, -\frac{11\pi}{4}, -\frac{15\pi}{4}, -\frac{19\pi}{4}$ or $-\frac{23\pi}{4}$

(d) $x = n\pi + \frac{\pi}{4}$, where $n \in \mathbb{Z}$.

2(a)



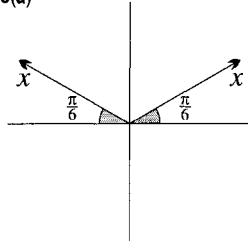
(b) $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$

or $\frac{17\pi}{3}$

(c) $-\frac{\pi}{3}, -\frac{5\pi}{3}, -\frac{7\pi}{3}, -\frac{11\pi}{3}, -\frac{13\pi}{3}$ or $-\frac{17\pi}{3}$

(d) $x = 2n\pi + \frac{\pi}{3}$ or $2n\pi - \frac{\pi}{3}$, where $n \in \mathbb{Z}$.

3(a)



(b) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}$

or $\frac{29\pi}{6}$

(c) $-\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, -\frac{23\pi}{6}, -\frac{31\pi}{6}$ or $-\frac{35\pi}{6}$

(d) $x = \frac{\pi}{6} + 2n\pi$ or $\frac{5\pi}{6} + 2n\pi$, where $n \in \mathbb{Z}$. [Alternatively, $x = m\pi + (-1)^m \frac{\pi}{6}$, where $m \in \mathbb{Z}$.]

4(a) $x = n\pi + \frac{\pi}{3}, n \in \mathbf{Z}$ (b) $x = 2n\pi \pm \frac{\pi}{4}, n \in \mathbf{Z}$ (c) $x = 2n\pi + \frac{\pi}{3}$ or $x = 2n\pi + \frac{2\pi}{3}, n \in \mathbf{Z}$.

[Alternatively, $x = m\pi + (-1)^m \frac{\pi}{3}, m \in \mathbf{Z}$.]

(d) $x = n\pi - \frac{\pi}{4}, n \in \mathbf{Z}$ (e) $x = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbf{Z}$

(f) $x = 2n\pi - \frac{\pi}{6}$ or $x = 2n\pi + \frac{7\pi}{6}$. [Alternatively, $x = m\pi - (-1)^m \frac{\pi}{6} = m\pi + (-1)^{m+1} \frac{\pi}{6}, m \in \mathbf{Z}$.]

5(a) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in \mathbf{Z}$ (b) $\theta = n\pi + \frac{\pi}{4}, n \in \mathbf{Z}$

(c) $\theta = 2n\pi + \frac{\pi}{5}$ or $\theta = 2n\pi + \frac{4\pi}{5}$.

[Alternatively, $\theta = m\pi + (-1)^m \frac{\pi}{5}, m \in \mathbf{Z}$.]

(d) $\theta = 2n\pi + \frac{4\pi}{3}$ or $\theta = 2n\pi - \frac{\pi}{3}$.

[Alternatively, $\theta = m\pi + (-1)^m \frac{4\pi}{3}, m \in \mathbf{Z}$.]

(e) $\theta = n\pi - \frac{\pi}{3}, n \in \mathbf{Z}$ (f) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in \mathbf{Z}$

6(a) $x = n\pi, n \in \mathbf{Z}$ (b) $x = 2n\pi, n \in \mathbf{Z}$

(c) $x = n\pi, n \in \mathbf{Z}$ (d) $x = 2n\pi + \frac{\pi}{2}, n \in \mathbf{Z}$

(e) $x = 2n\pi + \frac{\pi}{2}, n \in \mathbf{Z}$ (f) $x = 2n\pi - \frac{\pi}{2}, n \in \mathbf{Z}$

7(a)(i) $x = n\pi, n \in \mathbf{Z}$ (ii) $x = -\pi, 0$ or π

(b)(i) $x = \frac{\pi}{2} + 4n\pi$ or $x = \frac{3\pi}{2} + 4n\pi$.

[Alternatively, $x = 2m\pi + (-1)^m \frac{\pi}{2}, m \in \mathbf{Z}$.]

(ii) $x = \frac{\pi}{2}$ (c)(i) $x = \frac{n\pi}{3} + \frac{\pi}{18}, n \in \mathbf{Z}$

(ii) $x = -\frac{17\pi}{18}, -\frac{11\pi}{18}, -\frac{5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}$ or $\frac{13\pi}{18}$

(d)(i) $x = n\pi + \frac{\pi}{4}, n \in \mathbf{Z}$ (ii) $x = -\frac{3\pi}{4}$ or $\frac{\pi}{4}$

(e)(i) $x = 2n\pi + \frac{7\pi}{12}$ or $2n\pi - \frac{11\pi}{12}, n \in \mathbf{Z}$

(ii) $x = -\frac{11\pi}{12}$ or $\frac{7\pi}{12}$ (f)(i) $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbf{Z}$

(ii) $x = -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}$ or $\frac{11\pi}{12}$

(g)(i) $x = n\pi \pm \frac{\pi}{10}, n \in \mathbf{Z}$

(ii) $x = -\frac{9\pi}{10}, -\frac{\pi}{10}, \frac{\pi}{10}$ or $\frac{9\pi}{10}$

(h)(i) $x = \frac{\pi}{6} + \frac{2}{3}n\pi$.

[Alternatively, $x = \frac{m\pi}{3} + (-1)^m \frac{\pi}{6}, m \in \mathbf{Z}$.]

(ii) $x = -\frac{\pi}{2}, \frac{\pi}{6}$ or $\frac{5\pi}{6}$ (i)(i) $x = \frac{n\pi}{4} + \frac{\pi}{12}, n \in \mathbf{Z}$

(ii) $x = -\frac{11\pi}{12}, -\frac{2\pi}{3}, -\frac{5\pi}{12}, -\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}$ or $\frac{5\pi}{6}$

(j)(i) $x = n\pi + \frac{3\pi}{8}, n \in \mathbf{Z}$ (ii) $x = -\frac{5\pi}{8}$ or $\frac{3\pi}{8}$

(k)(i) $x = 2n\pi + \frac{5\pi}{7}$ or $2n\pi - \frac{3\pi}{7}, n \in \mathbf{Z}$

(ii) $x = -\frac{3\pi}{7}$ or $\frac{5\pi}{7}$

(l)(i) $x = -\frac{\pi}{5} + n\pi$ or $x = \frac{2\pi}{5} + n\pi$.

[Alternatively, if m is even, $x = \frac{m\pi}{2} - \frac{\pi}{5}$; if m is odd, $x = \frac{m\pi}{2} - \frac{\pi}{10}$.] (ii) $x = -\frac{3\pi}{5}, -\frac{\pi}{5}, \frac{2\pi}{5}$ or $\frac{4\pi}{5}$

8(a)(i) $\theta = n\pi$ or $\theta = \frac{3\pi}{2} + 2n\pi, n \in \mathbf{Z}$. [Alternatively, $\theta = m\pi$ or $\theta = m\pi + (-1)^{m+1} \frac{\pi}{2}, m \in \mathbf{Z}$.]

(ii) $\theta = -\pi, -\frac{\pi}{2}, 0$ or π (b)(i) $\theta = n\pi + \frac{\pi}{2}$ or

$\theta = \frac{\pi}{6} + 2n\pi$ or $\theta = \frac{5\pi}{6} + 2n\pi, n \in \mathbf{Z}$. [Alternatively, $\theta = 2m\pi \pm \frac{\pi}{2}$ or $m\pi + (-1)^m \frac{\pi}{6}, m \in \mathbf{Z}$.]

(ii) $\theta = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$ or $\frac{5\pi}{6}$ (c)(i) $\theta = n\pi$ or $n\pi + \frac{\pi}{3}, n \in \mathbf{Z}$

(ii) $\theta = -\pi, -\frac{2\pi}{3}, 0, \frac{\pi}{3}$ or π

(d)(i) $\theta = 2n\pi \pm \pi$ or $2n\pi \pm \frac{2\pi}{3}, n \in \mathbf{Z}$

(ii) $\theta = -\pi, -\frac{2\pi}{3}, \frac{2\pi}{3}$ or π (e)(i) $\theta = \frac{n\pi}{2} - \frac{\pi}{6}, n \in \mathbf{Z}$

(ii) $\theta = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}$ or $\frac{5\pi}{6}$

(f)(i) $\theta = \frac{n\pi}{2}$ or $\frac{n\pi}{2} + \frac{\pi}{8}, n \in \mathbf{Z}$

(ii) $\theta = -\pi, -\frac{7\pi}{8}, -\frac{\pi}{2}, -\frac{3\pi}{8}, 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}$ or π

9(c) $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ or 2π

10(b) $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$ or 2π

11(c) $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{9\pi}{8}, \frac{13\pi}{8}$ or $\frac{7\pi}{4}$

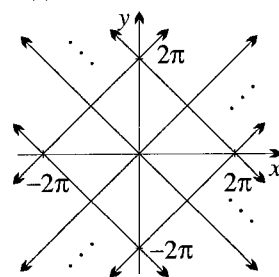
12(a) $x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or π

(b) $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$ or π

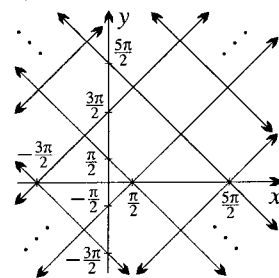
(c) $x = \frac{\pi}{12}, \frac{\pi}{8}, \frac{5\pi}{12}, \frac{5\pi}{8}$ or $\frac{3\pi}{4}$

(d) $x = \frac{\pi}{12}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{3\pi}{4}$ or $\frac{7\pi}{8}$

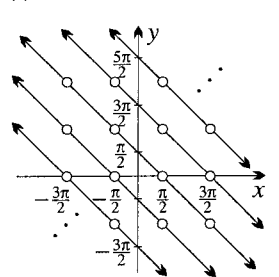
13(a)



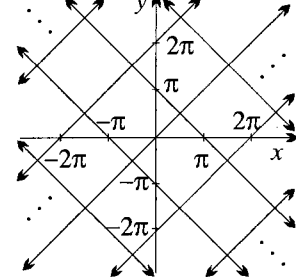
(c)



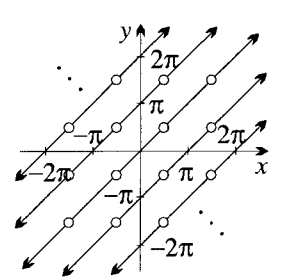
(e)



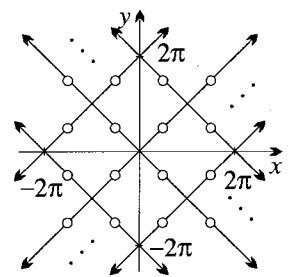
(b)



(d)



(f)



in x -axis: (a), (c), (f); in y -axis: (a), (f);

in $y = x$: (a), (b), (d), (f)

Chapter Two

Exercise 2A (Page 40)

- 1(a) $\cos 2\theta$ (b) $\sin 40^\circ$ (c) $\tan 50^\circ$
 (d) $\cos 70^\circ$ (e) $\sin 6\alpha$ (f) $\frac{1}{\tan \theta} = \cot \theta$
 2(a) $\sin 4\theta$ (b) $\cos x$ (c) $\cos 6\alpha$
 (d) $\tan 70^\circ$ (e) $\cos 50^\circ$ (f) $\tan 8x$
 3(a) $\frac{4}{5}$ (b) $\frac{7}{25}$ (c) $-\frac{16}{65}$ (d) $\frac{120}{169}$ (e) $\frac{24}{7}$ (f) $\frac{33}{56}$
 5(a) $\sin 30^\circ = \frac{1}{2}$ (b) $\cos 30^\circ = \frac{1}{2}\sqrt{3}$
 (c) $\tan 135^\circ = -1$ (d) $\cos 45^\circ = \frac{1}{2}\sqrt{2}$
 (e) $\frac{1}{2}\sin \frac{\pi}{6} = \frac{1}{4}$ (f) $\sin \frac{2\pi}{3} = \frac{1}{2}\sqrt{3}$
 (g) $\cos \frac{7\pi}{6} = -\frac{1}{2}\sqrt{3}$ (h) $\tan \frac{4\pi}{3} = \sqrt{3}$ (i) $\frac{1}{4}\sqrt{2}$
 (j) -1
 6(a) $\frac{1}{2}\sin \theta$ (b) $\sin^2 x$ (c) $\cos^2 2x$ (d) $2\sin^2 3\theta$
 (e) $\cos 20^\circ$ (f) $2\cos^2 \frac{\alpha}{2}$ (g) $\sin 5x$ (h) $\frac{1}{4}\sin^2 2\alpha$
 7(a) $\frac{3}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{10}}$ (c) $\frac{1}{3}$
 9(a) $-\frac{4}{7}\sqrt{2}$ (b) $-\frac{9}{10}$
 12(a) $\frac{1}{3}\sqrt{5}$ (b) $-\frac{1}{9}$ (c) $\frac{4}{9}\sqrt{5}$ (d) $\frac{7}{27}\sqrt{5}$ (e) $-\frac{8}{81}\sqrt{5}$
 (f) $-\frac{79}{81}$ (g) $-\frac{7}{22}\sqrt{5}$ (h) $\frac{8}{79}\sqrt{5}$ (i) $\frac{1}{6}\sqrt{30}$ (j) $\frac{1}{5}\sqrt{5}$
 15(a) $y = 2x^2 - 8x + 7$ (b) $y = \frac{2x-2}{2x-x^2}$ (c) $y = \frac{4-x^2}{4+x^2}$
 (d) $9y^2 = 16x^2(9-x^2)$
 16(a) $\frac{1}{2}\sqrt{2}$
 18(a) $a^4 = 2a^2 - b^2$ (b) $a^2 + b^2 = 2(c+1)$
 19(d) $\frac{\sqrt{5}-1}{4}$

$$20(b) \sin \frac{90^\circ}{2^n} = \frac{1}{2} \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{\dots + \sqrt{2}}}}}_{n \text{ terms}}}$$

$$\text{Let } T_n = 2^n \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{\dots + \sqrt{2}}}}}_{n \text{ terms}}}$$

Since $\frac{\sin \theta}{\theta} \rightarrow 1$ as $\theta \rightarrow 0$, $T_n \rightarrow \pi$ as $n \rightarrow \infty$.

$$T_4 \div 3.1365, T_8 \div 3.141573$$

Exercise 2B (Page 44)

- 1(a) $\frac{2t}{1+t^2}$ (b) $\frac{1-t^2}{1+t^2}$ (c) $\frac{2t}{1-t^2}$ (d) $\frac{1+t^2}{1-t^2}$ (e) $\frac{2t^2}{1+t^2}$
 (f) t
 2(a) $\frac{1-t^2}{1+t^2}$ (b) $\frac{(1-t)^2}{1+t^2}$ (c) $\frac{1+t}{1-t}$
 3(a) $\tan 20^\circ$ (b) $\sin 20^\circ$ (c) $\cos 20^\circ$ (d) $\sin 4x$
 (e) $\tan 4x$ (f) $\cos 4x$
 4(a) $\tan 30^\circ = \frac{1}{3}\sqrt{3}$ (b) $\sin 30^\circ = \frac{1}{2}$
 (c) $\cos 150^\circ = -\frac{1}{2}\sqrt{3}$ (d) $\sin 225^\circ = -\frac{1}{2}\sqrt{2}$
 (e) $\cos \frac{3\pi}{4} = -\frac{1}{2}\sqrt{2}$ (f) $\tan \frac{11\pi}{6} = -\frac{1}{3}\sqrt{3}$
 6(b)(ii) $\sqrt{2} - 1 = \tan 22\frac{1}{2}^\circ$,
 since $\tan 45^\circ = \tan 225^\circ = 1$.
 8(a) $-\frac{3}{4}$ (b) $-\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $3 + \sqrt{10}$

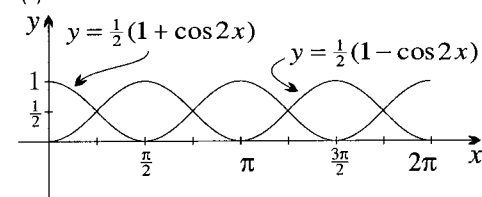
$$9(a)(i) \cos \theta = 2 \cos^2 \frac{1}{2}\theta - 1$$

$$(b)(i) \sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$$

Exercise 2C (Page 47)

- 1(a) $\frac{56}{65}$ (b) $\frac{33}{65}$
 2(a) $\frac{a}{c}$ (c) $\frac{a+b}{c}$
 3(a) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ (b) $\tan \theta = \frac{h}{30}$, $\tan 2\theta = \frac{h}{10}$
 4(a) $\frac{a}{x}$, $\frac{b}{x}$ (d) The expression under the square root in (c) is not positive unless $b > 2a$.
 5(a) $\frac{\pi}{2}$ (b) $\frac{1}{8}(\pi+2)$ (c) $\frac{1}{12}(\pi-3)$ (d) $\frac{1}{32}(\pi+2\sqrt{2})$
 (e) $\frac{1}{24}(4\pi+9)$ (f) $\frac{1}{24}(2\pi-3\sqrt{3})$

6(b)



$$7 \cos x \sin x = \frac{1}{2} \sin 2x, \text{ and } -\frac{1}{2} \leq \frac{1}{2} \sin 2x \leq \frac{1}{2}.$$

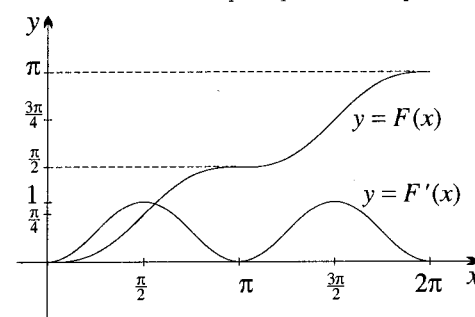
$$8(b) \frac{\sin \theta}{\sin \phi}$$

$$12(b) \cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$$

$$(c)(i) \frac{3\pi}{8} \quad (ii) \frac{1}{32}(3\pi+8)$$

16(b)(i) $x = 0, \pi$ or 2π (ii) $0 < x < \pi$ or $\pi < x < 2\pi$
 (iii) no values of x

(c) It is because $-\frac{1}{4} \leq \frac{1}{4} \sin 2x \leq \frac{1}{4}$.



(d) $(\frac{\pi}{2}, \frac{\pi}{4})$ and $(\frac{3\pi}{2}, \frac{3\pi}{4})$ are points of inflexion, while $(0, 0)$, $(\pi, \frac{\pi}{2})$ and $(2\pi, \pi)$ are stationary (or horizontal) points of inflexion.

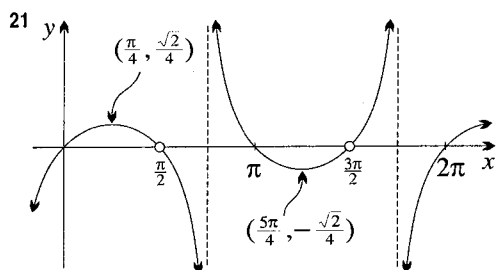
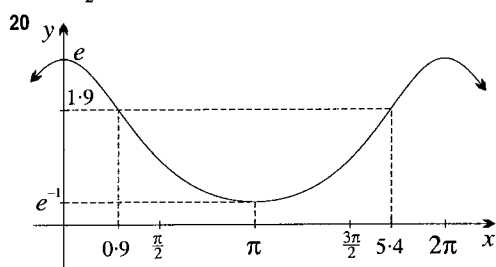
(f)(i) $k = 3\pi$ (ii) $k = n\pi$, where n is an integer.

Exercise 2D (Page 53)

- 1(a) $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ (b) $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$
 (c) $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ (d) $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
 (e) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ (f) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$
 2(a) $\alpha = 30^\circ, 120^\circ, 210^\circ$ or 300° (b) $\alpha = 0^\circ, 180^\circ$
 or 360° (c) $\alpha = 10^\circ, 50^\circ, 130^\circ, 170^\circ, 250^\circ$ or 290°
 (d) $\alpha = 45^\circ, 105^\circ, 165^\circ, 225^\circ, 285^\circ$ or 345°
 3(a) $\theta = \frac{\pi}{2}$ or $\frac{5\pi}{6}$ (b) $\theta = \frac{7\pi}{12}$ or $\frac{11\pi}{12}$

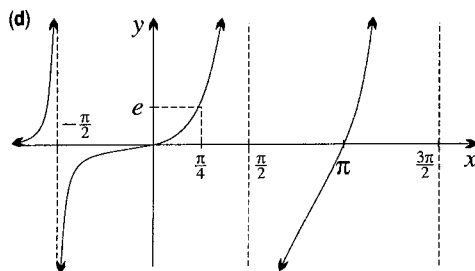
- (c) $\theta = \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}$ or $\frac{13\pi}{8}$
 (d) $\theta = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}$ or $\frac{19\pi}{12}$
 4(a) $x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (b) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
 (c) $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$ or $\frac{23\pi}{12}$
 (d) $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$
 5(a) $\alpha = 0^\circ, 90^\circ, 180^\circ$ or 360° (b) $\alpha = 60^\circ$ or 300°
 (c) $\alpha = 45^\circ, 90^\circ, 225^\circ$ or 270° (d) $\alpha = 75^\circ 58', 135^\circ, 255^\circ 58'$ or 315° (e) $\alpha = 90^\circ, 210^\circ$ or 330°
 (f) $\alpha = 0^\circ, 60^\circ, 300^\circ$ or 360° (g) $\alpha = 63^\circ 26', 135^\circ, 243^\circ 26'$ or 315° (h) $\alpha = 45^\circ$ or 225° (i) $\alpha = 15^\circ, 75^\circ, 105^\circ, 165^\circ, 195^\circ, 255^\circ, 285^\circ$ or 345°
 (j) $\alpha = 180^\circ$ or 240°
 6(a) $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (b) $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ (c) $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}$ or $\frac{17\pi}{9}$ (d) $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ or 2π
 7(a) $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ or 2π (b) $x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$
 (c) $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ or 2π
 (d) $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$ or 2π
 8(a) $0 < x < \pi$ (b) $0 < x < \frac{\pi}{2}$ or $\pi < x < \frac{3\pi}{2}$
 (c) $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$ (d) $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ or $\frac{7\pi}{6} \leq x \leq \frac{11\pi}{6}$
 (e) $\frac{7\pi}{12} \leq x \leq \frac{23\pi}{12}$ (f) $\frac{\pi}{8} \leq x < \frac{\pi}{4}$ or $\frac{5\pi}{8} \leq x < \frac{3\pi}{4}$ or $\frac{9\pi}{8} \leq x < \frac{5\pi}{4}$ or $\frac{13\pi}{8} \leq x < \frac{7\pi}{4}$
 9(a) $A = 120^\circ$ or 240° (b) $A = 0^\circ, 60^\circ, 300^\circ$ or 360° (c) $A = 45^\circ, 161^\circ 34', 225^\circ$ or $341^\circ 34'$
 (d) $A = 30^\circ, 60^\circ, 210^\circ$ or 240° (e) $A = 60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ$ or 300° (f) $A = 45^\circ, 60^\circ, 120^\circ, 135^\circ, 225^\circ, 240^\circ, 300^\circ$ or 315° (g) $A = 0^\circ, 180^\circ, 210^\circ, 330^\circ$ or 360° (h) $A = 60^\circ$ or 300°
 (i) $A = 71^\circ 34', 135^\circ, 251^\circ 34'$ or 315° (j) $A = 45^\circ, 60^\circ, 120^\circ, 135^\circ, 225^\circ, 240^\circ, 300^\circ$ or 315°
 10(a) $\theta = 90^\circ, 194^\circ 29', 270^\circ$ or $345^\circ 31'$
 (b) $\theta = 60^\circ, 120^\circ, 240^\circ$ or 300° (c) $\theta = 60^\circ$ or 300° (d) $\theta = 22^\circ 30', 67^\circ 30', 112^\circ 30', 157^\circ 30', 202^\circ 30', 247^\circ 30', 292^\circ 30'$ or $337^\circ 30'$
 (e) $\theta = 41^\circ 49', 138^\circ 11', 210^\circ$ or 330°
 (f) $\theta = 54^\circ 44', 125^\circ 16', 234^\circ 44'$ or $305^\circ 16'$
 (g) $\theta = 106^\circ 16'$ (h) $\theta = 0^\circ, 60^\circ, 300^\circ$ or 360°
 (i) $\theta = 30^\circ, 90^\circ, 150^\circ, 210^\circ$ or 330°
 (j) $\theta = 45^\circ, 63^\circ 26', 225^\circ$ or $243^\circ 26'$
 11(b) $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}$ or $\frac{5\pi}{4}$
 (c) $x = \frac{\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}$ or $\frac{7\pi}{4}$
 12(a) $x = 2n\pi + \frac{2\pi}{3}$ or $x = 2n\pi - \frac{2\pi}{3}$ or $2n\pi$, where $n \in \mathbf{Z}$. (b) $x = \frac{n\pi}{2}$ or $2n\pi + \frac{\pi}{6}$ or $2n\pi - \frac{\pi}{6}$, where $n \in \mathbf{Z}$. (c) $x = n\pi$ or $n\pi + (-1)^n \frac{\pi}{6}$, where $n \in \mathbf{Z}$.
 (d) $x = n\pi - \frac{\pi}{4}$, where $n \in \mathbf{Z}$.
 13(b) $x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$ or 2π
 14(a) $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$
 (b) $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$ or 2π

- (c) $x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}$ or $\frac{19\pi}{12}$ (d) $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ or $\frac{11\pi}{6}$
 (e) $x = \frac{\pi}{3}, \pi$ or $\frac{5\pi}{3}$ (f) $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$ or 2π
 15(b) $\theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$
 16(b) $x = 36^\circ, 108^\circ, 252^\circ$ or 324°
 17(b) $\theta = 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}$ or π
 18(a) $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ or $\frac{5\pi}{4} \leq x \leq \frac{7\pi}{4}$ (b) $0 < x < \frac{\pi}{4}$ or $\pi < x < \frac{5\pi}{4}$ (c) $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ or $\frac{5\pi}{4} \leq x \leq \frac{7\pi}{4}$
 (d) $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$ (e) $\pi \leq x \leq \frac{7\pi}{6}$ or $\frac{11\pi}{6} \leq x \leq 2\pi$
 (f) $\frac{\pi}{3} \leq x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \leq \pi$ or $\frac{4\pi}{3} \leq x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x \leq 2\pi$
 19(a) $k = \frac{n\pi}{2}$, where n is an integer.
 (b) $\frac{(2n-1)\pi}{2} < k < n\pi$, where n is an integer.



Because $\tan \frac{\pi}{2}$ and $\tan \frac{3\pi}{2}$ are undefined, and the function values do not approach ∞ or $-\infty$ as x approaches $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

- 22(b) 0 and π . The respective gradients are 1 and e^π .



23 $\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(-1 + \sqrt{5})$,
 $\sin 36^\circ = \cos 54^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$,
 $\sin 54^\circ = \cos 36^\circ = \frac{1}{4}(1 + \sqrt{5})$,
 $\sin 72^\circ = \cos 18^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$.

$$\tan 18^\circ = \frac{1}{5}\sqrt{25 - 10\sqrt{5}}, \tan 36^\circ = \sqrt{5 - 2\sqrt{5}},$$

$$\tan 54^\circ = \frac{1}{5}\sqrt{25 + 10\sqrt{5}}, \tan 72^\circ = \sqrt{5 + 2\sqrt{5}}$$

$$24(b) \theta = 160^\circ 55' \text{ or } 289^\circ 5'$$

$$25(c) x \doteq -2.571, -1.368 \text{ or } 3.939$$

$$26(e) x = \tan \frac{\pi}{10}, -\tan \frac{\pi}{10}, \tan \frac{3\pi}{10} \text{ or } -\tan \frac{3\pi}{10}$$

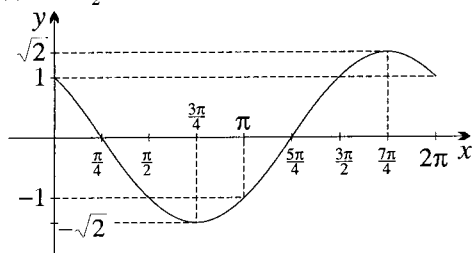
Exercise 2E (Page 60)

$$1(a) R = 2, \alpha = \frac{\pi}{3} \quad (b) R = 3\sqrt{2}, \alpha = \frac{\pi}{4}$$

$$2(a) R = 13, \alpha \doteq 22^\circ 37' \quad (b) R = 2\sqrt{5}, \alpha \doteq 63^\circ 26'$$

$$3(b) A = \sqrt{2} \quad (c) \alpha = \frac{\pi}{4} \quad (d) \text{Maximum is } \sqrt{2}, \text{ when } x = \frac{7\pi}{4}. \text{ Minimum is } -\sqrt{2}, \text{ when } x = \frac{3\pi}{4}.$$

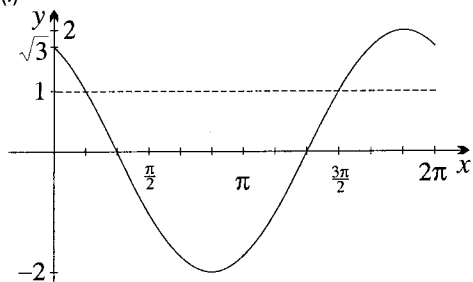
$$(e) x = \frac{\pi}{2} \text{ or } \pi \quad (f) \text{amplitude: } \sqrt{2}, \text{ period: } 2\pi$$



$$5(b) B = 2 \quad (c) \theta = \frac{\pi}{6} \quad (d) \text{Maximum is } 2, \text{ when } x = -\frac{\pi}{6}. \text{ Minimum is } -2, \text{ when } x = \frac{5\pi}{6}.$$

$$(e) x = \frac{\pi}{6}, \frac{3\pi}{2}$$

(f)



$$6(c) x \doteq 126^\circ 52'$$

$$7(b) x = 90^\circ \text{ or } x \doteq 323^\circ 8'$$

$$8(b) x = 270^\circ \text{ or } x \doteq 306^\circ 52'$$

$$9(a) 3\sin(x + \tan^{-1} \frac{2}{\sqrt{5}})$$

$$(b) x = 180^\circ \text{ or } x \doteq 276^\circ 23'$$

$$10(a) x \doteq 77^\circ 39' \text{ or } 344^\circ 17'$$

$$(b) x \doteq 103^\circ 29' \text{ or } 156^\circ 8'$$

$$(c) x \doteq 30^\circ 41' \text{ or } 297^\circ 26'$$

$$(d) x \doteq 112^\circ 37' \text{ or } 323^\circ 8'$$

$$11(c) x = 0, \frac{3\pi}{2}, 2\pi$$

$$12(b) x = 0, \frac{2\pi}{3}, 2\pi$$

$$13(b) x = 90^\circ \text{ or } x \doteq 298^\circ 4'$$

$$14(b) x = 180^\circ \text{ or } x \doteq 67^\circ 23'$$

$$16(a) x = 90^\circ \text{ or } x \doteq 12^\circ 41'$$

$$(b) x \doteq 36^\circ 52' \text{ or } 241^\circ 56'$$

$$(c) x \doteq 49^\circ 48' \text{ or } 197^\circ 35'$$

$$(d) x = 180^\circ \text{ or } x \doteq 280^\circ 23'$$

$$17(a) A = 2, \alpha = \frac{5\pi}{6} \quad (b) A = 5\sqrt{2}, \alpha = \frac{5\pi}{4}$$

$$18(a) A = \sqrt{41}, \alpha \doteq 321^\circ 20'$$

$$(b) A = 5\sqrt{5}, \alpha \doteq 259^\circ 42'$$

$$19(a)(i) 2\cos(x + \frac{11\pi}{6}) \quad (ii) x = \frac{\pi}{2} \text{ or } \frac{11\pi}{6}$$

$$(b)(i) \sqrt{2}\sin(x + \frac{3\pi}{4}) \quad (ii) x = 0 \text{ or } \frac{3\pi}{2}$$

$$(c)(i) 2\sin(x + \frac{5\pi}{3}) \quad (ii) x = \frac{\pi}{6} \text{ or } \frac{3\pi}{2}$$

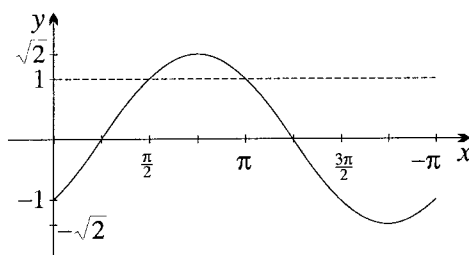
$$(d)(i) \sqrt{2}\cos(x - \frac{5\pi}{4}) \quad (ii) x = \pi \text{ or } \frac{3\pi}{2}$$

$$20(a)(i) \sqrt{5}\sin(x + 116^\circ 34')$$

$$(ii) x = 270^\circ \text{ or } x \doteq 36^\circ 52'$$

$$(b)(i) 5\cos(x - 3.7851) \quad (ii) x \doteq 2.63 \text{ or } 4.94$$

$$21(a)(ii) \quad (iii) \frac{\pi}{2} < x < \pi \quad (b)(i) \frac{\pi}{2} \leq x \leq \frac{11\pi}{6}$$



$$(ii) 0 < x < \frac{\pi}{6} \text{ or } \frac{3\pi}{2} < x < 2\pi \quad (iii) \frac{2\pi}{3} < x < \pi \text{ or } \frac{5\pi}{3} < x < 2\pi \quad (iv) 0 \leq x \leq \frac{\pi}{12} \text{ or } \frac{17\pi}{12} \leq x \leq 2\pi$$

$$22(a) x = \frac{7\pi}{12}, \frac{11\pi}{12} \quad (b) x = \frac{\pi}{3}, \frac{4\pi}{3}$$

$$(c) x = 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \pi, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}, 2\pi$$

$$23(a) x \doteq 313^\circ 36' \quad (b) x \doteq 79^\circ 6' \text{ or } 218^\circ 59'$$

$$24(b) x \doteq 36^\circ 52'$$

$$25 \theta = 0, \frac{3\pi}{4}, \frac{3\pi}{2} \text{ or } \frac{7\pi}{4}$$

$$26(b) x = n\pi + \frac{\pi}{6} \text{ or } n\pi - \frac{\pi}{12}, n \in \mathbb{Z}$$

$$27(b) \sin x + \sqrt{3}\cos x = 2\sin(x - \frac{5\pi}{3})$$

$$\text{or } 2\cos(x - \frac{\pi}{6}) \text{ or } 2\cos(x + \frac{11\pi}{6})$$

$$(c) \cos x - \sin x = \sqrt{2}\cos(x - \frac{7\pi}{4}) \text{ or } \sqrt{2}\sin(x + \frac{3\pi}{4})$$

$$\text{or } \sqrt{2}\sin(x - \frac{5\pi}{4})$$

$$30(b) x = 2n\pi \pm \cos^{-1} \frac{c}{r} + \theta, n \in \mathbb{Z}$$

$$(d)(i) \angle MOP = \theta \quad (iii) \angle MOQ \text{ is obtained from } 2n\pi + \cos^{-1} \frac{c}{r} + \theta, \text{ while } \angle MOQ' \text{ is obtained from } 2n\pi - \cos^{-1} \frac{c}{r} + \theta. \quad (e) ON > OP$$

Exercise 2F (Page 65)

$$1(b)(i) \cos 50^\circ + \cos 20^\circ \quad (ii) \sin 80^\circ - \sin 16^\circ$$

$$(iii) \sin 4\alpha + \sin 2\alpha \quad (iv) \cos 2y - \cos 2x$$

$$2(b)(i) 2\cos 14^\circ \cos 2^\circ \quad (ii) 2\cos 38^\circ \sin 18^\circ$$

$$(iii) 2\sin 5x \cos x \quad (iv) -2\sin 2x \sin 3y$$

$$3(a)(ii) x = 0, \frac{\pi}{2} \text{ or } \pi \quad (b) x = \frac{\pi}{4}, \frac{\pi}{2} \text{ or } \frac{3\pi}{4}$$

$$4(a)(ii) -\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x + C$$

$$(b) \frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x + C$$

$$6(a) \frac{1}{24}(3\sqrt{3} - 4) \quad (b) \frac{1}{48}(3 - 2\sqrt{2})$$

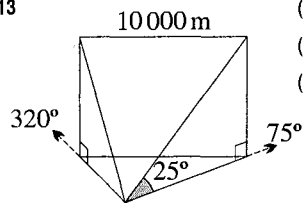
$$7(b)(i) 0 \quad (ii) 30\pi \quad (iii) 30\pi$$

$$9(a) 2\sin 2x \cos x \quad (b) x = 0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3} \text{ or } \frac{3\pi}{2}$$

- 10(a) $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ or $\frac{5\pi}{6}$
 (b) $x = 0, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{2\pi}{3}$ or π
 (c) $x = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}$ or $\frac{7\pi}{8}$
 (d) $x = 0, \frac{2\pi}{5}, \frac{\pi}{2}$ or $\frac{4\pi}{5}$ (e) $x = 0, \frac{2\pi}{5}, \frac{\pi}{2}$ or $\frac{4\pi}{5}$
 (f) $x = 0, \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{\pi}{2}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}$ or π
 11(a) $x = \frac{\pi}{12}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{3\pi}{4}$ or $\frac{7\pi}{8}$
 (b) $x = \frac{2n\pi}{5} + \frac{\pi}{10}$ or $2n\pi + \frac{\pi}{2}$, where $n \in \mathbf{Z}$.
 13(d) No. For example, substitute the values $n = 1$ and $\lambda = 1.1$.

Exercise 2G (Page 70)

- 1(a) $56^\circ 19'$ (b) 8.8 cm (c) $27^\circ 7'$
 2(a) $39^\circ 52', 35^\circ 33'$ (b) 72 metres
 3(b) 110 metres (c) 14°
 4(a) x (g) $35^\circ 16'$
 5(b) $35^\circ 16'$
 6(a) $63^\circ 26'$ (b) $54^\circ 44'$ (c) $53^\circ 8'$
 7 $54^\circ 44'$
 9(d) 5040 metres
 12(c) $67^\circ 23'$
 13



- (a) 10 000 metres
 (c) 3941 metres
 (d) 54°

- 14(a)(ii) $\cos^2 \alpha + \cos^2 \beta = 1$, where $\alpha + \beta = 90^\circ$.
 (b)(ii) $\sin^2 \theta + \sin^2 \phi = 1$, where $\theta + \phi = 90^\circ$.
 15 $16^\circ 16'$

Exercise 2H (Page 75)

- 1(a) $h \cot 55^\circ$
 (b) It is the angle between south and east.
 (d) 114 metres
 2(b) 13 metres
 3(a) $x \cot 27^\circ$
 4(c) 129 metres
 5(a) $AT = h \operatorname{cosec} 55^\circ$, $BT = h \operatorname{cosec} 40^\circ$ (b) 90°
 (d) 52 metres
 6(b) $PL = h \cot 9^\circ$, $QL = h \cot 12^\circ$
 8(a) $y = h \cot \beta$
 9(b)(i) $\sqrt{3}h, h, h$
 (ii) $\cos \alpha = \frac{100}{h}$ or $\frac{80000-h^2}{400h}$, $h = 200$ metres
 10(a) $BD = \sqrt{3}h$, $CD = h$
 11(a) $AC = 2\sqrt{a^2 + b^2}$, $AM = \sqrt{a^2 + b^2}$,
 $AT = \sqrt{a^2 + b^2 + h^2}$ (b) $\cos \alpha = \frac{-a^2 + b^2 + h^2}{a^2 + b^2 + h^2}$,
 $\cos \beta = \frac{a^2 - b^2 + h^2}{a^2 + b^2 + h^2}$, $\cos \theta = \frac{-a^2 - b^2 + h^2}{a^2 + b^2 + h^2}$

- 12(c) 17 metres
 14(b) 535 metres
 15(a) $PC = h$, $PD = \frac{1}{3}h\sqrt{3}$ (c) 305°
 16(b) $13^\circ 41'$
 17(b)(i) The foot of the tower is equidistant from P , Q and R , the distance being $h \cot 30^\circ$.

Chapter Three

Exercise 3A (Page 82)

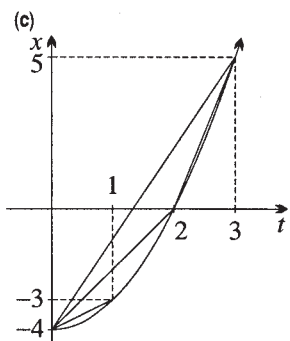
1(a) $x = -4, -3, 0, 5$

(b)(i) 1 m/s

(ii) 2 m/s

(iii) 3 m/s

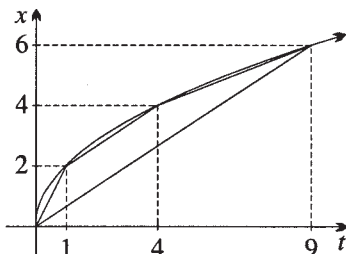
(iv) 5 m/s

2(a) $t = 0, 1, 4, 9, 16$

(b)(i) 2 cm/s

(ii) $\frac{2}{3}$ cm/s(iii) $\frac{2}{5}$ cm/s(iv) $\frac{2}{3}$ cm/s

(c) They are parallel.

3(a) $x = 0, 3, 4, 3, 0$

(b)(i) 2 m/s (ii) -2 m/s

(iii) 0 m/s

(d) The total distance travelled is 8 metres.

All three average speeds are 2 m/s.

4(a)(i) -1 m/s

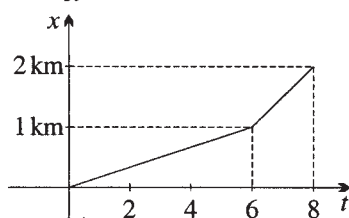
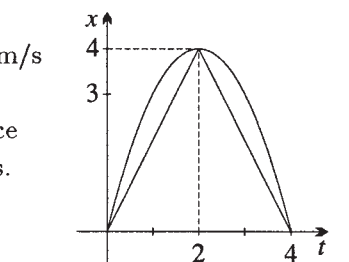
(ii) 4 m/s (iii) -2 m/s (b) 40 metres, $1\frac{1}{3}$ m/s(c) 0 metres, 0 m/s (d) $2\frac{2}{19}$ m/s

5(a)(i) 6 minutes

(ii) 2 minutes

(c) 15 km/hr

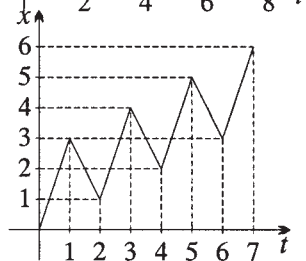
(d) 20 km/hr



6(b) 7 hours

(c) $2\frac{4}{7}$ m/hr (d) those

between 1 and 2 metres high or between 4 and 5 metres high



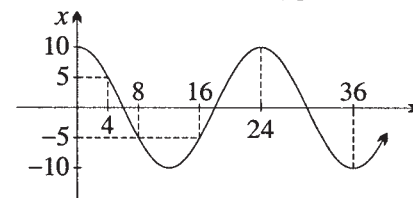
7(a)(i) once (ii) three times (iii) twice

(b)(i) when $t = 4$ and when $t = 14$ (ii) when $0 \leq t < 4$ and when $4 < t < 14$ (c) It rises 2 metres, at $t = 8$.(d) It sinks 1 metre, at $t = 17$.(e) As $t \rightarrow \infty$, $x \rightarrow 0$, meaning that eventually it ends up at the surface.(f)(i) -1 m/s (ii) $\frac{1}{2}$ m/s (iii) $-\frac{1}{3}$ m/s (g)(i) 4 metres

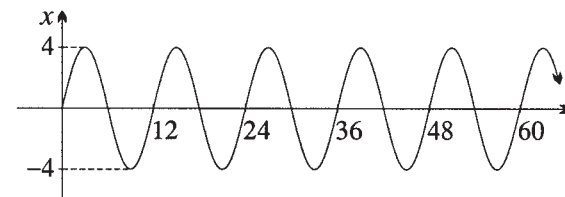
(ii) 6 metres (iii) 9 metres (iv) 10 metres

(h)(i) 1 m/s (ii) $\frac{3}{4}$ m/s (iii) $\frac{9}{17}$ m/s8(b) $t = 4, t = 20$ (c) $8 < t < 16$ (d) 12 cm, $\frac{3}{4}$ cm/s (e)(i) $t = \frac{8}{\pi} \sin^{-1} \frac{1}{3} \doteq 0.865$,and $t = 8 - \frac{8}{\pi} \sin^{-1} \frac{1}{3} \doteq 7.13$ (ii) 0.865 seconds and 7.13 seconds, 4 metres, 0.638 m/s

9 amplitude: 10 metres, period: 24 seconds

(c) The maximum is 20 metres, when $t = 12, 36$ and 60. (d) 100 metres, $1\frac{2}{3}$ m/s (e) It is at $x = 0$ when $t = 6, 18, 30, 42$ and 54. (f) 10, 5, -5,-10, -5, 5, 10 (g) $-1\frac{1}{4}$ m/s, $-2\frac{1}{2}$ m/s, $-1\frac{1}{4}$ m/s(h) $x = -5$ when $t = 8$ or $t = 16$, $x < -5$ when $8 < t < 16$.

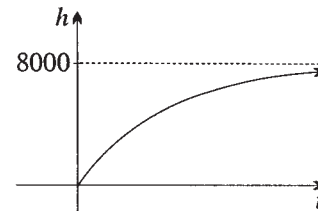
10(a) amplitude: 4 metres, period: 12 seconds

(b) 10 times (c) $t = 3, 15, 27, 39, 51$ (d) It travels 16 metres with average speed $1\frac{1}{3}$ m/s.(e) $x = 0$, $x = 2$ and $x = 4$, 2 m/s and 1 m/s11(a) When $t = 0$, $h = 0$.As $t \rightarrow \infty$, $h \rightarrow 8000$.

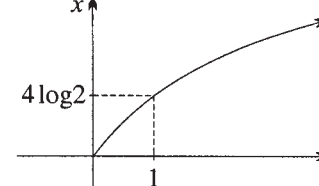
(b) 0, 3610, 5590, 6678

(d) 361 m/min, 198 m/min, 109 m/min

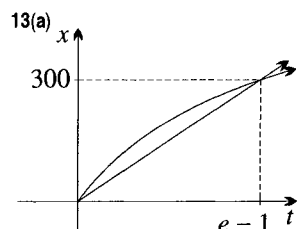
(f) 77 minutes



12(a)

(b) When $t = 2$, $x = 4 \log 3 \doteq 4.394$,the average speed is $2 \log 3 \doteq 2.197$ m/s,

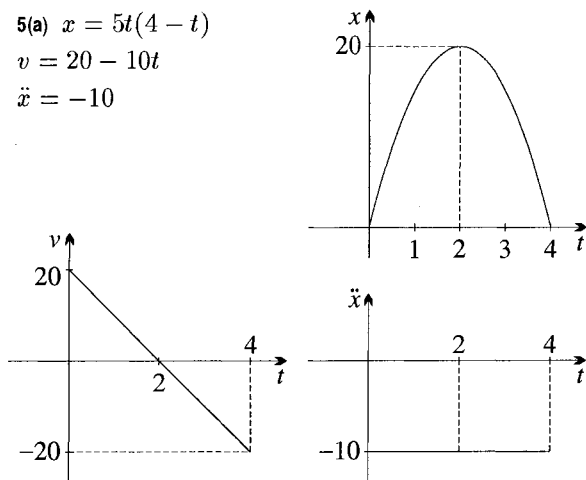
- $\angle AOP = 2 \log 3 \doteq 2.197$ radians,
 $AP^2 = 8(1 - \cos(2 \log 3))$, $AP \doteq 3.562$ metres.
 (c) $\angle AOB = 2 \log(t+1)$. The train is at A when
 $t = e^\pi - 1 \doteq 22$, when $t = e^{2\pi} - 1 \doteq 534$,
 and when $t = e^{3\pi} - 1 \doteq 12391$.
 (d) Since $2 \log(t+1) \rightarrow \infty$ as $t \rightarrow \infty$, the train
 will return to A infinitely many times.



- (c) The maximum distance is
 $300 \log(e-1) - \frac{300(e-2)}{e-1} \doteq 37$ metres
 when $t = e-2 \doteq 43''$.

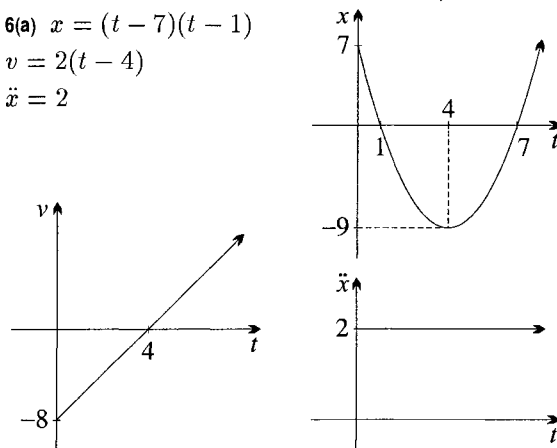
Exercise 3B (Page 89)

- 1(a) $v = 2t - 8$, $\ddot{x} = 2$, which is constant.
 (b) $x = -15$ metres, $v = 2$ m/s, $\ddot{x} = 2$ m/s²
 (c) When $t = 4$, $v = 0$ and $x = -16$.
 2(a) $v = 3t^2 - 12t - 1$, $\ddot{x} = 6t - 12$ (b) When $t = 0$,
 $x = 2$ cm, $|v| = 1$ cm/s, $\ddot{x} = -12$ cm/s². (c)(i) left
 $(x = -28)$ (ii) left ($v = -10$) (iii) right ($\ddot{x} = 6$)
 (d) When $t = 2$, $\ddot{x} = 0$ and $|v| = 13$ cm/s.
 3 $v = 2\pi \cos \pi t$, $\ddot{x} = -2\pi^2 \sin \pi t$ (a) When $t = 1$,
 $x = 0$, $v = -2\pi$ and $\ddot{x} = 0$. (b)(i) right ($v = \pi$)
 (ii) left ($\ddot{x} = -\pi^2 \sqrt{3}$)
 4 $v = -4e^{-4t}$, $x = 16e^{-4t}$ (a) e^{-4t} is positive
 for all t , so v is always negative, and $\ddot{x}$ is always
 positive. (b)(i) $x = 1$ (ii) $x = 0$ (c)(i) $v = -4$,
 $\ddot{x} = 16$ (ii) $v = 0$, $\ddot{x} = 0$
 5(a) $x = 5t(4-t)$
 $v = 20 - 10t$
 $\ddot{x} = -10$

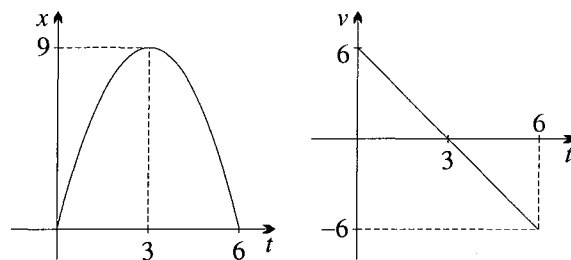


- (b) It returns at $t = 4$; both speeds are 20 m/s.
 (c) 20 metres after 2 seconds
 (d) -10 m/s². Although the ball is stationary, its
 velocity is changing, meaning that its acceleration
 is nonzero. (e) After 1 second, when $v = 10$ m/s,
 and after 3 seconds, when $v = -10$ m/s.

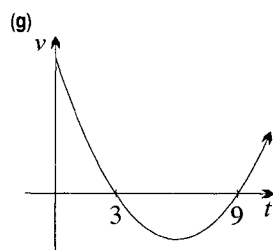
6(a) $x = (t-7)(t-1)$
 $v = 2(t-4)$
 $\ddot{x} = 2$



- (b)(i) $t = 1$ and $t = 7$ (ii) $t = 4$ (c)(i) 7 metres when
 $t = 0$ (ii) 9 metres when $t = 4$ (iii) 27 metres
 when $t = 10$ (d) -1 m/s, $t = 3\frac{1}{2}$, $x = -8\frac{3}{4}$
 (e) 25 metres, $3\frac{4}{7}$ m/s
 7(a) $x = t(6-t)$, $v = 2(3-t)$, $\ddot{x} = -2$



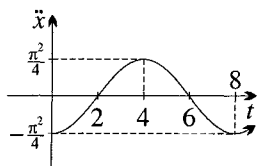
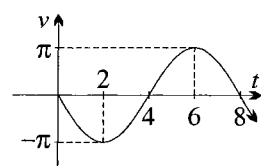
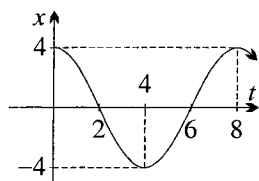
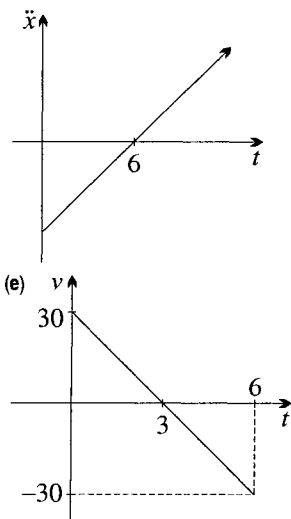
- (b)(i) When $t = 2$, it is moving upwards and accel-
 erating downwards. (ii) When $t = 4$, it is moving
 downwards and accelerating downwards.
 (c) $v = 0$ when $t = 3$. It is stationary for zero
 time, 9 metres up the plane, and is accelerating
 downwards at 2 m/s².
 (d) 4 m/s. When $v = 4$, $t = 1$ and $x = 5$.
 (e) All three average speeds are 3 m/s.
 8(a) 8 metres when $t = 3$ (b)(i) when $t = 3$ and
 $t = 9$ (ii) when $0 \leq t < 3$ and when $t > 9$
 (iii) when $3 < t < 9$ (c) $t = 9$, $v = 0$, accelerating
 forwards (d) $t = 6$, $x = 4$, moving backwards
 (e) $0 \leq t < 6$
 (f)(i) $t \doteq 4, 12$ (ii) $t \doteq 10$ (iii) $t \doteq 4, 8, 10$



- 9(a) 45 metres,
3 seconds, 15 m/s
(b) 30 m/s, 20, 10, 0,
-10, -20, -30
(c) 0 seconds
(d) The acceleration was
always negative.

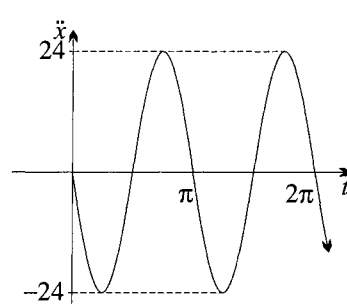
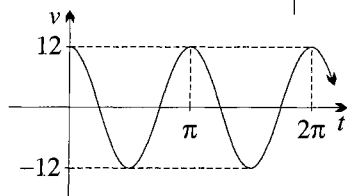
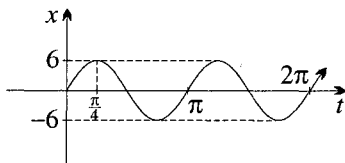
The velocity was decreasing at a constant rate of 10 m/s every second.

10(a) $x = 4 \cos \frac{\pi}{4}t$
 $v = -\pi \sin \frac{\pi}{4}t$
 $\ddot{x} = -\frac{1}{4}\pi^2 \cos \frac{\pi}{4}t$

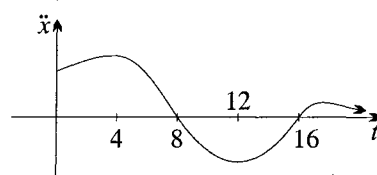
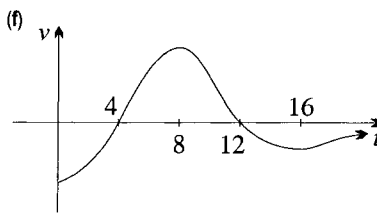


- (b) maximum displacement: $x = 4$ when $t = 0$
and $t = 8$, maximum velocity: π m/s when $t = 6$,
maximum acceleration: $\frac{1}{4}\pi^2$ m/s² when $t = 4$
(c) 40 metres, 2 m/s (d)(i) after $1\frac{1}{3}$ and $6\frac{2}{3}$ seconds
(ii) $1\frac{1}{3} < t < 6\frac{2}{3}$ (e)(i) after $4\frac{2}{3}$ and $7\frac{1}{3}$ seconds
(ii) $4\frac{2}{3} < t < 7\frac{1}{3}$

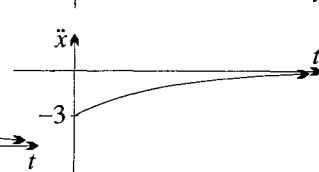
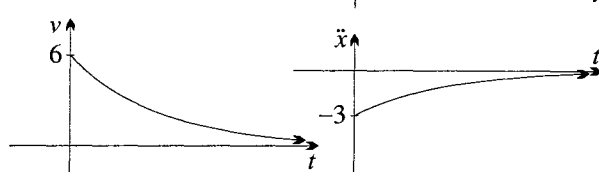
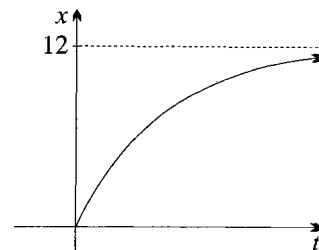
11(a) $x = 6 \sin 2t$
 $v = 12 \cos 2t$
 $\ddot{x} = -24 \sin 2t$



- (b) $\ddot{x} = -4x$ (c)(i) $x = 0$ when $t = 0, \frac{\pi}{2}$ or π .
(ii) $v = 0$ when $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. (iii) same as (i)
(d)(i) $x < 0$ when $\frac{\pi}{2} < t < \pi$.
(ii) $v < 0$ when $\frac{\pi}{4} < t < \frac{3\pi}{4}$.
(iii) $\ddot{x} < 0$ when $0 < t < \frac{\pi}{2}$. (e)(i) $t = \frac{\pi}{12}$ (ii) $t = \frac{\pi}{6}$
12(a)(i) $0 \leq t < 8$ (ii) $0 \leq t < 4$ and $t > 12$
(iii) roughly $8 < t < 16$ (b) roughly $t = 8$
(c)(i) $t \doteq 5, 11, 13$ (ii) $t \doteq 13, 20$
(iii) $t \doteq 5, 11, 13, 20$ (d) twice (e) 17 units



13(a) $x = 12 - 12e^{-0.5t}$
 $v = 6e^{-0.5t}$
 $\ddot{x} = -3e^{-0.5t}$



- (b)(i) downwards (Downwards is positive here.)
(ii) upwards (c) The velocity and acceleration tend
to zero, and the position tends to 12 metres be-
low ground level. (d) $x = 6$ when $e^{-0.5t} = \frac{1}{2}$,
that is, $t = 2 \log 2$. The speed then is 3 m/min

(half the initial speed of 6 m/min) and the acceleration is $-1\frac{1}{2}$ m/min² (half the initial acceleration of -3 m/min²). (e) 19 minutes

14(a) The displacement of M is the average of x_A and x_B , $v_M = 2e^{-t}(t^2 - 3t + 1)$, M returns to O after 1 second, when M is moving left at a speed of $2/e$. (b) The particle is furthest right at $t = \frac{1}{2}(3 - \sqrt{5})$, and furthest left at $t = \frac{1}{2}(3 + \sqrt{5})$. (c) They all move towards O . (d) $t = \frac{1}{2}(1 + \sqrt{5})$

15(a) $0 \leq x \leq 2r$ (b)(i) $dx/d\theta = \frac{2r \sin \theta}{\sqrt{5 - 4 \cos \theta}}$.

M is travelling upwards when $0 < \theta < \pi$.

(ii) M is travelling downwards when $\pi < \theta < 2\pi$.

(c) The speed is maximum when $\theta = \frac{\pi}{3}$ (when $\frac{dx}{d\theta} = r$) and when $\theta = \frac{5\pi}{3}$ (when $\frac{dx}{d\theta} = -r$).

(d) When $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$, $\angle APC$ is a right angle, so AP is a tangent to the circle. At these places, P is moving directly towards A or directly away from A , and so the distance AP is changing at the maximum rate. Again because AP is a tangent, $dx/d\theta$ at these points must equal the rate of change of arc length with respect to θ , which is r or $-r$ when $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ respectively.

16 $\sin \alpha = 2/g \doteq 0.20408$, $\alpha \doteq 11^\circ 47'$

Exercise 3C (Page 96)

1(a) $v = -4t$, $x = -2t^2$ (b) $v = 3t^2$, $x = t^3$

(c) $v = 2e^{\frac{1}{2}t} - 2$, $x = 4e^{\frac{1}{2}t} - 2t - 4$

(d) $v = -\frac{1}{3}e^{-3t} + \frac{1}{3}$, $x = \frac{1}{9}e^{-3t} + \frac{1}{3}t - \frac{1}{9}$

(e) $v = -4 \cos 2t + 4$, $x = -2 \sin 2t + 4t$

(f) $v = \frac{1}{\pi} \sin \pi t$, $x = -\frac{1}{\pi^2} \cos \pi t + \frac{1}{\pi^2}$

(g) $v = \frac{2}{3}t^{\frac{3}{2}}$, $x = \frac{4}{15}t^{\frac{5}{2}}$

(h) $v = -12(t+1)^{-1} + 12$, $x = -12 \log(t+1) + 12t$

2(a) $\ddot{x} = 0$, $x = -4t - 2$ (b) $\ddot{x} = 6$, $x = 3t^2 - 2$

(c) $\ddot{x} = \frac{1}{2}e^{\frac{1}{2}t}$, $x = 2e^{\frac{1}{2}t} - 4$

(d) $\ddot{x} = -3e^{-3t}$, $x = -\frac{1}{3}e^{-3t} - \frac{1}{3}$

(e) $\ddot{x} = 16 \cos 2t$, $x = -4 \cos 2t + 2$

(f) $\ddot{x} = -\pi \sin \pi t$, $x = \frac{1}{\pi} \sin \pi t - 2$

(g) $\ddot{x} = \frac{1}{2}t^{-\frac{1}{2}}$, $x = \frac{2}{3}t^{\frac{3}{2}} - 2$

(h) $\ddot{x} = -24(t+1)^{-3}$, $x = -12(t+1)^{-1} + 10$

3(a) $v = 10t$, $x = 5t^2$ (b)(i) 4 seconds (ii) 40 m/s

(c) After 2 seconds, it has fallen 20 metres, and its speed is 20 m/s. (d) It is halfway down after $2\sqrt{2}$ seconds, and its speed then is $20\sqrt{2}$ m/s.

4(a) $\ddot{x} = -10$, $v = -10t - 25$, $x = -5t^2 - 25t + 120$

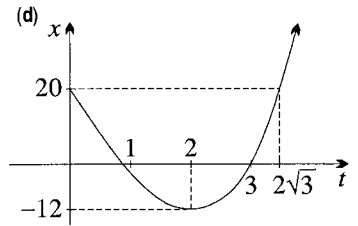
(i) 3 seconds (ii) 55 m/s (b) $\ddot{x} = 10$, $v = 10t + 25$, $x = 5t^2 + 25t$. Put $5t^2 + 25t = 120$.

5(a) $v = 6t^2 - 24$,
 $x = 2t^3 - 24t + 20$

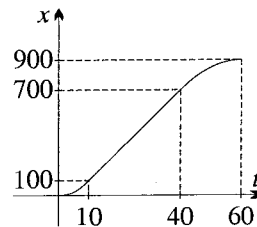
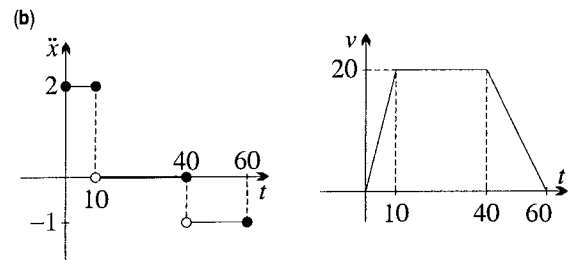
(b) $t = 2\sqrt{3}$,

$|v| = 48$ m/s

(c) $x = -12$ when $t = 2$.



6(a) 20 m/s, 900 metres



7(a)(i) $\int_1^2 \frac{4}{t+1} dt = 4(\log 3 - \log 2)$

(ii) $\int_1^2 \frac{4}{\log(t+1)} dt \doteq 4.54$

(b)(i) $\int_1^2 \sin \pi t dt = -\frac{2}{\pi}$

(ii) $\int_1^2 t \sin \pi t dt = -\frac{1}{4}(1 + 2\sqrt{2}) \doteq -0.957$

8(a) $\ddot{x} = 6t$, $v = 3t^2 - 9$

(b) $x = t^3 - 9t + C_1$, 3 seconds

10(a) $v = -5 + 20e^{-2t}$, $x = -5t + 10 - 10e^{-2t}$,
 $t = \log 2$ seconds (b) It rises $7\frac{1}{2} - 5 \log 2$ metres,
when acceleration is 10 m/s² downwards.

(c) The velocity approaches a limit of 5 m/s downwards, called the *terminal velocity*.

11 $e - 1$ seconds, $v = 1/e$, $\ddot{x} = -1/e^2$.

The velocity and acceleration approach zero, but the particle moves to infinity.

12(a) $x = t^2(t-6)^2$, after 6 seconds, 0 cm/s

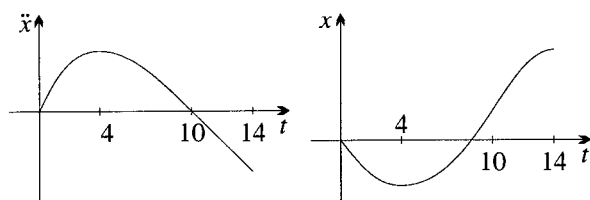
(b) 162 cm, 27 cm/s (c) $\ddot{x} = 12(t^2 - 6t + 6)$,
 $24\sqrt{3}$ cm/s after $3 - \sqrt{3}$ and $3 + \sqrt{3}$ seconds.

(d) The graphs of x , v and $\ddot{x}$ are all unchanged by reflection in $t = 3$, but the mouse would be running backwards!

13(a) $4 < t < 14$ (b) $0 < t < 10$ (c) $t = 14$

(d) $t = 4$ (e) $t \doteq 8$

(f)



- 14(a) $\ddot{x} = -4$, $x = 16t - 2t^2 + C$ (b) $x = C$ after 8 seconds, when the speed is 16 cm/s. (c) $v = 0$ when $t = 4$. Maximum distance right is 32 cm when $t = 4$, maximum distance left is 40 cm when $t = 10$. The acceleration is -4 cm/s^2 at all times.
(d) 104 cm, 10.4 cm/s

15(a) $v = 1 - 2 \sin t$, $x = t + 2 \cos t$

- (b) $\frac{\pi}{2} < t < \frac{3\pi}{2}$ (c) $t = \frac{\pi}{6}$ when $x = \frac{\pi}{6} + \sqrt{3}$, and $\frac{5\pi}{6}$ when $x = \frac{5\pi}{6} - \sqrt{3}$, $\frac{\pi}{6} < t < \frac{5\pi}{6}$.

(d) 3 m/s when $t = \frac{3\pi}{2}$

and $x = \frac{3\pi}{2}$,

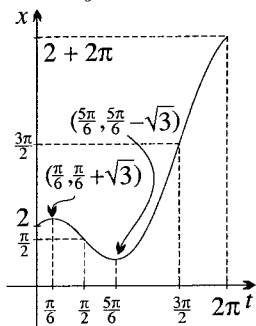
-1 m/s when $t = \frac{\pi}{2}$

and $x = \frac{\pi}{2}$.

(e) 2 π metres, 1 m/s

(f) $4\sqrt{3} + \frac{2\pi}{3}$ metres,

$\frac{1}{3} + \frac{2}{\pi}\sqrt{3} \text{ m/s}$



16(a) $x_1 = 2 + 6t + t^2$, $x_2 = 1 + 4t - t^2$,

$D = 1 + 2t + 2t^2$ (b) D is never zero, the minimum distance is 1 metre at $t = 0$ (t cannot be negative).

(c) $v_M = 5 \text{ m/s}$, $12\frac{1}{2}$ metres

17(a) Thomas, by 15 m/s (b) $x_T = 20 \log(t+1)$,

$x_H = 5t$ (c) during the 10th second, $3\frac{2}{11} \text{ m/s}$

(d) after 3 seconds, by 13 metres

18(a) For $V \geq 30 \text{ m/s}$, they collide after $180/V$ seconds, $\frac{180}{V^2}(V^2 - 900)$ metres above the valley floor.

(b) $V = 30\sqrt{2} \text{ m/s}$, $3\sqrt{2}$ seconds

19(a) $v = 5(e^{-2t} - 1)$, $x = \frac{5}{2}(1 - e^{-2t}) - 5t$

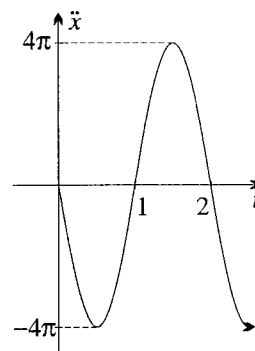
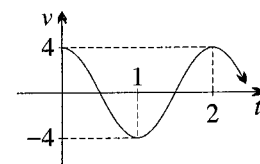
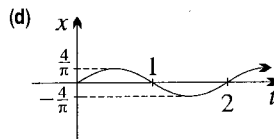
(b) The speed gradually increases with limit 5 m/s (the terminal velocity).

Exercise 3D (Page 105)

1(a) $v = 4 \cos \pi t$, $\ddot{x} = -4\pi \sin \pi t$

(b) $a = \frac{4}{\pi}$, $T = 2$ seconds, centre at O

(c) 4 m/s, $4\pi \text{ m/s}$, $\frac{4}{\pi}$ metres



- (e) $t = 1$ (when $v = -4 \text{ m/s}$) and $t = 2$ (when $v = 4 \text{ m/s}$) (f) $t = \frac{1}{2}$ (when $\ddot{x} = -4\pi \text{ m/s}^2$) and $t = 1\frac{1}{2}$ (when $\ddot{x} = 4\pi \text{ m/s}^2$)

2(a) $n = \frac{\pi}{2}$ and $a = 12$, so $x = 12 \cos \frac{\pi}{2}t$.

(b) $v = -6\pi \sin \frac{\pi}{2}t$, $\ddot{x} = -3\pi^2 \cos \frac{\pi}{2}t$ (c) 2 seconds

3(a) $x = 4 \sin 3t$ (b) $x = 2 \sin 6t$ (c) $x = \frac{3}{2} \cos 8t$

(d) $x = 16 \cos \frac{3}{4}t$

4(a) $x = 2 \sin 2t$, $v = 4 \cos 2t$, $-2 \leq x \leq 2$

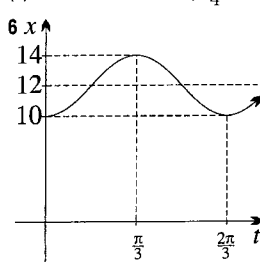
(b) $x = 6 \sin \frac{2}{3}t$, $v = 4 \cos \frac{2}{3}t$, 3π seconds

5(a) $v = bn \cos nt - cn \sin nt$,

$\ddot{x} = -bn^2 \sin nt - cn^2 \cos nt = -n^2x$

(b) $c = 3$ and $b = 0$, so $x = 3 \cos nt$.

(c) $x = 5 \cos 2\pi t$, $\frac{1}{4} \text{ s}$



(a) $v = 6 \sin 3t$,

$\ddot{x} = 18 \cos 3t$

(b) $a = 2$,

$T = \frac{2\pi}{3}$ seconds,

centre $x = 12$

(c) $10 \leq x \leq 14$,

$\frac{\pi}{3}$ seconds

- (d) $t = \frac{2\pi}{3}$ and $t = \frac{4\pi}{3}$. At both times, $|v| = 0$ and $\ddot{x} = 18 \text{ cm/s}^2$. (e) $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$.

At both times, $|v| = 6 \text{ cm/s}$ and $\ddot{x} = 0 \text{ cm/s}^2$.

7(a) amplitude: 6, period: π , initial phase: $\frac{\pi}{2}$

(b) $\dot{x} = 12 \cos(2t + \frac{\pi}{2})$, $\ddot{x} = -24 \sin(2t + \frac{\pi}{2})$,

$\ddot{x} = -4x$, so $n = 2$.

(c) $t = \frac{\pi}{4}$ when $v = -12$, $t = \frac{3\pi}{4}$ when $v = 12$

(d) $t = \frac{3\pi}{4}$ and $t = \frac{7\pi}{4}$, when $x = 0$

(e) $t = \pi$ and $t = 2\pi$, when $v = 0$ and $\ddot{x} = -24$

8(a)(i) Use expansions of $\sin(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

(ii) The graph of $x = \sin t$ shifted left $\frac{\pi}{2}$ is

$x = \cos t$. The graph of $x = \cos t$ shifted right $\frac{\pi}{2}$ is $x = \sin t$.

(b)(i) $\sin(t - \frac{\pi}{2}) = -\cos t$, $\cos(t + \frac{\pi}{2}) = -\sin t$

(ii) The graph of $x = \sin t$ shifted right $\frac{\pi}{2}$ is $x = -\cos t$. The graph of $x = \cos t$ shifted left $\frac{\pi}{2}$ is $x = -\sin t$.

- 9(a) $x = 120 \sin \frac{\pi}{12}t$, $v = 10\pi \cos \frac{\pi}{12}t$, 10π m/s
 (b)(i) $\frac{12}{\pi} \sin^{-1} \frac{1}{4} \doteq 0.9652$ seconds
 (ii) $12 + \frac{12}{\pi} \sin^{-1} \frac{1}{4} \doteq 12.97$ seconds (c) 4 seconds and 8 seconds

- 10(a) $x = 4 \cos 4t$, $v = -16 \sin 4t$ (b)(i) $\frac{\pi}{12}$ s (ii) $\frac{\pi}{6}$ s
 (c) $\frac{\pi}{24}$ seconds and $\frac{5\pi}{24}$ seconds

- 11 $x = \frac{1}{2} - \frac{1}{2} \cos 2t$, $x_0 = \frac{1}{2}$, $\frac{1}{2}$, $0 \leq x \leq 1$, π

- 12(a) $x = 2 - \cos 4t$ (b) $x_0 = 2$, 1 cm, $1 \leq x \leq 3$, $\frac{\pi}{2}$ s (c) 4 cm/s when $t = \frac{\pi}{8}$

- 13 $v = bn \cos nt - cn \sin nt$

- (a) $x = 6 \sin \frac{1}{2}t + 6 \cos \frac{1}{2}t$, $\frac{3\pi}{2}$ s and $\frac{7\pi}{2}$ s

- (b) $x = \frac{9}{\pi} \sin \frac{\pi}{3}t - 2 \cos \frac{\pi}{3}t$, about 0.582 s and 3.582 s

- 14(a) 4 times, $a = 4$,

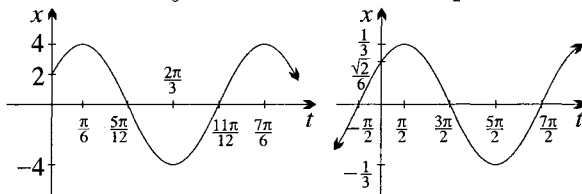
- $T = \pi$, $x = 4 \cos 2t$

- shifted right $\frac{\pi}{6}$

- (b) once, $a = \frac{1}{3}$,

- $T = 4\pi$, $x = \frac{1}{3} \sin \frac{1}{2}t$

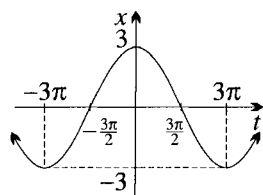
- shifted left $\frac{\pi}{2}$



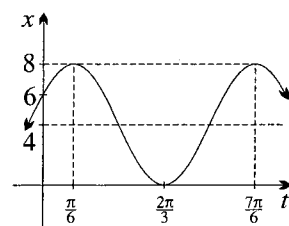
- (c) once, $a = 3$,

- $T = 6\pi$, $x = -3 \cos \frac{1}{3}t$

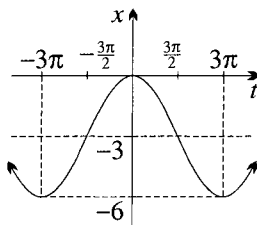
- shifted left 3π



- 15(a) twice



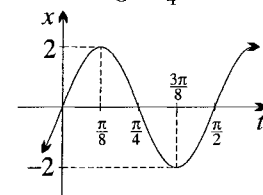
- (c) once



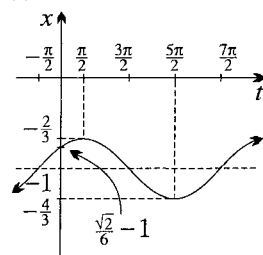
- (d) 9 times, $a = 2$,

- $T = \frac{\pi}{2}$, $x = -2 \sin 4t$

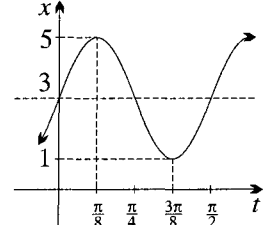
- shifted right $\frac{\pi}{4}$



- (b) never



- (d) never



- 16 $v = an \cos(nt + \alpha)$ (a) $n = \frac{\pi}{3}$, $\alpha = 0$, $a = \frac{15}{\pi}$
 (b) $n = \frac{2}{3}$, $\alpha = \frac{3\pi}{2}$, $a = 5$ (c) $n = 1$, $\alpha = \frac{3\pi}{4}$, $a = \sqrt{2}$

- 17 $v = -2a \sin(2t - \varepsilon)$ (a) $\varepsilon = \frac{\pi}{2}$, $a = 3$

- (b) $\varepsilon = \frac{5\pi}{3}$, $a = 2$

- 18 $a = \frac{32}{\pi} \sqrt{2}$, $\alpha = \frac{\pi}{4}$

- 19 $a = 5$, $\alpha \doteq 2.248$

- 20(a) $x = 4 \sin 4t + 3 \cos 4t$ (b) $x = 5 \cos(4t - \varepsilon)$, where $\varepsilon = \tan^{-1} \frac{4}{3}$ (c) 5 m, 20 m/s

- (d) $t = \frac{\pi}{4} - \frac{1}{4} \tan^{-1} \frac{3}{4}$, $t = \frac{\pi}{8} + \frac{1}{4} \tan^{-1} \frac{4}{3}$

- 21(a) $x = 24 \sin \frac{1}{4}t + 4 \cos \frac{1}{4}t$, $t = 8 \tan^{-1} 6 \doteq 11.2$

- (b) $x = 4\sqrt{37} \cos(\frac{1}{4}t - \alpha)$, where $\alpha = \tan^{-1} 6$

- 22(a) $x = \frac{2}{77}t(18 - t)$, 2, $\frac{162}{77}$

- (b) $x = -4 \sin \frac{\pi}{6}t$, 2, 4

- 23(a) 10:00 am (b) 7:33 am (c) 12:27 pm

- 24 11:45 am to 8:15 pm

- 25(a) $x = 2\sqrt{5} \cos(3\pi t - \varepsilon)$, where $\varepsilon = \pi - \tan^{-1} \frac{1}{2} \doteq 2.678$. (b)(i) 0.195 seconds (ii) 0.287 seconds

- 26 $x = \frac{v(0)}{n} \sin nt + x(0) \cos nt$,

- $\frac{1}{n} \sqrt{n^2 x(0)^2 + v(0)^2}$

- 28(a) $\sin nt + \sin(nt + \alpha) \equiv 2 \cos \frac{1}{2}\alpha \sin(nt + \frac{1}{2}\alpha)$

Exercise 3E (Page 112)

- 1(a) $t = \frac{1}{6}(x - 1)$, $x = 6t + 1$

- (b) $t = \frac{1}{18}(1 - x^3)$, $x = (1 - 18t)^{\frac{1}{3}}$

- (c) $t = \frac{1}{2} \log(2x - 1)$, $x = \frac{1}{2}(e^{2t} + 1)$

- (d) $t = \frac{1}{6}(x^{-1} - 1)$, $x = (1 + 6t)^{-1}$

- (e) $t = \frac{1}{12}(x^{-2} - 1)$, $x = (12t + 1)^{-\frac{1}{2}}$

- (f) $t = \frac{1}{2}(e^{2x} - e^2)$, $x = \frac{1}{2} \log(2t + e^2)$

- (g) $t = \tan^{-1} x - \frac{\pi}{4}$, $x = \tan(t + \frac{\pi}{4})$

- (h) $t = \tan x - \tan 1$, $x = \tan^{-1}(t + \tan 1)$

- 2(a) $\ddot{x} = 0$ (b) $\ddot{x} = -72x^{-5}$ (c) $\ddot{x} = 2(2x - 1)$

- (d) $\ddot{x} = 72x^3$ (e) $\ddot{x} = 108x^5$ (f) $\ddot{x} = -2e^{-4x}$

- (g) $\ddot{x} = 2x(1 + x^2)$ (h) $\ddot{x} = -2 \cos^3 x \sin x$

- 3(a) $v^2 = 4x^3$ (b) $v^2 = 2(1 - e^{-x})$

- (c) $v^2 = 12x$ (d) $v^2 = \log(2x + 1)$

- (e) $v^2 = \frac{1}{3}(1 - \cos 6x)$ (f) $v^2 = \tan^{-1} \frac{1}{2}x$

- 4(a) 100 m/s (b) Downwards is positive, so while the stone is falling, its velocity is positive.

- (c) $x = 5t^2$, 10 seconds

- 5(a) 20 metres (b) Upwards is positive, so while the stone is rising, its velocity is positive.

- (c) $t = \frac{1}{10}(20 - \sqrt{400 - 20x})$, $x = 20t - 5t^2$, 2 seconds

7(a) $v^2 = 2(4 - x^2)$, $\sqrt{6}$ cm/s. It starts at $x = 2$, so on the first occasion it reaches $x = 1$, it must be moving backwards.

(b) $-2 \leq x \leq 2$, $2\sqrt{2}$ cm/s

8(a) $t = \frac{1}{8}x^3$, $x = 2t^{\frac{1}{3}}$ (b) $v = \frac{2}{3}t^{-\frac{2}{3}}$, $\ddot{x} = -\frac{4}{9}t^{-\frac{5}{3}}$
(c) $\ddot{x} = -\frac{128}{9}x^{-5}$

9(b) $v^2 = 2x \log x$ (c) $\ddot{x}$ is initially positive, and the particle moves off in the positive direction; thus x remains greater than 1, $\ddot{x}$ remains positive, and the particle continues moving in that direction. $v = 2e$.

10(a) $v^2 = \frac{1}{3} \tan^{-1} \frac{1}{6}x$. The acceleration is always positive, and the velocity is initially zero, so for $t > 0$ the velocity is always positive.

(b)(i) $\sqrt{\frac{\pi}{12}}$ (ii) $\sqrt{\frac{\pi}{6}}$

11(a) $\ddot{x} = -\frac{5}{2}$ m/s, $v^2 = 10\,000 - 5x$

(b)(i) $v = 50\sqrt{2}$ m/s (ii) $x = 1500$ metres

(c) The plane is still moving forwards while it is braking. (d) $x = 100t - \frac{5}{4}t^2$, 40 seconds

12(a) $v^2 = 6 - 2e^{-x}$. The acceleration is always positive, and the velocity is initially 2. Hence the velocity is always increasing with minimum 2. The particle continues to accelerate, but with limiting velocity $\sqrt{6}$.

(b) $v^2 = V^2 + 2(1 - e^{-x})$, $V = -\sqrt{2(e-1)}$. The particle has limiting velocity $\sqrt{2e}$.

13(a) The velocity $\cos^2 2x$ can never be negative, so the particle can never be moving backwards, so it can never return to anywhere it has already been. (b) $x = \frac{1}{2} \tan^{-1} 2t$, $x \rightarrow \frac{\pi}{4}$, $v = \frac{1}{1+4t^2}$

(c) $t = \frac{1}{2}$, $v = \frac{1}{2}$, $\ddot{x} = -1$

14(a) $\ddot{x} = -12$ (b) $x = 3(1 - e^{-2t})$ (c) As $t \rightarrow \infty$, the particle moves to the limiting position $x = 3$.

15(a) The velocity is positive everywhere, so it can never be on the negative side of its initial position. $\ddot{x} = 2x^3 e^{-2x^2}(1 - x^2)$, maximum $1/e$ at $x = 1$
(b) 1.695

16(a) $v^2 = e^{-x}$ (b) v is initially positive, and is never zero. $x = 2 \log \frac{1}{2}(t+2)$. As $t \rightarrow \infty$, $x \rightarrow \infty$ (slowly) and $v \rightarrow 0$.

17(a) $v^2 = 2(x-5)(x-4)$. v^2 cannot be negative. (b) $x = 6$ ($x = -5$ is impossible, because the particle can never pass through the origin). The particle moves forwards with increasing velocity.

18 $x_A = \frac{\pi}{16} - \frac{1}{4} \tan^{-1} \frac{1}{4}x$, they take $\frac{\pi}{16}$ seconds, B is released from $x = 2\sqrt{2}$, $x_B = \frac{\pi}{16} - \frac{1}{4} \sin^{-1} \frac{1}{4}x$.

19(a) $v^2 = 2x^3$ (b) Initially, v is negative. Since $v^2 = 2x^3$, v can only be zero at the origin. But since $\ddot{x} = 3x^2$ the acceleration at the origin would also be zero. Hence if the particle ever arrived at the origin, it would then be permanently at rest. Thus the velocity can never change from negative to positive, and must always be negative or zero. $x = \frac{2}{(t+\sqrt{2})^2}$. The particle starts at $x = 1$ and moves backwards towards the origin, its speed having limit zero, and position having limit the origin.

20(a) $v^2 = 49 - (x-7)^2$, $0 \leq x \leq 14$, maximum speed 7 at $x = 7$ (b) $\ddot{x} = 7 - x$, $4 \leq x \leq 10$

21(b) $v = 0$, $\ddot{x} = -15$. It moves off in the negative direction. (c) $6\sqrt{2}$ at the origin. It oscillates between $x = 3$ and $x = -2$.

22(a) $v^2 = 10^{14} + 2k(1 - \cos \pi x)$. Since $1 - \cos \pi x$ is never negative, v^2 never drops below the square of its initial value. The velocity is minimum at $x = 0, 2, 4, \dots$ and maximum at $x = 1, 3, 5, \dots$

(b) $k = \frac{3}{4} \times 10^{14}$, $\frac{3}{4}\pi \times 10^{14}$ m/s² at $x = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$

23 $x = 2^{\frac{1}{10}t+1}$, $x = 8\sqrt{2}$

25(b) $v^2 = V^2 + 2gR^2(1/x - 1/R)$,

$H = 2gR^2/(2gR - V^2)$ (c) 11.2 km/s

26(a) $x = 150 - \frac{50 \log v}{\log 10}$, $x = 150$ metres

(b) $t = \frac{1000}{99} \left(\frac{1}{v} - \frac{1}{1000} \right)$, $t = 10\frac{1}{11}$ s

27(a) $v = 500 - 5x$, $x = 100(1 - e^{-5t})$

(b) The bullet moves to a limiting position of $x = 100$ as the velocity decreases to zero.

Exercise 3F (Page 120)

1(a) $v = -6 \sin 2t$, $\ddot{x} = -12 \cos 2t$, $v^2 = 4(9 - x^2)$, $\ddot{x} = -4x$ (b) $2\sqrt{5}$ m/s or $-2\sqrt{5}$ m/s, -8 m/s²

2(a) $v^2 = 9(25 - x^2)$ (b) $v = 12$ m/s or $v = -12$ m/s, $\ddot{x} = -27$ m/s² (c) 15 m/s, $\frac{2\pi}{3}$ seconds

3(a) $v^2 = 16(36 - x^2)$ (b) 6 cm, $\frac{\pi}{2}$ seconds

(c) $|v| = 16\sqrt{2}$ cm/s, $\ddot{x} = -32$ cm/s²

4(a) $v^2 = 4(36 - x^2)$, π seconds, 12 m/s

(b)(i) $x = 6 \cos 2t$ (ii) $x = -6 \cos 2t$ (iii) $x = 6 \sin 2t$

(iv) $x = -6 \sin 2t$

5(a) 32 cm/min (b) 8 cm

6(a) $x = \sin 4t$, $v = 4 \cos 4t$, $a = 1$ metre

- (b) $\frac{4\pi}{3}$ m/s, $\frac{8\pi^2}{9}$ m/s²
 7(a) $\frac{5\pi}{2}$ cm/s, $\frac{5\pi^2}{8}$ cm/s²
 (b) 2π cm/s, $\frac{3\pi^2}{8}$ or $-\frac{3\pi^2}{8}$ cm/s²
 8 $5\sqrt{2}$ m/s, $3\sqrt{2}$ m/s
 9(a) $a = 4$, $2\sqrt{7}$ or $-2\sqrt{7}$ m/s
 (b) 4 m/s, 6 or -6 m/s²
 10(a) $v^2 = \frac{225}{4}(4 - x^2)$, $v = 10\sqrt{2}$ or $-10\sqrt{2}$ m/s,
 $\ddot{x} = -37\frac{1}{2}$ m/s²
 (b) $v^2 = -\frac{5}{3}x^2 + \frac{1280}{3}$, amplitude 16
 11(b)(i) When $x = 0$, $|v| = an$.
 (ii) When $x = \frac{1}{2}a$, $|v| = \frac{1}{2}\sqrt{3}an$ and $\ddot{x} = -\frac{1}{2}an^2$.
 13 15 cm/s
 14(a) $\ddot{x} = -9(x - 1)$, centre: $x = 1$, period: $\frac{2\pi}{3}$,
 amplitude: 2 (b)(i) $\ddot{x} = -16(x - 2)$, centre: $x = 2$,
 period: $\frac{\pi}{2}$, amplitude: 3 (ii) $\ddot{x} = -9(x - 6)$,
 centre: $x = 6$, period: $\frac{2\pi}{3}$, amplitude: 4
 (iii) $\ddot{x} = -2(x + 2)$, centre: $x = -2$, period: $\pi\sqrt{2}$,
 amplitude: 1 (iv) $\ddot{x} = -3(x + \frac{5}{3})$, centre: $x = -\frac{5}{3}$,
 period: $2\pi/\sqrt{3}$, amplitude: $2\frac{1}{3}$
 15(a)(i) $\ddot{x} = 50 \cos 10t = 100(\frac{1}{2} - x)$
 (ii) $\ddot{x} = 50(1 - 2\sin^2 5t) = 100(\frac{1}{2} - x)$
 (b) centre: $x = \frac{1}{2}$, range: $0 \leq x \leq 1$,
 period: $\frac{\pi}{5}$ minutes, $t = \frac{\pi}{5}$
 16(a) centre: $x = 7$. Since the amplitude is 7, the
 extremes of motion are $x = 0$ and $x = 14$, and the
 particle is stationary there.
 (b) $v^2 = 9(49 - (x - 7)^2)$, 21 cm/s
 (c) Although the particle is stationary for an in-
 stant, its acceleration at that time is positive (it
 is actually 63 m/s²), and so the speed immediately
 changes and the particle moves away.
 17(a) $\ddot{x} = -9x$ (b) period: $\frac{2\pi}{3}$, amplitude: $2\sqrt{13}$,
 maximum speed $6\sqrt{13}$, $|\ddot{x}| = 9\sqrt{13}$
 18(a) $x = 3$, $\frac{\pi}{2}$ (b) $x = 3 + 2\sin(4t + \frac{\pi}{3})$
 (c) $t = \frac{\pi}{12}(3n - 1)$, where n is a positive integer,
 $|v| = 8$
 19(a) $\ddot{x} = -4(x - 10)$, centre: $x = 10$, period: π ,
 amplitude: 10
 (b) $\frac{3\pi}{4} - \frac{1}{2}\tan^{-1}\frac{3}{4}$ ($= \pi - \tan^{-1}2 \doteq 2.034$)
 20(b) $\dot{x} = -16\pi \sin 2\pi t$, $\dot{y} = 16\pi \cos 2\pi t$,
 $\ddot{x} = -32\pi^2 \cos 2\pi t$, $\ddot{y} = -32\pi^2 \sin 2\pi t$
 (c)(i) $\frac{\pi}{6}$ or $\frac{7\pi}{6}$ (ii) $\frac{\pi}{3}$ or $\frac{4\pi}{3}$ (iii) $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$
 21(b) $a = \sqrt{\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_1^2 - v_2^2}}$
 (c) 5 cm, π seconds, 10 cm/s
 22 $v = \frac{1}{2}V\sqrt{3}$ or $v = -\frac{1}{2}V\sqrt{3}$,
 $x = \frac{1}{2}a\sqrt{3}$ or $x = -\frac{1}{2}a\sqrt{3}$

23(b) When $\alpha = \pi$, $A = 3$ and $x = 3 \sin t$.

When $\alpha = 0$, $A = 1$ and $x = -\sin t$.

(c) twice (d) When $\alpha = \frac{\pi}{3}$, $x = \sqrt{3} \cos t$.

When $\alpha = \frac{5\pi}{3}$, $x = -\sqrt{3} \cos t$.

Exercise 3G (Page 127)

- 1(a) $\dot{x} = 6\sqrt{3}$, $\dot{y} = 6$ (b) $\dot{x} = 4\sqrt{2}$, $\dot{y} = -4\sqrt{2}$
 (c) $\dot{x} = 12$, $\dot{y} = 16$
 2(a) $v = 6\sqrt{2}$, $\theta = 45^\circ$ (b) $v = 14$, $\theta = -60^\circ$
 (c) $v = \sqrt{74}$, $\theta = \tan^{-1}\frac{7}{5}$ [$\doteq 54^\circ 28'$]
 3(a) Initially, $\dot{x} = \dot{y} = 10$ (b) $\dot{x} = 10$, $x = 10t$,
 $\dot{y} = -10t + 10$, $y = -5t^2 + 10t$
 (c) 5 metres, 1 second (d) 20 metres, 2 seconds
 (e) $\dot{x} = 10$, $x = 5$, $\dot{y} = 5$ and $y = 3.75$
 (i) 6.25 metres (ii) $5\sqrt{5} \doteq 11$ m/s, $\tan^{-1}\frac{1}{2} \doteq 27^\circ$
 4(a) Initially, $\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5}$. (b) $\dot{x} = \sqrt{5}$,
 $x = t\sqrt{5}$, $\dot{y} = -10t + 2\sqrt{5}$, $y = -5t^2 + 2t\sqrt{5}$
 (d) 1 metre (e) 2 metres (f) $\dot{x} = \sqrt{5}$, $\dot{y} = -2\sqrt{5}$,
 $v = 5$ m/s, $\theta = -\tan^{-1}2$ (g) $y = 2x - x^2$
 5(a) $x = 20t$, $y = -5t^2 + 20\sqrt{3}t$ (b) $4\sqrt{3}$ seconds,
 $80\sqrt{3}$ metres (c) $2\sqrt{3}$ seconds, 60 metres
 (d) It is false. The horizontal range would not have
 changed, although the flight time would have been
 4 seconds and the maximum height would have
 been 20 metres.
 6(a) $x = 10\sqrt{3}t$, $y = -5t^2 + 10t$
 (b) 5 s, $50\sqrt{3}$ metres (c) 80 metres
 (d) 44 m/s, 67° (e) $y = -\frac{1}{60}x^2 + \frac{1}{\sqrt{3}}x$
 7(a) 101 m/s (b) $x = 101t$, $y = -5t^2$ (d) 149 m/s,
 $\tan \theta = \frac{20}{101}\sqrt{30}$, $\theta \doteq 47^\circ 19'$ (e) 1.106 km
 8(c) $V = 36$, $\theta \doteq 41^\circ 49'$ (d) 129 metres
 9(a) $\dot{x} = V \cos \alpha$, $x = Vt \cos \alpha$, $\dot{y} = -gt + V \sin \alpha$,
 $y = -\frac{1}{2}gt^2 + Vt \sin \alpha$
 (b)(i) $H = \frac{V^2}{2g} \sin^2 \alpha$, $\frac{V}{g} \sin \alpha$ seconds (ii) $\frac{V^2}{2g}$
 when $\alpha = 90^\circ$, half this value when $\alpha = 45^\circ$.
 (c)(i) $R = \frac{v^2}{g} \sin 2\alpha$, $T = \frac{2V}{g} \sin \alpha$ (ii) V^2/g when
 $\alpha = 45^\circ$, half this value when $\sin 2\alpha = \frac{1}{2}$, that is,
 $\alpha = 15^\circ$ or 75° .
 10(c) 50 metres
 11(d) $60^\circ 15'$ or $72^\circ 54'$
 12(b) 0.36 s (c) 12° (d) No, it lands 4.72 metres
 in front of him.
 13(b)(ii) 16 metres (iii) 112°
 14(d) 15 metres (e) 10 m/s, $63^\circ 26'$
 15(c) $T = 4$, $\theta = 30^\circ$

16(b)(ii) Yes. The vertical components of their initial velocities are equal, and they are both subject to the same force (gravity) acting in the vertical direction.

18(b)(iv) 52°

Exercise 3H (Page 134)

1(a) 40 metres (b) 10 metres (c) $\alpha = 45^\circ$
 (d) When $x = 10$, $y = 7\frac{1}{2}$, so the ball goes under A.
 (e) $V = 20$ m/s

2(a) $\dot{x} = 5\sqrt{2}$, $x = 5t\sqrt{2}$, $\dot{y} = -10t + 5\sqrt{2}$,
 $y = -5t^2 + 5t\sqrt{2}$ (b) range: 10 metres, maximum
 height: 2.5 metres when $x = 5$ (c)(i) 1.6 metres
 (ii) $y' = -\frac{1}{5}x + 1$, $\theta = -\tan^{-1} \frac{3}{5}$ (d)(i) $x = 3$
 (ii) $\theta = \tan^{-1} \frac{2}{5}$

3(b) $R = 21.6$ metres, $H = 4.05$ metres
 (c) $\tan^{-1} \frac{3}{4}$ (d) 15 m/s
 (e) $t = 0.8$, when $x = 9.6$, and $t = 1$, when $x = 12$
 4(a) $\dot{x} = 200$, $\dot{y} = 0$ (b) $\dot{x} = 200$, $x = 200t$,
 $\dot{y} = -10t$, $y = -5t^2$, $y = -\frac{1}{8000}x^2$ (c) 600 metres
 (d) $8^\circ 32'$

5(c)(i) $\alpha = 15^\circ$ or 75°
 (ii) It will if $\alpha = 75^\circ$, but not if $\alpha = 15^\circ$.
 6(b) $62^\circ 22'$ or $37^\circ 5'$

7(b) range: 38.4 metres, height: 12.8 metres
 (c)(ii) 33.3 metres
 8 $\alpha = 45^\circ$, $V = 4\sqrt{5}$ m/s, $\frac{2}{5}\sqrt{10}$ seconds.

The ground is the latus rectum of the parabola.

9(a) $x = \frac{1}{2}Vt\sqrt{2}$, $y = -\frac{1}{2}gt^2 + \frac{1}{2}Vt\sqrt{2}$
 10(c)(ii) $R = 18$ metres

11(c) For $0^\circ < \alpha < 45^\circ$, $0 < \tan \alpha < 1$. Hence if
 α_1 and α_2 are both less than 45° , then the two
 roots of the quadratic both lie between 0 and 1.
 But the product of these roots is greater than 1,
 so α_1 and α_2 cannot both be less than 45° .

13(b) $\frac{1}{2}(1 + \sqrt{2})$ (c) $\frac{1}{2}(1 - \sqrt{2})$
 (d) The two speeds are equal.

14 $(x - k \sin 2\alpha)^2 = -4k \cos^2 \alpha (y - k \sin^2 \alpha)$,
 vertex: $(k \sin 2\alpha, k \sin^2 \alpha)$,

focus: $(k \sin 2\alpha, -k \cos 2\alpha)$, directrix: $y = k$
 (a) k is the maximum height when the projectile
 is fired vertically upwards.

(b) focus: the circle $x^2 + y^2 = k^2$ with centre O and
 radius k , vertex: the ellipse $\frac{x^2}{k^2} + \frac{(y - \frac{1}{2}k)^2}{(\frac{1}{2}k)^2} = 1$

(c) The focus is $(k \cos(2\alpha - \frac{\pi}{2}), k \sin(2\alpha - \frac{\pi}{2}))$.
 This is a general property of parabolas.

Chapter Four

Exercise 4A (Page 141)

1(a) yes (b) no (c) no (d) no (e) yes (f) no
 (g) yes (h) yes (i) no (j) yes (k) yes (l) no

2(a) 3, 4, $4x^3$, -11 , not monic
 (b) 3, -6 , $-6x^3$, 10, not monic
 (c) 0, 2, 2, 2, not monic (d) 12, 1, x^{12} , 0, monic
 (e) 3, 1, x^3 , 0, monic (f) 5, -1 , $-x^5$, 0, not monic
 (g) no degree, no leading coefficient, no leading
 term, 0, not monic

(h) 2, -3 , $-3x^2$, 0, not monic
 (i) 6, -4 , $-4x^6$, -5 , not monic
 3(a) $x^2 + 2x + 3$ (b) $x^2 + 2x + 3$ (c) $-x^2 + 8x + 1$
 (d) $x^2 - 8x - 1$ (e) $5x^3 - 13x^2 - x + 2$
 (f) $5x^3 - 13x^2 - x + 2$

5(a) $2(x^2 - \frac{3}{2}x + 2)$ (b) $3(x^3 - 2x^2 - \frac{5}{3}x + \frac{1}{3})$
 (c) $-2(x^5 - \frac{7}{2}x^4 + 2x - \frac{11}{2})$ (d) $\frac{2}{3}(x^3 - 6x + 24)$

6(a) $x(x - 10)(x + 2)$, 0, 10, -2
 (b) $x^2(2x + 1)(x - 1)$, 0, 1, $-\frac{1}{2}$
 (c) $(x - 3)(x + 3)(x^2 + 4)$, 3, -3
 (d) $(x - 2)(x^2 + 2x + 4)$, 2
 (e) $(x - 3)(x + 3)(x^2 + 9)$, 3, -3
 (f) $(x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1)$, 1, -1

7(a)(i) $p + q$ (ii) the maximum of p and q
 (b) $P(x)Q(x)$ still has degree $p + p = 2p$, but
 $P(x) + Q(x)$ may have degree less than p (if the
 leading terms cancel out), or it could be the zero
 polynomial. (c) $x^2 + 2$ and $-x^2 + 3$. Do not
 choose two opposite polynomials, like $x^2 + 1$ and
 $-x^2 - 1$, because their sum is the zero polynomial,
 which does not have a degree.

8 $x + 1$
 9(a) $a = 3$, $b = -4$ and $c = 1$ (b) $a = 2$ and $b = 3$
 (c) $a = 1$, $b = 2$ and $c = 1$
 (d) $a = 1$, $b = 2$ and $c = -1$

10(a) $a = 4$, $b \neq \frac{2}{3}$ and c arbitrary (b) $a = 4$,
 $b = \frac{2}{3}$ and c arbitrary (c) $a = 5$, b and c arbitrary
 (d) $a = 4$, $b = \frac{2}{3}$ and $c = \frac{1}{5}$

12(c) A polynomial is even if and only if the coef-
 ficients of the odd powers of x are zero. A poly-
 nomial is odd if and only if the coefficients of the
 even powers of x are zero.

13(a) True. If $P(x)$ is even, then the terms are
 of the form $a_n x^{2n}$, where $n \geq 0$ is an integer.
 Therefore $P'(x)$ has terms of the form $2na_n x^{2n-1}$,
 so all powers of x will be odd.

(b) False. For example, $Q(x) = x^3 + 1$ is not odd but $Q'(x) = 3x^2$ is even.

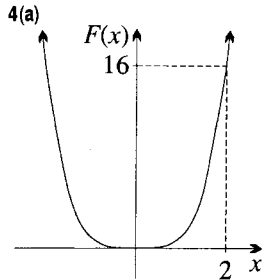
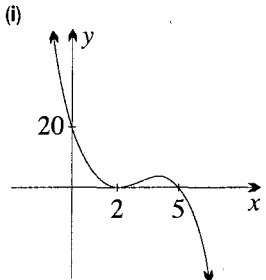
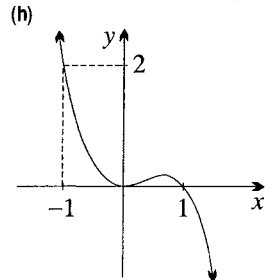
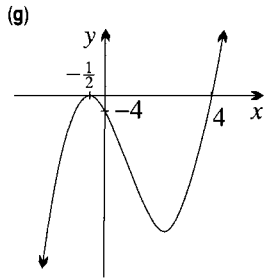
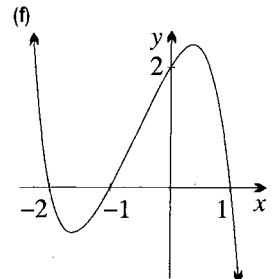
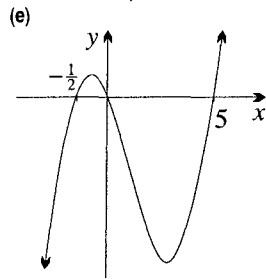
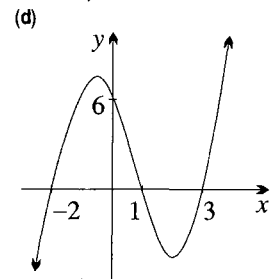
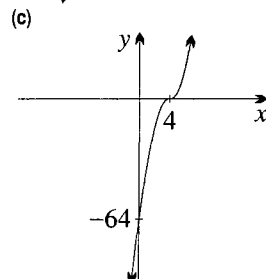
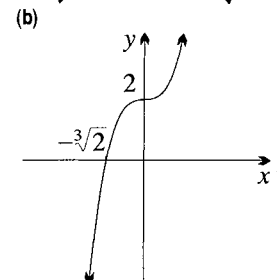
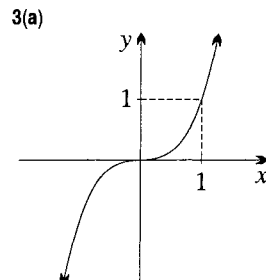
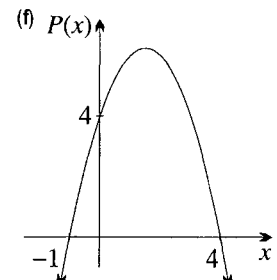
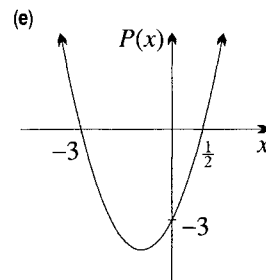
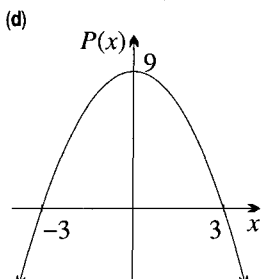
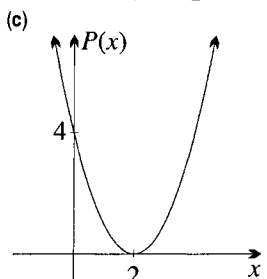
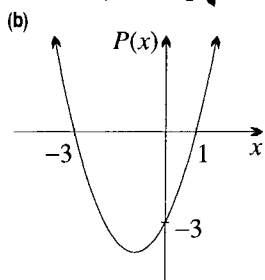
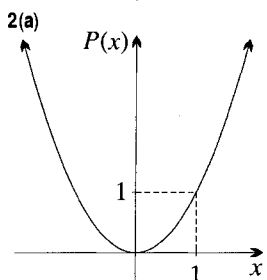
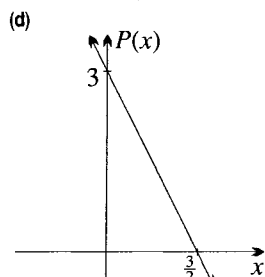
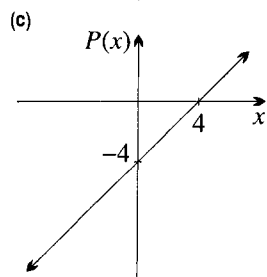
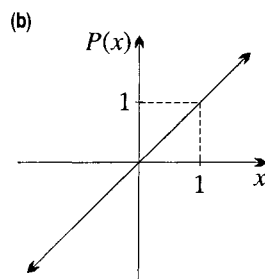
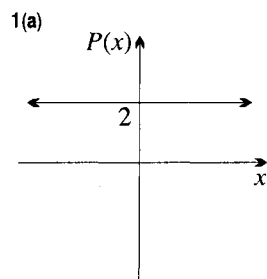
(c) True. If $R(x)$ is odd, then the terms are of the form $a_n x^{2n+1}$, where $n \geq 0$ is an integer. Therefore $R'(x)$ has terms of the form $(2n+1)a_n x^{2n}$, so all powers of x will be even.

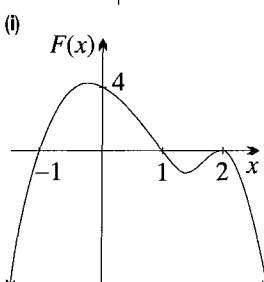
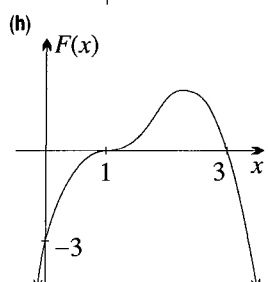
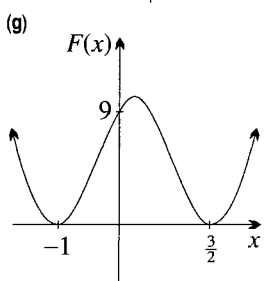
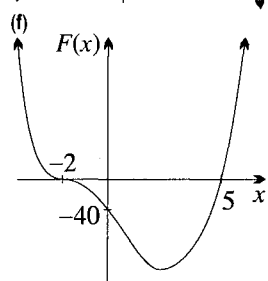
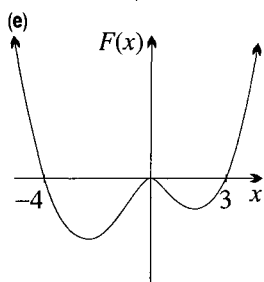
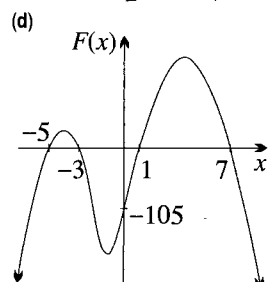
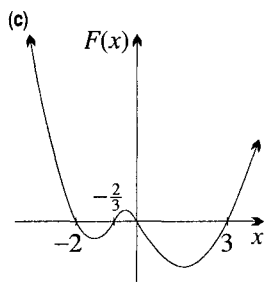
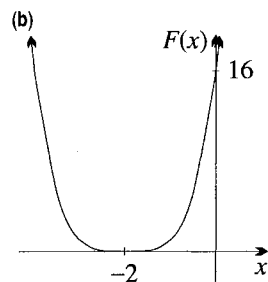
(d) True. As $S'(x)$ is odd, it has no constant term, and all powers of x are odd. Therefore all the terms in $S(x)$ will have even powers.

14(a) They are both nonzero constants.

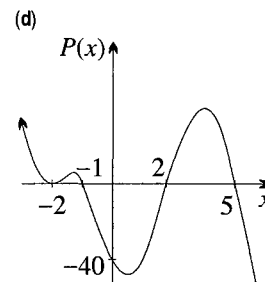
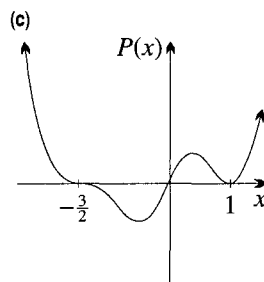
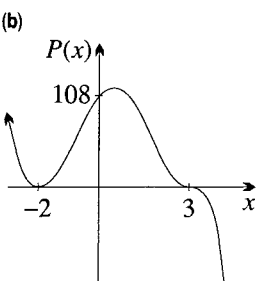
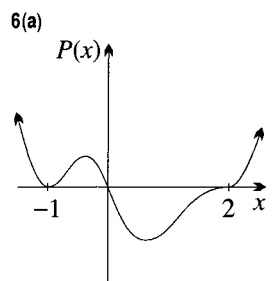
(b) If $f(x)$ were a polynomial, $\frac{1}{f(x)}$ would not be a polynomial.

Exercise 4B (Page 145)

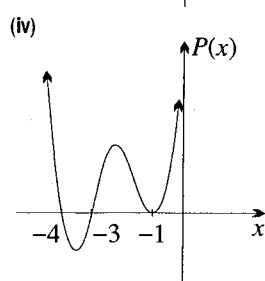
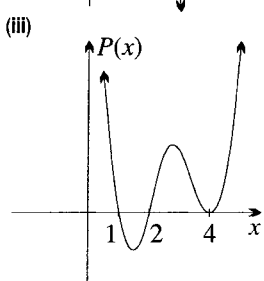
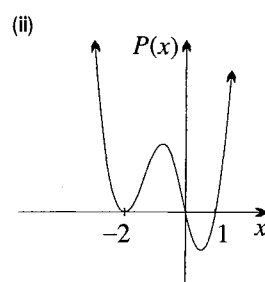
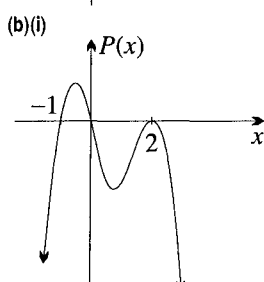
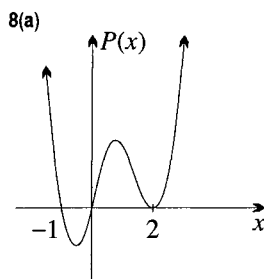




5(a) There are two zeroes, one between 0 and 1, and one between 2 and 3. (b) There are three zeroes, one between -2 and -1, one between -1 and 0, and one between 1 and 2.



7(a) $x > 2$ or $x < 0$, $x \neq -1$ (b) $x \leq 3$
 (c) $x \leq -\frac{3}{2}$ or $x \geq 0$ (d) $x > 5$ or $-1 < x < 2$



9(a) $P(x) = x^2 - x - 6$ (b) $P(x) = x^2 + 4x + 1$
 (c) $P(x) = 2x^3 - 6x^2 + 4x$

10(a)(i) $a = c = e = 0$ (ii) $b = d = f = 0$

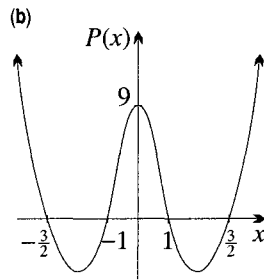
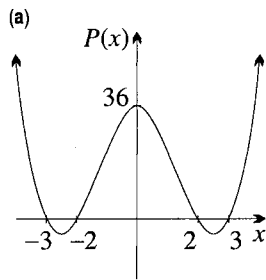
(b) $P(x) = x^4 - 10x^2 + 9$

(c) $P(x) = -3x^5 + 15x^3 - 12x$

11(c) $P(x) = x^3 - 4x$

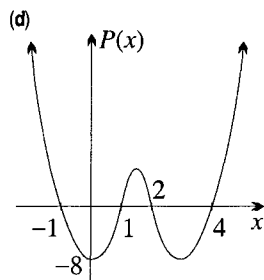
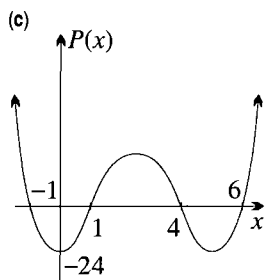
12(a) $(x + 3)(x - 3)(x + 2)(x - 2)$

(b) $(2x - 3)(2x + 3)(x + 1)(x - 1)$

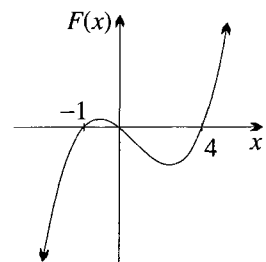


(c) $(x-6)(x+1)(x-4)(x-1)$

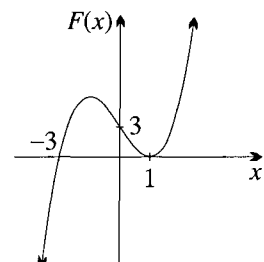
(d) $(x-4)(x+1)(x-1)(x-2)$



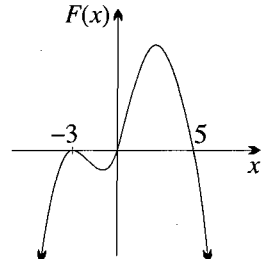
13(a)(i)



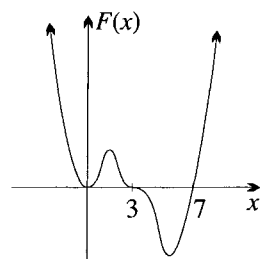
(ii)



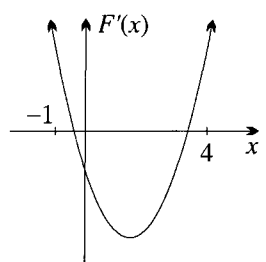
(iii)



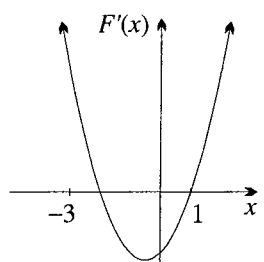
(iv)



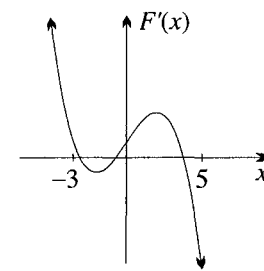
(b)(i)



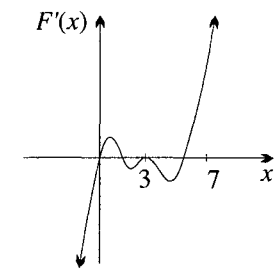
(ii)



(iii)

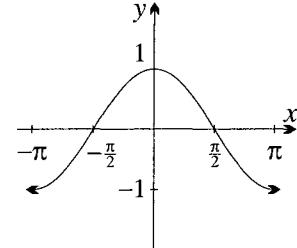


(iv)

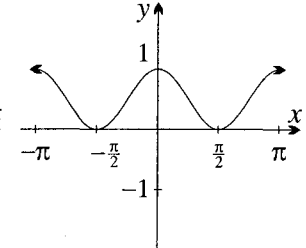


(c) (i) is increasing when $x > 4$ or $-1 < x < 0$ and decreasing when $x < -1$ or $0 < x < 4$. (ii) is increasing when $x > -3$, $x \neq 1$ and decreasing when $x < -3$. (iii) is increasing when $0 < x < 5$ and decreasing when $x > 5$ or $x < 0$, $x \neq -3$. (iv) is increasing when $x < 3$, $x \neq 0$ or $x > 7$ and decreasing when $3 < x < 7$.

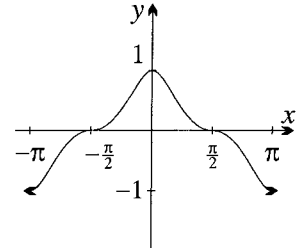
14(a)



(b)



(c)



15(a)(i) $f''(x) = 6ax + 2b$ and $f''(0)$, so $b = 0$.

Substituting $x = 0$ gives $d = 0$.

(ii) The curve has point symmetry in the origin.

(b) Translate the curve so that the point of inflexion is at the origin.

(c) We know that the curve has point symmetry in the point of inflexion, so the image of one turning point must be the other one. Now use part (b).

16 The graphs always intersect at $(0, 1)$ and at $(-1, 0)$. If m and n are both even, they also intersect at $(-2, 1)$, and if m and n are both odd, they also intersect at $(-2, -1)$.

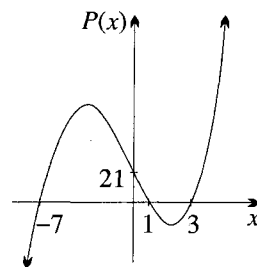
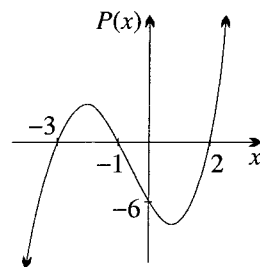
Exercise 4C (Page 149)

- 1(a) $63 = 5 \times 12 + 3$ (b) $125 = 8 \times 15 + 5$
 (c) $324 = 11 \times 29 + 5$ (d) $1857 = 23 \times 80 + 17$
 2(a) $x^2 - 4x + 1 = (x+1)(x-5) + 6$
 (b) $x^2 - 6x + 5 = (x-5)(x-1)$
 (c) $x^3 - x^2 - 17x + 24 = (x-4)(x^2 + 3x - 5) + 4$
 (d) $2x^3 - 10x^2 + 15x - 14 = (x-3)(2x^2 - 4x + 3) - 5$
 (e) $4x^3 - 4x^2 + 7x + 14 = (2x+1)(2x^2 - 3x + 5) + 9$
 (f) $x^4 + x^3 - x^2 - 5x - 3 = (x-1)(x^3 + 2x^2 + x - 4) - 7$
 (g) $6x^4 - 5x^3 + 9x^2 - 8x + 2$
 $= (2x-1)(3x^3 - x^2 + 4x - 2)$
 (h) $10x^4 - x^3 + 3x^2 - 3x - 2$
 $= (5x+2)(2x^3 - x^2 + x - 1)$
 3(a) $\frac{x^2 - 4x + 1}{x+1} = x - 5 + \frac{6}{x+1}$,
 $\frac{1}{2}x^2 - 5x + 6 \log(x+1) + C$
 (b) $\frac{x^2 - 6x + 5}{x-5} = x - 1, \frac{1}{2}x^2 - 5x + C$
 (c) $\frac{x^3 - x^2 - 17x + 24}{x-4} = x^2 + 3x - 5 + \frac{4}{x-4}$,
 $\frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + 4 \log(x-4) + C$
 (d) $\frac{2x^3 - 10x^2 + 15x - 14}{x-3} = 2x^2 - 4x + 3 - \frac{5}{x-3}$,
 $\frac{2}{3}x^3 - 2x^2 + 3x - 5 \log(x-3) + C$
 4(a) $x^3 + x^2 - 7x + 6 = (x^2 + 3x - 1)(x-2) + 4$
 (b) $x^3 - 4x^2 - 2x + 3 = (x^2 - 5x + 3)(x+1)$
 (c) $x^4 - 3x^3 + x^2 - 7x + 3$
 $= (x^2 - 4x + 2)(x^2 + x + 3) + (3x - 3)$
 (d) $2x^5 - 5x^4 + 12x^3 - 10x^2 + 7x + 9$
 $= (x^2 - x + 2)(2x^3 - 3x^2 + 5x + 1) + (7 - 2x)$
 5(a) 0, 1 or 2 (b) $D(x)$ has degree 3 or higher.
 6(a) $x^3 - 5x + 3 = (x-2)(x^2 + 2x - 1) + 1$
 (b) $2x^3 + x^2 - 11 = (x+1)(2x^2 - x + 1) - 12$
 (c) $x^3 - 3x^2 + 5x - 4 = (x^2 + 2)(x-3) + (3x+2)$,
 $\frac{x^3 - 3x^2 + 5x - 4}{x^2 + 2} = x - 3 + \frac{3x+2}{x^2+2}$,
 $\frac{1}{2}x^2 - 3x + \frac{3}{2} \log(x^2 + 2) + \sqrt{2} \tan^{-1}(x/\sqrt{2}) + C$
 (d) $2x^4 - 5x^2 + x - 2$
 $= (x^2 + 3x - 1)(2x^2 - 6x + 15) + (13 - 50x)$
 (e) $2x^3 - 3 = (2x-4)(x^2 + 2x + 4) + 13$
 (f) $x^5 + 3x^4 - 2x^2 - 3$
 $= (x^2 + 1)(x^3 + 3x^2 - x - 5) + (x+2)$,
 $\frac{x^5 + 3x^4 - 2x^2 - 3}{x^2 + 1} = x^3 + 3x^2 - x - 5 + \frac{x+2}{x^2+1}$,
 $\frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 - 5x + \frac{1}{2} \log(x^2 + 1) + 2 \tan^{-1} x + C$
 7(a) quotient: $\frac{1}{2}x + \frac{7}{4}$, remainder: $\frac{21}{4}$
 (b) quotient: $2x^2 + \frac{1}{3}x + \frac{13}{9}$, remainder: $-\frac{5}{9}$
 (c) quotient: $\frac{1}{2}x^2 + \frac{1}{4}x + \frac{7}{8}$, remainder: $\frac{29}{8}$

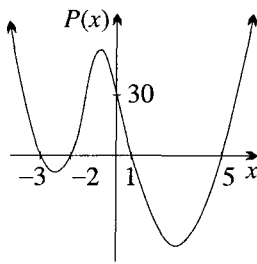
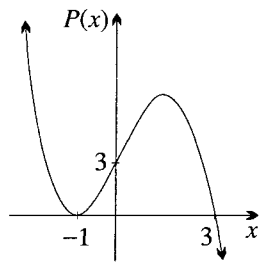
- 8(a) $P(x) = (x-3)(x+1)(x+4)$
 (b) $x > 3$ or $-4 < x < -1$
 9(a) $(x-2)(x+1)(2x-1)(x+3)$
 (b) $-3 \leq x \leq -1$ or $\frac{1}{2} \leq x \leq 2$
 10(a) $30 = 4 \times 7 + 2$, $30 = 7 \times 4 + 2$ (b) $D(x)$
 11(a) quotient: $x^2 - 3x + 5$, remainder: $12 - 13x$
 (b) $a = 8$ and $b = -5$ (c) $(x^2 + x - 1)(x^2 - 3x + 5)$
 12(a) $x^4 - x^3 + x^2 - x + 1$
 $= (x^2 + 4)(x^2 - x - 3) + (3x + 13)$
 (b) $a = -4$ and $b = -12$ (c) $(x^2 + 4)(x^2 - x - 3)$
 13 $a = 41$ and $b = -14$
 14(a) $a = 7$ and $b = -32$
 (b) $A(x) = \frac{1}{20}(x^3 - 4x^2 - 24)$, $B(x) = -\frac{1}{20}(x-5)$

Exercise 4D (Page 153)

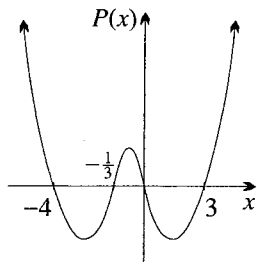
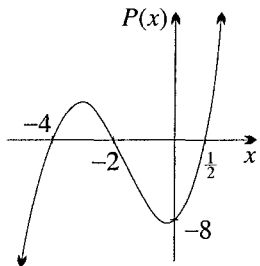
- 1(a) 3 (b) 25 (c) -15 (d) -3 (e) 111 (f) -41
 2(a) yes (b) no (c) no (d) yes (e) no (f) yes
 3(a) $3\frac{5}{8}$ (b) $12\frac{1}{8}$ (c) $3\frac{14}{27}$
 4(a) $k = 4$ (b) $m = -\frac{1}{2}$ (c) $p = -14$ (d) $a = -1$
 5(b) $x \geq 6$ or $-1 \leq x \leq 3$
 6(b) $x \leq -2$ or $-\frac{1}{2} \leq x \leq 3$
 7(a) $(x-2)(x+1)(x+3)$ (b) $(x-1)(x-3)(x+7)$



- (c) $(x+1)^2(3-x)$ (d) $(x-1)(x-2)(x-3)(x+5)$



- (e) $(x+2)(2x-1)(x+4)$ (f) $x(x-3)(3x+1)(x+4)$



- 8(a) $-1, -4$ or 2 (b) 3 or -2 (c) $2, \frac{1}{2}(-3 \pm \sqrt{13})$
 (d) 3 (e) $2, -\frac{2}{3}$ or $-\frac{1}{2}$ (f) $-2, \frac{1}{4}(-3 \pm \sqrt{17})$
 9(a) $(2x-1)(x+3)(x-2)$ (b) $(3x+2)(2x+1)(x-1)$
 10(a) $P(x) = (x+1)(x-2)^2$,
 $Q(x) = (x+1)(x-2)(x+3)$
 and $R(x) = (x+1)^2(x-2)$
 (b) $(x+1)(x-2)$ (c) $(x+1)^2(x-2)^2(x+3)$
 11 $\frac{1}{4}, 1$ and 3
 12(a) $a = 4$ and $b = 11$ (b) $a = 2$ and $b = -9$
 (c) $P(x) = -x^3 + 16x$ (d) $p = 2$ or $p = 3$
 13(a) $3 - x$ (b) $3 - 2x$
 14(a) $b = 0, c = -9$ and $d = 0$
 (c) $-3 < x < 0$ or $x > 3$
 15(b) $t = 1$ or $t = -2$
 16(a) $x + 1$ is a factor when n is odd.
 (b) $x + a$ is a factor when n is odd.
 17(a) $P(x) = (x-1)(x+3)Q(x) + (2x-1)$ (b) 1
 18(a) $\frac{1}{2}$ (b) $a = -2$ and $b = -7$ (c) 8
 19(b) $a = -2$
 20 $(p^2 + pq + q^2)x - pq(p + q)$
 21(a) $(12, 0)$ (b) $g(x) = a(x-k)(x-12)^2$ (c) 8
 (e) $a = \frac{2}{27}, k = 3, g(x) = \frac{2}{27}(x-3)(x-12)^2$
 (f) $f(x) = \frac{2}{27}(x-3)(x-12)^2 + 4$
 23(a) $a^2 + b^2 + c^2 - ab - ac - bc$
 (b) $(b-c)(c-a)(a-b)(a+b+c)$

Exercise 4E (Page 159)

- 1(a) $(x+1)(x-3)(x-4)$ (b) $x(x+2)(x-3)(x-1)$
 (c) $(3x-1)(2x+1)(x-1)$
 2(c) $(x-2)(x+3)(x+1)(x-5)$
 3(a) $P(x) = (x-1)(x+1)(x-3)(2x+1)$
 (b) $P(x) = (x-1)(x-2)(x+2)(2x-3)$
 (c) $P(x) = (2x-5)(3x-2)(x+1)(x-2)$
 (d) $P(x) = (x-2)(x-3)(3x-1)^2$
 4(a) $a = 2, b = \frac{1}{3}$ and $c = \frac{5}{2}$
 (b) $a = -1, b = 3, c = \frac{1}{2}$ and $d = \frac{5}{4}$
 5(a) $a = 3, b = -16$ and $c = 27$
 (b) $a = 2, b = -2, c = -7$ and $d = -7$
 (c) $(x+1)^3 - (x+1)^2 - 4(x+1) + 5$
 (d) $a = 3, b = -2$ and $c = 1$
 6(a) $P(x) = (x-2)^2(x+5)$
 (b) $P(x) = (x-1)(x+3)(2x-7)$
 8 The maximum of m and n .
 9 There must be one turning point between each of the consecutive zeroes.
 10 $a = b = c = 0, d = k$

- 11(a) maximum at $(2, 19)$, minimum at $(-2, -13)$, three zeroes (b) maximum at $(1, 10)$, minimum at $(-\frac{5}{3}, \frac{14}{27})$, one zero (c) maximum at $(3, 26)$, horizontal point of inflexion at $(0, -1)$, two zeroes (d) minimum at $(1, 1)$, horizontal point of inflexion at $(0, 2)$, no zeroes
 12(a) The curves are tangent at $x = 3$ and cross at $x = -1$. (b) The curves are tangent at $x = 2$ and cross at $x = 3$. (c) The curves cross at $x = -5, x = -2$ and $x = 3$. (d) The curves are tangent to one another and cross at $x = 1$ and cross at $x = -2$. (e) The curves cross at the origin and cross and are tangent to each other at $x = -1$.
 13(b) $a = -5$ and $b = 8$ (c) $2(x-2)^2(x-3)^2$

Exercise 4F (Page 165)

- 1(a) 4 (b) 2 (c) 8 (d) 2 (e) 14 (f) 12 (g) 6
 (h) 24 (i) $\frac{17}{2}$
 2(a) -2 (b) -11 (c) 12 (d) $-\frac{11}{12}$ (e) $-\frac{1}{6}$ (f) 0
 (g) -132 (h) 26 (i) $\frac{13}{72}$
 The roots are $-1, -4$ and 3 .
 3(a) 5 (b) 2 (c) 4 (d) -3 (e) $-\frac{4}{3}$ (f) $-\frac{2}{3}$ (g) $-\frac{5}{3}$
 (h) 21
 4(a) $x^2 + x - 6 = 0$ (b) $x^3 - 7x + 6 = 0$
 (c) $x^4 + x^3 - 7x^2 - x + 6 = 0$
 5(a) 3 (b) $-\frac{1}{2}$
 (c) $-3, 1$. Hence 1 is a double zero. (d) $\frac{2}{3}, 2$
 6(a) -1 (triple zero), and 3 (b) $-4, -1, 2$ and 3
 (c) -1 and 5 (both double zeroes)
 (d) -2 and 2 (No real numbers are solutions of $\alpha + \beta = -1$ and $\alpha\beta = 1$.)
 7(a) $-\frac{5}{2}$ (b) -2 (c) $\frac{41}{4}$ (d) $-30\frac{5}{8}$ (e) $\frac{1}{2}\sqrt{57}$
 8(c) $\alpha^2 - 3\alpha + 5 = 0$, which has negative discriminant. (d) once
 9(a) $a = 5, b = 12, (x-3)(x+1)(x-4)$ (b) $a = -5, b = 8, \frac{1}{2}$ is the other zero.
 10(a) $a = 3$ and $b = -24, (x-3)(x+4)(x+2)$
 (b) $a = -1$ and $b = 3, 5, -4, \sqrt{3}, -\sqrt{3}$
 13(a) $\frac{3}{2}, \frac{3}{2}$ and -1 (b) $\frac{1}{3}, -4$ and 4
 (c) $6, \frac{1}{2}$ and -4 (d) $4, \frac{1}{2}$ and 2
 14(a) $-\frac{1}{3}, 1$ and $\frac{7}{3}$. The inflexion is $(1, 0)$. (b) $\frac{1}{4}, \frac{1}{2}$ and 1 (c) $x = \frac{1}{2}, -1$ or 2
 15(a) $a = -12$ and the roots are $-2, 2$ and -3 .
 (b) $a = -5$ and the roots are $4, \frac{1}{4}$ and -3 .
 (c) $-1, -2, 2$ and 4
 18(a)(i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{4}$
 19(c) $\cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{8\pi}{9}$

(d)(i) 0 (ii) $\frac{1}{8}$ (iii) 6 (iv) $\frac{3}{2}$

20 12

21 0, because 1 is one of the roots, and so 0 is one of the factors of the expression.

23 $d = \frac{1}{8}b(4c - b^2)$, $e = \frac{1}{64}(4c - b^2)^2$

Exercise 4G (Page 169)

1(b) The equation is $(x - 4)^2 = 0$, so $x = 4$ is a double root, and the line is a tangent at $T(4, -8)$.

2(b)(i) $\alpha + \alpha = 4$ (ii) $b = -4$.

(iii) $y = -4 - 2x$, $T = (2, -8)$.

3(b) $\alpha + \beta = 4$, $M = (2, 3)$

(d) Since the gradient is 1, rise = run.

$AB = 2\sqrt{10}$.

4(b) The roots are 1, 1 and 3. (c) The line is a tangent at $(1, 2)$ because $x = 1$ is a double root of the equation. The curves also intersect at $(3, 0)$.

5(b)(i) $\alpha + \alpha + 0 = 5$ (ii) $m = -\frac{1}{4}$.

(iii) $y = -\frac{1}{4}x$, $T = (2\frac{1}{2}, -\frac{5}{8})$.

6(b) $\alpha + \beta + 2 = 5$, $M = (1\frac{1}{2}, -\frac{1}{2})$ (d) $\sqrt{26}$

7(a) The zeroes of $F'(x)$ satisfy $3x^2 + 2ax + b = 0$.

(b) Since $F''(x) = 6x + 2a$, the inflexion is at $x = -\frac{1}{3}a = \frac{1}{2}(\alpha + \beta)$.

8(a) $y = mx + (m - 7)$

(b) $x^3 - 3x^2 + (4 - m)x + (8 - m) = 0$

(c) The line is a tangent at $x = \alpha$ and meets the curve at $A(-1, -7)$. $\alpha = 2$, $T = (2, 5)$, $m = 4$

9(a) $y = mx - mp + p^3$

(c) $x = -\frac{1}{2}p$, so M lies on $x = -\frac{1}{2}p$.

10(a) $m = 2$, $\alpha = 1$ (b) $y + 3 = m(x + 2)$

(c) $y = 2x + 1$

11(a) $y = (x + 1)(x - 2)(x - 5)(x + 2)$ (c) $\alpha + \beta = 2$, $\alpha^2 + \beta^2 + 4\alpha\beta = -9$, $2\alpha^2\beta + 2\alpha\beta^2 = m - 16$, $\alpha^2\beta^2 = 20 - b$

(d) $m = -10$, $b = -22\frac{1}{4}$, $y = -10x - 22\frac{1}{4}$

12(a) $k = -\frac{1}{4}$, $(\frac{1}{2}\sqrt{2}, \frac{1}{4})$ and $(-\frac{1}{2}\sqrt{2}, \frac{1}{4})$

(b) $c = \frac{1}{2}$, $(\frac{1}{2}\sqrt{2}, \frac{1}{4}\sqrt{2})$ and $(-\frac{1}{2}\sqrt{2}, -\frac{1}{4}\sqrt{2})$

(c) $k = 0$ and $T(0, 0)$, or $k = \frac{4}{27}$ and $(\frac{2}{3}, \frac{8}{27})$

(d) $\alpha = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$

13(a) $y = -\frac{1}{3}x$ (b) $x^2 + (y - 1)^2 = 1$. It is the circle with diameter OF . (c) $x = 2$ and $y \geq 0$

(d)(i) $y = -mx$ (ii) $y = \frac{1}{2}b$ (e) $x = -\frac{1}{2}$

14(a) $a = \frac{5}{4}$, $(\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and $(-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$

(b) Either $a = 1$ and the other points are $(1, 0)$ and $(-1, 0)$, or $a = -1$ and there are no other points.

(c) $a = \frac{1}{2}$ (d) If α is a triple root of $x^3 - 6x^2 - (2 + m)x + (1 - b) = 0$, then $3\alpha = 6$ and

$\alpha = 2$, so the point is $(2, -19)$. (e) $y = -16x - 4$, $|\alpha - \beta| = 2\sqrt{6}$, $AB = 2\sqrt{6} \times \sqrt{16^2 + 1} = 2\sqrt{1542}$

15 Let the roots be $\alpha - k$, α and $\alpha + k$.

17(c) $M = \left(\frac{\lambda + 2}{2\lambda}, -\frac{\lambda + 2}{2\lambda}\right)$, locus: $y = -x$

(d) $\lambda = 2(\sqrt{2} + 1)$ (e) $\lambda < -2(\sqrt{2} - 1)$ or $\lambda > 2(\sqrt{2} + 1)$, but $\lambda \neq -1$

18(b) $x = 1/b$

Chapter Five

Exercise 5A (Page 176)

- 2(a) $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
 (b) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$
 (c) $1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$ (d) $1 - 9x + 36x^2 - 84x^3 + 126x^4 - 126x^5 + 84x^6 - 36x^7 + 9x^8 - x^9$ (e) $1 + 5c + 10c^2 + 10c^3 + 5c^4 + c^5$ (f) $1 + 8y + 24y^2 + 32y^3 + 16y^4$
 (g) $1 + \frac{7}{3}x + \frac{7}{3}x^2 + \frac{35}{27}x^3 + \frac{35}{81}x^4 + \frac{7}{81}x^5 + \frac{7}{729}x^6 + \frac{1}{2187}x^7$ (h) $1 - 9z + 27z^2 - 27z^3$

(i) $1 - \frac{8}{x} + \frac{28}{x^2} - \frac{56}{x^3} + \frac{70}{x^4} - \frac{56}{x^5} + \frac{28}{x^6} - \frac{8}{x^7} + \frac{1}{x^8}$

(j) $1 + \frac{10}{x} + \frac{40}{x^2} + \frac{80}{x^3} + \frac{80}{x^4} + \frac{32}{x^5}$

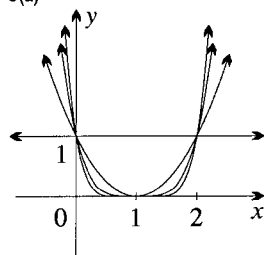
(k) $1 + \frac{5y}{x} + \frac{10y^2}{x^2} + \frac{10y^3}{x^3} + \frac{5y^4}{x^4} + \frac{y^5}{x^5}$

(l) $1 + \frac{12x}{y} + \frac{54x^2}{y^2} + \frac{108x^3}{y^3} + \frac{81x^4}{y^4}$

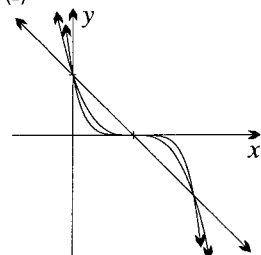
4(a)(i) $55x^2$ (ii) $165x^8$ (b)(i) $-35x^3$ (ii) $-21x^5$

(c)(i) $240x^4$ (ii) $192x^5$ (d)(i) $-\frac{12}{x}$ (ii) $\frac{54}{x^2}$

5(a)



(b)



7(a) x^3 (b) x^6

8 21

9(a) $a = 76, b = 44$ (b) $a = 16, b = -8$

(c) $a = 433, b = 228$ (d) $a = 4069, b = -2220$

10(a) 152 (b) $88\sqrt{3}$

12(a) 1.018 14 (b) 0.815 37 (c) 0.032 00

13(a)(i) $1 + 4x + 6x^2 + \dots$ (ii) -14

(b)(i) $1 + 10x + 40x^2 + 80x^3 + \dots$ (ii) 40

(c)(i) $1 - 12x + 54x^2 - 108x^3 + \dots$ (ii) -228

14(a) -12 (b) 0 (c) 380 (d) $-\frac{5}{3}$ (e) 750 (f) -8

15(a) 97 (b) $1\frac{10}{27}$

16(a)(i) $15x^2$ (ii) $20x^3$ (iii) $3 : 4x$

(iv) 135, 540, 1 : 4 (b)(i) $\frac{224}{81x^5}$ (ii) $\frac{448}{729x^6}$

(iii) $9x : 2$ (iv) $\frac{7}{81}, \frac{7}{729}, 9 : 1$

17(a) $x = 0$ or $\frac{1}{2}$ (b) $x = \frac{5}{2}$ (c) $x = 0, 1$ or 5

18(a) $k = 5$ (b) $k = \frac{1}{2}$ (c) $k = -2$

19(a) $\frac{1}{42}u^2$ (b) $\frac{1}{630}u^2$ (c) $\frac{\pi}{20}u^3$

20(b) \$1124.86

21 1.0634

23(a) 3 points, 3 segments, 1 triangle (b) 4 points, 6 segments, 4 triangles, 1 quadrilateral

(c) 5 points, 10 segments, 10 triangles, 5 quadrilaterals, 1 pentagon (d) 21

24 $(1+x+y)^0=0, (1+x+y)^1=1+x+y,$

$(1+x+y)^2=1+2x+2y+2xy+x^2+y^2,$

$(1+x+y)^3=1+3x+3y+6xy+3x^2+3y^2+3x^2y+3xy^2+x^3+y^3,$

$(1+x+y)^4=1+4x+4y+12xy+6x^2+6y^2+6x^2y+12x^2y+12xy^2+4x^3+4y^3+4x^3y+4xy^3+x^4+y^4$

The coefficients form a triangular pyramid, with 1s on the edges, and each face a copy of Pascal's triangle.

Exercise 5B (Page 183)

1(a) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(b) $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

(c) $r^6 - 6r^5s + 15r^4s^2 - 20r^3s^3 + 15r^2s^4 - 6rs^5 + s^6$

(d) $p^{10} + 10p^9q + 45p^8q^2 + 120p^7q^3 + 210p^6q^4 + 252p^5q^5 + 210p^4q^6 + 120p^3q^7 + 45p^2q^8 + 10pq^9 + q^{10}$

(e) $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6 - 36a^2b^7 + 9ab^8 - b^9$

(f) $32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$

(g) $p^7 - 14p^6q + 84p^5q^2 - 280p^4q^3 + 560p^3q^4$

$-672p^2q^5 + 448pq^6 - 128q^7$

(h) $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$

(i) $a^3 - \frac{3}{2}a^2b + \frac{3}{4}ab^2 - \frac{1}{8}b^3$

(j) $\frac{1}{32}r^5 + \frac{5}{48}r^4s + \frac{5}{36}r^3s^2 + \frac{5}{54}r^2s^3 + \frac{5}{162}rs^4 + \frac{1}{243}s^5$

(k) $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

2(a) $1 + 4x^2 + 6x^4 + 4x^6 + x^8$ (b) $1 - 9x^2 + 27x^4 - 27x^6$

(c) $x^{12} + 12x^{10}y^3 + 60x^8y^6 + 160x^6y^9 + 240x^4y^{12} + 192x^2y^{15} + 64y^{18}$

(d) $x^9 - 9x^7 + 36x^5 - 84x^3 + 126x$

$-\frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

(e) $x^3\sqrt{x} + 7x^3\sqrt{y} + 21x^2y\sqrt{x} + 35x^2y\sqrt{y}$

$+ 35xy^2\sqrt{x} + 21xy^2\sqrt{y} + 7y^3\sqrt{x} + y^3\sqrt{y}$

(f) $\frac{32}{x^5} + \frac{240}{x^2} + 720x + 1080x^4 + 810x^7 + 243x^{10}$

3(a) ${}^4C_0 = 1, {}^4C_1 = 4, {}^4C_2 = 6, {}^4C_3 = 4, {}^4C_4 = 1$

(b)(i) 16 (ii) 0

4(a) 32 (b) 32 (c) 20 (d) 252

5(a) x^5 (b) b^4 (c) $8y^3$ (d) $64y^6$

6(a)(i) $1024 + 1280x + 640x^2 + 160x^3 + \dots$ (ii) -160

(b)(i) $1 - 12x + 60x^2 - 160x^3 + 240x^4 - \dots$ (ii) 720

(c)(i) $2187 - 5103y + 5103y^2 - 2835y^3 + 945y^4 - \dots$

(ii) 11 718

7(a) $2x^6 + 30x^4y^2 + 30x^2y^4 + 2y^6$

8(a) 540 (b) 48 (c) -960 (d) -8

9(a)(i) $x^3 + 3x^2h + 3xh^2 + h^3$ (ii) $3x^2h + 3xh^2 + h^3$

(iii) $3x^2$ (b) $5x^4$

10(b) 466 (c) $7258\sqrt{2}$ (d) 42

11(a) $\frac{7}{2}$ (b) $\frac{131}{4}\sqrt{7}$

12 $x^3 + y^3 + z^3 + 6xyz + 3x^2y + 3xy^2 + 3xz^2 + 3x^2z + 3y^2z + 3yz^2$

13(a) 1.104 08 (b) 0.903 92 (c) 51.54

14(a)(i) $\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

(ii) $\left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$

(iii) $\left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right)$

$+ 35\left(x + \frac{1}{x}\right)$ (b)(i) 2 (ii) 2 (iii) 2

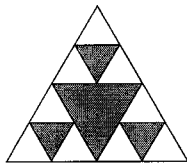
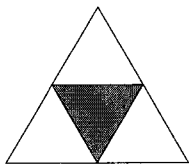
15 $a = 3$ or $a = -3$

17(a) $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

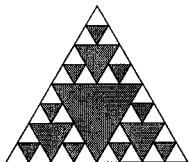
(b) $A = -6$, $B = 9$ and $C = -2$.

18 19

19(a) The limiting figure for this process is called the Sierpinski Gasket. It is one of the classic regular fractals.



(b) Sierpinski's triangle is formed.



Exercise 5C (Page 187)

1(a) 6 (b) 5040 (c) 3 628 800 (d) 1 (e) 1 (f) 15 120

(g) 15 (h) 6720 (i) 45 (j) 220 (k) 70 (l) 2520

(m) 5005 (n) 13 860

2(a) $6x^5$ (b) $30x^4$ (c) $120x^3$ (d) $360x^2$ (e) $720x$

(f) 720 (g) 0

3(a) n (b) $n!$ (c) 1 (d) $n(n+1)$ (e) $(n+1)(n+2)$

(f) $\frac{1}{n(n-1)}$ (g) $\frac{n-2}{n}$ (h) $\frac{(n-1)!}{n+1}$

4(a) $7 \times 7!$ (b) $n \times n!$ (c) $57 \times 6!$

(d) $(n^2 + n + 1) \times (n-1)!$ (e) $9^2 \times 7!$

(f) $(n+1)^2 \times (n-1)!$

5(a) $\frac{1+n}{n!}$ (b) $\frac{n}{(n+1)!}$ (c) $\frac{1-n-n^2}{(n+1)!}$

6(a)(i) nx^{n-1} (ii) $n(n-1)x^{n-2}$ (iii) $n!$

(iv) $n(n-1)(n-2)\dots(n-k+1)x^{n-k}$

$= \frac{n!}{(n-k)!} x^{n-k}$

(b)(i) $-1!x^{-2}$ (ii) $2!x^{-3}$ (iii) $-5!x^{-6}$

(iv) $(-1)^n n! x^{-(n+1)}$

7(b) $(n+1)! - 1$

9(a)(i) 2^8 (ii) 10^2 (b)(i) 2^{97} (ii) 5^{24} (iii) 7^{16} (iv) 13^7

11(a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}, \frac{1}{144}$ (b) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}, \frac{719}{720}$

(c) $\sum_{k=1}^n \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$. The limit is 1.

(d) The sequence can be written as $\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \dots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$.

12(a) $2^{15} \times 15!$ (b) $\frac{30!}{2^{15} \times 15!}$ (c) $\frac{2^{30} \times (15!)^2}{30!}$

13(a)(i) $1+x+x^2+x^3+\dots$ (ii) It is an infinite GP, so for $-1 < x < 1$ it converges to $\frac{1}{1-x}$, as expected.

(b) $-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots$ (c)(i) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

(ii) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(iii) $x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{2!3!} + \frac{1}{5!}\right)x^5$

14 0.14%

Exercise 5D (Page 193)

1(a) 10 (b) 210 (c) 20 (d) 1287 (e) 792 (f) 1

(g) 9 (h) 11 (i) 35 (j) 10 (k) 14 (l) 132

2(a)(i) 56 (ii) 35 (b) 5

3(b)(i) 6 (ii) 10 (iii) 30 (iv) $n = 4$ or $n = 8$

4(a)(i) $672x^2$ (ii) $280x^4$ (b)(i) $\frac{1001}{16}x^9y^5$

(ii) $\frac{1001}{256}x^5y^9$ (c)(i) $-\frac{33}{1024}x^{10}y^2$ (ii) $\frac{168399}{16}x^5y^{12}$

(d)(i) $-1140a^3b^{\frac{17}{2}}$ (ii) $190a^2b^9$

5(a)(i) 1 (ii) n (iii) $\frac{1}{2}n(n-1)$ (iv) $\frac{1}{6}n(n-1)(n-2)$

(b)(i) 16 (ii) 9 (iii) 4 (iv) 6 (v) 4 (vi) 7

6(a) $5x^2 : 39$ (b) $5 : 2$ (c) $18304 : 1$

7(b)(i) 126 (ii) 36 (iii) 84

8(b)(i) ${}^{10}C_4 2^6 3^4 = 2^7 \times 3^5 \times 5 \times 7$

(ii) ${}^{10}C_7 2^3 3^7 = 2^6 \times 3^8 \times 5$

(iii) ${}^{10}C_6 2^4 3^6 = 2^5 \times 3^7 \times 5 \times 7$

9(b)(i) ${}^{15}C_2 5^2 2^{-13}$ (ii) $-{}^{15}C_7 5^7 2^{-8}$

(iii) ${}^{15}C_{10} 5^{10} 2^{-5}$

- 10(a) ${}^8C_4 \times 3^4 = 5670$ (b) $-{}^{12}C_9 \times 2^3 = -1760$
 (c) ${}^{10}C_2 \times 5^2 \times 2^8 = 288\,000$
 (d) ${}^6C_4 \times a^2 \times (\frac{1}{2})^4 = \frac{15}{16}a^2$
 11(a) -672 (b) $\frac{969}{2}$ (c) $-112\,266$ (d) $21\,875$ (e) $\frac{40}{49}$
 (f) $-{}^{19}C_9 (\frac{3}{5})^{10}$
 12(a)(i) 3640 (ii) 140 (b)(i) -385 (ii) 66
 (c)(i) $-2\,379\,520$ (ii) $10\,920$ (d)(i) -1241
 (ii) $161\,838$
 13(a)(i) $-1\,959\,552x^5y^5$ (ii) $924x^3y^2$
 (iii) $-\frac{9724}{390\,625}x^9y^{18}$ (iv) $\frac{160x^3}{27y^3}$
 (b)(i) $90a^3b^2$, $270a^2b^3$ (ii) $-\frac{77}{2592}a^6b^5$, $\frac{77}{3888}a^5b^6$
 (iii) $6435a^{\frac{8}{3}}b^{\frac{7}{2}}$, $6435a^{\frac{7}{3}}b^4$ (iv) $\frac{63b^4}{8a^5}$, $-\frac{63b^5}{16a^4}$
 14(a) $x = \frac{11}{2}$ (b) $x = -\frac{7}{3}$
 15(a) 6 (b) 45 (c) 84
 16(a) $a = -24$, $b = 158$ (b) $n = 13$, 286
 17(a) $a = 2$ and $n = 14$ (b) $a = -\frac{1}{3}$ and $n = 10$
 18(a) $n = 14$ (b) $n = 13$ (c) $n = 9$
 20(a) $0.877\,52$ (b) 1.1157 (c) $0.985\,10$
 21(a) $n = 10$ (b) $n = 7$ (c)(i) $n = 14$ or $n = 7$
 22 $1 - 4x + 10x^2 - 16x^3 + 19x^4 - \dots$
 23 ${}^{3n}C_n (= {}^{3n}C_{2n})$
 24(a) ${}^{12}C_r (-1)^r a^{12-r} b^r x^r$ (b) $\frac{5}{8}$
 26(a) ${}^nC_0 x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots$
 $+ {}^nC_n h^n$ (b) nx^{n-1}
 28(a) $\frac{9-9n}{2}$ (b) $-\frac{9}{2}n(n-1)(2n-1)$
 29(b) $1001, 2002, 3003$
 30(b) 12
 31(c) If you add any column downwards from the top to any point, then the sum is diagonally below and to the right.
 32(a) $\frac{n-r+1}{r}$ (b) $\frac{n(n+1)}{2}$
 33(b)(i) $1-x+x^2-x^3+\dots$ (ii) $1+2x+3x^2+4x^3+\dots$
 (iii) $1-2x+3x^2-4x^3+\dots$
 (iv) $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{3}{48}x^3-\dots$

Exercise 5E (Page 199)

- 1(a) $t_{k+1} = {}^{12}C_{k+1} 2^{11-k} 3^{k+1}$, $t_k = {}^{12}C_k 2^{12-k} 3^k$
 (c) ${}^{12}C_7 2^5 3^7$
 2(a) $c_{k+1} = {}^{25}C_{k+1} 7^{24-k} 3^{k+1}$, $c_k = {}^{25}C_k 7^{25-k} 3^k$
 (c) ${}^{25}C_7 7^{18} 3^7$
 3(a) $T_{k+1} = {}^{13}C_{k+1} 3^{12-k} 4^{k+1} x^{k+1}$,
 $T_k = {}^{13}C_k 3^{13-k} 4^k x^k$ (c) ${}^{13}C_5 3^8 2^5$
 4(a) $T_{k+1} = {}^{21}C_{k+1} 5^{k+1} x^{k+1}$, $T_k = {}^{21}C_k 5^k x^k$
 (c) ${}^{21}C_{16} 3^{16}$

- 5(a)(i) $t_{k+1} = {}^{15}C_{k+1} 5^{14-k} 2^{k+1}$,
 $t_k = {}^{15}C_k 5^{15-k} 2^k$ (iii) ${}^{15}C_4 5^{11} 2^4$
 (b)(i) $T_{k+1} = {}^{15}C_{k+1} 5^{14-k} 2^{k+1} x^{k+1}$,
 $T_k = {}^{15}C_k 5^{15-k} 2^k x^k$ (iii) ${}^{15}C_6 5^{15} (\frac{2}{3})^6$
 6(a)(i) ${}^{11}C_9 4^9$ (ii) $T_8 = {}^{11}C_8 (\frac{8}{3})^8$
 (b)(i) ${}^9C_2 2^3 3^2$ (ii) $T_1 = T_2 = 1152$
 (c)(i) ${}^{12}C_8 3^4 5^8$ (ii) $T_6 = {}^{12}C_6 10^6$
 (d)(i) ${}^{11}C_6 5^5 6^6$, (ii) $T_5 = {}^{11}C_5 5^6 4^5$
 7(a)(i) ${}^9C_8 7^8$ (ii) $T_8 = {}^9C_8 (\frac{14}{3})^8$
 (b)(i) $-{}^{14}C_3 7^{11} 2^3$ (ii) $T_2 = {}^{14}C_2 7^{12} (\frac{4}{3})^2$
 (c)(i) ${}^{12}C_8 2^8$ (ii) $T_{11} = -{}^{12}C_{11} \times 6^{11}$
 (d)(i) $-{}^{15}C_5 2^{10}$ (ii) $T_{11} = -{}^{15}C_{11} 4^4 3^7$
 8(a)(i) ${}^{10}C_6 2^4 3^6$ (ii) ${}^{10}C_3 9^7 2^{-4}$ (b)(i) ${}^{12}C_7 2^5 3^7$
 (ii) Two terms have the greatest value, which is $10\,264\,320$.
 9 The equal terms are ${}^{14}C_6 (\frac{2}{3})^6$ and ${}^{14}C_5 (\frac{2}{3})^5$.
 10(a) $T_{r+1} = {}^nC_{r+1} x^{n-r-1} y^{r+1}$,
 $T_r = {}^nC_r x^{n-r} y^r$ (c) $T_5 = \frac{63}{256}$
 11(a) $n = 17$, $r = 2$ (b) ${}^{17}C_8 = {}^{17}C_9$
 12 $r = 8$
 13 $\theta = 22^\circ$
 16 2

Exercise 5F (Page 204)

- 1(a) The five numbers on the row indexed by $n = 4$ have a sum of $2^4 = 16$.
 (b)(i) The sum of the first, third and fifth terms on the row equals the sum of the second and fourth terms.
 (ii) The sum of the first, third and fifth terms on the row is half the sum of the whole row.
 (c)(i) $4(1+x)^3 = {}^4C_1 + 2{}^4C_2 x + 3{}^4C_3 x^2 + 4{}^4C_4 x^3$
 6(b) $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + ({}^{2n}C_{2n})^2 = (-1)^n {}^{2n}C_n$
 7(a) $4 \leq r \leq n$ (b) $r \leq p$ and $r \leq q$
 9(a) $\frac{{}^{3n}C_{n+r}}{5^{n+1} - 1}$
 10(a) $\frac{n+1}{n+1}$
 12(a) $k \times {}^5C_k = 5 \times {}^4C_{k-1}$, and more generally,
 $k \times {}^nC_k = n \times {}^{n-1}C_{k-1}$. (b) $\frac{{}^5C_k}{k+1} = \frac{{}^6C_{k+1}}{6}$,
 and more generally, $\frac{{}^nC_k}{k+1} = \frac{{}^{n+1}C_{k+1}}{n+1}$.
 15(d) $\frac{8}{15}$
 20(c) ${}^7C_1$, ${}^7C_2$, ${}^7C_3$, and ${}^7C_4$, ${}^7C_5$, ${}^7C_6$;
 ${}^{14}C_4$, ${}^{14}C_5$, ${}^{14}C_6$, and ${}^{14}C_8$, ${}^{14}C_9$, ${}^{14}C_{10}$;
 ${}^{23}C_8$, ${}^{23}C_9$, ${}^{23}C_{10}$, and ${}^{23}C_{13}$, ${}^{23}C_{14}$, ${}^{23}C_{15}$

Chapter Six

Exercise 6A (Page 211)

1(a) $\sec x \tan x$ (b) $-\operatorname{cosec} x \cot x$ (c) $-\operatorname{cosec}^2 x$

(d) $-3 \operatorname{cosec} 3x \cot 3x$ (e) $\operatorname{cosec}^2(1-x)$

(f) $5 \sec(5x-2) \tan(5x-2)$

2(a) $4\sqrt{3}$ (b) $-\frac{8}{3}$

3(a) $12x+2y=\pi+2$ (b) $\sqrt{2}x+y=\sqrt{2}(1+\frac{\pi}{4})$

(c) $3\sqrt{3}x-2y=\pi\sqrt{3}-5$ (d) $y=-1$

4(a) $-\operatorname{cosec}^2 x e^{\cot x}$ (b) $\tan x$

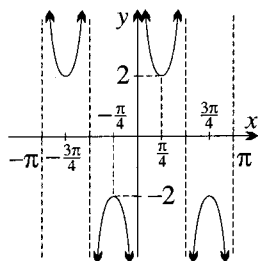
(c) $\operatorname{cosec} x(1-x \cot x)$ (d) $-2 \cot x \operatorname{cosec}^2 x$

(e) $4 \sec^4 x \tan x$ (f) $-\operatorname{cosec} x \sec x$

(g) $2e^{2x} \sec 2x(1+\tan 2x)$ (h) $\frac{-2 \operatorname{cosec}^2 x(x \cot x+1)}{x^3}$

5(a) $-\frac{\pi}{2}, 0$ and $\frac{\pi}{2}$ (b) odd (c) $y > 0$ in quadrants 1 and 3, $y < 0$ in quadrants 2 and 4. (e) minimum turning points at $(-\frac{3\pi}{4}, 2)$ and $(\frac{\pi}{4}, 2)$, maximum turning points at $(-\frac{\pi}{4}, -2)$ and $(\frac{3\pi}{4}, -2)$

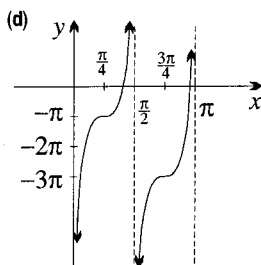
(f)



8(b) $\frac{\pi}{2} < x < \frac{3\pi}{2}$

9(b) $x = \frac{\pi}{2}$ (c) $(\frac{\pi}{4}, -\pi)$

and $(\frac{3\pi}{4}, -3\pi)$ are horizontal points of inflexion.

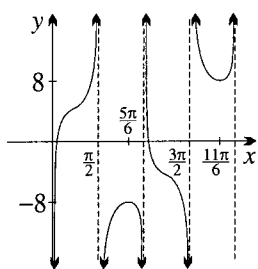


10(a) $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

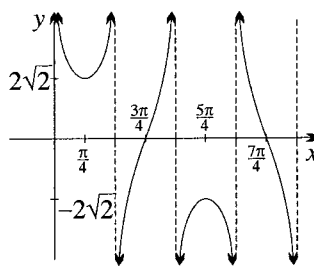
(c) $(\frac{5\pi}{6}, -8)$ is a maximum turning point,

$(\frac{11\pi}{6}, 8)$ is a minimum turning point. (d) $y \rightarrow \infty$ as $x \rightarrow \frac{\pi}{2}^-$, $x \rightarrow \pi^+$, $x \rightarrow \frac{3\pi}{2}^+$, $x \rightarrow 2\pi^-$ and $y \rightarrow -\infty$ as $x \rightarrow 0^+$, $x \rightarrow \frac{\pi}{2}^+$, $x \rightarrow \pi^-$, $x \rightarrow \frac{3\pi}{2}^-$.

(e)



11



12(c)(i) 0 (ii) π

13(a) $\frac{1}{x^2} \operatorname{cosec}^2 \frac{1}{x}$ (b) $\frac{\tan x}{\log(\sec x)}$

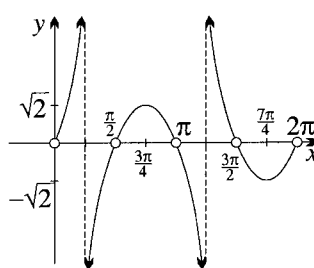
(c) $\frac{3 \sec 3x}{\tan 3x - \sec 3x}$

14(b) $3\sqrt{2}x - 2y = 6$

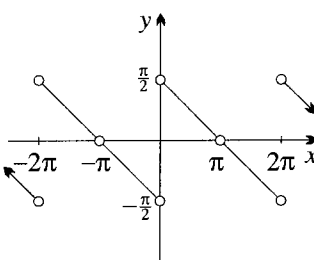
16(b) $\theta = \frac{\pi}{6}$

18(a) $\frac{\operatorname{cosec} x \cot x}{\operatorname{cosec}^2 y}$ (b) $\frac{y - \sec(x+y) \tan(x+y)}{\sec(x+y) \tan(x+y) - x}$

19



20



Exercise 6B (Page 216)

1(a) $\frac{1}{2} \sin 2x + C$ (b) $-\frac{1}{2} \cos 2x + C$ (c) $\frac{1}{2} \tan 2x + C$

(d) $-\frac{1}{2} \cot 2x + C$ (e) $\frac{1}{2} \sec 2x + C$

(f) $-\frac{1}{2} \operatorname{cosec} 2x + C$

2(a) $3 \sin \frac{1}{3}x + C$ (b) $2 \cos \frac{1}{2}(1-x) + C$

(c) $-\frac{1}{3} \tan(4-3x) + C$ (d) $-\frac{5}{2} \cot \frac{1}{5}(2x+3) + C$

(e) $\frac{1}{a} \sec(ax+b) + C$ (f) $\frac{1}{b} \operatorname{cosec}(a-bx) + C$

3(a) $(2-\sqrt{2}) \text{ units}^2$ (b) $\frac{1}{6} \sqrt{3} \text{ units}^2$

(c) $3(2-\sqrt{2}) \text{ units}^2$ (d) $\frac{1}{2} \log 2 \text{ units}^2$

4(a) $\ln 2$ (b) $\frac{1}{6} \ln 2$ (c) $\ln(\sqrt{2}+1)$ (d) $\frac{1}{2} \ln 3$

5 $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ (a) $\frac{1}{2}x - \frac{1}{4} \sin 2x + C$

(b) $\frac{1}{2}x - \frac{1}{8} \sin 4x + C$ (c) $\frac{1}{2}x - \sin \frac{1}{2}x + C$ (d) $\frac{\pi}{6}$

6 $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ (a) $\frac{1}{2}x + \frac{1}{4} \sin 2x + C$

(b) $\frac{1}{2}x + \frac{1}{24} \sin 12x + C$ (c) $\frac{1}{2}x + \frac{1}{2} \sin x + C$ (d) $\frac{\pi}{8}$

7(a)(i) $\frac{1}{2} \tan 2x - x + C$ (ii) $-2 \cot \frac{1}{2}x - x + C$

(b)(i) $\sqrt{3}-1-\frac{\pi}{12}$ (ii) $\frac{1}{4}\sqrt{3}-\frac{\pi}{12}$

8(a) $\frac{1}{8}(14-\pi)$ (b) $\frac{1}{2}(3+\sqrt{3})$

9(a) $\frac{\pi}{2}(\sqrt{3}-1) \text{ units}^3$ (b) $\frac{\pi}{2}(4-\pi) \text{ units}^3$

- (c) $\frac{\pi}{4}$ units³ (d) $\pi \ln 2$ units³
 (e) $\frac{1}{2}(4 - \pi)$ units³ (f) $\pi\sqrt{3}$ units³
 10(a) $\frac{1}{4}\sin^4 x + C$ (b) $-\frac{1}{5}\cot^5 x + C$
 (c) $\frac{1}{7}\sec^7 x + C$ (d) $\frac{2}{7}$ (e) $\frac{1}{3}$ (f) $\frac{2}{3}(4 - \sqrt{2})$
 11(a) $\sec x^2 + C$ (b) $-\ln(1 + \cot x) + C$
 (c) $\ln \sin e^x + C$ (d) $\frac{1}{2}e^{\sec 2x} + C$
 12(a) $\frac{1}{4}\sqrt{3}$ (b) $1 + \sqrt{2} - \sqrt{3}$ (c) $\frac{1}{2} + \frac{1}{3}\sqrt{3}$ (d) $\ln \frac{3}{2}$
 13(a) $(\pi + 2)$ units², $\frac{\pi}{2}(3\pi + 8)$ units³
 (b) $\frac{1}{2}(\sqrt{3} + 1)$ units², $\frac{\pi}{12}(4\pi + 9)$ units³
 (c) $\frac{1}{4}\sqrt{2}$ units², $\frac{\pi}{32}(\pi + 2)$ units³
 (d) $\frac{1}{2}\ln 3$ units², $\frac{2\pi}{\sqrt{3}}$ units³
 (e) $\frac{2\pi}{3} + \ln(7 + 4\sqrt{3})$ units²,
 $\frac{\pi}{3}(2\pi + 6\sqrt{3} + 6\ln(7 + 4\sqrt{3}))$ units³
 14(a) $\frac{1}{6}(\pi\sqrt{3} + 6\ln 2)$ (b) $\sqrt{3} - 1$ (c) $\frac{58}{15}$
 (d) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2} + 1)$ (e) $\frac{2}{3}(3\sqrt{3} - 2)$ (f) $\frac{3\pi}{16}$
 15 $\frac{1}{2}$
 17(b) $\frac{32}{3}\sqrt{2} + \frac{15}{2}\ln(\sqrt{2} + 1)$

Exercise 6C (Page 220)

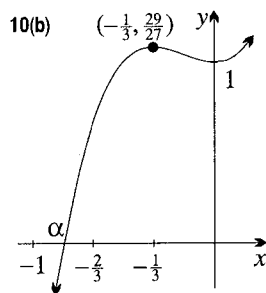
- 1(c) $\frac{1}{4}(1 + x^2)^4 + C$
 2(a) $\frac{1}{4}(2x + 3)^4 + C$ (b) $\frac{1}{5}(1 + x^3)^5 + C$
 (c) $\frac{-1}{1+x^2} + C$ (d) $2\sqrt{3x-5} + C$
 (e) $\frac{1}{4}\sin^4 x + C$ (f) $\log_e(1 + x^4) + C$
 3(c) $-\sqrt{1 - x^2} + C$
 4(a) $\frac{1}{24}(x^4 + 1)^6 + C$ (b) $\frac{2}{9}(x^3 - 1)^{\frac{3}{2}} + C$ (c) $\frac{1}{3}e^{x^3} + C$
 (d) $\frac{-1}{(1+\sqrt{x})^2} + C$ (e) $\frac{1}{6}\tan^3 2x + C$ (f) $-e^{\frac{1}{x}} + C$
 5(a) $\frac{65}{12}$ (b) $\sqrt{2} - 1$ (c) $\frac{1}{3}$ (d) $\frac{1}{24}$ (e) 2 (f) $\frac{1}{2}(e^2 - 1)$
 (g) $\frac{1}{10}$ (h) $\frac{\pi^4}{64}$ (i) 3 (j) $\frac{1}{2}\ln 3$
 6(a) $\frac{\pi}{12}$ units² (b) $\frac{\pi^2}{6}$ units³
 7(a) $\log_e \frac{3}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$
 8(a) $\sqrt{1 + e^{2x}} + C$ (b) $\ln(\ln x) + C$
 (c) $-\ln(\ln \cos x) + C$ (d) $\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C$
 9(a) $y = \frac{1}{2}\tan^{-1} e^{2x}$ (b) $y = \sin^{-1} \frac{x}{2} + \frac{x}{2} + \frac{1}{2}$
 10(b)(i) $\frac{2}{\ln 2}$ (ii) $\frac{1}{5}(4\sqrt{2} - 1)$
 11(a) $\ln 2$ (b) $\frac{e}{e+1}$
 12 $\tan^{-1} \sqrt{x-1} + C$
 13(a) $(3 - 2\ln 2)$ units² (b) $\frac{\pi}{4}(15 - 16\ln 2)$ units³
 14(a) $2\sin^{-1} \sqrt{x} + C_1$ (b) $\sin^{-1}(2x - 1) + C_2$

Exercise 6D (Page 223)

- 1(c) $\frac{1}{7}(x-1)^7 + \frac{1}{6}(x-1)^6 + C$
 2(a) $\frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$ (b) $\ln(x-1) - \frac{1}{x-1} + C$
 3(c) $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$
 4(a) $\frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$
 (b) $\frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + C$
 5(a) $(x+2) - 4\ln(x+2) + C$
 (b) $\frac{1}{3}(2x-1)^{\frac{3}{2}} + 2(2x-1)^{\frac{1}{2}} + C$
 (c) $\frac{3}{40}(4x-5)^{\frac{5}{2}} + \frac{5}{8}(4x-5)^{\frac{3}{2}} + C$
 (d) $2(1 + \sqrt{x}) - 2\ln(1 + \sqrt{x}) + C$
 6(a) $\frac{49}{20}$ (b) $2\ln 2 - \frac{1}{2}$ (c) $\frac{8}{9}$ (d) $\frac{1}{9}$ (e) $\frac{128}{15}$ (f) $\frac{4}{3}$
 (g) $4 - 6\ln \frac{5}{3}$ (h) $\frac{2517}{40}$
 7(a) $\sin^{-1} \frac{x+2}{3} + C$ (b)(i) $\frac{1}{\sqrt{3}}\tan^{-1} \frac{x+1}{\sqrt{3}} + C$
 (ii) $\sin^{-1} \frac{x+1}{\sqrt{5}} + C$ (iii) $\frac{\pi}{6}$ (iv) $\frac{\pi}{16}$
 8(b)(i) $\frac{1}{3}\tan^{-1} \frac{x}{3} + C$ (ii) $\cos^{-1} \frac{x}{\sqrt{3}} + C$
 (iii) $\frac{1}{2}\sin^{-1} 2x + C$ (iv) $\frac{1}{4}\tan^{-1} 4x + C$ (v) $\frac{\pi}{6}$ (vi) $\frac{\pi}{24}$
 9(b)(i) $\frac{x}{4\sqrt{4+x^2}} + C$ (ii) $\frac{\pi}{12} - \frac{\sqrt{3}}{8}$ (iii) π
 (iv) $-\frac{\sqrt{25-x^2}}{25x} + C$ (v) $-\frac{\sqrt{9+x^2}}{9x} + C$ (vi) $\frac{\sqrt{3}}{8}$
 10(b) $\frac{\pi}{8}(\pi + 2)$ units³
 11 $y = \sqrt{x^2 - 9} - 3\tan^{-1} \frac{\sqrt{x^2 - 9}}{3}$
 12 $\frac{1}{3}(6\sqrt{3} - 7\sqrt{2})$ units²
 14(a) $\ln(\sec \theta + \tan \theta) + C$
 15(b) $\frac{8}{3}$

Exercise 6E (Page 230)

- 1(b) $P(\frac{5}{2}) = \frac{1}{4}$ (c) The root is closer to $2\frac{1}{2}$.
 2(a)(ii) The root is between $\frac{3}{4}$ and 1. (b)(iii) 1.2
 3(a)(ii) 0.9 (b)(iii) 2.2 (c)(iii) 1.1
 4(a) 2.3 (c) 2.236 067 98
 5(a) $x_1 = 3.1$, $x_5 = 3.105 482 62$
 (b) $x_1 = 1.9$, $x_5 = 1.903 813 69$
 (c) $x_1 = 1.9$, $x_5 = 1.895 494 27$
 6(a) 2.42, 2.414 213 56 (b) 0.84, 0.843 734 28
 (c) 1.21, 1.203 947 57 (d) 0.85, 0.850 651 21
 (e) 2.22, 2.219 107 15 (f) 1.14, 1.141 890 55
 7(b) The root is between 2.5 and 2.625. (c) no
 8(a) 3.61, 3.605 551 28
 (b) 3.27, 3.271 066 31
 (c) 2.75, 2.752 525 92
 9 3.162 277 66

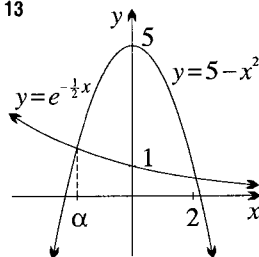


(c) 4 (d) The curve has negative gradient at $x = -\frac{1}{4}$, so the tangent at $x = -\frac{1}{4}$ will cut the x -axis further away from α .

11 No. The tangent at $x = a$ has positive gradient, and so will cut the x -axis further away from r .

12(a) A (b) A (c) A (d) D (e) C (f) B (g) B (h) A (i) D (j) C (k) C (l) C

13



14(a)(ii) It is an AP, with $a = d = \frac{1}{k}$.

(b) $x_{n+1} = x_n(1 + \frac{1}{k})$, and so the sequence is a GP with $a = r = 1 + \frac{1}{k}$. (c) e^{-kx} is steeper, and so approaches zero more quickly as $x \rightarrow \infty$.

16(e) $x_1 = \cot \theta_1 = \cot \left(\frac{m\pi}{2^n - 1} \right)$, where m and n are integers with $n \geq 1$ and $0 \leq m \leq n$.

Exercise 6F (Page 235)

1(b) $y'' = -\frac{1}{4}x^{-\frac{3}{2}}$, which is negative for $x > 0$.

2(a)(i) $\frac{\sqrt{3}}{4}$ square units (ii) $\frac{3\sqrt{3}}{2}$ square units

(b)(i) $\frac{1}{\sqrt{3}}$ square units (ii) $2\sqrt{3}$ square units

3(b)(i) e^{-1} (ii) $\frac{1}{2}(1 + e^{-1})$

4(a) $y = x$ (b) $y = \frac{2x}{\pi}$ (c) $y = \frac{3x}{\pi}$

5(a)(i) $\frac{1}{2}r^2 \sin x$ (ii) $\frac{1}{2}r^2 x$ (iii) $\frac{1}{2}r^2 \tan x$

6(b) $\frac{1}{3}$

7(a) $-1 < x < 1$ (b) $f'(x) = \frac{2}{1-x^2}$

8(b) $M(1, \frac{8}{9})$, $N(2, \frac{4}{9})$

(c) The area of $ABDC$ is $\frac{3}{4}$ square units and the area of $MNDC$ is $\frac{2}{3}$ square units.

9(e) zero, one, two, three, four, five

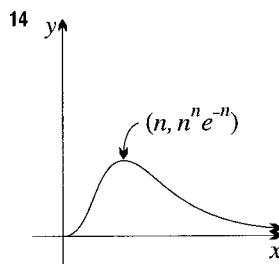
10(a) $y'' = -\frac{1}{x^2}$, which is negative for all $x > 0$.

(c) $(\frac{a+2b}{3}, \frac{\ln a + 2 \ln b}{3})$

11(a) $x = \frac{\pi}{4}$

12 $\frac{1}{2}$

13(a) $\frac{1}{6}$



(c) For $x > n$,
 $f(x) < f(n)$,
that is,
 $x^n e^{-x} < n^n e^{-n}$.

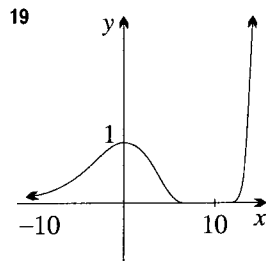
15(b) $\frac{4900}{3321}$ (c) $\frac{1}{2}(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1})$, limiting sum is $\frac{3}{4}$.

16(b) 3

17(b) $f(x)$ is stationary at $x = 1$ and increasing for all other positive values of x . Also, $f(0) = 0$. Hence the graph of $f(x)$ lies completely above the x -axis for $x > 0$.

18(c) From part (b), $\sqrt[n]{n}$ is not an integer. Therefore $\sqrt[n]{n}$ is not rational.

19



(b) $(0, 1)$ is a maximum turning point,
 $(10, 0)$ is a minimum turning point.

(c) As $x \rightarrow \infty$, $y \rightarrow \infty$,
and as $x \rightarrow -\infty$, $y \rightarrow 0$.

20(b)(i) $x > 1 + \sqrt{2}$ or $x < 1 - \sqrt{2}$ (c)(i) $a^n - b^n$

26(c) when $x_1 = x_2 = x_3 = \dots = x_n$

27(a) $y'''' = u''''v + 5u''''v' + 10u''''v'' + 10u''''v''' + 5u'v'''' + v''''$ (b) $(x^2 - 9x + 16)e^{-x}$

(c) $y^{(n)} = \sum_{k=0}^n {}^nC_k u^{(n-k)} v^{(k)}$

Chapter Seven

Exercise 7A (Page 245)

- 1(a) 300 500 (b) 125 (c)(i) $d = -3$ (ii) $T_{35} = -2$
 (iii) $S_n = \frac{1}{2}n(203 - 3n)$
 2(a)(i) $\frac{3}{2}$ (ii) 26 375 (iii) $|r| = \frac{3}{2} > 1$
 (b)(i) $\frac{1}{3}$ (ii) $|r| = \frac{1}{3} < 1$, $S_\infty = 27$
 3(a) \$48 000, \$390 000 (b) the 7th year
 4(a) \$62 053, \$503 116 (b) the 13th year
 5(a) \$25 000, \$27 500, \$30 000, $d = \$2500$
 (b) \$20 000, \$23 000, \$26 450, $r = 1.15$ (c) \$2727
 6(a) the 18th year (b) the 19th year
 7(a) SC50: 50%, SC75: 25%, SC90: 10% (c) 4
 (d) at least 7
 8(a) 2000 (b) 900 (c) 10 years
 9(a) 18 times (b) 1089 (c) Monday
 10(a) 6 metres, 36 metres, 66 metres (b) $30n - 24$
 (c) 6 (d) 486 metres
 11(a) $T_n = 3 \times (\frac{2}{3})^{n-1}$ (b) 4.5 metres (c)(ii) 16
 12 4 units
 13(a) $D = 3200$ (b) $D = 3800$ (c) the 15th year
 (d) $S_{13} = \$546 000$, $S_{14} = 602 000$
 14(a) the 10th year (b) the 7th year
 15(a) $(\frac{1}{2})^{\frac{1}{4}}$ (b) $S_\infty = \frac{F}{1 - (\frac{1}{2})^{\frac{1}{4}}} \doteq 6.29F$
 16(a)(i) $x \neq n\pi$, where $n \in \mathbf{Z}$, $S_\infty = \operatorname{cosec}^2 x$
 (ii) $x \neq \frac{\pi}{2} + n\pi$, where $n \in \mathbf{Z}$, $S_\infty = \sec^2 x$
 (b)(i) $\frac{1+t^2}{(1+t)^2}$ (ii) $\frac{1+t^2}{(1-t)^2}$
 17(b) at $x = 16$ (c)(i) at $x = 18$, halfway between
 the original positions (ii) 36 metres, the original
 distance between the bulldozers
 18(a) 125 metres (b) 118.75 metres
 (d) $a = 118.75$, $d = -6.25$, $\ell = 6.25$ and $n = 19$
 (e) $2 \times S_{19} + 125 = 20 \times 125$, which is $2\frac{1}{2}$ km.
 19(a) $\frac{1}{2} \cos \theta \sin \theta$ (b) $\sin^2 \theta$
 20(a) $\frac{1}{\sqrt{n}}$
 (d) No — the spiral keeps turning without bound.

Exercise 7B (Page 251)

- 1(a) \$6050 (b) \$25 600 (c) 11 (d) 5.5%
 2(a) \$59 750 (b) \$13 250
 3(a) Howard — his is \$21 350, and hers is \$21 320.
 (b) Juno — hers is now \$21 360.67.
 4(a) \$16 830.62 (b) \$8000 (c)(ii) 3 years (d) 7.0%
 5(a) \$1120 (b) \$1123.60 (c) \$1125.51 (d) \$1126.83
 (e) \$1127.34 (f) \$1127.47 The values in the
 previous parts are converging towards $1000 \times e^{0.12}$

- 6 \$101 608.52
 7 $A_n = P(1 + 0.12n)$ (a) 9 years (b) 17 years
 (c) 25 years (d) 75 years
 8 $A_n = P \times 1.12^n$ (a) 7 years (b) 10 years
 (c) 13 years (d) 21 years
 9(a) \$40 988 (b) \$42 000
 10(a) \$12 209.97 (b) 4.4% per annum
 11 8 years and 9 months
 12 \$1 110 000
 13 Sid
 14(a)(i) $A_n = P(1 + r)^n$ (ii) $n = \frac{\log 2}{\log(1+r)}$
 (b)(i) $B_n = P(1 + Rn)$ (ii) $R = \frac{1}{n}((1 + r)^n - 1)$
 15(a) $A_n = P + PRn$ (c) P is the principal, PRn is
 the simple interest and $\sum_{k=2}^n {}^nC_k R^k$ is the result of
 compound interest over and above simple interest.
 16(a) $\sum_{k=0}^n {}^nC_k PR^k$ (b) ${}^nC_k$ is the number of ways
 of choosing k years from the n years of the invest-
 ment. PR^k is the contribution to the interest for
 each of those sets of k years. In particular, PRn ,
 the simple interest, is the contribution to the total
 interest of applying the interest rate to all possi-
 ble combinations of one year. (c) In the life of a
 loan, more interest is earned from this term than
 from any other.

Exercise 7C (Page 255)

- 1(a)(i) $M \times 1.065^n$ (ii) $M \times 1.065^{n-1}$
 (iii) $M \times 1.065$
 (iv) $A_n = 1.065M + 1.065^2M + \dots + 1.065^nM$
 (c) \$188 146 and \$75 000 (d)(i) $\frac{300\,000}{188\,146} \times M \doteq \$4784$
 (ii) \$1784
 2(a) M , $1.04M$, $1.04^{n-1}M$ (c) \$893 342
 3(a) \$200 000 (b) \$67 275 (c) \$630 025
 4(a) \$360 (b) \$970.27
 5(a) \$31 680 (b) \$394 772 (c) \$1 398 905
 6(a) \$67 168.92 (b) \$154 640.32
 7 \$3086
 8(a) \$25 718.41
 (b) $\$25\,718.41 + \$23\,182.17 = \$48\,900.58$
 9(a) \$286 593 (b)(i) \$107 355 (ii) \$152 165
 10(a) \$424 195.23 (b) \$431 235.13
 11 \$55 586.38
 12(c) $A_2 = 1.01M + 1.01^2M$,
 $A_3 = 1.01M + 1.01^2M + 1.01^3M$,
 $A_n = 1.01M + 1.01^2M + \dots + 1.01^nM$

(e) \$4350.76 (f) \$363.70

13(b) $A_2 = 1.002 \times 100 + 1.002^2 \times 100$, $A_3 = 1.002 \times 100 + 1.002^2 \times 100 + 1.002^3 \times 100$, $A_n = 1.002 \times 100 + 1.002^2 \times 100 + \dots + 1.002^n \times 100$

(d) about 549 weeks

14(b) exponential (c) \$10 436 (d)(i) They are the same. (ii) Superannuation is an approximation

for the area under the graph of compound interest.

15(a) The function FV calculates the value just after the last premium has been paid, not at the end of that year.**Exercise 7D (Page 259)**1(a)(i) $P \times 1.015^n$ (ii) $M \times 1.015^{n-1}$ (iii) $M \times 1.015^{n-2}$ and M (iv) $A_n = P \times 1.015^n - (M + 1.015M + 1.015^2M + \dots + 1.015^{n-1}M)$ (c) 0 (d) $M = \frac{P \times 1.015^n \times 0.015}{1.015^n - 1}$ (e) \$2542(a) $A_n = P \times 1.006^n - (M + 1.006M + 1.006^2M + \dots + 1.006^{n-1}M)$

(c) \$162 498, which is more than half.

(d) -\$16 881 (f) 8 months

3(a) The loan is repaid in 25 years. (b) \$1226.64

(c) \$367 993 (d) \$187 993 and 4.2%

4 \$345

5(a) \$4202 (b) $A_{10} = \$6.65$ (c) Each instalment is approximately 48 cents short because of rounding.

6 \$216 511

7 It will take 57 months, but the final payment will be lower than usual.

8(a) \$160 131.55 (b) $\$1633.21 < \$1650$, so the couple can afford the loan.

9 \$44 131.77

10(a) \$2915.90 (b) \$84.10

11(b) zero balance after 20 years (c) \$2054.25

12(c) $A_2 = 1.005^2 P - M - 1.005 M$, $A_3 = 1.005^3 P - M - 1.005 M - 1.005^2 M$, $A_n = 1.005^n P - M - 1.005 M - \dots - 1.005^{n-1} M$

(e) \$1074.65 (f) \$34 489.78

13(b) $A_2 = 1.008^2 P - M - 1.008 M$, $A_3 = 1.008^3 P - M - 1.008 M - 1.008^2 M$, $A_n = 1.008^n P - M - 1.008 M - \dots - 1.008^{n-1} M$

(d) \$136 262

(e) $n = \log_{1.008} \frac{125M}{125M - P}$, 202 months

14(a) \$542 969.89 (b) \$285 151.16

15(b) 10.5% per annum

16(a) \$839 343 (b) \$6478

Exercise 7E (Page 264)1(b) $1 \text{ m}^2/\text{s}$ (c) 7 metres (d) 9 m^2 2(a) $A = \frac{1}{2} \ell^2$ (c)(i) $5 \text{ m}^2/\text{s}$ (ii) $3 \text{ m}^2/\text{s}$

(d) 34 metres

3(a) $15.1 \text{ m}^3/\text{s}$ (b) $30.2 \text{ m}^2/\text{s}$ 4(b) $\frac{2}{9\pi} \text{ cm/s}$ (c) $\frac{10}{\sqrt{\pi}} \text{ cm}$, $\frac{4000}{3\sqrt{\pi}} \text{ cm}^3$ 5(a) $90\,000\pi \text{ mm}^3/\text{min}$ (b) the rate is constant at $6\pi \text{ mm/min}$.6(b) $\frac{1}{6} \text{ cm/s}$

7(b) 5 degrees per second

8(a) $V = \frac{4}{3}\pi h^3$ (b) $\frac{1}{32\pi} \text{ m/s}$

9 2 degrees per minute

11(a) $-2\sqrt{1-x^2}$ (b) -2 m/s — as the point crosses the y -axis it is travelling horizontally at a speed of 2 m/s .12(a) $\frac{2CV^2}{L^2} \text{ m/s}^2$ (c) As L decreases, the speed passing the truck increases, so the driver should wait as long as possible before beginning to accelerate. A similar result is obtained if the distance between car and truck is increased. Optimally, the driver should allow both L to decrease and C to increase.

(d) 950 metres

13(b) This is just two applications of the chain rule.

(d) 6

14(c) $x = h = 50(\sqrt{3} + 1) \text{ metres}$ (d) 200 km/h **Exercise 7F (Page 268)**1(a) 25 minutes (b) $V = 5(t^2 - 50t) + 3145$

(c) 3145 litres

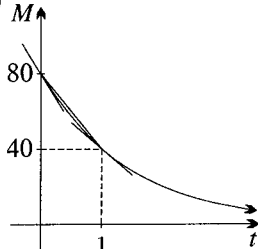
2(a) $P = 6.8 - 2 \log(t + 1)$

(b) approximately 29 days

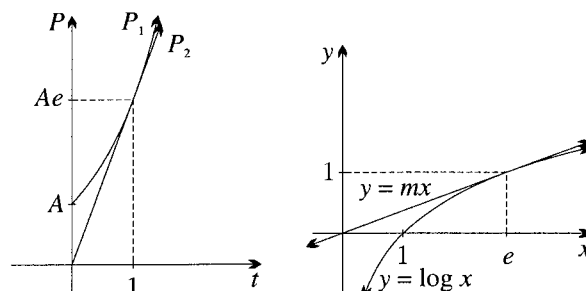
3(a) $-2 \text{ m}^3/\text{s}$ (b) 20 s (c) $V = 520 - 2t + \frac{1}{20}t^2$ (d) 20 m^3 (e) 2 minutes and 20 seconds4(a) no (b) $x = \frac{5}{2}(1 - e^{-0.4t})$ (c) $t \doteq 1.28$ (d) $x = \frac{5}{2}$ 5(a) 0 (b) 250 m/s (c) $x = 1450 - 250(5e^{-0.2t} + t)$ 6(a) $I = 18\,000 - 5t + \frac{48}{\pi} \sin \frac{\pi}{12}t$ (b) $\frac{dI}{dt}$ has a maximum of -1 , so it is always negative. (c) There will be 3600 tonnes left.7(a) $\theta = \tan^{-1}t + \frac{\pi}{4}$ (b) $t = \tan(\theta - \frac{\pi}{4})$ (c) As $t \rightarrow \infty$, $\tan^{-1}t \rightarrow \frac{\pi}{2}$, and so $\theta \rightarrow \frac{3\pi}{4}$.

- 8(a) 1200 m^3 per month at the beginning of July
 (b) $W = 0.7t - \frac{3}{\pi} \sin \frac{\pi}{6}t$
 9(b) $r = k(t - 12)$ (c) $k = -\frac{1}{48}$
 10(a) $V = \frac{1}{3}\pi r^3$ (b) $\frac{dr}{dt} = \frac{1}{2\pi r^2}$
 (c) $t = \frac{2\pi}{3}(r^3 - 1000)$ (d) 25 minutes 25 seconds
 11(a) $V = \frac{\pi}{3}(128 - 48h + h^3)$ (b)(i) $A = \pi(16 - h^2)$
 (iii) 1 hour 20 minutes

Exercise 7G (Page 273)

- 1(b) 1350 (c) 135 per hour (d) 23 hours
 2(c) 6.30 grams, 1.46 grams per minute
 (d) 6 minutes 58 seconds (e) 20 g, $20e^{-k} \doteq 15.87$ g,
 $20e^{-2k} \doteq 12.60$ g, $20e^{-3k} = 10$ g,
 $r = e^{-k} = 2^{\frac{1}{3}} \doteq 0.7937$
 3(b) $-\frac{1}{5} \log \frac{7}{10}$ (c) 10 290 (d) At $t \doteq 8.8$, that is,
 some time in the fourth year from now.
 4(b) 30 (c)(i) 26 (ii) $\frac{1}{5} \log \frac{15}{13}$ (or $-\frac{1}{5} \log \frac{13}{15}$)
 5(a) 80 g, 40 g, 20 g, 10 g (e) 
 (b) 40 g, 20 g, 10 g.
 During each hour, the average mass loss is 50%.
 (c) $M_0 = 80$,
 $k = \log 2 \doteq 0.693$
 (d) 55.45 g/hr, 27.73 g/hr, 13.86 g/hr, 6.93 g/hr
 6(a) $C = C_0 \times 1.01^t$ (i) $1.01^{12} - 1 \doteq 12.68\%$
 (ii) $\log_{1.01} 2 \doteq 69.66$ months (b) $k = \log 1.01$
 (i) $e^{12k} - 1 \doteq 12.68\%$ (ii) $\frac{1}{k} \log 2 \doteq 69.66$ months
 7(b) $L = \frac{1}{2}$
 8(c) 25 (d) $\frac{k}{A} = \frac{1}{2} \log \frac{5}{3}$ (or $-\frac{1}{2} \log \frac{3}{5}$)
 (e) 6 hours 18 minutes
 9(b) $C_0 = 20\,000$, $k = \frac{1}{5} \log \frac{9}{8} \doteq 0.024$
 (c) 64 946 ppm (d)(i) 330 metres from the cylinder
 (ii) If it had been rounded down, then the concentration would be above the safe level.
 10(a) $y(3) = A_0 e^{3k} = A_0(e^k)^3$ and we know that $e^k = \frac{3}{4}$. (b) $y(3) = \frac{27}{64}A_0$
 11(a)(ii) $k = \frac{1}{12} \log \frac{122}{105}$ (b)(ii) $\ell = \frac{1}{12} \log \frac{217}{100}$
 (c) At $t = \frac{\log \frac{525}{100}}{\ell - k} \doteq 31.85$, that is, in the 32nd month.
 (d) $\ell C = \ell \times 100 \times e^{32\ell} \doteq 51$ cents per month
 12(a) $\frac{dV}{dt} = -0.15V$ (b) $V = 12\,000 e^{-0.15t}$
 (c) \$10 328.50, a decrease of about 13.9%
 (d) \$1549.27 per year (e) At $t \doteq 15.4$, that is, during the 16th year.

- 13(a) $A = 5000 \times 1.07^t$ (c) $A = 5000 \times e^{t \log 1.07}$
 (e) $A = \$7503.65$ (f) \$507.69 per year
 14(a)(ii) (b)(i)



- (c) In part (a), changing the base is equivalent to stretching the graph horizontally. Since both curve and straight line are equally stretched, the straight line will still pass through the origin. The same is true in part (b) except that the stretch is vertical.
 (d) The graph in part (b) is just a reflection in the line $y = x$ of the graph in part (a).
 15(a) $B = \frac{2N_0^2}{N_c}$ and $C = \left(\frac{N_0}{N_c}\right)^2$ (b) $\frac{B}{C} = 2N_c$

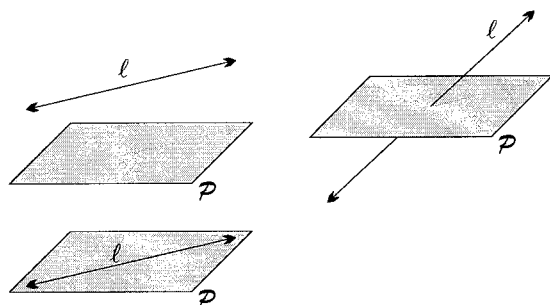
Exercise 7H (Page 279)

- 1(a)(ii) 12 000, $P \rightarrow \infty$ as $t \rightarrow \infty$
 (b)(ii) 12 000, $P \rightarrow 10\,000$ as $t \rightarrow \infty$
 (c)(ii) 8000, $P \rightarrow 10\,000$ as $t \rightarrow \infty$
 2(b) $A = 1000$, $k = \frac{1}{3} \log 6$ (c) 67 420 bugs
 (d) 10.4 weeks
 3(b) $B = 970\,000$, $k = -\frac{1}{10} \log \frac{47}{97} = \frac{1}{10} \log \frac{97}{47}$
 (c) 158 000 flies (d) 73 days
 4(b) $T_e = 20$, $A = 70$ (c) $k = \frac{1}{6} \log \frac{7}{3}$ [Alternatively, $k = -\frac{1}{6} \log \frac{3}{7}$.] (d) 13 minutes 47 seconds
 5(a) $A = 34$ (b) $\frac{1}{45} \log 2$ (or $-\frac{1}{45} \log \frac{1}{2}$) (c) 16.5°C
 6(a) $1 - e^{-\frac{1}{16}t}$ is always positive for $t > 0$. The body is falling. (b) It is the acceleration of the body. (c) -160 m/s (d) $16 \log \frac{8}{7} \doteq 2.14 \text{ s}$
 7(b)(i) 15 cm (ii) 15 is the average (iii) 15
 (c) $\frac{1}{5} \log \frac{5}{3}$
 8(b) $-\frac{V}{R}$ (c) $I \rightarrow \frac{V}{R}$ (d) $4.62 \times 10^{-4} \text{ s}$
 9(b) $M \rightarrow a$ as $t \rightarrow \infty$ (c) $k = \frac{1}{120} \log_e 100$
 (d) 2 minutes 45 seconds
 10(a) $2w \text{ g/min}$ (b) $\frac{Q}{1000} \text{ g/L}$ (c) $\frac{Qw}{1000} \text{ g/min}$
 (f) -2000 (g) $Q \rightarrow 2000$
 (h) $w = \frac{1000}{345} \log 2 \doteq 2 \text{ L/min}$
 12(b) $A = 1000$, $I = 9000$ and $k = \frac{1}{7} \log 3$
 (c) 36 000
 13 The man's coffee is cooler.

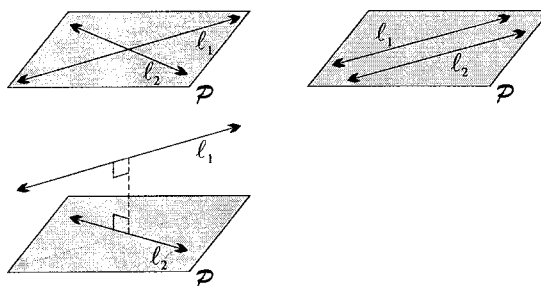
Chapter Eight

Exercise 8A (Page 288)

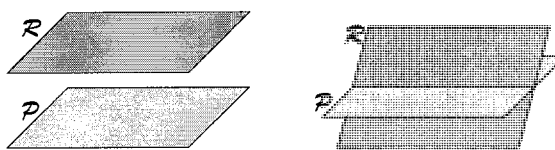
- 1(a) 70° (b) 45° (c) 60° (d) 50° (e) 22°
 (f) $\alpha = 153^\circ$, $\beta = 27^\circ$ (g) 34°
 (h) $\alpha = 70^\circ$, $\beta = 70^\circ$
 2(a) 35° (b) 43° (c) 60° (d) $\alpha = 130^\circ$, $\beta = 50^\circ$
 (e) $\alpha = 123^\circ$, $\beta = 123^\circ$ (f) 60°
 (g) $\alpha = 65^\circ$, $\beta = 65^\circ$ (h) $\alpha = 90^\circ$, $\beta = 90^\circ$
 3(a) equal alternate angles (b) equal corresponding angles (c) supplementary co-interior angles (d) supplementary co-interior angles
 5(a) $\alpha = 52^\circ$, $\beta = 38^\circ$ (b) $\alpha = 30^\circ$, $\beta = 60^\circ$
 (c) 24° (d) 36° (e) $\alpha = 15^\circ$, $\beta = 105^\circ$, $\gamma = 60^\circ$, $\delta = 105^\circ$ (f) 24° (g) 15° (h) 22°
 6(a) $\alpha = 75^\circ$, $\beta = 105^\circ$ (b) $\alpha = 252^\circ$, $\beta = 72^\circ$
 (c) 32° (d) 62° (e) 60° (f) 135° (g) 48° (h) 35°
 10(a) $\theta = 58^\circ$ (b) $\theta = 37^\circ$, $\phi = 15^\circ$ (c) $\theta = 10^\circ$
 (d) $\theta = 12^\circ$, $\phi = 41^\circ$
 11(a) $\angle DOB$ and $\angle COE$ are straight angles; $\angle BOC$ and $\angle DOE$ are vertically opposite angles, and so are $\angle BOE$ and $\angle COD$.
 (b) $GA \parallel BD$ (alternate angles are equal)
 (c) $\angle BOE = 90^\circ$
 12(a) $\alpha = 60^\circ$ (b) $\alpha = 90^\circ$ (c) $\alpha = 105^\circ$
 14 $\angle FBE = \angle FBD + \angle DBE$
 $= \frac{1}{2}(\angle ABD + \angle BDC) = \frac{1}{2}180^\circ = 90^\circ$
 16 $\angle FBE = \frac{k}{k+l} \times 180^\circ$
 17(a) two walls and the ceiling of a room
 (b) three pages of a book
 (c) the floors of a multi-storey building
 (d) the sides of a simple tent and the ground
 (e) the floor, ceiling and one wall of a room
 (f) a curtain rod in front of a window pane
 (g) the corner post of a soccer field
 18(a) The line is parallel to the plane, or intersects with it at a point, or lies in the plane.



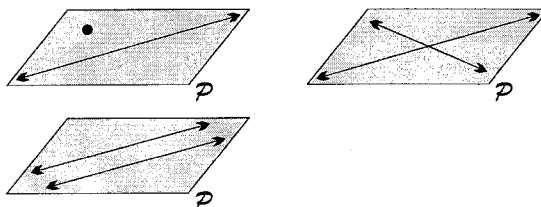
- (b) The lines lie in one plane and intersect, or lie in one plane and are parallel, or are skew.



- (c) The two planes are parallel, or intersect.



- 19(a) a point and a line, or two intersecting lines, or two parallel lines



- (b) AB and CD , BC and AD , CA and BD

Exercise 8B (Page 295)

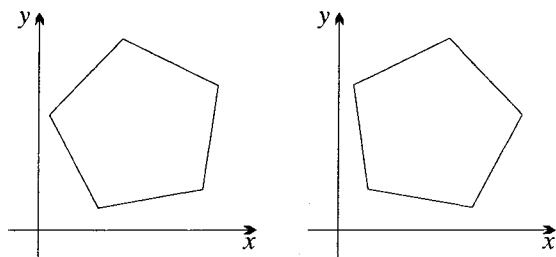
- 1(a) 55° (b) 55° (c) 52° (d) 70° (e) 60° (f) 30°
 (g) 18° (h) 20°
 2(a) 108° (b) 129° (c) 24° (d) 74°
 3(a) 99° (b) 138° (c) 65° (d) 56° (e) 60° (f) 80°
 (g) 36° (h) 24°
 5(a)(i) 108° (ii) 72° (b)(i) 120° (ii) 60° (c)(i) 135°
 (ii) 45° (d)(i) 140° (ii) 40° (e)(i) 144° (ii) 36°
 (f)(i) 150° (ii) 30°
 6(a)(i) 8 (ii) 10 (iii) 45 (iv) 180 (b)(i) 5 (ii) 9
 (iii) 20 (iv) 720
 (c) Solving for n does not give an integer value.
 (d) Solving for n does not give an integer value.
 8(a) $\alpha = 59^\circ$, $\beta = 108^\circ$ (b) $\alpha = 45^\circ$, $\beta = 60^\circ$
 (c) $\alpha = 76^\circ$, $\beta = 106^\circ$ (d) $\alpha = 110^\circ$, $\beta = 50^\circ$
 (e) $\alpha = 106^\circ$, $\beta = 40^\circ$ (f) $\alpha = 51^\circ$, $\beta = 43^\circ$
 (g) $\alpha = 104^\circ$, $\beta = 48^\circ$ (h) $\alpha = 87^\circ$
 9(a) 50° (b) $\theta = 35^\circ$, $\phi = 40^\circ$ (c) $\theta = 40^\circ$, $\phi = 50^\circ$
 (d) $\theta = 108^\circ$, $\phi = 144^\circ$
 10(a) 23° (b) 17° (c) 22° (d) 31° (e) 44° (f) 38°
 (g) 60° (h) 45°

- 11(a) 140° (b) 68° (c) 47° (d) 50°
 16(a) $\frac{n-2}{2}$ (b)(i) No, because $n = 3\frac{1}{3}$, which is not an integer. (ii) Yes, $n = 9$ and it is a nonagon.
 17(a) 360° (b) They are the same.
 19(a) $\theta > 60^\circ$ (b) $\theta < 120^\circ$
 20(a) $\frac{720}{n}$ (b) $m = \frac{2n}{n-4}$ (c) $n = 5$ gives a pentagon and decagon, $n = 6$ gives a hexagon with itself, $n = 8$ gives an octagon and a square, $n = 12$ gives a dodecagon and an equilateral triangle.
 (d) For $n = 5$, $2 \cos 36^\circ \doteq 1.62$. For $n = 6$, 1.
 For $n = 8$, $\frac{1}{\sqrt{2}}$. For $n = 12$, $\frac{1}{\sqrt{3}}$.

$$24(a) \frac{(\cos \alpha - 1) + \sqrt{2(1 - \cos \alpha)}}{(\cos \alpha + 1)},$$

$$\text{where } \alpha = \frac{(n-2)180^\circ}{n}. \quad (b)(i) \frac{1}{3} \quad (ii) \sqrt{2} - 1$$

- 25(a) exterior angle of triangle theorem
 (b) One angle is obtuse. (c) Nothing. $m_1 m_2 m_3$ may be positive, negative, zero, or even undefined if one line is vertical.
 26 If n is even, then the product must be positive because opposite sides are parallel. If n is odd, then the product could be positive or negative, depending on its orientation.



- 27(a) 1 (b) 0 (c) 2

Exercise 8C (Page 304)

- 1(a) $\triangle ABC \equiv \triangle RQP$ (AAS)
 (b) $\triangle ABC \equiv \triangle CDA$ (SSS)
 (c) $\triangle ABC \equiv \triangle CDE$ (RHS)
 (d) $\triangle PQR \equiv \triangle GEF$ (SAS)
 2(a) $\triangle ABC \equiv \triangle DEF$ (AAS), $x = 4$
 (b) $\triangle GHI \equiv \triangle LKJ$ (RHS), $x = 20$
 (c) $\triangle QRS \equiv \triangle UTV$ (SAS), $x = \sqrt{61}$
 (d) $\triangle MLN \equiv \triangle MPN$ (AAS), $x = 12$
 3(a) $\triangle ABC \equiv \triangle FDE$ (SSS), $\theta = 67^\circ$
 (b) $\triangle XYZ \equiv \triangle XVW$ (SAS), $\theta = 86^\circ$
 (c) $\triangle ABC \equiv \triangle BAD$ (SSS), $\theta = 49^\circ$
 (d) $\triangle PQR \equiv \triangle HIG$ (RHS), $\theta = 71^\circ$
 4(a) $\theta = 64^\circ$ (b) $\theta = 69^\circ$ (c) $\theta = 36^\circ$ (d) $\theta = 84^\circ$
 (e) $\theta = 64^\circ$ (f) $\theta = 90^\circ$ (g) $\theta = 45^\circ$ (h) $\theta = 120^\circ$

- 5(a) The diagram does not show a pair of equal sides. The correct reason is AAS.
 (b) The diagram does not show a pair of equal hypotenuses. The correct reason is SAS.
 6(a) $\triangle AXB \equiv \triangle CXD$ (SAS)
 (b) $\triangle ABD \equiv \triangle CBD$ (SSS)
 (c) $\triangle ABC \equiv \triangle ADC$ (RHS)
 (d) $\triangle ABF \equiv \triangle DEC$ (AAS)
 7 This is the spurious ASS test — the angles are not the included angles.
 8 In both cases, two sides are given but not the included angle.
 9(a) It has reflection symmetry in its altitude.
 (b) It has reflection symmetry in each of its altitudes. It has rotational symmetry of 120° and 240° about the point where the altitudes are concurrent.
 10(a)(ii) AM (b)(i) Each altitude is an axis of symmetry. (ii) There is 60° rotational symmetry about the point where the three altitudes meet.
 12 $AB = AC = BC$ by construction
 13(a) SSS
 16(b) SAS
 17(b) SAS
 18(a) SSS (b) The base angles are equal.
 (c) $CX = AC - AX = BD - BX = DX$
 (d) equal alternate angles
 19(a)(i) 66° (ii) 24° (b)(i) $\angle A = \angle C = 2\alpha$
 (ii) $\triangle ABC$ is equilateral, and in triangle ABE , $\angle A = \angle E = 30^\circ$.
 20(a) $\angle BXY = \alpha + \beta = \angle BYX$,
 hence $\triangle BXY$ is isosceles (base angles equal).
 (b)(i) $\triangle ADX \equiv \triangle CDX$ (AAS),
 hence $AD = CD$. (ii) $\triangle CDB \equiv \triangle ADB$ (SAS)
 21(a) exterior angle of $\triangle ACD$ (b) $180^\circ - (2\alpha + \beta)$
 (c) $\angle EDB = \frac{1}{2}\beta$
 22(a) Two equal radii form two sides of each triangle. (b) SSS (c) AAS or SAS (d) matching sides and matching angles, $\triangle AMO \equiv \triangle BMO$
 23(a) $AB = BC$ (given), $\angle CAB = 36^\circ$ (b) SAS
 (c) 36° (d) area = $x^2 \sin 108^\circ + \frac{1}{4}x^2 \tan 72^\circ$
 24(a) SAS (b) AAS
 (c) $\triangle OQM \equiv \triangle OSM$ (SAS or SSS)
 25(a) SAS (b) SAS
 26(a) exterior angle of $\triangle ABP$
 (b) base angles of isosceles $\triangle PBC$

28 $\triangle BDM$ is an isosceles right-angled triangle with the right angle at M .

Exercise 8D (Page 311)

1(a) $\alpha = 115^\circ$, $\beta = 72^\circ$ (b) $\alpha = 128^\circ$, $\beta = 52^\circ$

(c) $\alpha = 90^\circ$, $\beta = 102^\circ$ (d) $\alpha = 47^\circ$, $\beta = 133^\circ$

2(a) $\alpha = 27^\circ$, $\beta = 99^\circ$ (b) $\alpha = 41^\circ$, $\beta = 57^\circ$

(c) $\alpha = 40^\circ$, $\beta = 100^\circ$ (d) $\alpha = 30^\circ$, $\beta = 150^\circ$

3 Test for a parallelogram: two opposite sides are equal and parallel.

4 Test for a parallelogram: diagonals bisect each other.

5 No. It could be a trapezium with a pair of equal but non-parallel sides.

6(a) 180° about the intersection of the diagonals

(b) A trapezium with equal non-parallel sides has reflection symmetry.

7 $\sin(180^\circ - \theta) = \sin \theta$, that is, the sine of an angle and its supplement are equal.

8(a) Co-interior angles are supplementary.

(b)(i) AAS (c)(i) AAS

9(a)(i) angle sum of a quadrilateral (ii) Co-interior angles are supplementary. (b)(i) SSS (iii) Test for a parallelogram: opposite angles are equal.

(c)(i) SAS (ii) Test for a parallelogram: opposite sides are equal. (d)(i) SAS

10(a) SAS (b) matching angles, $\triangle BAD \equiv \triangle ABC$ (d) Co-interior angles are supplementary.

11(a) Properties of a parallelogram: opposite angles are equal. (b) Properties of a parallelogram: opposite sides are equal. (c) SAS

(d) A quadrilateral with equal opposite sides is a parallelogram.

12(a) SAS (b) SAS (c) Test for a parallelogram: opposite sides are equal. Alternatively, use the equality of alternate angles to prove that the opposite sides are parallel.

13(a) AAS (b) Z is the midpoint of AC , and this is where BD meets AC .

14 In the first question, $AD = CB$ (opposite sides of parallelogram $ABCD$), and hence $AY = CX$. Thus the sides AY and CX are equal and parallel, and so $AYCX$ is a parallelogram.

In the second question, the diagonals BD and AC of the parallelogram $ABCD$ bisect each other. Hence the intervals BD and PQ bisect each other, and so $BPDQ$ is also a parallelogram.

In the third question, $AXCY$ is a parallelogram because the sides AX and CY are equal and parallel. Hence XY meets AC at the midpoint of AC , which is where BD meets AC .

16 In both parts, $DABF$ is a parallelogram, so $\angle F = \alpha$ (opposite angles are equal) and $FB = DA$ (opposite sides are equal). Also, $\angle ABC = \angle BCF$ (alternate angles, $DF \parallel AB$).

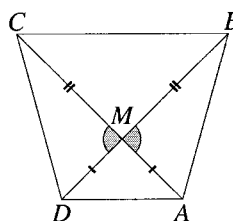
(a)(ii) $\triangle BFC$ is isosceles, so $\angle BCF = \alpha$.

(b)(ii) $\angle BCF = \angle F = \alpha$, hence $\triangle BCF$ is isosceles.

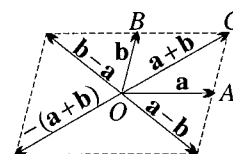
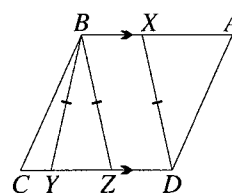
17(b) $\triangle AMD$ is isosceles.

18(b) a trapezium

(c) $DX > DA$



19



20(a) Test for a parallelogram: one pair of opposite sides are equal and parallel. (b) AAS (c) equal corresponding angles (d) Repeated use of the result in part (c) yields $PQ \parallel SR$ and $QR \parallel PS$. Hence $PQRS$ is a parallelogram by definition.

Exercise 8E (Page 316)

1(a) 45° (b) 76° (c) 15° (d) 9°

2(a) $\alpha = 15^\circ$, $\phi = 105^\circ$

3(a)(i) A rectangle has horizontal and vertical reflection symmetries. It has rotation symmetry of 180° about the intersection of its diagonals.

(ii) A rhombus has reflection symmetries in each diagonal. It has rotation symmetry of 180° about the intersection of its diagonals.

(iii) A square has horizontal and vertical reflection symmetries, as well as those in each diagonal. It has rotation symmetries of 90° , 180° and 270° about the intersection of its diagonals.

(b) A circle has reflection symmetry in every diagonal,

and rotation symmetry of every number of degrees about the centre.

4(a) The diagonals bisect each other at right angles. **(b)** The diagonals are equal and bisect each other. **(c)** By parts (a) and (b), $ABCD$ is both a rectangle and a rhombus.

5(a) Test for a rhombus: all sides are equal.

(b) Property of a rhombus: diagonals bisect vertex angles.

6(a) base angles of isosceles triangle ABD

(b) alternate angles, $AB \parallel DC$ **(d)** angle sum of triangles

7(a) Since opposite sides are equal, it is a parallelogram and it has a pair of equal adjacent sides.

(b)(i) Test for a parallelogram: the diagonals bisect each other. **(ii)** SAS **(c)(i)** half the angle sum of a quadrilateral **(iii)** Test for a parallelogram: opposite angles are equal. **(iv)** The base angles of $\triangle ABD$ are equal.

8(b)(ii) SAS

9(b)(i) Test for a parallelogram: diagonals bisect each other.

(ii) base angles of isosceles triangle ABM

(iii) base angles of isosceles triangle BCM

10(a)(i) SAS **(b)(i)** SAS

11(a) Test for a rhombus: all sides are equal.

(b) Properties of a rhombus: diagonals bisect each other at right angles.

12(a) Test for a rhombus: all sides are equal.

(b) Properties of a rhombus: diagonals bisect each other at right angles.

13(a) Test for a rhombus: all sides are equal.

14(a) Test for a rhombus: all sides are equal.

(b) Properties of a rhombus: diagonals bisect each other at right angles.

15(a)(i) SAS **(ii)** Test for a rhombus: all sides are equal. **(b)(i)** Opposite sides are parallel by construction. **(ii)** Test for a rhombus: diagonals bisect vertex angles.

16(a) AAS **(b)** By definition: $PQRS$ is a parallelogram with a pair of adjacent sides equal.

17(a) SAS

18(a) Property of a rhombus: diagonals bisect vertex angles. **(b)** SAS

(c) alternate angles, $AD \parallel BC$ **(d)** 90°

19(a) alternate angles, $BC \parallel AR$ **(b)** The diagonals of a rectangle are equal and bisect each other.

20(a)(i) SSS **(b)(i)** SAS

Exercise 8F (Page 322)

1(a) 33 **(b)** 50 **(c)** 28 **(d)** 72

2(a) $A = 36$, $P = 24$ **(b)** $A = 18$, $P = 12\sqrt{2}$

(c) $A = 60$, $P = 32$ **(d)** $A = 48$, $P = 28$

3(a) $A = 54$, $P = 18 + 4\sqrt{13}$, diagonals: $\sqrt{205}$ and $\sqrt{61}$ **(b)** $A = 264$, $P = 72$, diagonals: 30 and $4\sqrt{37}$ **(c)** $A = 120$, $P = 52$, diagonals: 24 and 10

(d) $A = 600$, $P = 100$

NOTE: The second diagonal, which is 30, is most easily obtained from the area of the rhombus.

4(a) A square is a rhombus, so the result follows from the area formula for a rhombus. **(b)** $s^2 = ab$, so $s = \sqrt{ab}$. **(c)** Since $C = 90^\circ$, $\sin C = 1$, so the trigonometric formula becomes $A = \frac{1}{2}ab$, and a and b are the base and the perpendicular height.

6(a) Both triangles have the same base and altitude — the distance between the parallel lines.

(b) $\triangle BCX = \triangle ABC - \triangle ABX$
 $= \triangle ABD - \triangle ABX = \triangle ADX$

7(a) sums of equal areas **(b)** Triangles with the same base and area have the same altitude.

8 Any two adjacent triangles have the same height and equal bases. They will all be congruent when the parallelogram is also a rhombus.

9(a) Properties of a parallelogram: the diagonals bisect each other. **(b)** $a^2 : b^2$

(c) When the parallelogram is a rectangle.

10(a) $2x^2 - 2x + 1$ **(b)** $\frac{1}{2} \text{ m}^2$ when $x = \frac{1}{2} \text{ metre}$.

11(a) 20 m^2 **(b)** 76 m^2

12(a)(i) $\frac{\sqrt{3}}{4}$ **(ii)** $\frac{3\sqrt{3}}{2}$ **(b)(i)** $\frac{1}{\sqrt{3}}$ **(ii)** $2\sqrt{3}$

(c) The area of the inscribed hexagon is smaller than the circle, which is in turn smaller than the escribed hexagon.

13(a) AAS **(b)** $AG = \frac{7}{4}$ **(c)** $37\frac{1}{2}$

14(a) the area of the two triangles formed by one diagonal **(b)** $a^2 + b^2 - 2ab \cos \theta$, $a^2 + b^2 + 2ab \cos \theta$ **(c)** area of annulus $= \pi ab |\cos \theta|$

15(a) In the trapezium $DBRC$, the areas of $\triangle CPR$ and $\triangle DBP$ are equal, proven in question 15 of Section 8D. The result then follows. **(b)** $\frac{1}{5}$

16(a) The fat rhombus has angles of 72° and 108° . The thin rhombus has angles of 36° and 144° .

17(a) AQ dissects $\triangle OQX$ into two triangles of equal area, hence AQ bisects the base OX .

(c) $\frac{\pi}{6\sqrt{2-\sqrt{2}}}$ **(d)** $\frac{\pi}{12}(\sqrt{2} + 1)$

$$(e) AR = \frac{1}{12} (\pi(\sqrt{2} + 1) - 6),$$

$$RX = \frac{1}{12} (12 - \pi(\sqrt{2} + 1)) \quad (g) 162^\circ 48'$$

Exercise 8G (Page 326)

1 a, c, d

2(a) $c = 13$ (b) $c = \sqrt{41}$ (c) $a = 5\sqrt{7}$

(d) $b = 2\sqrt{10}$

3 The cosine rule is $c^2 = a^2 + b^2 - 2ab \cos C$, but here $C = 90^\circ$ and $\cos 90^\circ = 0$, so the third term disappears, giving Pythagoras' theorem.

4(a) $29'49''$ (b) $20'7''$

5(a) $a^2 = s^2 - b^2$ (i) 108 cm^2 (ii) $40\sqrt{14} \text{ cm}^2$

(b) This is an equilateral triangle with $a = b\sqrt{3}$ and area $= b^2\sqrt{3}$.

6(a)(i) 17 cm (ii) $56^\circ 9', 123^\circ 51'$ (b) $4\sqrt{11} \text{ cm}$

(c)(i) 10 cm (ii) $5\sqrt{5} \text{ cm}$

7(b) $x = 3$ or 4 , so the diagonals are 6 cm and 8 cm .8(a)(ii) 10 cm when $t = 3$, and 17 cm when $t = 4$.

(b)(i) $3, 4, 5$ (ii) $5, 12, 13$ (iii) $7, 24, 25$

(iv) $33, 56, 65$

9(b)(i) c^2 (ii) $(b-a)^2$ (iii) Each is $\frac{1}{2}ab$.

10(a) $p^4 + p^2q^2, q^4 + p^2q^2$

11(a) $\angle PRS = 15^\circ$ (b) $RS = 2, QS = \sqrt{3}$

12(b) $x^2 + y^2 = 25, (x+10)^2 + y^2 = 169$

(c) $x = \frac{11}{5}, \cos \alpha = \frac{61}{65}$

13(a) 4

19(a) $a^2 + b^2, b^2 + c^2, c^2 + d^2, a^2 + d^2$

21(a) $(c+x)^2 + h^2 = a^2, (c-x)^2 + h^2 = b^2,$
 $h^2 + x^2 = d^2$

23(c) There are two possible configurations:
 $\theta = 77^\circ$ for P outside and $\theta = 166^\circ$ for P inside.

25(b) The possible remainders are $0, 1$ and 4 .

A simple addition table for the LHS shows that there are only six cases, and in each case one of the integers has remainder 0 .

Exercise 8H (Page 333)

1(a) $\triangle ABC \parallel \triangle QPR$ (AA)

(b) $\triangle ABC \parallel \triangle CAD$ (SSS)

(c) $\triangle ABD \parallel \triangle DBC$ (RHS)

(d) $\triangle ABC \parallel \triangle ACD$ (SAS)

2(a) $\triangle ABC \parallel \triangle DEF$ (AA), $x = 4\frac{4}{5}$

(b) $\triangle GHI \parallel \triangle LKJ$ (RHS), $x = 9$

(c) $\triangle QRS \parallel \triangle UTV$ (SAS), $x = 61$

(d) $\triangle LMN \parallel \triangle LPM$ (AA), $x = 18$

3(a) $\triangle ABC \parallel \triangle FDE$ (SSS), $\theta = 67^\circ$

(b) $\triangle XYZ \parallel \triangle X VW$ (SAS), $\theta = 86^\circ$

$VW \parallel ZY$ because alternate angles are equal.

(c) $\triangle PQR \parallel \triangle PRS$ (SSS), $\theta = 52^\circ$

(d) $\triangle PQR \parallel \triangle HIG$ (RHS), $\theta = 71^\circ$

4(a) SAS (b) AA (c) RHS (d) AA

5(a) 64 metres. Use the AA similarity test.

(b) $5 \text{ cm}, 15 \text{ cm}^2, 15 \text{ cm}^3$ (c) $2 \div 4^{\frac{1}{3}} \doteq 1.26 \text{ cm}$

6(a) SSS. Alternate angles $\angle BAC$ and $\angle ACD$ are equal. (b) AA, $ON = 21, PN = 17$ (c) SAS,

trapezium with $AB \parallel KL$ (alternate angles $\angle BAL$ and $\angle ALK$ are equal) (d) AA, $AB = 16, FB = 7$

(e) AA, $FQ = 6, GQ = 8, PQ = 3\sqrt{5}, RQ = 4\sqrt{5}$

(f) AA, $RL = 6$

7(a) $\frac{1}{2}ab \sin C$ (b) $\frac{1}{2}k^2ab \sin C$

10(a) AA, $AD = 15, DC = 20, BC = 16$

(b) $AM = 12, BM = 16, DM = 9$

12(a) $BD = \frac{a^2}{c}$ (b) $AD = \frac{b^2}{c}$

13(a) $\sin^2 \alpha$ (b) $\cos^2 \alpha$

14(a) AA, $\frac{c^2}{b}$ (b) AA

15(a) Yes, the similarity factor is the ratio of their side lengths. (b) No, the ratio of side lengths

may differ in the two rectangles. (c) No, the ratio of diagonals may differ in the two rhombuses.

(d) Yes, the similarity factor is the ratio of their side lengths. (e) No, the ratio of leg to base may

differ in the two triangles. (f) Yes, the similarity factor is the ratio of their radii. (g) Yes, the

similarity factor is the ratio of their focal lengths. (h) Yes, the similarity factor is the ratio of their side lengths.

16(a) AA (b) $\frac{1}{2}(\sqrt{5} - 1)$ (c) $\cos 72^\circ = \frac{1}{4}(\sqrt{5} - 1)$

17(a) Properties of a parallelogram: opposite sides are equal — used twice. (b) SAS

(c) $\triangle QPB \parallel \triangle QCA$ (SAS)

18(a)(i) AA (ii) matching sides of similar triangles

(iii) Since $ad = bc$, it follows that

$$\frac{1}{2}ad \sin \theta = \frac{1}{2}bc \sin \theta. \quad (b)(ii) \frac{27\sqrt{15}}{4}$$

19(a) AA (b) $k : \ell$ (d) multiples of 10

20(a) AA (b) similar triangles in the ratio of $2 : 1$

(c) Square part (b), then use Pythagoras' theorem and the given data.

21(a) SAS (d) The diagonals of a parallelogram bisect each other.

22(a) SAS

$$23(a) 3 : 4 : 5 \quad (b) \sqrt{\sqrt{5} - 1} : \sqrt{2} : \sqrt{\sqrt{5} + 1}$$

$$= 2 : \sqrt{2(\sqrt{5} + 1)} : \sqrt{5} + 1$$

24(a) Reflections preserve distances.

(b) SSS (c) AA

25(a) SAS (c) circumcircle of $\triangle CDO$

Exercise 8I (Page 340)

1(a) $x = 7\frac{1}{2}$ (b) $x = 11$ (c) $x = 15$, $y = 7$

(d) $x = 5$, $y = 18$, $z = 12$

2(a) $x = 7$ (b) $x = 5$ (c) $x = 2$ (d) $x = 6$

(e) $x = 12\frac{1}{2}$ (f) $x = 10\frac{1}{2}$ (g) $x = 4$ (h) $x = 13$

3(a) $x = 6$, $y = 4\frac{1}{2}$, $z = \frac{2}{3}$

(b) $x = 2\frac{2}{3}$, $y = 4\frac{1}{2}$, $z = 2$

(c) $x = 7\frac{1}{2}$, $y = 15$, $z = 3\frac{1}{2}$

(d) $x = 7$, $y = 6$

4(a) $x = 12$ (b) $x = 2$ (c) $x = 1$, $y = 2\frac{1}{4}$

(d) $x = 1$, $y = 1\frac{2}{3}$

5(a) $x = 2$ (b) $x = 4$ (c) $x = 5$ (d) $x = 1 + \sqrt{22}$

6(a) $x = 12$, $y = 6\frac{2}{5}$, $z = 9\frac{3}{5}$ (b) $1 : 2$

(c) parallelogram, $1 : 2$

7(a) SAS

(b) $PQ \parallel BC$ (corresponding angles are equal)

8(a) AA

9(a) SAS (b) Test for a parallelogram: a pair of opposite sides are equal and parallel.

11(a) A line parallel to the base divides the other two sides in the same ratio. Since $AB = AC$, it follows that $DB = EC$. (b) SAS

12(a) EG divides two sides of $\triangle DFC$ in half and hence is parallel to AC .

(b) $FC = 2 \times EG = 2 \times EB$

13(b) 6 cm

15(a) The opposite triangles formed by the wires and the poles are similar, with ratio $2 : 3$. Now using horizontal intercepts, the crossover point is 6 metres above the ground. The height is unchanged when the distance apart changes.

(b) The ladders reach 3.2 metres and 1.8 metres respectively up the wall. The opposite triangles formed by the ladders and the walls have similarity ratio $9 : 16$. The crossover point is 1.152 metres above the ground.

16(a) The base angles are equal.

18 Q describes a circle with centre B and radius $\frac{R(k+\ell)}{k}$.

Chapter Nine

Exercise 9A (Page 349)

1(a) $OA = OB$ (radii) (b) $OF = OG$ (radii)

(c) All sides are equal, being radii.

(d) They subtend equal angles at the centre.

(e) The diagonals bisect each other.

(f) The diagonals are equal and bisect each other.

2(a) $\alpha = 35^\circ$, $\beta = 10^\circ$, $\gamma = 45^\circ$

(b) $\alpha = 100^\circ$, $\beta = 120^\circ$, $\gamma = 20^\circ$ (c) $\alpha = 40^\circ$

(d) $\alpha = 30^\circ$ (e) $\alpha = 80^\circ$, $\beta = 40^\circ$

(f) $\alpha = 100^\circ$, $\beta = 27\frac{1}{2}^\circ$ (g) $\alpha = 50^\circ$, $\beta = 65^\circ$, both arcs and both chords subtend 100° at the centre. (h) $\alpha = \beta = \gamma = 110^\circ$, $\delta = 30^\circ$, both arcs subtend 220° at the centre, both chords subtend 140° at the centre.

3(a) 13 (b) $4\sqrt{10}$, $\frac{2}{11}\sqrt{10}$ (c) $\sqrt{51}$, $2\sqrt{51}$, 0.7

4(b) The perpendicular bisector of AB is a diameter, so its midpoint is the centre.

6(a) SSS test (b) matching angles (c) Matching altitudes of congruent triangles are equal.

7(a) SAS test (b) matching sides

8(a) RHS test (b) RHS test (c) Using matching lengths, $AB = AM + MB = XN + NY = XY$.

9 $5\sqrt{15} + 10\sqrt{3}$ or $5\sqrt{15} - 10\sqrt{3}$

10(a) Use exterior angles at Q , then at P . (c) The alternate angles $\angle ODA$ and $\angle DAP$ are equal.

11(a) $\triangle OAF \equiv \triangle OBG$ (AAS). Notice that $\triangle OFG$ and $\triangle OAG$ are each isosceles, with equal base angles. (b) $\triangle AOF \equiv \triangle BOF$ (AAS)

(c) $PZ = ZQ$, so $FO = OG$ (intercepts).

12(a) $\triangle FGJ \equiv \triangle KJG$ (SSS),

so $\angle MGJ = \angle MJG$. (b) $\triangle PAB \equiv \triangle QAB$ (SSS), arc $BPA =$ arc BQA .

(c) arc $SP =$ arc SQ , $\triangle PST \equiv \triangle QST$ (SSS), so ST bisects the apex angle of the isosceles $\triangle PSQ$.

13(a) SSS test (b) SAS test (d) $OAPB$ is a rhombus if and only if the circles have equal radii.

14(a) $\triangle OPA$ and $\triangle OPB$ are both equilateral.

(b) $\sqrt{3} : 1$ (c) $4\pi - 3\sqrt{3} : 6\pi$

15(a) $\triangle CAO \equiv \triangle CBO$ (SSS)

(b) $\triangle CAM \equiv \triangle CBM$ (SAS or AAS)

16 $x^2 + h^2 = 1$ and $9x^2 + h^2 = 4$, $x = \frac{1}{4}\sqrt{6}$, $h = \frac{1}{4}\sqrt{10}$ (a) $\frac{3}{2}\sqrt{6}$ (b) $\frac{1}{4}\sqrt{10}$

17(a) Use the cosine rule. (b) Use simple trigonometry. (c) $\cos \theta = 1 - 2\sin^2 \frac{1}{2}\theta$

18(a) $(2, 4)$, $2\sqrt{5}$ (b) $(2, 5\frac{5}{8})$, $6\frac{1}{6}$ (c) $(2, \frac{2}{3}\sqrt{3})$ or $(2, -\frac{2}{3}\sqrt{3})$, $\frac{4}{3}\sqrt{3}$ (d) $(2\frac{1}{2}, 2\frac{1}{2})$, $\frac{5}{2}\sqrt{2}$

19 The perpendicular distance of the chord from the centre remains constant. $\frac{1}{4}(4 - \lambda^2) : 1$

20(a) $\lambda = \frac{3}{2\pi}\sqrt{3}$, $\frac{2}{\pi}\sqrt{2}$, $\frac{3}{\pi}$, $\frac{4}{\pi}\sqrt{2 - \sqrt{2}}$

$\mu = \frac{3}{4\pi}\sqrt{3}$, $\frac{2}{\pi}$, $\lambda = \frac{3}{2\pi}\sqrt{3}$, $\frac{2}{\pi}\sqrt{2}$

(b) $\mu = \frac{n}{\pi} \sin \frac{\pi}{n}$, $\mu = \frac{n}{2\pi} \sin \frac{2\pi}{n}$, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

(i) $n = 82$ (ii) $n = 41$

Exercise 9B (Page 354)

1(a) $\alpha = 90^\circ$, $\beta = 55^\circ$ (b) $\alpha = 80^\circ$, $\beta = 40^\circ$

(c) $\alpha = 55^\circ$, $\beta = 35^\circ$ (d) $\alpha = 40^\circ$, $\beta = 20^\circ$

(e) $\alpha = \beta = \gamma = 22^\circ$ (f) $\alpha = 90^\circ$, $\beta = 40^\circ$,

$\gamma = 20^\circ$ (g) $\alpha = 140^\circ$, $\beta = 20^\circ$ (h) $\alpha = 70^\circ$,

$\beta = 220^\circ$, $\gamma = 110^\circ$ (i) $\alpha = 90^\circ$, $\beta = 40^\circ$,

$\gamma = 60^\circ$, $\delta = 30^\circ$ (j) $\alpha = 70^\circ$, $\beta = 20^\circ$

(k) $\alpha = 18^\circ$, $\beta = 36^\circ$, $\gamma = 54^\circ$ (l) $\alpha = 22^\circ$,

$\beta = 30^\circ$, $\gamma = 38^\circ$ (m) $\alpha = 45^\circ$, $\beta = 70^\circ$

(n) $\alpha = 90^\circ$, $\beta = 55^\circ$, $\gamma = 35^\circ$ (o) $\alpha = 110^\circ$,

$\beta = 140^\circ$, $\gamma = 70^\circ$ (p) $\alpha = 100^\circ$, $\beta = 140^\circ$,

$\gamma = 70^\circ$

2 In all parts, the interval named as diameter subtends a right angle at the other two points on the circle.

(a) $ACBD$, diameter AB (b) $FGHI$,

diameter FI (c) $OMXN$, diameter OX

(d) $OXFY$, diameter OF

3 The photographer must stand on the circle with the building as diameter, so that the midpoint of the building is the centre of the circle.

6(a)(i) exterior angles, base angles of isosceles triangles (b) If the diagonals of a quadrilateral are equal and bisect each other, then it is a rectangle.

(c) $OM \perp AB$ because OM bisects the chord AP . $OM \parallel BP$ by intercepts. Hence $\angle P = \angle OMA = 90^\circ$ (corresponding angles, $OM \parallel BP$).

7(a)(i) intercepts (ii) SAS (iii) $AO = PO$ by matching angles. (b)(i) It is an angle in a semicircle.

(ii) The corresponding angles are equal.

(iii) Parallel lines through the common point B are the same line.

8(a) $\angle B = \alpha$ (opposite angles of parallelogram), reflex $\angle O = 360^\circ - \alpha$, reflex $\angle O = 2 \times \angle B$.

(b) $\alpha = 32^\circ$, $\beta = 48^\circ$, $\gamma = 66^\circ$

(c) $\alpha = 56^\circ$, $\beta = 34^\circ$, $\angle PAQ = 34^\circ$, so $AP \parallel BQ$ (alternate angles are equal).

9(a) $\triangle BCA \equiv \triangle BCP$ (RHS) (b) Use intercepts.

(c) Opposite sides of a parallelogram are equal.

10(a) $\alpha = 30^\circ$ (angles on the same arc BC),

$\beta = 60^\circ$ (base angles of isosceles $\triangle OBC$),

$\gamma = 30^\circ$ (angle sum of $\triangle MOC$),

$\triangle OCM \equiv \triangle BCM$ (AAS)

(b) $\alpha = 60^\circ$ (equilateral $\triangle OFG$),

$\beta = 30^\circ$ (angles on the same arc FG),

$\gamma = 90^\circ$ ($\triangle OPF \equiv \triangle GPF$, SSS)

(c) $PQRM$ is a parallelogram because the diagonals bisect each other, and is a rhombus because the adjacent sides OP and OR are equal. Hence

$\alpha = 90^\circ$ (diagonals of a rhombus are perpendicular). Also, $OR = QR$ ($\triangle OMR \equiv \triangle QMR$, SAS), hence $\beta = 60^\circ$ (equilateral $\triangle OQR$).

11(a) $\alpha = 115^\circ$, $\beta = 75^\circ$ (b) $\alpha = 110^\circ$, $\beta = 35^\circ$

(c) $\alpha = 120^\circ$, $\beta = 30^\circ$, $\gamma = 60^\circ$ (d) $\alpha = 40^\circ$,

$\beta = 70^\circ$, $\gamma = 110^\circ$, $\delta = 70^\circ$

12(a)(i) Each is an angle in a semicircle.

(ii) $\angle FBG = 180^\circ$ (iii) $\triangle AFB \equiv \triangle AGB$ (RHS)

(b)(i) SSS (ii) Use matching angles.

(iii) angles standing on the same arc AB

(iv) Use angle sums of $\triangle BPQ$ and $\triangle BOZ$.

13(a)(i) The intervals FH and HG subtend right angles at M , the interval GF subtends a right angle at H . (ii) Use Pythagoras' theorem.

(b)(i) angles on the same arc BD (ii) angle sum of $\triangle MCD$ (iii) BD subtends 90° at O and M .

14(b) $\frac{\pi}{2}$ (c) $\lambda = 2 + \sqrt{3}$ (when $\lambda > 1$) or $\lambda = 2 - \sqrt{3}$ (when $0 < \lambda < 1$)

16 The rate of turning of the binoculars is half the angular velocity of the horse.

Exercise 9C (Page 359)

1(a) $\alpha = \beta = 25^\circ$ (b) $\alpha = 90^\circ$, $\beta = 110^\circ$

(c) $\alpha = 15^\circ$, $\beta = 100^\circ$ (d) $\alpha = 25^\circ$, $\beta = 65^\circ$

(e) $\alpha = 40^\circ$, $\beta = 85^\circ$ (f) $\alpha = 142^\circ$, $\beta = 95^\circ$

(g) $\alpha = 60^\circ$, $\beta = 70^\circ$ (h) $\alpha = 100^\circ$, $\beta = 110^\circ$,

$\gamma = 80^\circ$ (i) $\alpha = 35^\circ$, $\beta = 30^\circ$, $\gamma = 30^\circ$

(j) $\alpha = 25^\circ$, $\beta = 32^\circ$, $\gamma = 57^\circ$ (k) $\alpha = 50^\circ$,

$\beta = 130^\circ$, $\gamma = 25^\circ$ (l) $\alpha = 32^\circ$, $\beta = 34^\circ$, $\gamma = 34^\circ$

2(a) $\alpha = 72^\circ$, $\beta = 58^\circ$ (b) $\alpha = 22^\circ$, $\beta = 52^\circ$

(c) $\alpha = 100^\circ$, $\beta = 100^\circ$, $\gamma = 100^\circ$ (d) $\alpha = 76^\circ$,

$\beta = 34^\circ$, $\gamma = 76^\circ$ (e) $\alpha = 64^\circ$, $\beta = 36^\circ$, $\gamma = 40^\circ$

(f) $\alpha = 66^\circ$, $\beta = 114^\circ$ (g) $\alpha = 60^\circ$, $\beta = 25\frac{5}{7}^\circ$

(h) $\alpha = 20^\circ$, $\beta = 50^\circ$, $\gamma = 44^\circ$

3 If two angles are supplementary, then the sines of the two angles are equal.

4(a) Using exterior angles of the cyclic quadrilateral, $\angle ECD = \alpha$ and $\angle EDC = \alpha$. Then the corresponding angles $\angle ECD$ and $\angle A$ are equal, and the base angles of $\triangle BCD$ are equal. (b) Use angles on the same arcs BD and AC , and alternate angles. Then $\triangle AMB$ and $\triangle CMD$ are isosceles. (c) angles on the same arc AB , angles on equal arcs AB and AD (d) $\angle C = 60^\circ$ (equilateral $\triangle BCD$), $\angle A = 120^\circ$ (opposite angles of cyclic quadrilateral), $\angle ABC = 30^\circ$ (angle sum of isosceles $\triangle ABD$)

5(a) angles on the same arc DC , angles on the same arc BC (b) Use the angle sum of $\triangle BCD$.

6(a) opposite angles of cyclic quadrilateral

(b) angles on the same arc AB , supplements of equal angles $\angle APB$ and $\angle AQB$, angle sum of quadrilateral $PMQN$ (c) angles on equal arcs AB and AC , alternate angles are equal (d) angles on the same arc PB , angle sums of $\triangle AMP$ and $\triangle QBP$ ($\angle QBP = 90^\circ$, being an angle in a semi-circle) (e) angles on the same arc BY , angle sums of $\triangle XMB$ and $\triangle AYB$ (f) exterior angle of cyclic quadrilateral $BADE$, $\angle DAC = \angle DEC = \angle XEY$

7(a) Exterior angle of cyclic quadrilateral, $\angle C = 180^\circ - \alpha$ and so the co-interior angles are supplementary. (b) Both are 90° (angles in semi-circles), they add to a straight angle. (c) Both are 90° (opposite angle of cyclic quadrilateral, and angle in a semicircle), they add to a straight angle. (d) $\angle FGP = 90^\circ = \angle GFQ$. If the radii are equal, $\triangle FGP \equiv \triangle GFQ$ (RHS). (e) Angles on the same arc FA , opposite angles are equal. (f) Angles on the same arc AB . Angles on the same arc FP , vertically opposite angles at B , angles on the same arc GQ .

8(a) 10 (b) $5\sqrt{3}$ (c) $13\frac{1}{3}$ (d) $13\frac{1}{2}$

9(a) opposite interior angle of cyclic quadrilateral $ABCD$ (d) The bisector of the angle formed by the other two sides is perpendicular to the pair of parallel sides. (What if both pairs of opposite sides are parallel?)

10(a) Each side is an equal chord of the circumcircle, and so subtends the same angle at P . Since $\angle APG = 135^\circ$, each angle is $22\frac{1}{2}^\circ = \frac{1}{8} \times 180^\circ$.

14(a) OC subtends right angles at P and Q , AB subtends right angles at P and Q . (b) OQ subtends equal angles at C and P , AQ subtends

equal angles at P and B . (c) Use angle sums of $\triangle OQC$ and $\triangle ORB$.

16(a) α stands on the fixed arc KL .

(c) $\frac{dy}{d\theta} = \frac{a}{\sin \alpha} (\cos \theta + \cos(\theta + \alpha))$ (d) $a \sec \frac{1}{2}\alpha$

18(a) angles on the same arc CX

20(a) SAS similarity test (b) $\angle AOM = \angle PMO$ (matching angles), hence $AO \parallel MP$ (alternate angles are equal), hence $AO \perp BC$ (corresponding angles, $AO \parallel MP$). (c) The same construction can be done from B and the midpoint of AC , and again from C and the midpoint of AB . Hence O lies on all three altitudes.

Exercise 9D (Page 366)

1(a) Opposite angles are supplementary.

(b) Exterior angle equals opposite interior angle.

(c) Arc AD subtends equal angles at B and C .

2(a) Arc BC subtends equal angles at P and Q .

(b) $\angle A = 40^\circ$, $\alpha = 70^\circ$, exterior angle of $PQCB$ equals opposite interior angle. (c) $\angle AJK = \beta$ (corresponding angles, $JK \parallel BC$), $\angle AKJ = \beta$ (base angles of isosceles $\triangle AJK$), exterior angle of $JKCB$ equals opposite interior angle.

(d) $\angle BDE = 180^\circ - \angle A$ (opposite angles of parallelogram, angles on a straight line), $\angle E = \angle BDE$ (base angles of isosceles $\triangle BDE$), opposite angles of $ABEC$ are supplementary.

(e) $\angle AMO = \angle ONA = 90^\circ$ (intervals joining centre to midpoint of chord) (f) $\angle A$ and $\angle C$ are supplementary and $\angle BGH = \angle C$.

(g) $\angle A$ and $\angle Q$ are supplementary and $\angle A = \angle PCD$.

(h) $BP \perp AQ$ (angle in a semicircle)

3(a) BC subtends equal angles at E and D , so $BEDC$ is cyclic. The angle equalities then follow by angles on the same arc ED and exterior angle of the cyclic quadrilateral. (b) $\angle BMD = 2\theta$, because the opposite interior angles in $\triangle AMB$ are both θ . Now BD subtends equal angles of 2θ at O and M .

4(a) The opposite angles are supplementary.

(b) Since $\angle A + \angle C = \angle B + \angle D$ and the sum of the angles is 360° , it follows that $\angle A + \angle C = 180^\circ$.

9(a) intercepts in $\triangle MAB$ and $\triangle MBC$ (b) $\angle A + \angle C = 180^\circ$ (opposite angles of cyclic quadrilateral $MABC$), and $\angle QPM = \angle A$ and $\angle QRM = \angle C$ (corresponding angles on parallel lines), so

$\angle QPM + \angle QRM = \angle C$. Hence $MPQR$ is cyclic (opposite angles are supplementary).

11(a) $\angle P$ is subtended by the fixed chord AB .

(b) PM subtends right angles at X and Y .

(c) The locus of M is an arc of a circle through AB .

13(a) The circle has radius c . (b) Opposite angles are right angles. (c) angles on the same arc BG , and exterior opposite angle of cyclic quadrilateral

(d) angles on the same arc FC (f) The point C now lies outside the circle, but $CMFB$ is still cyclic. Being opposite angles of a cyclic quadrilateral, $\angle FBC$ and $\angle FMC$ are now supplementary,

so $\tan \angle FBC = -\tan \angle FMC$.

15(a) the midpoint of the diameter AB

(b) The interval XY is a chord of the circle with diameter AB , and subtends a constant angle at A .

(c) a circle with centre the midpoint of AB

16(a) Use the circles $ROPB$, $QOPC$ and $RQPB$.

(b) $\angle RAO = \angle RQO = \angle PQO = \angle PCO = \beta$, $\angle QAO = \angle QRO = \angle PRO = \angle PBO = \gamma$

(c) $\triangle ABC$ and $\triangle PQR$ have the same angle sum $2\alpha + 2\beta + 2\gamma$. (d) The arc RQ subtends equal angles at P and L . Since BC is the diameter of the circle $BRQC$, RQ subtends 2α at the midpoint of BC . The interval RQ can only subtend 2α at two points on BC (the locus of such points forms a circle — these points are P and L . (e) $NHLP$ is a cyclic quadrilateral. Thus $\angle LHC = 90^\circ + \alpha$, so $\angle HLC = \gamma = \angle OBL$, so $OB \parallel HL$, so H bisects OC .

17(a) When P is inside the square, $\angle APB = 135^\circ$. When P is outside the square, $\angle APB = 45^\circ$.

(b) Construct the square $PQAB$ adjacent to the square $ABCD$, and construct the circle $PQAB$. Then any angle at the circumference subtended by AB is 45° if the point is outside $ABCD$, and 135° if it is inside. Construct X and Y dividing AB internally and externally respectively in the ratio $1 : 2$, and construct the circle with diameter XY . Then the two points of intersection of the circles are the two positions of P .

Exercise 9E (Page 372)

1(a) $\alpha = 36^\circ$ (b) $\alpha = 41^\circ$, $\beta = 49^\circ$ (c) $\alpha = 45^\circ$,

$\beta = 67\frac{1}{2}^\circ$ (d) $\alpha = 55^\circ$ (e) $\alpha = 44^\circ$ (f) $\alpha = 54^\circ$

(g) $\alpha = 50^\circ$ (h) $\alpha = 100^\circ$

2(a) $x = 12$ cm (b) $x = 6$ (c) $x = 4\sqrt{2} - 4$

(d) $x = 6$ (e) $x = 7$ (f) $x = 4$ (g) $x = 12$

(h) $x = 9$

3(a) $\alpha = 60^\circ$, $\beta = 30^\circ$, $\gamma = 30^\circ$ (b) $\alpha = 140^\circ$,

$\beta = 80^\circ$ (c) $\alpha = 130^\circ$, $\beta = 115^\circ$, $\gamma = 80^\circ$

(d) $\alpha = 100^\circ$, $\beta = 30^\circ$

7(a) $r\sqrt{2}$ (b) $2r$

8(c) 28

9(b) $r(\sqrt{3} + 2)$

10(a) $x = 2$ (b) $x = 10$ (c) $x = 5\frac{1}{3}$

(d) $x = \frac{1}{2}(7 + \sqrt{17})$ or $x = \frac{1}{2}(7 - \sqrt{17})$

14(b) $a = \ell + m$, $b = \frac{m^2 + \ell^2}{m - \ell}$, $c = \frac{2\ell m}{m - \ell}$

17(a) $2\sqrt{15}$ cm (b) $14\sqrt{2}$ cm (c) $\sqrt{143}$ cm, $3\sqrt{7}$ cm

19(a) AAS test

20(a) $TP = 17\frac{1}{7}$, $BP = 12\frac{6}{7}$

(b) $21\sqrt{3} - 36 - \frac{21\pi}{2} + 6\pi\sqrt{3} \doteq 0.0347$ m²

22(a) $r^2(\sqrt{3} - \frac{\pi}{2})$, $r(\frac{2}{3}\sqrt{3} - 1)$ (b) $\frac{2}{3}r(3 + \sqrt{6})$

Exercise 9F (Page 378)

2(a) $\alpha = 70^\circ$, $\beta = 50^\circ$ (b) $\alpha = \beta = \gamma = 65^\circ$

(c) $\alpha = \beta = \gamma = 60^\circ$ (d) $\alpha = \beta = 70^\circ$, $\gamma = 40^\circ$

(e) $\alpha = 68^\circ$, $\beta = 50^\circ$ (f) $\alpha = 70^\circ$, $\beta = 55^\circ$

(g) $\alpha = \beta = 44^\circ$, $\gamma = 92^\circ$ (h) $\alpha = 50^\circ$, $\beta = 40^\circ$

(i) $\alpha = 55^\circ$, $\beta = 66^\circ$, $\gamma = 59^\circ$

(j) $\alpha = 50^\circ$, $\beta = 55^\circ$, $\gamma = 50^\circ$, $\delta = 25^\circ$

(k) $\alpha = \beta = 30^\circ$ (l) $\alpha = 85^\circ$, $\beta = \gamma = 25^\circ$

3(a) $\alpha = \theta$, $\beta = \theta$, $\gamma = 180^\circ - 3\theta$

(b) $\alpha = \theta$, $\beta = 180^\circ - 2\theta$, $\gamma = 2\theta$

(c) $\alpha = 90^\circ - \frac{1}{2}\theta$, $\beta = 90^\circ - \frac{1}{2}\theta$

(d) $\alpha = \theta$, $\beta = \theta$, $\gamma = \theta$, $\delta = 180^\circ - 2\theta$

4(a) $\angle A = \alpha$ (alternate segment theorem)

(b) alternate angles, $PQ \parallel AB$, and alternate segment theorem (c) alternate segment theorem and base angles of isosceles triangle (d) alternate segment theorem and base angles of isosceles triangle

5(a) $\alpha = 40^\circ$, $\beta = 30^\circ$

(b) $\alpha = 65^\circ$, $\beta = 50^\circ$, $\gamma = 25^\circ$

6(a) $\angle OAB = 90^\circ - \alpha$ (radius and tangent)

(b) $\angle OBA = 90^\circ - \alpha$ ($OA = OB$, radii),

$\angle AOB = 2\alpha$ (angles in the triangle OAB)

(c) The angle at the centre is twice the angle at the circumference.

7(a) $\angle P = \alpha$, using the alternate segment theorem and vertically opposite angles.

(b) Using alternate angles and the alternate segment theorem, $\angle BTY = \angle QTX$. But XTY is a straight line, so by the converse of vertically opposite angles, QTB is a straight line.

8(a)(i) $\angle AST = \beta$ and $\angle BST = \alpha$ (alternate segment theorem) (ii) AA similarity test

(iii) matching sides

(b) The argument is similar to that in part (a).

9(a) $\angle CTS = \alpha$ (alternate segment theorem), so $\angle CBS = 180^\circ - \alpha$ (cyclic quadrilateral $CBST$), so $CB \parallel TA$ (co-interior angles are supplementary). (b) $\angle BST = \alpha = \angle TBQ$ (alternate segment theorem), making the alternate angles equal.

10(a) $\angle ETA = \beta$ (alternate segment theorem), and $\angle TGA = \alpha + \beta$ (exterior angle of $\triangle TGB$)

(b) $\angle ABQ = 180^\circ - \alpha$ (opposite angles of cyclic quadrilateral $PABQ$), $\angle ATS = \alpha$ (alternate angles, $PQ \parallel ST$) and $\angle ABT = \alpha$ (alternate segment theorem), so $\angle ABQ$ and $\angle ABT$ are supplementary.

12(a) $\angle XSU = \angle SUX = \alpha$, $\angle XTU = \angle XUT = \beta$, using base angles of isosceles triangles and the alternate segment theorem. Hence $\alpha + \beta = 90^\circ$ (angle sum of $\triangle SUT$). (b) AS and BT are diameters, because they both subtend right angles at U . They are parallel because of the equal alternate angles $\angle ASB$ and $\angle B$. (c) If AB were a tangent to either circle, then it would be parallel to AT , so $ABTS$ would be a rectangle.

14(a) Because the two circles have equal radii, the chord AB subtends equal angles at the circumferences of the two circles. (b) $\triangle BPQ$ is isosceles, so $\triangle PMB \equiv \triangle QMB$ (SSS). (c) Since AB always subtends a right angle at M , the locus is the circle with diameter AB . (d) The proof is the same, except that $\angle APB$ and $\angle AQB$ are now supplementary.

Exercise 9G (Page 384)

1(a) $x = 9$ (b) $x = 2\sqrt{7}$ (c) $x = 3$ (d) $x = 9\sqrt{2}$
(e) $x = 15$ (f) $x = 3$ (g) $x = 4$ (h) $x = 6$

2(a)(i) The perpendicular to a chord from the centre bisects the chord. (ii) $x = 4$ (iii) 40 (b)(i) The line through the centre bisecting a chord is perpendicular to the chord. (ii) $x = 7\frac{1}{4}$ (iii) $72\frac{1}{2}$

3(a) tangent and secant from an external point

(b) $SK^2 = KA \times KB = TK^2$

4(b) the SAS similarity test (c) matching angles

(d) The interval BC subtends equal angles at A and D .

5(a) the SAS similarity test (b) matching angles

(c) The external angle at D equals the internal angle at A .

6(a) The perpendicular from the centre to a chord bisects the chord. (b) the intersecting chord theorem (c) $a + b$ is the diameter.

(d) The semichord x is less than the radius.

7(a) tangent and secant from A (b) Pythagoras' theorem in $\triangle ABT$ (c) angle in the semicircle

BMT (d) M (e) T (f) secant and tangent from B to new circle (g) AA similarity test (h) matching sides, $\triangle ATM \parallel \triangle TBM$ (i) matching sides, $\triangle ATM \parallel \triangle TBM$

8(a) AA similarity test (b) matching sides

(c) $OP = OM + MP$ (d) Use Pythagoras' theorem in $\triangle OMT$.

9(a)(i) alternate segment theorem

(ii) $\angle OTF' = \alpha$ ($OT = OF'$), $\angle FTM = \alpha$ because $TM \perp F'F$ and $F'T \perp FT$.

(iii) $F'T \perp FT$ (b)(i) $TM \perp F'F$ and $F'T \perp FT$

(ii) $\angle F'TO = \alpha$ ($OT = OF'$), $2\alpha + \angle OTM = 90^\circ$

(iii) TP is perpendicular to the radius OT .

10(a) The result still holds, because

$\sin(180^\circ - \theta) = \sin \theta$.

18(a) $\triangle BMA$, $\triangle CMB$, $\triangle CBA$, $\triangle ADC$, $\triangle BAF$

(c) $AG : GB = (r^2 - 1) : r^2$, $r = \sqrt{2}$ (d) Yes. Choose $r = \sqrt{2}$. (e) No. Were $DFGB$ a circle, then $\angle FDG = \angle FBG$. But $\angle ADG = \angle FBG$, so this is impossible unless G and B coincide.

Chapter Ten

Exercise 10A (Page 395)

- 1(a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 0
 2(a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{3}$
 3(a) $\frac{4}{9}$ (b) $\frac{5}{9}$ (c) $\frac{11}{18}$
 4(a) $\frac{4}{9}$ (b) $\frac{5}{9}$ (c) $\frac{11}{18}$ (d) $\frac{7}{18}$ (e) $\frac{1}{3}$ (f) $\frac{1}{6}$
 5(a) $\frac{1}{26}$ (b) $\frac{5}{26}$ (c) $\frac{21}{26}$ (d) 0 (e) $\frac{3}{26}$ (f) $\frac{5}{26}$
 6(a) $\frac{1}{20}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2}$ (e) $\frac{2}{5}$ (f) $\frac{1}{5}$ (g) $\frac{1}{4}$
 (h) 0 (i) 1
 7(a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{13}$ (d) $\frac{1}{52}$ (e) $\frac{1}{4}$ (f) $\frac{3}{13}$ (g) $\frac{1}{2}$
 (h) $\frac{1}{13}$ (i) $\frac{3}{13}$ (counting an ace as a one)
 8(a) $\frac{1}{15}$ (b) $\frac{7}{150}$ (c) $\frac{1}{2}$ (d) $\frac{4}{25}$ (e) $\frac{1}{75}$ (f) $\frac{17}{50}$
 9(a) $\frac{1}{5}$ (b) $\frac{3}{40}$ (c) $\frac{9}{20}$ (d) $\frac{7}{100}$ (e) $\frac{7}{50}$ (f) $\frac{1}{200}$
 10(a) $\frac{3}{4}$ (b) $\frac{1}{4}$
 11 AB, AC, AD, BC, BD, CD (a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$
 (d) $\frac{1}{6}$
 12 HH, HT, TH, TT (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$
 13(a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{4}$ (d) $\frac{1}{4}$
 14(a) 23, 32, 28, 82, 29, 92, 38, 83, 39, 93, 89, 98
 (b)(i) $\frac{1}{12}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{6}$ (v) $\frac{1}{4}$ (vi) 0
 15(a) The captain is listed first and the vice-captain second. AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, BA, CA, DA, EA, CB, DB, EB, DC, EC, ED (b)(i) $\frac{1}{20}$ (ii) $\frac{2}{5}$ (iii) $\frac{3}{5}$ (iv) $\frac{1}{5}$
 16(a)(i) $\frac{2}{5}$ (ii) $\frac{3}{5}$ (iii) $\frac{1}{5}$ (b) $\frac{9}{25}$, $\frac{16}{25}$, $\frac{1}{5}$
 17 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66 (a) $\frac{1}{6}$ (b) $\frac{1}{6}$
 (c) $\frac{1}{36}$ (d) $\frac{1}{6}$ (e) $\frac{1}{6}$ (f) $\frac{1}{4}$ (g) $\frac{11}{36}$ (h) $\frac{4}{9}$ (i) $\frac{5}{36}$
 (j) $\frac{1}{6}$
 18 187 or 188
 19(a) $\frac{227}{300}$ (b) Since a probability of $\frac{3}{4}$ would predict about 225 heads and $\frac{7}{9}$ would predict about 233 heads, both these fractions seem consistent with the experiment. Probabilities of $\frac{1}{2}$ and $\frac{5}{8}$ can safely be excluded.
 20 HHH, HHT, HTH, HTT, THH, THT, TTH, TTT (a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2}$ (e) $\frac{1}{2}$ (f) $\frac{1}{2}$
 21(a)(i) $\frac{1}{4}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{2}$ (b)(i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$
 22(a) $\frac{1}{16}$ (b) $\frac{1}{4}$ (c) $\frac{11}{16}$ (d) $\frac{5}{16}$ (e) $\frac{3}{8}$ (f) $\frac{5}{16}$
 23(a) $\frac{2}{9}$ (b) $\frac{\pi}{18}$
 24(a) The argument is invalid, because on any one day the two outcomes are not equally likely. The argument really can't be corrected.
 (b) The argument is invalid. One team may be significantly better than the other, the game may

be played in conditions that suit one particular team, and so on. Even when the teams are evenly matched, the high-scoring nature of this game makes a draw an unlikely event. The three outcomes are not equally likely. The argument really can't be corrected.

(c) The argument is invalid, because we would presume that Peter has some knowledge of the subject, and is therefore more likely to choose one answer than another. The argument would be valid if the questions were answered at random.

(d) The argument is only valid if there are equal numbers of red, white and black beads, otherwise the three outcomes are not equally likely.

(e) This argument is valid. He is as likely to pick the actual loser of the semi-final as he is to pick any of the other three players.

$$25(a) \frac{1}{2^n} \quad (b) 1 - 2^{1-n}$$

26(b) Throw the needle 1000 times, say, and let S be the number of times it lies across a crack. Then $\pi \doteq \frac{2000}{S}$. (c) $\frac{1}{\pi}$ (d) $\frac{2}{3\pi}(6 - 3\sqrt{3} + \pi) \doteq 0.837$

Exercise 10B (Page 400)

- 1(a)(i) $A \cup B = \{1, 3, 5, 7\}$, $A \cap B = \{3, 5\}$
 (ii) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
 $A \cap B = \{4, 9\}$
 (iii) $A \cup B = \{h, o, b, a, r, t, i, c, e, n\}$,
 $A \cap B = \{h, o, b\}$
 (iv) $A \cup B = \{1, 2, 3, 5, 7, 9\}$, $A \cap B = \{3, 5, 7\}$
 (b)(i) $\{2, 4, 5, 6, 8, 9\}$ (ii) $\{1, 2, 3, 5, 8, 10\}$
 (iii) $\{7\}$ (iv) $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$
 (v) $\{1, 3, 4, 6, 7, 9, 10\}$ (vi) $\{2, 5, 8\}$
 2 78%
 3(a) $\frac{4}{7}$ (b) 32
 4(a) $\frac{14}{15}$ (b) 8
 5(a) $\frac{1}{6}$ (b) $\frac{5}{6}$ (c) $\frac{1}{3}$ (d) 0 (e) 1 (f) 0 (g) $\frac{1}{6}$ (h) $\frac{2}{3}$
 6(a) $\frac{1}{13}$ (b) $\frac{1}{13}$ (c) $\frac{2}{13}$ (d) 0 (e) $\frac{11}{13}$ (f) $\frac{1}{2}$ (g) $\frac{3}{13}$
 (h) $\frac{3}{26}$ (i) $\frac{8}{13}$ (j) $\frac{5}{13}$
 7(a) no (b)(i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{3}$ (iv) $\frac{5}{6}$
 8(a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$ (e) $\frac{1}{4}$ (f) $\frac{1}{6}$ (g) $\frac{1}{6}$
 (h) $\frac{1}{36}$ (i) $\frac{11}{36}$ (j) $\frac{25}{36}$
 9(a)(i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{3}$ (iv) $\frac{1}{2}$ (v) $\frac{1}{2}$ (b)(i) $\frac{3}{5}$ (ii) $\frac{4}{5}$
 (iii) $\frac{3}{5}$ (iv) 0 (v) 1 (c)(i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{2}{3}$ (iv) $\frac{1}{3}$
 (v) $\frac{5}{6}$
 10(a) $\frac{7}{15}$ (b) 0 (c) $\frac{3}{5}$ (d) $\frac{5}{7}$

- 11(a)(i) no (ii) $\frac{1}{2}, \frac{1}{4}, \frac{3}{20}, \frac{3}{5}$ (b)(i) no (ii) $\frac{1}{2}, \frac{3}{10}, \frac{3}{20}, \frac{13}{20}$ (c)(i) yes (ii) $\frac{1}{4}, \frac{9}{20}, 0, \frac{7}{10}$
 12 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
 (a) $\frac{1}{4}$ (b) $\frac{6}{25}$ (c) $\frac{11}{100}$ (d) $\frac{19}{50}$
 13(a) $\frac{9}{25}$ (b) $\frac{7}{50}$ (c) $\frac{17}{50}$
 14(a) 10 (b)(i) $\frac{4}{21}$ (ii) $\frac{1}{3}$
 15 $\frac{1}{4}$
 16(a) $\frac{7}{12}$ (b) $\frac{13}{60}$ (c) $\frac{3}{10}$ (d) $\frac{7}{60}$
 17 $\frac{7}{43}$

18(a) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
 (b) 207 (c) $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$.
 It is possible to draw a Venn diagram with four sets, but only if the fourth set is represented not by a circle, but by a complicated loop — the final diagram must have 16 regions.

Exercise 10C (Page 406)

- 1(a) $\frac{1}{24}$ (b) $\frac{1}{28}$ (c) $\frac{1}{12}$ (d) $\frac{1}{96}$ (e) $\frac{1}{42}$ (f) $\frac{1}{336}$
 2(a) $\frac{1}{12}$ (b) $\frac{1}{12}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$
 3(a) $\frac{1}{25}$ (b) $\frac{2}{25}$ (c) $\frac{3}{25}$ (d) $\frac{3}{25}$ (e) $\frac{4}{25}$ (f) $\frac{2}{25}$ (g) $\frac{1}{25}$
 4(a) $\frac{15}{49}$ (b) $\frac{8}{49}$ (c) $\frac{6}{49}$
 5(a) $\frac{1}{10}$ (b) $\frac{3}{10}$ (c) $\frac{3}{10}$ (d) $\frac{3}{10}$
 6(a) $\frac{1}{7}$ (b) $\frac{180}{1331}$
 7(a)(i) $\frac{13}{204}$ (ii) $\frac{1}{17}$ (iii) $\frac{4}{663}$ (iv) $\frac{1}{2652}$
 (b) $\frac{1}{16}, \frac{1}{16}, \frac{1}{169}, \frac{1}{2704}$
 8(a)(i) $\frac{2}{3}$ (ii) $\frac{1}{3}$ (b)(i) $\frac{8}{27}$ (ii) $\frac{1}{27}$ (iii) $\frac{4}{27}$

9(a) The argument is invalid, because the events 'liking classical music' and 'playing a classical instrument' are not independent. One would expect that most of those playing a classical instrument would like classical music, whereas a smaller proportion of those not playing a classical instrument would like classical music. The probability that a student does both cannot be discovered from the given data — one would have to go back and do another survey. (b) The argument is invalid, because the events 'being prime' and 'being odd' are not independent — two out of the three odd numbers less than 7 are prime, but only one out of the three such even numbers is prime. The correct argument is that the odd prime numbers amongst the numbers 1, 2, 3, 4, 5 and 6 are 3

and 5, hence the probability that the die shows an odd prime number is $\frac{2}{6} = \frac{1}{3}$. (c) The teams in the competition may not be of equal ability, and factors such as home-ground advantage may also affect the outcome of a game, hence assigning a probability of $\frac{1}{2}$ to winning each of the seven games is unjustified. Also, the outcomes of successive games are not independent — the confidence gained after winning a game may improve a team's chances in the next one, a loss may adversely affect their chances, or a team may receive injuries in one game leading to a depleted team in the next. The argument really can't be corrected. (d) This argument is valid. The coin is normal, not biased, and tossed coins do not remember their previous history, so the next toss is completely unaffected by the previous string of heads.

- 10 HHH, HHM, HMH, MHH, HMM, MHM, MMH, MMM (a) $p(\text{HHH}) = 0.9^3 = 0.729$
 (b) 0.001 (c) $p(\text{HMM}) = 0.9 \times 0.1^2 = 0.009$
 (d) $p(\text{HMM}) + p(\text{MHM}) + p(\text{MMH}) = 3 \times 0.009 = 0.027$ (e) 0.081 (f) 0.243
 11(a) $\frac{1}{36}$ (b) $\frac{1}{6}$ (c) $\frac{1}{4}$ (d) $\frac{1}{36}$ (e) $\frac{1}{36}$ (f) $\frac{1}{18}$ (g) $\frac{1}{12}$
 (h) $\frac{1}{12}$ (i) $\frac{1}{6}$
 12(a) $p(\text{CCCCC}) = (\frac{1}{5})^5 = \frac{1}{3125}$ (b) $\frac{1024}{3125}$ (c) $\frac{16}{3125}$
 (d) $\frac{256}{3125}$ (e) $\frac{256}{625}$ (f) $\frac{4}{625}$
 13(a) $\frac{1}{46656}$ (b) $\frac{5}{7776}$
 14(a) $\frac{1}{6}$ (b) $\frac{5}{6}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$
 15(a) $\frac{3}{64}$ (b) $\frac{17}{64}$ (c) $\frac{5}{17}$
 16(a)(i) $\frac{3}{4}$ (ii) $\frac{31}{32}$ (iii) $\frac{1023}{1024}$ (b) $1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

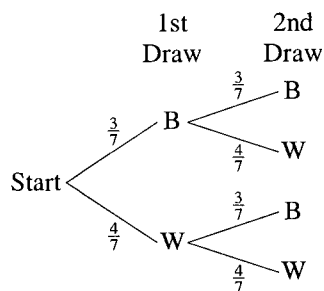
(c) 14
 17(a) $(\frac{5}{6})^n$ (b) $a = \frac{5}{6}, r = \frac{5}{6}$ (c) 13
 18(a) $\frac{9}{25}$ (b) 11
 19(a) $\frac{1}{12960000}$ (b) 233
 20(a) $\frac{1}{9}$ (b) $\frac{1}{9}$. Retell as 'Nick begins by picking out two socks for the last morning and setting them aside'. (c) $\frac{1}{9}$. Retell as 'Nick begins by picking out two socks for the third morning and setting them aside'. (d) $\frac{1}{63}$ (e) $\frac{1}{9 \times 7 \times 5 \times 3}$ (f) zero
 21 In each part, retell the process of selection as 'First choose a court for Kia, then choose one of the remaining 11 positions for Abhishek'.
 (a) $\frac{3}{11}$ (b) $\frac{1}{11}$ (c) $\frac{4}{33}$
 22 Suppose first that the contestant changes her choice. If her original choice was correct, she loses,

otherwise she wins, so her chance of winning is $\frac{2}{3}$. Suppose now that the contestant does not change her choice. If her original choice was correct, she wins, otherwise she loses, so her chance of winning is $\frac{1}{3}$. Thus the strategy of changing will double her chance of winning.

23 $\frac{11}{19}$

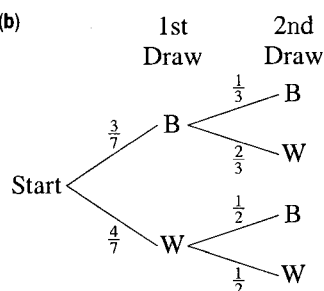
Exercise 10D (Page 410)

1(a)



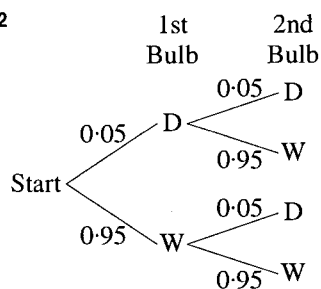
(i) $\frac{16}{49}$
(ii) $\frac{24}{49}$

(b)



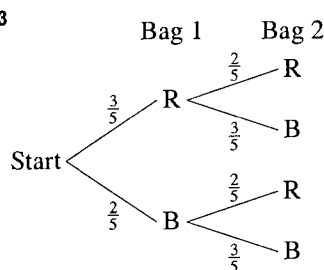
(i) $\frac{3}{7}$
(ii) $\frac{4}{7}$

2



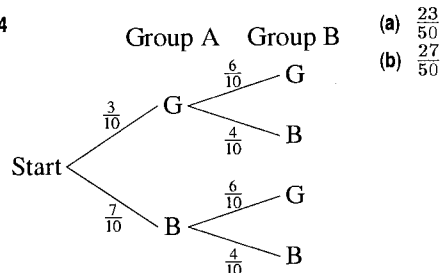
(a) 0.25%
(b) 99.75%

3



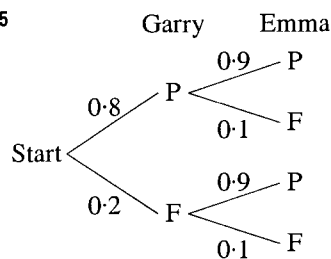
(a) $\frac{12}{25}$
(b) $\frac{13}{25}$

4



(a) $\frac{23}{50}$
(b) $\frac{27}{50}$

5



(a) 8%
(b) 18%
(c) 26%
(d) 28%

6(a) $\frac{1}{11}$ (b) $\frac{14}{33}$ (c) $\frac{10}{33}$ (d) $\frac{19}{33}$

7(a) 0.28 (b) 0.50

8(a) 0.01 (b) 0.23

9(a)(i) $\frac{1}{25}$ (ii) $\frac{3}{5}$ (b) $\frac{1}{20}, \frac{3}{5}$

10 4.96%

11 0.35

12(a) $\frac{21}{3980}$ (b) $\frac{144}{995}$

13(a) $\frac{9}{25}$ (b) $\frac{21}{25}$

14(a) $\frac{3}{10}$ (b) $\frac{7}{24}$ (c) $\frac{21}{40}$ (d) $\frac{11}{60}$

15(a)(i) $\frac{5}{33}$ (ii) $\frac{5}{22}$ (iii) $\frac{19}{33}$ (iv) $\frac{1}{4}$ (v) $\frac{19}{66}$ (vi) $\frac{47}{66}$

(b) $\frac{25}{144}, \frac{5}{24}, \frac{5}{9}, \frac{1}{4}, \frac{25}{72}, \frac{47}{72}$

16(a) $\frac{5}{6}$ (b) $\frac{5}{12}$ (c) $\frac{1}{6}$ (d) $\frac{3}{4}$

17 The chance that at least one of them will shoot a basket is $1 - p$ (they both miss). The boy missing and the girl missing are independent events. The correct answer is 0.895.

18(a) $\frac{4}{9}$ (b) $\frac{65}{81}$ (c) 4

19 $\frac{4}{11}$

20(a) $\frac{1}{36}$ (b) $\frac{1}{46656}$ (c) $\frac{11}{36}$ (d) $\frac{2}{11}$

21(a) $\frac{1}{6}$ (b) $\frac{5}{6}$ (c) $\frac{1}{5}$

22(a) $\frac{1}{20}$ (b)(i) $\frac{57}{8000}$ (ii) $\frac{3971}{4000}$

23(a) $\frac{1}{216}$ (b) $\frac{5}{72}$ (c) $\frac{5}{12}$ (d) $\frac{5}{9}$

24(a) $\frac{5}{36}$ (b) $\frac{13}{18}$ (c) $\frac{5}{36} + \frac{13}{18} \times \frac{5}{36} + \frac{13^2}{18^2} \times \frac{5}{36}$ (d) $\frac{1}{2}$

25(a) $\frac{3}{7}$ (b) $\frac{16}{49}$ (c) $\frac{9399}{16807}$ (d) $\frac{21}{37}$

26(a) $\frac{1}{3}$ (b) $\frac{1}{4}$

27(a) $\frac{1}{25}$ (b) $\frac{3}{25}$ (c) $\frac{6}{25}$ (d) $\frac{19}{25}$

28(a) $\frac{2}{5}$ (b) $\frac{9}{25}$ (c) $\frac{1622}{3125}$ (d) $\frac{10}{19}$

29(a) $\frac{15}{16}$ (b) $\frac{2}{3}$

- 30(a) $\frac{g}{g+b} \times \frac{g-1}{g+b-1} \times \frac{g-2}{g+b-2}$ (b)(iii) maximum turning point at $g = 5 + 3\sqrt{2}$ and minimum turning point at $g = 5 - 3\sqrt{2}$ (iv) 3 green and 2 blue or 4 green and 10 blue marbles

Exercise 10E (Page 417)

- 1(a) 30 (b) 90 (c) 6 (d) 6 (e) 120 (f) 24 (g) 15 120 (h) 1 814 400
 2 There are 6: DOG, DGO, ODG, OGD, GOD, GDO
 3 FEG, FGE, FEH, FHE, FEI, FIE, FGH, FHG, FGI, FIG, FHI, FIH
 4(a) 360 (b) 720
 5(a) 120 (b) 625
 6 60, 36
 7 5040
 8(a) 40 320 (b) 336
 9(a) 12 (b) 864
 10 720
 11 48
 12(a) 10^7 (b) 5×10^6 (c) 5^7 (d) 32 000
 13(a) 10 000 (b) 5040 (c) 625 (d) 1000.
 14(a) 3024 (b) 336 (c) 1344 (d) 336 (e) 1008
 15(a) 6561 (b) 729 (c) 2916 (d) 729 (e) 2187
 16(a) 6 760 000 (b) 3 276 000 (c) 26 000 (d) 48
 17(a) 720 (b) 120 (c) 24 (d) 360 (half of them)
 18(a) 144 (b) 120 (c) 144 (d) 2520 (half of the total)
 19(a) 720 (b) 720 (c) 4320
 20 622 080
 21(a) 24 (b) 240
 22 1728
 23 24
 24 2046
 25(a) 1152 (b) 1152
 26(a) 720 (b) 120 (c) 1680 (d) 4200 (e) 960 (f) 480
 27(a) 5040 (b) 4320 (c) 720 (d) 144 (e) 720 (f) 960 (g) 1440
 28(a) 7^7 (b) 6×7^6 (c) 7^6
 (d) $3^4 \times 4^3 + 4^4 \times 3^3 = 7 \times 12^3$
 29(a)(i) 3 628 800 (ii) 725 760 (iii) 725 760
 (iv) 2 257 920 (b)(i) $2(n-1)!$ (ii) $2(n-1)!$
 (iii) $(n-2)(n-3)(n-2)!$
 30 8640
 31(a) 40 320 (b) 20 160 (c) 17 280

- 32(a) 120 (b) 24 (c) 95
 33(a) 5040 (b) 20 160
 34(a)(i) 64 (ii) 32 (b)(i) 340 (ii) 170
 35(a) 96 (b) 36 (c) 24
 36(a) 5^5 ways (b) $5! = 120$ ways
 (c) $5 \times 4^3 = 320$ ways
 37(a) 133 (b) 104 (c) 29 (d) 50
 38(a) 3 (b) 3
 39(a) 0, 1 and 2 (b)(i) $4!$ (ii) $4 \times 3!$ (iii) $6 \times 2!$
 (iv) $1 \times 4! - 4 \times 3! + 6 \times 2! - 4 \times 1! + 1 \times 0!$
 (c) $1 \times 5! - 5 \times 4! + 10 \times 3! - 10 \times 2! + 5 \times 1! - 1 \times 0!$
 (d) $\sum_{r=0}^n (-1)^r {}^n C_r (n-r)!$
 40(a) There are $n-1$ choices for r , and for each such choice the remaining $n-2$ objects must undergo a derangement. (b) $(n-1)D(n-1)$
 (c) $D(n) = (n-1)D(n-1) + (n-1)D(n-2)$, where $D(1) = 0$ and $D(2) = 1$.

Exercise 10F (Page 423)

- 1(a) 3 (b) 12 (c) 120 (d) 6720 (e) 10 080 (f) 90 720 (g) 4 989 600 (h) 45 360 (i) 25 740
 2 60
 3(a) 6 (b) 15 (c) 20
 4(a) 40 320 (b) 8 (c) 56 (d) 560
 5(a) 56 (b) 20
 6(a) 56 (b) 5
 7(a) 60 (b) 24 (c) 36 (d) 30 (half of them)
 8(a)(i) 180 (ii) 60 (iii) 120 (iv) 24 (b) 40
 9(a) 90 720 (b) 720 (c) 720 (d) 45 360 (half of them)
 10 2 721 600
 11(a) 1024 (b) 256 (c) 45 (d) 252 (e) 56 (f) 512
 (g) 8 (h) 70
 12(a) 60 (b) 60
 13(a) 120 (b) 60
 15(a) 453 600 (b) 90 720 (c) 5040 (d) 10 080 (e) 80 640 (f) 282 240
 16(a) 3 628 800 (b) 4
 17 864
 18(a) 2520 (b) 720 (c)(i) 600 (ii) 480 (iii) 360
 (iv) 240 (v) 120 (d) 840 (e) 210 (f) 420
 19 1 995 840

Exercise 10G (Page 430)

- 1 There are ${}^5C_2 = 10$ possible combinations:
PQ, PR, PS, PT, QR, QS, QT, RS, RT and ST.
- 2(a) 21 (b) 35 (c) 15 (d) 126
- 3(a)(i) 45 (ii) 45
(b) ${}^{10}C_2 = {}^{10}C_8$, and in general ${}^nC_r = {}^nC_{n-r}$.
- 4(a) 44 352 (b) 34 650
- 5(a) 70 (b) 36 (c) 16 (d) 1 (e) 69
- 6(a) 126 (b) 45 (c) 51 (d) 75
- 7(a) 2002 (b) 56 (c) 6 (d) 840 (e) 420 (f) 1316
(g) 715 (h) 1287
- 8(a) 70 (b) 5 (c) 35
- 9(a) 792 (b) 462 (c) 120 (d) 210 (e) 420
- 10(a)(i) 126. The number cannot begin with a zero.
(ii) 252 (b) In each part, once the five numbers have been selected, they can only be arranged in one way.
- 11 13860
- 12(a) 1745 944 200 (b) 413 513 100
- 13(a) 45 (b) 120 (c) 36 (d) 8
- 14(a) 10 (b) 110
- 15 21
- 16(a) 65 780 (b) 1287 (c) 48 (d) 22 308 (e) 288
(f) 3744
- 17(a) 2 (b) 5 (c) 35 (d) ${}^nC_2 - n$
- 18(a) 220 (b) 9240 (c)(i) 2772 (ii) 6468
- 19(a) 1024 (b) 968 (c) 466 (d) 247
- 20(a) 16 (b) 20 (c) 12 (d) 8 (e) 5
- 21(a) 252 (b) 126
- 22(a) 315 (b) 210
- 23 120
- 25(a) 210 (b) 90 (c) 126 (d) 126
- 26(a) 30 (b) 24
- 27 5151
- 28 1360
- 29(a) 3 (b) 315 (c)(i) 155 925 (ii) 10 800

Exercise 10H (Page 435)

- 1(a) 84 (b) $\frac{5}{42}$
- 2(a) $\frac{1}{210}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{4}{15}$
- 3(a) $\frac{1}{13}$ (b) $\frac{46}{455}$ (c) $\frac{3}{91}$ (d) $\frac{3}{13}$
- 4(a) $\frac{8}{429}$ (b) $\frac{1}{143}$ (c) $\frac{140}{429}$ (d) $\frac{421}{429}$ (e) $\frac{2}{11}$ (f) $\frac{1}{3}$
- 5(a) $\frac{1}{22100}$ (b) $\frac{1}{5525}$ (c) $\frac{11}{850}$ (d) $\frac{22}{425}$ (e) $\frac{11}{1105}$
(f) $\frac{13}{34}$ (g) $\frac{16}{5525}$ (h) $\frac{6}{5525}$ (i) $\frac{741}{1700}$ (j) $\frac{64}{425}$
- 6(a) $\frac{3}{70304}$ (b) $\frac{1}{2197}$ (c) $\frac{1}{64}$ (d) $\frac{1}{16}$ (e) $\frac{27}{2197}$ (f) $\frac{3}{8}$
(g) $\frac{6}{2197}$ (h) $\frac{3}{2197}$ (i) $\frac{27}{64}$ (j) $\frac{5}{32}$
- 7(a) $\frac{1}{10}$ (b) $\frac{1}{10}$ (c) $\frac{1}{3}$

- 8(a) $\frac{1}{10}$ (b) $\frac{2}{5}$
- 9(a) $\frac{1}{15}$ (b) $\frac{2}{3}$
- 10(a) $\frac{1}{42}$ (b) $\frac{2}{7}$ (c) $\frac{2}{7}$ (d) $\frac{1}{35}$ (e) $\frac{1}{7}$
- 11(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{5}$ (d) $\frac{1}{60}$ (e) $\frac{2}{3}$
- 12(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{1}{7}$ (d) $\frac{2}{7}$
- 13(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{2}{15}$ (d) $\frac{1}{5}$
- 14(a) $\frac{1}{26}$ (b) $\frac{5}{13}$ (c) $\frac{15}{26}$ (d) $\frac{1}{26}$
- 15(a) 0.403 (b) 0.597 (c) 0.001 74 (d) 0.291
- 16(a) $\frac{1}{22}$ (b) $\frac{125}{1728}$ (c) $\frac{5}{144}$
- 17(a) $\frac{10}{31}$ (b) $\frac{15}{31}$ (c) $\frac{6}{31}$
- 18(a) $\frac{1}{60}$ (b) $\frac{3}{5}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$ (e) $\frac{1}{20}$ (f) $\frac{3}{5}$ (g) $\frac{3}{10}$
(h) $\frac{9}{10}$ (i) $\frac{1}{10}$ (j) $\frac{3}{5}$
- 19(a) $\frac{3}{8}$ (b) $\frac{1}{2}$ (c) $\frac{21}{32}$ (d) $\frac{3}{32}$ (e) $\frac{17}{64}$
- 20(a) $\frac{281}{462}$ (b) 8
- 21(a) $\frac{1}{27417}$ (b) $\frac{28}{703}$
- 22(a) $\frac{352}{833}$ (b) $\frac{198}{4165}$ (c) $\frac{88}{4165}$ (d) $\frac{1}{4165}$ (e) $\frac{6}{4165}$
(f) $\frac{192}{54145}$ (g) $\frac{33}{16660}$ (h) $\frac{1}{649740}$
- 23(a) $\frac{1}{125}$ (b) $\frac{4}{125}$ (c) $\frac{16}{125}$ (d) $\frac{108}{125}$
 $\frac{n^2(n-1)^2(n-2)!}{2n^n}$
- 24(a) $\frac{48}{125}$ (b) $\frac{n^2(n-1)^2(n-2)!}{2n^n}$
- 25(a) 0.0082
 $\frac{365P_n}{365^n}$
(b) $1 - \frac{365P_n}{365^n}$
- (d) 23 (e) 45
- 26(a) $\frac{1}{25}$ (b) $\frac{3}{25}$ (c) $\frac{19}{25}$
- 27(b) $\frac{1}{8}$ (c) 2^{1-n}

Exercise 10I (Page 441)

- 1(a)(i) 120 (ii) 24 (b)(i) 3 628 800 (ii) 362 880
- 2(a) 10 080 (b) 1440
- 3(a) 24 (b) 24 (c) 6 (d) 4 (e) 12
- 4(a) 5040 (b) 144 (c) 576 (d) 1440
(e) 3600 (f) 240
- 5(a) $\frac{3}{10}$ (b) $\frac{1}{5}$ (c) $\frac{1}{10}$ (d) $\frac{9}{10}$
- 6(a) 5040 (b) 576 (c) 144 (d) 2304 (e) 1440
- 7(a) $\frac{1}{12}$ (b) $\frac{1}{9}$
- 8(a) $(n-1)!$ (b) $2 \times (n-2)!$ (c) $(n-3) \times (n-2)!$
(d) $6 \times (n-3)!$
- 9(a) 39 916 800 (b) 165
- 10 145 152
- 11(a) 288 (b) $\frac{1}{4}$
- 12 $\frac{n!(n+1)!}{(2n)!}$
- 13(a) 60 (b) 181 440 (c) 9

Exercise 10J (Page 446)

- 1(a) $\frac{1}{32}$ (b) $\frac{5}{16}$ (c) $\frac{5}{32}$ (d) $\frac{31}{32}$
 2 ${}^6C_2(\frac{4}{5})^4(\frac{1}{5})^2$
 3(a) ${}^7C_3(\frac{2}{3})^3(\frac{1}{3})^4$ (b) ${}^7C_5(\frac{2}{3})^5(\frac{1}{3})^2$ (c) $(\frac{1}{3})^7$
 (d) $1 - (\frac{1}{3})^7$
 4(a) ${}^4C_3(\frac{5}{6})^3(\frac{1}{6})$ (b) ${}^4C_2(\frac{5}{6})^2(\frac{1}{6})^2$
 5 $(0.65)^{12}$
 6(a) ${}^{12}C_3(\frac{5}{6})^9(\frac{1}{6})^3$ (b) ${}^{12}C_8(\frac{5}{6})^4(\frac{1}{6})^8$
 (c) ${}^{12}C_{10}(\frac{5}{6})^2(\frac{1}{6})^{10} + {}^{12}C_{11}(\frac{5}{6})(\frac{1}{6})^{11} + (\frac{1}{6})^{12}$
 7(a) 0.2009 (b) 0.7368 (c) 0.2632
 8 $(\frac{5}{6})^{15} + {}^{15}C_1(\frac{5}{6})^{14}(\frac{1}{6}) + {}^{15}C_2(\frac{5}{6})^{13}(\frac{1}{6})^2$
 9(a) $(\frac{9}{10})^{20} + {}^{20}C_1(\frac{9}{10})^{19}(\frac{1}{10}) + {}^{20}C_2(\frac{9}{10})^{18}(\frac{1}{10})^2$
 (b) $1 - (\frac{9}{10})^{20}$
 10(a) $0.91^{10} + {}^{10}C_1 \times 0.91^9 \times 0.09$
 $+ {}^{10}C_2 \times 0.91^8 \times 0.09^2$
 (b) $1 - 0.91^{10} - {}^{10}C_1 \times 0.91^9 \times 0.09$
 11 0.593
 12 0.000 786
 13 ${}^{31}C_3 \times 0.95^{28} \times 0.05^3$
 14(a)(i) 0.107 64 (ii) 0.113 72 (b) 0.785 49
 15(a) 17 (b) 7
 16(a)(i) 0.487 (ii) 0.031 (b) We have assumed that boys and girls are equally likely. We have also assumed that in any one family, the events 'having a boy' and 'having a girl' are independent.
 17(a) The argument is invalid. Normally, mathematics books are grouped together, so once the shelf is chosen, one would expect all or none of the books to be mathematics books, thus the five stages are not independent events. The result would be true if the books were each chosen at random from the library.
 (b) The argument is invalid. People in a particular neighbourhood tend to vote more similarly than the population at large, so the four events are not independent. The result would be true if one chose four streets at random, and then chose a voter randomly from each street.
 18(a) 0.0124 (b) 0.7102
 19(a) 0.409 600 (b) 0.001 126 (c) 0.000 869
 20(a) 0.0060 (b) 0.0303
 21(a) $\frac{3}{250}$ (b)(i) $(\frac{3}{250})^{10}$ (ii) ${}^{10}C_5(\frac{3}{250})^5(\frac{247}{250})^5$
 (iii) $(\frac{247}{250})^{10} + 10(\frac{247}{250})^9(\frac{3}{250})$
 22(a) $\frac{7}{16}$ (b)(i) ${}^8C_3(\frac{9}{16})^5(\frac{7}{16})^3$
 (ii) $1 - (\frac{9}{16})^8 - 8(\frac{9}{16})^7(\frac{7}{16}) - {}^8C_2(\frac{9}{16})^6(\frac{7}{16})^2$
 23(a) 34 (b) 22
 24(a) 33, ${}^{200}C_{33}(\frac{1}{6})^{33}(\frac{5}{6})^{167} \doteq 0.075 65$

- (b) 20 and 21, ${}^{41}C_{20}(\frac{1}{2})^{41} \doteq 0.1224$
 (c) 2, ${}^{35}C_2(\frac{1}{13})^2(\frac{12}{13})^{33} \doteq 0.2509$ (d) 8 and 9,
 ${}^{35}C_9(\frac{1}{4})^9(\frac{3}{4})^{26} = {}^{35}C_8(\frac{1}{4})^8(\frac{3}{4})^{27} \doteq 0.1520$
 26(a) 0.2048 (b) 0.737 28
 (c)(i) $\frac{n(n-1)(n-2)(n-3)}{20 \times 19 \times 18 \times 17}$
 27(a) $a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 3b^2c + 3bc^2 + 6abc$ (b)(i) 0.102 96 (ii) 0.131 33
 (iii) 0.897 04

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