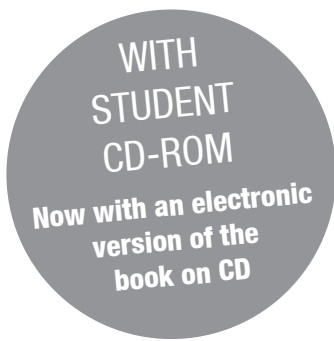




YEAR
11

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Mathematics

3unit

BILL PENDER

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Contents

Preface	vii
How to Use This Book	ix
About the Authors	xiii
Chapter One — Methods in Algebra	1
1A Terms, Factors and Indices	1
1B Expanding Brackets	3
1C Factorisation	5
1D Algebraic Fractions	7
1E Four Cubic Identities	11
1F Linear Equations and Inequations	13
1G Quadratic Equations	16
1H Simultaneous Equations	18
1I Completing the Square	21
1J The Language of Sets	23
Chapter Two — Numbers and Functions	29
2A Cardinals, Integers and Rational Numbers	29
2B The Real Numbers	35
2C Surds and their Arithmetic	39
2D Rationalising the Denominator	42
2E Equality of Surdic Expressions	45
2F Relations and Functions	48
2G Review of Known Functions and Relations	52
2H Inverse Relations and Functions	59
2I Shifting and Reflecting Known Graphs	64
2J Further Transformations of Known Graphs	69
Chapter Three — Graphs and Inequations	73
3A Inequations and Inequalities	73
3B Intercepts and Sign	78
3C Domain and Symmetry	82
3D The Absolute Value Function	85
3E Using Graphs to Solve Equations and Inequations	91
3F Regions in the Number Plane	96
3G Asymptotes and a Curve Sketching Menu	100

Chapter Four — Trigonometry	107
4A Trigonometry with Right Triangles	107
4B Theoretical Exercises on Right Triangles	114
4C Trigonometric Functions of a General Angle	117
4D The Quadrant, the Related Angle and the Sign	121
4E Given One Trigonometric Function, Find Another	127
4F Trigonometric Identities and Elimination	129
4G Trigonometric Equations	133
4H The Sine Rule and the Area Formula	139
4I The Cosine Rule	146
4J Problems Involving General Triangles	150
Chapter Five — Coordinate Geometry	156
5A Points and Intervals	156
5B Gradients of Intervals and Lines	162
5C Equations of Lines	167
5D Further Equations of Lines	170
5E Perpendicular Distance	176
5F Lines Through the Intersection of Two Given Lines	180
5G Coordinate Methods in Geometry	184
Chapter Six — Sequences and Series	188
6A Indices	188
6B Logarithms	192
6C Sequences and How to Specify Them	196
6D Arithmetic Sequences	200
6E Geometric Sequences	203
6F Arithmetic and Geometric Means	207
6G Sigma Notation	211
6H Partial Sums of a Sequence	213
6I Summing an Arithmetic Series	215
6J Summing a Geometric Series	219
6K The Limiting Sum of a Geometric Series	223
6L Recurring Decimals and Geometric Series	227
6M Factoring Sums and Differences of Powers	229
6N Proof by Mathematical Induction	231
Chapter Seven — The Derivative	237
7A The Derivative — Geometric Definition	237
7B The Derivative as a Limit	241
7C A Rule for Differentiating Powers of x	245
7D The Notation $\frac{dy}{dx}$ for the Derivative	250
7E The Chain Rule	254
7F The Product Rule	260
7G The Quotient Rule	262
7H Rates of Change	265
7I Limits and Continuity	268
7J Differentiability	273
7K Extension — Implicit Differentiation	276

Chapter Eight — The Quadratic Function	280
8A Factorisation and the Graph	280
8B Completing the Square and the Graph	285
8C The Quadratic Formulae and the Graph	289
8D Equations Reducible to Quadratics	292
8E Problems on Maximisation and Minimisation	294
8F The Theory of the Discriminant	299
8G Definite and Indefinite Quadratics	304
8H Sum and Product of Roots	307
8I Quadratic Identities	311
Chapter Nine — The Geometry of the Parabola	316
9A A Locus and its Equation	316
9B The Geometric Definition of the Parabola	320
9C Translations of the Parabola	325
9D Parametric Equations of Curves	327
9E Chords of a Parabola	330
9F Tangents and Normals: Parametric Approach	333
9G Tangents and Normals: Cartesian Approach	338
9H The Chord of Contact	341
9I Geometrical Theorems about the Parabola	345
9J Locus Problems	350
Chapter Ten — The Geometry of the Derivative	357
10A Increasing, Decreasing and Stationary at a Point	357
10B Stationary Points and Turning Points	362
10C Critical Values	367
10D Second and Higher Derivatives	371
10E Concavity and Points of Inflexion	373
10F Curve Sketching using Calculus	378
10G Global Maximum and Minimum	380
10H Applications of Maximisation and Minimisation	383
10I Maximisation and Minimisation in Geometry	388
10J Primitive Functions	391
Chapter Eleven — Integration	397
11A Finding Areas by a Limiting Process	397
11B The Fundamental Theorem of Calculus	402
11C The Definite Integral and its Properties	407
11D The Indefinite Integral	412
11E Finding Area by Integration	415
11F Area of a Compound Region	419
11G Volumes of Solids of Revolution	423
11H The Reverse Chain Rule	429
11I The Trapezoidal Rule	432
11J Simpson's Rule	434

Chapter Twelve — The Logarithmic Function		438
12A Review of Logarithmic and Exponential Functions		438
12B The Logarithmic Function and its Derivative		441
12C Applications of Differentiation		450
12D Integration of the Reciprocal Function		454
12E Applications of Integration		459
Chapter Thirteen — The Exponential Function		462
13A The Exponential Function and its Derivative		462
13B Applications of Differentiation		467
13C Integration of the Exponential Function		472
13D Applications of Integration		476
13E Natural Growth and Decay		479
Chapter Fourteen — The Trigonometric Functions		487
14A Radian Measure of Angle Size		487
14B Mensuration of Arcs, Sectors and Segments		491
14C Graphs of the Trigonometric Functions in Radians		496
14D Trigonometric Functions of Compound Angles		504
14E The Angle Between Two Lines		509
14F The Behaviour of $\sin x$ Near the Origin		513
14G The Derivatives of the Trigonometric Functions		517
14H Applications of Differentiation		523
14I Integration of the Trigonometric Functions		528
14J Applications of Integration		534
Answers to Exercises		538
Index		633

Preface

This textbook has been written for students in Years 11 and 12 taking the course previously known as ‘3 Unit Mathematics’, but renamed in the new HSC as two courses, ‘Mathematics’ (previously called ‘2 Unit Mathematics’) and ‘Mathematics, Extension 1’. The book develops the content at the level required for the 2 and 3 Unit HSC examinations. There are two volumes — the present volume is roughly intended for Year 11, and the second for Year 12. Schools will, however, differ in their choices of order of topics and in their rates of progress.

Although these Syllabuses have not been rewritten for the new HSC, there has been a gradual shift of emphasis in recent examination papers.

- The interdependence of the course content has been emphasised.
- Graphs have been used much more freely in argument.
- Structured problem solving has been expanded.
- There has been more stress on explanation and proof.

This text addresses these new emphases, and the exercises contain a wide variety of different types of questions.

There is an abundance of questions in each exercise — too many for any one student — carefully grouped in three graded sets, so that with proper selection the book can be used at all levels of ability. In particular, those who subsequently drop to 2 Units of Mathematics, and those who in Year 12 take 4 Units of Mathematics, will both find an appropriate level of challenge. We have written a separate book, also in two volumes, for the 2 Unit ‘Mathematics’ course alone.

We would like to thank our colleagues at Sydney Grammar School and Newington College for their invaluable help in advising us and commenting on the successive drafts, and for their patience in the face of some difficulties in earlier drafts. We would also like to thank the Head Masters of Sydney Grammar School and Newington College for their encouragement of this project, and Peter Cribb and the team at Cambridge University Press, Melbourne, for their support and help in discussions. Finally, our thanks go to our families for encouraging us, despite the distractions it has caused to family life.

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How to Use This Book

This book has been written so that it is suitable for the full range of 3 Unit students, whatever their abilities and ambitions. The book covers the 2 Unit and 3 Unit content without distinction, because 3 Unit students need to study the 2 Unit content in more depth than is possible in a 2 Unit text. Nevertheless, students who subsequently move to the 2 Unit course should find plenty of work here at a level appropriate for them.

The Exercises: No-one should try to do all the questions! We have written long exercises so that everyone will find enough questions of a suitable standard — each student will need to select from them, and there should be plenty left for revision. The book provides a great variety of questions, and representatives of all types should be selected.

Each chapter is divided into a number of sections. Each of these sections has its own substantial exercise, subdivided into three groups of questions:

FOUNDATION: These questions are intended to drill the new content of the section at a reasonably straightforward level. There is little point in proceeding without mastery of this group.

DEVELOPMENT: This group is usually the longest. It contains more substantial questions, questions requiring proof or explanation, problems where the new content can be applied, and problems involving content from other sections and chapters to put the new ideas in a wider context. Later questions here can be very demanding, and Groups 1 and 2 should be sufficient to meet the demands of all but exceptionally difficult problems in 3 Unit HSC papers.

EXTENSION: These questions are quite hard. Some are algebraically challenging, some establish a general result beyond the theory of the course, some make difficult connections between topics or give an alternative approach, some deal with logical problems unsuitable for the text of a 3 Unit book. Students taking the 4 Unit course should attempt some of these.

The Theory and the Worked Exercises: The theory has been developed with as much rigour as is appropriate at school, even for those taking the 4 Unit course. This leaves students and their teachers free to choose how thoroughly the theory is presented in a particular class. It can often be helpful to learn a method first and then return to the details of the proof and explanation when the point of it all has become clear.

The main formulae, methods, definitions and results have been boxed and numbered consecutively through each chapter. They provide a summary only, and represent an absolute minimum of what should be known. The worked examples

have been chosen to illustrate the new methods introduced in the section, and should be sufficient preparation for the questions of the following exercise.

The Order of the Topics: We have presented the topics in the order we have found most satisfactory in our own teaching. There are, however, many effective orderings of the topics, and the book allows all the flexibility needed in the many different situations that apply in different schools (apart from the few questions that provide links between topics).

The time needed for the algebra in Chapter One will depend on students' experiences in Years 9 and 10. The same applies to other topics in the early chapters — trigonometry, quadratic functions, coordinate geometry and particularly curve sketching. The Study Notes at the start of each chapter make further specific remarks about each topic.

We have left Euclidean geometry and polynomials until Year 12 for two reasons. First, we believe as much calculus as possible should be developed in Year 11, ideally including the logarithmic and exponential functions and the trigonometric functions. These are the fundamental ideas in the course, and it is best if Year 12 is used then to consolidate and extend them (and students subsequently taking the 4 Unit course particularly need this material early). Secondly, the Years 9 and 10 Advanced Course already develops much of the work on polynomials and Euclidean geometry in Options recommended for those proceeding to 3 Unit, so that revisiting them in Year 12 with the extensions and far greater sophistication required seems an ideal arrangement.

The Structure of the Course: Recent examination papers have included longer questions combining ideas from different topics, thus making clear the strong interconnections amongst the various topics. Calculus is the backbone of the course, and the two processes of differentiation and integration, inverses of each other, dominate most of the topics. We have introduced both processes using geometrical ideas, basing differentiation on tangents and integration on areas, but the subsequent discussions, applications and exercises give many other ways of understanding them. For example, questions about rates are prominent from an early stage.

Besides linear functions, three groups of functions dominate the course:

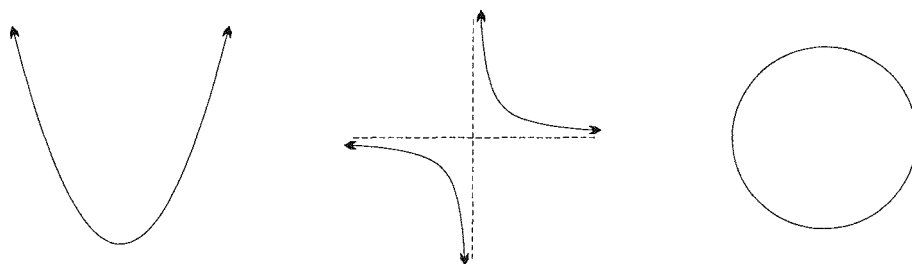
THE QUADRATIC FUNCTIONS: These functions are known from earlier years. They are algebraic representations of the parabola, and arise naturally in situations where areas are being considered or where a constant acceleration is being applied. They can be studied without calculus, but calculus provides an alternative and sometimes quicker approach.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS: Calculus is essential for the study of these functions. We have chosen to introduce the logarithmic function first, using definite integrals of the reciprocal function $y = 1/x$. This approach is more satisfying because it makes clear the relationship between these functions and the rectangular hyperbola $y = 1/x$, and because it gives a clear picture of the new number e . It is also more rigorous. Later, however, one can never overemphasise the fundamental property that the exponential function with base e is its own derivative — this is the reason why these functions are essential for the study of natural growth and decay, and therefore occur in almost every application of mathematics.

Arithmetic and geometric sequences arise naturally throughout the course. They are the values, respectively, of linear and exponential functions at integers, and these interrelationships need to be developed, particularly in the context of applications to finance.

THE TRIGONOMETRIC FUNCTIONS: Again, calculus is essential for the study of these functions, whose definition, like the associated definition of π , is based on the circle. The graphs of the sine and cosine functions are waves, and they are essential for the study of all periodic phenomena — hence the detailed study of simple harmonic motion in Year 12.

Thus the three basic functions of the course — x^2 , e^x and $\sin x$ — and the related numbers e and π are developed from the three most basic degree 2 curves — the parabola, the rectangular hyperbola and the circle. In this way, everything in the course, whether in calculus, geometry, trigonometry, coordinate geometry or algebra, is easily related to everything else.



The geometry of the circle is mostly studied using Euclidean methods, and the highly structured arguments used here contrast with the algebraic arguments used in the coordinate geometry approach to the parabola. In the 4 Unit course, the geometry of the rectangular hyperbola is given special consideration in the context of a coordinate geometry treatment of general conics.

Polynomials are a generalisation of quadratics, and move the course a little beyond the degree 2 phenomena described above. The particular case of the binomial theorem then becomes the bridge from elementary probability using tree diagrams to the binomial distribution with all its practical applications. Unfortunately the power series that link polynomials with the exponential and trigonometric functions are too sophisticated for a school course. Projective geometry and calculus with complex numbers are even further removed, so it is not really possible to explain that exponential and trigonometric functions are the same thing, although there are many clues.

Algebra, Graphs and Language: One of the chief purposes of the course, stressed in recent examinations, is to encourage arguments that relate a curve to its equation. Being able to predict the behaviour of a curve given only its equation is a constant concern of the exercises. Conversely, the behaviour of a graph can often be used to solve an algebraic problem. We have drawn as many sketches in the book as space allowed, but as a matter of routine, students should draw diagrams for almost every problem they attempt. It is because sketches can so easily be drawn that this type of mathematics is so satisfactory for study at school.

This course is intended to develop simultaneously algebraic agility, geometric intuition, and rigorous language and logic. Ideally then, any solution should

display elegant and error-free algebra, diagrams to display the situation, and clarity of language and logic in argument.

Theory and Applications: Elegance of argument and perfection of structure are fundamental in mathematics. We have kept to these values as far as is reasonable in the development of the theory and in the exercises. The application of mathematics to the world around us is an equally fundamental, and we have given many examples of the usefulness of everything in the course. Calculus is particularly suitable for presenting this double view of mathematics.

We would therefore urge the reader sometimes to pay attention to the details of argument in proofs and to the abstract structures and their interrelationships, and at other times to become involved in the interpretation provided by the applications.

Limits, Continuity and the Real Numbers: This is a first course in calculus, geometrically and intuitively developed. It is not a course in analysis, and any attempt to provide a rigorous treatment of limits, continuity or the real numbers would be quite inappropriate. We believe that the limits required in this course present little difficulty to intuitive understanding — really little more is needed than $\lim_{x \rightarrow \infty} 1/x = 0$ and the occasional use of the sandwich principle in proofs. Characterising the tangent as the limit of the secant is a dramatic new idea, clearly marking the beginning of calculus, and quite accessible. Continuity and differentiability need only occasional attention, given the well-behaved functions that occur in the course. The real numbers are defined geometrically as points on the number line, and provided that intuitive ideas about lines are accepted, everything needed about them can be justified from this definition. In particular, the intermediate value theorem, which states that a continuous function can only change sign at a zero, is taken to be obvious.

These unavoidable gaps concern only very subtle issues of ‘foundations’, and we are fortunate that everything else in the course can be developed rigorously so that students are given that characteristic mathematical experience of certainty and total understanding. This is the great contribution that mathematics brings to all our education.

Technology: There is much discussion, but little agreement yet, about what role technology should play in the mathematics classroom or what machines or software may be effective. This is a time for experimentation and diversity. We have therefore given only a few specific recommendations about technology, but we encourage such investigation, and the exercises give plenty of scope for this. The graphs of functions are at the centre of the course, and the more experience and intuitive understanding students have, the better able they are to interpret the mathematics correctly. A warning here is appropriate — any machine drawing of a curve should be accompanied by a clear understanding of why such a curve arises from the particular equation or situation.

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*The mathematician's patterns, like the painter's or the poet's,
must be beautiful. The ideas, like the colours or the words,
must fit together in a harmonious way. Beauty is the first test.*

— The English mathematician G. H. Hardy (1877–1947)

CHAPTER ONE

Methods in Algebra

Mathematics is the study of structure, pursued using a highly refined form of language in which every word has an exact meaning, and in which the logic is expressed with complete precision. As the structures and the logic of their explanation become more complicated, the language describing them in turn becomes more specialised, and requires systematic study for the meaning to be understood. The symbols and methods of *algebra* are one aspect of that special language, and fluency in algebra is essential for work in all the various topics of the course.

STUDY NOTES: Several topics in this chapter will probably be quite new — the four cubic identities of Section 1E, solving a set of three simultaneous equations in three variables in Section 1G, and the language of sets in Section 1J. The rest of the chapter is a concise review of algebraic work which would normally have been carefully studied in previous years, and needs will therefore vary as to the amount of work required on these exercises.

1A Terms, Factors and Indices

A *pronumeral* is a symbol that stands for a number. The pronumeral may stand for a known number, or for an unknown number, or it may be a *variable*, standing for any one of a whole set of possible numbers. Pronumerals, being numbers, can therefore be subjected to all the operations that are possible with numbers, such as addition, subtraction, multiplication and division (except by zero).

Like and Unlike Terms: An *algebraic expression* is an expression such as

$$x^2 + 2x + 3x^2 - 4x - 3,$$

in which pronumerals and numbers and operations are combined. The five *terms* in the above expression are x^2 , $2x$, $3x^2$, $-4x$ and -3 . The two *like terms* x^2 and $3x^2$ can be combined to give $4x^2$, and the like terms $2x$ and $-4x$ can be combined to give $-2x$. This results in three *unlike terms* $4x^2$, $-2x$ and -3 , which cannot be combined.

WORKED EXERCISE: $x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$

Multiplying Terms: To simplify a product like $3xy \times (-6x^2y) \times \frac{1}{2}y$, it is best to work systematically through the signs, the numerals, and the pronumerals.

WORKED EXERCISE: (a) $4ab \times 7bc = 28ab^2c$ (b) $3xy \times (-6x^2y) \times \frac{1}{2}y = -9x^3y^3$

Index Laws: Here are the standard laws for dealing with indices (see Chapter Six for more detail).

1	INDEX LAWS:	$a^x a^y = a^{x+y}$	$(ab)^x = a^x b^x$
		$\frac{a^x}{a^y} = a^{x-y}$	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
		$(a^x)^n = a^{xn}$	

WORKED EXERCISE:

(a) $3x^4 \times 4x^3 = 12x^7$

(d) $(-5x^2)^3 \times (2xy)^4 = -125x^6 \times 16x^4y^4$
 $= -2000x^{10}y^4$

(b) $(48x^7y^3) \div (16x^5y^3) = 3x^2$

(c) $(3a^4)^3 = 27a^{12}$

(e) $\frac{(6x^4y)^2}{3(x^2y^3)^3} = \frac{36x^8y^2}{3x^6y^9}$
 $= \frac{12x^2}{y^7}$

Exercise 1A

1. Simplify:

(a) $3x - 2y + 5x + 6y$

(c) $9x^2 - 7x + 4 - 14x^2 - 5x - 7$

(b) $2a^2 + 7a - 5a^2 - 3a$

(d) $3a - 4b - 2c + 4a + 2b - c + 2a - b - 2c$

2. Find the sum of:

(a) $x + y + z$, $2x + 3y - 2z$ and $3x - 4y + z$

(b) $2a - 3b + c$, $15a - 21b - 8c$ and $24b + 7c + 3a$

(c) $5ab + bc - 3ca$, $ab - bc + ca$ and $-ab + 2ca + bc$

(d) $x^3 - 3x^2y + 3xy^2$, $-2x^2y - xy^2 - y^3$ and $x^3 + 4y^3$

3. Subtract:

(a) x from $3x$

(b) $-x$ from $3x$

(c) $2a$ from $-4a$

(d) $-b$ from $-5b$

4. From:

(a) $7x^2 - 5x + 6$ take $5x^2 - 3x + 2$

(c) $3a + b - c - d$ take $6a - b + c - 3d$

(b) $4a - 8b + c$ take $a - 3b + 5c$

(d) $ab - bc - cd$ take $-ab + bc - 3cd$

5. Subtract:

(a) $x^3 - x^2 + x + 1$ from $x^3 + x^2 - x + 1$

(b) $3xy^2 - 3x^2y + x^3 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$

(c) $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$

(d) $x^4 + 5 + x - 3x^3$ from $5x^4 - 8x^3 - 2x^2 + 7$

6. Multiply:

(a) $5a$ by 2

(c) $-3a$ by a

(e) $4x^2$ by $-2x^3$

(b) $6x$ by -3

(d) $-2a^2$ by $-3ab$

(f) $-3p^2q$ by $2pq^3$

7. Simplify:

(a) $2a^2b^4 \times 3a^3b^2$

(b) $-6ab^5 \times 4a^3b^3$

(c) $(-3a^3)^2$

(d) $(-2a^4b)^3$

8. If $a = -2$, find the value of: (a) $3a^2 - a + 4$ (b) $a^4 + 3a^3 + 2a^2 - a$
9. If $x = 2$ and $y = -3$, find the value of: (a) $8x^2 - y^3$ (b) $x^2 - 3xy + 2y^2$
10. Simplify: (a) $\frac{5x}{x}$ (b) $\frac{-7x^3}{x}$ (c) $\frac{-12a^2b}{-ab}$ (d) $\frac{-27x^6y^7z^2}{9x^3y^3z}$
11. Divide:
- (a) $-2x$ by x (c) x^3y^2 by x^2y (e) $14a^5b^4$ by $-2a^4b$
- (b) $3x^3$ by x^2 (d) a^6x^3 by $-a^2x^3$ (f) $-50a^2b^5c^8$ by $-10ab^3c^2$
- DEVELOPMENT**
12. Simplify: (a) $\frac{3a \times 3a \times 3a}{3a + 3a + 3a}$ (b) $\frac{3c \times 4c^2 \times 5c^3}{3c^2 + 4c^2 + 5c^2}$ (c) $\frac{ab^2 \times 2b^2c^3 \times 3c^3a^4}{a^3b^3 + 2a^3b^3 + 3a^3b^3}$
13. Simplify: (a) $\frac{(-2x^2)^3}{-4x}$ (b) $\frac{(3xy^3)^3}{3x^2y^4}$ (c) $\frac{(-ab)^3 \times (-ab^2)^2}{-a^5b^3}$ (d) $\frac{(-2a^3b^2)^2 \times 16a^7b}{(2a^2b)^5}$
14. What must be added to $4x^3 - 3x^2 + 2$ to give $3x^3 + 7x - 6$?
15. Take the sum of $2a - 3b - 4c$ and $-4a + 7b - 5c$ from the sum of $4c - 2b$ and $5b - 2a - 2c$.
16. If $X = 2b + 3c - 5d$ and $Y = 4d - 7c - b$, take $X - Y$ from $X + Y$.
17. Divide the product of $(-3x^7y^5)^4$ and $(-2xy^6)^3$ by $(-6x^3y^8)^2$.
- EXTENSION**
18. For what values of x is it true that: (a) $x \times x \leq x + x$? (b) $x \times x \times x \leq x + x + x$?

1B Expanding Brackets

The laws of arithmetic tell us that $a(x + y) = ax + ay$, whatever the values of a , x and y . This enables expressions with brackets to be *expanded*, meaning that they can be written in a form without brackets.

WORKED EXERCISE:

$$\begin{aligned}
 \text{(a)} \quad 3x(x - 2xy) &= 3x^2 - 6x^2y & \text{(c)} \quad (4x - 2)(4x - 3) \\
 & & &= 4x(4x - 3) - 2(4x - 3) \\
 \text{(b)} \quad a^2(a - b) - b^2(b - a) & & &= 16x^2 - 12x - 8x + 6 \\
 &= a^3 - a^2b - b^3 + ab^2 & &= 16x^2 - 20x + 6
 \end{aligned}$$

Special Quadratic Identities: These three identities are so important that they need to be memorised rather than worked out each time.

2	SQUARE OF A SUM:	$(A + B)^2 = A^2 + 2AB + B^2$
	SQUARE OF A DIFFERENCE:	$(A - B)^2 = A^2 - 2AB + B^2$
	DIFFERENCE OF SQUARES:	$(A + B)(A - B) = A^2 - B^2$

WORKED EXERCISE:

$$\begin{aligned}
 \text{(a)} \quad (4x + 5y)^2 &= 16x^2 + 40xy + 25y^2 \text{ (square of a sum)} \\
 \text{(b)} \quad \left(t - \frac{1}{t}\right)^2 &= t^2 - 2 + \frac{1}{t^2} \text{ (square of a difference)} \\
 \text{(c)} \quad (x^2 + 3y)(x^2 - 3y) &= x^4 - 9y^2 \text{ (difference of squares)}
 \end{aligned}$$

Exercise 1B

1. Expand:

(a) $4(a + 2b)$	(d) $-a(a + 4)$	(g) $-2x(x^3 - 2x^2 - 3x + 1)$
(b) $x(x - 7)$	(e) $5(a + 3b - 2c)$	(h) $3xy(2x^2y - 5x^3)$
(c) $-3(x - 2y)$	(f) $-3(2x - 3y + 5z)$	(i) $-2a^2b(a^2b^3 - 2a^3b)$

2. Expand and simplify:

- (a) $3(x - 2) - 2(x - 5)$
 (b) $-7(2a - 3b + c) - 6(-a + 4b - 2c)$
 (c) $x^2(x^3 - 5x^2 + 6x - 1) - 2x(x^4 + 10x^3 - 2x^2 - 7x + 3)$
 (d) $-2x^3y(3x^2y^4 - 4xy^5 + 5y^7) - 3xy^2(x^2y^6 + 2x^4y^3 - 2x^3y^4)$

3. Expand and simplify:

(a) $(x + 2)(x + 3)$	(c) $(x - 4)(x + 2)$	(e) $(3x + 8)(4x - 5)$
(b) $(2a + 3)(a + 5)$	(d) $(2b - 7)(b - 3)$	(f) $(6 - 7x)(5 - 6x)$

4. (a) By expanding $(A + B)(A + B)$, prove the special expansion $(A + B)^2 = A^2 + 2AB + B^2$.

(b) Similarly, prove the special expansions:

(i) $(A - B)^2 = A^2 - 2AB + B^2$ (ii) $(A - B)(A + B) = A^2 - B^2$

5. Expand, using the special expansions:

(a) $(x - y)^2$	(c) $(n - 5)^2$	(e) $(2a + 1)^2$	(g) $(3x + 4y)(3x - 4y)$
(b) $(a + 3)^2$	(d) $(c - 2)(c + 2)$	(f) $(3p - 2)^2$	(h) $(4y - 5x)^2$

6. Multiply:

(a) $a - 2b$ by $a + 2b$	(c) $4x + 7$ by itself	(e) $a + b - c$ by $a - b$
(b) $2 - 5x$ by $5 + 4x$	(d) $x^2 + 3y$ by $x^2 - 4y$	(f) $9x^2 - 3x + 1$ by $3x + 1$

7. Expand and simplify:

(a) $\left(t + \frac{1}{t}\right)^2$	(b) $\left(t - \frac{1}{t}\right)^2$	(c) $\left(t + \frac{1}{t}\right)\left(t - \frac{1}{t}\right)$

DEVELOPMENT

8. (a) Subtract $a(b + c - a)$ from the sum of $b(c + a - b)$ and $c(a + b - c)$.

(b) Subtract the sum of $2x^2 - 3(x - 1)$ and $2x + 3(x^2 - 2)$ from the sum of $5x^2 - (x - 2)$ and $x^2 - 2(x + 1)$.

9. Simplify: (a) $14 - (10 - (3x - 7) - 8x)$ (b) $4(a - 2(b - c) - (a - (b - 2)))$

10. Use the special expansions to find the value of: (a) 102^2 (b) 999^2 (c) 203×197

11. Expand and simplify:

(a) $(a - b)(a + b) - a(a - 2b)$	(d) $(p + q)^2 - (p - q)^2$
(b) $(x + 2)^2 - (x + 1)^2$	(e) $(2x + 3)(x - 1) - (x - 2)(x + 1)$
(c) $(a - 3)^2 - (a - 3)(a + 3)$	(f) $3(a - 4)(a - 2) - 2(a - 3)(a - 5)$

12. If $X = x - a$ and $Y = 2x + a$, find the product of $Y - X$ and $X + 3Y$ in terms of x and a .

13. Expand and simplify:

(a) $(x - 2)^3$	(c) $(x + y - z)(x - y + z)$
(b) $(x + y + z)^2 - 2(xy + yz + zx)$	(d) $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

14. Prove the identities:

$$(a) (a + b + c)(ab + bc + ca) - abc = (a + b)(b + c)(c + a)$$

$$(b) (ax + by)^2 + (ay - bx)^2 + c^2(x^2 + y^2) = (x^2 + y^2)(a^2 + b^2 + c^2)$$

EXTENSION

15. If $2x = a + b + c$, show that $(x - a)^2 + (x - b)^2 + (x - c)^2 + x^2 = a^2 + b^2 + c^2$.

16. If $(a + b)^2 + (b + c)^2 + (c + d)^2 = 4(ab + bc + cd)$, prove that $a = b = c = d$.

1C Factorisation

Factorisation is the reverse process of expanding brackets, and will be needed on a routine basis throughout the course. The various methods of factorisation are listed systematically, but in every situation common factors should always be taken out first.

METHODS OF FACTORISATION:

3

HIGHEST COMMON FACTOR: Always try this first.

DIFFERENCE OF SQUARES: This involves two terms.

QUADRATICS: This involves three terms.

GROUPING: This involves four or more terms.

Factoring should continue until each factor is *irreducible*, meaning that it cannot be factored further.

Factoring by Highest Common Factor and Difference of Squares: In every situation, look for any common factors of all the terms, and then take out the highest common factor.

WORKED EXERCISE: Factor: (a) $18a^2b^4 - 30b^3$ (b) $80x^4 - 5y^4$

SOLUTION:

(a) The highest common factor of $18a^2b^4$ and $30b^3$ is $6b^3$,
so $18a^2b^4 - 30b^3 = 6b^3(3a^2b - 5)$.

(b) $80x^4 - 5y^4 = 5(16x^4 - y^4)$ (highest common factor)
 $= 5(4x^2 - y^2)(4x^2 + y^2)$ (difference of squares)
 $= 5(2x - y)(2x + y)(4x^2 + y^2)$ (difference of squares again)

Factoring Monic Quadratics: A quadratic is called *monic* if the coefficient of x^2 is 1. Suppose we are required to factor a monic quadratic expression like $x^2 - 13x + 36$. We look for two numbers whose sum is -13 (the coefficient of x) and whose product is 36 (the constant).

WORKED EXERCISE: Factor: (a) $x^2 - 13x + 36$ (b) $a^2 + 12ac - 28c^2$

SOLUTION:

(a) The numbers with sum -13
and product 36 are -9 and -4 ,
so $x^2 - 13x + 36$
 $= (x - 9)(x - 4)$.

(b) The numbers with sum 12
and product -28 are 14 and -2 ,
so $a^2 + 12ac - 28c^2$
 $= (a + 14c)(a - 2c)$.

Factoring Non-monic Quadratics: In a *non-monic* quadratic like $2x^2 + 11x + 12$, where the coefficient of x^2 is not 1, we look for two numbers whose sum is 11 (the coefficient of x), and whose product is 24 (the product of the constant term and the coefficient of x^2).

WORKED EXERCISE: Factor: (a) $2x^2 + 11x + 12$ (b) $6s^2 - 11st - 10t^2$

SOLUTION:

(a) The numbers with sum 11 and product 24 are 8 and 3, so $2x^2 + 11x + 12$ $= (2x^2 + 8x) + (3x + 12)$ $= 2x(x + 4) + 3(x + 4)$ $= (2x + 3)(x + 4).$	(b) The numbers with sum -11 and product -60 are -15 and 4, so $6s^2 - 11st - 10t^2$ $= (6s^2 - 15st) + (4st - 10t^2)$ $= 3s(2s - 5t) + 2t(2s - 5t)$ $= (3s + 2t)(2s - 5t).$

Factoring by Grouping: When there are four or more terms, it is sometimes possible to split the expression into groups, factor each group in turn, and then factor the whole expression by taking out a common factor or by some other method.

WORKED EXERCISE: Factor: (a) $12xy - 9x - 16y + 12$ (b) $s^2 - t^2 + s - t$

SOLUTION:

- (a) $12xy - 9x - 16y + 12 = 3x(4y - 3) - 4(4y - 3)$

$$= (3x - 4)(4y - 3)$$
- (b) $s^2 - t^2 + s - t = (s + t)(s - t) + (s - t)$

$$= (s - t)(s + t + 1)$$

Exercise 1C

1. Write as a product of two factors:

(a) $ax - ay$	(c) $3a^2 - 6ab$	(e) $6a^3 + 2a^4 + 4a^5$
(b) $x^2 + 3x$	(d) $12x^2 + 18x$	(f) $7x^3y - 14x^2y^2 + 21xy^2$

2. Factor by grouping in pairs:

(a) $ax - ay + bx - by$	(c) $x^2 - 3x - xy + 3y$	(e) $ab + ac - b - c$
(b) $a^2 + ab + ac + bc$	(d) $2ax - bx - 2ay + by$	(f) $2x^3 - 6x^2 - ax + 3a$

3. Factor each difference of squares:

(a) $x^2 - 9$	(c) $4x^2 - y^2$	(e) $1 - 49k^2$
(b) $1 - a^2$	(d) $25x^2 - 16$	(f) $81a^2b^2 - 64$

4. Factor each of these quadratic expressions:

(a) $x^2 + 8x + 15$	(f) $p^2 + 9p - 36$	(k) $x^2 - 5xy + 6y^2$
(b) $x^2 - 4x + 3$	(g) $u^2 - 16u - 80$	(l) $x^2 + 6xy + 8y^2$
(c) $a^2 + 2a - 8$	(h) $x^2 - 20x + 51$	(m) $a^2 - ab - 6b^2$
(d) $y^2 - 3y - 28$	(i) $t^2 + 23t - 50$	(n) $p^2 + 3pq - 40q^2$
(e) $c^2 - 12c + 27$	(j) $x^2 - 9x - 90$	(o) $c^2 - 24cd + 143d^2$

5. Write each quadratic expression as a product of two factors:

(a) $2x^2 + 5x + 2$	(f) $6x^2 - 7x - 3$	(k) $24x^2 - 50x + 25$
(b) $3x^2 + 8x + 4$	(g) $6x^2 - 5x + 1$	(l) $2x^2 + xy - y^2$
(c) $6x^2 - 11x + 3$	(h) $3x^2 + 13x - 30$	(m) $4a^2 - 8ab + 3b^2$
(d) $3x^2 + 14x - 5$	(i) $12x^2 - 7x - 12$	(n) $6p^2 + 5pq - 4q^2$
(e) $9x^2 - 6x - 8$	(j) $12x^2 + 31x - 15$	(o) $18u^2 - 19uv - 12v^2$

6. Write each expression as a product of three factors:

(a) $3a^2 - 12$	(e) $25y - y^3$	(i) $x^4 - 3x^2 - 4$
(b) $x^4 - y^4$	(f) $16 - a^4$	(j) $ax^2 - a - 2x^2 + 2$
(c) $x^3 - x$	(g) $4x^2 + 14x - 30$	(k) $16m^3 - mn^2$
(d) $5x^2 - 5x - 30$	(h) $x^3 - 8x^2 + 7x$	(l) $ax^2 - a^2x - 20a^3$

DEVELOPMENT

7. Factor as fully as possible:

(a) $72 + x - x^2$	(h) $a^2 - bc - b + a^2c$	(o) $12x^2 - 8xy - 15y^2$
(b) $(a - b)^2 - c^2$	(i) $9x^2 + 36x - 45$	(p) $x^2 + 2ax + a^2 - b^2$
(c) $a^3 - 10a^2b + 24ab^2$	(j) $4x^4 - 37x^2 + 9$	(q) $9x^2 - 18x - 315$
(d) $a^2 - b^2 - a + b$	(k) $x^2y^2 - 13xy - 48$	(r) $x^4 - x^2 - 2x - 1$
(e) $x^4 - 256$	(l) $x(x - y)^2 - xz^2$	(s) $10x^3 - 13x^2y - 9xy^2$
(f) $4p^2 - (q + r)^2$	(m) $20 - 9x - 20x^2$	(t) $x^2 + 4xy + 4y^2 - a^2 + 2ab - b^2$
(g) $6x^4 - x^3 - 2x^2$	(n) $4x^3 - 12x^2 - x + 3$	(u) $(x + y)^2 - (x - y)^2$

EXTENSION

8. Factor fully:

(a) $a^2 + b(b + 1)a + b^3$	(f) $(a^2 - b^2 - c^2)^2 - 4b^2c^2$
(b) $a(b + c - d) - c(a - b + d)$	(g) $(ax + by)^2 + (ay - bx)^2 + c^2(x^2 + y^2)$
(c) $(a^2 - b^2)^2 - (a - b)^4$	(h) $x^2 + (a - b)xy - aby^2$
(d) $4x^4 - 2x^3y - 3xy^3 - 9y^4$	(i) $a^4 + a^2b^2 + b^4$
(e) $(x^2 + xy)^2 - (xy + y^2)^2$	(j) $a^4 + 4b^4$

1D Algebraic Fractions

An *algebraic fraction* is a fraction containing pronumerals. They are manipulated in the same way as arithmetic fractions, and factorisation plays a major role.

Addition and Subtraction of Algebraic Fractions: A common denominator is required, but finding the lowest common denominator can involve factoring all the denominators.

4

ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS: First factor all denominators. Then work with the lowest common denominator.

WORKED EXERCISE:

$$\begin{aligned}
 \text{(a)} \quad \frac{1}{x-4} - \frac{1}{x} &= \frac{x - (x-4)}{x(x-4)} \\
 &= \frac{4}{x(x-4)} \\
 \text{(b)} \quad \frac{2}{x^2-x} - \frac{5}{x^2-1} &= \frac{2}{x(x-1)} - \frac{5}{(x-1)(x+1)} \\
 &= \frac{2(x+1) - 5x}{x(x-1)(x+1)} \\
 &= \frac{2-3x}{x(x-1)(x+1)}
 \end{aligned}$$

Multiplication and Division of Algebraic Fractions: The key step here is to factor all numerators and denominators completely before cancelling factors.

5

MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS: First factor all numerators and denominators completely. Then cancel common factors.

To divide by an algebraic fraction, multiply by its reciprocal in the usual way.

WORKED EXERCISE:

$$\begin{aligned}
 \text{(a)} \quad \frac{2a}{9-a^2} \times \frac{a-3}{a^3+a} &= \frac{2a}{(3-a)(3+a)} \times \frac{a-3}{a(a^2+1)} \\
 &= -\frac{2}{(a+3)(a^2+1)} \\
 \text{(b)} \quad \frac{6abc}{ab+bc} \div \frac{6ac}{a^2+2ac+c^2} &= \frac{6abc}{b(a+c)} \times \frac{(a+c)^2}{6ac} \\
 &= a+c
 \end{aligned}$$

Simplifying Compound Fractions: A *compound fraction* is a fraction in which either the numerator or the denominator is itself a fraction.

6

SIMPLIFYING COMPOUND FRACTIONS: Multiply top and bottom by something that will clear fractions from numerator and denominator together.

WORKED EXERCISE:

$$\begin{aligned}
 \text{(a)} \quad \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} &= \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} \times \frac{12}{12} \\
 &= \frac{6-4}{3+2} \\
 &= \frac{2}{5} \\
 \text{(b)} \quad \frac{\frac{1}{t} + \frac{1}{t+1}}{\frac{1}{t} - \frac{1}{t+1}} &= \frac{\frac{1}{t} + \frac{1}{t+1}}{\frac{1}{t} - \frac{1}{t+1}} \times \frac{t(t+1)}{t(t+1)} \\
 &= \frac{(t+1)+t}{(t+1)-t} \\
 &= 2t+1
 \end{aligned}$$

Exercise 1D

1. Simplify:

(a) $\frac{x}{2x}$

(c) $\frac{3x^2}{9xy}$

(e) $\frac{12xy^2z}{15x^2yz^2}$

(b) $\frac{a}{a^2}$

(d) $\frac{12ab}{4a^2b}$

(f) $\frac{uvw^2}{u^3v^2w}$

2. Simplify:

$$\begin{array}{llll}
 \text{(a)} \quad \frac{x}{3} \times \frac{3}{x} & \text{(d)} \quad \frac{a^2}{2b} \times \frac{b^2}{a^2} & \text{(g)} \quad \frac{5}{a} \div 10 & \text{(j)} \quad \frac{2a}{3b} \times \frac{5c^2}{2a^2b} \times \frac{3b^2}{2c} \\
 \text{(b)} \quad \frac{a}{4} \div \frac{a}{2} & \text{(e)} \quad \frac{3x^2}{4y^2} \times \frac{2y}{x} & \text{(h)} \quad \frac{2ab}{3c} \times \frac{c^2}{ab^2} & \text{(k)} \quad \frac{12x^2yz}{8xy^3} \times \frac{24xy^2}{36yz^2} \\
 \text{(c)} \quad x \times \frac{3}{x^2} & \text{(f)} \quad \frac{x^2}{3ay^3} \div \frac{x^2}{3ay^3} & \text{(i)} \quad \frac{8a^3b}{5} \div \frac{4ab}{15} & \text{(l)} \quad \frac{3a^2b}{4b^3c} \times \frac{2c^2}{8a^3} \div \frac{6ac}{16b^2}
 \end{array}$$

3. Write as a single fraction:

$$\begin{array}{llll}
 \text{(a)} \quad \frac{x}{2} + \frac{x}{5} & \text{(d)} \quad \frac{2a}{3} + \frac{3a}{2} & \text{(g)} \quad \frac{1}{x} + \frac{1}{2x} & \text{(j)} \quad x + \frac{1}{x} \\
 \text{(b)} \quad \frac{a}{3} - \frac{a}{6} & \text{(e)} \quad \frac{7b}{10} - \frac{19b}{30} & \text{(h)} \quad \frac{3}{4x} + \frac{4}{3x} & \text{(k)} \quad a + \frac{b}{a} \\
 \text{(c)} \quad \frac{x}{8} - \frac{y}{12} & \text{(f)} \quad \frac{xy}{30} - \frac{xy}{18} & \text{(i)} \quad \frac{1}{a} - \frac{1}{b} & \text{(l)} \quad \frac{1}{2x} - \frac{1}{x^2}
 \end{array}$$

4. Simplify:

$$\begin{array}{llll}
 \text{(a)} \quad \frac{x+1}{2} + \frac{x+2}{3} & \text{(e)} \quad \frac{x-5}{3x} - \frac{x-3}{5x} & \text{(i)} \quad \frac{2}{x+3} - \frac{2}{x-2} \\
 \text{(b)} \quad \frac{2x-1}{5} - \frac{x+3}{2} & \text{(f)} \quad \frac{1}{x} - \frac{1}{x+1} & \text{(j)} \quad \frac{x}{x+y} + \frac{y}{x-y} \\
 \text{(c)} \quad \frac{2x+1}{3} - \frac{x-5}{6} + \frac{x+4}{4} & \text{(g)} \quad \frac{1}{x+1} - \frac{1}{x+1} & \text{(k)} \quad \frac{a}{x+a} - \frac{b}{x+b} \\
 \text{(d)} \quad \frac{3x-7}{5} + \frac{4x+3}{2} - \frac{2x-5}{10} & \text{(h)} \quad \frac{2}{x-3} + \frac{3}{x-2} & \text{(l)} \quad \frac{x}{x-1} - \frac{x}{x+1}
 \end{array}$$

5. Factor where possible and then simplify:

$$\begin{array}{lll}
 \text{(a)} \quad \frac{a}{ax+ay} & \text{(d)} \quad \frac{a^2-9}{a^2+a-12} & \text{(g)} \quad \frac{ac+ad+bc+bd}{a^2+ab} \\
 \text{(b)} \quad \frac{3a^2-6ab}{2a^2b-4ab^2} & \text{(e)} \quad \frac{x^2+2xy+y^2}{x^2-y^2} & \text{(h)} \quad \frac{y^2-8y+15}{2y^2-5y-3} \\
 \text{(c)} \quad \frac{x^2+2x}{x^2-4} & \text{(f)} \quad \frac{x^2+10x+25}{x^2+9x+20} & \text{(i)} \quad \frac{9ax+6bx-6ay-4by}{9x^2-4y^2}
 \end{array}$$

6. Simplify:

$$\begin{array}{ll}
 \text{(a)} \quad \frac{3x+3}{2x} \times \frac{x^2}{x^2-1} & \text{(d)} \quad \frac{x^2-x-20}{x^2-25} \times \frac{x^2-x-2}{x^2+2x-8} \div \frac{x+1}{x^2+5x} \\
 \text{(b)} \quad \frac{a^2+a-2}{a+2} \times \frac{a^2-3a}{a^2-4a+3} & \text{(e)} \quad \frac{ax+bx-2a-2b}{3x^2-5x-2} \times \frac{9x^2-1}{a^2+2ab+b^2} \\
 \text{(c)} \quad \frac{c^2+5c+6}{c^2-16} \div \frac{c+3}{c-4} & \text{(f)} \quad \frac{2x^2+x-15}{x^2+3x-28} \div \frac{x^2+6x+9}{x^2-4x} \div \frac{6x^2-15x}{x^2-49}
 \end{array}$$

7. Simplify:

$$\begin{array}{ll}
 \text{(a)} \quad \frac{1}{x^2+x} + \frac{1}{x^2-x} & \text{(d)} \quad \frac{3}{x^2+2x-8} - \frac{2}{x^2+x-6} \\
 \text{(b)} \quad \frac{1}{x^2-4} + \frac{1}{x^2-4x+4} & \text{(e)} \quad \frac{x}{a^2-b^2} - \frac{x}{a^2+ab} \\
 \text{(c)} \quad \frac{1}{x-y} + \frac{2x-y}{x^2-y^2} & \text{(f)} \quad \frac{1}{x^2-4x+3} + \frac{1}{x^2-5x+6} - \frac{1}{x^2-3x+2}
 \end{array}$$

8. Simplify:

$$\begin{array}{lll}
 \text{(a)} \frac{b-a}{a-b} & \text{(c)} \frac{x^2-5x+6}{2-x} & \text{(e)} \frac{m}{m-n} + \frac{n}{n-m} \\
 \text{(b)} \frac{v^2-u^2}{u-v} & \text{(d)} \frac{1}{a-b} - \frac{1}{b-a} & \text{(f)} \frac{x-y}{y^2+xy-2x^2}
 \end{array}$$

DEVELOPMENT

9. Study the worked exercise on compound fractions and then simplify:

$$\begin{array}{lllll}
 \text{(a)} \frac{1-\frac{1}{2}}{1+\frac{1}{2}} & \text{(c)} \frac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{10}} & \text{(e)} \frac{\frac{1}{x}}{1+\frac{2}{x}} & \text{(g)} \frac{1}{\frac{1}{b}+\frac{1}{a}} & \text{(i)} \frac{1-\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+1}} \\
 \text{(b)} \frac{2+\frac{1}{3}}{5-\frac{2}{3}} & \text{(d)} \frac{\frac{17}{20}-\frac{3}{4}}{\frac{4}{5}-\frac{3}{10}} & \text{(f)} \frac{t-\frac{1}{t}}{t+\frac{1}{t}} & \text{(h)} \frac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}} & \text{(j)} \frac{\frac{3}{x+2}-\frac{2}{x+1}}{\frac{5}{x+2}-\frac{4}{x+1}}
 \end{array}$$

10. If $x = \frac{1}{\lambda}$ and $y = \frac{1}{1-x}$ and $z = \frac{y}{y-1}$, show that $z = \lambda$.

11. Simplify:

$$\begin{array}{ll}
 \text{(a)} \frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2+y^2}{x-y} & \text{(b)} \frac{8x^2+14x+3}{8x^2-10x+3} \times \frac{12x^2-6x}{4x^2+5x+1} \div \frac{18x^2-6x}{4x^2+x-3} \\
 \text{(c)} \frac{(a-b)^2-c^2}{ab-b^2-bc} \times \frac{c}{a^2+ab-ac} \div \frac{ac-bc+c^2}{a^2-(b-c)^2} & \text{(d)} \frac{x-y}{x} + \frac{x^3+y^3}{xy^2} - \frac{x^2+y^2}{x^2} \\
 \text{(e)} \frac{x+4}{x-4} - \frac{x-4}{x+4} & \text{(f)} \frac{4y}{x^2+2xy} - \frac{3x}{xy+2y^2} + \frac{3x-2y}{xy} \\
 \text{(g)} \frac{8x}{x^2+5x+6} - \frac{5x}{x^2+3x+2} - \frac{3x}{x^2+4x+3} & \text{(h)} \frac{1}{x-1} + \frac{2}{x+1} - \frac{3x-2}{x^2-1} - \frac{1}{x^2+2x+1}
 \end{array}$$

12. (a) Expand $\left(x + \frac{1}{x}\right)^2$.(b) Suppose that $x + \frac{1}{x} = 3$. Use part (a) to evaluate $x^2 + \frac{1}{x^2}$ without attempting to find the value of x .

EXTENSION

13. Simplify these algebraic fractions:

$$\begin{array}{ll}
 \text{(a)} \frac{1}{(a-b)(a-c)} + \frac{1}{(b-c)(b-a)} + \frac{1}{(c-a)(c-b)} \\
 \text{(b)} \left(1 + \frac{45}{x-8} - \frac{26}{x-6}\right) \left(3 - \frac{65}{x+7} + \frac{8}{x-2}\right) \\
 \text{(c)} \left(2 - \frac{3n}{m} + \frac{9n^2-2m^2}{m^2+2mn}\right) \div \left(\frac{1}{m} - \frac{1}{m-2n-\frac{4n^2}{m+n}}\right) \\
 \text{(d)} \frac{1}{x+\frac{1}{x+2}} \times \frac{1}{x+\frac{1}{x-2}} \div \frac{x-\frac{4}{x}}{x^2-2+\frac{1}{x^2}}
 \end{array}$$

1E Four Cubic Identities

The three special quadratic identities will be generalised later to any degree. For now, here are the cubic versions of them. They will be new to most people.

7

CUBE OF A SUM:	$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
CUBE OF A DIFFERENCE:	$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$
DIFFERENCE OF CUBES:	$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
SUM OF CUBES:	$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

The proofs of these identities are left to the first two questions in the following exercise.

WORKED EXERCISE: Here is an example of each identity.

(a) $(x + 5)^3 = x^3 + 15x^2 + 75x + 125$

(b) $(2x - 3y)^3 = 8x^3 - 36x^2y + 54xy^2 - 27y^3$

(c) $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

(d) $4^3 + 5^3 = (4 + 5)(16 - 20 + 25) = 9 \times 21 = 3^3 \times 7$

WORKED EXERCISE: (a) Simplify $\frac{a^3 + 1}{a + 1}$. (b) Factor $a^3 - b^3 + a - b$.

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \frac{a^3 + 1}{a + 1} &= \frac{(a + 1)(a^2 - a + 1)}{a + 1} \\ &= a^2 - a + 1 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad a^3 - b^3 + a - b &= (a - b)(a^2 + ab + b^2) + (a - b) \\ &= (a - b)(a^2 + ab + b^2 + 1) \end{aligned}$$

Exercise 1E

- (a) Prove the factorisation $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ by expanding the RHS.

(b) Similarly, prove the factorisation $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$.
- (a) Prove the identity $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ by writing $(A + B)^3 = (A + B)(A^2 + 2AB + B^2)$ and expanding.

(b) Similarly, prove the identity $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$.
- Expand:

(a) $(a + b)^3$	(c) $(b - 1)^3$	(e) $(1 - c)^3$	(g) $(2x + 5y)^3$
(b) $(x - y)^3$	(d) $(p + 2)^3$	(f) $(t - 3)^3$	(h) $(3a - 4b)^3$
- Factor:

(a) $x^3 + y^3$	(d) $g^3 - 1$	(g) $27 - t^3$	(j) $u^3 - 64v^3$
(b) $a^3 - b^3$	(e) $b^3 - 8$	(h) $125 + a^3$	(k) $a^3b^3c^3 + 1000$
(c) $y^3 + 1$	(f) $8c^3 + 1$	(i) $27h^3 - 1$	(l) $216x^3 + 125y^3$
- Write as a product of three factors:

(a) $2x^3 + 16$	(c) $24t^3 + 81$	(e) $250p^3 - 432q^3$	(g) $5x^3y^3 - 5$
(b) $a^4 - ab^3$	(d) $x^3y - 125y$	(f) $27x^4 + 1000xy^3$	(h) $x^6 + x^3y^3$

6. Simplify:

(a) $\frac{x^3 - 1}{x^2 - 1}$

(b) $\frac{a^2 - 3a - 10}{a^3 + 8}$

(c) $\frac{a^3 + 1}{6a^2} \times \frac{3a}{a^2 + a}$

(d) $\frac{x^2 - 9}{x^4 - 27x} \div \frac{x + 3}{x^2 + 3x + 9}$

7. Simplify:

(a) $\frac{3}{a - 2} - \frac{3a}{a^2 + 2a + 4}$

(b) $\frac{1}{x^3 - 1} + \frac{x + 1}{x^2 + x + 1}$

(c) $\frac{1}{x^2 - 2x - 8} - \frac{1}{x^3 + 8}$

(d) $\frac{a^2}{a^3 + b^3} + \frac{a - b}{a^2 - ab + b^2} + \frac{1}{a + b}$

DEVELOPMENT

8. Factor as fully as possible:

(a) $a^3 + b^3 + a + b$

(b) $x^6 - 64$

(c) $2a^4 - 3a^3 + 16a - 24$

(d) $(x + y)^3 - (x - y)^3$

(e) $s^3 - t^3 + s^2 - t^2$

(f) $(t - 2)^3 + (t + 2)^3$

(g) $(a - 2b)^3 + (2a - b)^3$

(h) $x^6 - 7x^3 - 8$

(i) $u^7 + u^6 + u + 1$

(j) $2 + x^3 - 3x^6$

(k) $x^7 - x^3 + 8x^4 - 8$

(l) $a^5 + a^4 + a^3 + a^2 + a + 1$

9. Simplify:

(a) $\frac{6a^2 + 6}{a^2 + a + 1} \times \frac{a^3 - 1}{a^3 - 3a^2} \times \frac{a^3 + a^2}{a^4 - 1}$

(b) $\frac{x^4 - 8x}{x^2 - 4x - 5} \times \frac{x^2 + 2x + 1}{x^3 - x^2 - 2x} \div \frac{x^2 + 2x + 4}{x - 5}$

(c) $\frac{(a + 1)^3 - (a - 1)^3}{3a^3 + a}$

(d) $\frac{1}{x - 3} - \frac{8x}{x^3 - 27} - \frac{x - 3}{x^2 + 3x + 9}$

(e) $\frac{3x^2 + 2x + 4}{x^3 - 1} - \frac{x + 1}{x^2 + x + 1} - \frac{2}{x - 1}$

(f) $\frac{1 + x + x^2}{1 - x^3} + \frac{x - x^2}{(1 - x)^3}$

EXTENSION

10. Find the four quartic identities that correspond to the cubic identities in this exercise. That is, find the expansions of $(A + B)^4$ and $(A - B)^4$ and find factorisations of $A^4 + B^4$ and $A^4 - B^4$.

11. Factor as fully as possible: (a) $x^7 + x$ (b) $x^{12} - y^{12}$

12. If $x + y = 1$ and $x^3 + y^3 = 19$, find the value of $x^2 + y^2$.

13. Simplify $(x - y)^3 + (x + y)^3 + 3(x - y)^2(x + y) + 3(x + y)^2(x - y)$.

14. If $a + b + c = 0$, show that $(2a - b)^3 + (2b - c)^3 + (2c - a)^3 = 3(2a - b)(2b - c)(2c - a)$.

15. Simplify $\frac{a^4 - b^4}{a^2 - 2ab + b^2} \div \frac{a^2b + b^3}{a^3 - b^3} \times \frac{a^2b - ab^2 + b^3}{a^4 + a^2b^2 + b^4}$.

16. Simplify $(1 + a)^2 \div \left(1 + \frac{a}{1 - a + \frac{a}{1 + a + a^2}} \right)$.

1F Linear Equations and Inequations

The rules for solving equations and for solving inequations are the same, except for a qualification about multiplying or dividing an inequation by a negative:

- 8** **LINEAR EQUATIONS:** Any number can be added to or subtracted from both sides. Both sides can be multiplied or divided by any nonzero number.
- LINEAR INEQUATIONS:** The rules for inequations are the same as those for equations, except that when both sides are multiplied or divided by a negative number, the inequality sign is reversed.

WORKED EXERCISE: Solve: (a) $\frac{4-7x}{4x-7} = 1$ (b) $x - 12 < 5 + 3x$

SOLUTION:

(a) $\frac{4-7x}{4x-7} = 1$ $\times (4x-7)$ $4-7x = 4x-7$ $+ 7x$ $4 = 11x-7$ $+ 7$ $11 = 11x$ $\div 11$ $x = 1$	(b) $x - 12 < 5 + 3x$ $- 3x$ $-2x - 12 < 5$ $+ 12$ $-2x < 17$ $\div (-2)$ $x > -8\frac{1}{2}$ Because of the division by the negative, the inequality was reversed.

Changing the Subject of a Formula: Similar sequences of operations allow the subject of a formula to be changed from one pronumeral to another.

WORKED EXERCISE: Given the formula $y = \frac{x+1}{x+a}$:

- (a) change the subject to a , (b) change the subject to x .

SOLUTION:

(a) $y = \frac{x+1}{x+a}$ $\times (x+a)$ $xy + ay = x + 1$ $- xy$ $ay = x + 1 - xy$ $\div y$ $a = \frac{x+1-xy}{y}$	(b) $y = \frac{x+1}{x+a}$ $\times (x+a)$ $xy + ay = x + 1$ $xy - x = 1 - ay$ $x(y-1) = 1 - ay$ $\div (y-1)$ $x = \frac{1-ay}{y-1}$

Exercise 1F

1. Solve:

(a) $-2x = -20$	(c) $-a = 5$	(e) $-1 - x = 0$	(g) $2t < t$
(b) $3x > 2$	(d) $\frac{x}{-4} \leq -1$	(f) $0.1y = 5$	(h) $-\frac{1}{2}x = 8$

2. Solve:

(a) $3x - 5 = 22$	(c) $1 - 2x < 9$	(e) $-13 \leq 5a - 6$	(g) $19 = 3 - 7y$
(b) $4x + 7 \geq -13$	(d) $6x = 3x - 21$	(f) $-2 > 4 + \frac{t}{5}$	(h) $23 - \frac{u}{3} \geq 7$

3. Solve:

- (a) $5x - 2 < 2x + 10$ (h) $7x - (3x + 11) = 6 - (15 - 9x)$
 (b) $5 - x = 27 + x$ (i) $4(x + 2) = 4x + 9$
 (c) $16 + 9a > 10 - 3a$ (j) $3(x - 1) < 2(x + 1) + x$
 (d) $13y - 21 \leq 20y - 35$ (k) $(x - 3)(x + 6) \leq (x - 4)(x - 5)$
 (e) $13 - 12x \geq 6 - 3x$ (l) $(1 + 2x)(4 + 3x) = (2 - x)(5 - 6x)$
 (f) $3(x + 7) = -2(x - 9)$ (m) $(x + 3)^2 > (x - 1)^2$
 (g) $8 + 4(2 - x) > 3 - 2(5 - x)$ (n) $(2x - 5)(2x + 5) = (2x - 3)^2$

4. Solve:

- (a) $\frac{x}{8} = \frac{1}{2}$ (e) $\frac{2}{a} = 5$ (i) $\frac{7 - 4x}{6} < 1$ (m) $\frac{1}{a} + 4 = 1 - \frac{2}{a}$
 (b) $\frac{a}{12} = \frac{2}{3}$ (f) $3 = \frac{9}{2y}$ (j) $\frac{5 + a}{a} = -3$ (n) $\frac{4}{x - 1} = -5$
 (c) $\frac{y}{20} < \frac{4}{5}$ (g) $\frac{2x + 1}{5} \geq -3$ (k) $\frac{9 - 2t}{t} = 13$ (o) $\frac{3x}{1 - 2x} = 7$
 (d) $\frac{1}{x} = 3$ (h) $\frac{5a}{3} - 1 \geq \frac{3a}{5} + 1$ (l) $6 - \frac{c}{3} > c$ (p) $\frac{11t}{8t + 13} = -2$

5. Solve:

- (a) $\frac{x}{3} - \frac{x}{5} \geq 2$ (i) $\frac{x + 1}{x + 2} = \frac{x - 3}{x + 1}$
 (b) $\frac{a}{10} - \frac{a}{6} < 1$ (j) $\frac{(3x - 2)(3x + 2)}{(3x - 1)^2} = 1$
 (c) $\frac{x}{6} + \frac{2}{3} = \frac{x}{2} - \frac{5}{6}$ (k) $\frac{a + 5}{2} - \frac{a - 1}{3} > 1$
 (d) $\frac{1}{x} - 3 = \frac{1}{2x}$ (l) $\frac{3}{4} - \frac{x + 1}{12} \leq \frac{2}{3} - \frac{x - 1}{6}$
 (e) $\frac{1}{2x} - \frac{2}{3} = 1 - \frac{1}{3x}$ (m) $\frac{2x}{5} + \frac{2 - 3x}{4} < \frac{3}{10} - \frac{3 - 5x}{2}$
 (f) $\frac{x}{3} - 2 < \frac{x}{2} - 3$ (n) $\frac{3}{4}(x - 1) - \frac{1}{2}(3x + 2) = 0$
 (g) $\frac{x - 2}{3} > \frac{x + 4}{4}$ (o) $\frac{4x + 1}{6} - \frac{2x - 1}{15} = \frac{3x - 5}{5} - \frac{6x + 1}{10}$
 (h) $\frac{3}{x - 2} = \frac{4}{2x + 5}$ (p) $\frac{7(1 - x)}{12} - \frac{3 + 2x}{9} \geq \frac{5(2 + x)}{6} - \frac{4 - 5x}{18}$

6. (a) If $v = u + at$, find a when $t = 4$, $v = 20$ and $u = 8$.
 (b) Given that $v^2 = u^2 + 2as$, find the value of s when $u = 6$, $v = 10$ and $a = 2$.
 (c) Suppose that $\frac{1}{u} + \frac{1}{v} = \frac{1}{t}$. Find v , given that $u = -1$ and $t = 2$.
 (d) If $S = -15$, $n = 10$ and $a = -24$, find ℓ , given that $S = \frac{n}{2}(a + \ell)$.
 (e) Temperatures in degrees Fahrenheit and degrees Celsius are related by the formula $F = \frac{9}{5}C + 32$. Find the value of C that corresponds to $F = 95$.
 (f) Suppose that the variables c and d are related by the formula $\frac{3}{c + 1} = \frac{5}{d - 1}$. Find c when $d = -2$.

DEVELOPMENT

7. Solve each of the following inequations for the given domain of the variable, and graph each solution on the real number line:
- $2x - 3 < 5$, where x is a positive integer.
 - $1 - 3x \leq 16$, where x is a negative integer.
 - $4x + 5 > 2x - 3$, where x is a real number.
 - $7 - 2x \geq x + 1$, where x is a real number.
 - $4 \leq 2x < 14$, where x is an integer.
 - $-12 < 3x < 9$, where x is an integer.
 - $1 < 2x + 1 \leq 11$, where x is a real number.
 - $-10 \leq 2 - 3x \leq -1$, where x is a real number.
8. Solve each of these problems by constructing and then solving a linear equation:
- Five more than twice a certain number is one more than the number itself. What is the number?
 - I have \$175 in my wallet, consisting of \$10 and \$5 notes. If I have twice as many \$10 notes as \$5 notes, how many \$5 notes do I have?
 - My father is 24 years older than me, and 12 years ago he was double my age. How old am I now?
 - The fuel tank in my new car was 40% full. I added 28 litres and then found that it was 75% full. How much fuel does the tank hold?
 - A certain tank has an inlet valve and an outlet valve. The tank can be filled via the inlet valve in 6 minutes and emptied (from full) via the outlet valve in 10 minutes. If both valves are operating, how long would it take to fill the tank if it was empty to begin with?
 - A basketball player has scored 312 points in 15 games. How many points must he average per game in his next 3 games to take his overall average to 20 points per game?
 - A cyclist rides for 5 hours at a certain speed and then for 4 hours at a speed 6 km/h greater than her original speed. If she rides 294 km altogether, what was her initial speed?
 - Two trains travel at speeds of 72 km/h and 48 km/h respectively. If they start at the same time and travel towards each other from two places 600 km apart, how long will it be before they meet?

9. Rearrange each formula so that the pronumeral written in the brackets is the subject:

(a) $a = bc - d$ [b]	(e) $\frac{a}{2} - \frac{b}{3} = a$ [a]	(h) $a = \frac{b+5}{b-4}$ [b]
(b) $t = a + (n-1)d$ [n]	(f) $\frac{1}{f} + \frac{2}{g} = \frac{5}{h}$ [g]	(i) $c = \frac{7+2d}{5-3d}$ [d]
(c) $\frac{p}{q+r} = t$ [r]	(g) $x = \frac{y}{y+2}$ [y]	(j) $u = \frac{v+w-1}{v-w+1}$ [v]
(d) $u = 1 + \frac{3}{v}$ [v]		

10. Solve: (a) $\frac{x}{x-2} + \frac{3}{x-4} = 1$ (b) $\frac{3a-2}{2a-3} - \frac{a+17}{a+10} = \frac{1}{2}$

EXTENSION

11. (a) Show that $\frac{x-1}{x-3} = 1 + \frac{2}{x-3}$.
- (b) Hence solve $\frac{x-1}{x-3} - \frac{x-3}{x-5} = \frac{x-5}{x-7} - \frac{x-7}{x-9}$.

1G Quadratic Equations

This section reviews the solution of quadratic equations by factorisation and by the quadratic formula. The third method, completing the square, will be reviewed in Section 1I.

Solving a Quadratic by Factorisation: This method is the simplest, but it normally only works when the roots are rational numbers.

SOLVING A QUADRATIC BY FACTORING:

9

1. Get all the terms on the left, then factor the left-hand side.
2. Use the principle that if $AB = 0$, then $A = 0$ or $B = 0$.

WORKED EXERCISE: Solve $5x^2 + 34x - 7 = 0$.

SOLUTION: $5x^2 + 34x - 7 = 0$

$$(5x - 1)(x + 7) = 0 \quad (\text{factoring the LHS})$$

$$5x - 1 = 0 \text{ or } x + 7 = 0 \quad (\text{one of the factors must be zero})$$

$$x = \frac{1}{5} \text{ or } x = -7$$

Solving a Quadratic by the Formula: This method works whether the solutions are rational numbers or involve surds.

10

THE QUADRATIC FORMULA: The solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Always calculate $b^2 - 4ac$ first.

The formula is proven by completing the square, as discussed in Chapter Eight.

WORKED EXERCISE: Use the quadratic formula to solve:

(a) $5x^2 + 2x - 7 = 0$

(b) $3x^2 + 4x - 1 = 0$

SOLUTION:

(a) $5x^2 + 2x - 7 = 0$

Here $b^2 - 4ac = 2^2 + 140$

$$= 144$$

$$= 12^2,$$

$$\text{so } x = \frac{-2 + 12}{10} \text{ or } \frac{-2 - 12}{10}$$

$$= 1 \text{ or } -1\frac{2}{5}.$$

(b) $3x^2 + 4x - 1 = 0$

Here $b^2 - 4ac = 4^2 + 12$

$$= 28$$

$$= 4 \times 7,$$

$$\text{so } x = \frac{-4 + 2\sqrt{7}}{6} \text{ or } \frac{-4 - 2\sqrt{7}}{6}$$

$$= \frac{1}{3}(-2 + \sqrt{7}) \text{ or } \frac{1}{3}(-2 - \sqrt{7}).$$

Exercise 1G

1. Solve:

(a) $x^2 = 9$

(c) $1 - t^2 = 0$

(e) $4x^2 - 1 = 0$

(b) $a^2 - 4 = 0$

(d) $x^2 = \frac{9}{4}$

(f) $25y^2 = 16$

2. Solve by factoring:

(a) $x^2 - 5x = 0$

(c) $t^2 = t$

(e) $2b^2 - b = 0$

(g) $3y^2 = 2y$

(b) $c^2 + 2c = 0$

(d) $3a = a^2$

(f) $3u^2 + u = 0$

(h) $12u + 5u^2 = 0$

3. Solve by factoring:

(a) $x^2 - 3x + 2 = 0$

(e) $p^2 = p + 6$

(i) $u^2 + u = 56$

(b) $x^2 + 6x + 8 = 0$

(f) $a^2 = a + 132$

(j) $50 + 27h + h^2 = 0$

(c) $a^2 + 2a - 15 = 0$

(g) $c^2 + 18 = 9c$

(k) $k^2 = 60 + 11k$

(d) $y^2 + 4y = 5$

(h) $8t + 20 = t^2$

(l) $\alpha^2 + 20\alpha = 44$

4. Solve by factoring:

(a) $3a^2 - 7a + 2 = 0$

(e) $5x^2 - 26x + 5 = 0$

(i) $25x^2 + 9 = 30x$

(b) $2x^2 + 11x + 5 = 0$

(f) $4t^2 + 9 = 15t$

(j) $6x^2 + 13x + 6 = 0$

(c) $3b^2 - 4b - 4 = 0$

(g) $t + 15 = 2t^2$

(k) $12b^2 + 3 + 20b = 0$

(d) $2y^2 + 5y = 12$

(h) $10u^2 + 3u - 4 = 0$

(l) $6k^2 + 13k = 8$

5. Solve using the quadratic formula, giving exact answers followed by approximations to four significant figures where appropriate:

(a) $x^2 - x - 1 = 0$

(e) $c^2 - 6c + 2 = 0$

(i) $2b^2 + 3b = 1$

(b) $y^2 + y = 3$

(f) $4x^2 + 4x + 1 = 0$

(j) $3c^2 = 4c + 3$

(c) $a^2 + 12 = 7a$

(g) $2a^2 + 1 = 4a$

(k) $4t^2 = 2t + 1$

(d) $u^2 + 2u - 2 = 0$

(h) $5x^2 + 13x - 6 = 0$

(l) $x^2 + x + 1 = 0$

6. Solve by factoring:

(a) $x = \frac{x+2}{x}$

(c) $y + \frac{2}{y} = \frac{9}{2}$

(e) $\frac{5k+7}{k-1} = 3k+2$

(b) $a + \frac{10}{a} = 7$

(d) $(5b-3)(3b+1) = 1$

(f) $\frac{u+3}{2u-7} = \frac{2u-1}{u-3}$

7. Find the exact solutions of:

(a) $x = \frac{1}{x} + 2$

(c) $a = \frac{a+4}{a-1}$

(e) $\frac{y+1}{y+2} = \frac{3-y}{y-4}$

(b) $\frac{4x-1}{x} = x$

(d) $\frac{5m}{2} = 2 + \frac{1}{m}$

(f) $2(k-1) = \frac{4-5k}{k+1}$

8. (a) If $y = px - ap^2$, find p , given that $a = 2$, $x = 3$ and $y = 1$.

(b) Given that $(x-a)(x-b) = c$, find x when $a = -2$, $b = 4$ and $c = 7$.

(c) Suppose that $S = \frac{n}{2}(2a + (n-1)d)$. Find the positive value of n if $S = 80$, $a = 4$ and $d = 6$.

9. Find a in terms of b if:

(a) $a^2 - 5ab + 6b^2 = 0$

(b) $3a^2 + 5ab - 2b^2 = 0$

10. Find y in terms of x if:

(a) $4x^2 - y^2 = 0$

(b) $x^2 - 9xy - 22y^2 = 0$

DEVELOPMENT

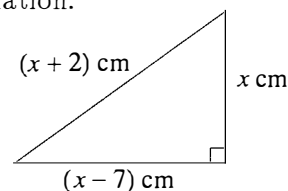
11. Solve each problem by forming and solving a suitable quadratic equation:

(a) Find the value of x in the diagram opposite.

(b) Find a positive integer which when increased by 30 is 12 less than its square.

(c) Two positive numbers differ by 3 and the sum of their squares is 117. Find the numbers.

(d) A rectangular area can be completely tiled with 200 square tiles. If the side length of each tile was increased by 1 cm, it would only take 128 tiles to tile the area. Find the side length of each tile.



- (e) The numerator of a certain fraction is 3 less than its denominator. If 6 is added to the numerator and 5 to the denominator, the value of the fraction is doubled. Find the fraction.
- (f) A photograph is 18 cm by 12 cm. It is to be surrounded by a frame of uniform width whose area is equal to that of the photograph. Find the width of the frame.
- (g) A certain tank can be filled by two pipes in 80 minutes. The larger pipe by itself can fill the tank in 2 hours less than the smaller pipe by itself. How long does each pipe take to fill the tank on its own?
- (h) Two trains each make a journey of 330 km. One of the trains travels 5 km/h faster than the other and takes 30 minutes less time. Find the speeds of the trains.

12. Solve each of these equations:

(a) $\frac{2}{a+3} + \frac{a+3}{2} = \frac{10}{3}$

(c) $\frac{3t}{t^2-6} = \sqrt{3}$

(b) $\frac{k+10}{k-5} - \frac{10}{k} = \frac{11}{6}$

(d) $\frac{3m+1}{3m-1} - \frac{3m-1}{3m+1} = 2$

EXTENSION

13. (a) Find x in terms of c , given that $\frac{2}{3x-2c} + \frac{3}{2x-3c} = \frac{7}{2c}$.

(b) Find x in terms of a and b if $\frac{a^2b}{x^2} + \left(1 + \frac{b}{x}\right)a = 2b + \frac{a^2}{x}$.

1H Simultaneous Equations

This section will review the two algebraic approaches to simultaneous equations — substitution and elimination (graphical interpretations will be discussed in Chapters Two and Three). Both linear and non-linear simultaneous equations will be reviewed, and the methods extended to systems of three equations in three unknowns.

Solution by Substitution: This method can be applied whenever one of the equations can be solved for one of the variables.

11

SIMULTANEOUS EQUATIONS BY SUBSTITUTION: Solve one of the equations for one of the variables, then substitute it into the other equation.

WORKED EXERCISE: Solve these simultaneous equations by substitution:

(a) $3x - 2y = 29$ (1) (b) $y = x^2$ (1)

$4x + y = 24$ (2) $y = x + 2$ (2)

SOLUTION:

(a) Solving (2) for y , $y = 24 - 4x$. (2A)

Substituting (2A) into (1), $3x - 2(24 - 4x) = 29$

$x = 7$.

Substituting $x = 7$ into (1), $21 - 2y = 29$

$y = -4$.

So $x = 7$ and $y = -4$.

(b) Substituting (1) into (2),

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } -1.$$

From (1), when $x = 2$, $y = 4$, and when $x = -1$, $y = 1$.

So $x = 2$ and $y = 4$, or $x = -1$ and $y = 1$.

Solution by Elimination: This method, when it can be used, is more elegant, and can involve less algebraic manipulation with fractions.

12

SIMULTANEOUS EQUATIONS BY ELIMINATION: Take suitable multiples of the equations so that one variable is eliminated when the equations are added or subtracted.

WORKED EXERCISE: Solve these simultaneous equations by elimination:

(a) $3x - 2y = 29$ (1) $4x + 5y = 8$ (2)	(b) $x^2 + y^2 = 53$ (1) $x^2 - y^2 = 45$ (2)

SOLUTION:

(a) Taking $4 \times (1)$ and $3 \times (2)$, $12x - 8y = 116 \quad (1A)$ $12x + 15y = 24. \quad (2A)$ Subtracting (1A) from (2A), $23y = -92$ $y = -4.$ Substituting into (1), $3x + 8 = 29$ $x = 7.$ So $x = 7$ and $y = -4$.	(b) Adding (1) and (2), $2x^2 = 98$ $x^2 = 49.$ Subtracting (2) from (1), $2y^2 = 8$ $y^2 = 4.$ So $x = 7$ and $y = 2$, or $x = 7$ and $y = -2$, or $x = -7$ and $y = 2$, or $x = -7$ and $y = -2$.

Systems of Three Equations in Three Variables: The key step here is to reduce the system to two equations in two variables.

13

SOLVING THREE SIMULTANEOUS EQUATIONS: Using either substitution or elimination, produce two simultaneous equations in two of the variables.

WORKED EXERCISE: Solve simultaneously:

$3x - 2y - z = -8$	(1)
$5x + y + 3z = 23$	(2)
$4x + y - 5z = -18$	(3)

SOLUTION: Subtracting (3) from (2),

$$x + 8z = 41. \quad (4)$$

Doubling (3),

$$8x + 2y - 10z = -36 \quad (3A)$$

and adding (1) and (3A),

$$11x - 11z = -44$$

$$x - z = -4. \quad (5)$$

Equations (4) and (5) are now two equations in two unknowns.

Subtracting (5) from (4),

$$9z = 45$$

$$z = 5.$$

Substituting $z = 5$ into (5),

$$x = 1$$

and substituting into (2),

$$y = 3.$$

So $x = 1$, $y = 3$ and $z = 5$ (which should be checked in the original equations).

Exercise 1H

1. Solve by substitution:

- (a) $y = 2x$ and $3x + 2y = 14$
 (b) $y = -3x$ and $2x + 5y = 13$
 (c) $y = 4 - x$ and $x + 3y = 8$
 (d) $x = 5y + 4$ and $3x - y = 26$

- (e) $2x + y = 10$ and $7x + 8y = 53$
 (f) $2x - y = 9$ and $3x - 7y = 19$
 (g) $4x - 5y = 2$ and $x + 10y = 41$
 (h) $2x + 3y = 47$ and $4x - y = 45$

2. Solve by elimination:

- (a) $2x + y = 1$ and $x - y = -4$
 (b) $2x + 3y = 16$ and $2x + 7y = 24$
 (c) $3x + 2y = -6$ and $x - 2y = -10$
 (d) $5x - 3y = 28$ and $2x - 3y = 22$
 (e) $3x + 2y = 7$ and $5x + y = 7$
 (f) $3x + 2y = 0$ and $2x - y = 56$

- (g) $15x + 2y = 27$ and $3x + 7y = 45$
 (h) $7x - 3y = 41$ and $3x - y = 17$
 (i) $2x + 3y = 28$ and $3x + 2y = 27$
 (j) $3x - 2y = 11$ and $4x + 3y = 43$
 (k) $4x + 6y = 11$ and $17x - 5y = 1$
 (l) $8x = 5y$ and $13x = 8y + 1$

3. Solve by substitution:

- (a) $y = 2 - x$ and $y = x^2$
 (b) $y = 2x - 3$ and $y = x^2 - 4x + 5$
 (c) $y = 3x^2$ and $y = 4x - x^2$
 (d) $x - y = 5$ and $y = x^2 - 11$

- (e) $x - y = 2$ and $xy = 15$
 (f) $3x + y = 9$ and $xy = 6$
 (g) $x^2 - y^2 = 16$ and $x^2 + y^2 = 34$
 (h) $x^2 + y^2 = 117$ and $2x^2 - 3y^2 = 54$

DEVELOPMENT

4. Solve each of these problems by constructing and then solving a pair of simultaneous equations:

- (a) If 7 apples and 2 oranges cost \$4, while 5 apples and 4 oranges cost \$4.40, find the cost of each apple and orange.
 (b) Twice as many adults as children attended a certain concert. If adult tickets cost \$8 each, child tickets cost \$3 each and the total takings were \$418, find the numbers of adults and children who attended.
 (c) A man is 3 times as old as his son. In 12 years time he will be twice as old as his son. How old is each of them now?
 (d) At a meeting of the members of a certain club, a proposal was voted on. If 357 members voted and the proposal was carried by a majority of 21, how many voted for and how many voted against the proposal?
 (e) The value of a certain fraction becomes $\frac{1}{5}$ if one is added to its numerator. If one is taken from its denominator, its value becomes $\frac{1}{7}$. Find the fraction.
 (f) Kathy paid \$320 in cash for a CD player. If she paid in \$20 notes and \$10 notes and there were 23 notes altogether, how many of each type were there?
 (g) Two people are 16 km apart on a straight road. They start walking at the same time. If they walk towards each other, they will meet in 2 hours, but if they walk in the same direction (so that the distance between them is decreasing), they will meet in 8 hours. Find their walking speeds.

- (h) A certain integer is between 10 and 100. Its value is 8 times the sum of its digits and if it is reduced by 45, its digits are reversed. Find the integer.

5. Solve simultaneously:

(a) $\frac{y}{4} - \frac{x}{3} = 1$ and $\frac{x}{2} + \frac{y}{5} = 10$

(b) $4x + \frac{y-2}{3} = 12$ and $3y - \frac{x-3}{5} = 6$

6. Solve simultaneously:

(a) $x = 2y$

$y = 3z$

$x + y + z = 10$

(c) $2a - b + c = 10$

$a - b + 2c = 9$

$3a - 4c = 1$

(e) $2x - y - z = 17$

$x + 3y + 4z = -20$

$5x - 2y + 3z = 19$

(b) $x + 2y - z = -3$

$3x - 4y + z = 13$

$2x + 5y = -1$

(d) $p + q + r = 6$

$2p - q + r = 1$

$p + q - 2r = -9$

(f) $3u + v - 4w = -4$

$u - 2v + 7w = -7$

$4u + 3v - w = 9$

7. Solve simultaneously:

(a) $x + y = 15$ and $x^2 + y^2 = 125$

(d) $x + y = 9$ and $x^2 + xy + y^2 = 61$

(b) $x - y = 3$ and $x^2 + y^2 = 185$

(e) $x + 2y = 5$ and $2xy - x^2 = 3$

(c) $2x + y = 5$ and $4x^2 + y^2 = 17$

(f) $3x + 2y = 16$ and $xy = 10$

EXTENSION

8. Solve simultaneously:

(a) $\frac{7}{x} - \frac{5}{y} = 3$ and $\frac{2}{x} + \frac{25}{2y} = 12$

(b) $9x^2 + y^2 = 52$ and $xy = 8$

9. Consider the equations $12x^2 - 4xy + 11y^2 = 64$ and $16x^2 - 9xy + 11y^2 = 78$.

(a) By letting $y = mx$, show that $7m^2 + 12m - 4 = 0$.

(b) Hence, or otherwise, solve the two equations simultaneously.

11 Completing the Square

We will see in Chapter Eight that completing the square, because it can be done in all situations, is more important for the investigation of quadratics than factoring. For example, the quadratic formula reviewed earlier is proven by completing the square. The review in this section will be restricted to *monic* quadratics, in which the coefficient of x^2 is 1.

Perfect Squares: When the quadratic $(x + \alpha)^2$ is expanded,

$$(x + \alpha)^2 = x^2 + 2\alpha x + \alpha^2,$$

the coefficient of x is twice α and the constant is the square of α . Reversing the process, the constant term in a perfect square can be found by taking half the coefficient of x and squaring it.

14

COMPLETING THE SQUARE IN AN EXPRESSION: To complete the square in a given expression $x^2 + bx + \dots$, halve the coefficient b of x and square it.

WORKED EXERCISE: Complete the square in: (a) $a^2 + 16a + \dots$ (b) $x^2 - 3x + \dots$

SOLUTION:

(a) The coefficient of a is 16, half of 16 is 8, and $8^2 = 64$,
so $a^2 + 16a + 64 = (a + 8)^2$.

(b) The coefficient of x is -3 , half of -3 is $-1\frac{1}{2}$, and $(-1\frac{1}{2})^2 = 2\frac{1}{4}$,
so $x^2 - 3x + 2\frac{1}{4} = (x - 1\frac{1}{2})^2$.

Solving Quadratic Equations by Completing the Square: This is the process underlying the quadratic formula.

15

SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE:

Complete the square in the quadratic by adding the same to both sides.

WORKED EXERCISE: Solve: (a) $t^2 + 8t = 20$ (b) $x^2 - x - 1 = 0$ (c) $x^2 + x + 1 = 0$

SOLUTION:

(a) $t^2 + 8t = 20$

$$t^2 + 8t + 16 = 36$$

$$(t + 4)^2 = 36$$

$$t + 4 = 6 \text{ or } t + 4 = -6$$

$$t = 2 \text{ or } -10$$

(b) $x^2 - x - 1 = 0$

$$x^2 - x + \frac{1}{4} = 1\frac{1}{4}$$

$$(x - \frac{1}{2})^2 = \frac{5}{4}$$

$$x - \frac{1}{2} = \frac{1}{2}\sqrt{5} \text{ or } -\frac{1}{2}\sqrt{5}$$

$$x = \frac{1}{2} + \frac{1}{2}\sqrt{5} \text{ or } \frac{1}{2} - \frac{1}{2}\sqrt{5}$$

(c) $x^2 + x + 1 = 0$

$$x^2 + x + \frac{1}{4} = -\frac{3}{4}$$

$$(x + \frac{1}{2})^2 = -\frac{3}{4}$$

This is impossible, because a square can't be negative,

so the equation has no solutions.

Exercise 11

1. Write down the constant which must be added to each expression in order to create a perfect square:

(a) $x^2 + 2x$

(c) $a^2 + 10a$

(e) $c^2 + 3c$

(g) $b^2 + 5b$

(b) $y^2 - 6y$

(d) $m^2 - 18m$

(f) $x^2 - x$

(h) $t^2 - 9t$

2. Factor:

(a) $x^2 + 4x + 4$

(c) $p^2 + 14p + 49$

(e) $t^2 - 16t + 64$

(g) $x^2 + 20xy + 100y^2$

(b) $y^2 + 2y + 1$

(d) $m^2 - 12m + 36$

(f) $400 - 40u + u^2$

(h) $a^2b^2 - 24ab + 144$

3. Copy and complete:

(a) $x^2 + 6x + \dots = (x + \dots)^2$

(e) $u^2 + u + \dots = (u + \dots)^2$

(b) $y^2 + 8y + \dots = (y + \dots)^2$

(f) $t^2 - 7t + \dots = (t + \dots)^2$

(c) $a^2 - 20a + \dots = (a + \dots)^2$

(g) $m^2 + 50m + \dots = (m + \dots)^2$

(d) $b^2 - 100b + \dots = (b + \dots)^2$

(h) $c^2 - 13c + \dots = (c + \dots)^2$

4. Solve each of the following quadratic equations by completing the square:

(a) $x^2 - 2x = 3$

(d) $y^2 + 3y = 10$

(g) $x^2 - 10x + 20 = 0$

(b) $x^2 - 6x = 0$

(e) $b^2 - 5b - 14 = 0$

(h) $y^2 - y + 2 = 0$

(c) $a^2 + 6a + 8 = 0$

(f) $x^2 + 4x + 1 = 0$

(i) $a^2 + 7a + 7 = 0$

5. Complete the square for each of the given expressions:

(a) $p^2 - 2pq + \dots$	(c) $x^2 - 6xy + \dots$	(e) $u^2 - uv + \dots$
(b) $a^2 + 4ab + \dots$	(d) $c^2 + 40cd + \dots$	(f) $m^2 + 11mn + \dots$

DEVELOPMENT

6. Solve by dividing both sides by the coefficient of x^2 and then completing the square:

(a) $3x^2 - 15x + 18 = 0$	(d) $2x^2 + 8x + 3 = 0$	(g) $3x^2 - 8x - 3 = 0$
(b) $2x^2 - 4x - 1 = 0$	(e) $4x^2 + 4x - 3 = 0$	(h) $2x^2 + x - 15 = 0$
(c) $3x^2 + 6x + 5 = 0$	(f) $4x^2 - 2x - 1 = 0$	(i) $2x^2 - 10x + 7 = 0$

7. (a) If $x^2 + y^2 + 4x - 2y + 1 = 0$, show that $(x + 2)^2 + (y - 1)^2 = 4$.

(b) Show that the equation $x^2 + y^2 - 6x - 8y = 0$ can be written in the form $(x - a)^2 + (y - b)^2 = c$, where a , b and c are constants. Hence write down the values of a , b and c .

(c) If $x^2 + 1 = 10x + 12y$, show that $(x - 5)^2 = 12(y + 2)$.

(d) Find values for A , B and C if $y^2 - 6x + 16y + 94 = (y + C)^2 - B(x + A)$.

EXTENSION

8. (a) Write down the expansion of $(x + a)^3$ and hence complete the cube in

$$x^3 + 12x^2 + \dots = (x + \dots)^3.$$

(b) Hence use a suitable substitution to change the equation $x^3 + 12x^2 + 30x + 4 = 0$ into a cubic equation of the form $u^3 + cu + d = 0$.

1J The Language of Sets

We will often want to speak about collections of things such as numbers, points and lines. In mathematics, these collections are called *sets*, and this section will introduce or review some of the language associated with sets. Logic is very close to the surface when talking about sets, and particular attention should be given to the words ‘if’, ‘if and only if’, ‘and’, ‘or’ and ‘not’.

Listing Sets and Describing Sets: A *set* is a collection of things. When a set is specified, it needs to be made absolutely clear what things are its members. This can be done by *listing* the members inside curly brackets:

$$S = \{1, 3, 5, 7, 9\},$$

read as ‘ S is the set whose members are 1, 3, 5, 7 and 9’. It can also be done by *writing a description* of its members inside curly brackets, for example,

$$T = \{\text{odd integers from 0 to 10}\},$$

read as ‘ T is the set of odd integers from 0 to 10’.

Equal Sets: Two sets are called *equal* if they have exactly the same members. Hence the sets S and T in the previous paragraph are equal, which is written as $S = T$. The order in which the members are written doesn’t matter at all, neither does repetition, so, for example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\} = \{1, 3, 1, 5, 1, 7, 9\}.$$

Members and Non-members: The symbol $\in$ means ‘is a member of’, and the symbol $\notin$ means ‘is not a member of’, so if $A = \{3, 4, 5, 6\}$, then

$$3 \in A \quad \text{and} \quad 2 \notin A \quad \text{and} \quad 6 \in A \quad \text{and} \quad 9 \notin A,$$

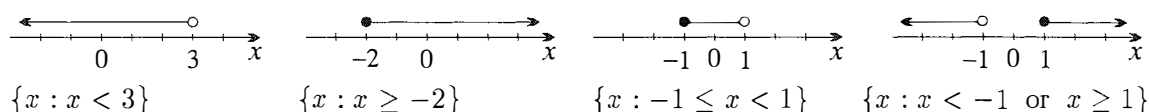
which is read as ‘3 is a member of A ’ and ‘2 is not a member of A ’, and so on.

Set-builder Notation: A third way to specify a set is to write down, using a colon (:), all the conditions something must fulfil to be a member of the set. For example,

$$\{n : n \text{ is a positive integer, } n < 5\} = \{1, 2, 3, 4\},$$

which is read as ‘The set of all n such that n is a positive integer and n is less than 5’.

If the type of number is not specified, real numbers are normally intended. Here are some sets of real numbers, in set-builder notation, with their graphs on the number line:



The Size of a Set: A set may be *finite* or *infinite*. If a set S is finite, then $|S|$ is the symbol for the number of members in S . For example,

$$\{\text{positive even numbers}\} \text{ is infinite} \quad \text{and} \quad |\{a, e, i, o, u\}| = 5.$$

Some sets have only one member, for example $\{3\}$ is ‘the set whose only member is 3’. The set $\{3\}$ is a different object from the number 3:

$$3 \in \{3\} \quad \text{and} \quad 3 \neq \{3\} \quad \text{and} \quad |\{3\}| = 1.$$

The Empty Set: The symbol $\emptyset$ represents the *empty set*, which is the set with no members:

$$|\emptyset| = 0 \quad \text{and} \quad x \notin \emptyset, \text{ whatever } x \text{ is.}$$

There is only one empty set, because any two empty sets have the same members (that is, none at all) and so are equal.

Subsets of Sets: A set A is called a *subset* of a set B if every member of A is a member of B . This relation is written as $A \subset B$. For example,

$$\begin{aligned} \{\text{men in Australia}\} &\subset \{\text{people in Australia}\} \\ \{2, 3, 4\} &\not\subset \{3, 4, 5\}. \end{aligned}$$

Because of the way subsets have been defined, every set is a subset of itself. Also the empty set is a subset of every set. For example,

$$\{1, 3, 5\} \subset \{1, 3, 5\}, \quad \emptyset \subset \{1, 3, 5\} \quad \text{and} \quad \{3\} \subset \{1, 3, 5\}.$$

‘If’ means Subset, ‘If and Only If’ means Equality: The word ‘if’ and the phrase ‘if and only if’ are fundamental to mathematical language. They have an important interpretation in the language of sets, the first in terms of subsets of sets, the second in terms of equality of sets:

$$A \subset B \text{ means ‘If } x \in A, \text{ then } x \in B\text{’}.$$

$$A = B \text{ means ‘} x \in A \text{ if and only if } x \in B\text{’}.$$

Union and Intersection: The *union* $A \cup B$ of two sets A and B is the set of everything belonging to A or to B or to both. Their *intersection* $A \cap B$ is the set of everything belonging to both A and B . For example,

$$\begin{aligned} \text{if } A &= \{0, 1, 2, 3\} \text{ and } B = \{1, 3, 6\}, \\ \text{then } A \cup B &= \{0, 1, 2, 3, 6\} \text{ and } A \cap B = \{1, 3\}. \end{aligned}$$

Two sets A and B are called *disjoint* if they have no elements in common, that is, if $A \cap B = \emptyset$. For example, the sets $\{2, 4, 6, 8\}$ and $\{1, 3, 5, 7\}$ are disjoint.

‘Or’ means Union, ‘And’ means Intersection: The definitions of union and intersection can be written in set-builder notation using the words ‘and’ and ‘or’:

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} \\ A \cap B &= \{x : x \in A \text{ and } x \in B\}. \end{aligned}$$

This connection between the words ‘and’ and ‘or’ and set notation should be carefully considered. The word ‘or’ in mathematics always means ‘and/or’, and never means ‘either, but not both’.

The Universal Set and the Complement of a Set: A *universal set* is the set of everything under discussion in a particular situation. For example, if $A = \{1, 3, 5, 7, 9\}$, then possible universal sets are the set of all positive integers less than 11, or the set of all real numbers.

Once a universal set E is fixed, then the *complement* $\bar{A}$ of any set A is the set of all members of that universal set which are not in A . For example,

$$\begin{aligned} \text{if } A &= \{1, 3, 5, 7, 9\} \text{ and } E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \\ \text{then } \bar{A} &= \{2, 4, 6, 8, 10\}. \end{aligned}$$

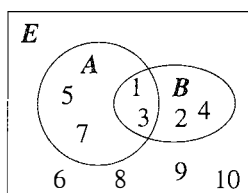
Notice that every member of the universal set is either in A or in $\bar{A}$, but never in both A and $\bar{A}$:

$$A \cup \bar{A} = E \quad \text{and} \quad A \cap \bar{A} = \emptyset.$$

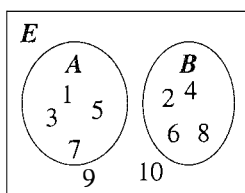
‘Not’ means Complement: There is an important connection between the word ‘not’ and the complement of a set. If the definition of the complementary set is written in set-builder notation,

$$\bar{A} = \{x \in E : x \text{ is not a member of } A\}.$$

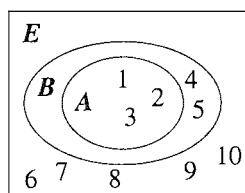
Venn Diagrams: A *Venn diagram* is a diagram used to represent the relationship between sets. For example, the four diagrams below represent the four different possible relationships between two sets A and B . In each case, the universal set is again $E = \{1, 2, 3, \dots, 10\}$.



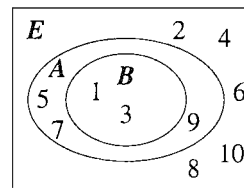
$$\begin{aligned} A &= \{1, 3, 5, 7\} \\ B &= \{1, 2, 3, 4\} \end{aligned}$$



$$\begin{aligned} A &= \{1, 3, 5, 7\} \\ B &= \{2, 4, 6, 8\} \end{aligned}$$



$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{1, 2, 3, 4, 5\} \end{aligned}$$



$$\begin{aligned} A &= \{1, 3, 5, 7, 9\} \\ B &= \{1, 3\} \end{aligned}$$

Compound sets such as $A \cup B$, $A \cap B$ and $\overline{A \cap B}$ can be visualised by shading regions of a Venn diagram, as is done in several questions in the following exercise.

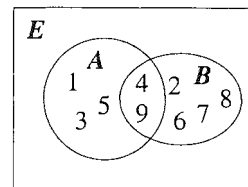
The Counting Rule for Sets: To calculate the size of the union $A \cup B$ of two sets, adding the sizes of A and of B will not do, because the members of the intersection $A \cap B$ would be counted twice. Hence $|A \cap B|$ needs to be subtracted again, and the rule is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, the Venn diagram on the right shows the two sets:

$$A = \{1, 3, 4, 5, 9\} \quad \text{and} \quad B = \{2, 4, 6, 7, 8, 9\}.$$

From the diagram, $|A \cup B| = 9$, $|A| = 5$, $|B| = 6$ and $|A \cap B| = 2$, and the formula works because $9 = 5 + 6 - 2$.



When two sets are disjoint, there is no overlap between A and B to cause any double counting. With $A \cap B = \emptyset$ and $|A \cap B| = 0$, the counting rule becomes

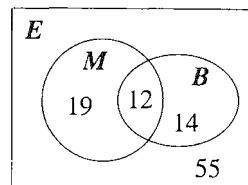
$$|A \cup B| = |A| + |B|.$$

Problem Solving Using Venn Diagrams: A Venn diagram is often the most convenient way to sort out problems involving overlapping sets of things. In the following exercise, the number of members of each region is written inside the region, rather than the members themselves.

WORKED EXERCISE: 100 Sydneysiders were surveyed to find out how many of them had visited the cities of Melbourne and Brisbane. 31 people had visited Melbourne, 26 people had visited Brisbane and 12 people had visited both cities. Find how many people had visited:

- Melbourne or Brisbane,
- Brisbane but not Melbourne,
- only one of the two cities,
- neither city.

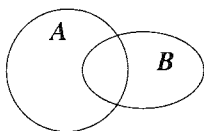
SOLUTION: Let M be the set of people who have visited Melbourne, let B be the set of people who have visited Brisbane, and let E be the universal set of all people surveyed. Calculations should begin with the 12 people in the intersection of the two regions. Then the numbers shown in the other three regions of the Venn diagram can easily be found, and so:



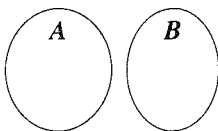
- $|\{\text{visited Melbourne or Brisbane}\}| = 19 + 14 + 12 = 45$
- $|\{\text{visited Melbourne only}\}| = 19$
- $|\{\text{visited only one city}\}| = 19 + 14 = 33$
- $|\{\text{visited neither city}\}| = 100 - 45 = 55$

Exercise 1J

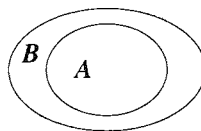
- State whether each set is finite or infinite. If it is finite, state its number of members:
 - $\{1, 3, 5, \dots\}$
 - $\{0, 1, 2, \dots, 9\}$
 - $\emptyset$
 - $\{\text{points on a line}\}$
 - $\{n : n \text{ is a positive integer and } 1 < n < 20\}$
 - $\{x : 3 \leq x \leq 5\}$
 - $\{a, l, g, e, b, r, a\}$
 - $\{\text{multiples of 7 that are less than 100}\}$
- Decide whether each of the following statements is true or false:
 - If two sets have the same number of members, then they are equal.
 - If two sets are equal, then they have the same number of members.
 - If $A = \{0, 0\}$, then $|A| = 1$.
 - $|\{0\}| = 0$
 - $1\,000\,000 \in \{1, 2, 3, \dots\}$
 - $|\{40, 41, 42, \dots, 60\}| = 20$
- State in each case whether or not $A \subset B$ (that is, whether A is a subset of B):
 - $A = \{2, 4, 5\}$, $B = \{n : n \text{ is an even positive integer and } n < 10\}$
 - $A = \{2, 3, 5\}$, $B = \{\text{prime numbers less than 10}\}$
 - $A = \{d, a, n, c, e\}$, $B = \{e, d, u, c, a, t, i, o, n\}$
 - $A = \emptyset$, $B = \{51, 52, 53, \dots, 99\}$
 - $A = \{3, 6, 9, \dots\}$, $B = \{6, 12, 18, \dots\}$
- Answer true or false:
 - If $A \subset B$ and $B \subset A$, then $A = B$.
 - If $A \subset B$ and $B \subset C$, then $A \subset C$.
- List all the subsets of each of these sets:
 - $\{a\}$
 - $\{a, b\}$
 - $\{a, b, c\}$
 - $\emptyset$
- Find $A \cup B$ and $A \cap B$ for each pair of sets:
 - $A = \{m\}$, $B = \{m, n\}$
 - $A = \{2, 4, 6\}$, $B = \{4, 6, 8\}$
 - $A = \{1, 3, 4, 6, 9\}$, $B = \{2, 4, 5, 7, 8, 9\}$
 - $A = \{c, o, m, p, u, t, e, r\}$, $B = \{s, o, f, t, w, a, r, e\}$
 - $A = \{\text{prime numbers less than 12}\}$, $B = \{\text{odd numbers less than 12}\}$
- If $A = \{\text{students who study Japanese}\}$ and $B = \{\text{students who study History}\}$, carefully describe each of the following sets:
 - $A \cap B$
 - $A \cup B$
- Copy and complete:
 - If $P \subset Q$, then $P \cup Q = \dots$
 - If $P \subset Q$, then $P \cap Q = \dots$
- Suppose $A = \{1, 3, 6, 8\}$ and $B = \{3, 4, 6, 7, 10\}$, with universal set $\{1, 2, 3, \dots, 10\}$. List the members of:
 - $\overline{A}$
 - $\overline{B}$
 - $\overline{A \cup B}$
 - $\overline{A \cap B}$
 - $\overline{A} \cap \overline{B}$
 - $\overline{A \cap B}$
- Select the Venn diagram that best shows the relationship between each pair of sets A and B :



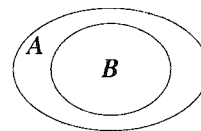
I



II

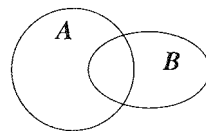


III



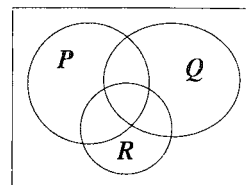
IV

- (a) $A = \{\text{positive integers}\}$, $B = \{\text{positive real numbers}\}$
 (b) $A = \{7, 1, 4, 8, 3, 5\}$, $B = \{2, 9, 0, 7\}$
 (c) $A = \{\text{multiples of } 3\}$, $B = \{\text{multiples of } 5\}$
 (d) $A = \{l, e, a, r, n\}$, $B = \{s, t, u, d, y\}$
 (e) $A = \{\text{politicians in Australia}\}$, $B = \{\text{politicians in NSW}\}$.
11. In each of the following, A and B represent sets of real numbers. For each part, graph on separate number lines: (i) A , (ii) B , (iii) $A \cup B$, (iv) $A \cap B$.
 (a) $A = \{x : x > 0\}$, $B = \{x : x \leq 3\}$ (b) $A = \{x : x \leq -1\}$, $B = \{x : x > 2\}$
 (c) $A = \{x : -3 \leq x < 1\}$, $B = \{x : -1 \leq x \leq 4\}$
12. (a) Explain the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$ by making reference to the Venn diagram opposite.
 (b) If $|A \cup B| = 17$, $|A| = 12$ and $|B| = 10$, find $|A \cap B|$.
 (c) Show that the relationship in part (a) is satisfied when $A = \{3, 5, 6, 8, 9\}$ and $B = \{2, 3, 5, 6, 7, 8\}$.
13. Use a Venn diagram to solve each of these problems:
 (a) In a group of 20 people, there are 8 who play the piano, 5 who play the violin and 3 who play both. How many people play neither?
 (b) Each person in a group of 30 plays either tennis or golf. 17 play tennis, while 9 play both. How many play golf?
 (c) In a class of 28 students, there are 19 who like geometry and 16 who like trigonometry. How many like both if there are 5 students who don't like either?



DEVELOPMENT

14. Shade each of the following regions on the given diagram (use a separate diagram for each part).
 (a) $P \cap Q \cap R$
 (b) $(P \cap R) \cup (Q \cap R)$
 (c) $\overline{P} \cup \overline{Q} \cup \overline{R}$ (where $\overline{P}$ denotes the complement of P)



15. A group of 80 people was surveyed about their approaches to keeping fit. It was found that 20 jog, 22 swim and 18 go to the gym on a regular basis. If 10 people both jog and swim, 11 people both jog and go to the gym, 6 people both swim and go to the gym and 43 people do none of these activities on a regular basis, how many people do all three?

EXTENSION

16. (a) Explain why a five-member set has twice as many subsets as a four-member set.
 (b) Hence find a formula for the number of subsets of an n -member set.
17. How many different possibilities for shading are there, given a Venn diagram with three overlapping sets within a universal set?
18. Express in words: ' $\{\emptyset\} \neq \emptyset$ because $\emptyset \in \{\emptyset\}$ '. Is the statement true or false?
19. Decide whether or not the following statement is true:
 $A \subset B$ if and only if, if $x \notin B$ then $x \notin A$.
20. Simplify $(A \cap (A \cap \overline{B})) \cup ((A \cap B) \cup (B \cap \overline{A}))$.
21. The definition $A = \{\text{sets that are not members of themselves}\}$ is impossible. Explain why, by considering whether or not A is a member of itself.

Numbers and Functions

The principal purpose of this course is the study of functions of real numbers, and the first task is therefore to make it clear what numbers are and what functions are. The first five sections of this chapter review the four number systems, with particular attention to the arithmetic of surds. The last five sections develop the idea of functions and relations and their graphs, with a review of known graphs, and a discussion of various ways in which the graph of a known function or relation can be transformed, allowing a wide variety of new graphs to be obtained.

STUDY NOTES: Although much of the detail here may be familiar, the systematic exposition of numbers and functions in this chapter will be new and demanding for most pupils. Understanding is vital, and the few proofs that do occur are worth emphasising. In the work on surds, the exact value of the number must constantly be distinguished from its decimal approximation produced on the calculator. In the work on functions, computer sketching can make routine the understanding that a function has a graph — an understanding fundamental for the whole course but surprisingly elusive — and computers are particularly helpful in understanding transformations of graphs and how they can be effected algebraically, because a large number of similar examples can be examined in a short time. Nevertheless, pupils must eventually be able to construct a graph from its equation on their own.

2A Cardinals, Integers and Rational Numbers

Our experience of numbers arises from the two quite distinct fields of counting and geometry, and we shall need to organise these contrasting insights into a unified view. This section concerns the cardinal numbers, the integers and the rational numbers, which are all based on counting.

The Cardinal Numbers: Counting things requires the numbers 0, 1, 2, 3, ... These numbers are called the *cardinal numbers*, and the symbol $\mathbf{N}$ is conventionally used for the set of all cardinal numbers.

1 **DEFINITION:** $\mathbf{N} = \{ \text{cardinal numbers} \} = \{ 0, 1, 2, 3, \dots \}$

This is an infinite set, because no matter how many cardinal numbers are listed, there will always be more. The number 0 is the smallest cardinal, but there is no largest cardinal, because given any cardinal n , the cardinal $n + 1$ is bigger.

Closure of $\mathbf{N}$: If two cardinals a and b are added, the sum $a + b$ and the product ab are still cardinals. We therefore say that the set $\mathbf{N}$ of cardinals is *closed* under

addition and multiplication. But the set of cardinals is not closed under either subtraction or division. For example,

$$5 - 7 \text{ is not a cardinal} \quad \text{and} \quad 6 \div 10 \text{ is not a cardinal.}$$

Divisibility — HCF and LCM: Division of cardinals sometimes does result in a cardinal. A cardinal a is called a *divisor* of the cardinal b if the quotient $b \div a$ is a cardinal. For example,

$$\begin{aligned} \{\text{divisors of } 24\} &= \{1, 2, 3, 4, 6, 8, 12, 24\} \\ \{\text{divisors of } 30\} &= \{1, 2, 3, 5, 6, 10, 15, 30\} \end{aligned}$$

The *highest common factor* or HCF of two or more cardinals is the largest cardinal that is a divisor of each of them, so the HCF of 24 and 30 is 6. The key to cancelling a fraction down to its *lowest terms* is dividing the numerator and denominator by their HCF:

$$\frac{24}{30} = \frac{24 \div 6}{30 \div 6} = \frac{4}{5}$$

If a is a divisor of b , then b is a *multiple* of a . For example,

$$\begin{aligned} \{\text{multiples of } 24\} &= \{24, 48, 72, 96, 120, 144, \dots\} \\ \{\text{multiples of } 30\} &= \{30, 60, 90, 120, 150, \dots\} \end{aligned}$$

The *lowest common multiple* or LCM of two or more cardinals is the smallest positive cardinal that is a multiple of each of them, so the LCM of 24 and 30 is 120. The key to adding and subtracting fractions is finding the LCM of their denominators, called the *lowest common denominator*:

$$\frac{5}{24} + \frac{7}{30} = \frac{5 \times 5}{120} + \frac{7 \times 4}{120} = \frac{53}{120}$$

Prime Numbers: A *prime number* is a cardinal number greater than 1 whose only divisors are itself and 1. The primes form a sequence whose distinctive pattern has confused every mathematician since Greek times:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \dots$$

A cardinal greater than 1 that has factors other than itself and 1 is called a *composite number*, and without giving its rather difficult proof, we shall assume the ‘unique factorisation theorem’:

2 THEOREM: Every positive cardinal number can be written as a product of prime numbers in one and only one way, apart from the order of the factors.

So, for example, $24 = 2^3 \times 3$, and $30 = 2 \times 3 \times 5$. This theorem means that as far as multiplication is concerned, the prime numbers are the building blocks for all cardinal numbers, no matter how big or complicated they might be. (No primes divide 1, and so the factorisation of 1 into primes requires the qualification that a product of no factors is 1.)

The Greeks were able to prove that there are infinitely many prime numbers, and the proof of this interesting result is given here because it is a clear example of ‘proof by contradiction’, where one assumes the theorem to be false and then works towards a contradiction.

3 **THEOREM:** There are infinitely many prime numbers.

PROOF: Suppose by way of contradiction that the theorem were false.

Then there would be a finite list $p_1, p_2, p_3, \dots, p_n$ of all the primes.

Form the product $N = p_1 p_2 p_3 \dots p_n$ of all of them.

Then $N + 1$ has remainder 1 after division by each of the primes $p_1, p_2, p_3, \dots, p_n$.

So $N + 1$ is not divisible by any of the primes $p_1, p_2, p_3, \dots, p_n$.

So the prime factorisation of $N + 1$ must involve primes other than $p_1, p_2, p_3, \dots, p_n$.

But $p_1, p_2, p_3, \dots, p_n$ is supposed to be a complete list of primes.

This is a contradiction, so the theorem is true.

In case you are tempted to think that all theorems about primes are so easily proven, here is the beginning of the list of *prime pairs*, which are pairs of prime numbers differing by 2:

3, 5; 5, 7; 11, 13; 17, 19; 29, 31; 41, 43; 59, 61; 71, 73;

No-one has yet been able to prove either that this list of prime pairs is finite, or that it is infinite. Computers cannot answer this question, because no computer search for prime pairs could possibly establish whether this list of prime pairs terminates or not.

The Integers: The desire to give a meaning to calculations like $5 - 7$ leads to *negative numbers* $-1, -2, -3, -4, \dots$, and the positive and negative numbers together with zero are called the *integers*, from ‘integral’ meaning ‘whole’. The symbol **Z** (from the German word *zahlen* meaning *numbers*) is conventionally used for the set of integers.

4 **DEFINITION:** $\mathbf{Z} = \{\text{integers}\} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

This set **Z** is another infinite set containing the set **N** of cardinal numbers. There is neither a greatest nor a least integer, because given any integer n , the integer $n + 1$ is greater than n , and the integer $n - 1$ is less than n .

Closure of Z: The set **Z** of integers is closed not only under addition and multiplication, but also under subtraction. For example,

$$7 + (-11) = -4 \quad (-8) \times 3 = -24 \quad (-16) - (-13) = -3$$

but the set is still not closed under division. For example, $12 \div 10$ is not an integer.

The Rational Numbers: The desire to give meaning to a calculation like ‘divide 7 into 3 equal parts’ leads naturally to fractions and the system of *rational numbers*. Positive rational numbers were highly developed by the Greeks, for whom ratio was central to their mathematical ideas.

5 **DEFINITION:** A *rational number* is a number that can be written as a *ratio* or fraction a/b , where a and b are integers and $b \neq 0$:

$$\mathbf{Q} = \{\text{rational numbers}\} \quad (\mathbf{Q} \text{ stands for quotient.})$$

For example, $2\frac{1}{2} = \frac{5}{2}$, $-\frac{1}{3} = \frac{-1}{3}$, $30 \div 24 = \frac{5}{4}$, $4 = \frac{4}{1}$ and $-7 = \frac{-7}{1}$.

Because every integer a is also a fraction $a/1$, the set $\mathbf{Q}$ of rational numbers contains the set $\mathbf{Z}$ of integers. So we now have three successively larger systems of numbers, $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$.

Lowest Terms: Multiplication or division of the numerator and denominator by the same nonzero number doesn't change the value of a fraction. So division of the numerator and denominator by their HCF always cancels a fraction down to lowest terms, in which the numerator and denominator have no common factor greater than 1. Multiplying the numerator and denominator by -1 reverses both their signs, so the denominator can always be made positive.

6

A STANDARD FORM: Every rational number can be written in the form a/b , where a and b are integers with highest common factor 1, and $b \geq 1$.

Closure of $\mathbf{Q}$: The rational numbers are closed under all four operations of addition, multiplication, subtraction and division (except by 0). The *opposite* of a rational number a/b is obtained by taking the opposite of either the numerator or denominator. The sum of a number and its opposite is zero:

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \quad \text{and} \quad \frac{a}{b} + \left(\frac{-a}{b}\right) = 0.$$

The *reciprocal* (inverse is not the correct word) of a nonzero rational number a/b is obtained by exchanging the numerator and denominator. The product of a number and its reciprocal is 1:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \quad \left(\text{or } \frac{1}{a/b} = \frac{b}{a}\right) \quad \text{and} \quad \frac{a}{b} \times \frac{b}{a} = 1.$$

The reciprocal x^{-1} of a rational number x is analogous to its *opposite* $-x$:

$$x \times x^{-1} = 1 \quad \text{and} \quad x + (-x) = 0.$$

Terminating and Recurring Decimals: A *terminating decimal* is an alternative notation for a rational number that can be written as a fraction with a power of 10 as the denominator:

$$1\frac{2}{5} = \frac{14}{10} = 1.4 \quad 3\frac{1}{8} = \frac{3125}{1000} = 3.125 \quad 578\frac{3}{50} = 578 + \frac{6}{100} = 578.06$$

Other than this rather narrow purpose, decimals are useful for approximating numbers so that they can be compared or placed roughly on a number line. For this reason, decimals are used when a quantity like distance or time is being physically measured — the act of measuring can never produce an exact answer.

A rational number that cannot be written with a power of 10 as its denominator can, however, be written as a *recurring decimal*, in which the digits to the right of a certain point cycle endlessly. In the division process, the cycling begins when a remainder occurs which has occurred before.

$$\begin{aligned} \frac{2}{3} &= 0.666\,666\,\dots = 0.\dot{6} & 6\frac{3}{7} &= 6.428\,571\,428\,571\,\dots = 6.\dot{4}2857\dot{1} \\ 13\frac{10}{11} &= 13.909\,09\,\dots = 13.\dot{9}\dot{0} & 24\frac{35}{44} &= 24.795\,454\,545\,45\,\dots = 24.79\dot{5}\dot{4} \end{aligned}$$

Conversely, every recurring decimal can be written as a fraction, by the following method.

WORKED EXERCISE: Write each recurring decimal as a fraction in lowest terms:

(a) $0.\dot{5}\dot{1}$

(b) $7.\dot{3}\dot{4}8\dot{6}$

SOLUTION:

(a) Let $x = 0.\dot{5}\dot{1}$.

Then $x = 0.515151 \dots$

$\boxed{\times 100} \quad 100x = 51.515151 \dots$

Subtracting the last two lines,

$$99x = 51$$

$$x = \frac{17}{33}$$

So $0.\dot{5}\dot{1} = \frac{17}{33}$.

(b) Let $x = 7.\dot{3}\dot{4}8\dot{6}$.

Then $x = 7.348648 \dots$

$\boxed{\times 1000} \quad 1000x = 7348.648648 \dots$

Subtracting the last two lines,

$$999x = 7341.3$$

$$x = \frac{73413}{9990}$$

So $7.\dot{3}\dot{4}8\dot{6} = \frac{2719}{370}$.

7 **METHOD:** If the cycle length is n , multiply by 10^n and subtract.

NOTE: Some examples in the exercises below show that every terminating decimal has an alternative representation as a recurring decimal with endlessly cycling 9s. For example,

$1 = 0.\dot{9}$

$7 = 6.\dot{9}$

$5.2 = 5.1\dot{9}$

$11.372 = 11.371\dot{9}$

Exercise 2A

- Find all primes: (a) less than 100, (b) between 150 and 200.
- Find the prime factorisations of:

(a) 24	(c) 72	(e) 104	(g) 189	(i) 315
(b) 60	(d) 126	(f) 135	(h) 294	(j) 605
- Find the HCF of the numerator and denominator of each fraction, then express the fraction in lowest terms:

(a) $\frac{72}{64}$	(c) $\frac{78}{104}$	(e) $\frac{168}{216}$
(b) $\frac{84}{90}$	(d) $\frac{112}{144}$	(f) $\frac{294}{315}$
- Find the LCM of the two denominators, and hence express as a single fraction:

(a) $\frac{1}{8} + \frac{1}{12}$	(c) $\frac{3}{8} - \frac{13}{36}$	(e) $\frac{55}{72} - \frac{75}{108}$
(b) $\frac{5}{18} - \frac{2}{15}$	(d) $\frac{37}{42} + \frac{23}{30}$	(f) $\frac{7}{60} + \frac{31}{78}$
- Express each number as a recurring or terminating decimal. Do not use a calculator.

(a) $\frac{5}{8}$	(c) $\frac{7}{16}$	(e) $\frac{3}{20}$	(g) $4\frac{16}{25}$	(i) $\frac{23}{8}$
(b) $\frac{2}{3}$	(d) $\frac{5}{9}$	(f) $\frac{7}{12}$	(h) $5\frac{4}{11}$	(j) $\frac{17}{6}$
- Express each decimal as a rational number in lowest terms:

(a) 0.15	(c) 0.108	(e) $3.\dot{1}2$	(g) $1.\dot{6}$	(i) $0.\dot{2}\dot{1}$
(b) $0.\dot{7}$	(d) $0.\dot{1}\dot{8}$	(f) $5.\dot{4}\dot{5}$	(h) $1.\dot{2}\dot{1}$	(j) $6.\dot{5}\dot{3}$

DEVELOPMENT

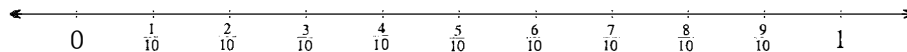
7. Express each of the following as a recurring or terminating decimal:
- | | | | | |
|---------------------|---------------------|---------------------|---------------------|---------------------|
| (a) $\frac{11}{6}$ | (c) $\frac{7}{15}$ | (e) $\frac{2}{27}$ | (g) $3\frac{1}{7}$ | (i) $\frac{27}{13}$ |
| (b) $\frac{13}{12}$ | (d) $\frac{16}{37}$ | (f) $\frac{13}{24}$ | (h) $1\frac{3}{14}$ | (j) $1\frac{5}{21}$ |
8. Express each decimal as a rational number in lowest terms:
- | | | | | |
|-------------------------|--------------------------|------------------|-------------------|--------------------|
| (a) $0.\dot{7}\dot{5}$ | (c) $4.\dot{5}6\dot{7}$ | (e) $1.\dot{9}$ | (g) $1.5\dot{2}$ | (i) $7.13\dot{8}$ |
| (b) $1.\dot{0}3\dot{7}$ | (d) $0.\dot{4}35\dot{6}$ | (f) $2.4\dot{9}$ | (h) $2.34\dot{5}$ | (j) $0.113\dot{6}$ |
9. Write down the recurring decimals for $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$. What is the pattern?
10. Find the prime factorisation of the following numbers and hence determine the square root of each by halving the indices:
- | | | | |
|---------|---------|----------|----------|
| (a) 256 | (b) 576 | (c) 1225 | (d) 1936 |
|---------|---------|----------|----------|
11. Write the HCF and LCM of each pair of numbers in prime factor form:
- | | | | |
|------------------|-------------------|-------------------|-------------------|
| (a) 792 and 1188 | (b) 1183 and 1456 | (c) 2646 and 3087 | (d) 3150 and 5600 |
|------------------|-------------------|-------------------|-------------------|
12. (a) In order to determine whether a given number is prime, it must be tested for divisibility by smaller primes. Given a number between 200 and 250, which primes need to be tested?
- (b) A student finds that none of the primes less than 22 is a factor of 457. What can be said about the number 457?
- (c) Which of 247, 329, 451, 503, 727 and 1001 are primes?

EXTENSION

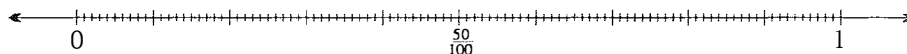
13. Prove that if x is the HCF of a and b , then x must be a factor of $a - b$.
14. The factors of a perfect number, other than itself, add up to that number.
- (a) Show that 28 is a perfect number. (b) Euclid knew that if $2^n - 1$ is a prime number, then $2^{n-1}(2^n - 1)$ is a perfect number. Test this proposition for $n = 2, 3, 4$ and 5.
15. (a) Evaluate $\frac{1}{3}$ as a decimal on your calculator.
- (b) Subtract $0.333\ 333\ 33$ from this, multiply the result by 10^8 and then take the reciprocal.
- (c) Show arithmetically that the final answer in part (b) is 3. Is the answer on your calculator also equal to 3? What does this tell you about the way fractions are stored on a calculator?
16. Two numbers m and n are called *relatively prime* if the HCF of m and n is 1. The Euler function $\phi(n)$ of n is the number of integers less than or equal to n that are relatively prime to n .
- (a) Confirm the following by listing the integers that are relatively prime to the given number:
- | | |
|----------------------|-----------------------|
| (i) $\phi(9) = 6$ | (iii) $\phi(32) = 16$ |
| (ii) $\phi(25) = 20$ | (iv) $\phi(45) = 24$ |
- (b) It is known that $\phi(p^k) = p^k - p^{k-1}$ for a prime p and a positive integer k . Show that this is true for $p = 2$ and $k = 1, 2, 3, 4$.
- (c) Prove that $\phi(3^n) = 2 \times 3^{n-1}$. Generalise this result to $\phi(p^n)$, where p is prime and n is a positive integer.

2B The Real Numbers

Whereas the integers are regularly spaced 1 unit apart, the rational numbers are packed infinitely closely together. For example, between 0 and 1 there are 9 rationals with denominator 10:



and there are 99 rationals with denominator 100 between 0 and 1:

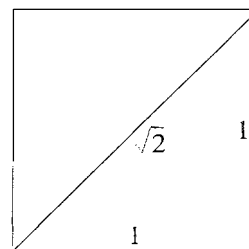


and so on, until the rationals are spread ‘as finely as we like’ along the whole number line.

8

THE RATIONAL NUMBERS ARE DENSE: This means that within any small interval on the number line, there are infinitely many rational numbers.

There are Numbers that are Not Rational: Although the rational numbers are dense, there are many more numbers on the number line. In fact ‘most numbers’ are not rational. In particular, some of the most important numbers you will meet in this course are *irrational*, like $\sqrt{2}$, π and the number e , which we will define later. The proof by contradiction that $\sqrt{2}$ is irrational was found by the Greeks, and the result is particularly important, since $\sqrt{2}$ arises so easily in geometry from the very simple and important process of constructing the diagonal of the unit square. It is a surprising result, and shows very clearly how fractions cannot form a number system sufficient for studying geometry.



9

THEOREM: The number $\sqrt{2}$ is irrational.

PROOF: Suppose, by way of contradiction, that $\sqrt{2}$ were rational. Then $\sqrt{2}$ could be written as

$$\sqrt{2} = \frac{a}{b}, \text{ where } a \text{ and } b \text{ are integers with no common factors, and } b \geq 1.$$

Multiplying both sides by b and then squaring both sides gives

$$a^2 = 2b^2.$$

Since $2b^2$ is even, then the left-hand side a^2 must also be even.

Hence a must be even, because if a were odd, then a^2 would be odd.

So $a = 2k$ for some integer k , and so $a^2 = 4k^2$ is divisible by 4.

So the right-hand side $2b^2$ is divisible by 4, and so b^2 is even.

Hence b must be even, because if b were odd, then b^2 would be odd.

But now both a and b are even, and so have a common factor 2.

This is a contradiction, so the theorem is true.

It now follows immediately that every multiple of $\sqrt{2}$ by a rational number must be irrational. The exercises ask for similar proofs by contradiction that numbers like $\sqrt{3}$, $\sqrt[3]{2}$ and $\log_2 3$ are irrational, and the following worked example shows that $\log_2 5$ is irrational. Unfortunately the proofs that π and e are irrational are considerably more difficult.

WORKED EXERCISE: Show that $\log_2 5$ is irrational (remember that $x = \log_2 5$ means that $2^x = 5$).

SOLUTION: Suppose, by way of contradiction, that $\log_2 5$ were rational.

Then $\log_2 5$ could be written $\log_2 5 = \frac{a}{b}$,

where a and b are integers with no common factors, and $b \geq 1$.

Writing this using powers, $2^{\frac{a}{b}} = 5$,

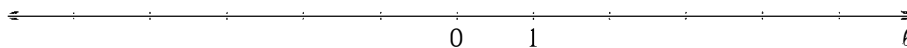
and taking the b th power of both sides, $2^a = 5^b$.

Now the LHS is even, being a positive integer power of 2,

and the RHS is odd, being a positive integer power of 5.

This is a contradiction, so the theorem is true.

The Real Numbers and the Number Line: The existence of irrational numbers means that the arithmetic of Section 2A, based on the cardinals and their successive extension to the integers and the rationals, is inadequate, and we require a more general idea of number. It is at this point that we turn away from counting and make an appeal to geometry to define a still larger system called the *real numbers* as the points on the number line. Take a line ℓ and turn it into a number line by choosing two points on it called 0 and 1:



The exercises review the standard methods of using ruler and compasses to construct a point on ℓ corresponding to any rational number, and the construction of further points on ℓ corresponding to the square roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, ... Even the number π can notionally be placed on the line by rolling a circle of diameter 1 unit along the line. It seems reasonable therefore to make the following definition.

10

DEFINITION: The *real numbers* are the points on the number line:

$$\mathbf{R} = \{ \text{real numbers} \}$$

The real numbers are often referred to as the *continuum*, because the rationals, despite being dense, are in a sense scattered along the number line like specks of dust, but do not ‘join up’. For example, the rational multiples of $\sqrt{2}$, which are all irrational, are just as dense on the number line as the rational numbers. It is only the real line itself which is completely joined up, to be the continuous line of geometry rather than falling apart into an infinitude of discrete points.

NOTE: There is, as one might expect, a great deal more to be done here. First, the operations of addition, subtraction, multiplication and division need to be defined, and shown to be consistent with these operations in the rationals. Secondly, one needs to explain why all other irrationals, like $\log_2 5$ and $\sqrt[3]{2}$, really do have their place amongst the real numbers. And thirdly and most fundamentally, our present notion of a line is far too naive and undeveloped as yet to carry the rigorous development of our definitions. These are very difficult questions, which were resolved only towards the end of the 19th century, and then incompletely. Interested readers may like to pursue these questions in a more advanced text.

The Real Numbers and Infinite Decimals: The convenience of approximating a real number by a terminating decimal, for example $\pi \doteq 3.14159$, leads to the intuitive idea of representing a real number by an infinite decimal.

11

REAL NUMBERS AND DECIMALS:

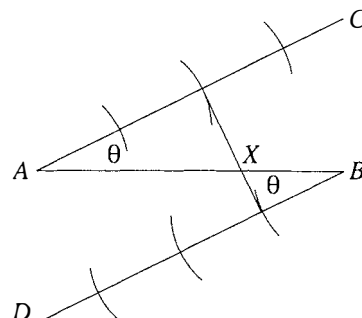
- (a) Every real number can be represented by one and only one infinite decimal, and every infinite decimal corresponds to one and only one real number (excluding decimals with recurring 9s).
- (b) A decimal represents a rational number if and only if it is either terminating or recurring.

Exercise 2B

- Copy the proof that $\log_2 5$ is irrational, given above, and modify it to prove that $\log_2 3$, $\log_2 7$ and $\log_3 5$ are irrational.
- State which numbers are rational, and express those that are rational as fractions $\frac{p}{q}$ in lowest terms:
 (a) $4\frac{1}{2}$, 5 , $-5\frac{3}{4}$, 0 , π , $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$ (b) $\sqrt[3]{27}$, $\sqrt[3]{14}$, $\sqrt{\frac{4}{9}}$, -3 , $16^{\frac{1}{2}}$, $7^{\frac{1}{2}}$
- Given that a , b , c and d are integers, with b and c nonzero, simplify the average of $\frac{a}{b}$ and $\frac{c}{d}$ and explain why it is rational.

DEVELOPMENT

- Use the proof that $\sqrt{2}$ is irrational as a guide to prove that $\sqrt{3}$, $\sqrt{5}$ and $\sqrt[3]{2}$ are irrational.
- Why does it follow from the previous question that $2 + \sqrt{3}$ is irrational? [HINT: Begin by writing, 'Suppose $x = 2 + \sqrt{3}$ were rational', then subtract 2 from both sides.]
- [This is a ruler and compasses construction to divide a given interval in the ratio 2 : 1. The method is easily generalised to any ratio.]
 - About half way down a fresh page construct a horizontal line segment AB of length about 10 cm.
 - At A construct a ray AC at an acute angle to AB (about 45° will do) and about 15 cm long.
 - At B construct a second ray BD , parallel with the first and on the opposite side of AB , by copying $\angle BAC$ to $\angle ABD$.
 - Set the compasses to a fixed radius of about 4 cm and mark off three equal lengths starting from A along AC . Do the same on the other ray starting at B .
 - Join the second mark on AC with the first mark on BD , which intersects with AB at X . The point X now divides AB in the ratio 2 : 1, or, to put it another way, X is $\frac{2}{3}$ of the way along AB . Confirm this by measurement.
- Use similar constructions to the one described in the previous question to find the point X on AB that represents the rational number: (a) $\frac{3}{5}$ (b) $\frac{5}{6}$



8. Lengths representing each of the surds may be constructed using right triangles.
- Use a scale of 4 cm = 1 unit to construct a right-angled triangle with base 3 units and altitude 1 unit. Pythagoras' theorem asserts that the hypotenuse has length $\sqrt{10}$ units. Measure the hypotenuse to one decimal place to confirm this.
 - Use this hypotenuse as the base of another right-angled triangle with altitude 1 unit. What will the length of the hypotenuse of this second triangle be? Measure it to one decimal place to confirm your answer.
9. Use compasses and ruler to construct lengths representing $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{6}$.

————— EXTENSION —————

10. (a) Prove that $\sqrt{6}$ is irrational. (b) Hence prove that $\sqrt{2} + \sqrt{3}$ is irrational.
[HINT: Begin by writing, 'Suppose $x = \sqrt{2} + \sqrt{3}$ were rational', then square both sides.]
11. [Continued fractions and approximations for π]
While π is irrational, we can find good approximations that are rational using the continued fraction expansion of π on the right. The first step is to calculate the first few terms of the continued fraction.
- $$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \dots}}}}$$
- Let $\pi = 3 + \frac{1}{a_1}$. Then use the calculator to obtain the value of $a_1 = 7 \dots$ by subtracting 3 from π and taking the reciprocal. Now let $a_1 = 7 + \frac{1}{a_2}$, and obtain the value of $a_2 = 15 \dots$ by a similar sequence of operations. Then continue the process twice more (the calculator's approximation to π may not be good enough to obtain 292).
 - Truncating the continued fraction at 7 yields the familiar result, $\pi \div 3 + \frac{1}{7} = \frac{22}{7}$. Show that this approximation is accurate to two decimal places.
 - Truncate the continued fraction one step further, simplify the resulting fraction, and find how many decimal places it is accurate to.
 - Truncate one step further again. Show that the resulting fraction is $\frac{355}{113} = 3\frac{16}{113}$, and that this approximation differs from π by less than 3×10^{-7} .
12. Use the calculator to find the continued fractions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.
13. The series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$ is known to converge to $\frac{\pi^2}{8}$. Use your calculator to add the first twelve terms, and hence approximate π to three significant figures.
14. Suppose a and b are positive irrational numbers, where $a < b$. Choose any positive integer n such that $\frac{1}{n} < b - a$, and let p be the greatest integer such that $\frac{p}{n} < a$.
- Prove that the rational number $\frac{p+1}{n}$ lies between a and b .
 - If $a = \frac{1}{\sqrt{1001}}$ and $b = \frac{1}{\sqrt{1000}}$, find the least possible value of n and the corresponding value of p .
 - Hence use part (a) to construct a rational number between $\frac{1}{\sqrt{1001}}$ and $\frac{1}{\sqrt{1000}}$.

2C Surds and their Arithmetic

Numbers like $\sqrt{2}$ and $\sqrt{3}$ occur constantly in our work, because they are required for the solution of quadratic equations. This section and the next two review the various methods of dealing with them.

Square Roots and Positive Square Roots: The square of any real number is positive, except that $0^2 = 0$. This means that negative numbers cannot have square roots, and that the only square root of 0 is 0 itself. Positive numbers, however, have two square roots which are the opposites of each other; for example, the square roots of 9 are 3 and -3 . Consequently the well-known symbol $\sqrt{}$ *does not mean square root, but is defined to mean the positive square root* (or zero, if $x = 0$).

12

DEFINITION: For $x > 0$, $\sqrt{x}$ means the positive square root of x .
 For $x = 0$, $\sqrt{0} = 0$.
 For $x < 0$, $\sqrt{x}$ is undefined.

For example, $\sqrt{25} = 5$, even though 25 has two square roots, -5 and 5 . The symbol for the negative square root of 25 is $-\sqrt{25}$.

On the other hand every number, positive or negative or zero, has exactly one cube root, and so the symbol $\sqrt[3]{}$ simply means cube root. For example, $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$.

What is a Surd: The word *surd* is often used to refer to any expression involving a square or higher root. It is better, however, to use a definition that excludes expressions like $\sqrt{\frac{4}{9}}$ and $\sqrt[3]{8}$, which can be simplified to rational numbers.

13

DEFINITION: An expression $\sqrt[n]{x}$, where x is a rational number and $n \geq 2$ is an integer, is called a *surd* if it is not itself a rational number.

It was proven in the last section that $\sqrt{2}$ was irrational, and in the same way, most roots of rational numbers are irrational. Here is the precise result for square roots, which won't be proven formally, and which is easily generalised to higher roots:

'If a and b are positive integers with no common factor, then $\sqrt{a/b}$ is rational if and only if both a and b are squares of integers.'

Simplifying Expressions Involving Surds: Here are some laws from earlier years for simplifying expressions involving square roots. The first pair restate the definition of square root, and the second pair are easily proven by squaring.

14

LAWS CONCERNING SURDS: Suppose that a and b are non-negative real numbers:

(a) $\sqrt{a^2} = a$

(c) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(b) $(\sqrt{a})^2 = a$

(d) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ (provided $b \neq 0$)

Taking Out Square Divisors: A surd like $\sqrt{50}$, in which 50 is divisible by the square 25, is not regarded as being simplified, because it can be expressed as

$$\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}.$$

15

METHOD: Always check the number inside the square root for divisibility by one of the squares 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...

If there is a square divisor, then it is quickest to divide by the largest possible square divisor.

WORKED EXERCISE: Simplify: (a) $\sqrt{72} - \sqrt{50} + \sqrt{12}$ (b) $(\sqrt{15} - \sqrt{6})^2$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \sqrt{72} - \sqrt{50} + \sqrt{12} &= \sqrt{36 \times 2} - \sqrt{25 \times 2} + \sqrt{4 \times 3} \\ &= 6\sqrt{2} - 5\sqrt{2} + 2\sqrt{3} \\ &= \sqrt{2} + 2\sqrt{3} \end{aligned} \qquad \begin{aligned} \text{(b)} \quad (\sqrt{15} - \sqrt{6})^2 &= 15 - 2\sqrt{90} + 6 \\ &= 21 - 2\sqrt{9 \times 10} \\ &= 21 - 6\sqrt{10} \end{aligned}$$

Exercise 2C

1. Complete the following table of values for $y = \sqrt{x}$ correct to two decimal places, and graph the points. Use a scale of 1 unit = 4 cm on both axes. Join the points with a smooth curve to obtain a graph of the function $y = \sqrt{x}$.

x	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.5	2
y										

Why were more points chosen near $x = 0$?

2. Simplify the following (assume all pronumerals are positive):

(a) $\sqrt{16}$	(e) $\sqrt{27}$	(i) $2\sqrt{121}$	(m) $\sqrt[3]{64}$
(b) $\sqrt{81}$	(f) $\sqrt{20}$	(j) $5\sqrt{x^2}$	(n) $\sqrt[3]{343}$
(c) $\sqrt{36}$	(g) $\sqrt{6x^2}$	(k) $2\sqrt{18}$	(o) $\sqrt[3]{8x^3}$
(d) $\sqrt{12}$	(h) $\sqrt{8y^2}$	(l) $\sqrt{y^3}$	(p) $\sqrt[3]{4y^3}$

3. Express the following in simplest form without the use of a calculator:

(a) $\sqrt{2} \times \sqrt{3}$	(f) $4\sqrt{7} \times 3\sqrt{7}$	(k) $3\sqrt{12} \times 2\sqrt{18}$
(b) $\sqrt{6} \times \sqrt{2}$	(g) $3\sqrt{5} \times \sqrt{15}$	(l) $7\sqrt{24} \times 5\sqrt{18}$
(c) $\sqrt{3} \times \sqrt{15}$	(h) $(2\sqrt{3})^2$	(m) $\sqrt{\pi^2} \times \sqrt{72}$
(d) $(\sqrt{5})^2$	(i) $3\sqrt{6} \times \sqrt{10}$	(n) $\sqrt{2a^4} \times \sqrt{2\pi^3}$
(e) $2\sqrt{3} \times 3\sqrt{5}$	(j) $\sqrt{56} \times 5\sqrt{6}$	(o) $6\sqrt{44} \times 7\sqrt{48x^4}$

4. Rewrite each value as a single surd, that is, in the form $\sqrt{n}$:

(a) $2\sqrt{5}$	(d) $5\sqrt{6}$	(g) $9\sqrt{7}$	(j) $6\pi\sqrt{6}$
(b) $3\sqrt{3}$	(e) $4\sqrt{3}$	(h) $2\sqrt{17}$	(k) $3y\sqrt{13y}$
(c) $6\sqrt{2}$	(f) $2\sqrt{8}$	(i) $5x\sqrt{11}$	(l) $12a^2\sqrt{6}$

5. Simplify the following:

(a) $\sqrt{\frac{9}{4}}$

(c) $\sqrt{\frac{7}{25}}$

(e) $\sqrt{5\frac{4}{9}}$

(g) $\sqrt[3]{\frac{27}{8}}$

(b) $\sqrt{\frac{9}{16}}$

(d) $\sqrt{6\frac{1}{4}}$

(f) $\sqrt[3]{\frac{7}{64}}$

(h) $\sqrt[3]{2\frac{10}{27}}$

6. Use the result $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ to simplify these fractions:

(a) $\frac{\sqrt{6}}{\sqrt{3}}$

(d) $\frac{\sqrt{156}}{\sqrt{13}}$

(g) $\frac{\sqrt{28}}{\sqrt{63}}$

(j) $\frac{\sqrt{21}}{\sqrt{75}}$

(b) $\frac{\sqrt{42}}{\sqrt{7}}$

(e) $\frac{\sqrt{10}}{\sqrt{40}}$

(h) $\frac{\sqrt{72}}{\sqrt{98}}$

(k) $\frac{\sqrt{66}}{\sqrt{54}}$

(c) $\frac{\sqrt{60}}{\sqrt{12}}$

(f) $\frac{\sqrt{8}}{\sqrt{50}}$

(i) $\frac{\sqrt{96}}{\sqrt{12}}$

(l) $\frac{\sqrt{91}}{\sqrt{52}}$

7. Simplify each surd, then use the approximations $\sqrt{2} = 1.41$, $\sqrt{3} = 1.73$ and $\sqrt{5} = 2.24$ to evaluate the following to two decimal places:

(a) $\sqrt{8}$

(c) $\sqrt{20}$

(e) $\sqrt{27}$

(g) $\sqrt{50}$

(b) $\sqrt{12}$

(d) $\sqrt{18}$

(f) $\sqrt{45}$

(h) $\sqrt{75}$

DEVELOPMENT

8. Find a pair of values a and b for which $\sqrt{a^2 + b^2} \neq a + b$. Are there any values that make the LHS and RHS equal?

9. Simplify each expression, then collect like terms:

(a) $\sqrt{50} - \sqrt{18}$

(b) $3\sqrt{75} + 5\sqrt{3}$

(c) $\sqrt{7} + \sqrt{28}$

(d) $\sqrt{54} - \sqrt{24}$

10. Simplify each surdic expression completely:

(a) $\sqrt{12} + \sqrt{49} - \sqrt{64}$

(d) $\sqrt{90} - \sqrt{40} + \sqrt{10}$

(g) $\sqrt{6} + \sqrt{24} + \sqrt{72}$

(b) $\sqrt{96} - \sqrt{24} - \sqrt{54}$

(e) $\sqrt{45} + \sqrt{80} - \sqrt{125}$

(h) $\sqrt{27} - \sqrt{117} + \sqrt{52}$

(c) $\sqrt{18} + \sqrt{8} - \sqrt{50}$

(f) $\sqrt{27} - \sqrt{50} + \sqrt{3}$

(i) $\sqrt{63} + 2\sqrt{18} - 5\sqrt{7}$

11. Find the value of each pronumeral by first simplifying the surdic terms:

(a) $\sqrt{75} + \sqrt{27} = \sqrt{a}$

(c) $\sqrt{240} - \sqrt{135} = \sqrt{y}$

(b) $\sqrt{44} + \sqrt{99} = \sqrt{x}$

(d) $\sqrt{150} + \sqrt{54} - \sqrt{216} = \sqrt{m}$

12. Expand the following, expressing your answers in simplest form:

(a) $\sqrt{3}(\sqrt{2} + \sqrt{3})$

(e) $\sqrt{a}(\sqrt{a} + \sqrt{b})$

(b) $\sqrt{5}(\sqrt{5} + \sqrt{15})$

(f) $4\sqrt{a}(1 - \sqrt{a})$

(c) $3\sqrt{2}(\sqrt{6} - \sqrt{8})$

(g) $\sqrt{x}(\sqrt{x+2} + \sqrt{x})$

(d) $\sqrt{7}(3\sqrt{3} - \sqrt{14})$

(h) $\sqrt{x-1}(\sqrt{x-1} + \sqrt{x+1})$

13. Expand and simplify:

(a) $(\sqrt{5} + \sqrt{2})(\sqrt{3} - \sqrt{2})$

(e) $(2\sqrt{6} + 1)(2\sqrt{6} + 2)$

(b) $(\sqrt{2} + \sqrt{3})(\sqrt{5} + 1)$

(f) $(3\sqrt{7} - 2)(\sqrt{7} + 1)$

(c) $(\sqrt{3} - 1)(\sqrt{2} - 1)$

(g) $(2\sqrt{5} + \sqrt{3})(2 - \sqrt{3})$

(d) $(\sqrt{6} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

(h) $(3\sqrt{2} - \sqrt{5})(\sqrt{6} - \sqrt{5})$

14. Expand and simplify:

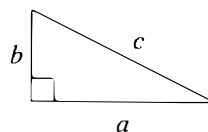
(a) $(\sqrt{2} + 1)^2$	(h) $(\sqrt{6} - 2\sqrt{2})^2$
(b) $(\sqrt{3} + 1)(\sqrt{3} - 1)$	(i) $(2\sqrt{a} - 1)^2$
(c) $(1 - \sqrt{5})(1 + \sqrt{5})$	(j) $(2 + \sqrt{a+2})^2$
(d) $(1 - \sqrt{3})^2$	(k) $(\sqrt{x-1} - 1)(\sqrt{x-1} + 1)$
(e) $(\sqrt{3} + \sqrt{2})^2$	(l) $(\sqrt{x+1} + \sqrt{x-2})^2$
(f) $(\sqrt{5} + \sqrt{7})(\sqrt{5} - \sqrt{7})$	(m) $(\frac{1}{2} + \frac{1}{2}\sqrt{5})^2$
(g) $(2\sqrt{6} - 5)(2\sqrt{6} + 5)$	(n) $(\frac{1}{2} - \frac{1}{2}\sqrt{5})^2$

15. Fully simplify these fractions:

(a) $\frac{\sqrt{3} \times 2\sqrt{5}}{\sqrt{15}}$	(c) $\frac{5\sqrt{7} \times \sqrt{3}}{\sqrt{28}}$	(e) $\frac{\sqrt{15} \times \sqrt{20}}{\sqrt{12}}$	(g) $\frac{\sqrt{8} \times 3\sqrt{7}}{\sqrt{21} \times \sqrt{12}}$
(b) $\frac{2\sqrt{6} \times \sqrt{5}}{\sqrt{10}}$	(d) $\frac{\sqrt{10} \times 3\sqrt{5}}{5\sqrt{2}}$	(f) $\frac{6\sqrt{3} \times 5\sqrt{2}}{\sqrt{12} \times \sqrt{18}}$	(h) $\frac{5\sqrt{44} \times \sqrt{14}}{\sqrt{24} \times 3\sqrt{33}}$

16. In the right triangle opposite, find the third side if:

(a) $a = \sqrt{2}$ and $b = \sqrt{7}$	(c) $a = \sqrt{7} + 1$ and $b = \sqrt{7} - 1$
(b) $b = \sqrt{5}$ and $c = 2\sqrt{5}$	(d) $a = 2\sqrt{3} + 3\sqrt{2}$ and $b = 2\sqrt{3} - 3\sqrt{2}$



17. The roots of the quadratic equation $ax^2 + bx + c = 0$ are known to be $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$. Find: (a) the sum of the roots, (b) the product of the roots.

18. Given that x and y are positive, simplify:

(a) $\sqrt{x^2 y^3}$	(c) $\sqrt{x^2 + 6x + 9}$	(e) $\sqrt{x^2 y^4 (x^2 + 2x + 1)}$
(b) $x \sqrt{x^2 y^6}$	(d) $\sqrt{x^3 + 2x^2 + x}$	(f) $\sqrt{x^4 + 2x^3 + x^2}$

EXTENSION

19. (a) Show that if $a = 1 + \sqrt{2}$, then $a^2 - 2a - 1 = 0$. (b) Hence show that $a = 2 + \frac{1}{a}$ and $\sqrt{2} = 1 + \frac{1}{a}$. (c) Show how these results can be used to construct the continued fraction for $\sqrt{2}$ found in question 12 of Exercise 2B.

2D Rationalising the Denominator

When dealing with surdic expressions, it is usual to remove any surds from the denominator, a process called *rationalising the denominator*. There are two quite distinct cases.

The Denominator has a Single Term: In the first case, the denominator is a surd or a multiple of a surd.

16 METHOD: In an expression like $\frac{5\sqrt{7}}{2\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.

WORKED EXERCISE:

$$\begin{aligned} \text{(a)} \quad \frac{5\sqrt{7}}{2\sqrt{3}} &= \frac{5\sqrt{7}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{5\sqrt{21}}{6} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{55}{\sqrt{11}} &= \frac{55}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} \\ &= 5\sqrt{11} \end{aligned}$$

The Denominator has Two Terms: The second case involves a denominator with two terms, one or both of which contain a surd.

17 METHOD: In an expression like $\frac{\sqrt{3}}{5 + 2\sqrt{3}}$, multiply top and bottom by $5 - 2\sqrt{3}$.

WORKED EXERCISE:

$$\begin{aligned} \text{(a)} \quad \frac{\sqrt{3}}{5 + 2\sqrt{3}} &= \frac{\sqrt{3}}{5 + 2\sqrt{3}} \times \frac{5 - 2\sqrt{3}}{5 - 2\sqrt{3}} \\ &= \frac{5\sqrt{3} - 6}{25 - 4 \times 3} \\ &= \frac{5\sqrt{3} - 6}{13} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{1}{2\sqrt{3} - 3\sqrt{2}} &= \frac{1}{2\sqrt{3} - 3\sqrt{2}} \times \frac{2\sqrt{3} + 3\sqrt{2}}{2\sqrt{3} + 3\sqrt{2}} \\ &= \frac{2\sqrt{3} + 3\sqrt{2}}{4 \times 3 - 9 \times 2} \\ &= -\frac{2\sqrt{3} + 3\sqrt{2}}{6} \end{aligned}$$

The method works because the identity $(x - y)(x + y) = x^2 - y^2$ (the difference of squares) guarantees that all surds will disappear from the denominator. The examples above involved

$$(5 + 2\sqrt{3}) \times (5 - 2\sqrt{3}) = 25 - 12 \quad \text{and} \quad (2\sqrt{3} - 3\sqrt{2}) \times (2\sqrt{3} + 3\sqrt{2}) = 12 - 18.$$

Comparing Expressions Involving Surds: When comparing two compound expressions, find whether the difference between them is positive or negative.

WORKED EXERCISE: Compare: (a) $4\sqrt{6}$ and $\frac{14}{\sqrt{2}}$ (b) $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad 4\sqrt{6} &= \sqrt{16 \times 6} \\ &= \sqrt{96} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \frac{14}{\sqrt{2}} &= 7\sqrt{2} \\ &= \sqrt{98}. \end{aligned}$$

$$\text{Hence } 4\sqrt{6} < \frac{14}{\sqrt{2}}.$$

$$\begin{aligned} \text{(b)} \quad (5 - 3\sqrt{2}) - (6\sqrt{2} - 8) &= 13 - 9\sqrt{2} \\ &= \sqrt{169} - \sqrt{162} \\ &> 0. \end{aligned}$$

$$\text{Hence } 5 - 3\sqrt{2} > 6\sqrt{2} - 8.$$

Exercise 2D

1. Express the following with rational denominators, in lowest terms:

$$\text{(a)} \quad \frac{1}{\sqrt{3}}$$

$$\text{(c)} \quad \frac{5}{\sqrt{11}}$$

$$\text{(e)} \quad \frac{2}{\sqrt{8}}$$

$$\text{(g)} \quad \frac{5}{\sqrt{15}}$$

$$\text{(b)} \quad \frac{2}{\sqrt{7}}$$

$$\text{(d)} \quad \frac{5}{\sqrt{5}}$$

$$\text{(f)} \quad \frac{3}{\sqrt{6}}$$

$$\text{(h)} \quad \frac{14}{\sqrt{10}}$$

2. Find the sum and the product of each pair:

- (a) $2 + \sqrt{3}$ and $2 - \sqrt{3}$ (c) $1 - \sqrt{2}$ and $1 + \sqrt{2}$ (e) $1 + 2\sqrt{3}$ and $1 - 2\sqrt{3}$
 (b) $3 - \sqrt{5}$ and $3 + \sqrt{5}$ (d) $5 + 2\sqrt{6}$ and $5 - 2\sqrt{6}$ (f) $\sqrt{7} - 1$ and $-\sqrt{7} - 1$

3. Rewrite each fraction with an integer denominator:

- (a) $\frac{1}{\sqrt{2} - 1}$ (c) $\frac{3}{\sqrt{5} - 1}$ (e) $\frac{1}{\sqrt{3} + \sqrt{2}}$ (g) $\frac{3}{4 - \sqrt{7}}$
 (b) $\frac{1}{1 - \sqrt{2}}$ (d) $\frac{3}{1 - \sqrt{5}}$ (f) $\frac{1}{\sqrt{5} + \sqrt{3}}$ (h) $\frac{3}{\sqrt{11} - \sqrt{6}}$

DEVELOPMENT

4. Express each of the following as a fraction with rational denominator:

- (a) $\frac{2}{3\sqrt{2}}$ (c) $\frac{2\sqrt{3}}{\sqrt{2}}$ (e) $\sqrt{\frac{3}{7}}$ (g) $\sqrt{4\frac{1}{4}}$
 (b) $\frac{5}{2\sqrt{5}}$ (d) $\sqrt{\frac{4}{3}}$ (f) $\frac{\sqrt{2}}{5\sqrt{7}}$ (h) $\sqrt{3\frac{3}{11}}$

5. Simplify the following by rationalising the denominator:

- (a) $\frac{3}{2\sqrt{2} - \sqrt{5}}$ (g) $\frac{\sqrt{3} - 1}{2 - \sqrt{3}}$ (m) $\frac{2}{\sqrt{x+1} + \sqrt{x-1}}$
 (b) $\frac{2}{3\sqrt{2} - 4}$ (h) $\frac{\sqrt{3} - \sqrt{7}}{3\sqrt{6} - \sqrt{7}}$ (n) $\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$
 (c) $\frac{4}{\sqrt{5} - \sqrt{3}}$ (i) $\frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}}$ (o) $\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$
 (d) $\frac{3\sqrt{3}}{\sqrt{5} + \sqrt{3}}$ (j) $\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$ (p) $\frac{4}{\sqrt{2}(\sqrt{5} - 1)}$
 (e) $\frac{2\sqrt{7}}{5 + 2\sqrt{7}}$ (k) $\frac{3}{\sqrt{x} + 2}$ (q) $\frac{6}{\sqrt{3}(\sqrt{7} + \sqrt{5})}$
 (f) $\frac{1 - \sqrt{2}}{1 + \sqrt{2}}$ (l) $\frac{1}{q - \sqrt{p}}$ (r) $\frac{2x}{\sqrt{x}(\sqrt{x+2} + \sqrt{x})}$

6. Show that each expression is rational: (a) $\frac{3}{2 + \sqrt{2}} + \frac{3}{\sqrt{2}}$ (b) $\frac{1}{3 + \sqrt{6}} + \frac{2}{\sqrt{6}}$

7. Simplify: (a) $\frac{1}{1 + \sqrt{3}} - \frac{1}{1 - \sqrt{3}}$ (b) $\frac{1}{2(\sqrt{5} + 1)} + \frac{1}{2(\sqrt{5} - 1)}$ (c) $\frac{1}{3\sqrt{2} + 1} + \frac{1}{1 - 3\sqrt{2}}$

8. Rationalise the denominator of $\frac{1}{\sqrt{x+h} + \sqrt{x}}$.

9. Evaluate $a + \frac{1}{a}$ for these values of a :

- (a) $1 + \sqrt{2}$ (b) $2 - \sqrt{3}$ (c) $\frac{3 - \sqrt{3}}{3 + \sqrt{3}}$ (d) $\frac{\sqrt{x} + \sqrt{2-x}}{\sqrt{x} - \sqrt{2-x}}$

10. (a) Show that $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$. (b) Given that $y = 2 + \sqrt{5}$, simplify $y + \frac{1}{y}$.

(c) Use the result in part (a) to evaluate $y^2 + \frac{1}{y^2}$ without determining y^2 .

(d) Similarly find $y^2 + \frac{1}{y^2}$ for: (i) $y = 1 + \sqrt{2}$ (ii) $y = 2 - \sqrt{3}$

11. Determine, without using a calculator, which is the greater number in each pair:

- (a) $2\sqrt{3}$ or $\sqrt{11}$ (b) $7\sqrt{2}$ or $3\sqrt{11}$ (c) $3 + 2\sqrt{2}$ or $15 - 7\sqrt{2}$ (d) $2\sqrt{6} - 3$ or $7 - 2\sqrt{6}$

EXTENSION

12. Express with integer denominator: (a) $\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ (b) $\frac{1}{\sqrt[3]{2}}$ (c) $\frac{1}{\sqrt[3]{2} - 1}$

13. (a) If $\sqrt{5} = 2 + \frac{1}{a}$, show that $a = 4 + \frac{1}{a}$.

(b) Hence deduce the continued fraction for $\sqrt{5}$ as found in question 12 of Exercise 2B.

14. The value of $\sqrt{17}$ is approximately 4.12 to two decimal places.

(a) Substitute this value to determine an approximation for $\frac{1}{\sqrt{17} - 4}$.

(b) Show that $\frac{1}{\sqrt{17} - 4} = \sqrt{17} + 4$, and that this last result gives a more accurate value for the approximation than that found in part (a).

15. It is given that $\frac{a}{b + \sqrt{c}} + \frac{d}{\sqrt{c}}$ is rational, where a, b, c and d are positive integers and c is not a square. Show that as a consequence $db^2 = c(a + d)$. Use this result to show that $\frac{a}{1 + \sqrt{c}} + \frac{d}{\sqrt{c}}$ is not rational.

16. (a) Let $x = \sqrt{2}$, show that $x = \frac{x^2 + 2}{2x}$.

(b) Let $\sqrt{2}$ be approximated by the slightly larger fraction $\frac{p}{q}$, that is $\frac{p}{q} = \sqrt{2} + \varepsilon$, where

ε is small and positive. Show that $\frac{p^2 + 2q^2}{2pq} = \frac{\sqrt{2} + \varepsilon}{2} + \frac{1}{\sqrt{2} + \varepsilon}$ and hence show that $\frac{p^2 + 2q^2}{2pq} > \sqrt{2}$.

(c) Show that $\frac{p^2 + 2q^2}{2pq} - \sqrt{2}$ is smaller than ε , that is, $\frac{p^2 + 2q^2}{2pq}$ is a better approximation for $\sqrt{2}$. NOTE: These results come from Newton's method for solving $x^2 = 2$ by approximation.

2E Equality of Surdic Expressions

There is only one way to write an expression like $3 + \sqrt{7}$ as the sum of a rational number and a surd. Although this may seem obvious, the result is surprising in that it generates two equations in rational numbers from just one surdic equation. Here are the precise statements:

THEOREM:

- 18 (a) Suppose that $a + b\sqrt{x} = A + B\sqrt{x}$, where a, b, A, B and x are rational, and $x \geq 0$ is not the square of a rational. Then $a = A$ and $b = B$.
- (b) Suppose that $a + \sqrt{b} = A + \sqrt{B}$, where a, b, A and B are rational with $b \geq 0$ and $B \geq 0$, and b is not the square of a rational. Then $a = A$ and $b = B$.

PROOF:

(a) Rearranging, $(b - B)\sqrt{x} = A - a$.

Now if $b \neq B$, then $\sqrt{x} = \frac{A - a}{b - B}$, which is rational.

This contradicts $\sqrt{x}$ being irrational, so $b = B$, and hence $a = A$.

(b) Given $a + \sqrt{b} = A + \sqrt{B}$,
then $\sqrt{b} - \sqrt{B} = A - a$, which is a rational number. (1)

Multiplying both sides by $\sqrt{b} + \sqrt{B}$,

$$b - B = (A - a)(\sqrt{b} + \sqrt{B}).$$

Now if $a \neq A$, then we could divide both sides by $A - a$, and

$$\sqrt{b} + \sqrt{B} = \frac{b - B}{A - a}, \text{ which is also rational,} \quad (2)$$

but then adding (1) and (2), $2\sqrt{b}$ would be a rational number, contradicting the fact that b is not the square of a rational.

Hence $a = A$, and so also $b = B$.

WORKED EXERCISE: Find rational values of x and y if:

(a) $x + y\sqrt{7} = (3 - 2\sqrt{7})^2$

(b) $(x + y\sqrt{5})^2 = 14 - 6\sqrt{5}$

SOLUTION:

(a) $\text{RHS} = 9 - 12\sqrt{7} + 28$
 $= 37 - 12\sqrt{7}$

Using part (a) of
the theorem above,

$$x = 37 \text{ and } y = -12.$$

(b) $\text{LHS} = x^2 + 5y^2 + 2xy\sqrt{5}$,
so $x^2 + 5y^2 = 14$
and $xy = -3$.

By inspection, $x = 3$ and $y = -1$,
or $x = -3$ and $y = 1$.

Closure of Sets of Surdic Expressions: The last question in Exercise 2E proves that

$$F = \{a + b\sqrt{2} : a \text{ and } b \text{ are rational}\}$$

is closed under the four operations of addition, multiplication, subtraction and division. In this way the set F forms a self-contained system of numbers that is larger than the set of rationals. More generally, replacing 2 by any non-square positive integer produces a similar system of numbers.

Exercise 2E

1. Find the values of the pronumerals a and b , given that they are rational:

(a) $a + b\sqrt{5} = 7 - 2\sqrt{5}$

(c) $-a + b\sqrt{3} = 7 - 4\sqrt{3}$

(e) $a + b\sqrt{x} = \frac{5}{7} + \frac{1}{2}\sqrt{x}$

(b) $a - b\sqrt{7} = 2 - 3\sqrt{7}$

(d) $a - b\sqrt{x} = 3 + 2\sqrt{x}$

(f) $a + b\sqrt{x} = \frac{2}{3} + 3\sqrt{x}$

2. Determine the rational numbers a and b :

(a) $2 + \sqrt{b} = a + 3\sqrt{2}$

(e) $\frac{4}{3} - b\sqrt{3} = a - \sqrt{\frac{3}{4}}$

(b) $a + \sqrt{12} = -1 + b\sqrt{3}$

(c) $a + b\sqrt{7} = 3 - \sqrt{28}$

(f) $a + b\sqrt{x} = -\frac{1}{5} - \sqrt{\frac{9x}{16}}$

(d) $-a + 2\sqrt{5} = -5 + \sqrt{b}$

3. Simplify the right-hand expressions in order to determine the rational numbers x and y :

$$(a) \ x + \sqrt{y} = \sqrt{7}(\sqrt{7} + 2)$$

$$(d) \ x + y\sqrt{5} = (\sqrt{5} + 2)(\sqrt{5} + 3)$$

$$(b) \ x + \sqrt{y} = (1 + \sqrt{3})^2$$

$$(e) \ x - y\sqrt{2} = (\sqrt{6} - \sqrt{3})^2$$

$$(c) \ x + y\sqrt{3} = (6 + \sqrt{3})^2$$

$$(f) \ x - \sqrt{y} = (3 - \sqrt{5})^2$$

DEVELOPMENT

4. Find the values of the integers x , y and z , given that z has no squares as factors:

$$(a) \ x + y\sqrt{z} = (\sqrt{6} - 2\sqrt{3})(\sqrt{6} + \sqrt{3})$$

$$(c) \ x + y\sqrt{z} = (\sqrt{5} - 2\sqrt{2})(\sqrt{5} + 3\sqrt{2})$$

$$(b) \ x + y\sqrt{z} = (\sqrt{15} + \sqrt{5})^2$$

$$(d) \ x + y\sqrt{z} = (\sqrt{15} - \sqrt{5})^2$$

5. Find rational numbers a and b such that:

$$(a) \ a + b\sqrt{3} = \frac{1}{2 - \sqrt{3}}$$

$$(c) \ a + b\sqrt{5} = \frac{2}{\sqrt{5} - 1}$$

$$(e) \ a + b\sqrt{2} = \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$$

$$(b) \ a + b\sqrt{5} = \frac{1}{2 + \sqrt{5}}$$

$$(d) \ a + b\sqrt{3} = \frac{\sqrt{3}}{3 + \sqrt{3}}$$

$$(f) \ a + b\sqrt{6} = \frac{2\sqrt{6} + 1}{2\sqrt{6} - 3}$$

6. Form a pair of simultaneous equations and solve them to find x and y , given that they are rational:

$$(a) \ x - 3 + \sqrt{y + 2} = -1 + \sqrt{5}$$

$$(e) \ x - 2y + \sqrt{x + y} = \frac{1}{2}\sqrt{6}$$

$$(b) \ x + 1 + \sqrt{7} = \frac{7}{2} + \sqrt{y - 1}$$

$$(f) \ 1\frac{1}{2} + \sqrt{x + 2y} = 3x + y + \frac{2}{3}\sqrt{3}$$

$$(c) \ x - y + \sqrt{x + y} = 3 + \sqrt{6}$$

$$(g) \ xy + \sqrt{3} = 10 + \sqrt{x - y}$$

$$(d) \ 6 + \sqrt{x - y} = x + y + 3\sqrt{2}$$

$$(h) \ xy + \sqrt{x + y} = \frac{5}{4} + \sqrt{3}$$

7. Find the rational values of a and b , with $a > 0$, by forming two simultaneous equations and solving them by inspection (part (d) may need substitution):

$$(a) \ (a + b\sqrt{2})^2 = 3 + 2\sqrt{2}$$

$$(c) \ (a + b\sqrt{3})^2 = 13 - 4\sqrt{3}$$

$$(b) \ (a + b\sqrt{5})^2 = 9 - 4\sqrt{5}$$

$$(d) \ (a + b\sqrt{7})^2 = 9\frac{1}{4} + 3\sqrt{7}$$

EXTENSION

8. (a) Let $\sqrt{15 - 6\sqrt{6}} = x - \sqrt{y}$. Square both sides and form a pair of simultaneous equations to find x and y , given that they are rational. Hence find $\sqrt{15 - 6\sqrt{6}}$.

(b) Similarly simplify: (i) $\sqrt{28 - 10\sqrt{3}}$ (ii) $\sqrt{66 + 14\sqrt{17}}$ (iii) $\sqrt{\frac{7}{12} - \frac{\sqrt{3}}{3}}$

9. Define the set $F = \{x + y\sqrt{2} : x, y \in \mathbf{Q}\}$. The parts of this question demonstrate that F is *closed* under the four algebraic operations. Let $a + b\sqrt{2}$ and $c + d\sqrt{2}$ be members of F .

(a) Show that $(a + b\sqrt{2}) + (c + d\sqrt{2})$ has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under addition.

(b) Show that $(a + b\sqrt{2}) - (c + d\sqrt{2})$ has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under subtraction.

(c) Show that $(a + b\sqrt{2}) \times (c + d\sqrt{2})$ has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under multiplication.

(d) Show that $(a + b\sqrt{2}) \div (c + d\sqrt{2})$, where c and d are not both zero, has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under division.

10. Prove that it is impossible to have $a + \sqrt{b} = A - \sqrt{B}$, where a, b, A and B are rational, with $b \geq 0$ and $B \geq 0$, and b not the square of a rational.

2F Relations and Functions

Having clarified some ideas about numbers, we turn now to the functions that will be the central objects of study in this course.

A Function and its Graph: When a quantity y is completely determined by some other quantity x as a result of any rule whatsoever, we say that y is a *function of x* . For example, the height y of a ball thrown vertically upwards will be a function of the time x after the ball is thrown. In units of metres and seconds, a possible such rule is

$$y = 5x(6 - x).$$

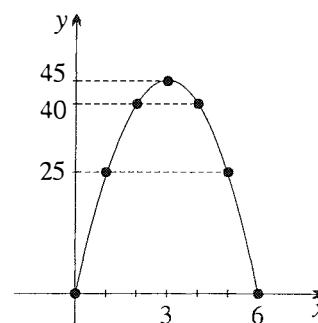
We can construct a *table of values* of this function by choosing just a few values of the time x and calculating the corresponding height y :

x	0	1	2	3	4	5	6
y	0	25	40	45	40	25	0

Each x -value and its corresponding y -value can then be put into an ordered pair ready to plot on a graph of the function. The seven ordered pairs calculated here are:

$$(0,0), (1,25), (2,40), (3,45), (4,40), (5,25), (6,0),$$

and the graph is sketched opposite. The seven representative points have been drawn, but there are infinitely many such ordered pairs, and they join up to make the nice smooth curve drawn in the graph.



We can take a more abstract approach to all this, and identify the function completely with the ordered pairs generated by the function rule. Notice that a y -value can occur twice: for example, the ordered pairs $(1,25)$ and $(5,25)$ show us that the ball is 25 metres high after 1 second, and again after 5 seconds when it is coming back down. But no x -value can occur twice because at any one time the ball can only be in one position. So the more abstract definition of a function is:

19

DEFINITION: A *function* is a set of ordered pairs in which no two ordered pairs have the same x -coordinate.

In this way the function is completely identified with its graph, and the rule and the graph are now regarded only as alternative representations of the set of ordered pairs.

Domain and Range: The time variable in our example cannot be negative, because the ball had not been thrown then, and cannot be greater than 6, because the ball hits the ground again after 6 seconds. The *domain* is the set of possible x -values, so the domain is the closed interval $0 \leq x \leq 6$. Again, the height of the ball never exceeds 45 metres and is never negative. The *range* is the set of possible y -values, so the range is the closed interval $0 \leq y \leq 45$:

20

DEFINITION: The *domain* of a function is the set of all x -coordinates of the ordered pairs.
The *range* of a function is the set of all y -coordinates.

The Natural Domain: Any restriction on the domain is part of the function, so the example of the ball thrown into the air should more correctly have been written

$$y = 5x(6 - x), \text{ where } 0 \leq x \leq 6.$$

When the equation of a function is given with no restriction, we assume by convention that the domain is as large as possible, consisting of all x -values that can validly be substituted into the equation. So, for example, ‘the function $y = \sqrt{4 - x^2}$ ’ means

$$\text{‘the function } y = \sqrt{4 - x^2}, \text{ where } -2 \leq x \leq 2\text{’,}$$

because one cannot take square roots of negative numbers. Again, ‘the function $y = \frac{1}{x-2}$ ’ implies the restriction $x \neq 2$, because division by 0 is impossible. This implied domain is called the *natural domain* of the function.

The Function Machine and the Function Rule: A function can be regarded as a ‘machine’ with inputs and outputs. For example, on the right are the outputs from the function $y = 5x(6 - x)$ when the seven numbers 0, 1, 2, 3, 4, 5 and 6 are the inputs. This sort of model for a function has of course become far more important in the last few decades because computers and calculators routinely produce output from a given input. If the name f is given to our function, we can write the results of the input/output routines as follows:

x		y
0	$\longrightarrow$	0
1	$\longrightarrow$	25
2	$\longrightarrow$	40
3	$\longrightarrow$	45
4	$\longrightarrow$	40
5	$\longrightarrow$	25
6	$\longrightarrow$	0

$$f(0) = 0, \quad f(1) = 25, \quad f(2) = 40, \quad f(3) = 45, \quad f(4) = 40, \quad \dots$$

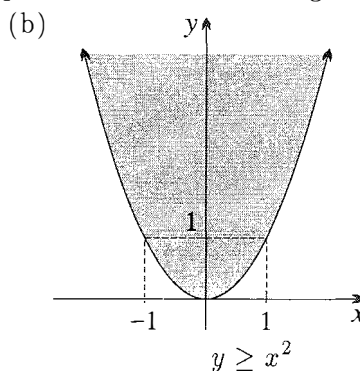
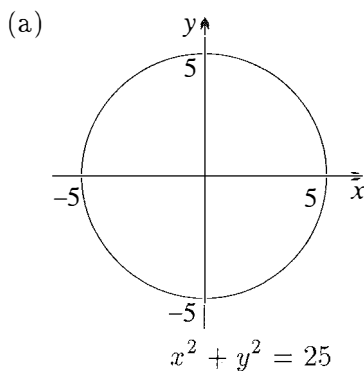
and since the output when x is input is $5x(6 - x)$, we can write the *function rule* using the well-known notation introduced by Leonhard Euler in 1735:

$$f(x) = 5x(6 - x), \text{ where } 0 \leq x \leq 6.$$

The pronumeral y is lost when the function is written this way, so a hybrid notation is sometimes used to express the fact that y is a function of x :

$$y(x) = 5x(6 - x), \text{ where } 0 \leq x \leq 6.$$

Relations: We shall often be dealing with graphs such as the following:



In case (a), the input $x = 3$ would result in the two outputs $y = 4$ and $y = -4$, because the vertical line $x = 3$ meets the graph at $(3, 4)$ and at $(3, -4)$. In case (b), the input $x = 1$ would give as output $y = 1$ and all numbers greater than 1. Such

objects are sometimes called ‘multi-valued functions’, but we will use the word *relation* to describe any curve or region in the plane, whether a function or not.

21 **DEFINITION:** Any set of ordered pairs is called a *relation*.

Once a relation is graphed, it is then quite straightforward to decide whether or not it is a function.

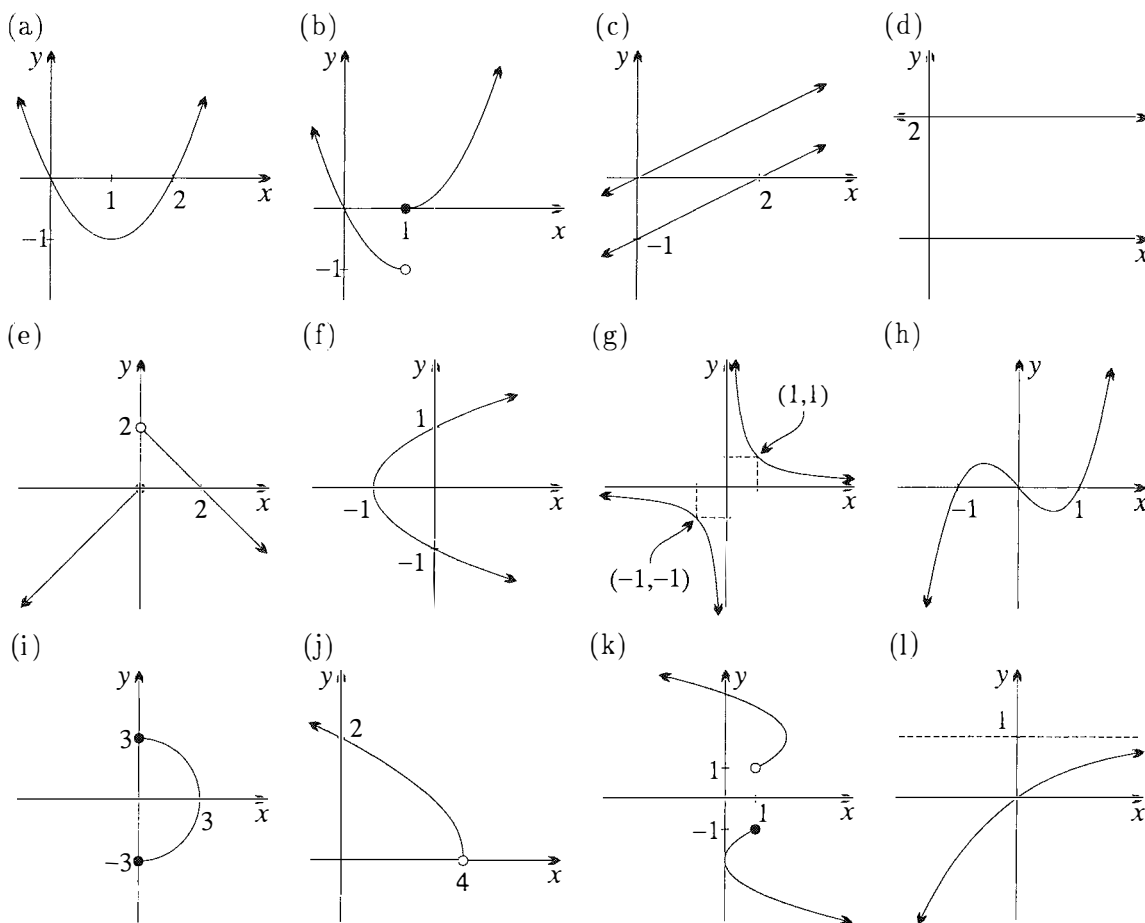
22 **VERTICAL LINE TEST:** A relation is a function if and only if there is no vertical line that crosses the graph more than once.

The ideas of domain and range apply not just to functions, but more generally to relations. In the examples given above:

- (a) For $x^2 + y^2 = 25$, the domain is $-5 \leq x \leq 5$, and the range is $-5 \leq y \leq 5$.
 (b) For $y \geq x^2$, the domain is the set $\mathbf{R}$ of all real numbers, and the range is $y \geq 0$.

Exercise 2F

1. Use the vertical line test to determine which of the following graphs represent functions:



2. What are the domain and range of each of the relations in question 1?

3. If $h(x) = x^2 - 2$, find:

- (a) $h(2)$ (c) $h(a)$ (e) $h(a+2)$ (g) $h(\frac{1}{2})$ (i) $h(t^2)$
 (b) $h(\sqrt{2})$ (d) $h(-a)$ (f) $h(x-1)$ (h) $h(3t+2)$ (j) $h(t+\frac{1}{t})$

4. If $g(x) = x^2 - 2x$, find:

(a) $g(0)$	(c) $g(-2)$	(e) $g(t)$	(g) $g(w-1)$	(i) $g(1-w)$
(b) $g(1)$	(d) $g(2)$	(f) $g(-t)$	(h) $g(w)-1$	(j) $g(2-x)$

5. (a) Let $f(x) = \begin{cases} x, & \text{for } x \leq 0, \\ 2-x, & \text{for } x > 0. \end{cases}$ Create a table of values for $-3 \leq x \leq 3$, and confirm that the graph is that in question 1(e) above.

(b) Let $f(x) = \begin{cases} (x-1)^2 - 1, & \text{for } x < 1, \\ (x-1)^2, & \text{for } x \geq 1. \end{cases}$ Create a table of values for $-1 \leq x \leq 3$, and confirm that the graph is that in question 1(b) above.

6. Find the natural domains of:

(a) $\ell(x) = x - 3$	(c) $s(x) = \sqrt{x}$	(e) $p(x) = \sqrt{2-x}$
(b) $r(x) = \frac{1}{x-3}$	(d) $S(x) = \frac{1}{\sqrt{x}}$	(f) $P(x) = \frac{1}{\sqrt{2-x}}$

DEVELOPMENT

7. Let $Q(x) = x^2 - 2x - 4$. Show that: (a) $Q(1 - \sqrt{5}) = 0$ (b) $Q(1 + \sqrt{5}) = 0$

8. Given that $f(x) = x^3 - x + 1$, evaluate and simplify:

(a) $f(h)$	(c) $\frac{f(h) - f(0)}{h}$	(e) $\frac{1}{4}(f(0) + 2f(\frac{1}{2}) + f(1))$
(b) $f(-h)$	(d) $\frac{f(h) - f(-h)}{2h}$	(f) $\frac{1}{6}(f(0) + 4f(\frac{1}{2}) + f(1))$

9. Create tables of values for these functions for $n = 1, 2, 3, 4, 5, 6$:

- (a) $S(n)$ = the sum of the positive numbers less than or equal to n
 (b) $d(n)$ = the number of positive divisors of n
 (c) $\sigma(n)$ = the sum of the positive divisors of n

10. Find the natural domains of:

(a) $c(x) = \sqrt{9-x^2}$	(c) $\ell(x) = \log_3 x$	(e) $p(x) = \frac{1}{x^2 - 5x + 6}$
(b) $h(x) = \sqrt{x^2 - 4}$	(d) $q(x) = \frac{x+2}{x+1}$	(f) $r(x) = \frac{x-3}{x^2 - 9}$

11. Given the functions $f(x) = x^2$, $F(x) = x + 3$, $g(x) = 2^x$ and $G(x) = 3x$, find:

(a) $f(F(5))$	(c) $f(F(x))$	(e) $g(G(2))$	(g) $g(G(x))$
(b) $F(f(5))$	(d) $F(f(x))$	(f) $G(g(2))$	(h) $G(g(x))$

12. (a) If $f(x) = 2^x$, show that $f(-x) = \frac{1}{f(x)}$.

(b) If $g(x) = \frac{x}{x^2 + 1}$, show that $g(\frac{1}{x}) = g(x)$ for $x \neq 0$.

(c) If $h(x) = \frac{x}{x^2 - 1}$, show that $h(\frac{1}{x}) = -h(x)$ for $x \neq 0$.

(d) If $f(x) = x + \frac{1}{x}$, show that $f(x) \times f(x + \frac{1}{x}) = f(x^2) + 3$.

13. For each of the following functions write out the equation $f(a) + f(b) = f(a+b)$. Then determine if there are any values of a and b for which $f(a) + f(b) = f(a+b)$.

- (a) $f(x) = x$ (b) $f(x) = 2x$ (c) $f(x) = x+1$ (d) $f(x) = 2x+1$ (e) $f(x) = x^2 + 3x$

EXTENSION

14. Evaluate $e(x) = \left(1 + \frac{1}{x}\right)^x$ on your calculator for $x = 1, 10, 100, 1000$ and 10000 , giving your answer to two decimal places. What do you notice happens as x gets large?
15. Let $c(x) = \frac{3^x + 3^{-x}}{2}$ and $s(x) = \frac{3^x - 3^{-x}}{2}$.
- Show that $(c(x))^2 = \frac{1}{2}(c(2x) + 1)$.
 - Find a similar result for $(s(x))^2$.
 - Hence show that $(c(x))^2 - (s(x))^2 = 1$.
16. Given that $\text{ath}(x) = \log_2 \left(\frac{1+x}{1-x} \right)$, show that $\text{ath} \left(\frac{2x}{1+x^2} \right) = 2\text{ath}(x)$.

NOTE: A function name does not have to be a single letter. In this case the function has been given the name 'ath' since it is related to the arc hyperbolic tangent which is studied in some university courses.

2G Review of Known Functions and Relations

This section will briefly review graphs that have been studied in previous years — linear graphs, quadratic functions, higher powers of x , circles and semicircles, half-parabolas, rectangular hyperbolas, exponential functions and log functions.

Linear Functions and Relations: Any equation that can be written in the form

$$ax + by + c = 0, \text{ where } a, b \text{ and } c \text{ are constants (and } a \text{ and } b \text{ are not both zero),}$$

is called a *linear relation*, because its graph is a straight line. Unless $b = 0$, the equation can be solved for y and is therefore a *linear function*.

- 23** **SKETCHING LINEAR FUNCTIONS:** Find the x -intercept by putting $y = 0$, and find the y -intercept by putting $x = 0$.

This method won't work when any of the three constants a , b and c is zero:

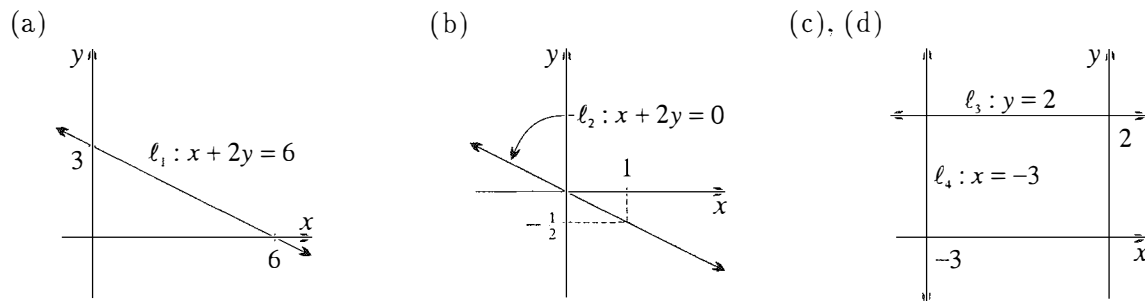
- 24** **SPECIAL CASES OF LINEAR GRAPHS:**
- If $a = 0$, then the equation has the form $y = k$, and its graph is a horizontal line with y -intercept k .
 - If $b = 0$, then the equation has the form $x = \ell$, and its graph is a vertical line with x -intercept ℓ .
 - If $c = 0$, both intercepts are zero and the graph passes through the origin. Find one more point on it, usually by putting $x = 1$.

WORKED EXERCISE: Sketch the following four lines:

- (a) $\ell_1: x + 2y = 6$ (b) $\ell_2: x + 2y = 0$ (c) $\ell_3: y = 2$ (d) $\ell_4: x = -3$

SOLUTION:

- (a) The line $\ell_1: x + 2y = 6$ has y -intercept $y = 3$ and x -intercept $x = 6$.
 (b) The line $\ell_2: x + 2y = 0$ passes through the origin, and $y = -\frac{1}{2}$ when $x = 1$.
 (c) The line ℓ_3 is horizontal with y -intercept 2.
 (d) The line ℓ_4 is vertical with x -intercept -3 .



Quadratic Functions: Sketches of quadratic functions will be required before their systematic treatment in Chapters 8 and 9. A *quadratic* is a function of the form

$$f(x) = ax^2 + bx + c, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

The graph of a quadratic function is a parabola with axis of symmetry parallel to the y -axis. Normally, four points should be shown on any sketch — the y -intercept, the two x -intercepts (which may coincide or may not exist), and the vertex. There are four steps for finding these points.

THE FOUR STEPS IN SKETCHING A QUADRATIC FUNCTION:

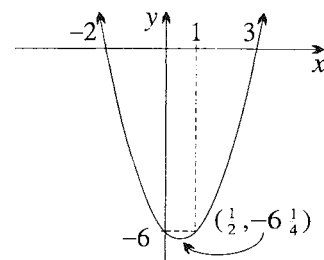
1. If a is positive, the parabola is concave up.
If a is negative, the parabola is concave down.
2. To find the y -intercept, put $x = 0$.
3. To find the x -intercepts:
 - (a) factor $f(x)$ and write down the x -intercepts, or
 - (b) complete the square, or
 - (c) use the formula, $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.
4. To find the vertex, first find the axis of symmetry:
 - (a) by finding the average of the x -intercepts, or
 - (b) by completing the square, or
 - (c) by using the formula for the axis of symmetry, $x = \frac{-b}{2a}$.

Then find the y -coordinate of the vertex by substituting back into $f(x)$.

WORKED EXERCISE: Sketch the graph of $y = x^2 - x - 6$, using the method of factoring.

SOLUTION: Factoring, $y = (x - 3)(x + 2)$.

1. Since $a = 1$, the parabola is concave up.
2. The y -intercept is -6 .
3. The x -intercepts are $x = 3$ and $x = -2$.
4. The axis of symmetry is $x = \frac{1}{2}$ (average of zeroes), and when $x = \frac{1}{2}$, $y = -6\frac{1}{4}$, so the vertex is $(\frac{1}{2}, -6\frac{1}{4})$.



WORKED EXERCISE: Sketch the graph of $y = x^2 + 2x - 3$, using the method of completing the square.

SOLUTION: The curve is concave up, with y -intercept $y = -3$.

Completing the square, $y = (x^2 + 2x + 1) - 1 - 3$

$$y = (x + 1)^2 - 4.$$

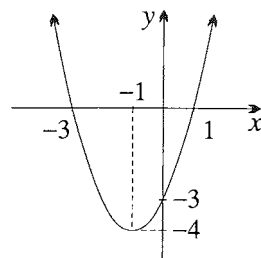
So the axis of symmetry is $x = -1$,

and the vertex is $(-1, -4)$.

Putting $y = 0$, $(x + 1)^2 = 4$

$$x + 1 = 2 \text{ or } -2,$$

so the x -intercepts are $x = 1$ and $x = -3$.



WORKED EXERCISE: Sketch the graph of $y = -x^2 + 4x - 5$, using the formulae for the zeroes and the axis of symmetry.

SOLUTION: The curve is concave down, with y -intercept $y = -5$.

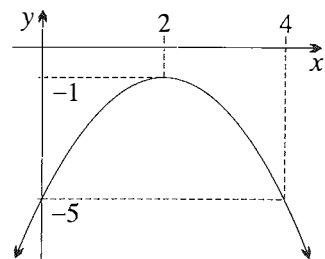
Since $b^2 - 4ac = -4$ is negative, there are no x -intercepts.

The axis of symmetry is $x = -\frac{b}{2a}$

$$x = 2.$$

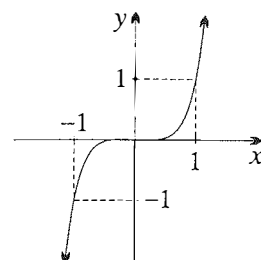
When $x = 2$, $y = -1$, so the vertex is $(2, -1)$.

By reflecting $(0, -5)$ about the axis $x = 2$, when $x = 4$, $y = -5$.



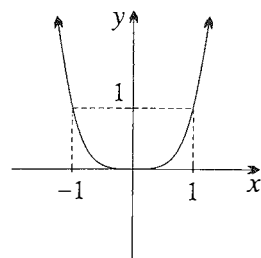
Higher Powers of x : On the right is the graph of $y = x^3$. All odd powers look similar, becoming flatter near the origin as the index increases, and steeper further away.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8



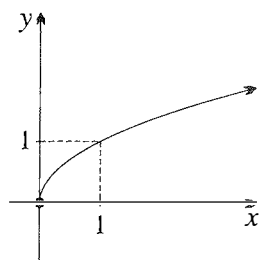
On the right is the graph of $y = x^4$. All even powers look similar — they are always positive, and become flatter near the origin as the index increases, and steeper further away.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16



The Function $y = \sqrt{x}$: The graph of $y = \sqrt{x}$ is the upper half of a parabola on its side, as can be seen by squaring both sides to give $y^2 = x$. Remember that the symbol $\sqrt{x}$ means the *positive* square root of x , so the lower half is excluded:

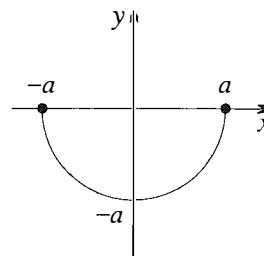
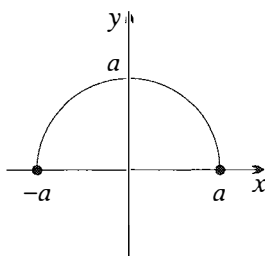
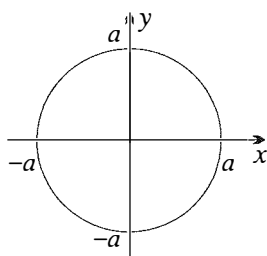
x	0	$\frac{1}{4}$	1	2	4
y	0	$\frac{1}{2}$	1	$\sqrt{2}$	2



Circles and Semicircles: The graph of $x^2 + y^2 = a^2$ is a circle with radius $a > 0$ and centre the origin, as sketched on the left below. This graph is not a function — solving for y yields

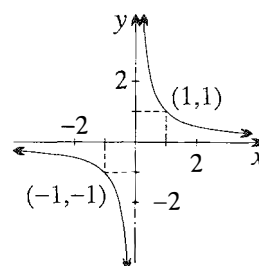
$$y = \sqrt{a^2 - x^2} \quad \text{or} \quad y = -\sqrt{a^2 - x^2},$$

which means there are two values of y for some values of x . The positive square root $y = \sqrt{a^2 - x^2}$, however, is a function, whose graph is the upper semicircle below. Similarly, the negative square root $y = -\sqrt{a^2 - x^2}$ is also a function, whose graph is the lower semicircle below:



The Rectangular Hyperbola: The reciprocal function $y = 1/x$ is well known, but it is worth careful attention because it is the best place to introduce some important ideas about limits and asymptotes. Here is a table of values and a sketch of the graph, which is called a *rectangular hyperbola*:

x	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
y	*	10	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$
x	-10	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$	
y	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{1}{2}$	-1	-2	-5	-10	



The star (*) at $x = 0$ indicates that the function is not defined there.

Limits and Asymptotes Associated with the Rectangular Hyperbola: Here is the necessary language and notation for describing the behaviour of $y = 1/x$ near $x = 0$ and for large x .

1. The domain is $x \neq 0$, because the reciprocal of 0 is not defined. The range can be read off the graph — it is $y \neq 0$.
2. (a) As x becomes very large positive, y becomes very small indeed. We can make y ‘as close as we like’ to 0 by choosing x sufficiently large. The formal notation for this is

$$y \rightarrow 0 \text{ as } x \rightarrow \infty \quad \text{or} \quad \lim_{x \rightarrow \infty} y = 0.$$

- (b) On the left, as x becomes very large negative, y also becomes very small:

$$y \rightarrow 0 \text{ as } x \rightarrow -\infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} y = 0.$$

- (c) The x -axis is called an *asymptote* of the curve (from the Greek word *asymptotos*, meaning ‘apt to fall together’), because the curve gets ‘as close as we like’ to the x -axis for sufficiently large x and for sufficiently large negative x .
3. (a) When x is a very small positive number, y becomes very large, because the reciprocal of a very small number is very large. We can make y ‘as large as we like’ by taking sufficiently small but still positive values of x . The formal notation is

$$y \rightarrow \infty \text{ as } x \rightarrow 0^+.$$

- (b) On the left-hand side of the origin, y is negative and can be made ‘as large negative as we like’ by taking sufficiently small negative values of x :

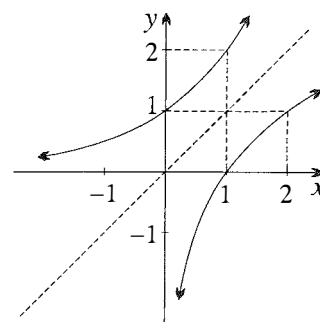
$$y \rightarrow -\infty \text{ as } x \rightarrow 0^-.$$

- (c) The y -axis is also an asymptote of the curve, because the curve gets ‘as close as we like’ to the y -axis when x is sufficiently close to zero.

Exponential and Logarithmic Functions: Functions of the form $y = a^x$, where the base a of the power is positive and not equal to 1, are called *exponential functions*, because the variable x is in the *exponent* or *index*. Functions which are of the form $y = \log_a x$ are called *logarithmic functions*. Here are the graphs of the two functions $y = 2^x$ and $y = \log_2 x$.

$y = 2^x$					
x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$y = \log_2 x$					
x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	-2	-1	0	1	2



The two graphs are reflections of each other in the line $y = x$. This is because the second table is just the first table turned upside down, which simply swaps the coordinates of each ordered pair in the function. The two functions are therefore *inverse functions* of each other, and inverse functions in general will be the subject of the next section. Corresponding to the reflection, the x -axis is an asymptote for $y = 2^x$, and the y -axis is an asymptote for $y = \log_2 x$.

Exercise 2G

1. Sketch the following special cases of linear graphs:

(a) $x = 1$	(c) $x = -1.5$	(e) $y = 2x$	(g) $x - y = 0$
(b) $y = -2$	(d) $y = 3$	(f) $y = -\frac{1}{2}x$	(h) $3x + 2y = 0$

2. For each linear function, find the y -intercept by putting $x = 0$, and the x -intercept by putting $y = 0$. Then sketch each curve.

(a) $y = x + 1$	(e) $x + y - 1 = 0$	(i) $2x - 3y - 12 = 0$
(b) $y = 4 - 2x$	(f) $2x - y + 2 = 0$	(j) $x + 4y + 6 = 0$
(c) $y = \frac{1}{2}x - 3$	(g) $x - 3y - 3 = 0$	(k) $5x + 2y - 10 = 0$
(d) $y = -3x - 6$	(h) $x - 2y - 4 = 0$	(l) $-5x + 2y + 15 = 0$

3. Determine the main features of each parabola — the vertex, intercepts and, where necessary, any additional points symmetric about the axis. Then sketch a graph showing these features.

(a) $y = x^2$	(c) $y = \frac{1}{2}x^2$	(e) $y = x^2 - 4$	(g) $y = 2 - \frac{1}{2}x^2$
(b) $y = -x^2$	(d) $y = x^2 + 1$	(f) $y = 9 - x^2$	(h) $y = -1 - x^2$

4. These quadratics are already factored. Find the x -intercepts and y -intercept. Then find the vertex by finding the average of the x -intercepts and substituting. Sketch the graph, then write down the range:

$$\begin{array}{lll} \text{(a)} y = (x - 4)(x - 2) & \text{(c)} y = -(x + 2)(x + 6) & \text{(e)} y = x(x - 3) \\ \text{(b)} y = (x - 4)(x + 2) & \text{(d)} y = -(x + 2)(x - 6) & \text{(f)} y = (x + 1)(x + 4) \end{array}$$

5. Factor these quadratics. Then find the x -intercepts, y -intercept and vertex, and sketch:

$$\begin{array}{lll} \text{(a)} y = x^2 + 6x + 8 & \text{(d)} y = -x^2 + 2x & \text{(g)} y = x^2 - 9 \\ \text{(b)} y = x^2 - 4x + 4 & \text{(e)} y = -x^2 + 2x + 3 & \text{(h)} y = x^2 + 9x + 14 \\ \text{(c)} y = x^2 - 10x - 11 & \text{(f)} y = -x^2 - 2x - 1 & \text{(i)} y = 4 - x^2 \end{array}$$

6. (a) Use a calculator and a table of values to plot the graphs of $y = x$, $y = x^2$, $y = x^3$ and $y = x^4$ on the same graph, for $-1.25 \leq x \leq 1.25$. Use a scale of 2 cm to 1 unit and plot points every 0.25 units along the x -axis.

- (b) Use a calculator and a table of values to plot the graph of $y = \sqrt{x}$ for $0 \leq x \leq 4$. Use a scale of 2 cm to 1 unit and plot points every 0.5 units along the x -axis. On the same number plane graph, plot $y = x^2$, for $0 \leq x \leq 2$, and confirm that the two curves are reflections of each other in the line $y = x$.

7. Identify the centre and radius of each of these circles and semicircles. Then sketch its graph, and write down its domain and range:

$$\begin{array}{lll} \text{(a)} x^2 + y^2 = 1 & \text{(d)} y^2 + x^2 = \frac{9}{4} & \text{(g)} y = \sqrt{\frac{25}{4} - x^2} \\ \text{(b)} x^2 + y^2 = 9 & \text{(e)} y = \sqrt{4 - x^2} & \text{(h)} x = \sqrt{9 - y^2} \\ \text{(c)} y^2 + x^2 = \frac{1}{4} & \text{(f)} y = -\sqrt{1 - x^2} & \text{(i)} x = -\sqrt{\frac{1}{4} - y^2} \end{array}$$

8. Use tables of values to sketch these hyperbolas. Then write down the domain and range of each:

$$\begin{array}{llll} \text{(a)} y = \frac{2}{x} & \text{(b)} y = -\frac{1}{x} & \text{(c)} xy = 3 & \text{(d)} xy = -2 \end{array}$$

9. Use tables of values to sketch these exponential and logarithmic functions. Then write down the domain and range of each:

$$\begin{array}{llll} \text{(a)} y = 3^x & \text{(c)} y = 10^x & \text{(e)} y = \left(\frac{1}{2}\right)^x & \text{(g)} y = \log_4 x \\ \text{(b)} y = 3^{-x} & \text{(d)} y = 10^{-x} & \text{(f)} y = \log_3 x & \text{(h)} y = \log_{10} x \end{array}$$

DEVELOPMENT

10. Factor these quadratics where necessary. Then find the intercepts and vertex and sketch:

$$\begin{array}{lll} \text{(a)} y = (2x - 1)(2x - 7) & \text{(c)} y = 9x^2 - 18x - 7 & \text{(e)} y = -4x^2 + 12x + 7 \\ \text{(b)} y = -x(2x + 9) & \text{(d)} y = 9x^2 - 30x + 25 & \text{(f)} y = -5x^2 + 2x + 3 \end{array}$$

11. Complete the square in each quadratic expression and hence find the coordinates of the vertex of the parabola. Use the completed square to find the x -intercepts, then graph:

$$\text{(a)} y = x^2 - 4x + 3 \quad \text{(b)} y = x^2 + 2x - 8 \quad \text{(c)} y = x^2 + 3x + 2 \quad \text{(d)} y = x^2 - x + 1$$

12. Use the formula to find the x -intercepts. Then use the formula for the axis of symmetry, and substitute to find the vertex. Sketch the graphs:

$$\text{(a)} y = x^2 + 2x - 5 \quad \text{(b)} y = x^2 - 7x + 3 \quad \text{(c)} y = 3x^2 - 4x - 1 \quad \text{(d)} y = 4 + x - 2x^2$$

13. Each equation below represents a half-parabola. Draw up a table of values and sketch them, then write down the domain and range of each:

$$\begin{array}{lll} \text{(a)} y = \sqrt{x} + 1 & \text{(c)} y = \sqrt{x - 4} & \text{(e)} x = \sqrt{y} \\ \text{(b)} y = 1 - \sqrt{x} & \text{(d)} y = \sqrt{4 - x} & \text{(f)} x = -\sqrt{y} \end{array}$$

14. Carefully graph each pair of equations on the same number plane. Hence find the intersection points, given that they have integer coordinates:

(a) $y = x, y = x^2$	(e) $x^2 + y^2 = 1, x + y = 1$
(b) $y = -x, y = -x^2 + 2x$	(f) $xy = -2, y = x - 3$
(c) $y = 2x, y = x^2 - x$	(g) $y = \frac{12}{x}, x^2 + y^2 = 25$
(d) $y = 1 - 2x, y = x^2 - 4x - 2$	(h) $y = -x^2 + x + 1, y = \frac{1}{x}$

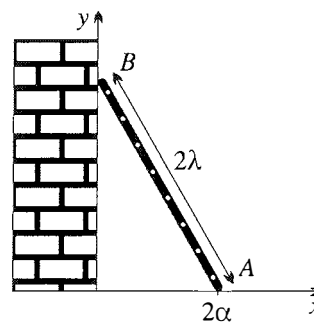
15. Write down the radius of each circle or semicircle and graph it. Also state any points on each curve whose coordinates are both integers:

(a) $x^2 + y^2 = 5$ (b) $y = -\sqrt{2 - x^2}$ (c) $x = \sqrt{10 - y^2}$ (d) $x^2 + y^2 = 17$

16. (a) Show that $(x + y)^2 - (x - y)^2 = 4$ is the equation of a hyperbola. Sketch it.
 (b) Show that $(x + y)^2 + (x - y)^2 = 4$ is the equation of a circle. Sketch it.
 (c) Solve these two equations simultaneously. Begin by subtracting the equation of the hyperbola from the equation of the circle.
 (d) Sketch both curves on the same number plane, showing the points of intersection.

————— EXTENSION —————

17. The diagram shows a ladder of length 2λ leaning against a wall so that the foot of the ladder is distant 2α from the wall.



- (a) Find the coordinates of B .
 (b) Show that the midpoint P of the ladder lies on a circle with centre at the origin. What is the radius of this circle?
18. (a) The line $y = -\frac{1}{4}b^2x + b$ has intercepts at A and B . Find the coordinates of P , the midpoint of AB .
 (b) Show that P lies on the hyperbola $y = \frac{1}{x}$.
 (c) Show that the area of $\triangle OAB$, where O is the origin, is independent of the value of b .
19. The curve $y = 2^x$ is approximated by the parabola $y = ax^2 + bx + c$ for $-1 \leq x \leq 1$. The values of the constants a, b and c are chosen so that the two curves intersect at $x = -1, 0, 1$.
 (a) Find the values of the constant coefficients.
 (b) Use this parabola to estimate the values of $\sqrt{2}$ and $1/\sqrt{2}$.
 (c) Compare the values found in part (b) with the values obtained by a calculator. Show that the percentage errors are approximately 1.6% and 2.8% respectively.
20. Consider the relation $y^2 = (1 - x^2)(4x^2 - 1)^2$.
 (a) Write down a pair of alternative expressions for y that are functions of x .
 (b) Find the natural domains of these functions.
 (c) Find any intercepts with the axes.
 (d) Create a table of values for each function. Select x values every 0.1 units in the domain. Plot the points so found. What is the familiar shape of the original relation? (A graphics calculator or computer may help simplify this task.)

2H Inverse Relations and Functions

At the end of the last section, the pair of inverse functions $y = 2^x$ and $y = \log_2 x$ were sketched from a table of values. We saw then how the two curves were reflections of each other in the diagonal line $y = x$, and how the two tables of values were the same except that the rows were reversed. Many functions similarly have a well-defined *inverse function* that sends any output back to the original input. For example, the inverse of the cubing function $y = x^3$ is the cube root function $y = \sqrt[3]{x}$.

x		y		x		y
2	→		→	8	→	2
1	→		→	1	→	1
$\frac{1}{2}$	→		→	$\frac{1}{8}$	→	$\frac{1}{2}$
0	→		→	0	→	0
$-\frac{1}{2}$	→		→	$-\frac{1}{8}$	→	$-\frac{1}{2}$
-1	→		→	-1	→	-1
-2	→		→	-8	→	-2

The exchanging of input and output can also be seen in the two tables of values, where the two rows are interchanged:

x	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2
x^3	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8

x	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8
$\sqrt[3]{x}$	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2

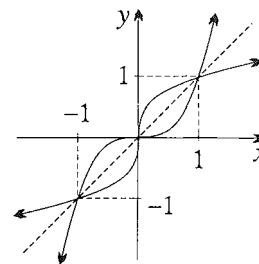
This exchanging of input and output means that the coordinates of each ordered pair are exchanged, so we are led to a definition that can be applied to any relation, whether it is a function or not:

26 DEFINITION: The inverse relation is obtained by reversing each ordered pair.

The exchanging of first and second components means that the domain and range are exchanged:

27 DOMAIN AND RANGE OF THE INVERSE: The domain of the inverse is the range of the relation, and the range of the inverse is the domain of the relation.

Graphing the Inverse Relation: Reversing an ordered pair means that the original first coordinate is read off the vertical axis, and the original second coordinate is read off the horizontal axis. Geometrically, this exchanging can be done by reflecting the point in the diagonal line $y = x$, as can be seen by comparing the graphs of $y = x^3$ and $y = \sqrt[3]{x}$, which are drawn here on the same pair of axes.



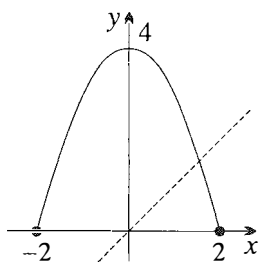
28 THE GRAPH OF THE INVERSE: The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.

Finding the Equations and Conditions of the Inverse Relation: When the coordinates are exchanged, the x -variable becomes the y -variable and the y -variable becomes the x -variable, so the method for finding the equation and conditions of the inverse is:

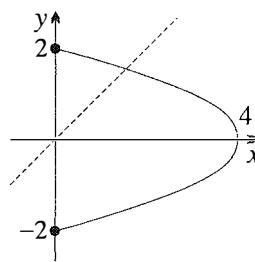
- 29** **THE EQUATION OF THE INVERSE:** To find the equations and conditions of the inverse relation, write x for y and y for x every time each variable occurs.

For example, the inverse of $y = x^3$ is $x = y^3$, which can then be solved for y to give $y = \sqrt[3]{x}$.

Testing whether the Inverse Relation is a Function: It is not true in general that the inverse of a function is a function. For example, the sketches below show the graphs of another function and its inverse:



$$y = 4 - x^2, \text{ where } -2 \leq x \leq 2$$



$$x = 4 - y^2, \text{ where } -2 \leq y \leq 2$$

The first is clearly a function, but the second fails the vertical line test, since the vertical line $x = 3$ crosses the graph at the two points $(3, 1)$ and $(3, -1)$. Before the second graph was even drawn, however, it was obvious from the first graph that the inverse would not be a function because the horizontal line $y = 3$ crossed the graph twice (and notice that reflection in $y = x$ exchanges horizontal and vertical lines).

- 30** **HORIZONTAL LINE TEST:** The inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.

Inverse Function Notation: If $f(x)$ is a function whose inverse is also a function, that function is written $f^{-1}(x)$. The index -1 used here means ‘inverse function’ and is not to be confused with its more common use for the reciprocal. To return to the original example,

$$\text{if } f(x) = x^3, \text{ then } f^{-1}(x) = \sqrt[3]{x}.$$

The inverse function sends each number back where it came from. Hence if the function and the inverse function are applied successively to a number, in either order, the result is the original number. For example, using cubes and cube roots,

$$(\sqrt[3]{8})^3 = 2^3 = 8 \quad \text{and} \quad \sqrt[3]{8^3} = \sqrt[3]{512} = 8.$$

- 31** **INVERSE FUNCTIONS:** If the inverse relation of a function $f(x)$ is also a function, the inverse function is written $f^{-1}(x)$. The composition of the function and its inverse sends every number back to itself:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

WORKED EXERCISE: Find the equations of the inverse relations of these functions. If the inverse is a function, find an expression for $f^{-1}(x)$, and then verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$(a) f(x) = 6 - 2x, \text{ where } x > 0 \qquad (c) f(x) = \frac{1-x}{1+x}$$

$$(b) f(x) = x^3 + 2 \qquad (d) f(x) = x^2 - 9$$

SOLUTION:

$$(a) \text{ Let } y = 6 - 2x, \text{ where } x > 0.$$

Then the inverse has the equation $x = 6 - 2y$, where $y > 0$

$$y = 3 - \frac{1}{2}x, \text{ where } y > 0.$$

The condition $y > 0$ means that $x < 6$,

$$\text{so } f^{-1}(x) = 3 - \frac{1}{2}x, \text{ where } x < 6.$$

Verifying, $f^{-1}(f(x)) = f^{-1}(6 - 2x) = 3 - \frac{1}{2}(6 - 2x) = 3 - 3 + x = x$

and $f(f^{-1}(x)) = f(3 - \frac{1}{2}x) = 6 - 2(3 - \frac{1}{2}x) = 6 - 6 + x = x$.

$$(b) \text{ Let } y = x^3 + 2.$$

Then the inverse has the equation $x = y^3 + 2$

$$y = \sqrt[3]{x-2}.$$

$$\text{So } f^{-1}(x) = \sqrt[3]{x-2}.$$

Verifying, $f^{-1}(f(x)) = \sqrt[3]{(x-2)^3} = x - 2$

and $f(f^{-1}(x)) = (\sqrt[3]{x-2})^3 = x - 2$.

$$(c) \text{ Let } y = \frac{1-x}{1+x}.$$

Then the inverse has equation $x = \frac{1-y}{1+y}$.

$$\boxed{\times (1+y)}$$

$$x + xy = 1 - y$$

$$y + xy = 1 - x \quad (\text{terms in } y \text{ on one side})$$

$$y(1+x) = 1-x \quad (\text{the key step})$$

$$y = \frac{1-x}{1+x}.$$

$$\text{So } f^{-1}(x) = \frac{1-x}{1+x}.$$

Notice that this function and its inverse are identical, so that if the function is applied twice, each number is sent back to itself.

For example, $f(f(2)) = f\left(-\frac{1}{3}\right) = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = 2$.

$$\text{In general, } f(f(x)) = \frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}} = \frac{(1+x)-(1-x)}{(1+x)+(1-x)} = \frac{2x}{2} = x.$$

(d) The function $f(x) = x^2 - 9$ fails the horizontal line test. For example, $f(3) = f(-3) = 0$, which means that the x -axis meets the graph twice. So the inverse relation of $f(x)$ is not a function. Alternatively, one can say that the inverse relation is $x = y^2 - 9$, which on solving for y gives

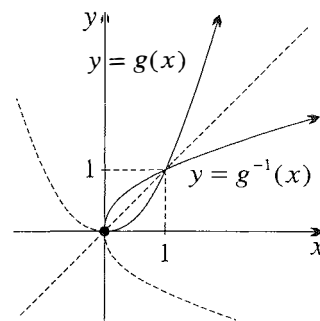
$$y = \sqrt{x+9} \text{ or } -\sqrt{x+9},$$

which is not unique, and so the inverse relation is not a function.

Restricting the Domain so the Inverse is a Function: When the inverse of a function is not a function, it is often convenient to restrict the domain of the function so that this restricted function has an inverse function. The example of taking squares and square roots should already be well known. The function $y = x^2$ does not have an inverse function, because, for example, 49 has two square roots, 7 and -7 . If, however, we restrict the domain to $x \geq 0$ and define a new restricted function

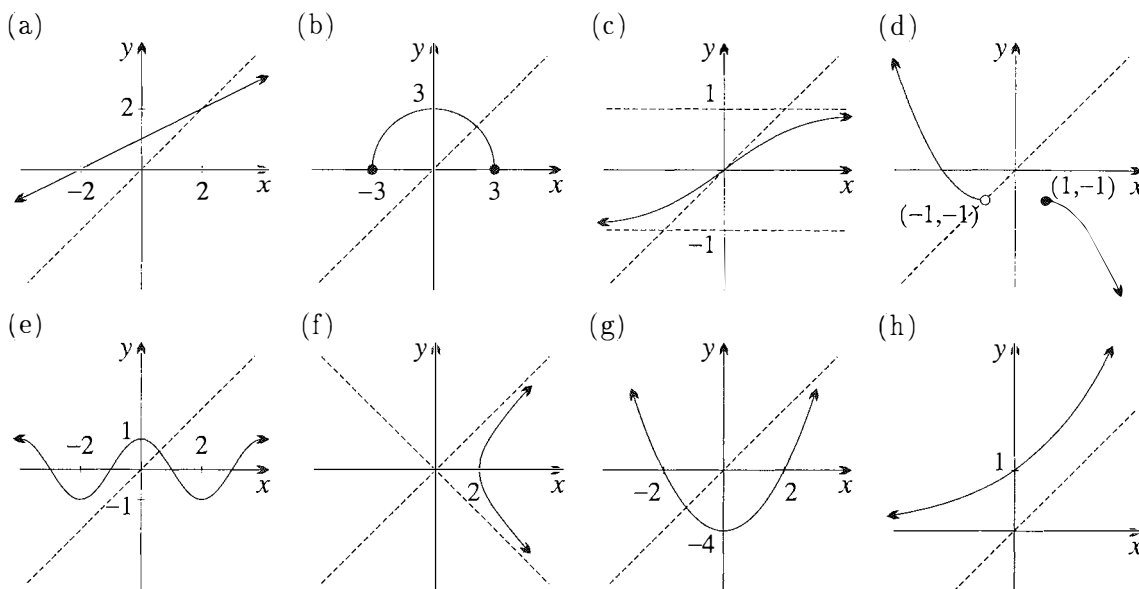
$$g(x) = x^2, \text{ where } x \geq 0,$$

then the inverse function is $g^{-1}(x) = \sqrt{x}$, where as explained earlier, the symbol $\sqrt{\quad}$ means ‘take the positive square root (or zero)’. On the right are the graphs of the restricted function and its inverse function, with the unrestricted function and its inverse relation shown dotted. These ideas will be developed in more general situations in the Year 12 Volume.



Exercise 2H

1. Draw the inverse relation for each of the following relations by reflecting in the line $y = x$:



2. Use the vertical and horizontal line tests to determine which relations and which inverse relations drawn in the previous question are also functions.
3. Determine the inverse algebraically by swapping x and y and then making y the subject:
- (a) $y = 3x - 2$ (c) $y = 3 - \frac{1}{2}x$ (e) $2x + 5y - 10 = 0$
 (b) $y = \frac{1}{2}x + 1$ (d) $x - y + 1 = 0$ (f) $y = 2$
4. For each function in the previous question, draw a graph of the function and its inverse on the same number plane to verify the reflection property. Draw a separate number plane for each part.
5. Find the inverse algebraically by swapping x and y and then making y the subject:
- (a) $y = \frac{1}{x} + 1$ (b) $y = \frac{1}{x + 1}$ (c) $y = \frac{x + 2}{x - 2}$ (d) $y = \frac{3x}{x + 2}$

6. Swap x and y and solve for y to find the inverse of each of the following functions. What do you notice, and what is the geometric significance of this?

$$(a) y = \frac{1}{x} \quad (b) y = \frac{2x-2}{x-2} \quad (c) y = \frac{-3x-5}{x+3} \quad (d) y = -x$$

DEVELOPMENT

7. Each pair of functions $f(x)$ and $g(x)$ are mutual inverses. Verify in each case by substitution that: (i) $f(g(2)) = 2$, and (ii) $g(f(2)) = 2$.

$$(a) f(x) = x + 13 \text{ and } g(x) = x - 13 \quad (c) f(x) = 2x + 6 \text{ and } g(x) = \frac{1}{2}(x - 6)$$

$$(b) f(x) = 7x \text{ and } g(x) = \frac{1}{7}x \quad (d) f(x) = x^3 - 6 \text{ and } g(x) = \sqrt[3]{x + 6}$$

Verify more generally in each case that: (iii) $f(g(x)) = x$, and (iv) $g(f(x)) = x$.

8. Graph each relation and its inverse. Find the equation of the inverse relation. In the cases where the inverse is a function, make y the subject of this equation:

$$(a) (x-3)^2 + y^2 = 4 \quad (c) (x+1)^2 + (y+1)^2 = 9 \quad (e) y = x^2 + 1$$

$$(b) y = 2^{-x} \quad (d) y = x^2 - 4 \quad (f) y = \log_3 x$$

9. Write down the inverse of each function, solving for y if it is a function. Sketch the function and the inverse on the same graph and observe the symmetry in the line $y = x$:

$$(a) y = x^2 \quad (c) y = -\sqrt{x} \quad (e) y = -\sqrt{4-x^2}$$

$$(b) y = 2x - x^2 \quad (d) y = 2^x \quad (f) y = \left(\frac{3}{2}\right)^x$$

10. Explain whether the inverse relation is a function. If it is a function, find $f^{-1}(x)$ and verify the two identities $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$(a) f(x) = x^2 \quad (e) f(x) = 9 - x^2 \quad (i) f(x) = x^2, x \leq 0$$

$$(b) f(x) = \sqrt{x} \quad (f) f(x) = 9 - x^2, x \geq 0 \quad (j) f(x) = x^2 - 2x, x \geq 1$$

$$(c) f(x) = x^4 \quad (g) f(x) = 3^{-x^2} \quad (k) f(x) = x^2 - 2x, x \leq 1$$

$$(d) f(x) = x^3 + 1 \quad (h) f(x) = \frac{1-x}{3+x} \quad (l) f(x) = \frac{x+1}{x-1}$$

11. (a) Show that the inverse function of $y = \frac{ax+b}{x+c}$ is $y = \frac{b-cx}{x-a}$.

(b) Hence show that $y = \frac{ax+b}{x+c}$ is its own inverse if and only if $a+c=0$.

12. Sketch on separate graphs: (a) $y = -x^2$ (b) $y = -x^2$, for $x \geq 0$

Draw the inverse of each on the same graph, then comment on the similarities and differences between parts (a) and (b).

EXTENSION

13. Suggest restrictions on the domains of the following in order that each have an inverse that is a function (there may be more than one answer). Draw the modified function and its inverse:

$$(a) y = -\sqrt{4-x^2} \quad (c) y = x^3 - x \quad (e) y = \sqrt{x^2}$$

$$(b) y = \frac{1}{x^2} \quad (d) y = \sin(90x^\circ) \quad (f) y = \tan(90x^\circ)$$

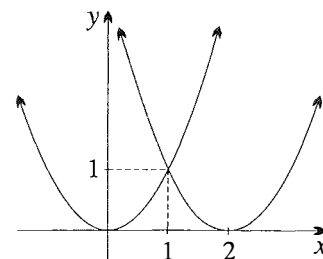
14. The logarithm laws indicate that $\log_3(x^n) = n \log_3(x)$. Explain why $y = \log_3(x^2)$ does not have an inverse that is a function, yet $y = 2 \log_3(x)$ does.

2I Shifting and Reflecting Known Graphs

There are various standard ways to manipulate given graphs to produce further graphs. For example, a graph can be shifted or stretched or reflected, or two graphs can be combined. Using these processes on known graphs can extend considerably the range of functions and relations whose graphs can be quickly recognised and drawn. This section deals with shifting and reflecting, and the next section deals with some further transformations.

Shifting Left and Right: The graphs of $y = x^2$ and $y = (x - 2)^2$ are sketched from their tables of values. They make it clear that the graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x - 2)^2$	16	9	4	1	0	1	4

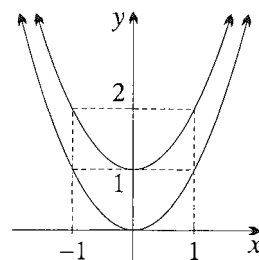


32

SHIFTING LEFT AND RIGHT: To shift k units to the right, replace x by $x - k$. Alternatively, if the graph is a function, the new function rule is $y = f(x - k)$.

Shifting Up and Down: The graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit, because the values in the table for $y = x^2 + 1$ are all 1 more than the values in the table for $y = x^2$:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 1$	10	5	2	1	2	5	10



Writing the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule completely analogous to that for horizontal shifting.

33

SHIFTING UP AND DOWN: To shift ℓ units upwards, replace y by $y - \ell$. Alternatively, if the graph is a function, the new function rule is $y = f(x) + \ell$.

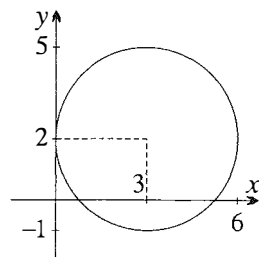
WORKED EXERCISE: Find the centre and radius of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$, then sketch it.

SOLUTION: Completing the square in both x and y ,

$$(x^2 - 6x + 9) + (y^2 - 4y + 4) + 4 = 9 + 4$$

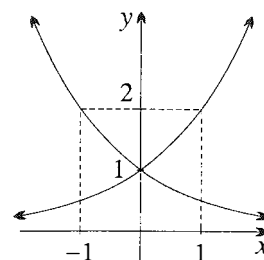
$$(x - 3)^2 + (y - 2)^2 = 9.$$

This is just $x^2 + y^2 = 9$ shifted right 3 and up 2, so the centre is $(3, 2)$ and the radius is 3.



Reflection in the y -axis: When the tables of values for $y = 2^x$ and $y = 2^{-x}$ are both written down, it is clear that the graphs of $y = 2^x$ and $y = 2^{-x}$ must be reflections of each other in the y -axis.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

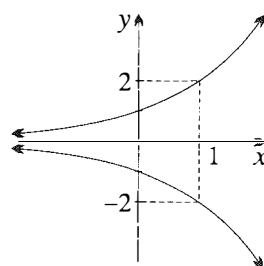


34

REFLECTION IN THE y -AXIS: To reflect in the y -axis, replace x by $-x$. Alternatively, if the graph is a function, the new function rule is $y = f(-x)$.

Reflection in the x -axis: All the values in the table below for $y = -2^x$ are the opposites of the values in the table for $y = 2^x$. This means that the graphs of $y = -2^x$ and $y = 2^x$ are reflections of each other in the x -axis.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8



Writing the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule completely analogous to that for reflection in the y -axis.

35

REFLECTION IN THE x -AXIS: To reflect in the x -axis, replace y by $-y$. Alternatively, if the graph is a function, the new function rule is $y = -f(x)$.

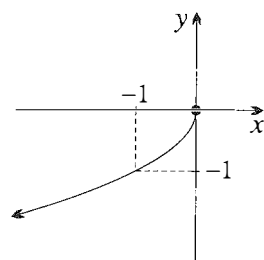
WORKED EXERCISE: From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$.

SOLUTION: The equation can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is reflected successively in both axes.

NOTE: Reflection in both the x -axis and the y -axis is the same as a rotation of 180° about the origin, and the order in which the reflections are done does not matter. This rotation of 180° about the origin is sometimes called the *reflection in the origin*, because every point in the plane is transformed along a line through the origin to a point on the opposite side of the origin and the same distance away.



Reflection in the Line $y = x$: The graphs of a relation and its inverse relation are reflections of each other in the diagonal line $y = x$, as discussed earlier in Section 2H.

36

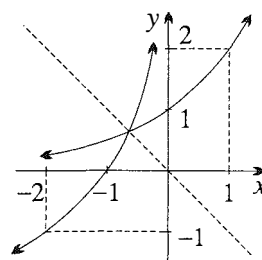
REFLECTION IN $y = x$: To reflect in the line $y = x$, replace x by y and y by x .

WORKED EXERCISE: [A harder example] What is the equation of the curve obtained when $y = 2^x$ is reflected in the line $y = -x$? Solve the resulting equation for y , and sketch the curves.

SOLUTION: Reflection in $y = -x$ is obtained by reflecting successively in $y = x$, in the x -axis, and in the y -axis (verify this with a square piece of paper). So the successive equations are

$$y = 2^x \longrightarrow x = 2^y \longrightarrow x = 2^{-y} \longrightarrow -x = 2^{-y}.$$

Solving the last equation for y gives $y = -\log_2(-x)$.



Exercise 2I

1. Write down the new equation for each function or relation after the given shift has been applied. Draw a graph of the image after the shift:

(a) $y = x^2$: right 1 unit	(e) $x^2 + y^2 = 9$: up 1 unit
(b) $y = 2^x$: down 3 units	(f) $y = x^2 - 4$: left 1 unit
(c) $y = \log_2 x$: left 2 units	(g) $xy = 1$: down 1 unit
(d) $y = \frac{1}{x}$: right 3 units	(h) $y = \sqrt{x}$: up 2 units

2. Repeat the previous question for a reflection in the given line:

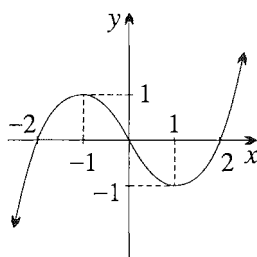
(a) x -axis	(c) $y = x$	(e) $y = x$	(g) x -axis
(b) y -axis	(d) x -axis	(f) $y = x$	(h) y -axis

3. Use the shifting results and completion of the square where necessary to determine the centre and radius of each circle:

(a) $(x + 1)^2 + y^2 = 4$	(d) $x^2 + 6x + y^2 - 8y = 0$
(b) $(x - 1)^2 + (y - 2)^2 = 1$	(e) $x^2 - 10x + y^2 + 8y + 32 = 0$
(c) $x^2 - 2x + y^2 - 4y - 4 = 0$	(f) $x^2 + 14x + 14 + y^2 - 2y = 0$

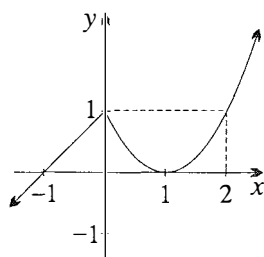
4. In each case an unknown function has been drawn. Draw the functions specified below:

(a)



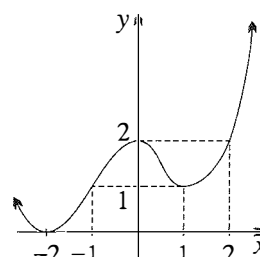
- (i) $y = f(x - 2)$ (ii) $y = f(x + 1)$

(c)



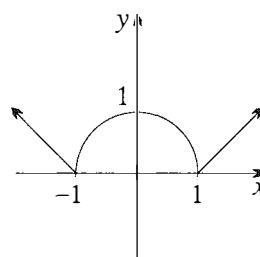
- (i) $y - 1 = h(x)$ (ii) $y = h(x) - 1$

(b)



- (i) $y = P(x + 2)$ (ii) $y = P(x + 1)$

(d)



- (i) $y - 1 = g(x)$ (ii) $y = g(x - 1)$

DEVELOPMENT

5. In each part explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each curve:

(a) From $y = 2x$:	(i) $y = 2x + 4$	(ii) $y = 2x - 4$	(iii) $y = -2x + 4$
(b) From $y = x^2$:	(i) $y = x^2 + 9$	(ii) $y = x^2 - 9$	(iii) $y = (x - 3)^2$
(c) From $y = x^2$:	(i) $y = (x + 1)^2$	(ii) $y = -(x + 1)^2$	(iii) $y = -(x + 1)^2 + 2$
(d) From $y = \sqrt{x}$:	(i) $y = \sqrt{x + 4}$	(ii) $y = -\sqrt{x + 4}$	(iii) $y = -\sqrt{x}$
(e) From $y = \frac{1}{x}$:	(i) $y = \frac{1}{x} + 1$	(ii) $y = \frac{1}{x + 2} + 1$	(iii) $y = -\frac{1}{x}$

6. Sketch $y = \frac{1}{x}$, then use the shifting procedures to sketch the following graphs. Find any x -intercepts and y -intercepts, and mark them on your graphs:

(a) $y = \frac{1}{x - 2}$	(d) $y = \frac{1}{x + 1} - 1$
(b) $y = 1 + \frac{1}{x - 2}$	(e) $y = 3 + \frac{1}{x + 2}$
(c) $y = \frac{1}{x - 2} - 2$	(f) $y = \frac{1}{x - 3} + 4$

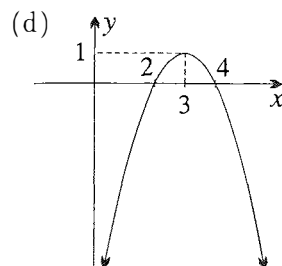
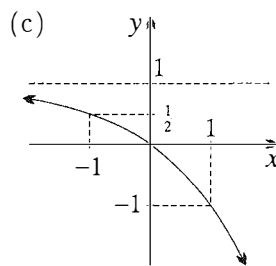
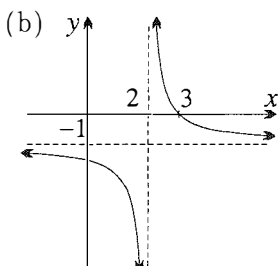
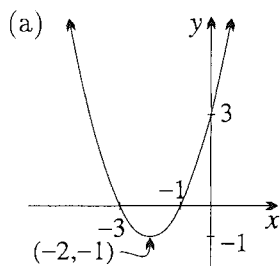
7. Complete squares, then sketch each of these circles, stating the centre and radius. By substituting $x = 0$ and then $y = 0$, find any intercepts with the axes:

(a) $x^2 - 4x + y^2 - 10y = -20$	(c) $x^2 + 4x + y^2 - 8y = 0$
(b) $x^2 + y^2 + 6y - 1 = 0$	(d) $x^2 - 2x + y^2 + 4y = 1$

8. (a) Find the equation of the image of the circle $(x - 2)^2 + (y + 1)^2 = 25$ after it has been reflected in the line $y = x$.

- (b) Find the coordinates of the points where the two circles intersect.

9. Describe each graph below as a standard curve transformed by shifts and reflections, and hence write down its equation:



10. [Revision — A medley of curve sketches] Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

(a) $y = 2x$,	$y = 2x + 3$,	$y = 2x - 1$.
(b) $y = -\frac{1}{2}x$,	$y = -\frac{1}{2}x + 1$,	$y = -\frac{1}{2}x - 2$.
(c) $y = x^2$,	$y = (x + 2)^2$,	$y = (x - 1)^2$.
(d) $x + y = 0$,	$x + y = 2$,	$x + y = -3$.
(e) $y = x^2$,	$y = 2x^2$,	$y = \frac{1}{2}x^2$.
(f) $x - y = 0$,	$x - y = 1$,	$x - y = -2$.
(g) $x^2 + y^2 = 4$,	$x^2 = 1 - y^2$,	$y^2 = 25 - x^2$.

(h) $y = 3x,$	$x = 3y,$	$y = 3x + 1,$	$x = 3y + 1.$
(i) $y = 2^x,$	$y = 3^x,$	$y = 4^x.$	
(j) $y = -x,$	$y = 4 - x,$	$y = x - 4,$	$x = -4 - y.$
(k) $y = x^2 - x,$	$y = x^2 - 4x,$	$y = x^2 + 3x.$	
(l) $(x - 1)^2 + y^2 = 1,$	$(x + 1)^2 + y^2 = 1,$	$x^2 + (y - 1)^2 = 1.$	
(m) $y = x^2 - 1,$	$y = 1 - x^2,$	$y = 4 - x^2,$	$y = -1 - x^2.$
(n) $y = (x + 2)^2,$	$y = (x + 2)^2 - 4,$	$y = (x + 2)^2 + 1.$	
(o) $y = x^2 - 1,$	$y = x^2 - 4x + 3,$	$y = x^2 - 8x + 15.$	
(p) $y = \sqrt{9 - x^2},$	$x = -\sqrt{4 - y^2},$	$y = -\sqrt{1 - x^2}.$	
(q) $y = \frac{1}{x},$	$y = 1 + \frac{1}{x},$	$y = -\frac{1}{x}.$	
(r) $y = \sqrt{x},$	$y = \sqrt{x} + 1,$	$y = \sqrt{x + 1}.$	
(s) $y = 2^x,$	$y = 2^x - 1,$	$y = 2^{x-1}.$	
(t) $y = \frac{1}{x},$	$y = \frac{1}{x - 2},$	$y = \frac{1}{x + 1}.$	
(u) $y = x^3,$	$y = x^3 + 1,$	$y = (x + 1)^3.$	
(v) $y = x^4,$	$y = (x - 1)^4,$	$y = x^4 - 1.$	
(w) $y = \sqrt{x},$	$y = -\sqrt{x},$	$y = 4 - \sqrt{x}.$	
(x) $y = 2^{-x},$	$y = 2^{-x} - 2,$	$y = 3 + 2^{-x}.$	
(y) $y = x^2,$	$x = y^2,$	$x = y^2 - 1.$	

11. Consider the straight line equation $x + 2y - 4 = 0$.

- The line is shifted 2 units left. Find the equation of the new line.
- The original line is shifted 1 unit down. Find the equation of this third line.
- Comment on your answers, and draw the lines on the same number plane.

12. Explain the point-gradient formula $y - y_1 = m(x - x_1)$ for a straight line in terms of shifts of the line $y = mx$.

EXTENSION

13. Suppose that $y = f(x)$ is a function whose graph has been drawn.

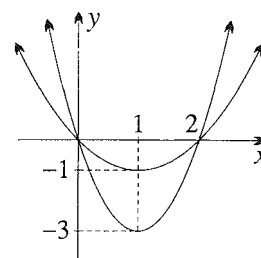
- Let $\mathcal{U}$ be shifting upwards a units and $\mathcal{H}$ be reflection in $y = 0$. Write down the equations of the successive graphs obtained by applying $\mathcal{U}$, then $\mathcal{H}$, then $\mathcal{U}$, then $\mathcal{H}$, and prove that the final graph is the same as the first. Confirm the equations by applying these operations successively to a square book or piece of paper.
- Let $\mathcal{R}$ be shifting right by a units. Write down the equations of the successive graphs obtained by applying $\mathcal{R}$, then $\mathcal{H}$, then $\mathcal{R}$, then $\mathcal{H}$, and describe the final graph. Confirm using the square book.
- Let $\mathcal{V}$ be reflection in $x = 0$ and $\mathcal{I}$ be reflection in $y = x$. Write down the equations of the successive graphs obtained by applying $\mathcal{I}$, then $\mathcal{V}$, then $\mathcal{I}$, then $\mathcal{H}$, and show that the final graph is the same as the first. Confirm using the square book.
- Write down the equations of the successive graphs obtained by applying the combination $\mathcal{I}$ -followed-by- $\mathcal{V}$ once, twice, $\dots$, until the original graph returns. Confirm using the square book.

2J Further Transformations of Known Graphs

This section deals with three further transformations of known graphs: stretching horizontally or vertically, graphing of the reciprocal of a function, and graphing of the sum or difference of two functions. These transformations are a little more difficult than shifting and reflecting, but they will prove very useful later on.

Stretching the Graph Vertically: Each value in the table below for $y = 3x(x - 2)$ is three times the corresponding value in the table for $y = x(x - 2)$. This means that the graph of $y = 3x(x - 2)$ is obtained from the graph of $y = x(x - 2)$ by stretching in the vertical direction by a factor of 3:

x	-2	-1	0	1	2	3	4
$x(x - 2)$	8	3	0	-1	0	3	8
$3x(x - 2)$	24	9	0	-3	0	9	24



Writing the transformed graph as $\frac{1}{3}y = x(x - 2)$ makes it clear that the stretching has been obtained by replacing y by $\frac{1}{3}y$.

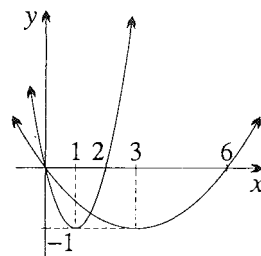
37 STRETCHING VERTICALLY: To stretch the graph in a vertical direction by a factor of a , replace y by y/a .
Alternatively, if the graph is a function, the new function rule is $y = af(x)$.

Stretching the Graph Horizontally: By analogy with the previous example, the graph of $y = x(x - 2)$ can be stretched horizontally by a factor of 3 by replacing x by $\frac{1}{3}x$, giving the new function

$$y = \frac{1}{3}x \left(\frac{1}{3}x - 2 \right) = \frac{1}{9}x(x - 6).$$

The following table of values should make this clear:

x	-3	0	3	6	9
$\frac{1}{3}x$	-1	0	1	2	3
$\frac{1}{3}x \left(\frac{1}{3}x - 2 \right)$	3	0	-1	0	3



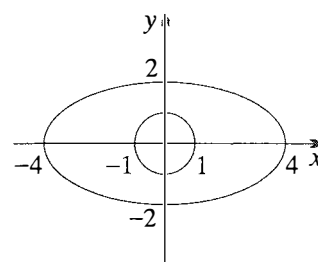
38 STRETCHING HORIZONTALLY: To stretch the graph in a horizontal direction by a factor of a , replace x by x/a .
Alternatively, if the graph is a function, the new function rule is $y = f(x/a)$.

WORKED EXERCISE: Obtain the graph of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$.

SOLUTION: The equation can be rewritten as

$$\left(\frac{x}{4} \right)^2 + \left(\frac{y}{2} \right)^2 = 1,$$

which is the unit circle stretched vertically by a factor of 2 and horizontally by a factor of 4.

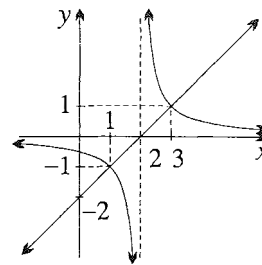


NOTE: Any curve of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called an *ellipse*. It can be obtained from the unit circle $x^2 + y^2 = 1$ by stretching horizontally by a factor of a and vertically by a factor of b , so that its x -intercepts are a and $-a$ and its y -intercepts are b and $-b$.

The Graph of the Reciprocal Function $y = 1/f(x)$: The graph of $y = x - 2$ is a line with x -intercept 2 and y -intercept -2 . The graph of the reciprocal function $y = \frac{1}{x-2}$ can be constructed from it by making a series of observations about the reciprocals of numbers:

1. When $x - 2 = 0$ (that is, when $x = 2$), then y is undefined.
2. When $x - 2 \neq 0$, then y has the same sign as $x - 2$.
3. (a) When $x - 2 = 1$, then $y = 1$.
(b) When $x - 2 = -1$, then $y = -1$.
4. (a) When $x - 2 \rightarrow \infty$, then $y \rightarrow 0^+$.
(b) When $x - 2 \rightarrow -\infty$, then $y \rightarrow 0^-$.
5. (a) When $x - 2 \rightarrow 0^+$, then $y \rightarrow \infty$.
(b) When $x - 2 \rightarrow 0^-$, then $y \rightarrow -\infty$.

x	0	1	2	3	4
$x - 2$	-2	-1	0	1	2
$\frac{1}{x - 2}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$



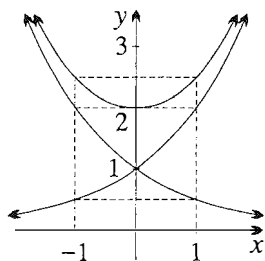
Adding and Subtracting Two Known Functions: Many functions can be written as the sum or difference of two simpler functions:

39

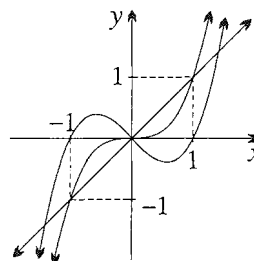
SUM AND DIFFERENCE: The graph of the sum or difference of two functions can be obtained from the graphs of the two functions by adding or subtracting the heights at each value of x .

Particularly significant are places where the heights are equal, where the heights are opposite, and where one of the heights is zero.

WORKED EXERCISE: Sketch, on one set of axes, the curves $y = 2^x$ and $y = 2^{-x}$. Hence sketch $y = 2^x + 2^{-x}$.



WORKED EXERCISE: Sketch, on one set of axes, $y = x^3$ and $y = x$. Hence sketch $y = x^3 - x$.



Exercise 2J

- On one set of axes, graph:
 - $y = x(3 + x)$, $y = 2x(3 + x)$, and $y = \frac{x}{2}(3 + \frac{x}{2})$.
 - $x^2 + y^2 = 36$, $(\frac{x}{2})^2 + (\frac{y}{3})^2 = 36$, and $(2x)^2 + (3y)^2 = 36$.
- In question 4 of Exercise 2I, the graphs of four unknown functions were drawn. Use stretching procedures to draw these new functions:

(a) (i) $y = f(2x)$	(b) (i) $y = P(\frac{x}{2})$	(c) (i) $\frac{y}{2} = h(x)$	(d) (i) $2y = g(x)$
(ii) $y = 2f(x)$	(ii) $y = \frac{1}{2}P(x)$	(ii) $y = h(\frac{x}{2})$	(ii) $y = g(2x)$
- In each part draw the first two functions on the same number plane, then add or subtract the heights as appropriate to sketch the third function. A table of values may be helpful:

(a) $y = x$, $y = x^3$, $y = x - x^3$	(c) $y = -x$, $y = 2^x$, $y = 2^x - x$
(b) $y = x^2$, $y = 2 - x$, $y = x^2 + x - 2$	(d) $y = x$, $y = \frac{1}{x}$, $y = x + \frac{1}{x}$

DEVELOPMENT

- Sketch $x + y = 1$. Then explain how each graph below may be obtained by stretchings of the first graph (there may be more than one answer), and sketch it:

(a) $\frac{x}{2} + y = 1$	(b) $\frac{x}{2} + \frac{y}{4} = 1$	(c) $2x + y = 1$
- By considering the reciprocals of numbers, graph:

(a) $y = x - 1$, and hence $y = \frac{1}{x - 1}$,
(b) $y = x + 2$, and hence $y = \frac{1}{x + 2}$,
(c) $y = 2x - 3$, and hence $y = \frac{1}{2x - 3}$.
- Sketch the first two functions on the same number plane and then add or subtract heights to sketch the remaining curves:

(a) $y = x$, $y = \frac{1}{x}$, $y = x + \frac{1}{x}$, $y = x - \frac{1}{x}$	(c) $y = x$, $y = \sqrt{x}$, $y = \sqrt{x} - x$
(b) $y = x^2$, $y = \frac{1}{x}$, $y = x^2 + \frac{1}{x}$, $y = x^2 - \frac{1}{x}$	(d) $y = \frac{1}{x}$, $y = \sqrt{x}$, $y = \frac{1}{x} - \sqrt{x}$
- Describe the transformations that have been applied to the curve $y = \sqrt{x}$ in order to obtain each equation, then sketch the given curve:

(a) $y = 2 - \sqrt{x}$	(b) $y = 2\sqrt{x} - 2$	(c) $y = \sqrt{4 - x}$
- (a) Sketch a graph of the parabola with equation $y = x^2 - 1$, showing the coordinates of the points where $y = -1$, $y = 0$ and $y = 1$.
 (b) Use the answers to part (a) and other observations about the reciprocals of numbers in order to graph $y = \frac{1}{x^2 - 1}$.
- Carefully graph $y = \sqrt{x + 1}$, and hence sketch $y = \frac{1}{\sqrt{x + 1}}$.

EXTENSION

- (a) Suggest two simple and distinct transformations for each of the following pairs of curves, by which the second equation may be obtained from the first:

(i) $y = 2^x$, $y = 2^{x+1}$	(ii) $y = \frac{1}{x}$, $y = \frac{k^2}{x}$	(iii) $y = 3^x$, $y = 3^{-x}$

 (b) Investigate other combinations of curves and transformations with similar ambiguity.
- Determine how the curve $y = x^3 - x$ must be transformed in order to obtain the graph of $y = x^3 - 3x$. [HINT: Only stretchings are involved.]

12. Consider the relation for the *Lemniscate of Bernoulli*, $(x^2 + y^2)^2 = x^2 - y^2$. Upon expansion this yields $y^4 + (2x^2 + 1)y^2 + (x^4 - x^2) = 0$, which is a quadratic equation in y^2 .

(a) Use the quadratic formula to find y^2 and hence show that y is one of the two functions

$$y = \sqrt{\frac{-(2x^2 + 1) + \sqrt{8x^2 + 1}}{2}} \quad \text{or} \quad y = -\sqrt{\frac{-(2x^2 + 1) + \sqrt{8x^2 + 1}}{2}}.$$

- (b) Explain why the domain is restricted to $-1 \leq x \leq 1$. (c) Find any x or y intercepts.
(d) Create a table of values for each function. Select x values every 0.1 units in the domain. Plot the points so found. What is the shape of the original relation? (A graphics calculator or computer may help simplify this task.)
(e) Write down the equation of this lemniscate if it is shifted right 1 unit and stretched vertically by a factor of 2, and then sketch it.

CHAPTER THREE

Graphs and Inequations

The interrelationship between algebra and graphs is the theme of this chapter. Graphs are used here to solve various equations and inequations, including those involving absolute value. Conversely, algebraic techniques are used to investigate unfamiliar graphs, resulting in a curve sketching menu that allows a systematic approach to the shapes of a wide variety of curves.

STUDY NOTES: Solving inequations and equations by using graphs is the theme of Sections 3A–3E, but Sections 3B and 3C also introduce algebraic approaches to the features of general curves. Section 3E also introduces the important absolute value function and its associated graphs. Section 3F relates regions in the coordinate plane to inequations in two variables.

The final Section 3G discusses asymptotes, and brings together four distinct features of graphs into a systematic method of sketching unknown graphs. After the derivative has been introduced, Chapter Ten will extend this menu with two further steps. If it proves too demanding at this stage, Section 3G could be delayed until Chapter Ten.

This chapter is probably the most useful place for machine drawing of curves to help clarify how the features of a graph are related to the algebraic properties of its equation, and to gain familiarity with the various graphs. An alternative approach to many questions requiring sketches would be to display the shape on a machine first and then give the required algebraic explanation. Sufficient questions should, however, be left to give practice in purely algebraic analysis.

3A Inequations and Inequalities

Statements involving the four symbols $<$ and $\leq$ and $>$ and $\geq$ occur frequently. This section begins a systematic approach to them.

There is a distinction between inequations and inequalities. A statement such as $x^2 \leq 16$ is an *inequation*; it has the solution $-4 \leq x \leq 4$, meaning that it is true for these numbers and not for any others. But a statement such as $x^2 + y^2 \geq 0$ is an *inequality*; it is true for all real numbers x and y , in the same way that an identity such as $(x - y)^2 = x^2 - 2xy + y^2$ is true for all real numbers.

The Meaning of ‘Less than’: There are both a geometric and an algebraic interpretation of the phrase ‘less than’. Suppose that a and b are real numbers.

THE GEOMETRIC INTERPRETATION OF $a < b$:

We say that $a < b$ if a is to the left of b on the number line:

1



THE ALGEBRAIC INTERPRETATION OF $a < b$:

We say that $a < b$ if $b - a$ is positive.

The first interpretation is geometrical, relying on the idea of a 'line' and of one point being 'on the left-hand side of' another. The second interpretation requires that the term 'positive number' be already understood. This second interpretation turns out to be very useful later in solving inequations and proving inequalities.

Solving Linear Inequations: As discussed in Chapter One, the rules for adding and subtracting from both sides, and for multiplying or dividing both sides, are exactly the same as for equations, with one qualification — the inequality symbol reverses when multiplying or dividing by a negative.

2

LINEAR INEQUATIONS: When multiplying or dividing both sides of an inequation by a negative, the inequality symbol is reversed.

WORKED EXERCISE:

(a) $3x - 7 \leq 8x + 18$

$+ (-8x + 7)$ $-5x \leq 25$

$\div (-5)$ $x \geq -5$



(b) $20 > 2 - 3x \geq 8$

-2 $18 > -3x \geq 6$

$\div (-3)$ $-6 < x \leq -2$



Solving Quadratic Inequations: The clearest way to solve a quadratic inequation is to sketch the graph of the associated parabola.

3

QUADRATIC INEQUATIONS: To solve a quadratic inequation, move everything to the LHS, sketch the graph of the LHS, showing the x -intercepts, then read the solution off the graph.

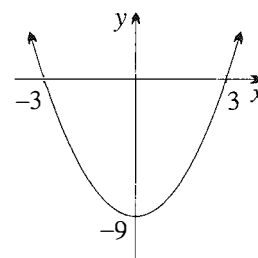
WORKED EXERCISE: Solve: (a) $x^2 > 9$ (b) $x + 6 \geq x^2$

SOLUTION:

(a) Moving everything onto the left, $x^2 - 9 > 0$
then factoring, $(x - 3)(x + 3) > 0$.

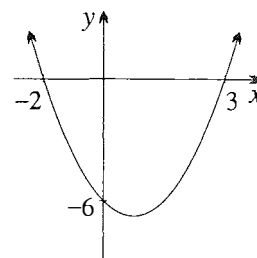
The graph must be above the x -axis,
so from the graph opposite, $x > 3$ or $x < -3$.

[This example is easy, and could be done at sight.]



(b) Moving everything onto the left, $x^2 - x - 6 \leq 0$
then factoring, $(x - 3)(x + 2) \leq 0$.

The graph must be below the x -axis,
so from the graph opposite, $-2 \leq x \leq 3$.



Solving Inequalities with a Variable in the Denominator: There is a problem with

$$\frac{5}{x-4} \geq 1.$$

The denominator $x - 4$ is sometimes positive and sometimes negative, so if both sides were multiplied by the denominator $x - 4$, the inequality symbol would reverse sometimes and not other times. The most straightforward approach is to multiply through instead by the square of the denominator, which must always be positive or zero.

4

VARIABLE IN THE DENOMINATOR: Multiply through by the *square* of the denominator, being careful to exclude the zeroes of the denominator.

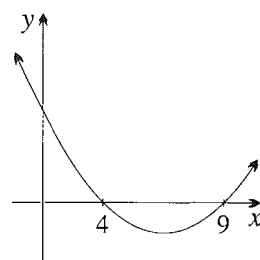
Once the fractions have been cleared, there will usually be common factors on both sides. These should *not* be multiplied out, because the factoring will be easier if they are left unexpanded.

WORKED EXERCISE: Solve $\frac{5}{x-4} \geq 1$.

SOLUTION: The key step is to multiply both sides by $(x - 4)^2$.

$$\begin{aligned} \boxed{\times (x-4)^2} \quad & 5(x-4) \geq (x-4)^2, \text{ and } x \neq 4, \\ & (x-4)^2 - 5(x-4) \leq 0, \text{ and } x \neq 4, \\ & (x-4)(x-4-5) \leq 0, \text{ } x \neq 4, \\ & (x-4)(x-9) \leq 0, \text{ } x \neq 4. \end{aligned}$$

From the diagram, $4 < x \leq 9$.



Solving Logarithmic and Exponential Inequalities: When the base is greater than 1, the exponential function and its inverse the logarithmic function are both increasing functions, so:

5

LOGARITHMIC AND EXPONENTIAL INEQUALITIES: The inequality symbol is unchanged when moving between exponential and logarithmic statements, provided the base is greater than 1.

WORKED EXERCISE: Solve: (a) $\log_5 x < 3$ (b) $-5 \leq \log_2 x \leq 5$

SOLUTION: Note that $\log x$ is only defined for $x > 0$.

(a) Changing to exponentials, (b) Changing to exponentials,

$$0 < x < 5^3$$

$$2^{-5} \leq x \leq 2^5$$

$$0 < x < 125.$$

$$\frac{1}{32} \leq x \leq 32.$$

Proving Inequalities — A. Standard Operations: There are many approaches to proving inequalities. The first of the three methods presented here uses only the standard operations.

6

PROVING INEQUALITIES (A): A proof of an inequality may proceed from a known result to the desired result using the standard operations.

WORKED EXERCISE: [A standard result which may be quoted] Prove that:

- (a) if $x > 1$, then $x^2 > x$, (b) if $0 < x < 1$, then $x^2 < x$.

SOLUTION:

- (a) Suppose that $x > 1$.

$$\boxed{\times x} \quad x^2 > x, \quad (\text{since we multiplied by a positive number}).$$

- (b) Suppose that $x < 1$, where $x > 0$.

$$\boxed{\times x} \quad x^2 < x, \quad (\text{since we multiplied by a positive number}).$$

Proving Inequalities — B. Everything on the Left: The second approach is based on the algebraic interpretation that $a < b$ means $b - a > 0$.

7

PROVING INEQUALITIES (B): To prove that $\text{LHS} < \text{RHS}$,
prove instead that $\text{RHS} - \text{LHS} > 0$.

WORKED EXERCISE: [A standard result which may be quoted]

If $0 < a < b$, prove that $\frac{1}{b} < \frac{1}{a}$, by proving that $\text{RHS} - \text{LHS} > 0$.

SOLUTION: $\text{RHS} - \text{LHS} = \frac{1}{a} - \frac{1}{b}$

$$= \frac{b - a}{ab}$$

 > 0 , since $b - a$ is positive and ab is positive.

Proving Inequalities — C. Squares Can't be Negative: The third approach uses the fact that a square can never be negative.

8

PROVING INEQUALITIES (C): Begin with a suitable statement that some square is positive or zero.

WORKED EXERCISE: Use the fact that $(x - y)^2 \geq 0$ to prove that $x^2 + y^2 \geq 2xy$, for all real numbers x and y .

SOLUTION: We know that $(x - y)^2 \geq 0$, for all x and y .

Expanding this, $x^2 - 2xy + y^2 \geq 0$, for all x and y ,

$$\boxed{+ 2xy} \quad x^2 + y^2 \geq 2xy, \quad \text{for all } x \text{ and } y.$$

Exercise 3A

1. Solve, and graph on the number line, the solutions of:

(a) $x > 1$	(d) $2x < 6$	(g) $3x - 1 < 5$	(j) $2 - 3x \geq 8$
(b) $x \leq 2$	(e) $x + 4 \geq 3$	(h) $5 - 2x \leq -1$	(k) $\frac{1}{3}x - 1 > -\frac{1}{3}$
(c) $-2x < 4$	(f) $3 - x > 1$	(i) $5x - 5 \geq 10$	(l) $\frac{1}{4}x + 2 \leq 1\frac{1}{2}$

2. Solve the following double inequations, and sketch the solutions on the number line:

(a) $-8 \leq 4x < 12$	(c) $-2 \leq 2x - 1 \leq 3$
(b) $4 < 3x \leq 15$	(d) $-1 \leq 4x - 3 < 13$

3. Solve these inequations:

(a) $2x + 3 > x + 7$

(c) $2 - x > 2x - 4$

(e) $2 < 3 - x \leq 5$

(b) $3x - 2 \leq \frac{1}{2}x + 3$

(d) $1 - 3x \geq 2 - 2x$

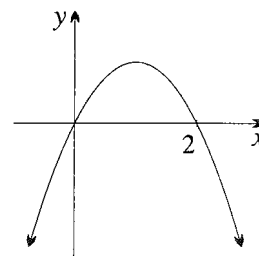
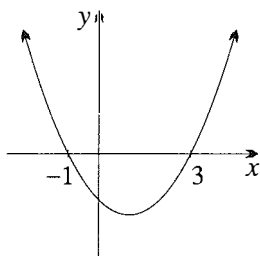
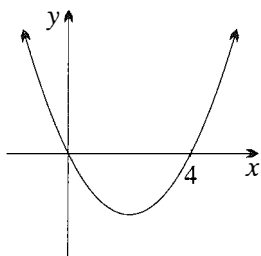
(f) $-4 \leq 1 - \frac{1}{3}x \leq 3$

4. Use the given graph of the LHS to help solve each inequation:

(a) $x(x - 4) < 0$

(b) $(x - 3)(x + 1) \geq 0$

(c) $x(2 - x) \leq 0$



5. Draw the associated parabola and hence solve:

(a) $(x + 2)(x - 4) < 0$

(c) $(2 - x)(x - 5) \geq 0$

(e) $(2x - 1)(x - 5) > 0$

(b) $(x - 3)(x + 1) > 0$

(d) $(x + 1)(x + 3) \geq 0$

(f) $(3x + 5)(x + 4) \leq 0$

6. Factor the LHS and draw an appropriate parabola in order to solve:

(a) $x^2 + 2x - 3 < 0$

(c) $x^2 + 6x + 8 > 0$

(e) $2x^2 - x - 3 \leq 0$

(b) $x^2 - 5x + 4 \geq 0$

(d) $x^2 - x - 6 \leq 0$

(f) $4 + 3x - x^2 > 0$

7. Collect terms on one side, factor and sketch the associated parabola, and hence solve:

(a) $x^2 \leq 1$

(c) $x^2 \geq 144$

(e) $x^2 + 9 \leq 6x$

(b) $x^2 > 3x$

(d) $x^2 > 0$

(f) $4x - 3 \geq x^2$

8. Multiply through by the square of the denominator and hence solve:

(a) $\frac{1}{x} \leq 2$

(c) $\frac{3}{x + 4} \geq 2$

(e) $\frac{2}{3 - x} > 1$

(b) $\frac{2}{x - 3} > 1$

(d) $\frac{5}{2x - 3} < 1$

(f) $\frac{4}{5 - 3x} \leq -1$

9. Draw a sketch of the curve $y = 2^x$ and the line $y = -1$. Hence explain why the inequation $2^x \leq -1$ has no solutions.

10. State whether these are true or false, and if false, give a counterexample:

(a) $x^2 > 0$

(c) $2^x > 0$

(e) $2x \geq x$

(g) $x \geq -x$

(b) $x^2 \geq x$

(d) $x \geq \frac{1}{x}$

(f) $x + 2 > x$

(h) $2x - 3 > 2x - 7$

11. Given that $x - y > y - z$, prove that $y < \frac{1}{2}(x + z)$.

12. If $a > b$ and $b \neq 0$, prove: (a) $-a < -b$ (b) $ab^2 > b^3$

DEVELOPMENT

13. Multiply through by the square of the denominator and hence solve:

(a) $\frac{5x}{2x - 1} \geq 3$

(b) $\frac{2x + 5}{x + 3} < 1$

(c) $\frac{x + 1}{x - 1} \leq 2$

(d) $\frac{4x + 7}{x - 2} > -3$

14. Draw $y = 2x - 1$ and $y = 2x + 3$ on the same number plane, and hence explain why the inequation $2x - 1 \leq 2x + 3$ is true for all real values of x .

15. (a) Draw $y = 1 - x$, $y = 2$ and $y = -1$ on the same number plane and find the points of intersection.

(b) Solve the inequation $-1 < 1 - x \leq 2$, and relate the answer to the graph.

16. Write down and solve a suitable inequation to find where the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.
17. Solve the following inequations involving logarithms and exponentials:
- (a) $3^x \geq 27$ (c) $\frac{1}{16} \leq 2^x \leq 16$ (e) $\log_2 x < 3$
 (b) $1 < 5^x \leq 125$ (d) $2^{-x} > 16$ (f) $-2 \leq \log_5 x \leq 4$
18. State whether these are true or false, and if false, give a counterexample:
- (a) If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$. (d) If $a < b$ and $a, b \neq 0$, then $\frac{1}{a} > \frac{1}{b}$.
 (b) If $a < b$, then $a^2 < b^2$. (e) If $a < b$, then $-a > -b$.
 (c) If $a^2 + b^2 = 0$, then $a = b = 0$. (f) If $0 < a < b$, then $\sqrt{a^2 + b^2} = a + b$.
19. If $-1 \leq t < 3$, what is the range of values of:
- (a) $4t$ (c) $t + 7$ (e) $\frac{1}{2}(t + 1)$ (g) 2^t
 (b) $-t$ (d) $2t - 1$ (f) $\frac{1}{2}(3t - 1)$ (h) $\sqrt{t + 1}$
20. What range of values may $x^2 + 3$ take if: (a) $2 < x < 4$ (b) $-1 < x \leq 3$
21. (a) Given that $x < y < 0$, show that $xy > y^2$.
 (b) Suppose $x > y > 0$. (i) Show that $x^2 > y^2$. (ii) For what values of n is $x^n > y^n$?
22. In the notes it was proven that $x^2 + y^2 \geq 2xy$. Use this result and appropriate substitutions to prove: (a) $a + \frac{1}{a} \geq 2$, for $a > 0$, (b) $\frac{a+b}{2} \geq \sqrt{ab}$, for a and b both positive.

EXTENSION

23. Prove that $x^2 + xy + y^2 > 0$ for any non-zero values of x and y .
24. (a) Prove that $(x + y)^2 \geq 4xy$. (b) Hence prove that $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{4}{x^2 + y^2}$.
25. (a) Expand $(a - b)^2 + (b - c)^2 + (a - c)^2$, and hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.
 (b) Expand $(a + b + c)((a - b)^2 + (b - c)^2 + (a - c)^2)$, and hence prove the identity $a^3 + b^3 + c^3 \geq 3abc$, for positive a, b and c .

3B Intercepts and Sign

When an unknown graph is being sketched, it is important to know the x -intercepts or zeroes — usually factoring is required for this. If the zeroes can be found, a table of test points can then determine where the graph is above the x -axis and where it is below the x -axis. Most functions in this section are *polynomials*, meaning that they can be written as a sum of multiples of powers of x , like $y = 3x^3 - 2x^2 + 7x + 1$.

The x - and y -intercepts: The places where the graph meets the x -axis and the y -axis are found by putting the other variable equal to zero.

9

THE x - AND y -INTERCEPTS: To find the y -intercept, substitute $x = 0$.
 To find the x -intercept, substitute $y = 0$.

The x -intercepts are also called the *zeroes* of the function. Finding them will usually involve factoring the function.

WORKED EXERCISE: Find the x - and y -intercepts of $y = x^3 - x^2 - x + 1$.

SOLUTION: When $x = 0$, $y = 1$ (this is the y -intercept).

Factoring by grouping, $y = x^2(x - 1) - (x - 1)$

$$= (x^2 - 1)(x - 1)$$

$$= (x + 1)(x - 1)^2,$$

so $y = 0$ when $x = -1$ or 1 (these are the x -intercepts).

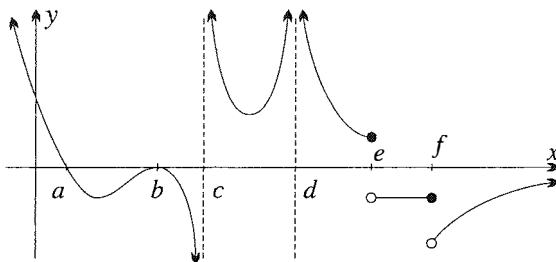
The complete graph is sketched in the next worked exercise.

The Sign of the Function: Given the zeroes, the sign of the function as x varies can be found by using a set of test values. This method requires a major theorem called the *intermediate value theorem*.

10

THE INTERMEDIATE VALUE THEOREM: The only places where a function may possibly change sign are zeroes and discontinuities.

The word *discontinuity* needs explanation. Informally, a function is said to be *continuous* at a point if its graph can be drawn through the point without lifting the pen off the paper — otherwise there is a *discontinuity* at the point. In the graph below, there are discontinuities at $x = c, d, e$ and f .



PROOF: This theorem goes to the heart of what the real numbers are and what continuity means, but for this course an example will be sufficient. The function sketched above changes sign at the zero at $x = a$ and at the discontinuities at $x = c$ and $x = e$. Notice that the function does not change sign at the zero at $x = b$ or at the discontinuities at $x = d$ and $x = f$.

11

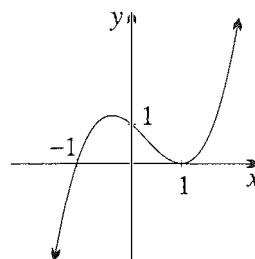
EXAMINING THE SIGN OF A FUNCTION: To examine the sign of a function, draw up a table of test values around any zeroes and discontinuities.

WORKED EXERCISE: Examine the sign of $y = (x + 1)(x - 1)^2$, and sketch the graph.

SOLUTION: There are zeroes at 1 and -1 , and no discontinuities.

x	-2	-1	0	1	2
y	-9	0	1	0	3
sign	-	0	+	0	+

So y is positive for $-1 < x < 1$ or $x > 1$,
and y is negative for $x < -1$.



WORKED EXERCISE: [A harder example] Examine the sign of $y = \frac{x-1}{x-4}$.

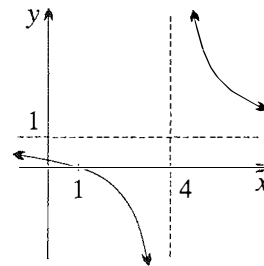
SOLUTION: Here y has a zero at $x = 1$,
but there is also a discontinuity at $x = 4$.

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	-	*	+

So y is positive for $x < 1$ or $x > 4$, and is negative for $1 < x < 4$.

[The graph is drawn opposite, but it won't be explained until Section 3G.]

NOTE: This procedure is unnecessary for many functions whose sign is more easily established. For example, the function $y = \frac{1}{1+x^2}$ must always be positive, since $x^2 + 1$ is always at least 1.



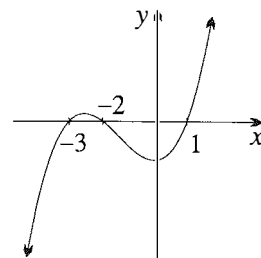
Solving Rational Inequalities Involving Cubics: The following rational inequation is solved in the usual way by multiplying through by $(x+2)^2$ to give a cubic inequation. Notice how the resulting common factor $(x+2)$ is never multiplied out.

WORKED EXERCISE: [A harder example involving a cubic graph]

Solve $\frac{3}{x+2} \leq x$.

SOLUTION: $\boxed{\times (x+2)^2}$ $3(x+2) \leq x(x+2)^2$, and $x \neq -2$,
 $x(x+2)^2 - 3(x+2) \geq 0$, and $x \neq -2$,
 $(x+2)(x^2 + 2x - 3) \geq 0$, and $x \neq -2$,
 $(x+2)(x+3)(x-1) \geq 0$, and $x \neq -2$.

From the diagram, $x \geq 1$ or $-3 \leq x < -2$.



An Alternative Approach to Rational Inequalities: The previous inequation could also be solved by collecting all terms on the left and using the methods of 'intercepts and sign'.

SOLUTION: The given inequation is $\frac{3}{x+2} \leq x$.

Collecting everything on the left, $\frac{3}{x+2} - x \leq 0$,

using a common denominator, $\frac{3 - x^2 - 2x}{x+2} \leq 0$,

and factoring, $\frac{(3+x)(1-x)}{x+2} \leq 0$.

The LHS has zeroes at $x = -3$ and $x = 1$, and a discontinuity at $x = -2$.

x	-4	-3	$-2\frac{1}{2}$	-2	0	1	2
LHS	$2\frac{1}{2}$	0	$-3\frac{1}{2}$	*	$1\frac{1}{2}$	0	$-1\frac{1}{4}$
sign	+	0	-	*	+	0	-

So the solution is $x \geq 1$ or $-3 \leq x < -2$.

Exercise 3B

1. Explain why the zeroes of $y = (x + 1)^2(1 - x)$ are $x = 1$ and $x = -1$. Then copy and complete the table of values and sketch the graph.

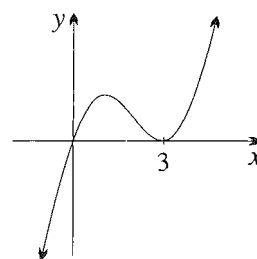
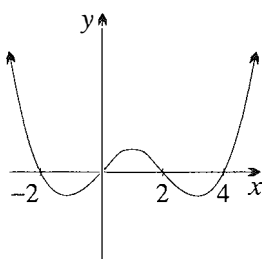
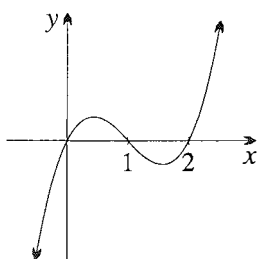
x	-2	-1	0	1	2
y					
sign					

2. Apply the methods used in the previous question to sketch the following quadratics, cubics and quartics. Mark all x - and y -intercepts.

(a) $y = (x + 1)(x + 3)$ (c) $y = (x - 1)(x + 2)^2$ (e) $y = (x - 2)x(x + 2)(x + 4)$
 (b) $y = x(x - 2)(x - 4)$ (d) $y = x(x - 2)(x + 2)$ (f) $y = (x - 1)^2(x - 3)^2$

3. Use the given graph of the LHS to help solve each inequation:

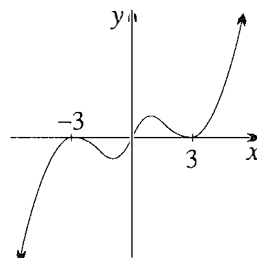
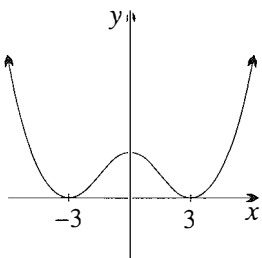
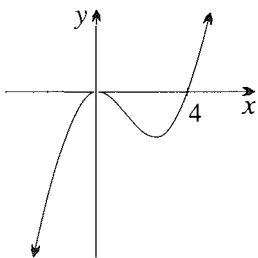
(a) $x(x - 1)(x - 2) \leq 0$ (b) $x(x + 2)(x - 2)(x - 4) < 0$ (c) $x(x - 3)^2 > 0$



(d) $x^2(x - 4) \geq 0$

(e) $(x - 3)^2(x + 3)^2 \leq 0$

(f) $x(x - 3)^2(x + 3)^2 \geq 0$



4. First factor each polynomial completely, then use the methods of the first two questions to sketch its graph (take out any common factors first):

(a) $f(x) = x^3 - 4x$ (b) $f(x) = x^3 - 5x^2$ (c) $f(x) = x^3 - 4x^2 + 4x$

5. From the graphs in the previous question, or from the tables of values used to construct them, solve the following inequations. Begin by getting all terms onto the one side:

(a) $x^3 > 4x$ (b) $x^3 < 5x^2$ (c) $x^3 + 4x \leq 4x^2$

DEVELOPMENT

6. If necessary, collect all terms on the LHS and factor. Then solve the inequation by finding any zeroes and discontinuities and drawing up a table of test values around them:

(a) $(x - 1)(x - 3)(x - 5) < 0$ (d) $x^3 > 9x$ (g) $x^4 \geq 5x^3$
 (b) $(x - 1)^2(x - 3)^2 > 0$ (e) $\frac{x + 3}{x + 1} < 0$ (h) $\frac{x^2 - 4}{x} \geq 0$
 (c) $\frac{x - 4}{x + 2} \leq 0$ (f) $\frac{x^2}{x - 5} < 0$ (i) $\frac{x - 2}{x^2 + 3x} \leq 0$

7. Factor each equation completely, and hence find the x -intercepts of the graph (factor parts (b) and (c) by grouping in pairs):

(a) $y = x^3 - x$ (b) $y = x^3 - 2x^2 - x + 2$ (c) $y = x^3 + 2x^2 - 4x - 8$

8. For each function in the previous question, examine the sign of the function around each zero and hence draw a graph of the function.

9. Find all zeroes of these functions, and any values of x where the function is discontinuous. Then analyse the sign of the function by taking test points around these zeroes and discontinuities:

$$(a) f(x) = \frac{x}{x-3}$$

$$(b) f(x) = \frac{x-4}{x+2}$$

$$(c) f(x) = \frac{x+3}{x+1}$$

10. Multiply through by the square of the denominator, collect all terms on one side and then factor to obtain a factored cubic. Sketch this cubic by examining the intercepts and the sign. Hence solve the original inequality:

$$(a) \frac{4}{x+3} \geq x$$

$$(b) \frac{2}{2x+3} < x$$

$$(c) \frac{8}{2x-3} \leq 2x-1$$

11. Solve these inequations by the alternative method of collecting everything on the LHS, finding a common denominator, factoring, and drawing up a table of test values. (This alternative method could also be applied to the relevant questions in Exercise 3A.)

EXTENSION

12. (a) Prove that $f(x) = 1 + x + x^2$ is positive for all x .
 (b) Prove that $f(x) = 1 + x + x^2 + x^3 + x^4$ is positive for all x . You will probably need to consider separately the three cases $x \geq 0$, $x \leq -1$ and $-1 < x < 0$.
 (c) Similarly prove that for all positive integers n , $f(x) = 1 + x + x^2 + \cdots + x^{2n-1} + x^{2n}$ is positive for all x .
 (d) Prove that $x = -1$ is the only zero of $f(x) = 1 + x + x^2 + \cdots + x^{2n-1}$.

3C Domain and Symmetry

A systematic investigation of an unknown curve should normally begin with a consideration of three features important in most graphs — the domain, whether the function has even or odd symmetry, and the analysis of its intercepts and sign explained in Section 3B. These features will become the first three steps of a systematic curve sketching menu at the end of this chapter.

Domain: Warning — no work should be done on an unfamiliar function without first finding its domain. If no specific restriction has been made, use the natural domain, which, as mentioned in Section 2F, is the set of all values of x that can be validly substituted into the equation of the function.

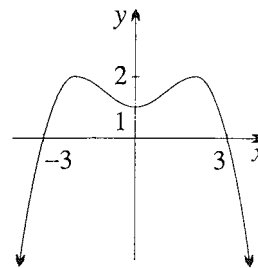
WORKED EXERCISE: Find the domain of $y = \frac{1}{\sqrt{4-x^2}}$.

SOLUTION: The square root can only exist when $-2 \leq x \leq 2$. Zero, however, has no reciprocal, so the domain of the function is $-2 < x < 2$.

Even Functions and Symmetry in the y -axis: It has been said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

First, a graph with line symmetry in the y -axis is called **even**. This means that the graph is unchanged by reflection in the y -axis, like the graph on the right.

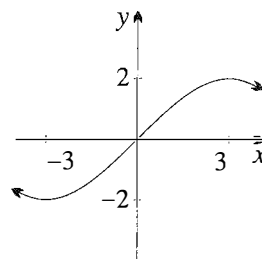
For any function $f(x)$, the graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the y -axis. So the function is even if the graphs of $f(x)$ and $f(-x)$ coincide, that is, if the function satisfies the identity $f(-x) = f(x)$.



- 12** **EVEN FUNCTIONS:** $f(x)$ is called **even** if $f(-x) = f(x)$, for all x in its domain.
A function is even if and only if its graph has line symmetry in the y -axis.

Odd Functions and Symmetry in the Origin: Secondly, a graph with point symmetry in the origin is called **odd**. This means that the graph is unchanged by a rotation of 180° about the origin, or equivalently by successive reflections in the x -axis and the y -axis.

Reflecting $y = f(x)$ in the x - and y -axis gives $y = -f(x)$ and $y = f(-x)$ respectively. So $f(x)$ is odd if the graphs of $f(-x)$ and $-f(x)$ coincide, that is, if $f(-x) = -f(x)$.



- 13** **ODD FUNCTIONS:** $f(x)$ is called **odd** if $f(-x) = -f(x)$, for all x in its domain.
A function is odd if and only if its graph has point symmetry in the origin.

WORKED EXERCISE: Test these functions for evenness or oddness, then sketch them:

- (a) $f(x) = x^4 - 3$ (b) $f(x) = x^3$ (c) $f(x) = x^2 - 2x$

- 14** **TESTING FOR EVENNESS AND ODDNESS:** To test whether a function is even or odd or neither, work out $f(-x)$ and compare it with $f(x)$.
Most functions are neither even nor odd.

SOLUTION:

$$\begin{aligned} \text{(a)} \quad f(x) &= x^4 - 3, \\ \text{so } f(-x) &= (-x)^4 - 3 \\ &= x^4 - 3 \\ &= f(x). \end{aligned}$$

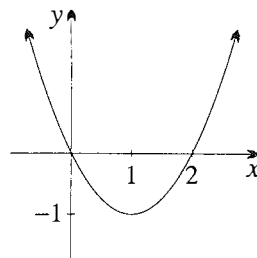
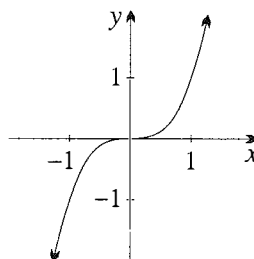
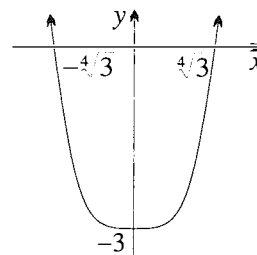
Hence $f(x)$ is an even function.

$$\begin{aligned} \text{(b)} \quad f(x) &= x^3, \\ \text{so } f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x). \end{aligned}$$

Hence $f(x)$ is an odd function.

$$\begin{aligned} \text{(c)} \quad f(x) &= x^2 - 2x, \\ \text{so } f(-x) &= (-x)^2 - 2(-x) \\ &= x^2 + 2x. \end{aligned}$$

Since $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.



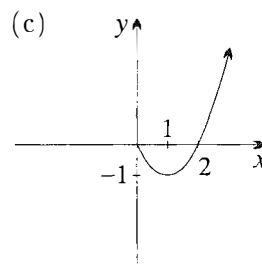
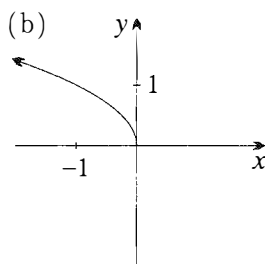
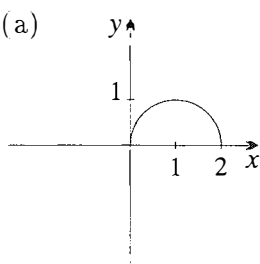
NOTE: The last curve is a parabola, and has line symmetry about its axis of symmetry, the line $x = 1$. The test described here, however, is only a test for two particular types of symmetry, not for symmetry in general. One other type of symmetry has been touched on in Section 2H, in that a curve is symmetric in the line $y = x$ if its inverse is the same as itself, which means algebraically that the equation is unchanged when x and y are exchanged.

Exercise 3C

1. What are the natural domains of the following functions of x ?

(a) $\frac{1}{x+1}$	(c) $x^3 + x + 1$	(e) $\sqrt{x}$	(g) $\sqrt{7-x}$
(b) $\frac{2}{2x-3}$	(d) $x^2 - x + 1$	(f) $\sqrt{x-1}$	(h) $\sqrt{x+4}$

2. For each graph below, complete the graph so that: (i) $f(x)$ is even, (ii) $f(x)$ is odd.



3. Pick up a book, reflect it in the vertical axis and then in the horizontal axis, and observe that it is now rotated 180° from its original position. Hence explain why the graph of $y = -f(-x)$ is the graph of $y = f(x)$ rotated 180° .

4. Simplify $f(-x)$ for each function, and hence determine whether it is odd, even or neither:

(a) $f(x) = x^2 - 9$	(e) $f(x) = x^3 + 5x^2$
(b) $f(x) = x^2 - 6x + 5$	(f) $f(x) = x^5 - 16x$
(c) $f(x) = x^3 - 25x$	(g) $f(x) = x^5 - 8x^3 + 16x$
(d) $f(x) = x^4 - 4x^2$	(h) $f(x) = x^4 + 3x^3 - 9x^2 - 27x$

Deduce a relationship between the powers of x and the answers for the previous parts.

5. Factor each polynomial in the previous question and write down its zeroes. Then use a table of test points to sketch its graph. Confirm that the graph exhibits the symmetry established above.

DEVELOPMENT

6. Determine the natural domains of the following functions of x :

(a) $\frac{x-4}{x-1}$	(c) $\frac{x^2-1}{x+1}$	(e) $\frac{1}{\sqrt{x+4}}$	(g) $3^x + 3$
(b) $\frac{x-1}{x-4}$	(d) $\frac{(x+1)(x-2)}{(x-2)}$	(f) $\frac{4}{\sqrt{x-1}}$	(h) $\frac{1}{2^x - 8}$

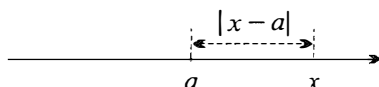
7. (a) Solve $x^2 - 4 \geq 0$, and hence write down the natural domain of $\sqrt{x^2 - 4}$.

(b) Solve $x^2 - 4 > 0$, and hence write down the natural domain of $\frac{5}{\sqrt{x^2 - 4}}$.

Distance Between Numbers: Replacing x by $x - a$ in the previous definition gives a measure of the distance from a on the number line.

16

DISTANCE BETWEEN NUMBERS: The distance from x to a on the number line is $|x - a|$.



Solving Equations and Inequalities on the Number Line: Most equations and inequalities involving absolute values in this course are simple enough to be solved using distances on the number line. More complicated equations may require the graphical methods of Section 3E.

17

METHOD FOR SOLVING SIMPLE ABSOLUTE VALUE (IN)EQUALITIES:

1. Force the equation or inequality into one of the following forms:

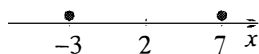
$$|x - a| = b, \text{ or } |x - a| < b, \text{ or } |x - a| \geq b \text{ or } \dots$$

2. Find the solution using distance on a number line.

WORKED EXERCISE:

(a) $|x - 2| = 5$

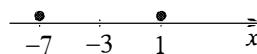
(distance from x to 2) = 5



so $x = -3$ or $x = 7$.

(b) $|x + 3| = 4$

(distance from x to -3) = 4

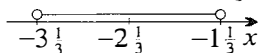


so $x = -7$ or $x = 1$.

(c) $|3x + 7| < 3$

$\boxed{\div 3} \quad |x + 2\frac{1}{3}| < 1$

(distance from x to $-2\frac{1}{3}$) < 1

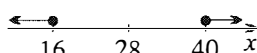


so $-3\frac{1}{3} < x < -1\frac{1}{3}$.

(d) $|7 - \frac{1}{4}x| \geq 3$

$\boxed{\times 4} \quad |28 - x| \geq 12$

(distance from x to 28) ≥ 12



so $x \leq 16$ or $x \geq 40$.

An Alternative Method: If $a \geq 0$ is a constant, then the statement $|f(x)| = a$ means that $f(x) = a$ or $f(x) = -a$. This gives a useful alternative method.

18

AN ALTERNATIVE METHOD: Suppose that $a \geq 0$ and $f(x)$ is a function of x .

1. To solve $|f(x)| = a$, write ' $f(x) = a$ or $f(x) = -a$ '.

2. To solve $|f(x)| < a$, write ' $-a < f(x) < a$ '.

3. To solve $|f(x)| > a$, write ' $f(x) > a$ or $f(x) < -a$ '.

WORKED EXERCISE:

(a) $|x - 2| = 5$

$x - 2 = 5$ or $x - 2 = -5$

$x = -3$ or $x = 7$

(b) $|3x + 7| < 3$

$-3 < 3x + 7 < 3$

$\boxed{-7} \quad -10 < 3x < -4$

$\boxed{\div 3} \quad -3\frac{1}{3} < x < -1\frac{1}{3}$

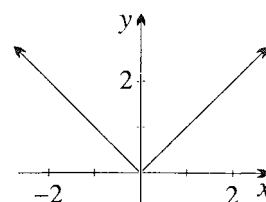
$$\begin{array}{ll}
 \text{(c)} & |7 - \frac{1}{4}x| \geq 3 \\
 & 7 - \frac{1}{4}x \geq 3 \text{ or } 7 - \frac{1}{4}x \leq -3 \\
 & \boxed{-7} \quad -\frac{1}{4}x \geq -4 \text{ or } -\frac{1}{4}x \leq -10 \\
 & \boxed{\times (-4)} \quad x \leq 16 \text{ or } x \geq 40 \\
 \text{(d)} & |\log_2 x| > 3 \\
 & \log_2 x > 3 \text{ or } \log_2 x < -3 \\
 & x > 8 \text{ or } 0 < x < \frac{1}{8}
 \end{array}$$

An Expression for Absolute Value Involving Cases: The absolute value of a negative number is the opposite of that number. Since the opposite of x is $-x$:

19 **ABSOLUTE VALUE INVOLVING CASES:** $|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$

This expression, with its two cases, allows us to draw the graph of $y = |x|$. Alternatively, a table of values makes clear the sharp point at the origin where the two branches meet at right angles:

x	-2	-1	0	1	2
$ x $	2	1	0	1	2



The domain is the set of all real numbers, and the range is $y \geq 0$. The function is even, the graph having line symmetry in the y -axis. The function has a zero at $x = 0$, and is positive for all other values of x .

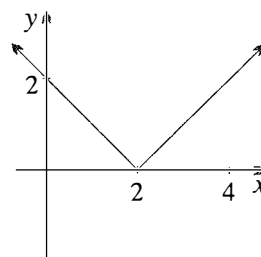
Graphing Functions with Absolute Value: The transformations of the last chapter can now be applied to the graph of $y = |x|$ in order to sketch many functions involving absolute value. The expression involving cases, however, is needed to establish the equations of the separate branches. More complicated functions often require an approach through cases.

WORKED EXERCISE: Sketch $y = |x - 2|$.

SOLUTION: This is just $y = |x|$ shifted 2 units to the right, or it is $y = x - 2$ with the bit under the x -axis reflected back above the x -axis.

Alternatively, from the expression using cases,

$$y = \begin{cases} x - 2, & \text{for } x \geq 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$

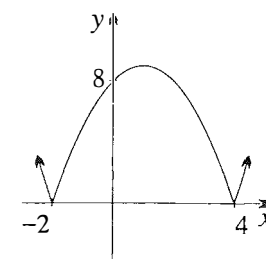


WORKED EXERCISE: Sketch $y = |x^2 - 2x - 8|$.

SOLUTION: Since $x^2 - 2x - 8 = (x - 4)(x + 2)$, this is $y = (x - 4)(x + 2)$, with the bit under the x -axis reflected back above the x -axis.

Alternatively, using cases,

$$y = \begin{cases} (x - 4)(x + 2), & \text{for } x \leq -2 \text{ or } x \geq 4, \\ -(x - 4)(x + 2), & \text{for } -2 < x < 4. \end{cases}$$



WORKED EXERCISE: [A harder example] Sketch $y = |x - 2| - |x + 4| + 1$.

SOLUTION: Considering the terms separately,

$$|x - 2| = \begin{cases} x - 2, & \text{for } x \geq 2, \\ -x + 2, & \text{for } x < 2, \end{cases}$$

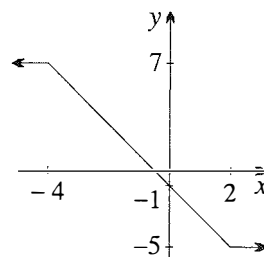
and $|x + 4| = \begin{cases} x + 4, & \text{for } x \geq -4, \\ -x - 4, & \text{for } x < -4. \end{cases}$

There are therefore three possible cases:

for $x \geq 2$,
$$y = (x - 2) - (x + 4) + 1 = -5,$$

for $-4 \leq x < 2$,
$$y = (-x + 2) - (x + 4) + 1 = -2x - 1,$$

and for $x < -4$,
$$y = (-x + 2) - (-x - 4) + 1 = 7.$$



Absolute Value as the Square Root of the Square: Taking the absolute value of a number can be thought of as separating the number into sign and size, stripping the sign, and replacing it by a positive sign. There already is an algebraic function capable of doing this job:

20

ABSOLUTE VALUE AS THE POSITIVE SQUARE ROOT OF THE SQUARE:

For all real numbers x , $|x|^2 = x^2$ and $|x| = \sqrt{x^2}$.

These examples with $x = -3$ should make this clear:

$$|-3|^2 = 9 = (-3)^2 \quad \text{and} \quad |-3| = \sqrt{9} = \sqrt{(-3)^2}.$$

Identities Involving Absolute Value: Here are some standard identities involving absolute value, to be proven in the following exercise.

21

IDENTITIES INVOLVING ABSOLUTE VALUE:

1. $|-x| = |x|$, for all x
2. $|x - y| = |y - x|$, for all x and y
3. $|xy| = |x||y|$, for all x and y
4. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, for all x , and for all $y \neq 0$

Proof is best done using $|x|^2 = x^2$. For example, here is the proof of the third identity:

$$\text{LHS}^2 = (xy)^2 = x^2y^2, \quad \text{RHS}^2 = x^2y^2 = \text{LHS}^2.$$

Since neither LHS nor RHS is negative, it follows that $\text{LHS} = \text{RHS}$.

Inequalities Involving Absolute Value: Here are some important inequalities.

22

INEQUALITIES INVOLVING ABSOLUTE VALUE:

1. $|x| \geq 0$, for all x
2. $-|x| \leq x \leq |x|$, for all x
3. $|x + y| \leq |x| + |y|$, for all x and y

The first is clear. The second needs separate consideration of positive and negative values of x . The third (called the *triangle inequality*) can be proven using $|x|^2 = x^2$:

$$\text{LHS}^2 = (x + y)^2 = x^2 + y^2 + 2xy, \quad \text{RHS}^2 = x^2 + y^2 + 2|xy|,$$

Since $xy \leq |xy|$ (by 2), so $\text{LHS}^2 \leq \text{RHS}^2$. But LHS and RHS are non-negative, so $\text{LHS} \leq \text{RHS}$.

Exercise 3D

1. (a) Copy and complete these tables of values of the functions $y = |x - 2|$ and $y = |x| - 2$:

x	-1	0	1	2	3
$ x - 2 $					

x	-1	0	1	2	3
$ x - 2$					

- (b) Draw the graphs of the two functions on the same number plane and observe the differences between them. How is each graph obtained by shifting $y = |x|$?

2. Evaluate:

(a) $ 5 $	(c) $ 14 - 9 - 12 $	(e) $ 4 - 7 $	(g) $ 3^2 - 5^2 $
(b) $ -3 $	(d) $ 7 - 4 $	(f) $\sqrt{(4 - 7)^2}$	(h) $ 11 - 16 - 8$

3. Solve the following equations:

(a) $ x = 3$	(c) $ x - 3 = 7$	(e) $ 2x - 1 = 11$	(g) $ 5x + 2 = 9$
(b) $ 2x = 10$	(d) $ x + 1 = 6$	(f) $ 3x + 2 = 8$	(h) $ 7x - 3 = 11$

4. State whether these are true or false, and if false, give a counterexample (difficulties will usually involve negative numbers):

(a) $ x > 0$	(c) $- x \leq x \leq x $	(e) $ 7x = 7 x $	(g) $ x ^3 = x^3$
(b) $ -x = x $	(d) $ x + 2 = x + 2$	(f) $ x ^2 = x^2$	(h) $ x - 7 = 7 - x $

5. Use either transformations or a table of values to obtain each graph from the graph of $y = |x|$. Also write down the equations of the two branches in each case.

(a) $y = 2x $	(c) $y = x - 1 $	(e) $y = x - 1$	(g) $y = 2 - x $
(b) $y = \frac{1}{2}x $	(d) $y = x + 3 $	(f) $y = x + 3$	(h) $y = 2 - x $

6. Explain why $|x| = c$ has no solution if the constant c is negative.

7. Use the fact that $|-x| = |x|$ to decide whether these functions are odd, even or neither:

(a) $f(x) = x + 1$	(b) $f(x) = x + x$	(c) $f(x) = x \times x $	(d) $f(x) = x^3 - x $

8. Solve the following inequations and graph the solutions on the number line:

(a) $ x - 2 < 3$	(c) $ x - 7 \geq 2$	(e) $ 6x - 7 > 5$
(b) $ 3x - 5 \leq 4$	(d) $ 2x + 1 < 3$	(f) $ 5x + 4 \geq 6$

DEVELOPMENT

9. (a) (i) Sketch a graph of $y = (x - 3)(x - 1)$. (ii) Hence obtain the graph of $y = |x^2 - 4x + 3|$ by reflecting in the x -axis those parts of the parabola that are below the x -axis.

- (b) Similarly sketch graphs of:

(i) $y = x^2 - x - 2 $	(ii) $y = 2x^2 - 5x - 3 $	(iii) $y = (x - 1)x(x + 1) $

10. (a) Explain why the double inequation $2 \leq |x| \leq 6$ is equivalent to $2 \leq x \leq 6$ or $-6 \leq x \leq -2$.
 (b) Similarly solve: (i) $2 < |x + 4| < 6$ (ii) $1 \leq |2x - 5| < 4$
11. (a) Where is $y = \frac{|x|}{x}$ undefined?
 (b) Use a table of values from $x = -3$ to $x = 3$ to sketch the graph.
 (c) Hence write down the equations of the two branches of $y = \frac{|x|}{x}$.
12. (a) Simplify $y = |x| + x$, for $x \geq 0$ and for $x < 0$, then sketch.
 (b) Simplify $y = |x| - x$, for $x \geq 0$ and for $x < 0$, then sketch.
13. State whether these are true or false. If false, give a counterexample. If true, provide examples with: (i) $x > 0$ and $y > 0$, (ii) $x > 0$ and $y < 0$, (iii) $x < 0$ and $y > 0$, (iv) $x < 0$ and $y < 0$.
 (a) $|x + y| = |x| + |y|$ (c) $|x - y| \leq |x| - |y|$ (e) $|x - y| \geq ||x| - |y||$
 (b) $|x + y| \leq |x| + |y|$ (d) $|x - y| \leq |x| + |y|$ (f) $2^{|x|} = 2^x$
14. Consider the function $y = \frac{1}{|x - 1|}$.
 (a) What is its natural domain?
 (b) Write down the equations of the two branches of the function and sketch its graph.
15. The identity $|xy| = |x||y|$ was proven in the notes above.
 (a) Noting that $-x = (-1)x$, prove that $|-x| = |x|$.
 (b) Prove the remaining identities from Table (21).
16. Use a similar proof to the one given in the text to prove:
 (a) $|x - y| \leq |x| + |y|$ (b) $|x - y| \geq ||x| - |y||$
17. Write down the equations of the two branches of the function, then sketch its graph:
 (a) $y = 2(x + 1) - |x + 1|$ (b) $y = x^2 - |2x|$
18. Consider the inequation $\left|x + \frac{1}{x}\right| < 2x$.
 (a) Explain why x must be positive. (b) Hence solve the inequation.
19. Carefully write down the equations of the branches of each function, then sketch its graph:
 (a) $y = |x + 1| - |x - 3|$ (b) $y = |x - 2| + |x + 1| - 4$ (c) $y = 2|x + 1| - |x - 1| - 1$

EXTENSION

20. The function $u(x)$ is defined by $u(x) = \begin{cases} \frac{1}{2} \left(1 + \frac{|x|}{x}\right), & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases}$
 (a) Sketch: (i) $u(x)$ (ii) $u(x - 1)$
 (b) Hence sketch $u(x) - u(x - 1)$.
21. Sketch the relation $|y| = |x|$ by considering the possible cases.

22. Consider the inequation $|x - a| + |x - b| < c$, where $a < b$.

(a) If $a \leq x \leq b$, show, using distances on a number line, that there can only be a solution if $b - a < c$.

(b) If $b < x$, show, using distances on a number line, that $x < \frac{a + b + c}{2}$.

(c) If $x < a$, show, using distances on a number line, that $x > \frac{a + b - c}{2}$.

(d) Hence show that either $\left|x - \frac{a + b}{2}\right| < \frac{c}{2}$ or there is no solution to the original problem.

(e) Hence find the solution to $|x + 2| + |x - 6| < 10$.

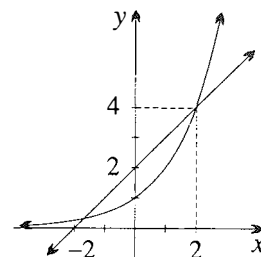
3E Using Graphs to Solve Equations and Inequations

In this section, graphs are used to solve equations and inequations. The advantage of this method is that once the graphs are drawn, it is usually obvious from the picture how many solutions there are, and indeed if there are any solutions at all, as well as their approximate values. Often exact solutions can then be calculated once the situation has been sorted out from the picture.

Constructing Two Functions from a Given Equation: Here is an equation that cannot be solved algebraically:

$$2^x = x + 2.$$

One solution of this equation is $x = 2$, but this is not the only solution. If we draw the graphs of the LHS, $y = 2^x$, and of the RHS, $y = x + 2$, then the situation becomes clear. From the graph, the LHS and RHS are equal at $x = 2$ (where they are both equal to 4), and at $x \doteq -1.69$ (where they are both about 0.31), and these two values of x are the solutions to the original equation.



23

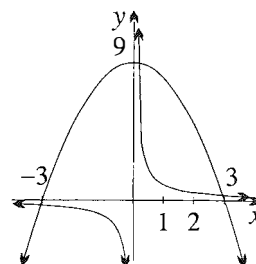
GRAPHICAL SOLUTION OF EQUATIONS: To solve an equation graphically, sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes, and read off the x -coordinates of any points of intersection.

The original equation may need to be rearranged first.

WORKED EXERCISE: Graph $y = 1/x$ and $y = 9 - x^2$ on the one set of axes. Hence find from your graph how many solutions the following equation has, and approximately where they are:

$$x^2 + \frac{1}{x} = 9.$$

SOLUTION: Transforming the equation to $\frac{1}{x} = 9 - x^2$, its solutions are the x -coordinates of the points of intersection of $y = 1/x$ and $y = 9 - x^2$. From the graph there are three solutions, one between -4 and -3, one between 0 and 1, and one between 2 and 3.



Solving an Inequality using Graphs: Now consider the inequality

$$2^x < x + 2.$$

From the sketch above, the curve $y = 2^x$ is only below the curve $y = x + 2$ between the two points of intersection, and so the solution of the inequality is approximately $-1.69 < x < 2$.

24

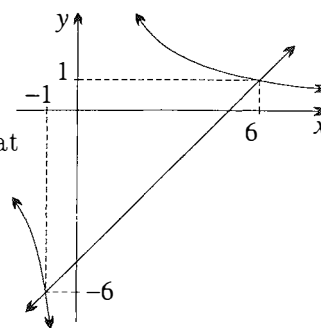
GRAPHICAL SOLUTION OF INEQUALITIES: Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes. Then examine which curve lies above the other at each value of x .

Inequalities with x in the Denominator — Graphical Solution: This is a way of avoiding the problem that an inequality cannot be multiplied through by a variable when the variable may be positive or negative.

WORKED EXERCISE: Solve $\frac{6}{x} \geq x - 5$.

SOLUTION: The graphs of $y = 6/x$ and $y = x - 5$ meet as shown at the points A and B whose x -coordinates are the solutions of

$$\begin{aligned} x - 5 &= \frac{6}{x} \\ x^2 - 5x - 6 &= 0 \\ x &= 6 \text{ or } -1. \end{aligned}$$



So from the graph, the solution of the inequality is $0 < x \leq 6$ or $x \leq -1$.

Absolute Value Equations — Graphical Solutions: If the graph can be used to sort out the situation, then the exact values can usually be found algebraically.

WORKED EXERCISE: Solve $|2x - 5| = x + 2$.

SOLUTION: The graphs $y = |2x - 5|$ and $y = x + 2$ intersect at P and at Q , and these points can be found algebraically.

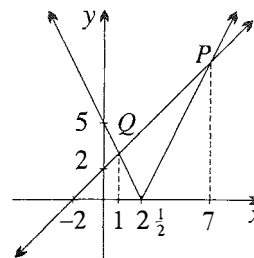
Here P is the intersection of $y = x + 2$ with $y = 2x - 5$:

$$\begin{aligned} x + 2 &= 2x - 5 \\ x &= 7, \end{aligned}$$

and Q is the intersection of $y = x + 2$ with $y = -2x + 5$:

$$\begin{aligned} x + 2 &= -2x + 5 \\ x &= 1. \end{aligned}$$

So $x = 7$ or $x = 1$.



Absolute Value Inequalities — Graphical Solution: The solutions of the inequality

$$|2x - 5| \geq x + 2$$

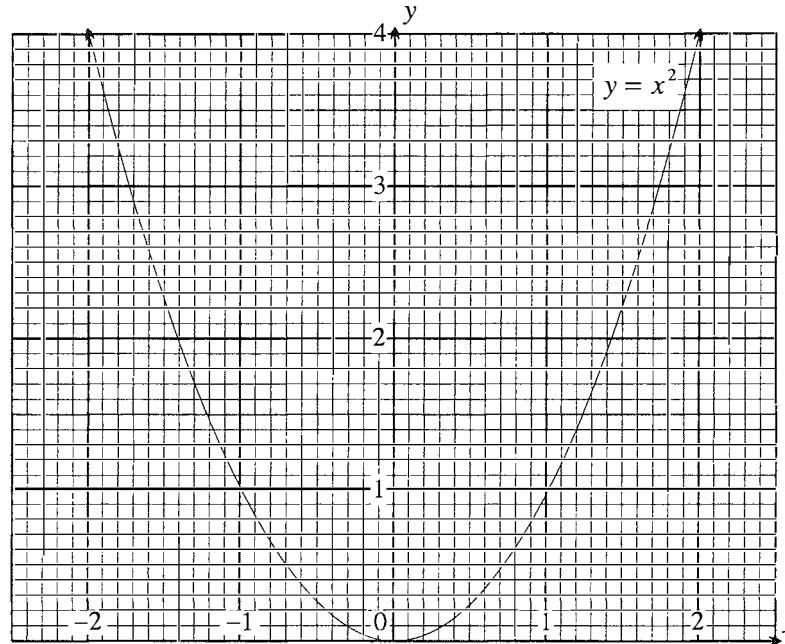
can be read off the graph sketched above. We look at where the graph of the LHS, $y = |2x - 5|$, is above the graph of the RHS, $y = x + 2$. This is to the right of P and to the left of Q , so the solution of the inequality is

$$x \leq 1 \text{ or } x \geq 7.$$

Exercise 3E

NOTE: Machine drawing of curves on a computer or a graphics calculator could be very helpful in this exercise.

1.



Photocopy the above sketch of the graph of $y = x^2$, for $-2 \leq x \leq 2$, in preparation for the following questions.

- Read $\sqrt{2}$ and $\sqrt{3}$ off the graph to one decimal place.
- What lines should be drawn on the graph to solve $x^2 = 2$ and $x^2 = 3$?
- Draw the line $y = x + 2$ on the graph, and hence read off the graph the solutions to $x^2 = x + 2$. Then check your solution by solving $x^2 = x + 2$ algebraically.
- From the graph, write down the solution of $x^2 > x + 2$.
- Draw a suitable line to solve $x^2 = 2 - x$ and $x^2 \leq 2 - x$.
- Draw $y = x + 1$, and hence solve $x^2 = x + 1$ approximately. Check your result algebraically.
- Find approximate solutions for these quadratic equations by rearranging them with x^2 as subject, and drawing a suitable line on the graph:

(i) $x^2 + x = 0$

(ii) $x^2 - x - \frac{1}{2} = 0$

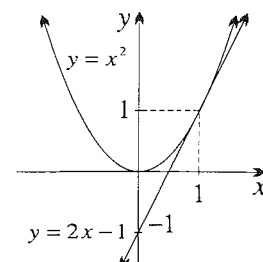
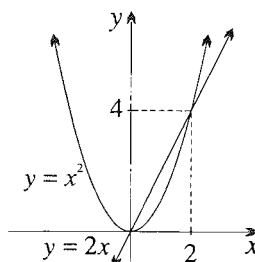
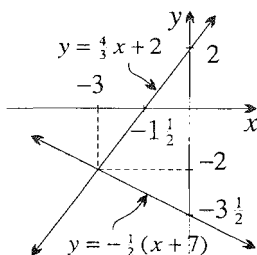
(iii) $2x^2 - x - 1 = 0$

2. Use the given graphs to help solve each inequality:

(a) $\frac{4}{3}x + 2 \leq -\frac{1}{2}(x + 7)$

(b) $x^2 \leq 2x$

(c) $x^2 \leq 2x - 1$

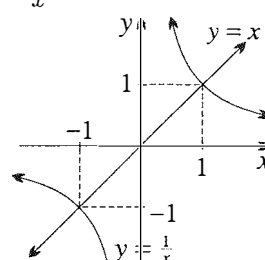
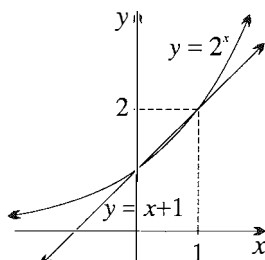
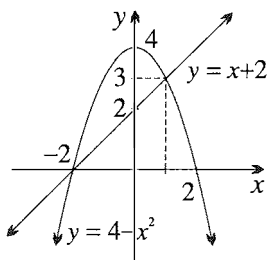


3. Solve each inequality, given the accompanying graphs of LHS and RHS:

(a) $4 - x^2 < x + 2$

(b) $2^x \leq x + 1$

(c) $\frac{1}{x} < x$



4. For each pair below: (i) Carefully sketch each pair of equations. (ii) Hence find the solution to the simultaneous equations, given that all points of intersection have integer coordinates. (iii) Write down the equation satisfied by the x -coordinates of the points of intersection.

(a) $y = x - 2$ and $y = 3 - \frac{1}{4}x$

(c) $y = \frac{2}{x}$ and $y = x - 1$

(b) $y = x$ and $y = 2x - x^2$

(d) $y = x^3$ and $y = x$

5. Use your graphs in the previous question to help solve the following inequalities:

(a) $x - 2 \geq 3 - \frac{1}{4}x$

(b) $x < 2x - x^2$

(c) $\frac{2}{x} > x - 1$

(d) $x^3 > x$

6. Draw graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them:

(a) $1 - \frac{1}{2}x = x^2 - 2x$

(c) $x^3 - x = \frac{1}{2}(x + 1)$

(e) $|x + 1| - 1 = \log_2 x$

(b) $|2x| = 2^x$

(d) $4x - x^2 = \frac{1}{x}$

(f) $2^{-x} - 1 = \frac{1}{x}$

7. (a) Sketch on the same number plane the functions $y = |x + 1|$ and $y = \frac{1}{2}x - 1$.

(b) Hence explain why all real numbers are solutions of the inequality $|x + 1| > \frac{1}{2}x - 1$.

8. Sketch each pair of equations and hence find the points of intersection:

(a) $y = |x + 1|$ and $y = 3$

(c) $y = |2x|$ and $2x - 3y + 8 = 0$

(b) $y = |x - 2|$ and $y = x$

(d) $y = |x| - 1$ and $y = 2x + 2$

9. Use your answers to the previous question to help solve:

(a) $|x + 1| \leq 3$

(c) $|2x| \geq \frac{2x + 8}{3}$

(b) $|x - 2| > x$

(d) $|x| > 2x + 3$

DEVELOPMENT

10. (a) Sketch $y = x^2 - 6$ and $y = |x|$ on one set of axes.

(b) Find the x -coordinates of the points of intersection.

(c) Hence solve $x^2 - 6 \leq |x|$.

11. (a) Draw $y = x^2 - 2$, $y = x$ and $y = -x$ on the same number plane and find all points of intersection of the three functions.

(b) Hence find the solutions of $x^2 - 2 = |x|$.

(c) Hence solve $x^2 - 2 > |x|$.

12. (a) Sketch $y = |2x + 1|$.
 (b) Draw on the same number plane $y = x + c$ for $c = -1$, $c = 0$ and $c = 1$.
 (c) For what values of c does $|2x + 1| = x + c$ have two solutions?
13. (a) Use a diagram and Pythagoras' theorem to show that for $b > 0$, the perpendicular distance from the line $x + y = 2b$ to the origin is $b\sqrt{2}$.
 (b) Hence find the range of values of b for which the line intersects the circle $x^2 + y^2 = 9$ twice.
14. (a) Sketch $y = |7x - 4|$ and $y = 4x + 3$ on the same number plane to find the number of solutions of $|7x - 4| = 4x + 3$.
 (b) Why is it inappropriate to use the graph to find the exact solutions?
 (c) Find the solutions by separately considering the two branches of the absolute value.
15. Sketch LHS and RHS on one pair of axes, then solve:
 (a) $|x - 1| \leq |x - 4|$ (c) $|2x| < |x - 2|$
 (b) $|x + 1| \geq |\frac{1}{2}x - 1|$ (d) $|x - 3| < |2x + 1|$
16. Draw appropriate graphs on graph or grid paper, or on a machine, in order to find the solutions, or estimates to one decimal place:
 (a) $x^3 = 2(x - 2)^2$ (c) $2^{-x} - (2x - x^2) = 0$
 (b) $x^3 = \sqrt{4 - x^2}$ (d) $x^2 - x - \log_2(x + 1) = 0$
17. Find the values of x for which the LHS and RHS are equal. Then sketch LHS and RHS on the same number plane and hence solve each inequation:
 (a) $x \geq \frac{2}{x - 1}$ (b) $\frac{x}{4} \leq -\frac{2}{x - 2}$ (c) $\frac{x + 2}{2} > \frac{2}{x - 1}$

EXTENSION

18. (a) Carefully sketch the graph of $y = |2x + 4| + |x - 1| - 5$ and write down the equation of each branch.
 (b) On the same number plane draw the lines $y = -1$ and $y = 2$. Hence solve the inequation $-1 \leq |2x + 4| + |x - 1| - 5 \leq 2$.
19. (a) Show that $y = mx + b$ must intersect $y = |x + 1|$ if $m > 1$ or $m < -1$.
 (b) Given that $-1 \leq m \leq 1$, find the relationship between b and m so that the two graphs do not intersect.
 (c) Generalise these results for $y = |px - q|$.
20. Sketch the LHS and RHS of $\frac{|2x - 3|}{4x - 6} \geq \frac{x}{4}$, paying attention to the branches of the LHS, and hence solve the inequality.
21. (a) Draw $y = a^x$ and $y = \log_a x$ for: (i) $a = 3$ (ii) $a = 2$ (iii) $a = \sqrt{2}$
 (b) Conclude how many solutions $a^x = \log_a x$ may have.

3F Regions in the Number Plane

The circle $x^2 + y^2 = 25$ divides the plane into two *regions* — inside the circle and outside the circle. The graph of the inequation $x^2 + y^2 > 25$ will be one of these regions. It remains to work out which of these regions should be shaded.

Graphing Regions: To sketch the region of an inequation, use the following method.

25

GRAPHING THE REGION CORRESPONDING TO AN INEQUATION:

1. **THE BOUNDARY:** Replace the inequality symbol by an equal symbol, and graph the curve. This will be the boundary of the region, and should be drawn broken if it is excluded, and unbroken if it is included.
2. **SHADING:** Determine which parts are included and which are excluded, and shade the parts that are included. This can be done in two ways:
 - (a) [Always possible] Take one or more test points not on any boundary, and substitute into the LHS and RHS of the original inequation. The origin is the easiest test point, otherwise try to choose points on the axes.
 - (b) [Quicker, but not always possible] Alternatively, solve the inequation for y if possible, and shade the region above or below the curve. Or solve for x , and sketch the region to the right or left of the curve.
3. **CHECKING BOUNDARIES AND CORNERS:** Check that boundaries are correctly broken or unbroken. Corner points must be marked clearly with a closed circle if they are included, or an open circle if excluded.

NOTE: [A nasty point] There may be points in the plane where the LHS or RHS of the inequation is undefined. For example, the RHS of $y > 1/x$ is undefined at all points on the y -axis, because $1/x$ is undefined when $x = 0$. If so, the set of all these points will usually be a boundary of the region too, and will be excluded.

WORKED EXERCISE: Sketch the region $x^2 + y^2 > 25$.

SOLUTION: The boundary is $x^2 + y^2 = 25$, and is excluded.

Take a test point $(0, 0)$. Then RHS = 25,

$$\text{LHS} = 0.$$

So $(0, 0)$ does not lie in the region.

WORKED EXERCISE: Sketch $y \geq x^2$.

SOLUTION: The boundary is $y = x^2$, and is included.

Because the inequation is $y \geq x^2$,

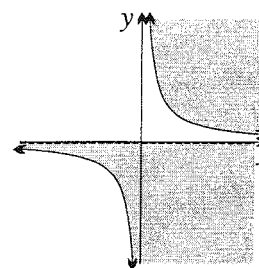
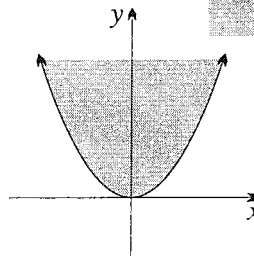
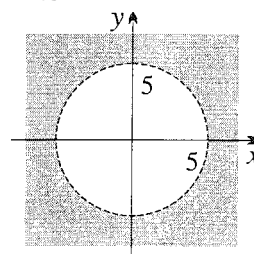
the region involved is the region *above* the curve.

WORKED EXERCISE: [A harder example] Sketch $x \geq \frac{1}{y}$.

SOLUTION: The boundary is $x = 1/y$, and is included.

Also, the x -axis $y = 0$ is a boundary, because the RHS is undefined when $y = 0$. This boundary is excluded.

Because the inequation is $x \geq 1/y$, the region to be shaded is the region to the right of the curve.



Intersections and Unions of Regions: Some questions will ask explicitly for the intersection or union of two regions. Other questions will implicitly ask for intersections. For example,

$$|2x + 3y| < 6$$

means $-6 < 2x + 3y < 6$, and so is the intersection of $2x + 3y > -6$ and $2x + 3y < 6$. Or there may be a restriction on x or on y , as in

$$x^2 + y^2 < 25, \text{ where } x \leq 3 \text{ and } y > -4,$$

which means the intersection of three different regions.

26

INTERSECTIONS AND UNIONS OF REGIONS: Draw each region, then sketch the intersection or union.

Pay particular attention to whether corner points are included or excluded.

WORKED EXERCISE: Graph the intersection and union of the regions $y > x^2$ and $x + y \leq 2$.

SOLUTION: The boundary of the first region is $y = x^2$, and the region lies above the curve (with the boundary excluded).

The boundary of the second region is $x + y = 2$. Solving for y gives $y \leq 2 - x$, and so the region lies below the curve (with the boundary included).

By inspection, or by simultaneous equations, the parabola and the line meet at $(1, 1)$ and $(-2, 4)$. These points are excluded from the intersection because they are not in the region $y > x^2$, but included in the union because they are in the region $x + y \leq 2$.

WORKED EXERCISE: Graph the region $|2x + 3y| < 6$.

SOLUTION: This is the region $-6 < 2x + 3y < 6$.

The boundaries are the parallel lines

$$2x + 3y = 6 \text{ and } 2x + 3y = -6,$$

both of which are excluded.

The required region is the region between these two lines.

WORKED EXERCISE: Graph the region $x^2 + y^2 \leq 25$, for $x \leq 3$ and $y > -4$, giving the coordinates of each corner point.

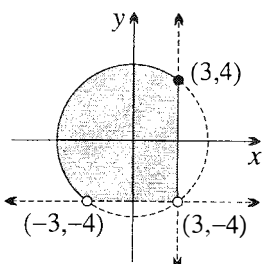
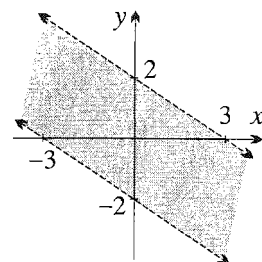
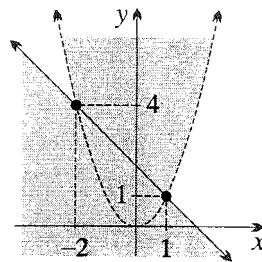
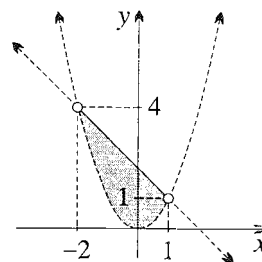
SOLUTION: The boundaries are $x^2 + y^2 = 25$ (included),

and the vertical and horizontal lines

$x = 3$ (included) and $y = -4$ (excluded).

The points of intersection are $(3, 4)$ (included),

$(3, -4)$ (excluded) and $(-3, -4)$ (excluded).



Exercise 3F

- For each inequation: (i) sketch the boundary, (ii) shade the region above or below the boundary, as required.

(a) $y < 1$	(c) $y > x - 1$	(e) $y \leq 2x + 2$
(b) $y \geq -3$	(d) $y \leq 3 - x$	(f) $y < \frac{1}{2}x - 1$
- For each inequation: (i) sketch the boundary, (ii) shade the region to the left or right of the boundary, as required.

(a) $x < -2$	(c) $x \geq y + 2$	(e) $x > 3 - y$
(b) $x > 1$	(d) $x < 2y - 1$	(f) $x \leq \frac{1}{2}y + 2$
- For each inequation, sketch the boundary line, then use a suitable test point to decide which side of the line to shade.

(a) $2x + 3y - 6 > 0$	(b) $x - y + 4 \geq 0$	(c) $y - 2x + 3 < 0$
- For each inequation, sketch the boundary circle, then use a suitable test point to decide which region to shade.

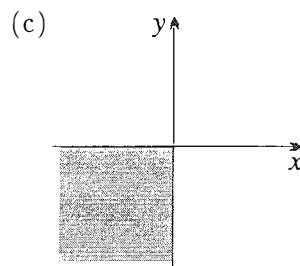
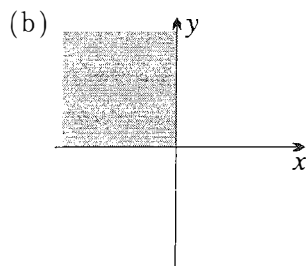
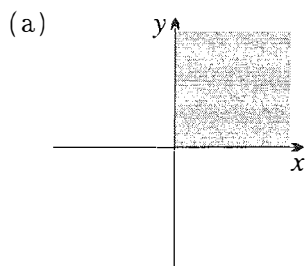
(a) $x^2 + y^2 < 4$	(c) $(x - 2)^2 + y^2 \leq 4$
(b) $x^2 + y^2 \geq 1$	(d) $(x + 1)^2 + (y - 2)^2 > 9$
- Sketch the following regions (some of the quadratics need factoring):

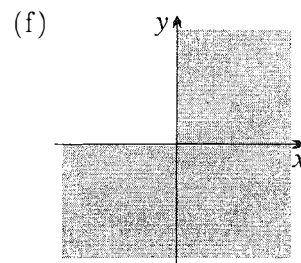
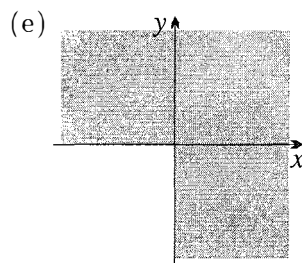
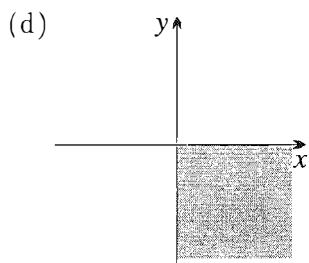
(a) $y \geq x^2 - 1$	(d) $y > 4 - x^2$	(g) $y < (5 - x)(1 + x)$
(b) $y < x^2 - 2x - 3$	(e) $y \leq x^2 + 3x$	(h) $y > (2x - 3)(x + 1)$
(c) $y \geq x^2 + 2x + 1$	(f) $y \leq 2 + x - x^2$	(i) $y \leq (2x + 1)(x - 3)$
- Draw the following regions of the number plane:

(a) $y > 2^x$	(c) $y \leq x + 1 $	(e) $y \leq \log_2 x$
(b) $y \geq x $	(d) $y > x^3$	(f) $y < \frac{1}{2}x - 1 $
- Find the intersection point of the lines $x = -1$ and $y = 2x - 1$.
 - Hence sketch the intersection of the regions $x > -1$ and $y \leq 2x - 1$, paying careful attention to the boundaries and their point of intersection.
 - Likewise sketch the union of the two regions.
- Sketch on separate number planes the two regions $y < x$ and $y \geq -x$. Hence sketch:
 - the union of these two regions,
 - the intersection of the regions.
 Pay careful attention to the boundaries and their points of intersection.
 - Similarly, graph the union and intersection of:
 - $y > x$ and $y \leq 2 - x$
 - $y > \frac{1}{2}x + 1$ and $y \leq -x - 2$

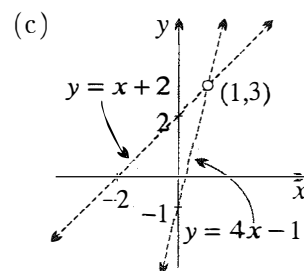
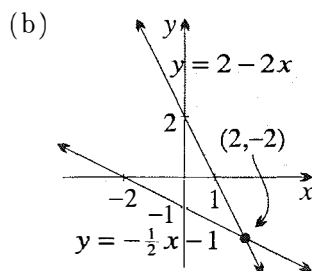
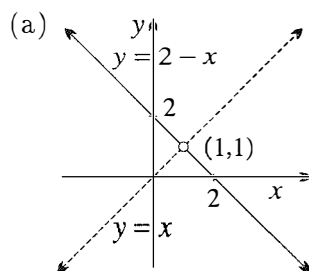
DEVELOPMENT

- Identify the inequations that correspond to the following regions:





10. Write down intersections or unions that correspond to the following regions:



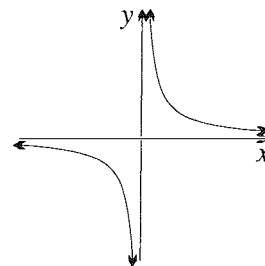
11. (a) Show that the lines $y = x + 1$, $y = -\frac{1}{2}x - 2$, and $y = 4x - 2$ intersect at $(-2, -1)$, $(0, -2)$ and $(1, 2)$. Then sketch all three on the same number plane.
- (b) Hence sketch the regions indicated by:
- $y < x + 1$ and $y \geq -\frac{1}{2}x - 2$
 - $y < x + 1$ and $y \geq -\frac{1}{2}x - 2$ and $y < 4x - 2$
 - $y > x + 1$ or $y < -\frac{1}{2}x - 2$ or $y < 4x - 2$
12. (a) Sketch the intersection of $x^2 + y^2 > 1$ and $x^2 + y^2 \leq 9$.
- (b) What is the union of these two regions?
13. (a) Sketch the union of $x^2 + y^2 \leq 1$ and $y > 2 - x$.
- (b) What is the intersection of these two regions?
14. (a) Find the intersection points of the line $y = 4 - x$ and the circle $x^2 + y^2 = 16$.
- (b) Hence sketch (i) the intersection and (ii) the union of $y \geq 4 - x$ and $x^2 + y^2 < 16$.
15. (a) The inequation $|x| < 2$ implies the intersection of two regions. Write down the equations of these two regions. Hence sketch the region $|x| < 2$.
- (b) The inequation $|x - y| \leq 2$ implies the intersection of two regions. Write down the equations of these two regions. Hence sketch $|x - y| \leq 2$.
16. (a) The inequation $|y| \geq 1$ implies the union of two regions. Write down the equations of these two regions. Hence sketch the region given by $|y| \geq 1$.
- (b) The inequation $|y + 2x| > 1$ implies the union of two regions. Write down the equations of these two regions. Hence sketch the region $|y + 2x| > 1$.
17. Sketch the region $x^2 + y^2 \geq 5$ for the domain $x > -1$ and range $y < 2$, and give the coordinates of each corner.
18. Sketch the region $y \leq x^2 - 2x + 2$ with $y \geq 0$ and $0 \leq x \leq 2$.
19. (a) Draw the curve $y = \sqrt{x}$.
- (b) Explain why the y -axis $x = 0$ is a boundary for $y < \sqrt{x}$.
- (c) Hence sketch the region $y < \sqrt{x}$.

20. (a) Explain why $x = 0$ is a boundary for $y > \frac{1}{x}$.
 (b) Hence sketch the region $y > \frac{1}{x}$.
21. Carefully sketch the following regions, paying attention to implied boundaries:
 (a) $y < \sqrt{4 - x^2}$ (b) $x > \sqrt{9 - y^2}$
22. Sketch the region defined by $x > |y + 1|$. [HINT: $x = |y + 1|$ is the inverse of what function?]
- EXTENSION —————
23. (a) How many regions do the coordinate axes and the hyperbola $y = \frac{1}{x}$ divide the number plane into? (b) Carefully sketch the following regions. [HINT: It may help to take test points in each of the regions found in the previous part.]
 (i) $y < \frac{1}{x}$ (ii) $xy < 1$ (iii) $1 > \frac{1}{xy}$
24. Graph the regions: (a) $|y| > |x|$ (b) $|xy| \geq 1$ (c) $\frac{1}{x} > \frac{1}{y}$
25. (a) Consider the region A with $x^2 + y^2 \geq 4$ and $x^2 + y^2 \leq 9$, and the region B which is the union of A with $x^2 + y^2 \leq 1$. Region A is called *connected* but region B is not connected. Discuss what might define a connected region.
 (b) Consider the region A with $x^2 + y^2 \leq 2$, and the region B which is the intersection of A with $y \leq |x|$. Region A is *convex* but region B is not convex. Discuss what might define a region that is not convex.

3G Asymptotes and a Curve Sketching Menu

The chapter concludes with a study of vertical and horizontal asymptotes, principally of rational functions. The three techniques of Sections 3B and 3C, together with asymptotes, are then combined into a systematic four-step approach to sketching an unknown graph. No such simple menu could possibly deal with the great variety of possible graphs; nevertheless, it will allow the main features of a surprising number of functions to be found. Two further steps involving calculus will be added in Chapter Ten.

Vertical Asymptotes: The rectangular hyperbola $y = 1/x$ has the y -axis as an asymptote, as discussed in Section 2G. This is because the values of y become very large in size, positive or negative, when x is near $x = 0$. In this course, discontinuities mostly arise from zeroes in the denominator. The test for a vertical asymptote is then very simple:



27

TESTING FOR VERTICAL ASYMPTOTES: If the denominator goes to zero as $x \rightarrow a$, and the numerator is *not* zero at $x = a$, then the vertical line $x = a$ is an asymptote.

Questions still remain about the behaviour of the function on each side of the asymptote. The table of signs is the easiest way to distinguish the two cases.

28

BEHAVIOUR NEAR A VERTICAL ASYMPTOTE: The choice between $y \rightarrow \infty$ and $y \rightarrow -\infty$ can be made by looking at a table of signs.

WORKED EXERCISE: Find any vertical asymptotes of the function $y = \frac{x-1}{x-4}$, and use a table of values to discuss the behaviour of the curve near them. (The curve itself is sketched in the next worked exercise).

SOLUTION: The vertical line $x = 4$ is an asymptote, because at $x = 4$ the denominator vanishes but the numerator does not. From the table of values opposite, around the zero at $x = 1$ and the discontinuity at $x = 4$ (or from the sketch below):

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	-	*	+

$y \rightarrow \infty$ as $x \rightarrow 4^+$, and $y \rightarrow -\infty$ as $x \rightarrow 4^-$.

NOTE: Functions like the one in the previous example, which are ratios of two polynomials, are called *rational functions*. Almost all the functions in this section are rational functions.

Behaviour as $x \rightarrow \infty$ and as $x \rightarrow -\infty$ — Horizontal Asymptotes: For most functions in this course, the following method will be sufficient.

29

BEHAVIOUR FOR LARGE x : Divide top and bottom by the highest power of x in the denominator. Then use the fact that

$$\frac{1}{x} \rightarrow 0, \text{ as } x \rightarrow \infty \text{ and as } x \rightarrow -\infty.$$

If $f(x)$ tends to a definite limit b as $x \rightarrow \infty$ or as $x \rightarrow -\infty$, then the horizontal line $y = b$ is a horizontal asymptote.

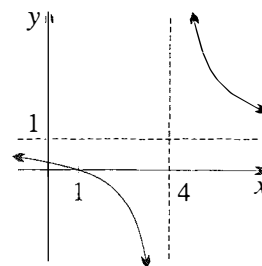
WORKED EXERCISE: Examine the behaviour of the earlier function $f(x) = \frac{x-1}{x-4}$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, noting horizontal asymptotes, then sketch the curve.

SOLUTION: Dividing top and bottom by x ,

$$f(x) = \frac{1 - \frac{1}{x}}{1 - \frac{4}{x}},$$

and so $f(x) \rightarrow 1$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

Hence $y = 1$ is a horizontal asymptote.



WORKED EXERCISE: Examine the behaviour of these functions as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, noting any horizontal asymptotes:

(a) $f(x) = \frac{3-5x-4x^2}{4-5x-3x^2}$

(c) $f(x) = \frac{x^2-1}{x-4}$

(b) $f(x) = \frac{x-1}{x^2-4}$

(d) $f(x) = \frac{1}{x} + \frac{1}{x-3}$

SOLUTION:

(a) Dividing top and bottom by x^2 , $f(x) = \frac{\frac{3}{x^2} - \frac{5}{x} - 4}{\frac{4}{x^2} - \frac{5}{x} - 3}$,

so $f(x) \rightarrow \frac{4}{3}$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$,
and $y = \frac{4}{3}$ is a horizontal asymptote.

(b) Dividing top and bottom by x^2 , $f(x) = \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$,

so $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$,
and the x -axis $y = 0$ is a horizontal asymptote.

(c) Dividing top and bottom by x , $f(x) = \frac{x - \frac{1}{x}}{1 - \frac{4}{x}}$,

so $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$,
and there are no horizontal asymptotes.

(d) Here $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$,
so $y = 0$ is a horizontal asymptote.

A Curve Sketching Menu: Here is a systematic approach to sketching a curve whose function is not easily analysed in terms of transformations of known curves. A 'sketch' of a graph is not an accurate plot. It is a neat diagram showing the main features of the curve, and unless there are major difficulties involved:

30 **SKETCHES:** A sketch should normally show any x - and y -intercepts, give some indication of scale on both axes, and have labels on both axes.

Suppose that $f(x)$ is an unfamiliar function, and a sketch of $y = f(x)$ is required:

A CURVE SKETCHING MENU:

- 0. PREPARATION:** Combine any fractions using a common denominator, then factor top and bottom as far as possible.
- 1. DOMAIN:** Find the domain (*always* do this first).
- 2. SYMMETRY:** Find whether the function is odd, or even, or neither.
- 31** **3. A. INTERCEPTS:** Find the y -intercept and the x -intercepts (zeroes).
B. SIGN: Find where the function is positive, and where it is negative.
- 4. A. VERTICAL ASYMPTOTES:** Examine the behaviour near any discontinuities, noting any vertical asymptotes.
B. HORIZONTAL ASYMPTOTES: Examine the behaviour of $f(x)$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, noting any horizontal asymptotes.

NOTE: Finding the domain and finding the zeroes may both require factorisation, which is the reason why the preparatory Step 0 is useful. Factorisation, however, may not always be possible, even with the formula for the roots of a quadratic, and in such cases approximation methods may be useful. Questions in exercises and examinations will normally give guidance as to what is required.

Putting it All Together — Example 1: All that remains is to give two examples of the whole process. First, here is the method applied to $f(x) = \frac{2x^2}{x^2 - 9}$.

SOLUTION:

0. PREPARATION: $f(x) = \frac{2x^2}{(x-3)(x+3)}$

1. DOMAIN: $x \neq 3$ and $x \neq -3$.

2. SYMMETRY: $f(-x) = \frac{2(-x)^2}{(-x)^2 - 9}$
 $= \frac{2x^2}{x^2 - 9}$
 $= f(x)$

so $f(x)$ is even, and has line symmetry in the y -axis.

3. INTERCEPTS AND SIGN: There is a zero at $x = 0$, and discontinuities at $x = 3$ and $x = -3$:

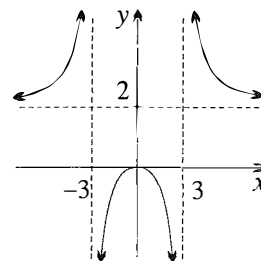
x	-4	-3	-1	0	1	3	4
$f(x)$	$\frac{32}{7}$	*	$-\frac{1}{4}$	0	$-\frac{1}{4}$	*	$\frac{32}{7}$
sign	+	*	-	0	-	*	+

4A. VERTICAL ASYMPTOTES: At $x = 3$ and $x = -3$ the denominator vanishes but the numerator does not, so $x = 3$ and $x = -3$ are vertical asymptotes. From the table of signs,

$$\begin{aligned} f(x) &\rightarrow \infty \text{ as } x \rightarrow 3^+ & \text{and} & & f(x) &\rightarrow -\infty \text{ as } x \rightarrow 3^-, \\ f(x) &\rightarrow -\infty \text{ as } x \rightarrow (-3)^+ & \text{and} & & f(x) &\rightarrow \infty \text{ as } x \rightarrow (-3)^-. \end{aligned}$$

4B. HORIZONTAL ASYMPTOTES: Dividing through by x^2 , $f(x) = \frac{2}{1 - \frac{9}{x^2}}$,

so $f(x) \rightarrow 2$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$,
and $y = 2$ is a horizontal asymptote.



Putting it All Together — Example 2: The second example is much more difficult and requires more algebraic manipulation. The calculations involving sign show an alternative approach using signs rather than numbers:

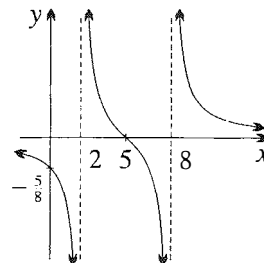
$$f(x) = \frac{1}{x-2} + \frac{1}{x-8}$$

SOLUTION:

0. PREPARATION: $f(x) = \frac{(x-8) + (x-2)}{(x-2)(x-8)}$
 $= \frac{2x-10}{(x-2)(x-8)}$
 $= \frac{2(x-5)}{(x-2)(x-8)}$

1. DOMAIN: $x \neq 2$ and $x \neq 8$.

2. SYMMETRY: $f(x)$ is neither even nor odd.



3. INTERCEPTS AND SIGN: There is a zero at $x = 5$, and discontinuities at $x = 2$ and $x = 8$:

x	0	2	3	5	6	8	9
$x - 2$	—	0	+	+	+	+	+
$x - 5$	—	—	—	0	+	+	+
$x - 8$	—	—	—	—	—	0	+
$f(x)$	—	*	+	0	—	*	+

- 4A. VERTICAL ASYMPTOTES: At $x = 2$ and $x = 8$ the denominator vanishes, but the numerator does not, so $x = 2$ and $x = 8$ are vertical asymptotes. From the table of signs,

$$\begin{aligned} f(x) &\rightarrow \infty \text{ as } x \rightarrow 2^+ \quad \text{and} \quad f(x) \rightarrow -\infty \text{ as } x \rightarrow 2^-, \\ f(x) &\rightarrow \infty \text{ as } x \rightarrow 8^+ \quad \text{and} \quad f(x) \rightarrow -\infty \text{ as } x \rightarrow 8^-. \end{aligned}$$

- 4B. HORIZONTAL ASYMPTOTES: From the original form of the given equation, $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, so $y = 0$ is a horizontal asymptote.

An Example with an Oblique Asymptote: Sometimes it becomes obvious from examination of a function for large x that the curve has an oblique asymptote, and although a systematic treatment is not appropriate, the following example is quite straightforward.

WORKED EXERCISE: Sketch the graph of $y = x - \frac{1}{x}$.

SOLUTION:

$$0. \quad y = \frac{x^2 - 1}{x} = \frac{(x - 1)(x + 1)}{x}$$

1. The domain is $x \neq 0$.

2. $f(-x) = -x + \frac{1}{x} = -f(x)$, so the function is odd.

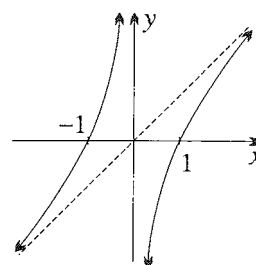
3. There are zeroes at $x = 1$ and $x = -1$, and a discontinuity at $x = 0$:

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-1\frac{1}{2}$	0	$1\frac{1}{2}$	*	$-1\frac{1}{2}$	0	$1\frac{1}{2}$

- 4A. The y -axis is a vertical asymptote.

As $x \rightarrow 0^+$, $y \rightarrow -\infty$, and as $x \rightarrow 0^-$, $y \rightarrow \infty$.

- 4B. As $|x| \rightarrow \infty$, $\frac{1}{x} \rightarrow 0$, so $y - x \rightarrow 0$ and $y = x$ is an oblique asymptote.



Extension — Long Division and Oblique Aymptotes: Systematic examination of oblique asymptotes is not required in this course, but is described here for those who may be interested. A rational function has an oblique asymptote when its numerator has degree one more than the degree of its denominator. The equation of the oblique asymptotes is then obtained by long division. For example,

$$\frac{2x^3 + 9x^2 + 8x + 1}{x^2 + 3x - 1} = 2x + 3 + \frac{x + 4}{x^2 + 3x - 1},$$

and so $y = 2x + 3$ is an oblique asymptote of the graph of $y = \frac{2x^3 + 9x^2 + 8x + 1}{x^2 + 3x - 1}$.

Exercise 3G

NOTE: Purely algebraic approaches to sketching curves like these can be rather demanding. As an alternative, some questions could be investigated first by machine drawing, followed by algebraic explanation of the features.

1. Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator, and taking the limit as $x \rightarrow \infty$ and as $x \rightarrow -\infty$:

$$\begin{array}{lll} \text{(a)} f(x) = \frac{1}{x+1} & \text{(c)} f(x) = \frac{2x+1}{3-x} & \text{(e)} \frac{1}{x^2+1} \\ \text{(b)} f(x) = \frac{x-3}{x+4} & \text{(d)} f(x) = \frac{5-x}{4-2x} & \text{(f)} \frac{x}{x^2+4} \end{array}$$

2. Sketch each rational function below after carrying out the following steps: (i) State the natural domain. (ii) Find the y -intercept. (iii) Explain why $y = 0$ is a horizontal asymptote. (iv) Draw up a table of values and examine the sign. (v) Identify any vertical asymptotes, and use the table of signs to write down its behaviour near any vertical asymptotes.

$$\begin{array}{llll} \text{(a)} y = \frac{1}{x-1} & \text{(b)} y = \frac{2}{3-x} & \text{(c)} y = -\frac{2}{x+2} & \text{(d)} y = \frac{5}{2x+5} \end{array}$$

3. Sketch the curve $y = \frac{x}{x-2}$ after performing the following steps:

- (a) Write down the natural domain. (b) Find the intercepts and examine the sign.
(c) Show that $y = 1$ is the horizontal asymptote.
(d) Investigate the behaviour near the vertical asymptote.

4. Consider $y = \frac{x-1}{x+3}$.

- (a) Where is the function undefined?
(b) Find the intercepts and examine the sign of the function.
(c) Identify and investigate the vertical and horizontal asymptotes.
(d) Hence sketch the curve.

5. Investigate the domain, intercepts, sign and asymptotes of the function $y = \frac{2}{(x-1)^2}$ and hence sketch its graph.

6. (a) Show that $y = \frac{3x}{x^2+9}$ is an odd function.

- (b) Show that it has only one intercept with the axes at the origin.
(c) Show that the x -axis is the horizontal asymptote.
(d) Hence sketch the curve.

7. (a) Investigate whether $y = \frac{10}{x^2+5}$ is even or odd. (b) What are its intercepts?

- (c) Show that $y = 0$ is a horizontal asymptote. (d) Hence sketch the curve.

8. Factor if necessary, and find any vertical and horizontal asymptotes:

$$\begin{array}{llll} \text{(a)} \frac{2(x+2)(x+3)}{(x-1)(x-3)} & \text{(b)} \frac{1-4x^2}{1-9x^2} & \text{(c)} \frac{x^2+2x+2}{x^2+5x+4} & \text{(d)} \frac{x-5}{x^2+3x-10} \end{array}$$

[Machine sketching of these curves would be useful to put these features in context.]

9. (a) Show that $y = \frac{4-x^2}{4+x^2}$ is even. (b) Find its three intercepts with the axes.
(c) Determine the equation of the horizontal asymptote. (d) Sketch the curve.

DEVELOPMENT

10. This question looks at graphs that have holes rather than vertical asymptotes.
- (a) Show that $\frac{x^2 - 4}{x - 2} = x + 2$, provided $x \neq 2$. Hence sketch the graph of $y = \frac{x^2 - 4}{x - 2}$.
- (b) Similarly sketch graphs of:
- (i) $y = \frac{(x + 1)(x - 3)}{x + 1}$ (ii) $y = \frac{x^3 - 1}{x - 1}$ (iii) $y = \frac{(x + 2)(x - 2)}{(x - 2)(x + 1)}$
11. Factor the numerator of $y = \frac{x^2 - 2x - 3}{x - 3}$ and hence show that the curve does not have a vertical asymptote at $x = 3$. Sketch the curve.
12. Factor the numerator of $y = \frac{x^3 - 2x^2 - x + 2}{x - 1}$ by grouping in pairs. Hence show that there is no vertical asymptote at $x = 1$. Sketch the curve.
13. (a) Show that $y = \frac{1}{x + 1} - \frac{1}{x}$ can be written as $y = -\frac{1}{x(x + 1)}$. Then identify the domain and any zeroes, examine the asymptotes and sign, and hence sketch the graph.
- (b) Likewise express $y = \frac{1}{x + 3} + \frac{1}{x - 3}$ with a common denominator and sketch it.
14. (a) Examine the sign and asymptotes of $y = \frac{1}{x(x - 2)}$ and hence sketch the curve.
- (b) Likewise sketch $y = \frac{2}{x^2 - 4}$.
15. [Two harder sketches with oblique asymptotes] (a) Identify the oblique asymptote of the function $y = x + \frac{1}{x}$. Then use the appropriate steps of the curve sketching menu to sketch it. (b) Similarly sketch $y = 2^{-x} - x - 3$, using the fact that $2^{-x} \rightarrow 0$ as $x \rightarrow \infty$ to identify the oblique asymptote.
16. Use the curve sketching menu as appropriate to obtain the graphs of:
- (a) $y = \frac{1 + x^2}{1 - x^2}$ (c) $y = \frac{x - 1}{(x + 1)(x - 2)}$ (e) $y = \frac{x^2 - 4}{(x + 2)(x - 1)}$
- (b) $y = \frac{x + 1}{x(x - 3)}$ (d) $y = \frac{x^2 - 2x}{x^2 - 2x + 2}$ (f) $y = \frac{x^2 - 2}{x}$

EXTENSION

17. (a) Show that $\frac{(x - 1)(x + 2)}{x - 3} = x + 4 + \frac{10}{x - 3}$, and deduce that $y = \frac{(x - 1)(x + 2)}{x - 3}$ has an oblique asymptote $y = x + 4$. Then sketch the graph.
- (b) Likewise sketch $y = \frac{x^2 - 4}{x + 1} = x - 1 - \frac{3}{x + 1}$, showing the oblique asymptote.
18. Consider carefully the asymptotes and intercepts of the following functions, and then sketch them:
- (a) $y = \frac{1 - 2^x}{1 + 2^x}$ (b) $y = \frac{1 + 2^x}{1 - 2^x}$
19. Investigate the asymptotic behaviour of the following functions, and graph them:
- (a) $y = \frac{x^3 - 1}{x}$ (b) $y = \frac{1}{x} + \sqrt{x}$ (c) $y = |x| + \frac{1}{x}$

Trigonometry

One of the major reasons why trigonometry is important is that the graphs of the sine and cosine functions are waves. Waves appear everywhere in the natural world, for example as water waves, as sound waves, or as the various electromagnetic waves responsible for radio, heat, light, ultraviolet radiation, X-rays and gamma rays. In quantum mechanics, a wave is associated with every particle. Trigonometry began, however, in classical times as the study of the relationships between angles and lengths in geometrical figures. Its name, from the Greek words *trigos* meaning ‘land’ and *metros* meaning ‘measurement’, reminds us that trigonometry is fundamental to surveying and navigation. This introductory chapter establishes the geometric context of the trigonometric functions and their graphs, developing them from the geometry of triangles and circles.

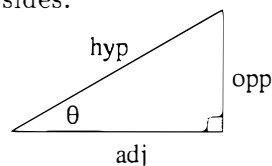
STUDY NOTES: Trigonometric problems involving right triangles (Sections 4A and 4B) and the sine, cosine and area rules (Sections 4H, 4I and 4J) should be familiar. On the other hand, the extension of the trigonometric functions to angles of any magnitude and the graphs of these functions (Sections 4C, 4D and 4E), and the work on trigonometric identities and trigonometric equations (Sections 4F and 4G) will mostly be new. Machine drawing of a variety of trigonometric graphs could be helpful in establishing familiarity with the graphs.

4A Trigonometry with Right Triangles

This section and the next will review the earlier definitions, based on triangles, of the six trigonometric functions for acute angles, and apply them to problems involving right triangles.

The Definition of the Trigonometric Functions: Suppose that θ is any acute angle (this means that $0 < \theta < 90^\circ$ — angles of 0° or 90° are not acute angles). Construct a right triangle with θ as one of the other two angles, and label the sides:

hyp — the *hypotenuse*, the side opposite the right angle,
opp — the side *opposite* the angle θ ,
adj — the third side, *adjacent* to θ but not the hypotenuse.



1

DEFINITION:	$\sin \theta = \frac{\text{opp}}{\text{hyp}}$	$\cos \theta = \frac{\text{adj}}{\text{hyp}}$	$\tan \theta = \frac{\text{opp}}{\text{adj}}$
	$\text{cosec } \theta = \frac{\text{hyp}}{\text{opp}}$	$\sec \theta = \frac{\text{hyp}}{\text{adj}}$	$\cot \theta = \frac{\text{adj}}{\text{opp}}$

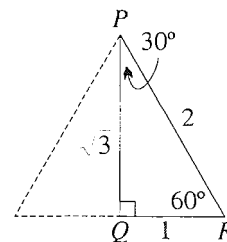
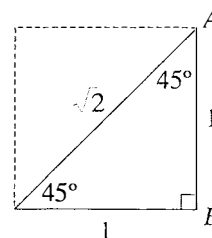
Any two triangles with angles of 90° and θ are similar, because they have the same three angles (this is the AA similarity test), and so their sides are in the same ratio. Hence the values of the six trigonometric functions at θ , defined above, are the same, whatever the size of the triangle. The full names of the six functions are:

sine, cosine, tangent, cosecant, secant, cotangent.

Special Angles: The values of the six trigonometric functions can be calculated exactly for the three acute angles 30° , 45° and 60° . The right triangle $\triangle ABC$ below, with two 45° angles, is formed by taking half of a square with side length 1. The right triangle $\triangle PQR$, whose other angles are 60° and 30° , is half of an equilateral triangle with side length 2. The third sides can then be calculated using Pythagoras' theorem, giving the exact values in the table below.

2

A TABLE OF EXACT VALUES			
θ	30°	45°	60°
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\operatorname{cosec} \theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec \theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot \theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$



Trigonometric Functions of Other Angles: The values of the trigonometric functions of other angles are rather complicated, but the calculator can be used to find approximations to them. Make sure you know how to use your particular machine to enter angles in degrees and minutes, and how to change angles given in decimals of a degree to angles given to the nearest minute. Here are two examples to try on your own calculator:

$$\sin 53^\circ 47' \doteq 0.8068 \quad \text{and} \quad \sin \theta = \frac{5}{8}, \text{ so } \theta \doteq 38^\circ 41'.$$

Finding an Unknown Side of a Triangle: The calculator does not have the secant, cosecant and cotangent functions, so it is best to use only sine, cosine and tangent in these problems.

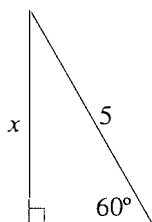
TO FIND AN UNKNOWN SIDE OF A RIGHT TRIANGLE:

3

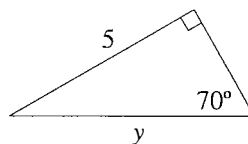
1. Start by writing $\frac{\text{unknown side}}{\text{known side}} = \dots$ (place the unknown top left).
2. Complete the RHS with sin, cos or tan, or the reciprocal of one of these.

WORKED EXERCISE: Find the sides marked with pronumerals in these triangles, in exact form if possible, or else correct to five significant figures.

(a)



(b)



SOLUTION:

$$\begin{aligned} \text{(a)} \quad \frac{x}{5} &= \sin 60^\circ \\ \boxed{\times 5} \quad x &= 5 \sin 60^\circ \\ &= \frac{5\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{y}{5} &= \frac{1}{\sin 70^\circ} \\ \boxed{\times 5} \quad y &= \frac{5}{\sin 70^\circ} \\ &\doteq 5.3209 \end{aligned}$$

Finding an Unknown Angle: As before, use only sine, cosine and tangent.

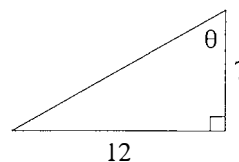
4

FINDING AN UNKNOWN ANGLE: To find an angle when given two sides of a right triangle, work out which one of $\cos \theta$, $\sin \theta$ or $\tan \theta$ is known.

WORKED EXERCISE: Find θ in the given triangle.

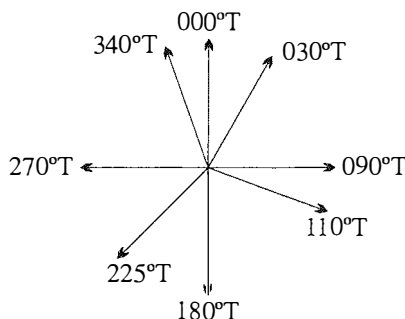
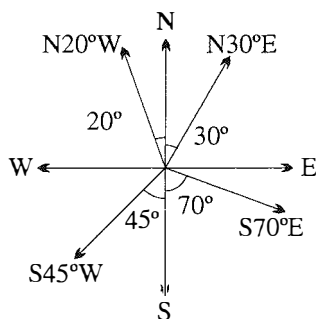
SOLUTION: The given sides are the opposite and the adjacent sides, so $\tan \theta$ is known.

$$\begin{aligned} \tan \theta &= \frac{12}{7} \\ \theta &\doteq 59^\circ 45'. \end{aligned}$$



Compass Bearings and True Bearings: *Compass bearings* specify direction in terms of the four cardinal directions north, south, east and west. Any other direction is given by indicating the deviation from north or south towards the east or west. The diagram on the left below gives four examples: N30°E, N20°W, S70°E and S45°W (which can also be written simply as SW).

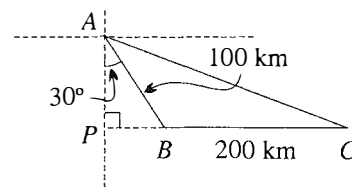
True bearings are measured clockwise from north. The diagram on the right below gives the same four directions expressed as true bearings: 030°T, 340°T, 110°T and 225°T. It is usual for three digits to be used even for numbers of degrees under 100.



WORKED EXERCISE: [Compass bearings and true bearings]

A plane flying at 400 km per hour flies from A to B in a direction $S30^\circ E$ for 15 minutes, then turns sharply to fly due east for 30 minutes to C .

- (a) Find how far south and east of A the point B is.
 (b) Find the true bearing of C from A , to the nearest degree.



SOLUTION:

- (a) The distances AB and BC are 100 km and 200 km respectively.

From the diagram on the right,

$$PB = 100 \cos 60^\circ$$

$$= 50 \text{ km,}$$

$$\text{and } AP = 100 \sin 60^\circ$$

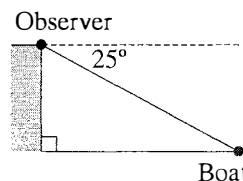
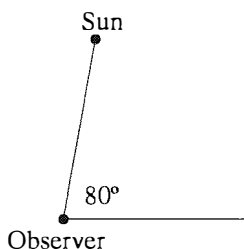
$$= 50\sqrt{3} \text{ km.}$$

$$\begin{aligned} \text{(b) } \tan \angle PAC &= \frac{PC}{AP} \\ &= \frac{50 + 200}{50\sqrt{3}} \\ &= \frac{5}{\sqrt{3}} \end{aligned}$$

$$\angle PAC \doteq 71^\circ,$$

so the bearing of C from A is about $109^\circ T$.

Angles of Elevation and Depression: Angles of elevation and depression are always measured from the horizontal. They are always acute angles.



The *angle of elevation* of the sun in the diagram above is 80° , because the angle at the observer between the sun and the horizontal is 80° .

For an observer on top of the cliff, the *angle of depression* of the boat is 25° , because the angle at the observer between boat and horizontal is 25° .

WORKED EXERCISE: [An example with two triangles] A walker walks on a flat plane directly towards a distant high rocky outcrop R . At point A the angle of elevation of the outcrop is 24° , and a kilometre closer at B the angle of elevation is 32° .

- (a) Find the horizontal distance from B to the outcrop, to the nearest metre.
 (b) Find the height of the outcrop above the plane, to the nearest metre.

SOLUTION: Let M be the point directly below R and level with the plane.

Let $x = BM$, and $h = RM$.

From $\triangle BMR$, $h = x \tan 32^\circ$,

and from $\triangle AMR$, $h = (x + 1) \tan 24^\circ$.

- (a) Equating these expressions for h ,

$$x \tan 32^\circ = (x + 1) \tan 24^\circ$$

$$x(\tan 32^\circ - \tan 24^\circ) = \tan 24^\circ$$

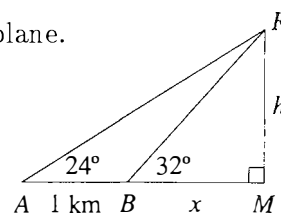
$$x = \frac{\tan 24^\circ}{\tan 32^\circ - \tan 24^\circ}$$

$$BM \doteq 2.478 \text{ km.}$$

- (b) Substituting,

$$h = \frac{\tan 24^\circ \tan 32^\circ}{\tan 32^\circ - \tan 24^\circ}$$

$$RM \doteq 1.549 \text{ km.}$$



Exercise 4A

1. Use your calculator to find, correct to four decimal places:

(a) $\sin 24^\circ$	(d) $\cos 32^\circ 24'$	(g) $\operatorname{cosec} 20^\circ$	(j) $\cot 28^\circ 30'$
(b) $\cos 61^\circ$	(e) $\tan 78^\circ 40'$	(h) $\sec 48^\circ$	(k) $\sec 67^\circ 43'$
(c) $\tan 35^\circ$	(f) $\cos 16^\circ 51'$	(i) $\cot 56^\circ$	(l) $\operatorname{cosec} 81^\circ 13'$

2. Use your calculator to find the acute angle θ correct to the nearest degree if:

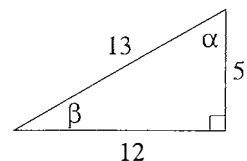
(a) $\tan \theta = 4$	(c) $\cos \theta = \frac{7}{9}$	(e) $\operatorname{cosec} \theta = 5.963$
(b) $\sin \theta = 0.456$	(d) $\sec \theta = 3$	(f) $\cot \theta = 2\frac{4}{7}$

3. Use your calculator to find the acute angle α correct to the nearest minute if:

(a) $\cos \alpha = \frac{3}{4}$	(c) $\sin \alpha = 0.7251$	(e) $\operatorname{cosec} \alpha = \frac{20}{13}$
(b) $\tan \alpha = 0.3$	(d) $\cot \alpha = 0.23$	(f) $\sec \alpha = 3.967$

4. From the diagram opposite, write down the value of:

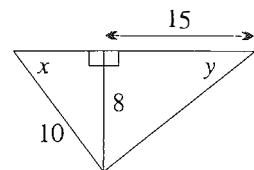
(a) $\sin \alpha$	(c) $\sec \beta$	(e) $\operatorname{cosec} \alpha$
(b) $\tan \beta$	(d) $\cot \alpha$	(f) $\sec \alpha$



5. (a) Use Pythagoras' theorem to find the third side in each of the right triangles in the diagram opposite.

(b) Write down the value of:

(i) $\cos y$	(iii) $\cot x$	(v) $\sec x$
(ii) $\sin x$	(iv) $\operatorname{cosec} y$	(vi) $\cot y$



6. Draw the two special triangles containing the acute angles 30° , 60° and 45° . Hence write down the exact value of:

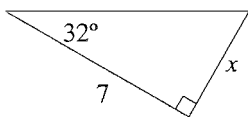
(a) $\sin 60^\circ$	(b) $\tan 30^\circ$	(c) $\cos 45^\circ$	(d) $\sec 60^\circ$	(e) $\operatorname{cosec} 45^\circ$	(f) $\cot 30^\circ$

7. Find, without using a calculator, the value of:

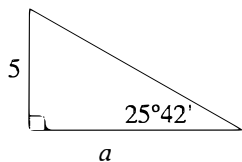
(a) $\sin 45^\circ \cos 45^\circ + \sin 30^\circ$	(c) $1 + \tan^2 60^\circ$
(b) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$	(d) $\operatorname{cosec}^2 30^\circ - \cot^2 30^\circ$

8. Find, correct to one decimal place, the lengths of the sides marked with pronumerals:

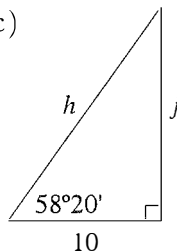
(a)



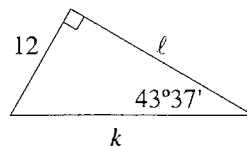
(b)



(c)

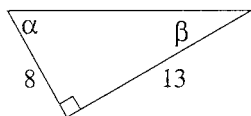


(d)

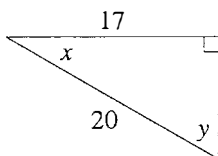


9. Find the sizes of the angles marked with pronumerals, correct to the nearest minute:

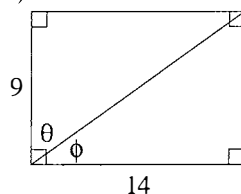
(a)



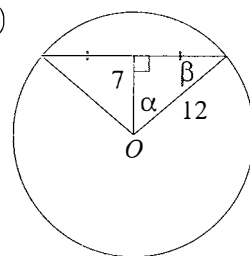
(b)



(c)



(d)



DEVELOPMENT

10. If $A = 17^\circ 25'$ and $B = 31^\circ 49'$, use your calculator to find, correct to two decimal places:

- (a) $\cos 3A$ (c) $\tan(B - A)$ (e) $\operatorname{cosec}(2A + B)$
 (b) $3 \cos A$ (d) $\tan B - \tan A$ (f) $\operatorname{cosec} 2A + \operatorname{cosec} B$

11. It is given that α is an acute angle and that $\tan \alpha = \frac{\sqrt{5}}{2}$.

- (a) Draw a right-angled triangle showing this information.
 (b) Use Pythagoras' theorem to find the length of the unknown side.
 (c) Hence: (i) write down the exact values of $\sin \alpha$ and $\cos \alpha$,
 (ii) show that $\sin^2 \alpha + \cos^2 \alpha = 1$.

12. Suppose that β is an acute angle and $\sec \beta = \frac{\sqrt{11}}{3}$.

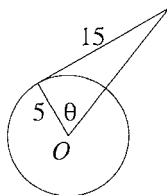
- (a) Find the exact value of: (i) $\operatorname{cosec} \beta$, (ii) $\cot \beta$. (b) Show that $\operatorname{cosec}^2 \beta - \cot^2 \beta = 1$.

13. Without using a calculator, show that:

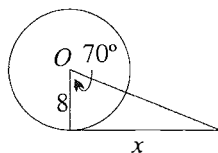
- (a) $1 + \tan^2 45^\circ = \sec^2 45^\circ$ (d) $\frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} = \tan 60^\circ$
 (b) $2 \sin 30^\circ \cos 30^\circ = \sin 60^\circ$ (e) $\sin 60^\circ \tan 45^\circ \tan 30^\circ = \cos 30^\circ \cot 45^\circ \cot 60^\circ$
 (c) $\cos^2 60^\circ - \cos^2 30^\circ = -\frac{1}{2}$ (f) $\sec^2 30^\circ \cot^2 60^\circ - 8 \cos^2 45^\circ \sin^2 60^\circ = -\frac{23}{9}$

14. Find each pronumeral, correct to four significant figures, or to the nearest minute:

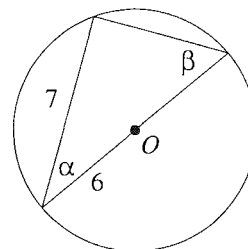
(a)



(b)

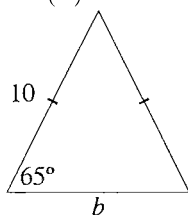


(c)

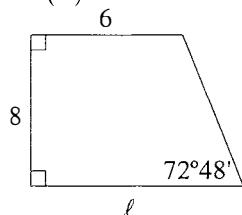


15. Find the value of each pronumeral, correct to three decimal places:

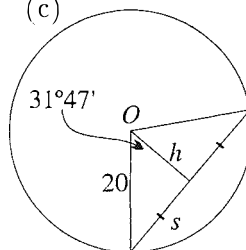
(a)



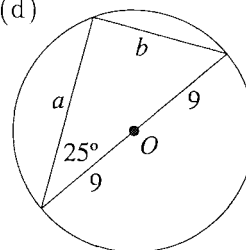
(b)



(c)



(d)

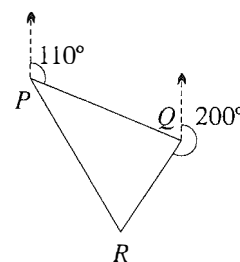


16. A ladder of length 5 metres is placed on level ground against a vertical wall. If the foot of the ladder is 1.5 metres from the base of the wall, find, to the nearest degree, the angle at which the ladder is inclined to the ground.

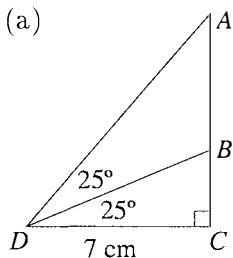
17. Find, to the nearest degree, the angle of depression of a boat 200 metres out to sea, from the top of a vertical cliff of height 40 metres.

18. A ship leaves port P and travels 150 nautical miles to port Q on a bearing of 110° . It then travels 120 nautical miles to port R on a bearing of 200° .

- (a) Explain why $\angle PQR = 90^\circ$.
 (b) Find, to the nearest degree, the bearing of port R from port P .

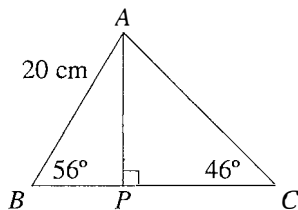


19. (a)



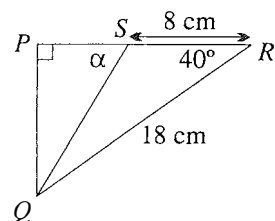
Show that $AC = 7 \tan 50^\circ$ and $BC = 7 \tan 25^\circ$, and hence find the length AB correct to 1 mm.

(b)



Show that $AP = 20 \sin 56^\circ$, and hence find the length of PC , giving your answer correct to 1 cm.

(c)



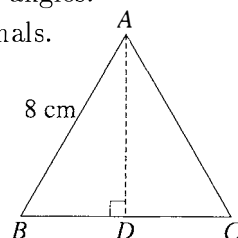
Show that $PR = 18 \cos 40^\circ$, find an expression for PQ , and hence find the angle α to the nearest minute.

20. Answer to four significant figures, or to the nearest minute:

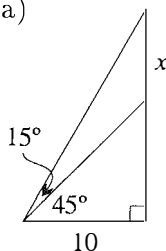
- A triangle has sides of 7 cm, 7 cm and 5 cm. What are the sizes of its angles?
- An isosceles triangle has base angles of 76° . What is the ratio of base to side length?
- A rectangle has dimensions 7 cm $\times$ 12 cm. At what acute angle do the diagonals meet?
- The diagonals of a rectangle meet at 35° . Find the ratio of the length and breadth.
- The diagonals of a rhombus are 16 cm and 10 cm. Find the vertex angles.
- One vertex angle of a rhombus is 25° . Find the ratio of the diagonals.

21. In the figure drawn on the right, $\triangle ABC$ is an equilateral triangle with side length 8 cm.

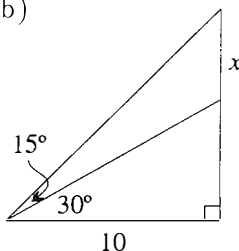
- Show that the perpendicular height AD is $4\sqrt{3}$ cm.
- Hence find the exact area of the triangle.



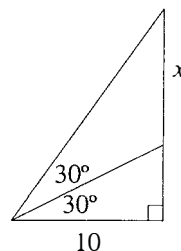
22. (a)



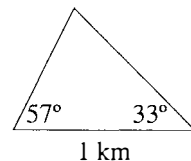
(b)



(c)

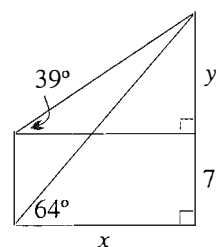


Show that $x = 10(\sqrt{3} - 1)$. Show that $x = \frac{10}{3}(3 - \sqrt{3})$. Show that $x = \frac{20}{3}\sqrt{3}$.

23. From the ends of a straight horizontal road 1 km long, a balloon directly above the road is observed to have angles of elevation of 57° and 33° respectively. Find, correct to the nearest metre, the height of the balloon above the road.24. From a ship sailing due north, a lighthouse is observed to be on a bearing of 42° . Later, when the ship is 2 nautical miles from the lighthouse, the bearing of the lighthouse from the ship is 148° . Find, correct to three significant figures, the distance of the lighthouse from the initial point of observation.25. (a) Use two right triangles in the diagram to write down two equations involving x and y .

(b) By solving the equations simultaneously, show that

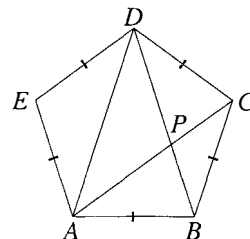
$$x = \frac{7}{\tan 64^\circ - \tan 39^\circ}.$$



EXTENSION

26. [The regular pentagon and the exact value of $\sin 18^\circ$] The regular pentagon $ABCDE$ has sides of length 1 unit. The diagonals AD , BD and AC have been drawn, and the diagonals BD and AC meet at P .

- (a) Find the size of each interior angle of the pentagon.
 (b) Show that $\angle DAB = 72^\circ$ and $\angle DAP = \angle BAP = 36^\circ$.
 (c) Show that the triangles DAB and ABP are similar.
 (d) Let $BP = x$, and show that $AB = AP = DP = 1$ and $DA = \frac{1}{x}$.
 (e) Show that $AD = \frac{1}{2}(\sqrt{5} + 1)$.
 (f) Hence show that $\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5} - 1)$.



4B Theoretical Exercises on Right Triangles

Many problems in trigonometry involve diagrams in which the sides and angles are given in terms of pronumerals. The two worked examples given here have been chosen because they give the classical definition of the sine function (first example), and explain the reason why the words ‘secant’ and ‘tangent’ are used (second example). They also show how close the connection is between the trigonometric functions and circles.

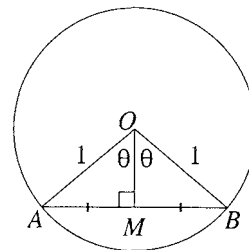
An Earlier Definition of the Sine Function: An earlier interpretation of $\sin \theta$ defined it as the length of the ‘semichord’ subtending an angle θ at the centre of a circle of radius 1. Suppose that a chord AB of a circle of radius 1 subtends an angle 2θ at the centre O . We need to prove that $\sin \theta = \frac{1}{2}AB$.

PROOF: Let M be the midpoint of AB ,
 then by circle geometry, $OM \perp AB$ and $\angle AOM = \theta$,
 so in the right triangle $\triangle AMO$,

$$\frac{AM}{AO} = \sin \theta$$

$$AM = \sin \theta$$

$$\sin \theta = \frac{1}{2}AB.$$



The Origin of the Words Secant and Tangent: The word ‘tangent’ comes from the Latin *tangens* meaning ‘touching’, and a tangent to a circle is a line touching it at one point. The word ‘secant’ comes from the Latin *secans* meaning ‘cutting’, and a secant to a circle is a line cutting it at two points. The following construction shows how an angle θ at the centre of a circle of radius 1 is associated with an interval on a tangent of length $\tan \theta$, and an interval on a secant of length $\sec \theta$.

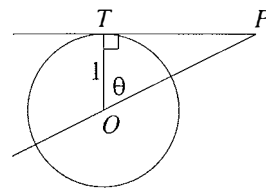
Suppose that P is a point outside a circle of radius 1. Let one of the tangents from P touch the circle at T , and let PT subtend an angle θ at the centre O . Construct the secant through P and O , and join the radius OT . Then

$$PT = \tan \theta \quad \text{and} \quad PO = \sec \theta.$$

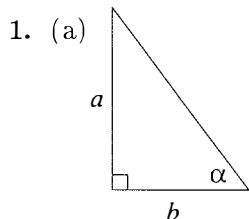
PROOF: By the radius and tangent theorem, the radius OT and the tangent PT are perpendicular, so in the right triangle $\triangle PTO$,

$$\frac{PT}{OT} = \tan \theta \quad \text{and} \quad \frac{PO}{TO} = \sec \theta.$$

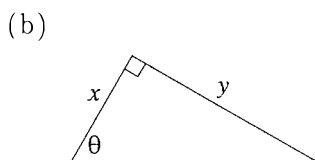
$$\text{Thus } PT = \tan \theta \quad \text{and} \quad PO = \sec \theta.$$



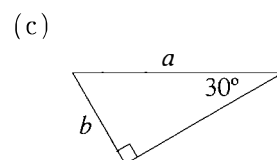
Exercise 4B



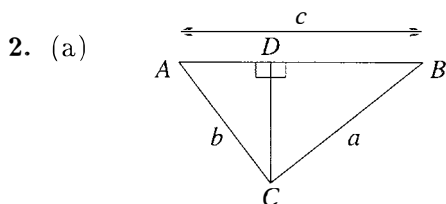
Show that $a = b \tan \alpha$.



$$\text{Show that } \sin^2 \theta = \frac{y^2}{x^2 + y^2}.$$



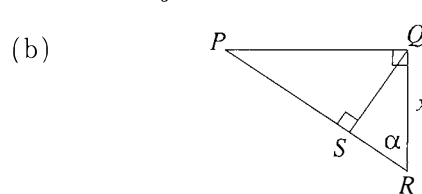
Show that $a = 2b$.



(i) Show that $AD = b \cos A$ and find a similar expression for BD .

(ii) Hence show that

$$c = a \cos B + b \cos A.$$



(i) Show that $PR = x \sec \alpha$.

(ii) Show that $SR = x \cos \alpha$.

(iii) Hence show that

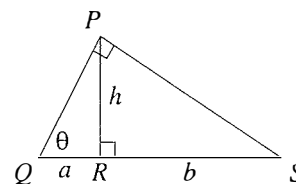
$$PS = x(\sec \alpha - \cos \alpha).$$

3. In the diagram opposite, $\triangle PQS$ is a right triangle, and PR is the altitude to the hypotenuse QS .

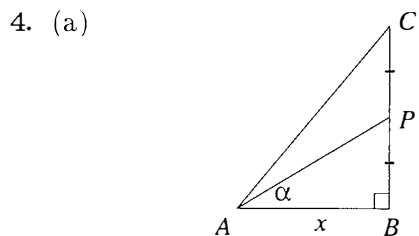
(a) Explain why $\angle RPS = \theta$.

(b) Find two expressions for $\tan \theta$.

(c) Hence show that $ab = h^2$.

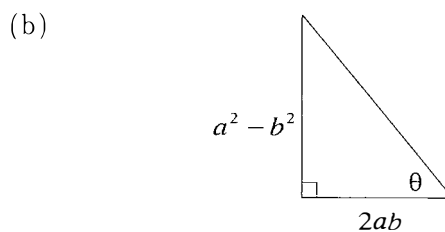


DEVELOPMENT



In the diagram above, $\triangle ABC$ is a right triangle and P is the midpoint of BC . If $\angle PAB = \alpha$, show that

$$BC = 2x \tan \alpha.$$



Prove the algebraic identity

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

Hence show that $\sin \theta = \frac{a^2 - b^2}{a^2 + b^2}$.

5. The tangent PT in the diagram opposite is perpendicular to the radius OT of a circle of radius r and centre O . Let A be the foot of the perpendicular from T to OP , and let $\angle TOP = \theta$.

- (a) Explain why $\angle ATP = \theta$.
 (b) Show that $AP = r \sin \theta \tan \theta$.

6. A rectangle $ABCD$ with length p and breadth q starts off lying flat on a horizontal plane. It is then rotated 60° clockwise about C until it reaches the position shown in the diagram opposite. Find the final heights of D , B and A above the plane.

7. PQR is an equilateral triangle of side length x , and PS is the perpendicular from P to QR . PS is produced to T so that $PT = x$.

- (a) Show that $\angle PQT = 75^\circ$ and hence that $\angle SQT = 15^\circ$.
 (b) Show that $QS = \frac{1}{2}x$ and that $PS = \frac{1}{2}x\sqrt{3}$.
 (c) Show that $ST = \frac{1}{2}x(2 - \sqrt{3})$.
 (d) Hence show that $\tan 15^\circ = 2 - \sqrt{3}$.

8. In triangle ABC , lines CP , PQ and QR are drawn perpendicular to AB , BC and AC respectively.

- (a) Explain why $\angle RBQ = \angle RQP = \angle QPC$.
 (b) Show that $QR = a \sin B \cos^2 B$.

9. AP , PQ and QR are three equal intervals inclined at angles α , 2α and 3α respectively to interval AB . Show that:

$$\tan \angle BAR = \frac{\sin \alpha + \sin 2\alpha + \sin 3\alpha}{\cos \alpha + \cos 2\alpha + \cos 3\alpha}.$$

10. In the diagram opposite, $ABCD$ is a rectangle in which $AB = x$ and $BC = y$. BP and CQ are drawn perpendicular to the interval AE , which is inclined at an angle θ to AB . Show that $AQ = x \cos \theta + y \sin \theta$.

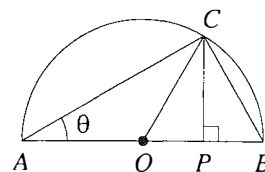
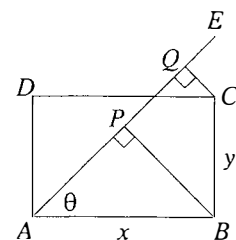
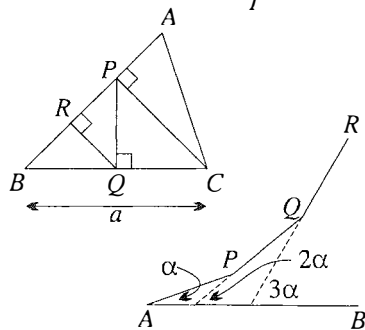
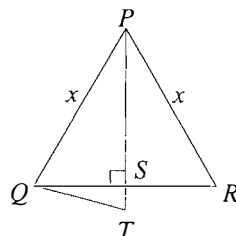
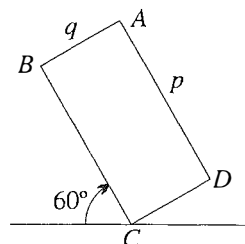
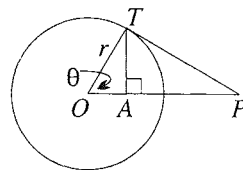
EXTENSION

11. In the diagram, O is the centre of the semicircle ACB , and P is the foot of the perpendicular from C to the diameter AB . Let $\angle OAC = \theta$.

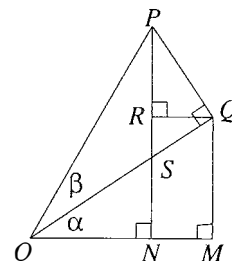
- (a) Show that $\angle POC = 2\theta$ and that $\angle PCB = \theta$.
 (b) Using the two triangles $\triangle APC$ and $\triangle ABC$, show that $\sin \theta \cos \theta = \frac{PC}{AB}$.
 (c) Hence show that $2 \sin \theta \cos \theta = \sin 2\theta$.

12. Using the same diagram as the previous question:

- (a) Explain why $AP - PB = 2 \times OP$.
 (b) Show that $\cos^2 \theta - \sin^2 \theta = \frac{AP}{AC} \times \frac{AC}{AB} - \frac{PB}{CB} \times \frac{CB}{AB}$.
 (c) Hence show that $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$.



13. In the given diagram, $\angle MOQ = \alpha$ and $\angle QOP = \beta$. Also, $PN \perp OM$, $PQ \perp OQ$ and $QR \perp PN$.
- (a) Explain why: (i) $\angle RPQ = \alpha$, (ii) $NP = MQ + RP$.
- (b) Hence use triangles OPN , MOQ , RPQ and POQ to show that $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$.



4C Trigonometric Functions of a General Angle

The definitions of the trigonometric functions given in Section 4A only work for acute angles, because only an angle between 0° and 90° can be put into a right triangle. This section introduces a set of more general definitions based on circles in the coordinate plane. The new definitions will apply to any angle, but will, of course, give the same values at acute angles as the previous definitions.

Putting a General Angle on the Cartesian Plane: Suppose that θ is any angle — possibly negative, possibly obtuse or reflex, possibly greater than 360° . Our first task is to establish a geometrical representation of the angle θ on the Cartesian plane so that we can work with the angle. We shall associate with θ a ray with vertex at the origin.

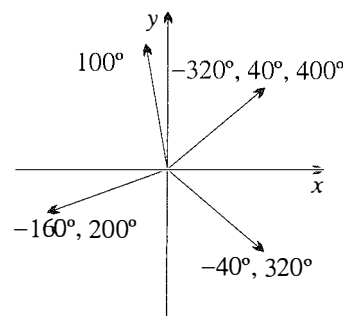
5

DEFINITION: To find the ray corresponding to θ , rotate the positive half of the x -axis through an angle θ in the anticlockwise direction.

Here are some examples of angles and the rays corresponding to them — notice how the angle is written at the end of the arrow representing the ray. If the angle is negative, then the ray is rotated backwards, which means clockwise. Hence one ray can correspond to many angles. For example, all the following angles have the same ray as 40° :

$$\dots, -680^\circ, -320^\circ, 400^\circ, 760^\circ, \dots$$

A given ray thus corresponds to infinitely many angles, all differing by multiples of 360° .



6

ANGLES AND RAYS: To each angle, there corresponds exactly one ray.

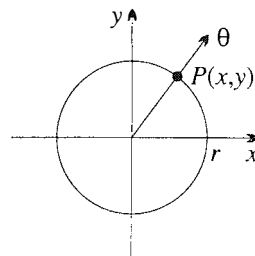
To each ray, there correspond infinitely many angles, all differing from each other by multiples of 360° .

The Definitions of the Trigonometric Functions: Suppose that θ is any angle. Construct the ray corresponding to θ , and construct a circle with centre the origin and any positive radius r . Let the ray and the circle intersect at the point $P(x, y)$. We now define the six trigonometric functions by:

7

DEFINITION:

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \operatorname{cosec} \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



NOTE: We chose r to be ‘any positive radius’. If a different radius had been chosen, x , y and r would change, but the two figures would be similar. Since the definitions depend only on the ratios of the lengths, the values of the trigonometric functions would not change.

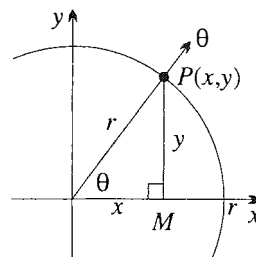
Agreement with the Earlier Definition: Suppose that θ is an acute angle ($0^\circ < \theta < 90^\circ$), and construct the ray corresponding to θ . Drop the perpendicular from P to meet the x -axis at M ; then $\theta = \angle POM$. Relating the sides to the angle θ ,

$$\text{hyp} = OP = r, \quad \text{opp} = PM = y, \quad \text{adj} = OM = x,$$

and so the old and the new definitions coincide.

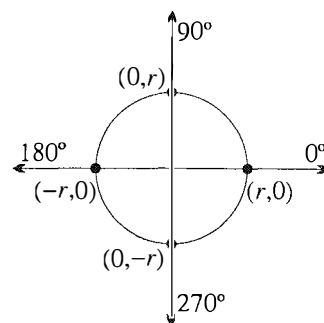
NOTE: Most people find that the diagram above is the easiest way to learn the new definitions of the trigonometric functions. Take the old definitions in terms of hypotenuse, opposite and adjacent sides, and make the replacements

$$\text{hyp} \longrightarrow r, \quad \text{opp} \longrightarrow y, \quad \text{adj} \longrightarrow x.$$



Boundary Angles: Integer multiples of 90° , that is $\dots, -90^\circ, 0^\circ, 90^\circ, 180^\circ, \dots$, are called *boundary angles* because they lie on the boundaries between quadrants. The values of the trigonometric functions at these boundary angles are not always defined, and are 0, 1 or -1 when they are defined. The accompanying diagram can be used to calculate them, and the results are shown in the table below. A star (*) means that the function is undefined at that value.

THE BOUNDARY ANGLES:				
θ	0°	90°	180°	270°
x	r	0	$-r$	0
y	0	r	0	$-r$
r	r	r	r	r
$\sin \theta$	0	1	0	-1
$\cos \theta$	1	0	-1	0
$\tan \theta$	0	*	0	*
$\text{cosec } \theta$	*	1	*	-1
$\sec \theta$	1	*	-1	*
$\cot \theta$	*	0	*	0



In practice, the answer to any question about the values of the trigonometric functions at these boundary angles should be read off the graphs of the functions, and these graphs need to be known very well indeed.

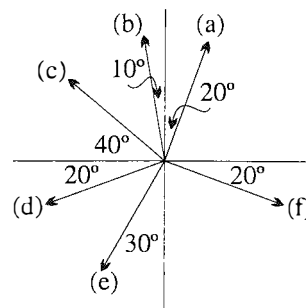
The Domains of the Trigonometric Functions: The trigonometric functions are defined everywhere except where the denominator is zero. Since y is zero at the angles $\dots, -180^\circ, 0^\circ, 180^\circ, 360^\circ, \dots$ and x is zero at $\dots, -90^\circ, 90^\circ, 270^\circ, 540^\circ, \dots$:

DOMAINS OF THE TRIGONOMETRIC FUNCTIONS:

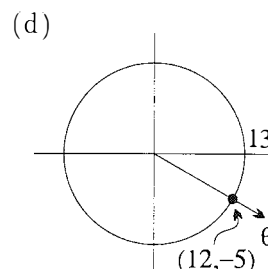
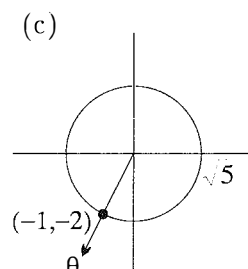
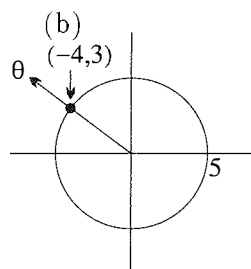
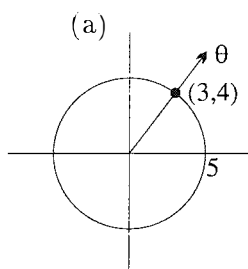
- 9 $\sin \theta$ and $\cos \theta$ are defined for all angles θ .
 $\tan \theta$ and $\sec \theta$ are undefined for $\theta = \dots, -90^\circ, 90^\circ, 270^\circ, 540^\circ, \dots$
 $\cot \theta$ and $\text{cosec } \theta$ are undefined for $\theta = \dots, -180^\circ, 0^\circ, 180^\circ, 360^\circ, \dots$

Exercise 4C

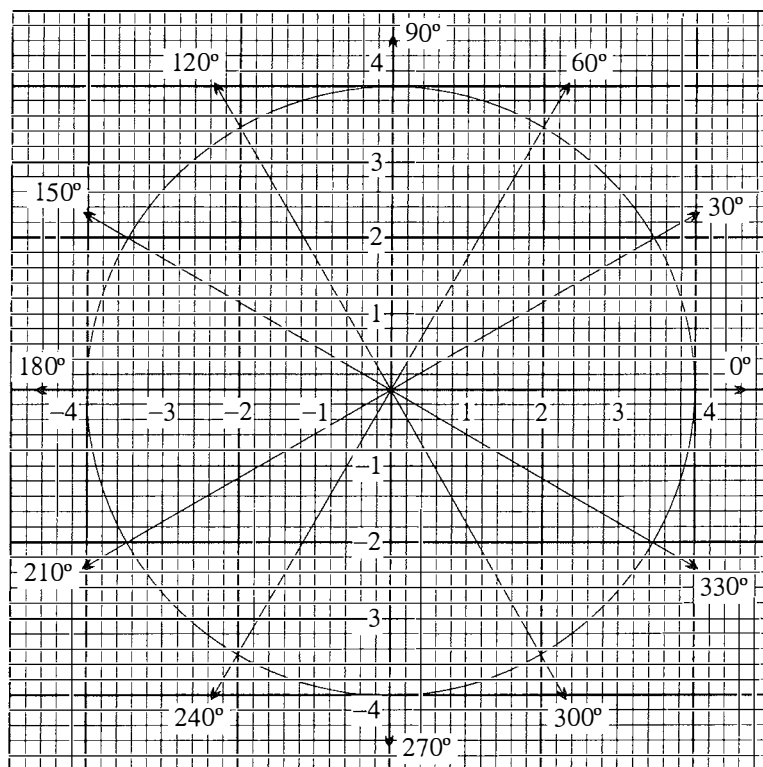
- On a number plane, draw rays representing the following angles:
 (a) 40° (b) 110° (c) 190° (d) 290° (e) 420° (f) 500°
- Repeat the previous question for these angles:
 (a) -50° (b) -130° (c) -250° (d) -350° (e) -440° (f) -550°
- For each of the angles in question 1, name the negative angle between -360° and 0° that is represented by the same ray.
- For each of the angles in question 2, name the positive angle between 0° and 360° that is represented by the same ray.
- Write down two positive angles between 0° and 720° and two negative angles between -720° and 0° that are represented by each of the rays in the diagram on the right.



- Write down the values of the six trigonometric ratios of the angle θ in each diagram:



- [The graphs of $\sin \theta$, $\cos \theta$ and $\tan \theta$] The diagram shows angles from 0° to 360° at 30° intervals. The circle has radius 4 units.



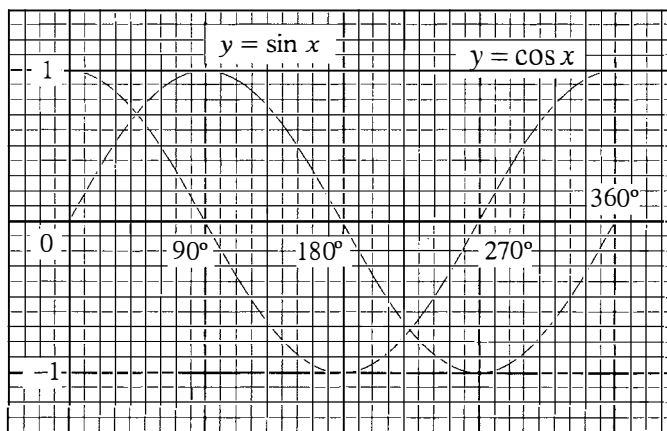
- (a) Use the diagram and the definitions of the three trigonometric ratios to complete the following table. Measure values of x and y correct to two decimal places, and use your calculator only to perform the necessary divisions.

θ	-30°	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°	390°
x															
y															
r															
$\sin \theta$															
$\cos \theta$															
$\tan \theta$															

- (b) Use your calculator to check the accuracy of the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ that you obtained in part (a).
- (c) Using the table of values in part (a), graph the curves $y = \sin \theta$, $y = \cos \theta$ and $y = \tan \theta$ as accurately as possible on graph paper. Use the following scales: 2 mm represents 10° on the horizontal axis and 2 cm represents 1 unit on the vertical axis.

DEVELOPMENT

8.



- (a) Read off the diagram above the value of:
- (i) $\cos 60^\circ$ (iii) $\sin 72^\circ$ (v) $\sin 144^\circ$ (vii) $\cos 153^\circ$ (ix) $\sin 234^\circ$
(ii) $\sin 210^\circ$ (iv) $\cos 18^\circ$ (vi) $\cos 36^\circ$ (viii) $\sin 27^\circ$ (x) $\cos 306^\circ$
- (b) Find from the graphs two values of x between 0° and 360° for which:
- (i) $\sin x = 0.5$ (iii) $\sin x = 0.9$ (v) $\sin x = 0.8$ (vii) $\sin x = -0.4$
(ii) $\cos x = -0.5$ (iv) $\cos x = 0.6$ (vi) $\cos x = -0.8$ (viii) $\cos x = -0.3$
- (c) Find two values of x between 0° and 360° for which $\sin x = \cos x$.
9. [The graphs of $\sec \theta$, $\operatorname{cosec} \theta$ and $\cot \theta$] From the definitions of the trigonometric functions,

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

- (a) Explain why the graph of $y = \operatorname{cosec} \theta$ has vertical asymptotes wherever $\sin \theta = 0$. Explain why the upper branches of $y = \operatorname{cosec} \theta$ have a minimum of 1 wherever $y = \sin \theta$ has a maximum of 1, and the lower branches have a maximum of -1 wherever $y = \sin \theta$ has a minimum of -1 . Hence sketch the graph of $y = \operatorname{cosec} \theta$.

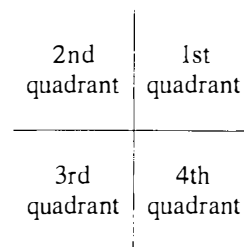
- (b) Use similar methods to produce the graph of $y = \sec \theta$ from the graph of $y = \cos \theta$, and the graph of $y = \cot \theta$ from the graph of $y = \tan \theta$.

EXTENSION

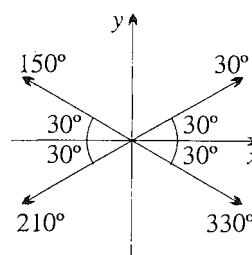
10. [The equation of a cone] The equation behind the definition of all the trigonometric functions is $x^2 + y^2 = r^2$, which is the equation of the circle, and is also Pythagoras' theorem. A third interpretation of this equation comes from regarding x , y and r all as variables, and plotting the resulting surface on a three-dimensional coordinate system with axes labelled x , y and r .
- Explain why the surface obtained in this way is a double cone, with vertex at the origin, and with a right angle at the vertex.
 - What sort of curve is obtained by fixing r at some nonzero value r_0 and letting x and y vary (that is, by cutting the surface with the plane $r = r_0$)?
 - What sort of curve is obtained by fixing x at some nonzero value x_0 and letting y and r vary (that is, by cutting the surface with the plane $x = x_0$)?

4D The Quadrant, the Related Angle and the Sign

It would have been obvious from the calculations in the previous exercise that symmetry in the x -axis and the y -axis plays a large role in the values taken by the trigonometric functions. This section examines that symmetry, and explains how the values of the trigonometric functions of any angle can easily be expressed in terms of the values of the trigonometric functions of acute angles. The diagram shows the conventional anticlockwise numbering of the four quadrants of the coordinate plane — acute angles are in the first quadrant and obtuse angles are in the second quadrant.



The Quadrant and the Related Angle: The diagram opposite shows the four rays corresponding to the four angles 30° , 150° , 210° , 330° . These four rays lie in each of the four quadrants of the plane, and they each make the same acute angle 30° with the x -axis. Consequently, the four rays are just the reflections of each other in the two axes.



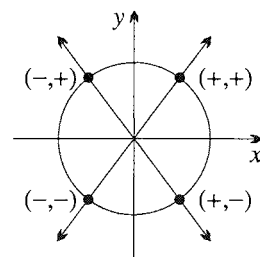
QUADRANT AND RELATED ANGLE: Suppose that θ is any angle.

- 10 The *quadrant* of θ is the quadrant (1, 2, 3 or 4) in which the ray lies.
The *related angle* of θ is the acute angle between the ray and the x -axis.

So each of the four angles in the diagram has the same related angle, 30° . The only time when θ and its related angle are the same is when θ is an acute angle, that is an angle between 0° and 90° .

The Signs of the Trigonometric Functions: The signs of the trigonometric functions depend only on the signs of x and y (the radius r is a positive constant). The signs of x and y depend in turn only on the quadrant in which the ray lies. Thus we can easily compute the signs of the trigonometric functions from the accompanying diagram and the definitions:

quadrant	1st	2nd	3rd	4th
x	+	-	-	+
y	+	+	-	-
r	+	+	+	+
$\sin \theta$	+	+	-	-
$\cos \theta$	+	-	-	+
$\tan \theta$	+	-	+	-
$\operatorname{cosec} \theta$	+	+	-	-
$\sec \theta$	+	-	-	+
$\cot \theta$	+	-	+	-

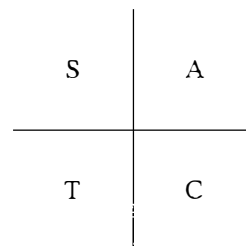


In NSW, these results are usually remembered by the phrase:

11 **SIGNS OF THE TRIGONOMETRIC FUNCTIONS:** ‘All Stations To Central’

indicating that the four letters A, S, T and C are placed successively in the four quadrants as shown. The significance of the letters is:

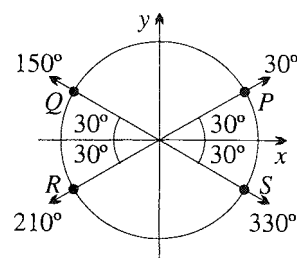
- A means all six functions are positive,
- S means only sine (and cosecant) are positive,
- T means only tangent (and cotangent) are positive,
- C means only cosine (and secant) are positive.



Study each of the graphs constructed in the previous exercise to see how the table of signs above, and the ASTC rule, agree with your observations about when the graph is above the x -axis and when it is below.

The Angle and the Related Angle: In the diagram on the right, a circle of radius r has been added to the earlier diagram that showed the four angles 30° , 150° , 210° and 330° all with the same related angle of 30° .

The four points P , Q , R and S where the rays meet the circle are all reflections of each other in the x and y axes. Because of this symmetry, the coordinates of these four points are identical apart from their sign. Hence the various trigonometric functions on these angles will all be the same too, except that the signs may be different.



12 **ANGLE AND RELATED ANGLE:** The trigonometric functions of any angle θ are the same as the trigonometric functions of its related angle, apart from a possible change of sign. (NOTE: The sign is found using the ASTC diagram.)

Evaluating the Trigonometric Functions at Any Angle: This gives a straightforward way of evaluating the trigonometric functions of any angle, and later, a very clear way of solving trigonometric equations.

13

TRIGONOMETRIC FUNCTIONS AT ANY ANGLE: Draw a quadrant diagram, then:

1. Place the ray in the correct quadrant, and use the ASTC rule to work out the sign of the answer.
2. Find the related angle, and work out the value of the trigonometric function at the related angle.

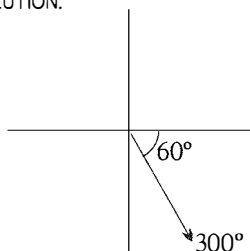
WORKED EXERCISE: Find the exact values of:

(a) $\tan 300^\circ$

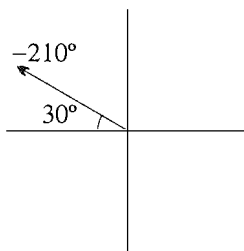
(b) $\sin(-210^\circ)$

(c) $\cos 570^\circ$

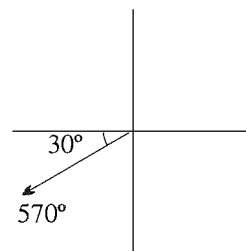
SOLUTION:



- (a) 300° is in quadrant 4, the related angle is 60° , so $\tan 300^\circ = -\tan 60^\circ = -\sqrt{3}$.



- (b) -210° is in quadrant 2, the related angle is 30° , so $\sin(-210^\circ) = +\sin 30^\circ = \frac{1}{2}$.



- (c) 570° is in quadrant 3, the related angle is 30° , so $\cos 570^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$.

NOTE: The calculator will give approximate values of the trigonometric functions without any need to find the related angle. But it will *not* give exact values when these values involve surds, and all calculators eventually cut out or become inaccurate for large angles.

General Angles With Pronumerals: This quadrant-diagram method can be used to generate formulae for expressions such as $\sin(180^\circ + A)$ or $\cot(360^\circ - A)$. The trick is to deal with A on the quadrant diagram as if it were acute.

14

SOME FORMULAE WITH GENERAL ANGLES:

$$\begin{array}{lll} \sin(180^\circ - A) = \sin A & \sin(180^\circ + A) = -\sin A & \sin(360^\circ - A) = -\sin A \\ \cos(180^\circ - A) = -\cos A & \cos(180^\circ + A) = -\cos A & \cos(360^\circ - A) = \cos A \\ \tan(180^\circ - A) = -\tan A & \tan(180^\circ + A) = \tan A & \tan(360^\circ - A) = -\tan A \end{array}$$

Some people prefer to learn this list of identities to evaluate trigonometric functions, but this seems unnecessary when the quadrant-diagram method is so clear.

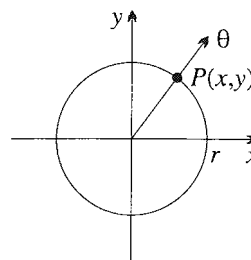
Specifying a Point in Terms of r and θ : If the definitions of $\sin \theta$ and $\cos \theta$ are rewritten with x and y as the subject:

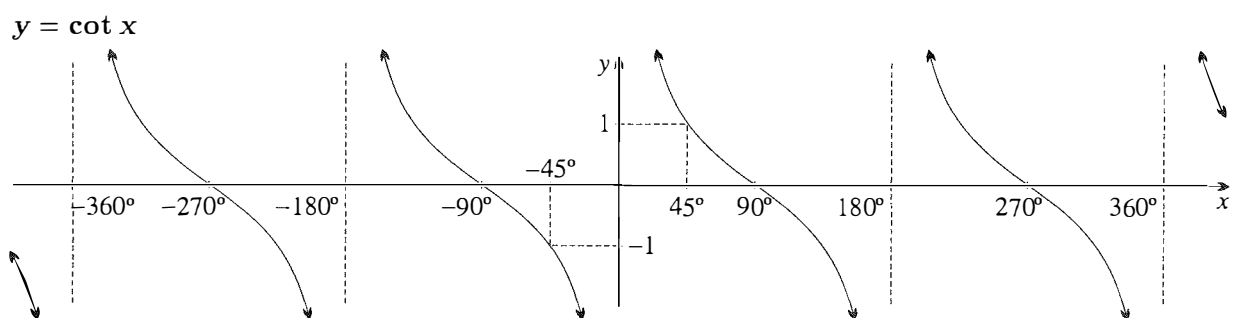
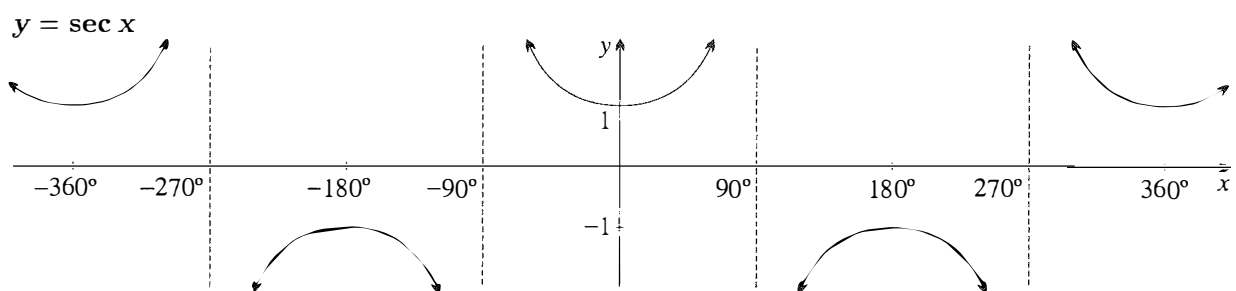
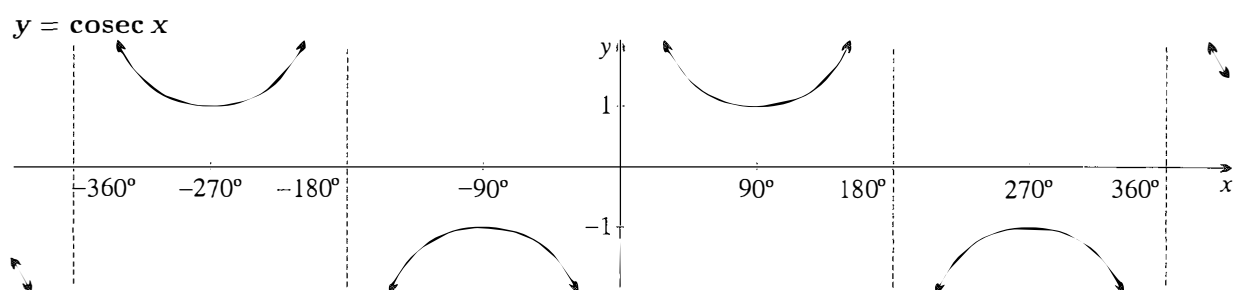
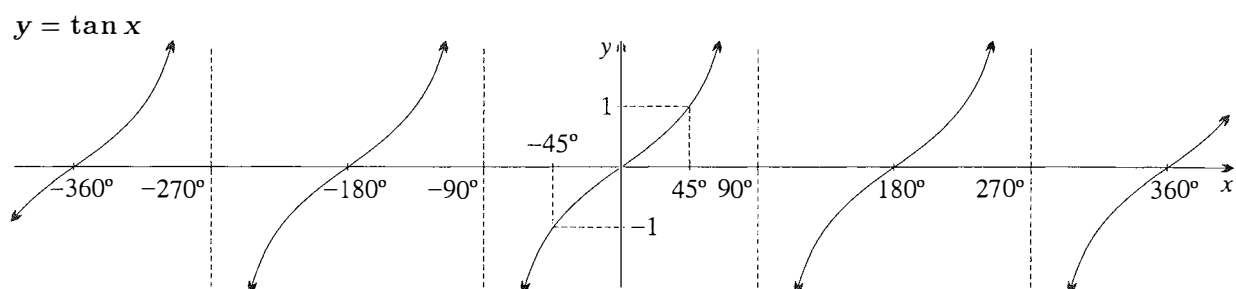
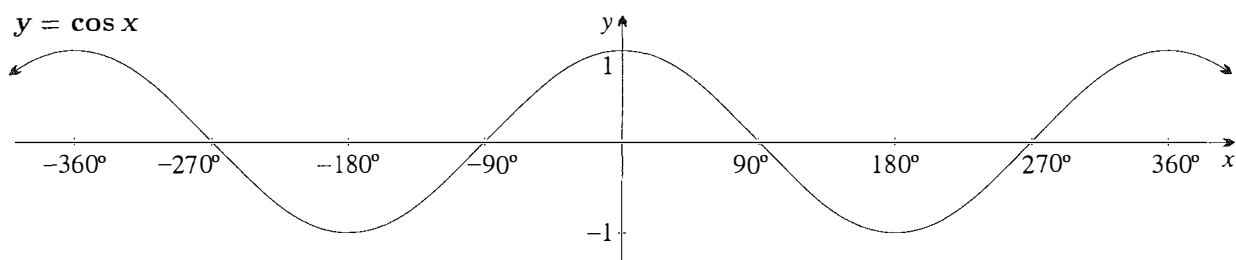
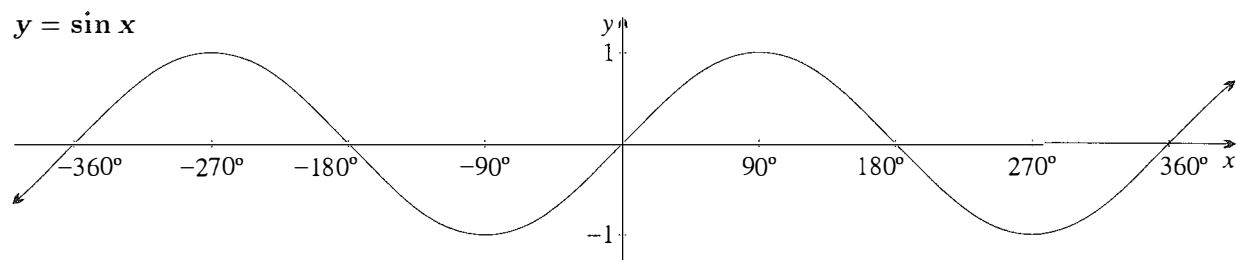
15

RECOVERING THE COORDINATES OF A POINT:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

This means that if a point P is specified in terms of its distance OP from the origin and the angle of the ray OP , then the x and y coordinates of P can be recovered by means of these formulae.



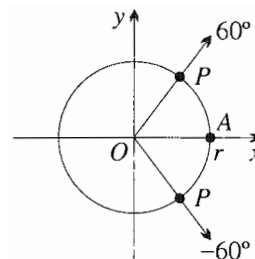


WORKED EXERCISE: The circle $x^2 + y^2 = 36$ meets the positive direction of the x -axis at A . Find the coordinates of the points P on the circle such that $\angle AOP = 60^\circ$.

SOLUTION: The circle has radius 6, so $r = 6$,
and the ray OP has angle 60° or -60° ,
so the coordinates (x, y) of P are

$$\begin{aligned} x &= 6 \cos 60^\circ & \text{or} & & x &= 6 \cos(-60^\circ) \\ &= 3 & & & &= 3 \\ y &= 6 \sin 60^\circ & \text{or} & & y &= 6 \sin(-60^\circ) \\ &= 3\sqrt{3}, & & & &= -3\sqrt{3}. \end{aligned}$$

So $P = (3, 3\sqrt{3})$ or $P = (3, -3\sqrt{3})$.



The Graphs of the Six Trigonometric Functions: In the diagrams on the previous page, the six trigonometric functions have been drawn over the extended range $-450^\circ \leq x \leq 450^\circ$ so that it becomes clear how the graphs are built up by infinite repetition of a simple element.

The sine and cosine graphs are waves. It turns out that these are the basic wave shapes, because any wave pattern, no matter how complicated, can always be reduced to a combination of various types of sine and cosine waves. Later in the course, these six graphs will become fundamental to our work in trigonometry. Their distinctive shapes and symmetries should be studied carefully and remembered. (A question in the following exercise discusses these things.)

Exercise 4D

1. Use the ASTC rule to determine the sign (+ or -) of each of these trigonometric ratios:

(a) $\sin 20^\circ$	(e) $\cot 140^\circ$	(i) $\sin 400^\circ$	(m) $\cot 600^\circ$
(b) $\sec 50^\circ$	(f) $\sin 310^\circ$	(j) $\sec(-30^\circ)$	(n) $\operatorname{cosec} 700^\circ$
(c) $\cos 100^\circ$	(g) $\operatorname{cosec} 200^\circ$	(k) $\tan(-130^\circ)$	(o) $\tan(-400^\circ)$
(d) $\tan 290^\circ$	(h) $\cos 320^\circ$	(l) $\cos 500^\circ$	(p) $\sec(-330^\circ)$

2. Find the related angle for each of the following:

(a) 36°	(c) 310°	(e) -60°	(g) -300°	(i) -500°
(b) 150°	(d) 200°	(f) -150°	(h) 430°	(j) 600°

3. Write each trigonometric ratio as the ratio of an acute angle with the correct sign attached:

(a) $\tan 130^\circ$	(d) $\cot 260^\circ$	(g) $\cos(-175^\circ)$	(j) $\sin(-455^\circ)$
(b) $\cos 310^\circ$	(e) $\sec 170^\circ$	(h) $\operatorname{cosec}(-235^\circ)$	(k) $\sec 1000^\circ$
(c) $\sin 220^\circ$	(f) $\operatorname{cosec} 320^\circ$	(i) $\tan 500^\circ$	(l) $\cot 2000^\circ$

4. Use the trigonometric graphs to find the values (if they exist) of these trigonometric ratios of boundary angles:

(a) $\sin 90^\circ$	(d) $\tan 360^\circ$	(g) $\operatorname{cosec} 270^\circ$	(j) $\cot 450^\circ$
(b) $\cos 180^\circ$	(e) $\tan 90^\circ$	(h) $\cot 270^\circ$	(k) $\cot 540^\circ$
(c) $\cos 270^\circ$	(f) $\sec 360^\circ$	(i) $\operatorname{cosec}(-270^\circ)$	(l) $\operatorname{cosec} 180^\circ$

5. Find the exact value of:

(a) $\cos 135^\circ$	(e) $\cot 210^\circ$	(i) $\sec 480^\circ$	(m) $\sin(-210^\circ)$
(b) $\sin 120^\circ$	(f) $\sec 150^\circ$	(j) $\cot 660^\circ$	(n) $\tan 1500^\circ$
(c) $\tan 225^\circ$	(g) $\operatorname{cosec} 330^\circ$	(k) $\operatorname{cosec}(-60^\circ)$	(o) $\operatorname{cosec}(-135^\circ)$
(d) $\cos 330^\circ$	(h) $\sin 405^\circ$	(l) $\cos(-135^\circ)$	(p) $\sec(-150^\circ)$

6. Given that $\sin 25^\circ \doteq 0.42$ and $\cos 25^\circ \doteq 0.91$, write down approximate values, without using a calculator, for:

(a) $\sin 155^\circ$	(c) $\cos 335^\circ$	(e) $\sin 205^\circ - \cos 155^\circ$
(b) $\cos 205^\circ$	(d) $\sin 335^\circ$	(f) $\cos 385^\circ - \sin 515^\circ$

7. Given that $\tan 35^\circ \doteq 0.70$ and $\sec 35^\circ \doteq 1.22$, write down approximate values for:

(a) $\tan 145^\circ$	(c) $\tan 325^\circ$	(e) $\sec 325^\circ + \tan 395^\circ$
(b) $\sec 215^\circ$	(d) $\tan 215^\circ + \sec 145^\circ$	(f) $\sec(-145^\circ) - \tan(-215^\circ)$

DEVELOPMENT

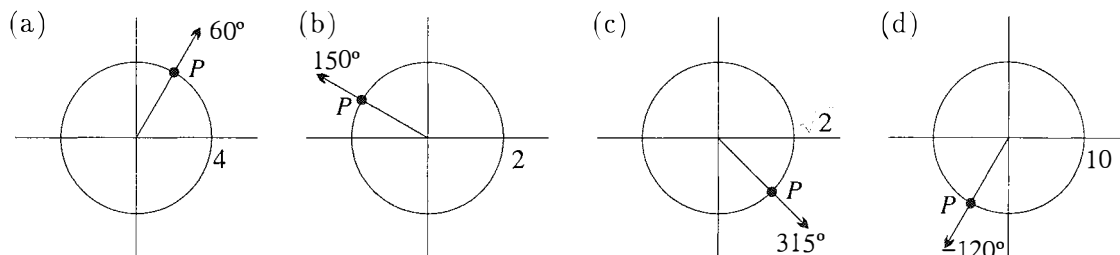
8. Find the value of:

- (a) $\sin 240^\circ \cos 150^\circ - \sin 150^\circ \cos 240^\circ$
 (b) $3 \tan 210^\circ \sec 210^\circ - \sin 330^\circ \cot 135^\circ - \cos 150^\circ \operatorname{cosec} 240^\circ$
 (c) $\sin^2 120^\circ \operatorname{cosec} 270^\circ - \cos^2 315^\circ \sec 180^\circ - \tan^2 225^\circ \cot 315^\circ$

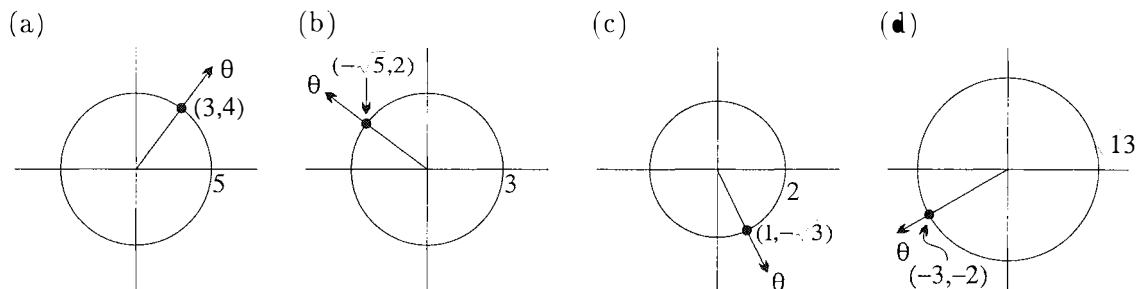
9. Prove:

- (a) $\sin 330^\circ \cos 150^\circ - \cos 390^\circ \sin 390^\circ = 0$
 (b) $\sin 420^\circ \cos 405^\circ + \cos 420^\circ \sin 405^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$
 (c) $\frac{\sin 135^\circ - \cos 120^\circ}{\sin 135^\circ + \cos 120^\circ} = 3 + 2\sqrt{2}$
 (d) $(\sin 150^\circ + \cos 270^\circ + \tan 315^\circ)^2 = \sin^2 135^\circ \cos^2 225^\circ$
 (e) $\frac{\sin 120^\circ}{\tan 300^\circ} - \frac{\cos 240^\circ}{\cot 315^\circ} = \tan^2 240^\circ - \operatorname{cosec}^2 330^\circ$

10. Find the coordinates of the point P in each of the following diagrams:



11. Find the angle θ , correct to the nearest minute where necessary, given that $0^\circ < \theta < 360^\circ$:



12. Show that the following relationships are satisfied by the given values:

(a) $\sin 2\theta = 2 \sin \theta \cos \theta$, when $\theta = 150^\circ$.

(b) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$, when $\theta = 135^\circ$.

(c) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, when $\theta = 225^\circ$.

(d) $\sin(A + B) = \sin A \cos B + \cos A \sin B$, when $A = 300^\circ$ and $B = 240^\circ$.

(e) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$, when $A = 330^\circ$ and $B = 210^\circ$.

13. Write as a trigonometric ratio of A , with the correct sign attached:

(a) $\sin(-A)$ (d) $\sec(-A)$ (g) $\cos(180^\circ - A)$ (j) $\operatorname{cosec}(360^\circ - A)$

(b) $\cos(-A)$ (e) $\sin(180^\circ - A)$ (h) $\tan(180^\circ + A)$ (k) $\cot(180^\circ - A)$

(c) $\tan(-A)$ (f) $\sin(360^\circ - A)$ (i) $\sec(180^\circ + A)$ (l) $\sec(360^\circ - A)$

14. Examine the graphs of the six trigonometric functions on page 124, then answer these questions.

(a) What are the ranges of the six functions?

(b) What is the *period* of each function, that is, how far does one move on the horizontal axis before the graph repeats itself? How is this period related to the identities

$$\sin(\theta + 360^\circ) = \sin \theta, \quad \sec(\theta + 360^\circ) = \sec \theta, \quad \tan(\theta + 180^\circ) = \tan \theta?$$

(c) Which functions are even and which are odd?

(d) More generally, about what points do the graphs have point symmetry? (That is, about what points are they unchanged by a rotation of 180° ?)

(e) What are the axes of symmetry of the graphs?

EXTENSION

15. Write as a trigonometric ratio of θ with the correct sign attached:

(a) $\sin(90^\circ + \theta)$ (c) $\cos(90^\circ + \theta)$ (e) $\cot(90^\circ + \theta)$

(b) $\sin(90^\circ - \theta)$ (d) $\sin(270^\circ - \theta)$ (f) $\sec(270^\circ - \theta)$

16. Simplify:

(a) $\cos(180^\circ - \alpha) \sec \alpha$ (c) $\sin(90^\circ - \alpha) \sec(90^\circ + \alpha)$

(b) $\sec \alpha \sin(180^\circ - \alpha)$ (d) $\cot(180^\circ + \alpha) \cos(270^\circ - \alpha)$

17. Show that:

(a) $\tan(90^\circ - A) \sec(180^\circ + A) \cos(90^\circ + A) = 1$

(b) $\tan(180^\circ - A) \sin(270^\circ + A) \operatorname{cosec}(360^\circ - A) = -1$

4E Given One Trigonometric Function, Find Another

When the exact value of one trigonometric function is known for an angle, the exact value of the other trigonometric functions can easily be found using the circle diagram and Pythagoras' theorem.

16

GIVEN ONE TRIGONOMETRIC FUNCTION, FIND ANOTHER: Draw a circle diagram, and use Pythagoras' theorem to find whichever of x , y and r is missing.

WORKED EXERCISE:

- (a) Given that $\sin \theta = \frac{1}{5}$, find $\cos \theta$.
 (b) Repeat if it is also known that $\tan \theta$ is negative.

SOLUTION:

- (a) First, the angle must be in quadrant 1 or 2.

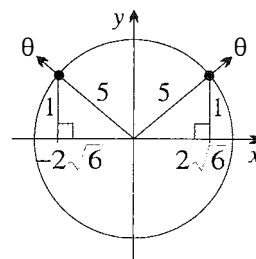
Since $\sin \theta = \frac{y}{r} = \frac{1}{5}$, we can take $y = 1$ and $r = 5$,

so by Pythagoras' theorem, $x = \sqrt{24} = 2\sqrt{6}$,

$$\text{so} \quad \cos \theta = \frac{2\sqrt{6}}{5} \quad \text{or} \quad -\frac{2\sqrt{6}}{5}.$$

- (b) Since $\tan \theta$ is negative, θ must be in quadrant 2,

$$\text{so} \quad \cos \theta = -\frac{2\sqrt{6}}{5}.$$

**Exercise 4E**

- (a) Given that $\cos \theta = \frac{3}{5}$ and θ is acute, find $\sin \theta$ and $\tan \theta$.
 (b) Given that $\tan \theta = -\frac{5}{12}$ and θ is obtuse, find $\sin \theta$ and $\sec \theta$.
- (a) Given that $\sin \alpha = \frac{8}{17}$, find the possible values of $\cos \alpha$ and $\cot \alpha$.
 (b) Given that $\cos x = -\frac{3}{4}$ and $90^\circ < x < 180^\circ$, find $\tan x$ and $\operatorname{cosec} x$.
- (a) Given that $\cot \beta = \frac{3}{2}$ and $\sin \beta < 0$, find $\cos \beta$.
 (b) If $\operatorname{cosec} \alpha = -\frac{5}{2}$ and $\cos \alpha > 0$, find $\cot \alpha$.
 (c) If $\tan \theta = 2$, find the possible values of $\operatorname{cosec} \theta$.
 (d) Suppose that $\sin A = 1$. Find $\sec A$.
- (a) Given that $\sec P = -3$ and $180^\circ < P < 360^\circ$, find $\operatorname{cosec} P$.
 (b) If $\cos \theta = -1$, find $\tan \theta$.
 (c) Suppose that $\cos \alpha = \frac{2}{3}$. Find the possible values of $\sin \alpha$ and $\cot \alpha$.
 (d) Given that $\cot x = -\frac{3}{5}$, find the possible values of $\operatorname{cosec} x$ and $\sec x$.

DEVELOPMENT

- Given that $\sin \theta = \frac{p}{q}$, with θ obtuse and p and q both positive, find expressions for $\cos \theta$ and $\tan \theta$.
- If $\tan \alpha = k$, where $k > 0$, find the possible values of $\sin \alpha$ and $\sec \alpha$.
- (a) Prove the algebraic identity $(1 - t^2)^2 + 4t^2 = (1 + t^2)^2$.
 (b) If $\cos x = \frac{1 - t^2}{1 + t^2}$ and x is acute, find expressions for $\sin x$ and $\tan x$.

EXTENSION

- If $\sin \theta = k$ and θ is obtuse, find an expression for $\tan(\theta + 90^\circ)$.
- If $\sec \theta = a + \frac{1}{4a}$, prove that $\sec \theta + \tan \theta = 2a$ or $\frac{1}{2a}$.

4F Trigonometric Identities and Elimination

Working with the trigonometric functions requires knowledge of a number of formulae called *trigonometric identities*, which relate trigonometric functions to each other. This section introduces eleven of these in four groups: the three *reciprocal identities*, the two *ratio identities*, the three *Pythagorean identities*, and the three *identities concerning complementary angles*.

The Three Reciprocal Identities: It follows immediately from the definitions of the trigonometric functions in terms of x , y and r that:

17

THE RECIPROCAL IDENTITIES: For all angles θ :

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad (\text{provided } \sin \theta \neq 0)$$

$$\sec \theta = \frac{1}{\cos \theta} \quad (\text{provided } \cos \theta \neq 0)$$

$$\cot \theta = \frac{1}{\tan \theta} \quad (\text{provided } \tan \theta \neq 0 \text{ and } \cot \theta \neq 0)$$

NOTE: The last identity needs attention. One cannot use the calculator to find $\cot 90^\circ$ or $\cot 270^\circ$ by first finding $\tan 90^\circ$ or $\tan 270^\circ$, because both of these are undefined. We already know, however, that $\cot 90^\circ = \cot 270^\circ = 0$.

The Two Ratio Identities: Again using the definitions of the trigonometric functions:

18

THE RATIO IDENTITIES: For any angle θ :

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (\text{provided } \cos \theta \neq 0)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (\text{provided } \sin \theta \neq 0)$$

The Three Pythagorean Identities: Since the point $P(x, y)$ lies on the circle with centre O and radius r , its coordinates satisfy

$$x^2 + y^2 = r^2.$$

Dividing through by r^2 gives $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$,

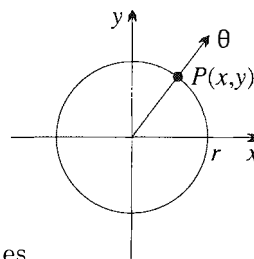
then by the definitions, $\sin^2 \theta + \cos^2 \theta = 1$.

Dividing through by $\cos^2 \theta$ and using the ratio and reciprocal identities,

$$\tan^2 \theta + 1 = \sec^2 \theta, \quad \text{provided } \cos \theta \neq 0.$$

Dividing through instead by $\sin^2 \theta$, $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$, provided $\sin \theta \neq 0$.

These identities are called the *Pythagorean identities* because they rely on the circle equation $x^2 + y^2 = r^2$, which is really just a restatement of Pythagoras' theorem.



19

THE PYTHAGOREAN IDENTITIES: For any angle θ :

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta \quad (\text{provided } \cos \theta \neq 0)$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta \quad (\text{provided } \sin \theta \neq 0)$$

The Three Identities for Complementary Angles: These identities relate the values of the trigonometric functions at any angle θ to their values at the complementary angle $90^\circ - \theta$.

20

THE COMPLEMENTARY IDENTITIES: For any angle θ :

$$\cos(90^\circ - \theta) = \sin \theta$$

$$\cot(90^\circ - \theta) = \tan \theta \quad (\text{provided } \tan \theta \text{ is defined})$$

$$\operatorname{cosec}(90^\circ - \theta) = \sec \theta \quad (\text{provided } \sec \theta \text{ is defined})$$

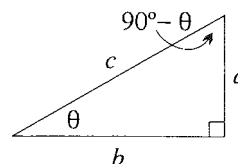
PROOF:

- A. [Acute angles] The triangle on the right shows that when θ is acute, viewing the right triangle from $90^\circ - \theta$ instead of from θ exchanges the opposite side and the adjacent side, so:

$$\cos(90^\circ - \theta) = \frac{a}{c} = \sin \theta,$$

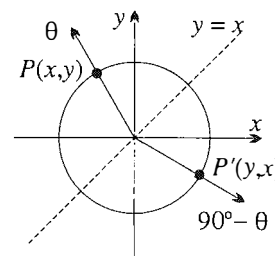
$$\cot(90^\circ - \theta) = \frac{a}{b} = \tan \theta,$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{c}{b} = \sec \theta.$$



- B. [General angles] For general angles, we take the full circle diagram, and reflect it in the diagonal line $y = x$. Let P' be the image of P under this reflection.

1. The image OP' of the ray OP corresponds with the angle $90^\circ - \theta$.
2. The image P' of $P(x, y)$ has coordinates $P'(y, x)$. We have seen before that reflection in the line $y = x$ reverses the coordinates of each point. So x and y are interchanged in passing from P to P' .



Applying the definitions of the trigonometric functions to the angle $90^\circ - \theta$:

$$\cos(90^\circ - \theta) = \frac{y}{r} = \sin \theta,$$

$$\cot(90^\circ - \theta) = \frac{y}{x} = \tan \theta, \quad \text{provided } x \neq 0,$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{r}{x} = \sec \theta, \quad \text{provided } x \neq 0.$$

Cosine, Cosecant and Cotangent: The complementary identities are the origin of the 'co-' prefix of cosine, cosecant and cotangent — the prefix is an abbreviation of the prefix 'com-' of complementary angle. The various identities can be easily remembered as:

21

CO-FUNCTIONS: The co-function of a complement is the function of an angle.
The co-function of an angle is the function of the complement.

Proving Identities: An *identity* is a statement that needs to be proven true for all values of θ for which both sides are defined. It is quite different from an equation, which needs to be solved and to have its solutions listed.

22

PROVING TRIGONOMETRIC IDENTITIES: Work separately on the LHS and the RHS until they are the same.

WORKED EXERCISE: Prove that $\sin A \sec A = \tan A$. NOTE: The necessary restriction to angles for which $\sec A$ and $\tan A$ are defined is implied by the statement.

$$\begin{aligned}\text{SOLUTION: LHS} &= \sin A \times \frac{1}{\cos A} \quad (\text{reciprocal identity}) \\ &= \tan A \quad (\text{ratio identity}) \\ &= \text{RHS}\end{aligned}$$

WORKED EXERCISE: Prove that $\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta \operatorname{cosec}^2 \theta$.

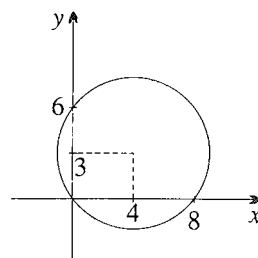
$$\begin{aligned}\text{PROOF: LHS} &= \frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} \quad (\text{common denominator}) \\ &= \frac{1}{\sin^2 \theta \cos^2 \theta} \quad (\text{Pythagorean identity}) \\ &= \sec^2 \theta \operatorname{cosec}^2 \theta \quad (\text{reciprocal identities}) \\ &= \text{RHS}\end{aligned}$$

Elimination: If x and y are given as functions of θ , then using the techniques of simultaneous equations, the θ can often be eliminated to give a relation (rarely a function) between x and y .

WORKED EXERCISE: Eliminate θ from the following pair, and describe the graph of the relation:

$$\begin{aligned}x &= 4 + 5 \cos \theta \\ y &= 3 - 5 \sin \theta\end{aligned}$$

SOLUTION: From the first equation, $5 \cos \theta = x - 4$,
and from the second equation, $5 \sin \theta = 3 - y$.
Squaring and adding, $25 \cos^2 \theta + 25 \sin^2 \theta = (x - 4)^2 + (3 - y)^2$
and since $\cos^2 \theta + \sin^2 \theta = 1$, $(x - 4)^2 + (y - 3)^2 = 25$,
which is a circle of radius 5 and centre $(4, 3)$.



Exercise 4F

1. Use your calculator to verify that:

$$\begin{array}{lll}(\text{a}) \sin 16^\circ = \cos 74^\circ & (\text{c}) \sec 7^\circ = \operatorname{cosec} 83^\circ & (\text{e}) 1 + \tan^2 55^\circ = \sec^2 55^\circ \\ (\text{b}) \tan 63^\circ = \cot 27^\circ & (\text{d}) \sin^2 23^\circ + \cos^2 23^\circ = 1 & (\text{f}) \operatorname{cosec}^2 32^\circ - 1 = \cot^2 32^\circ\end{array}$$

2. Simplify: (a) $\frac{1}{\sin \theta}$ (b) $\frac{1}{\tan \alpha}$ (c) $\frac{\sin \beta}{\cos \beta}$ (d) $\frac{\cos \phi}{\sin \phi}$

3. Simplify: (a) $\sin \alpha \operatorname{cosec} \alpha$ (b) $\cot \beta \tan \beta$ (c) $\cos \theta \sec \theta$

4. Prove: (a) $\tan \theta \cos \theta = \sin \theta$ (b) $\cot \alpha \sin \alpha = \cos \alpha$ (c) $\sin \beta \sec \beta = \tan \beta$

5. Prove: (a) $\cos A \operatorname{cosec} A = \cot A$ (b) $\operatorname{cosec} x \cos x \tan x = 1$ (c) $\sin y \cot y \sec y = 1$

6. Simplify: (a) $\frac{\cos \alpha}{\sec \alpha}$ (b) $\frac{\sin \alpha}{\operatorname{cosec} \alpha}$ (c) $\frac{\tan A}{\sec A}$ (d) $\frac{\cot A}{\operatorname{cosec} A}$

7. Simplify: (a) $\frac{1}{\sec^2 \theta}$ (b) $\sin^2 \alpha \operatorname{cosec}^2 \alpha$ (c) $\frac{\sin^2 \beta}{\cos^2 \beta}$ (d) $\frac{\cos^2 A}{\sin^2 A}$
8. Simplify: (a) $\sin(90^\circ - \theta)$ (b) $\sec(90^\circ - \alpha)$ (c) $\frac{1}{\cot(90^\circ - \beta)}$ (d) $\frac{\cos(90^\circ - \phi)}{\sin(90^\circ - \phi)}$
9. Simplify: (a) $\sin^2 \alpha + \cos^2 \alpha$ (b) $1 - \cos^2 \beta$ (c) $1 + \tan^2 \phi$ (d) $\sec^2 x - \tan^2 x$
10. Simplify: (a) $1 - \sin^2 \beta$ (b) $1 + \cot^2 \phi$ (c) $\operatorname{cosec}^2 A - 1$ (d) $\cot^2 \theta - \operatorname{cosec}^2 \theta$

DEVELOPMENT

11. Prove the identities:

(a) $(1 - \sin \theta)(1 + \sin \theta) = \cos^2 \theta$	(f) $3 \cos^2 \theta - 2 = 1 - 3 \sin^2 \theta$
(b) $(1 + \tan^2 \alpha) \cos^2 \alpha = 1$	(g) $2 \tan^2 A - 1 = 2 \sec^2 A - 3$
(c) $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$	(h) $1 - \tan^2 \alpha + \sec^2 \alpha = 2$
(d) $\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$	(i) $\cos^4 x + \cos^2 x \sin^2 x = \cos^2 x$
(e) $\tan^2 \phi \cos^2 \phi + \cot^2 \phi \sin^2 \phi = 1$	(j) $\cot \theta (\sec^2 \theta - 1) = \tan \theta$

12. Prove the identities:

(a) $\sin \theta \cos \theta \operatorname{cosec}^2 \theta = \cot \theta$	(f) $\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta} = 2 \sec^2 \theta$
(b) $(\cos \phi + \cot \phi) \sec \phi = 1 + \operatorname{cosec} \phi$	(g) $\sin \beta + \cot \beta \cos \beta = \operatorname{cosec} \beta$
(c) $\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha} = \sec \alpha \operatorname{cosec} \alpha$	(h) $\frac{1}{\sec \phi - \tan \phi} - \frac{1}{\sec \phi + \tan \phi} = 2 \tan \phi$
(d) $\frac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$	(i) $\frac{1 + \cot x}{1 + \tan x} = \cot x$
(e) $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$	(j) $\frac{\cos \alpha}{1 + \sin \alpha} = \sec \alpha (1 - \sin \alpha)$

13. (a) If $x = a \cos \alpha$ and $y = a \sin \alpha$, show that $x^2 + y^2 = a^2$.

(b) If $x = a \sec \theta$ and $y = b \tan \theta$, show that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(c) If $x = r \cos \theta \sin \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \phi$, show that $x^2 + y^2 + z^2 = r^2$.

(d) If $x = a \cos \theta - b \sin \theta$ and $y = a \sin \theta + b \cos \theta$, show that $x^2 + y^2 = a^2 + b^2$.

14. Eliminate θ from each pair of equations:

(a) $x = a \cos \theta$ and $y = b \sin \theta$	(c) $x = 2 + \cos \theta$ and $y = 1 + \sin \theta$
(b) $x = a \tan \theta$ and $y = b \sec \theta$	(d) $x = \sin \theta + \cos \theta$ and $y = \sin \theta - \cos \theta$

15. Prove that each expression is independent of θ :

(a) $\frac{\cos^2 \theta}{1 + \sin \theta} + \frac{\cos^2 \theta}{1 - \sin \theta}$	(c) $\frac{\tan \theta + \cot \theta}{\sec \theta \operatorname{cosec} \theta}$
(b) $\tan \theta (1 - \cot^2 \theta) + \cot \theta (1 - \tan^2 \theta)$	(d) $\frac{\tan \theta + 1}{\sec \theta} - \frac{\cot \theta + 1}{\operatorname{cosec} \theta}$

16. Prove the identities:

(a) $\frac{2 \cos^3 \theta - \cos \theta}{\sin \theta \cos^2 \theta - \sin^3 \theta} = \cot \theta$	(b) $\sec y + \tan y + \cot y = \frac{1 + \sin y}{\sin y \cos y}$
(c) $\frac{\cos A - \tan A \sin A}{\cos A + \tan A \sin A} = 1 - 2 \sin^2 A$	
(d) $(\sin \phi + \cos \phi)(\sec \phi + \operatorname{cosec} \phi) = 2 + \tan \phi + \cot \phi$	

$$\begin{aligned}
 \text{(e)} \quad & \frac{1}{1 + \tan^2 x} - \frac{1}{1 + \sec^2 x} = \frac{\cos^4 x}{1 + \cos^2 x} & \text{(f)} \quad & \frac{\cos \theta}{1 - \tan \theta} + \frac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta \\
 \text{(g)} \quad & (\tan \alpha + \cot \alpha - 1)(\sin \alpha + \cos \alpha) = \frac{\sec \alpha}{\operatorname{cosec}^2 \alpha} + \frac{\operatorname{cosec} \alpha}{\sec^2 \alpha} \\
 \text{(h)} \quad & \frac{1}{\sec \theta + \tan \theta} = \sec \theta - \tan \theta = \frac{\cos \theta}{1 + \sin \theta} \\
 \text{(i)} \quad & \frac{1}{\cot \theta - \cos \theta} = \frac{\tan \theta}{1 - \sin \theta} = \frac{\sin \theta + \sin^2 \theta}{\cos^3 \theta} \\
 \text{(j)} \quad & \sin^2 x(1 + n \cot^2 x) + \cos^2 x(1 + n \tan^2 x) = n + 1 \\
 & = \sin^2 x(n + \cot^2 x) + \cos^2 x(n + \tan^2 x) \\
 \text{(k)} \quad & \frac{(\sin^2 \alpha - \cos^2 \alpha)(1 - \sin \alpha \cos \alpha)}{\cos \alpha (\sec \alpha - \operatorname{cosec} \alpha)(\sin^3 \alpha + \cos^3 \alpha)} = \sin \alpha \\
 \text{(l)} \quad & \frac{1 + \operatorname{cosec}^2 A \tan^2 C}{1 + \operatorname{cosec}^2 B \tan^2 C} = \frac{1 + \cot^2 A \sin^2 C}{1 + \cot^2 B \sin^2 C}
 \end{aligned}$$

EXTENSION

17. Eliminate θ from each pair of equations:

- (a) $x = \operatorname{cosec}^2 \theta + 2 \cot^2 \theta$ and $y = 2 \operatorname{cosec}^2 \theta + \cot^2 \theta$
 (b) $x = \sin \theta - 3 \cos \theta$ and $y = \sin \theta + 2 \cos \theta$
 (c) $x = \sin \theta + \cos \theta$ and $y = \tan \theta + \cot \theta$ [HINT: Find $x^2 y$.]

18. (a) If $\frac{a}{\sin A} = \frac{b}{\cos A}$, show that $\sin A \cos A = \frac{ab}{a^2 + b^2}$.
 (b) If $\frac{a+b}{\operatorname{cosec} x} = \frac{a-b}{\cot x}$, show that $\operatorname{cosec} x \cot x = \frac{a^2 - b^2}{4ab}$.
 (c) If $\tan \theta + \sin \theta = x$ and $\tan \theta - \sin \theta = y$, prove that $x^4 + y^4 = 2xy(8 + xy)$.

4G Trigonometric Equations

This piece of work is absolutely vital, because so many problems in later work end up with a trigonometric equation that has to be solved. There are many small details and qualifications in the methods, and the subject needs a great deal of careful study.

Pay Attention to the Domain: To begin with a warning, before any other details:

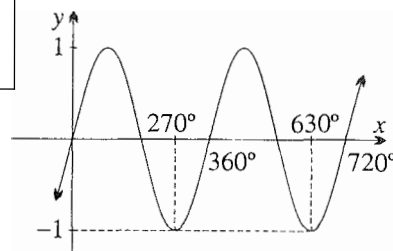
23 THE DOMAIN: Always pay attention to the domain in which the angle can lie.

Equations Involving Boundary Angles: The usual quadrants-and-related-angle method described below doesn't apply to boundary angles, which do not lie in any quadrant.

24 THE BOUNDARY ANGLES: If a trigonometric equation involves boundary angles, read the solutions off a sketch of the graph.

WORKED EXERCISE: Solve $\sin x = -1$, for $0^\circ \leq x \leq 720^\circ$.

SOLUTION: The graph of $y = \sin x$ is drawn on the right. Reading from this graph, $x = 270^\circ$ or 630° .



The Standard Method — Quadrants and Related Angle: Nearly all our trigonometric equations will eventually come down to something like

$$\sin x = -\frac{1}{2}, \text{ where } -180^\circ \leq x \leq 360^\circ.$$

As long as the angle is not a boundary angle, the method is:

THE QUADRANTS-AND-RELATED-ANGLE METHOD:

25

1. Draw a quadrant diagram, then draw a ray in each quadrant that the angle could be in.
2. Find the related angle (only work with positive numbers here):
 - (a) using special angles, or
 - (b) using the calculator to find an approximation.
3. Mark the angles on the ends of the rays, taking account of any restrictions on x , and write a conclusion.

WORKED EXERCISE: Solve each equation. Give the solution exactly if possible, or else to the nearest degree:

(a) $\sin x = -\frac{1}{2}, \quad -180^\circ \leq x \leq 360^\circ$ (b) $\tan x = -3, \quad 0^\circ \leq x \leq 360^\circ$

SOLUTION:

(a) $\sin x = -\frac{1}{2}, \text{ where } -180^\circ \leq x \leq 360^\circ$

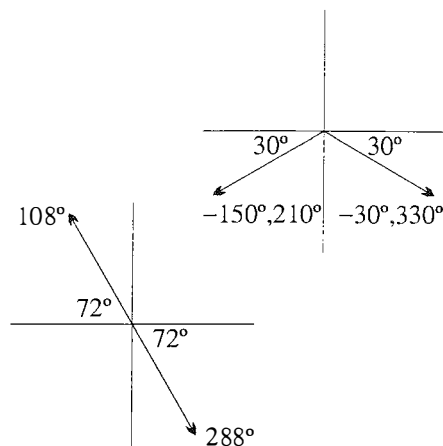
$$x = -150^\circ, -30^\circ, 210^\circ \text{ or } 330^\circ$$

(Since $\sin x$ is negative, x is in quadrants 3 or 4, the sine of the related angle is $+\frac{1}{2}$, so the related angle is 30° .)

(b) $\tan x = -3, \text{ where } 0^\circ \leq x \leq 360^\circ$

$$x \doteq 108^\circ \text{ or } 288^\circ$$

(Since $\tan x$ is negative, x is in quadrants 2 or 4, the tangent of the related angle is $+3$, so the related angle is about 72° .)



NOTE: When using the calculator, *never* enter a negative number and take an inverse trigonometric function of it. In the example above, the calculator was used to find the acute angle whose $\tan$ was 3, that is, $71^\circ 34'$. The positive number 3 was entered, not -3 .

The Three Reciprocal Functions: Because they are unfamiliar, and also because the calculator doesn't have specific keys for them:

26

THE RECIPROCAL FUNCTIONS: Try to change any of the three reciprocal functions secant, cosecant and cotangent to the three more common functions by taking reciprocals.

WORKED EXERCISE: Suppose we are given that $\operatorname{cosec} x = -2$.

Taking reciprocals of both sides gives $\sin x = -\frac{1}{2}$, which was solved in the previous worked example.

Equations with Compound Angles: These can cause trouble. Equations like

$$\tan 2x = \sqrt{3}, \text{ where } 0^\circ \leq x \leq 360^\circ, \quad \text{or}$$

$$\sin(x - 250^\circ) = \frac{\sqrt{3}}{2}, \text{ where } 0^\circ \leq x \leq 360^\circ,$$

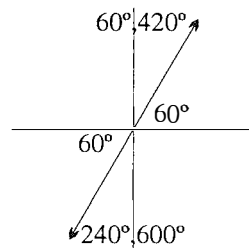
are really trigonometric equations in the compound angles $2x$ and $(x - 250^\circ)$ respectively. The secret lies in solving for the *compound angle*, and in *calculating first the domain for that compound angle*.

EQUATIONS WITH COMPOUND ANGLES:

- 27**
1. Let u be the compound angle.
 2. Find the restrictions on u from the given restrictions on x .
 3. Solve the trigonometric equation for u .
 4. Hence solve for x .

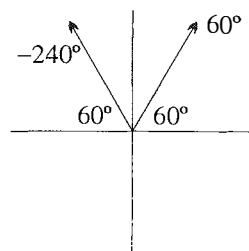
WORKED EXERCISE: Solve $\tan 2x = \sqrt{3}$, where $0^\circ \leq x \leq 360^\circ$.

SOLUTION: Let $u = 2x$.
 Then $\tan u = \sqrt{3}$, where $0^\circ \leq u \leq 720^\circ$,
 (the restriction on u is the key step here),
 so from the diagram, $u = 60^\circ, 240^\circ, 420^\circ$ or 600° .
 Since $x = \frac{1}{2}u$, $x = 30^\circ, 120^\circ, 210^\circ$ or 300° .



WORKED EXERCISE: Solve $\sin(x - 250^\circ) = \frac{\sqrt{3}}{2}$, where $0^\circ \leq x \leq 360^\circ$.

SOLUTION: Let $u = x - 250^\circ$.
 Then $\sin u = \frac{\sqrt{3}}{2}$, where $-250^\circ \leq u \leq 110^\circ$,
 (again, the restriction on u is the key step here),
 so from the diagram, $u = -240^\circ$ or 60° .
 Since $x = u + 250^\circ$, $x = 10^\circ$ or 310° .



Equations Requiring Algebraic Substitutions: If there are powers or reciprocals of the trigonometric function present, as in

$$5 \sin^2 x = \sin x, \text{ for } 0^\circ \leq x \leq 360^\circ, \quad \text{or}$$

$$\frac{4}{\cos x} - \cos x = 0, \text{ for } -180^\circ \leq x \leq 180^\circ,$$

but still only the one trigonometric function, then it is probably better to make a substitution so that the algebra can be done without interference by the trigonometric notation.

ALGEBRAIC SUBSTITUTION:

- 28**
1. Substitute u to obtain a purely algebraic equation.
 2. Solve the algebraic equation — it may have more than one solution.
 3. Solve each of the resulting trigonometric equations.

WORKED EXERCISE: Solve $5\sin^2 x = \sin x$, for $0^\circ \leq x \leq 360^\circ$. Give the exact value of the solutions if possible, otherwise approximate to the nearest minute.

SOLUTION: Let $u = \sin x$.

Then $5u^2 = u$

$$\boxed{-u} \quad 5u^2 - u = 0$$

$$u(5u - 1) = 0$$

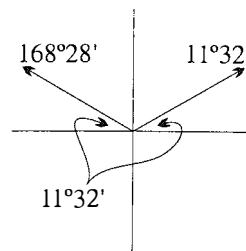
$$u = 0 \text{ or } u = \frac{1}{5},$$

so $\sin x = 0$ or $\sin x = \frac{1}{5}$.

Using the graph of $y = \sin x$ to solve $\sin x = 0$

and the quadrant-diagram method to solve $\sin x = \frac{1}{5}$,

$$x = 0^\circ, 180^\circ \text{ or } 360^\circ, \text{ or } x \doteq 11^\circ 32' \text{ or } 168^\circ 28'.$$



WORKED EXERCISE: Solve $\frac{4}{\cos x} - \cos x = 0$, for $-180^\circ \leq x \leq 180^\circ$.

SOLUTION: Let $u = \cos x$.

Then $\frac{4}{u} - u = 0$

$$\boxed{\times u} \quad 4 - u^2 = 0$$

$$u = 2 \text{ or } u = -2,$$

so $\cos x = 2$ or $\cos x = -2$.

Neither equation has a solution, because $\cos x$ lies between -1 and 1 ,

so there are no solutions.

Equations with More than One Trigonometric Function: Often a trigonometric equation will involve more than one trigonometric function, as, for example,

$$\sec^2 x + \tan x = 1, \text{ where } 180^\circ \leq x \leq 360^\circ.$$

29

EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION: Usually use trigonometric identities to produce an equation in only one trigonometric function, then proceed by substitution as before.

If all else fails, reduce everything to sines and cosines, and hope for the best!

WORKED EXERCISE: Solve $\sec^2 x + \tan x = 1$, where $180^\circ \leq x \leq 360^\circ$ (as above).

SOLUTION: Recognizing that $\sec^2 x = 1 + \tan^2 x$, the equation becomes

$$1 + \tan^2 x + \tan x = 1, \text{ where } 180^\circ \leq x \leq 360^\circ$$

$$\boxed{-1} \quad \tan^2 x + \tan x = 0,$$

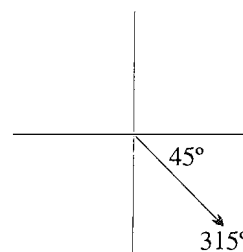
$$\tan x(\tan x + 1) = 0,$$

so $\tan x = 0$ or $\tan x = -1$.

Using the graph of $y = \tan x$ to solve $\tan x = 0$,

and the quadrant-diagram method to solve $\tan x = -1$,

$$x = 180^\circ, 360^\circ \text{ or } 315^\circ.$$



Homogeneous Equations: One special sort of equation which occurs quite often is called *homogeneous* in $\sin x$ and $\cos x$ because the sum of the indices of $\sin x$ and $\cos x$ in each term is the same. For example, the following equation is homogeneous of degree 2 in $\sin x$ and $\cos x$:

$$\sin^2 x - 3 \sin x \cos x + 2 \cos^2 x = 0, \text{ for } 0^\circ \leq x \leq 180^\circ.$$

30 HOMOGENEOUS EQUATIONS: To solve a homogeneous equation in $\sin x$ and $\cos x$, divide through by a power of $\cos x$ to produce an equation in $\tan x$.

WORKED EXERCISE: Continuing with the example above,

$$\boxed{\div \cos^2 x} \quad \tan^2 x - 3 \tan x + 2 = 0.$$

Let $u = \tan x$,

then $u^2 - 3u + 2 = 0$

$$(u - 2)(u - 1) = 0$$

$$u = 2 \text{ or } u = 1$$

$$\tan x = 2 \text{ or } \tan x = 1.$$

So $x \doteq 63^\circ 26' \text{ or } x = 45^\circ.$

Exercise 4G

- Solve each of these equations for $0^\circ \leq \theta \leq 360^\circ$ (each related angle is 30° , 45° or 60°):

(a) $\sin \theta = \frac{\sqrt{3}}{2}$	(c) $\cos \theta = -\frac{1}{\sqrt{2}}$	(e) $\operatorname{cosec} \theta = -2$
(b) $\tan \theta = 1$	(d) $\tan \theta = -\sqrt{3}$	(f) $\sec \theta = -\frac{2}{\sqrt{3}}$
- Solve each of these equations for $0^\circ \leq \theta \leq 360^\circ$ (the trigonometric graphs are helpful here):

(a) $\sin \theta = 1$	(c) $\cos \theta = 0$	(e) $\tan \theta = 0$
(b) $\cos \theta = -1$	(d) $\sec \theta = 1$	(f) $\cot \theta = 0$
- Solve each of these equations for $0^\circ \leq x \leq 360^\circ$. Use your calculator to find the related angle in each case, and give solutions correct to the nearest degree.

(a) $\cos x = \frac{3}{7}$	(c) $\tan x = -7$	(e) $\operatorname{cosec} x = -\frac{3}{2}$
(b) $\sin x = 0.1234$	(d) $\cot x = -0.45$	(f) $\sec x = 6$
- Solve each of these equations for $\bullet$ in the given domain. Give solutions correct to the nearest minute where necessary:

(a) $\sin \alpha = 0.1, 0^\circ \leq \bullet \leq 360^\circ$	(i) $\sqrt{3} \tan \alpha + 1 = 0, \alpha \text{ obtuse}$
(b) $\cos \alpha = -0.1, 0^\circ \leq \alpha \leq 360^\circ$	(j) $\operatorname{cosec} \bullet + 2 = 0, \alpha \text{ reflex}$
(c) $\tan \alpha = -1, -180^\circ \leq \alpha \leq 180^\circ$	(k) $2 \cos \alpha - 1 = 0, 0^\circ \leq \alpha \leq 360^\circ$
(d) $\operatorname{cosec} \alpha = -1, 0^\circ \leq \alpha \leq 360^\circ$	(l) $\cot \alpha = 3, 0^\circ \leq \alpha \leq 360^\circ$
(e) $\sin \alpha = 3, 0^\circ \leq \alpha \leq 360^\circ$	(m) $\tan \alpha = 0, -360^\circ \leq \bullet \leq 360^\circ$
(f) $\sec \alpha = \sqrt{2}, 0^\circ \leq \alpha \leq 360^\circ$	(n) $\tan \alpha = -0.3, -180^\circ \leq \alpha \leq 180^\circ$
(g) $\cos \alpha = 0, -180^\circ \leq \alpha \leq 180^\circ$	(o) $\sin \alpha = -0.7, 0^\circ \leq \bullet \leq 720^\circ$
(h) $\cot \alpha = \frac{1}{2}, \alpha \text{ reflex}$	(p) $\tan \alpha = 1 - \sqrt{2}, 0^\circ \leq \bullet \leq 360^\circ$

DEVELOPMENT

5. Solve for $0^\circ \leq \theta \leq 360^\circ$, giving solutions correct to the nearest degree where necessary:
 (a) $\cos^2 \theta = 1$ (b) $\sec^2 \theta = \frac{4}{3}$ (c) $\tan^2 \theta = 9$ (d) $\operatorname{cosec}^2 \theta = 2$
6. Solve for $0^\circ \leq x \leq 360^\circ$ (let u be the compound angle):
 (a) $\sin 2x = \frac{1}{2}$ (b) $\cos 2x = -\frac{1}{\sqrt{2}}$ (c) $\tan 3x = \sqrt{3}$ (d) $\sec 3x = 0$
7. Solve for $0^\circ \leq \alpha \leq 360^\circ$ (let u be the compound angle):
 (a) $\tan(\alpha - 45^\circ) = \frac{1}{\sqrt{3}}$ (c) $\cot(\alpha + 60^\circ) = 1$
 (b) $\sin(\alpha + 30^\circ) = -\frac{\sqrt{3}}{2}$ (d) $\operatorname{cosec}(\alpha - 75^\circ) = -2$
8. Solve for $0^\circ \leq \theta \leq 360^\circ$:
 (a) $\sin \theta = \cos \theta$ (c) $4 \sin \theta = 3 \operatorname{cosec} \theta$
 (b) $\sqrt{3} \sin \theta + \cos \theta = 0$ (d) $\sec \theta - 2 \cos \theta = 0$
9. Solve for $0^\circ \leq \theta \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:
 (a) $\cos^2 \theta - \cos \theta = 0$ (f) $\sec^2 \theta + 2 \sec \theta = 8$
 (b) $\cot^2 \theta = \sqrt{3} \cot \theta$ (g) $3 \cos^2 \theta + 5 \cos \theta = 2$
 (c) $2 \sin \theta \cos \theta = \sin \theta$ (h) $4 \operatorname{cosec}^2 \theta - 4 \operatorname{cosec} \theta - 15 = 0$
 (d) $\tan^2 \theta - \tan \theta - 2 = 0$ (i) $4 \sin^3 \theta = 3 \sin \theta$
 (e) $2 \sin^2 \theta - \sin \theta = 1$
10. Solve for $0^\circ \leq x \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:
 (a) $2 \sin^2 x + \cos x = 2$ (d) $6 \tan^2 x = 5 \sec x$
 (b) $\sec^2 x - 2 \tan x - 4 = 0$ (e) $6 \operatorname{cosec}^2 x = \cot x + 8$
 (c) $8 \cos^2 x = 2 \sin x + 7$
11. Solve for $0^\circ \leq \alpha \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:
 (a) $3 \sin \alpha = \operatorname{cosec} \alpha + 2$ (b) $3 \tan \alpha - 2 \cot \alpha = 5$
12. Solve for $0^\circ \leq A \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:
 (a) $\cot A + 4 \tan A = 4 \operatorname{cosec} A$ (b) $3(\tan A + \sec A) = 2 \cot A$
13. Solve for $0^\circ \leq x \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:
 (a) $\cos x \tan x + \tan x = \cos x + 1$ (b) $6 \sin x \cos x + 3 \sin x = 2 \cos x + 1$
14. Solve each of these homogeneous equations for $0^\circ \leq x \leq 360^\circ$ by dividing both sides by a suitable power of $\cos x$. Give solutions to the nearest minute where necessary.
 (a) $\sin x = 3 \cos x$ (c) $5 \sin^2 x + 8 \sin x \cos x = 4 \cos^2 x$
 (b) $\sin^2 x - 2 \sin x \cos x - 8 \cos^2 x = 0$ (d) $\sin^3 x + 2 \sin^2 x \cos x + \sin x \cos^2 x = 0$

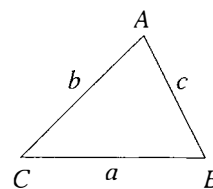
EXTENSION

15. Solve for $0^\circ \leq \theta \leq 360^\circ$, giving solutions correct to the nearest minute where necessary:
 (a) $4 \cos^2 \theta + 2 \sin \theta = 3$ (h) $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} + \cos \theta = 0$
 (b) $5 \sec^2 \theta + 7 \tan \theta = 7$ (i) $(\sqrt{3} + 1) \cos^2 \theta - 1 = (\sqrt{3} - 1) \sin \theta \cos \theta$
 (c) $\cos^2 \theta - 8 \sin \theta \cos \theta + 3 = 0$ (j) $\frac{1 + 2 \sin^2 \theta}{\cos^2 \theta} + 4 \tan \theta = 0$
 (d) $5 \sin^2 \theta - 4 \sin \theta \cos \theta + 3 \cos^2 \theta = 2$
 (e) $8 \cos^4 \theta - 10 \cos^2 \theta + 3 = 0$
 (f) $\sqrt{6} \cos \theta + \sqrt{2} \sin \theta + \sqrt{3} \cot \theta + 1 = 0$
 (g) $20 \cot \theta + 15 \cot \theta \operatorname{cosec} \theta - 4 \operatorname{cosec} \theta = 3(1 + \cot^2 \theta)$

4H The Sine Rule and the Area Formula

The sine rule, the area rule and the cosine rule belong both to trigonometry and to geometry. On the one hand, they extend the elementary trigonometry of Section 4A to non-right-angled triangles. On the other hand, they generalise Pythagoras' theorem, the isosceles triangle theorem, and some results about altitudes of triangles. They are also closely related to the four congruence tests, and the sine rule can be restated as a theorem about the diameter of the circum-circle of a triangle. These last three sections review the rules and their applications. Their proofs should now be given more attention, particularly because they involve connections between trigonometry and Euclidean Geometry.

Statement of the Sine Rule: We will often use the convention that each side of a triangle is given the lower-case letter of the opposite vertex, as in the diagram on the right. Using that convention, here are the verbal and symbolic statements of the sine rule.



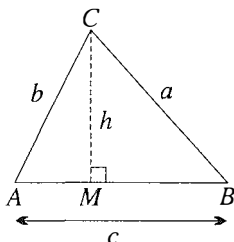
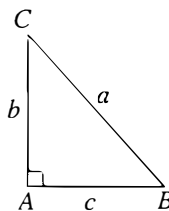
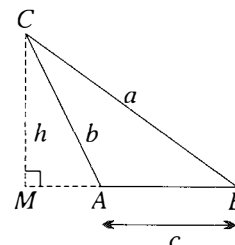
31

THEOREM — THE SINE RULE: In any triangle, the ratio of each side to the sine of the opposite angle is constant. That is, in any triangle $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proving the Sine Rule by Constructing an Altitude: So far we can only handle right triangles, so any proof of the sine rule must involve a construction with a right angle. The obvious approach is to construct an *altitude*, which is the perpendicular from one vertex to the opposite side.

GIVEN: Let ABC be any triangle. There are three cases, depending on whether $\angle A$ is an acute angle, a right angle, or an obtuse angle.

CASE 1: $\angle A$ is acuteCASE 2: $\angle A = 90^\circ$ CASE 3: $\angle A$ is obtuse

AIM: To prove that $\frac{a}{\sin A} = \frac{b}{\sin B}$.

In case 2, $\sin A = \sin 90^\circ = 1$, and $\sin B = \frac{b}{a}$, so the result is clear.

CONSTRUCTION: In the remaining cases 1 and 3, construct the altitude from C , meeting AB , produced if necessary, at M . Let h be the length of CM .

PROOF:

CASE 1 — Suppose $\angle A$ is acute.In the triangle ACM , $\frac{h}{b} = \sin A$

$$\boxed{\times b} \quad h = b \sin A.$$

In the triangle BCM , $\frac{h}{a} = \sin B$

$$\boxed{\times a} \quad h = a \sin B.$$

Equating these, $b \sin A = a \sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

CASE 3 — Suppose $\angle A$ is obtuse.In the triangle ACM , $\frac{h}{b} = \sin(180^\circ - A)$,and since $\sin(180^\circ - A) = \sin A$,

$$\boxed{\times b} \quad h = b \sin A.$$

In the triangle BCM , $\frac{h}{a} = \sin B$

$$\boxed{\times a} \quad h = a \sin B.$$

Equating these, $b \sin A = a \sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

The Area Formula: The well-known formula $\text{area} = \frac{1}{2} \times \text{base} \times \text{height}$ can be generalised to a formula involving two sides and the included angle.

32

THEOREM — THE AREA FORMULA: The area of a triangle is half the product of any two sides and the sine of the included angle. That is,

$$\text{area } \triangle ABC = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$$

PROOF: We use the same diagrams as in the proof of the sine rule.

In case 2, $\angle A = 90^\circ$ and $\sin A = 1$, so $\text{area} = \frac{1}{2}bc = \frac{1}{2}bc \sin A$, as required.Otherwise, $\text{area} = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times AB \times h$$

$$= \frac{1}{2} \times c \times b \sin A, \text{ since we proved before that } h = b \sin A.$$

Using the Sine Rule to Find a Side — The AAS Congruence Situation: For the sine rule to be applied to the problem of finding a side, one side and two angles must be known. This is the situation described by the AAS congruence test, so only one triangle will be possible. The sine rule should be learned in verbal form because the triangle being solved could have any names, or could be unnamed.

33

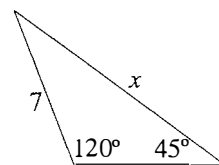
USING THE SINE RULE TO FIND A SIDE: In the AAS congruence situation:

$$\frac{\text{unknown side}}{\text{sine of opposite angle}} = \frac{\text{known side}}{\text{sine of opposite angle}}$$

WORKED EXERCISE: Find x in the given triangle.

SOLUTION:

$$\begin{aligned} \frac{x}{\sin 120^\circ} &= \frac{7}{\sin 45^\circ} \\ \boxed{\times \sin 120^\circ} \quad x &= \frac{7 \sin 120^\circ}{\sin 45^\circ} \\ &= 7 \times \frac{\sqrt{3}}{2} \times \sqrt{2} \\ &= \frac{7}{2} \sqrt{6} \end{aligned}$$



Using the Area Formula — The SAS Congruence Situation: The area formula requires the SAS situation where two sides and the included angle are known.

34

USING THE AREA FORMULA: In the SAS congruence situation:

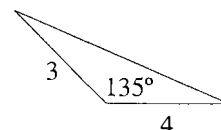
$$\text{area} = (\text{half the product of two sides}) \times (\text{sine of the included angle})$$

WORKED EXERCISE: Find the area of the given triangle.

SOLUTION: $\text{Area} = \frac{1}{2} \times 3 \times 4 \times \sin 135^\circ$

$$= 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 3\sqrt{2} \text{ square units.}$$



Using the Sine Rule to Find an Angle — The Ambiguous ASS Situation: It is well known that the SAS congruence test requires that the angle be included between the two sides. When two sides and a non-included angle are known, the situation is normally referred to as ‘the spurious ASS test’ or ‘the ambiguous ASS test’, because in many such situations the resulting triangle is not quite determined up to congruence, and two triangles may be possible.

When the sine rule is applied in the ASS situation, there is only one answer for the sine of an angle. Angles in triangles, however, can be acute or obtuse, and the sines of both acute and obtuse angles are positive, so there may be two possible solutions for the angle itself.

35

USING THE SINE RULE TO FIND AN ANGLE: If two sides and a non-included angle of the triangle are known, corresponding to the ambiguous ASS situation, then:

$$\frac{\text{sine of unknown angle}}{\text{opposite side}} = \frac{\text{sine of known angle}}{\text{opposite side}}$$

Always check the angle sum to see whether both answers are possible.

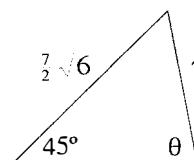
WORKED EXERCISE: Find θ in the given triangle.

SOLUTION: $\frac{\sin \theta}{\frac{7}{2}\sqrt{6}} = \frac{\sin 45^\circ}{7}$

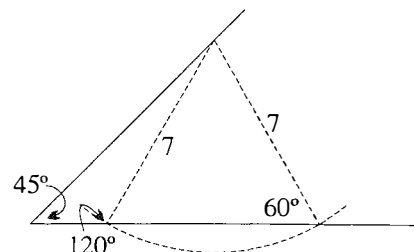
$$\sin \theta = \frac{7}{2}\sqrt{6} \times \frac{1}{7\sqrt{2}}$$

$$\sin \theta = \frac{1}{2}\sqrt{3},$$

so $\theta = 60^\circ \text{ or } 120^\circ.$



NOTE: There are two angles whose sine is $\frac{\sqrt{3}}{2}$, one acute and one obtuse. Moreover, $120^\circ + 45^\circ = 165^\circ$, leaving just 15° for the third angle in the obtuse case, so it all seems to work. Opposite is the ruler and compasses construction of the triangle, showing how two different triangles can be produced from the same given ASS measurements.



In many examples, however, the obtuse angle solution can be excluded by using the fact that the angle sum of the triangle cannot exceed 180° . In particular:

1. In a right triangle, both the other angles must be acute, giving rise to the well-known RHS congruence test.
2. If one angle is obtuse, then both other angles are acute. Hence there is a valid 'OSS' congruence test, which applies to the situation where two sides and a non-included obtuse angle are known.

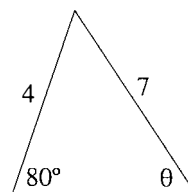
WORKED EXERCISE: Find θ in the given triangle.

SOLUTION:
$$\frac{\sin \theta}{4} = \frac{\sin 80^\circ}{7}$$

$$\sin \theta = \frac{4 \sin 80^\circ}{7}$$

$$\theta \doteq 34^\circ 15' \text{ or } 145^\circ 45'$$

But $\theta \doteq 145^\circ 45'$ is impossible, because the angle sum would then exceed 180° ,
so $\theta \doteq 34^\circ 15'$.



The Sine Rule and the Circumcircle: Now that the three ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ have been proven all to be equal, we obviously should be asking *what are they equal to?*

First, $\sin A$, $\sin B$ and $\sin C$ are all pure numbers, so the ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$, being lengths over numbers, must all be lengths. Secondly, the sine function cannot exceed 1, so each ratio $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ is a length greater than or equal to each of the sides.

The following theorem shows that the common value of these three ratios is the diameter of the *circumcircle*, which is the circle passing through all three vertices. This provides an alternative and far more enlightening proof of the sine rule, clearly illustrating connections between trigonometry and the geometry of circles.

36

THEOREM — THE SINE RULE AND THE CIRCUMCIRCLE: In any triangle, the ratio of each side to the sine of the opposite angle is constant, and this constant is equal to the diameter of the circumcircle of the triangle:

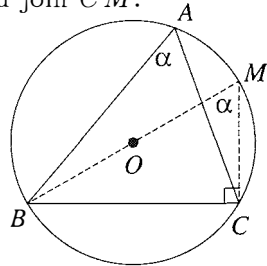
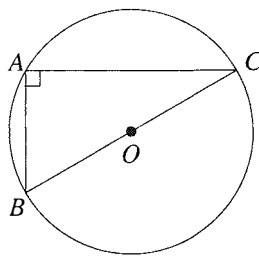
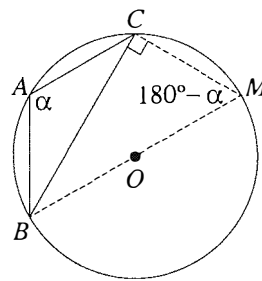
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of the circumcircle}$$

GIVEN: Let O be the centre of the circumcircle of $\triangle ABC$. Let d be the diameter of the circumcircle, and let $\angle A = \alpha$. There are three cases, according as to whether α is acute, obtuse, or a right angle.

AIM: To prove that $\frac{a}{\sin \alpha} = d$.

In case 2, $a = d$, and also $\sin \alpha = \sin 90^\circ = 1$ (angle in a semicircle).

CONSTRUCTION: In the remaining cases 1 and 3, construct the diameter BOM , and join CM .

CASE 1: α is acuteCASE 2: $\alpha = 90^\circ$ CASE 3: α is obtuse

PROOF: In case 1, $\angle M = \alpha$ (angles on the same arc BC),
and in case 3, $\angle M = 180^\circ - \alpha$ (cyclic quadrilateral $BMCA$).

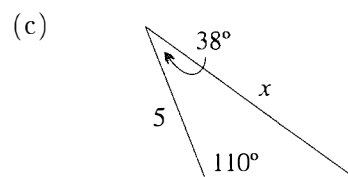
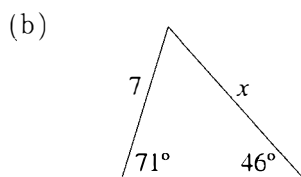
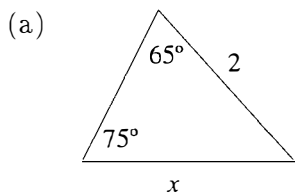
In both cases, $\sin \angle M = \sin \alpha$, since $\sin(180^\circ - \alpha) = \sin \alpha$.

Also $\angle BCM = 90^\circ$ (angle in a semicircle),

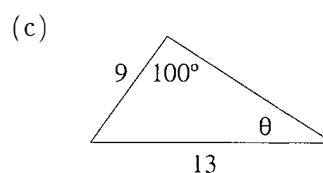
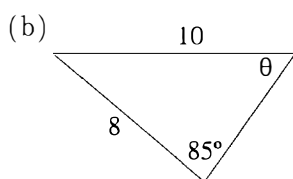
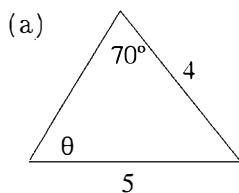
so in $\triangle BCM$, $\frac{a}{d} = \sin \alpha$, so that $\frac{a}{\sin \alpha} = d$, as required.

Exercise 4H

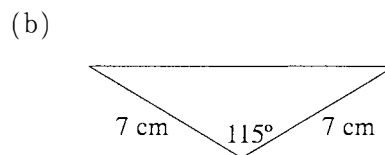
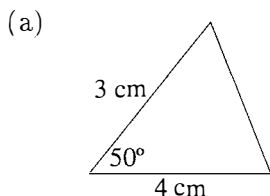
1. Find x , correct to one decimal place, in each triangle:



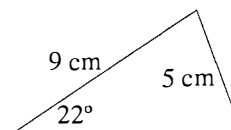
2. Find θ , correct to the nearest degree, in each triangle:



3. Find the area of each triangle, correct to the nearest square centimetre:



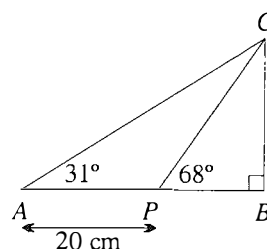
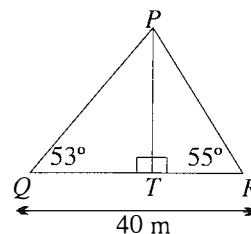
4. There are two triangles that have sides 9 cm and 5 cm, and in which the angle opposite the 5 cm side is 22° . Find, in each case, the size of the angle opposite the 9 cm side (answer correct to the nearest minute).



5. Sketch $\triangle ABC$ in which $a = 2.8$ cm, $b = 2.7$ cm and $A = 52^\circ 21'$.

- (a) Find B , holding the answer in memory, but writing it correct to the nearest minute.
(b) Hence find C , correct to the nearest minute, but hold the answer on the screen.
(c) Hence find the area of $\triangle ABC$ in cm^2 , correct to two decimal places.

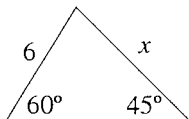
6. Sketch $\triangle PQR$ in which $p = 7$ units, $q = 15$ units and $\angle P = 25^\circ 50'$.
- (a) Find the two possible sizes of $\angle Q$, correct to the nearest minute.
- (b) For each of the possible sizes of $\angle Q$, find r , correct to one decimal place.
7. Find all the unknown sides (to one decimal place) and angles (to the nearest minute) of $\triangle ABC$ if $A = 40^\circ$, $a = 7.6$ and $b = 10.5$.
8. In $\triangle PQR$, $\angle Q = 53^\circ$, $\angle R = 55^\circ$ and $QR = 40$ metres. T is the point on QR such that $PT \perp QR$.
- (a) By using the sine rule in the triangle PQR , show that $PQ = \frac{40 \sin 55^\circ}{\sin 72^\circ}$.
- (b) Hence use $\triangle PQT$ to find PT , correct to the nearest metre.
9. In $\triangle ABC$, $\angle B = 90^\circ$ and $\angle A = 31^\circ$. P is a point on AB such that $AP = 20$ cm and $\angle CPB = 68^\circ$.
- (a) Show that $PC = \frac{20 \sin 31^\circ}{\sin 37^\circ}$.
- (b) Hence find PB , correct to the nearest centimetre.



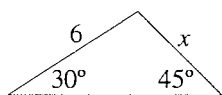
DEVELOPMENT

10. In $\triangle ABC$, $\sin A = \frac{1}{4}$, $\sin B = \frac{2}{3}$ and $a = 12$. Find the value of b .
11. Find the exact value of x in each diagram:

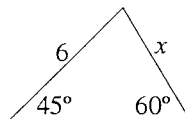
(a)



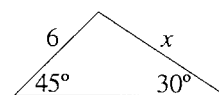
(b)



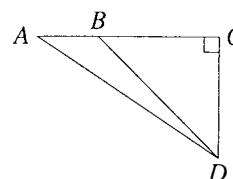
(c)



(d)



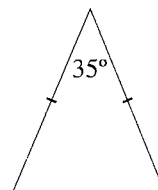
12. The points A , B and C lie on a horizontal line and D lies directly below C . The angles of depression of D from A and B are 34° and 62° respectively, and $AB = 75.4$ metres.



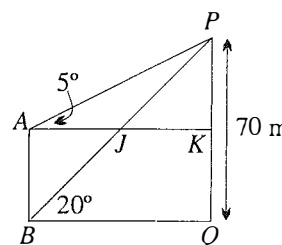
- (a) Show that $CD = \frac{75.4 \sin 34^\circ \sin 62^\circ}{\sin 28^\circ}$.

- (b) Hence find the height of C above D in metres, correct to one decimal place.

13. The vertical angle of an isosceles triangle is 35° , and its area is 35 cm^2 . Find the length of the equal sides, correct to the nearest millimetre.



14. Two towers AB and PQ stand on level ground. The angles of elevation of the top of the taller tower from the top and bottom of the shorter tower are 5° and 20° respectively. The height of the taller tower is 70 metres.



- (a) Explain why $\angle APJ = 15^\circ$.

- (b) Show that $AB = \frac{BP \sin 15^\circ}{\sin 95^\circ}$.

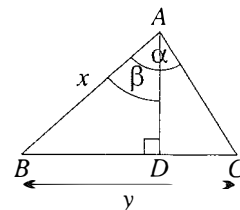
- (c) Show that $BP = \frac{70}{\sin 20^\circ}$.

- (d) Hence find the height of the shorter tower, correct to the nearest metre.

15. In the diagram opposite, $\angle BAC = \alpha$, $\angle BAD = \beta$, $BC = y$ and $AB = x$.

(a) Show that $y = \frac{x \sin \alpha}{\sin C}$.

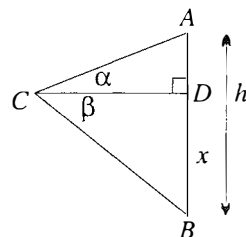
(b) Hence show that $y = \frac{x \sin \alpha}{\cos(\alpha - \beta)}$.



16. In the diagram opposite, $\angle ACD = \alpha$, $\angle BCD = \beta$, $AB = h$ and $BD = x$.

(a) Show that $BC = \frac{h \cos \alpha}{\sin(\alpha + \beta)}$.

(b) Hence show that $x = \frac{h \sin \beta \cos \alpha}{\sin(\alpha + \beta)}$.

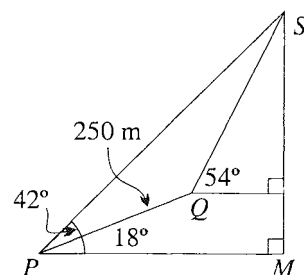


17. The summit S of a mountain is observed from two points P and Q 250 metres apart. PQ is inclined at 18° to the horizontal and the respective angles of elevation of S from P and Q are 42° and 54° .

(a) Explain why $\angle PSQ = 12^\circ$ and $\angle PQS = 144^\circ$.

(b) Show that $SP = \frac{250 \sin 144^\circ}{\sin 12^\circ}$.

(c) Hence find the vertical height SM , correct to the nearest metre.

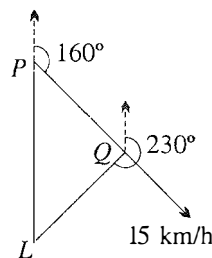


18. A ship is sailing at 15 km/h on a bearing of 160° . At 9:00 am it is at P , and lighthouse L is due south. At 9:40 am it is at Q , and the lighthouse is on a bearing of 230° .

(a) Show that $\angle PQL = 110^\circ$.

(b) Find the distance PL , correct to the nearest km.

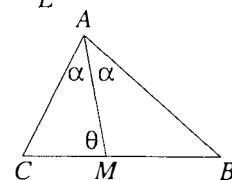
(c) Find the time, to the nearest minute, at which the lighthouse will be due west of the ship.



19. In a triangle ABC , the bisector of angle A meets the opposite side BC at M . Let $\alpha = \angle CAM = \angle BAM$, and let $\theta = \angle CMA$.

(a) Explain why $\sin \angle BMA = \sin \theta$.

(b) Hence show that $AC : AB = MC : MB$.



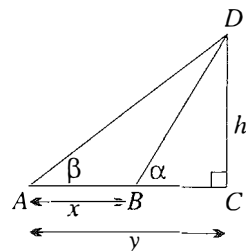
20. (a) Show that $h = \frac{x \sin \alpha \sin \beta}{\sin(\alpha - \beta)}$ in the diagram opposite.

(b) Use the fact that $\tan \alpha = \frac{h}{y - x}$ and $\tan \beta = \frac{h}{y}$ to show

that $h = \frac{x \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}$.

(c) Combine the expressions in parts (a) and (b) to show that $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

(d) Hence find the exact value of $\sin 15^\circ$.



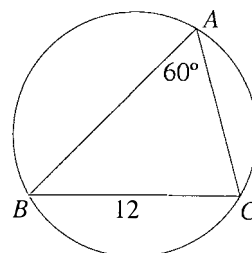
21. Suppose that the sine rule is being used in an ASS situation to find an angle θ in a triangle, and that $\sin \theta$ has been found. Explain why there is only one solution for θ if and only if $\theta = 90^\circ$ or the related angle of θ is less than the known angle.

EXTENSION

22. Two ships P and Q are observed to be NW and NE respectively of a port A . From a second port B , which is 1 km due east of A , the ships P and Q are observed to be WNW and NNE respectively. Show that the two ships are approximately 2.61 km apart.

23. [The circumcircle] In one of the proofs of the sine rule, we saw that $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ are each equal to the diameter of the circumcircle of $\triangle ABC$.

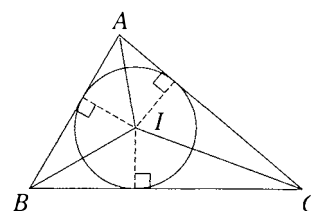
- (a) In $\triangle ABC$, $\angle A = 60^\circ$ and $BC = 12$. Find the diameter D_C of the circumcircle.
- (b) The triangle $\triangle ABC$ in part (a) is not determined up to congruence. Why does the diameter of the circumcircle nevertheless remain constant as the triangle varies?
- (c) A triangle $\triangle PQR$ with $\angle RPQ = 150^\circ$ is inscribed inside a circle of diameter D_C . Find the ratio $D_C : RQ$.



24. [The circumcircle and the incircle]

- (a) Let Δ be the area of $\triangle ABC$, and let D_C diameter of its circumcircle. Show that $D_C = \frac{abc}{2\Delta}$.

- (b) The *incircle* of triangle $\triangle ABC$ is the circle drawn inside the triangle and tangent to each side, as shown in the diagram. Let I , the *incentre*, be the centre of the incircle, and let D_I be its diameter. Find the area of the each of the three triangles $\triangle AIB$, $\triangle BIC$ and $\triangle CIA$. Hence show that $D_I = \frac{2\Delta}{s}$, where $s = \frac{1}{2}(a + b + c)$ is the *semiperimeter* of the triangle.



- (c) Hence find the ratio $D_C : D_I$ and the product $D_C D_I$.
- (d) Find also $\frac{\text{area of triangle}}{\text{area of circumcircle}}$ and $\frac{\text{area of triangle}}{\text{area of incircle}}$.

4I The Cosine Rule

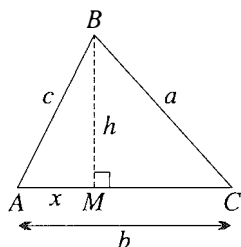
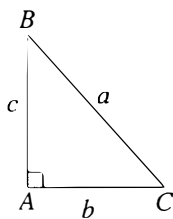
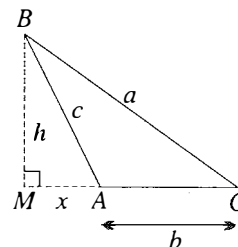
The cosine rule is a generalisation of Pythagoras' theorem to non-right-angled triangles, because it gives a formula for the square of any side in terms of the squares of the other two sides and the cosine of the opposite angle. The proof is based on Pythagoras' theorem, and again begins with the construction of an altitude.

37

THEOREM — THE COSINE RULE: The square of any side of a triangle equals the sum of the squares of the other two sides minus twice the product of those sides and the cosine of their included angle:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

GIVEN: Let ABC be any triangle. Again, there are three cases, according as to whether α is acute, obtuse, or a right angle.

CASE 1: $\angle A$ is acuteCASE 2: $\angle A = 90^\circ$ CASE 3: $\angle A$ is obtuse

AIM: To prove that $a^2 = b^2 + c^2 - 2bc \cos A$.

In case 2, $\cos A = 0$, and this is just Pythagoras' theorem.

CONSTRUCTION: In the remaining cases 1 and 3, construct the altitude from B , meeting AC , produced if necessary, at M . Let $BM = h$ and $AM = x$.

PROOF:

CASE 1 — Suppose $\angle A$ is acute.

By Pythagoras' theorem in $\triangle BMC$,

$$a^2 = h^2 + (b - x)^2.$$

By Pythagoras' theorem in $\triangle BMA$,

$$h^2 = c^2 - x^2,$$

$$\text{so } a^2 = c^2 - x^2 + (b - x)^2$$

$$= c^2 - x^2 + b^2 - 2bx + x^2$$

$$= b^2 + c^2 - 2bx. \quad (*)$$

Using trigonometry in $\triangle ABM$,

$$x = c \cos A.$$

$$\text{So } a^2 = b^2 + c^2 - 2bc \cos A.$$

CASE 3 — Suppose $\angle A$ is obtuse.

By Pythagoras' theorem in $\triangle BMC$,

$$a^2 = h^2 + (b + x)^2.$$

By Pythagoras' theorem in $\triangle BMA$,

$$h^2 = c^2 - x^2,$$

$$\text{so } a^2 = c^2 - x^2 + (b + x)^2$$

$$= c^2 - x^2 + b^2 + 2bx + x^2$$

$$= b^2 + c^2 + 2bx. \quad (*)$$

Using trigonometry in $\triangle ABM$,

$$x = c \cos(180^\circ - A)$$

$$= -c \cos A.$$

$$\text{So } a^2 = b^2 + c^2 - 2bc \cos A.$$

NOTE: The identity $\cos(180^\circ - A) = -\cos A$ is the key step in Case 3 of the proof. The cosine rule appears in Euclid's geometry book, but without any mention of the cosine ratio — the form given there is approximately the two statements in the proof marked with (*).

Using the Cosine Rule to Find a Side — The SAS Situation: For the cosine rule to be applied to find a side, two sides and the included angle must be known, which is the SAS congruence situation.

USING THE COSINE RULE TO FIND A SIDE: In the SAS congruence situation:

38

square of any side = (sum of squares of other two sides)

– (twice the product of those sides) $\times$ (cosine of their included angle)

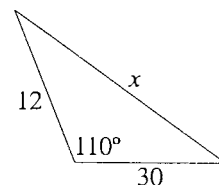
WORKED EXERCISE: Find x in the given triangle.

$$\text{SOLUTION: } x^2 = 12^2 + 30^2 - 2 \times 12 \times 30 \times \cos 110^\circ$$

$$= 144 + 900 - 720 \cos 110^\circ$$

$$= 1044 + 720 \cos 70^\circ$$

$$\text{So } x \doteq 35.92.$$



Using the Cosine Rule to Find an Angle — the SSS Situation: Solving the cosine rule above for $\cos A$ gives:

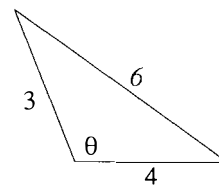
39 THE COSINE RULE WITH $\cos A$ AS SUBJECT: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Application of the cosine rule to find an angle requires that all three sides are known, which is the SSS congruence test. To use the rule, one can either substitute each time into the usual form of the cosine rule, or remember it verbally.

40 USING THE COSINE RULE TO FIND AN ANGLE: In the SSS congruence situation:
 $\cos \theta = \frac{(\text{sum of squares of two including sides}) - (\text{square of opposite side})}{2 \times (\text{product of the including sides})}$

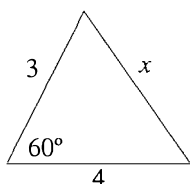
WORKED EXERCISE: Find θ in the given triangle.

SOLUTION: $\cos \theta = \frac{3^2 + 4^2 - 6^2}{2 \times 3 \times 4}$
 $= \frac{-11}{24}$
 so $\theta \doteq 117^\circ 17'$.



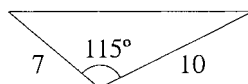
Exercise 4I

1. (a)



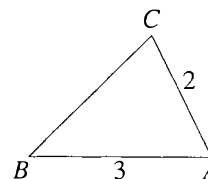
Find the length x as a surd.

(b)



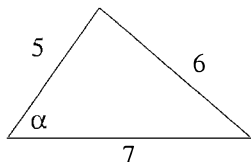
Find the unknown side to two decimal places.

(c)



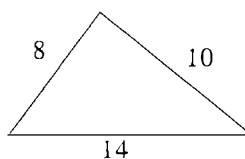
If $\cos A = \frac{1}{4}$, find the exact value of a .

2. (a)



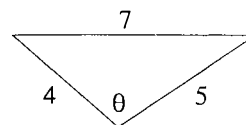
Find the angle α correct to the nearest degree.

(b)



Find the largest angle of the given triangle, to the nearest minute.

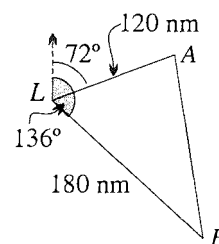
(c)



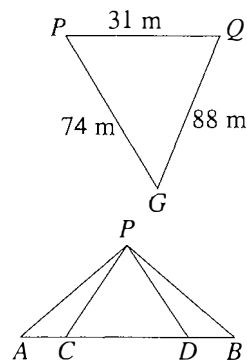
Find the value of $\cos \theta$.

3. P , Q and R are landmarks. It is known that R is 8.7 km from P and 9.3 km from Q , and that $\angle PRQ = 79^\circ 32'$. Find, in kilometres correct to one decimal place, the distance between P and Q .

4. Ship A is 120 nautical miles from a lighthouse L on a bearing of 72° , while ship B is 180 nautical miles from L on a bearing of 136° . Calculate the distance between the two ships, correct to the nearest nautical mile.

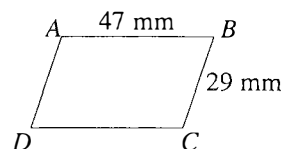


5. A golfer at G wishes to hit a shot between two trees P and Q . The trees are 31 metres apart, and the golfer is 74 metres from P and 88 metres from Q . Find the angle within which the golfer must play the shot (answer to the nearest degree).



6. Roof trusses AP , BP , CP and DP are nailed to a horizontal beam AB , as shown in the diagram opposite. Given that $AP = BP = 7.2$ metres, $CP = DP = 5.5$ metres, $AB = 10.6$ metres and $CD = 7.4$ metres, find, correct to the nearest minute:

- (a) $\angle APB$ (b) $\angle CPD$

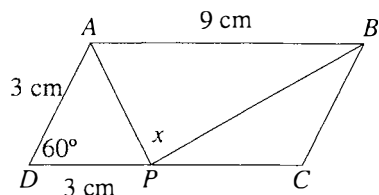
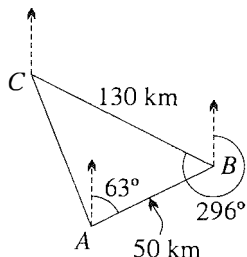


7. A parallelogram $ABCD$ has sides $AB = DC = 47$ mm and $AD = BC = 29$ mm. The longer diagonal BD is 60 mm.

- (a) Use the cosine rule to find the size of $\angle BCD$.
(b) Use cointerior angles on parallel lines to find the size of $\angle ABC$ (give each answer correct to the nearest minute).

DEVELOPMENT

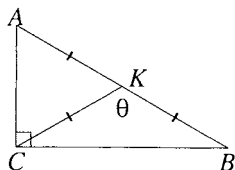
8. In $\triangle ABC$, $a = 31$ units, $b = 24$ units and $\cos C = \frac{59}{62}$. Show that:
(a) $c = 11$ units (b) $A = 120^\circ$
9. The sides of a triangle are in the ratio $5 : 16 : 19$. Find the smallest angle of the triangle, correct to the nearest minute.
10. In $\triangle PQR$, $p = 5\sqrt{3}$ cm, $q = 11$ cm and $R = 150^\circ$. Find: (a) r (b) $\cos P$
11. In $\triangle ABC$, $a = 4$ cm, $b = 5$ cm and $c = 6$ cm. Find $\cos A$, $\cos B$ and $\cos C$, and hence show that $6 \cos A \cos C = \cos B$.
12. A ship sails 50 km from port A to port B on a bearing of 63° , then sails 130 km from port B to port C on a bearing of 296° .
(a) Show that $\angle ABC = 53^\circ$.
(b) Find, to the nearest km, the distance of port A from port C .
(c) Use the cosine rule to find $\angle ACB$, and hence find the bearing of port A from port C , correct to the nearest degree.
13. $ABCD$ is a parallelogram in which $AB = 9$ cm, $AD = 3$ cm and $\angle ADC = 60^\circ$. The point P is the point on DC such that $DP = 3$ cm.



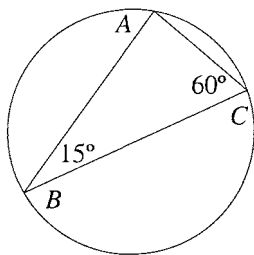
- (a) Explain why $\triangle ADP$ is equilateral and hence find AP .
(b) Use the cosine rule in $\triangle BCP$ to find the exact length of BP .
(c) Let $\angle APB = x$. Show that $\cos x = -\frac{1}{14}\sqrt{7}$.

EXTENSION

14. $\triangle ABC$ is right-angled at C , and K is the midpoint of AB . Also, CK has the same length as AK and BK . Prove that $\cos \theta = \frac{b^2 - a^2}{b^2 + a^2}$, where $\theta = \angle BKC$.



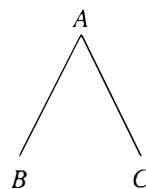
15. (a)



Two of the angles of a triangle ABC are 15° and 60° and the triangle is inscribed in a circle of radius 6 units.

- (i) Show that $AC^2 = 36(2 - \sqrt{3})$.
 (ii) Find the exact area of the triangle.

(b)



BA and CA are identical rods hinged at A . When $BC = 5$ cm, $\angle BAC = 45^\circ$ and when $BC = 6$ cm, $\angle BAC = \alpha$.

Show that $\cos \alpha = \frac{18\sqrt{2} - 11}{25}$.

16. [Heron's formula for the area of a triangle in terms of the side lengths]

- (a) By repeated application of factoring by the difference of squares, prove the identity

$$(2ab)^2 - (a^2 + b^2 - c^2)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c).$$

- (b) Let $\triangle ABC$ be any triangle, and let $s = \frac{1}{2}(a + b + c)$ be the *semiperimeter*. Prove that

$$(a + b + c)(a + b - c)(a - b + c)(-a + b + c) = 16s(s - a)(s - b)(s - c).$$

- (c) Write down the formula for $\cos C$ in terms of the sides a , b and c , then use (a) and

(b) and the Pythagorean identities to prove that $\sin C = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab}$.

- (d) Hence show that the area Δ of the triangle is $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

17. [The circumcircle and the incircle] In the previous Exercise 4H, formulae were developed for the diameters D_C and D_I of the circumcircle and incircle respectively of a triangle $\triangle ABC$.

- (a) Use these formulae, and the methods of the previous question, to find formulae for the diameters of these circles in terms of the side lengths of the triangle.

- (b) Show that area of circumcircle : area of incircle = $a^2b^2c^2 : 16(s-a)^2(s-b)^2(s-c)^2$.

4J Problems Involving General Triangles

A triangle has three lengths and three angles, and most triangle problems involve using three of these six measurements to calculate some of the others. The key to deciding which formula to use is to see which congruence situation applies.

Trigonometry and the Congruence Tests: The four congruence tests — RHS, AAS, SAS and SSS — can also be regarded as theorems about constructing triangles from given data. If you know three measurements including one length, then apart from the ambiguous ASS test, there is only one possible triangle with these three measurements, and you can construct it up to congruence.

THE SINE, COSINE AND AREA RULES AND THE CONGRUENCE TESTS:

In a right triangle, use simple trigonometry and Pythagoras' theorem. Otherwise:

AAS: Use the sine rule to find each of the other two sides.

41

ASS: Use the sine rule to find the unknown angle opposite a known side (possibly with two solutions).

SAS: Use the cosine rule to find the third side, and use the area rule to find the area.

SSS: Use the cosine rule to find any angle.

Problems Requiring Two Steps: Various situations with non-right-angled triangles require two steps for their solution, for example, finding the other two angles in an SAS situation, or finding the area given AAS, ASS or SSS situations.

WORKED EXERCISE: A boat sails 6 km due north from the harbour H to A , and a second boat sails 10 km from H to B on a bearing of 120° . What is the bearing of B from A , correct to the nearest minute?

SOLUTION: First, using the cosine rule to find AB ,

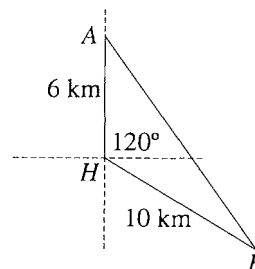
$$\begin{aligned} AB^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ \\ &= 36 + 100 - 120 \times \left(-\frac{1}{2}\right) \\ &= 196, \end{aligned}$$

so $AB = 14$ km.

Secondly, using the cosine rule to find $\angle A$,

$$\begin{aligned} \cos A &= \frac{6^2 + 14^2 - 10^2}{2 \times 6 \times 14} \\ &= \frac{11}{14}, \end{aligned}$$

so $A \doteq 38^\circ 13'$, and the bearing of B from A is about $141^\circ 47'$.



Finding the Third Side in the Ambiguous ASS Situation: The cosine rule in the form $a^2 = b^2 + c^2 - 2bc \cos A$ can also be rewritten as a quadratic in c :

$$c^2 - 2bc \cos A + (b^2 - a^2) = 0.$$

This allows the third side to be found in one step in the ambiguous ASS situation when two sides and a non-included angle are given. For there to be two solutions, the quadratic must have two positive solutions.

WORKED EXERCISE: A tree trunk grows at an angle of 30° to the ground, and a 4 metre rod hangs from a point P that is 6 metres along the trunk. Find (to the nearest centimetre) the maximum and minimum distances of the other end E of the rod from the base B of the tree when E is resting on the ground.

SOLUTION: We rearrange the cosine rule as a quadratic in $p = BE$:

$$4^2 = 6^2 + p^2 - 2 \times 6 \times p \times \cos 30^\circ$$

$$p^2 - 6p\sqrt{3} + 20 = 0.$$

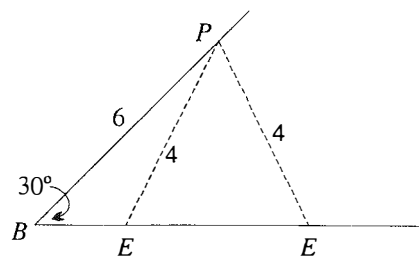
Using the quadratic formula,

$$b^2 - 4ac = 108 - 80$$

$$= 4 \times 7$$

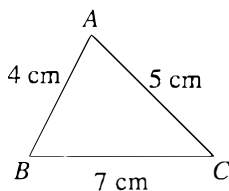
$$p = 3\sqrt{3} + \sqrt{7} \quad \text{or} \quad 3\sqrt{3} - \sqrt{7}$$

$$\doteq 7.84 \text{ metres or } 2.55 \text{ metres.}$$



Exercise 4J

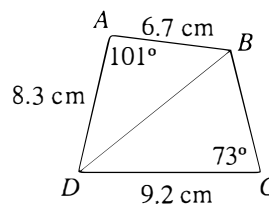
1. (a)



In $\triangle ABC$, $AB = 4$ cm, $BC = 7$ cm and $CA = 5$ cm.

- Find $\angle ABC$, correct to 1 minute.
- Hence calculate the area of $\triangle ABC$, correct to 0.1 cm^2 .

(b)



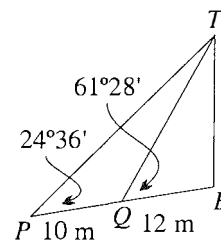
In the diagram above, $AB = 6.7$ cm, $AD = 8.3$ cm and $DC = 9.2$ cm. Also, $\angle A = 101^\circ$ and $\angle C = 73^\circ$.

- Find the diagonal BD , correct to the nearest millimetre.
- Hence find $\angle CBD$, correct to the nearest degree.

2. In the diagram opposite, TB is a vertical flagpole at the top of an inclined path PQB .

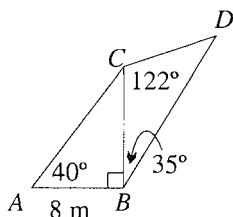
(a) Show that $TQ = \frac{10 \sin 24^\circ 36'}{\sin 36^\circ 52'}$.

- (b) Hence find the height of the flagpole in metres, correct to two decimal places.

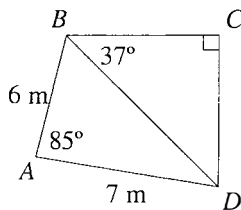


3. In each diagram, find CD correct to the nearest centimetre:

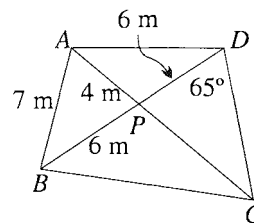
(a)



(b)



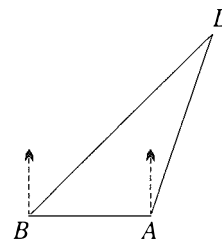
(c)



4. A ship at A is 10 nautical miles from a lighthouse L which is on a bearing of $N25^\circ E$. The ship then sails due west to B , from which the bearing of the lighthouse is $N55^\circ E$.

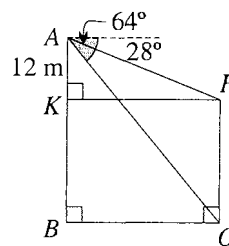
- (a) Show that $\angle ALB = 30^\circ$.

- (b) Using the sine rule, show that $AB = 5 \operatorname{cosec} 35^\circ$, and hence find the distance sailed by the ship from A to B . Give your answer in nautical miles, correct to one decimal place.



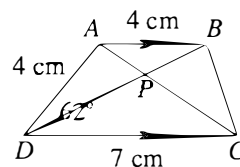
5. Two towers AB and PQ stand on level ground. Tower AB is 12 metres taller than tower PQ . From A , the angles of depression of P and Q are 28° and 64° respectively.

- Use $\triangle AKP$ to show that $KP = BQ = 12 \tan 62^\circ$.
- Use $\triangle ABQ$ to show that $AB = 12 \tan 62^\circ \tan 64^\circ$.
- Hence find the height of the shorter tower, correct to the nearest metre.
- Solve the problem again by finding AP using $\triangle AKP$ and then using the sine rule in $\triangle APQ$.



6. In the diagram opposite, $ABCD$ is a trapezium in which $AB \parallel DC$. The diagonals AC and BD meet at P . Also, $AB = AD = 4$ cm, $DC = 7$ cm and $\angle ADC = 62^\circ$.

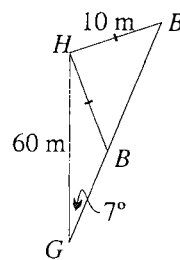
- (a) Find $\angle ACD$, correct to the nearest minute.
[HINT: Find AC first.]
(b) Explain why $\angle PDC = \frac{1}{2}\angle ADC$.
(c) Hence find, to the nearest minute, the acute angle between the diagonals of the trapezium.



DEVELOPMENT

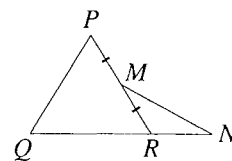
7. With his approach shot to the hole H , a golfer at G landed his ball B 10 metres from H . The direction of the shot was 7° away from the direct line between G and H .

- (a) Find, correct to the nearest minute, the two possible sizes of $\angle GBH$.
(b) Hence find the two possible distances the ball has travelled (answer in metres to one decimal place).

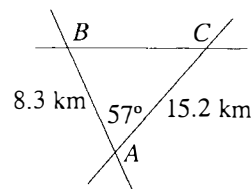


8. PQR is an equilateral triangle with side length 3 cm. M is the midpoint of PR and N is the point in QR produced such that $RN = 2$ cm.

- (a) Find MN .
(b) Hence calculate $\angle QNM$, correct to the nearest minute.

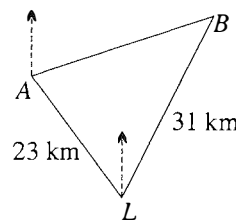


9. AB , BC and CA are straight roads. AB and AC intersect at 57° . $AB = 8.3$ km and $AC = 15.2$ km. Two cars P_1 and P_2 leave A at the same instant. P_1 travels along AB and then BC at 80 km/h while P_2 travels along AC at 50 km/h. Which car reaches C first, and by how many minutes does it do so (answer to one decimal place)?



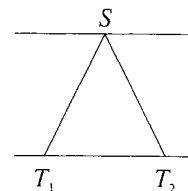
10. Town A is 23 km from landmark L in the direction $N56^\circ W$, and town B is 31 km from L in the direction $N46^\circ E$.

- (a) Find how far town B is from town A (answer to the nearest km).
(b) Find the bearing of town B from town A (answer to the nearest degree).



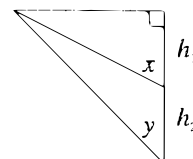
11. Two trees T_1 and T_2 on one bank of a river are 86 metres apart. A sign S on the opposite bank is between the trees and the angles ST_1T_2 and ST_2T_1 are $53^\circ 30'$ and $60^\circ 45'$ respectively.

- (a) Find ST_1 .
(b) Hence find the width of the river, correct to the nearest metre.



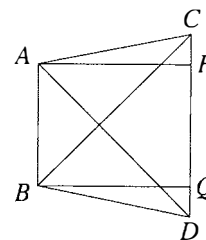
12. In the given diagram, prove that

$$h_1 = \frac{h_2 \cos x \sin y}{\sin(x - y)}.$$



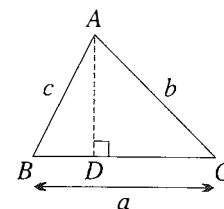
13. AB and CD are vertical lines, while AP and BQ are horizontal lines. From A and B , the angles of elevation of C are 37° and 48° respectively. From A , the angle of depression of D is 52° . Let $AP = BQ = x$.

- (a) Show that $CP = x \tan 37^\circ$, and write down similar expressions for CQ and PD .
 (b) Let α be the angle of elevation of B from D . Explain why $x \tan 48^\circ - x \tan 37^\circ = x \tan 52^\circ - x \tan \alpha$.
 (c) Hence find α , correct to the nearest minute.



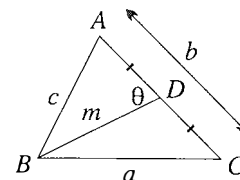
14. ABC is a triangle and D is the point on BC such that $AD \perp BC$.

- (a) Show that $BD = c \cos B$, and write down a similar expression for DC .
 (b) Hence show that $a = b \cos C + c \cos B$.
 (c) Show that $b \sin C = c \sin B$.
 (d) Use (b) and (c) to show that $\frac{a - b \cos C}{b \sin C} = \cot B$.



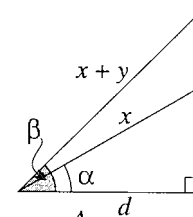
15. ABC is a triangle and D is the midpoint of AC . $BD = m$ and $\angle ADB = \theta$.

- (a) Simplify $\cos(180^\circ - \theta)$.
 (b) Show that $\cos \theta = \frac{4m^2 + b^2 - 4c^2}{4mb}$, and write down a similar expression for $\cos(180^\circ - \theta)$.
 (c) Hence show that $a^2 + c^2 = 2m^2 + \frac{1}{2}b^2$.



16. The sides of a triangle are $n^2 + n + 1$, $2n + 1$ and $n^2 - 1$, where $n > 1$. Find the largest angle of the triangle.

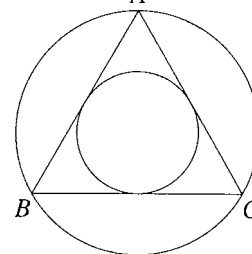
17. A ladder of length x cm is inclined at an angle α to the ground. The foot of the ladder is fixed. If the ladder were y cm longer, the inclination to the horizontal would be β . Show that the distance from the foot of the ladder to the wall is given by $\frac{y \cos \alpha \cos \beta}{\cos \alpha - \cos \beta}$ cm.



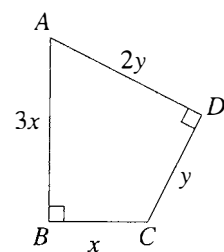
18. In $\triangle ABC$, $a \cos A = b \cos B$. Prove, using the cosine rule, that the triangle is either isosceles or right-angled.

19. The diagram opposite shows an equilateral triangle ABC whose sides are $2x$ units long.

- (a) Show that the inscribed circle tangent to all three sides has area $\frac{1}{3}\pi x^2$.
 (b) Show that the circumscribed circle passing through all three vertices has four times the area of the inscribed circle.



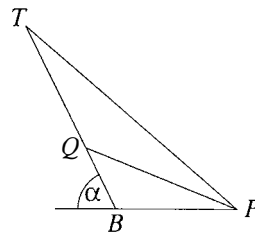
20. In the quadrilateral $ABCD$ sketched opposite, $\angle B$ and $\angle D$ are right angles, $AB = 3BC = 3x$ and $AD = 2DC = 2y$. Use Pythagoras' theorem and the cosine rule to show that $\angle A = 45^\circ$.



EXTENSION

21. From the top T of a hill BT inclined at α to the horizontal, the angle of depression of a point P on the plane below is 12° . From Q which is three-quarters of the way down the hill, the angle of depression of P is 6° .

- (a) Using the sine rule, show that $\sin \alpha = 3 \sin(\alpha - 12^\circ)$.
 (b) Apply the formula $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ to the result in (a) to find α to the nearest minute.



22. [Brahmagupta's formula for the area of a cyclic quadrilateral in terms of its sides]

- (a) Use repeated application of the difference of squares to prove the identity

$$4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 = (a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d).$$

- (b) Let $s = \frac{1}{2}(a+b+c+d)$ be the *semiperimeter* of the cyclic quadrilateral $PQRS$ in the diagram below. Prove that

$$(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d) = 16(s-a)(s-b)(s-c)(s-d).$$

- (c) Let $\angle P = \theta$, then by circle geometry the opposite angle is $\angle R = 180^\circ - \theta$. By equating expressions for the square of the diagonal $\ell = SQ$, prove that

$$\cos \theta = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

- (d) Hence show that

$$\sin \theta = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab + cd}.$$

- (e) Hence show that the area of the cyclic quadrilateral is

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

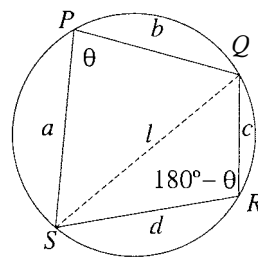
- (f) How can Heron's formula be generated as a special case of Brahmagupta's formula?

23. [Diagonals and diameter of a cyclic quadrilateral] Using the results established in the previous question, and with the same notation, let $m = PR$ be the other diagonal, let $\phi = \angle PQR$, and let D_C be the diameter of the circumcircle. Prove further that:

$$(i) \ell^2 = \frac{(ad+bc)(ac+bd)}{ab+cd} \quad (iii) \ell m = ac + bd$$

$$(ii) \frac{\ell}{m} = \frac{ad+bc}{ab+cd} = \frac{\sin \theta}{\sin \phi} \quad (iv) D_C = \frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}$$

$$(v) \frac{\text{area of quadrilateral}}{\text{area of circle}} = \frac{16(s-a)^{\frac{3}{2}}(s-b)^{\frac{3}{2}}(s-c)^{\frac{3}{2}}(s-d)^{\frac{3}{2}}}{\pi(ab+cd)(ac+bd)(ad+bc)}$$



CHAPTER FIVE

Coordinate Geometry

Coordinate geometry is geometry done in a number plane, where points are represented by ordered pairs of numbers, lines are represented by linear equations, and circles, parabolas and other curves are represented by more complicated equations. This chapter establishes the methods used in coordinate geometry to deal with intervals and lines.

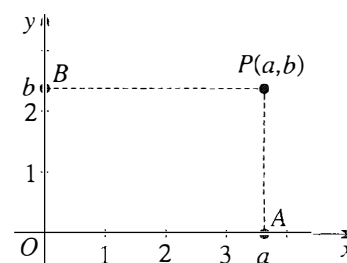
STUDY NOTES: Much of this work will be a consolidation of material from earlier years. Three topics, however, are quite new: the ratio division formula, including external division, in Section 5A, the perpendicular distance from a point to a line in Section 5E, and lines through the intersection of two given lines in Section 5F. The final Section 5G uses the methods of coordinate geometry to develop alternative proofs of theorems from geometry. The last two Sections 5F and 5G could be delayed if they seem too demanding at this stage.

5A Points and Intervals

The first task is to set up the coordinate plane, and to develop the distance formula, the midpoint formula and the ratio division formula for intervals.

Representing Points by Ordered Pairs: A blank plane in Euclidean geometry can be made into a *coordinate plane* by constructing a pair of axes in it:

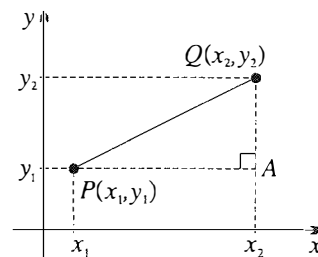
1. Any pair of perpendicular lines can be chosen as the axes. Their intersection is called the *origin*, and given the symbol O .
2. Each line must be made into a number line, with zero at the origin, and with the same scale on both axes.
3. The x -axis and the y -axis can be distinguished from each other, because a rotation of 90° anticlockwise about O rotates the x -axis onto the y -axis.



Any point P in the plane can now be given a unique pair of *coordinates*. Construct the rectangle $OAPB$ in which A lies on the x -axis and B lies on the y -axis, and let a and b be the real numbers on the axes associated with A and B respectively. Then the point P is identified with the ordered pair (a, b) .

Every point P now corresponds to a single ordered pair (a, b) of real numbers, and every ordered pair (a, b) of real numbers corresponds to a single point P . There is therefore no need to distinguish between the points and the ordered pairs, and we will write statements like ‘Let $P = (3, 5)$ ’.

The Distance Formula: The formula for distance on the number plane is Pythagoras' theorem. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane. Form the right triangle PQA , where A is the point (x_2, y_1) . Then $PA = |x_2 - x_1|$ and $QA = |y_2 - y_1|$, and so by Pythagoras' theorem the square of the hypotenuse PQ is given by:



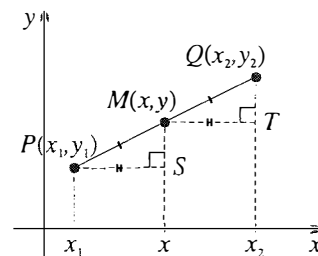
$$1 \quad \text{DISTANCE FORMULA: } PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

NOTE: The distance formula is better understood as a formula for the *square of the distance*, rather than for the distance itself. In applying the formula, first find the square of the distance, then write down the distance as the final step if it is required.

The Midpoint Formula: The midpoint of an interval can be found by averaging the coordinates of the two points. Congruence is the basis of the proof.

Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, and let $M(x, y)$ be the midpoint of PQ . Then $\triangle PMS$ is congruent to $\triangle MQT$, and so $PS = MT$. Algebraically,

$$\begin{aligned} x - x_1 &= x_2 - x \\ 2x &= x_1 + x_2 \\ x &= \frac{x_1 + x_2}{2}. \end{aligned}$$



The calculation for the y -coordinate is similar, so:

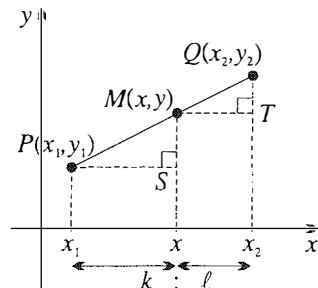
$$2 \quad \text{MIDPOINT FORMULA: } x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

WORKED EXERCISE: The interval joining $A(3, -7)$ and $B(-6, 2)$ is a diameter of a circle. Find the centre and radius of the circle.

$$\begin{aligned} \text{SOLUTION: } AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 & \text{Centre} &= \text{midpoint of } AB \\ &= (-9)^2 + 9^2 & &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= 2 \times 9^2 & &= \left(\frac{3 - 6}{2}, \frac{-7 + 2}{2} \right) \\ AB &= 9\sqrt{2}, & &= \left(-1\frac{1}{2}, -2\frac{1}{2} \right) \\ \text{so the radius is } &\frac{9}{2}\sqrt{2}. & & \end{aligned}$$

The Ratio Division Formula: Often an interval needs to be divided in some ratio other than $1:1$. Suppose then that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, and let $M(x, y)$ be the point dividing PQ in some ratio $k:\ell$. Then $\triangle PMS$ is similar to $\triangle MQT$, hence $PS:MT = k:\ell$, so that

$$\begin{aligned} \frac{x - x_1}{x_2 - x} &= \frac{k}{\ell} \\ \ell x - \ell x_1 &= kx_2 - kx \\ (k + \ell)x &= \ell x_1 + kx_2 \\ x &= \frac{\ell x_1 + kx_2}{k + \ell}. \end{aligned}$$



The calculation for the y -coordinate is similar, so:

3 **RATIO DIVISION FORMULA:** $x = \frac{\ell x_1 + k x_2}{k + \ell}$ and $y = \frac{\ell y_1 + k y_2}{k + \ell}$

WORKED EXERCISE: Given the points $A(1, 3)$ and $B(6, 28)$, find the point P dividing the interval AB in the ratio $2 : 3$.

SOLUTION: $x = \frac{\ell x_1 + k x_2}{k + \ell} = \frac{3 \times 1 + 2 \times 6}{5} = 3,$ $y = \frac{\ell y_1 + k y_2}{k + \ell} = \frac{3 \times 3 + 2 \times 28}{5} = 13,$

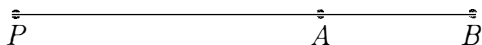
and so P is the point $(3, 13)$.

WORKED EXERCISE: The point $P(6, 13)$ divides the interval AB in the ratio $2 : 3$. Find the coordinates of A if $B = (-9, 25)$.

SOLUTION: $x = \frac{\ell x_1 + k x_2}{k + \ell}$ $y = \frac{\ell y_1 + k y_2}{k + \ell}$
 $6 = \frac{3x_1 + 2 \times (-9)}{5}$ $13 = \frac{3y_1 + 2 \times 25}{5}$
 $30 = 3x_1 - 18$ $65 = 3y_1 + 50$
 $x_1 = 16,$ $y_1 = 5,$

and so A is the point $(16, 5)$.

External Division of an Interval: The diagram below can be described by saying that ' A divides PB in the ratio $2 : 1$ '.



But since $AP : PB = 2 : 3$, we shall also describe it by the statement ' P divides AB externally in the ratio $2 : 3$ '. It turns out that if we use negative numbers in the ratio, and say

' P divides AB in the ratio $-2 : 3$ (or in the ratio $2 : -3$)',

then the formula for ratio division will give the coordinates of P , provided that a negative sign is first applied to one of the numbers in the ratio.

4 **EXTERNAL DIVISION:** If P divides AB externally in some ratio, for example $2 : 3$, then P divides AB in the ratio $-2 : 3$, or equivalently $2 : -3$.

WORKED EXERCISE: Find the point P which divides the interval AB externally in the ratio $2 : 5$, where $A = (-3, -5)$ and $B = (3, 7)$.

SOLUTION: The point P divides the interval AB in the ratio $-2 : 5$. Using the ratio division formula with $k = -2$ and $\ell = 5$, the point $P(x, y)$ is given by

$x = \frac{5 \times (-3) + (-2) \times 3}{-2 + 5} = -7,$ $y = \frac{5 \times (-5) + (-2) \times 7}{-2 + 5} = -13,$

and so P is the point $(-7, -13)$.

NOTE: We could equally well have taken the ratio as $2 : -5$, in which case the top and the bottom of each fraction would have been opposite, but the final result would be the same.

Testing for Special Quadrilaterals: Euclidean geometry will be reviewed in the Year 12 Volume, but many questions in this chapter ask for proofs that a quadrilateral is of a particular type. The most obvious way is to test the definition itself.

DEFINITIONS OF THE SPECIAL QUADRILATERALS:

- 5 A *trapezium* is a quadrilateral in which a pair of opposite sides are parallel.
 A *parallelogram* is a quadrilateral in which the opposite sides are parallel.
 A *rhombus* is a parallelogram with a pair of adjacent sides equal.
 A *rectangle* is a parallelogram with one angle a right angle.
 A *square* is both a rectangle and a rhombus.

There are, however, several further standard tests which the exercises assume (tests involving angles are omitted, being irrelevant here).

A QUADRILATERAL IS A PARALLELOGRAM:

- if the opposite sides are equal, or
- if one pair of opposite sides are equal and parallel, or
- if the diagonals bisect each other.

6 **A QUADRILATERAL IS A RHOMBUS:**

- if all sides are equal, or
- if the diagonals bisect each other at right angles.

A QUADRILATERAL IS A RECTANGLE:

- if the diagonals are equal and bisect each other.

Exercise 5A

NOTE: Diagrams should be drawn wherever possible.

1. Find the distance between each pair of points (find AB^2 first):

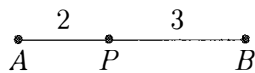
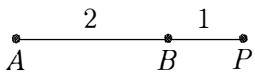
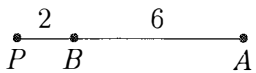
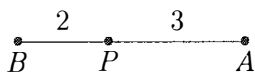
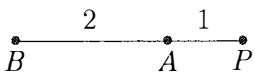
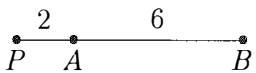
- (a) $A(1, 4)$, $B(5, 1)$ (c) $A(-5, -2)$, $B(3, 4)$ (e) $A(-4, -1)$, $B(4, 3)$
 (b) $A(-2, 7)$, $B(3, -5)$ (d) $A(3, 6)$, $B(5, 4)$ (f) $A(5, -12)$, $B(0, 0)$

2. Find the midpoint of each pair of points in the previous question.

3. Find the points dividing each interval AB in the given ratios:

- (a) $A(1, 2)$ and $B(7, 5)$ (i) 1 : 2 (ii) 2 : 1 (iii) 4 : -1 (iv) -4 : 1
 (b) $A(-1, 1)$ and $B(3, -1)$ (i) 1 : 3 (ii) 3 : 1 (iii) -1 : 3 (iv) -3 : 1
 (c) $A(-3, 2)$ and $B(7, -3)$ (i) 1 : 4 (ii) 3 : 2 (iii) 7 : -2 (iv) -4 : 3
 (d) $A(-7, 5)$ and $B(-1, -7)$ (i) 1 : 5 (ii) 1 : 1 (iii) 1 : -3 (iv) -1 : 5

4. Write down the ratio in which P divides each interval AB :

- (a)  (c)  (e) 
 (b)  (d)  (f) 

5. For each diagram in the previous question, write down the ratio in which:

- (i) A divides PB , (ii) B divides AP .

6. Given $A(1,1)$ and $B(5,3)$, find the coordinates of E if it divides the interval AB externally in the following ratios:
- (a) $3:2$ (b) $2:3$ (c) $1:3$ (d) $7:5$
7. The points $A(3,1)$, $B(2,4)$, $C(-1,3)$ and $D(-1,-2)$ are the vertices of a quadrilateral. Find the lengths of all four sides, and show that two pairs of adjacent sides are equal (you may know a common name for this sort of quadrilateral).
8. (a) Find the side lengths of the triangle formed by $X(0,-4)$, $Y(4,2)$ and $Z(-2,6)$, and show that it is a right isosceles triangle by showing that the side lengths satisfy Pythagoras' theorem.
- (b) Hence find the area of this triangle.

DEVELOPMENT

9. Each set of points given below comprises the vertices of: an isosceles triangle, an equilateral triangle, a right triangle, or none of these. Find the side lengths of each triangle and hence determine its type.
- (a) $A(-1,0)$, $B(1,0)$, $C(0,\sqrt{3})$ (c) $D(1,1)$, $E(2,-2)$, $F(-3,0)$
 (b) $P(-1,1)$, $Q(0,-1)$, $R(3,3)$ (d) $X(-3,-1)$, $Y(0,0)$, $Z(-2,2)$
10. (a) The interval joining $G(2,-5)$ and $H(-6,-1)$ is divided into four equal subintervals by the three points A , B and C . Find their coordinates by repeatedly taking midpoints.
- (b) Find the coordinates of the four points A , B , C and D which divide the interval joining $S(-2,3)$ and $T(8,18)$ into five equal subintervals. (You will need the ratio division formula.)
11. The quadrilateral $ABCD$ has vertices at $A(1,0)$, $B(3,1)$, $C(4,3)$ and $D(2,2)$.
- (a) Show that the intervals AC and BD bisect each other, by finding the midpoint of each and showing that these midpoints coincide. What can you now conclude about the type of quadrilateral $ABCD$ is?
- (b) Show that $AB = AD$. What can you now conclude about the quadrilateral $ABCD$?
12. Show that the points $A(1,4)$, $B(2,\sqrt{13})$, $C(3,2\sqrt{2})$ and $D(4,1)$ lie on a circle with centre the origin. What are the radius, diameter, circumference and area of this circle?
13. As discussed in Chapter Two, the circle with centre (h,k) and radius r has equation $(x-h)^2 + (y-k)^2 = r^2$. By identifying the centre and radius, find the equations of:
- (a) the circle with centre $(5,-2)$ passing through $(-1,1)$,
 (b) the circle with $K(5,7)$ and $L(-9,-3)$ as endpoints of a diameter.
14. (a) If $A(-1,2)$ is the midpoint of $S(x,y)$ and $T(3,6)$, find the coordinates of S .
- (b) The midpoint of PQ is $M(2,-7)$. Find P if: (i) $Q = (5,3)$, (ii) $Q = (-3,-7)$.
- (c) If AB is a diameter of a circle with centre $Q(4,5)$ and $A = (8,3)$, find B .
- (d) Given that $P(4,7)$ is one vertex of a square $PQRS$ and the centre of the square is $M(8,-1)$, find the coordinates of R .
15. (a) Given the point $A(7,8)$, find the coordinates of three points P with integer coordinates such that $AP = \sqrt{5}$.
- (b) If the distance from $U(3,7)$ to $V(1,y)$ is $\sqrt{13}$, find the two possible values of y .
- (c) Find a , if the distance from $A(a,0)$ to $B(1,4)$ is $\sqrt{18}$ units.

16. A triangle has vertices $A(-2, 2)$, $B(-4, -3)$ and $C(6, -2)$.
- Find the midpoint P of BC , then find the coordinates of the point M dividing the interval AP (called a *median*) in the ratio $2 : 1$.
 - Do likewise for the medians BQ and CR and confirm that the same point is obtained each time (this point is called the *centroid* of the triangle).
17. (a) Find the ratio in which the point $M(3, 5)$ divides the interval joining $A(-4, -9)$ and $B(5, 9)$. [HINT: Let M divide AB in the ratio $k : 1$, and find k .]
- (b) Given the collinear points $P(-2, -11)$, $Q(1, -2)$ and $R(3, 4)$, find: (i) the ratio in which P divides QR , (ii) the ratio in which Q divides RP , (iii) the ratio in which R divides PQ .
18. If the point M divides the interval AB in each ratio given below, draw a diagram and find in what ratio B divides AM :
- $AM : MB = 1 : 1$
 - $AM : MB = 1 : 3$
 - $AM : MB = 7 : 4$
 - $AM : MB = 2 : -1$
 - $AM : MB = -1 : 3$
 - $AM : MB = 4 : -3$
19. (a) Given the four collinear points $P(1, -8)$, $Q(5, -2)$, $R(7, 1)$ and $S(13, 10)$, show that:
- Q divides PR internally in the same ratio as S divides PR externally,
 - R divides QS internally in the same ratio as P divides QS externally.
- (b) Prove in general that if P , Q , R and S are four collinear points such that Q divides PR internally in the same ratio as S divides PR externally, then R divides QS internally in the same ratio as P divides QS externally. [HINT: Let $PQ = a$, $QR = b$ and $RS = c$.]
20. (a) Given $K(3, -1)$ and $L(-4, 2)$, find two positions of A on KL such that $KA = 2 \times KL$.
- (b) The point $Q(1, -2)$ divides the interval $R(x, y)$ to $S(4, 2)$ in the ratio $1 : 4$. Find R .
21. The point P divides the interval joining $A(-1, 4)$ and $B(2, -2)$ in the ratio $k : 1$.
- Write down the coordinates of P .
 - Given that P lies on the line $2y - x + 1 = 0$, find k , and hence find the coordinates of P .
22. (a) Given that $C(x, y)$ is equidistant from each of the points $P(1, 5)$, $Q(-5, -3)$ and $R(2, -2)$, use the distance formula to form two equations in x and y and solve them simultaneously to find the coordinates of C .
- (b) Find the coordinates of the point $M(x, y)$ which is equidistant from each of the points $P(4, 3)$ and $Q(3, 2)$, and is also equidistant from $R(6, 1)$ and $S(4, 0)$.
23. (a) The point $F(0, 1)$ divides PQ in the ratio $t : \frac{1}{t}$, where P is $(2t, t^2)$. Find Q .
- (b) The origin O divides RS externally in the ratio $r : \frac{1}{r}$, where $R(a, b)$.
- Find the coordinates of S .
 - What are these coordinates if $r = OR$?
24. Suppose A , B and P are the points $(0, 0)$, $(3a, 0)$ and (x, y) respectively. Use the distance formula to form an equation in x and y for the point P , and describe the curve so found if: (a) $PA = PB$, (b) $PA = 2PB$.

EXTENSION

25. The point M on the line through $P(x_1, y_1)$ and $Q(x_2, y_2)$ which divides PQ into the ratio $k : 1$ has coordinates $\left(\frac{x_1 + kx_2}{1 + k}, \frac{y_1 + ky_2}{1 + k} \right)$.
- Which point on the line PQ cannot be expressed in this manner?
 - What range of values of k will result in: (i) M between P and Q , (ii) M on the opposite side of Q from P , and (iii) M on the opposite side of P from Q . (iv) What happens as $k \rightarrow (-1)^+$ and as $k \rightarrow (-1)^-$?

26. The point $M(x, y)$ divides the interval joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $k : \ell$.
- (a) Show by geometry that $\frac{k}{\ell} = \frac{y - y_1}{y_2 - y}$.
- (b) Hence show that an equation of the line PQ is $(x - x_1)(y - y_2) = (y - y_1)(x - x_2)$.
- (c) Alternatively, justify the equation above by showing that P and Q satisfy it.

5B Gradients of Intervals and Lines

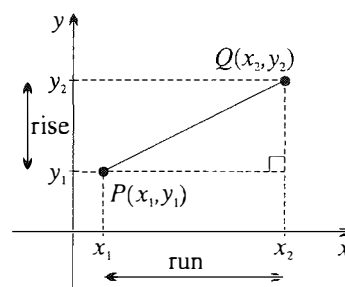
Gradient is the key to bringing lines and their equations into coordinate geometry. The gradient of an interval is easy to define, but we need to use similarity to define the gradient of a line.

The Gradient of an Interval: Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points in the number plane.

Define the *rise* from P to Q as the change $y_2 - y_1$ in y from P to Q , so that the rise is positive when Q is above P , and negative when Q is below P .

Define the *run* from P to Q as the change $x_2 - x_1$ in x , so that the run is positive when Q is on the right of P , and negative when Q is on the left of P .

The *gradient* of PQ is the ratio of these two changes.

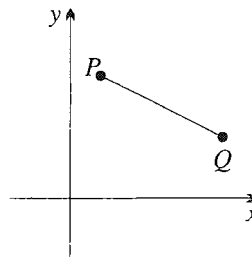
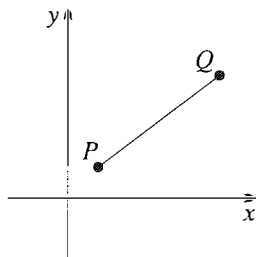


7 **GRADIENT FORMULA:** gradient of $PQ = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$

Intervals have gradient zero if and only if they are horizontal, because only horizontal intervals have zero rise. Vertical intervals, on the other hand, don't have a gradient, because their run is always zero and so the fraction is undefined.

Positive and Negative Gradients: If the rise and the run have the same sign, then the gradient will be positive, as in the first diagram below — in this case the interval slopes *upwards* as one moves from left to right.

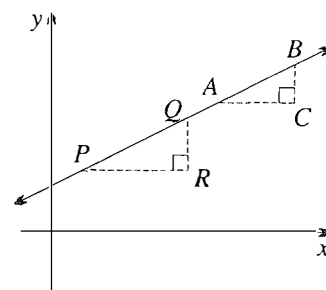
If the rise and run have opposite signs, then the gradient will be negative, as in the second diagram — now the interval slopes *downwards* as one moves from left to right.



Notice that if the points P and Q are interchanged, then both rise and run change signs, but the gradient remains the same.

The Gradient of a Line: The *gradient of a line* is defined to be the gradient of any interval within the line. This definition makes sense because any two intervals on the same line always have the same gradient.

To prove this, suppose that PQ and AB are two intervals on the same line ℓ . Construct right triangles PQR and ABC underneath the intervals, with sides parallel to the axes. Because these two triangles are similar (by the AA similarity test), the ratios of their heights and bases are the same, which means that the two intervals AB and PQ have the same gradient.



A Condition for Two Lines to be Parallel: The condition for two lines to be parallel is:

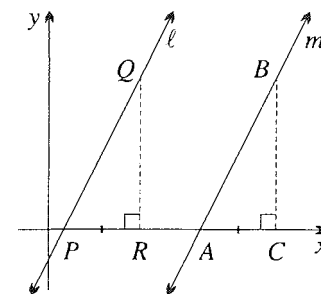
8

PARALLEL LINES: Two lines are parallel if and only if they have the same gradient (or are both vertical).

To prove this, let ℓ and m be two lines meeting the x -axis at P and A respectively, and construct the two triangles PQR and ABC as shown, with the two runs PR and AC equal.

If the lines are parallel, then the corresponding angles $\angle P$ and $\angle A$ are equal. Hence the two triangles are congruent by the AAS test, and so the rises RQ and CB must be equal.

Conversely, if the gradients are equal, then the rises RQ and CB are equal. Hence the triangles are congruent by the SAS test, so the corresponding angles $\angle P$ and $\angle A$ are equal, and so the lines must be parallel.



WORKED EXERCISE: Show that the points $A(3,6)$, $B(7,-2)$, $C(4,-5)$ and $D(-1,5)$ form a trapezium with $AB \parallel CD$.

$$\begin{aligned} \text{SOLUTION: gradient of } AB &= \frac{-2-6}{7-3} & \text{gradient of } CD &= \frac{5+5}{-1-4} \\ &= -2, & &= -2. \end{aligned}$$

So $AB \parallel CD$, and $ABCD$ is a trapezium.

Testing for Collinear Points: Three distinct points are called *collinear* if they all lie on the same line. To test whether three given points A , B and C are collinear, the most straightforward method is to find the gradients of AB and AC . If these gradients are equal, then the three points must be collinear, because then AB and AC are parallel lines passing through a common point A .

WORKED EXERCISE: Test whether $A(-2,5)$, $B(1,3)$ and $C(7,-1)$ are collinear.

$$\begin{aligned} \text{SOLUTION: gradient of } AB &= \frac{3-5}{1+2} & \text{gradient of } AC &= \frac{-1-5}{7+2} \\ &= -\frac{2}{3}, & &= -\frac{2}{3}. \end{aligned}$$

Since the gradients are equal, the points are collinear.

Gradient and the Angle of Inclination: Another very natural way of measuring the steepness of a line is to look at its *angle of inclination*, which is the angle between the upward direction of the line and the positive direction of the x -axis. The two diagrams below show that lines with positive gradient have acute angles of inclination, and lines with negative gradient have obtuse angles of inclination.



These two diagrams also illustrate the trigonometric relationship between the gradient and angle of inclination α .

9 **ANGLE OF INCLINATION:** gradient = $\tan \alpha$

PROOF: When α is acute, as in the first diagram, then the rise MP and the run OM are just the opposite and adjacent sides of the triangle POM , so

$$\tan \alpha = \frac{MP}{OM} = \text{gradient } OP.$$

When α is obtuse, as in the second diagram, then $\angle POM = 180^\circ - \alpha$, so

$$\tan \alpha = -\tan \angle POM = -\frac{MP}{OM} = \text{gradient } OP.$$

WORKED EXERCISE: Given the points $A(-3, 5)$, $B(-6, 0)$ and $O(0, 0)$, find the angles of inclination of AB and AO , and show that they are supplementary. What sort of triangle is $\triangle ABO$?

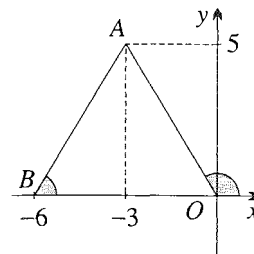
SOLUTION: gradient of $AB = \frac{0 - 5}{-6 + 3}$
 $= \frac{5}{3},$

so angle of inclination $\doteq 59^\circ$.

gradient of $AO = \frac{5 - 0}{-3 - 0}$
 $= -\frac{5}{3}.$

so angle of inclination $\doteq 121^\circ$.

Hence the angles of inclination are supplementary, and $\triangle ABO$ is isosceles.



A Condition for Lines to be Perpendicular: The condition for two lines to be perpendicular is:

10 **PERPENDICULARITY:** Two lines are perpendicular if and only if the product of their gradients is -1 (or one is vertical and the other horizontal).

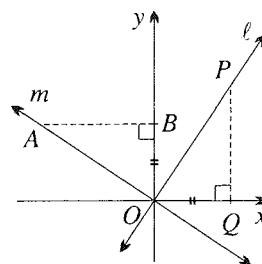
PROOF: We can shift each line sideways without rotating it so that it passes through the origin (remember that parallel lines have the same gradient). Also, one line must have positive gradient and the other negative gradient, otherwise one of the angles between them would be acute.

So let ℓ and m be two lines through the origin with positive and negative gradients respectively, and construct the two triangles POQ and AOB as shown, with the run OQ of ℓ equal to the rise OB of m . Then

$$\text{gradient of } \ell \times \text{gradient of } m = -\frac{QP \times OB}{OQ \times AB} = -\frac{QP}{AB}.$$

If the lines are perpendicular, then $\angle AOB = \angle POQ$. Hence the triangles are congruent by the AAS test, so $QP = AB$, with the result that the product of the gradients is -1 .

Conversely, if the product of the gradients is -1 , then $QP = AB$. Hence the triangles are congruent by the SAS test, so $\angle AOB = \angle POQ$ and the lines are perpendicular.



WORKED EXERCISE: Given the four points $A(-1, 1)$, $B(7, 11)$, $C(0, 8)$ and $D(a, -1)$, find a and the coordinates of D if $AB \perp CD$.

$$\begin{aligned} \text{SOLUTION: gradient of } AB &= \frac{11 - 1}{7 - (-1)} \\ &= \frac{5}{4} \\ \text{gradient of } CD &= \frac{-1 - 8}{a - 0} \\ &= -\frac{9}{a} \end{aligned}$$

Since AB and CD are perpendicular,
gradient of $CD \times$ gradient of $AB = -1$

$$\begin{aligned} -\frac{9}{a} \times \frac{5}{4} &= -1 \\ 4a &= 45 \\ a &= \frac{45}{4}. \end{aligned}$$

So D is the point $(11\frac{1}{4}, -1)$.

Exercise 5B

NOTE: Diagrams should be drawn wherever possible.

- Find the gradient of a line (i) parallel to, (ii) perpendicular to a line with gradient:
 - 2
 - 1
 - $\frac{3}{4}$
 - $-\frac{p}{q}$
- Find the gradient of each interval AB , then find the gradient of a line perpendicular to it:
 - $(1, 4), (5, 0)$
 - $(-5, -2), (3, 2)$
 - $(-1, -2), (1, 4)$
 - $(-2, -7), (3, 3)$
 - $(3, 6), (5, 5)$
 - $(-a, b), (3a, -b)$
- Find the gradient, to two decimal places, of a line with angle of inclination:
 - 15°
 - 135°
 - $22\frac{1}{2}^\circ$
 - 72°
- What angle of inclination (to the nearest degree where necessary) does a line with each gradient make with the x -axis? Does the line slope upwards or downwards?
 - 1
 - $-\sqrt{3}$
 - 4
 - $\frac{1}{\sqrt{3}}$
 - Find the acute angle made by each line in part (a) with the y -axis.
- Find the points A and B where each line meets the x -axis and y -axis respectively. Hence find the gradient of AB and its angle of inclination α (to the nearest degree):
 - $y = 3x + 6$
 - $3x + 4y + 12 = 0$
 - $4x - 5y - 20 = 0$
 - $y = -\frac{1}{2}x + 1$
 - $\frac{x}{3} - \frac{y}{2} = 1$
 - $\frac{x}{2} + \frac{y}{5} = 1$

6. Given $A = (2, 3)$, write down the coordinates of any three points P such that AP has gradient 2.

DEVELOPMENT

7. Find the gradient, to two decimal places, of a line sloping upwards if its acute angle with the y -axis is:
 (a) 15° (b) 45° (c) $22\frac{1}{2}^\circ$ (d) 72°
8. Find the gradients of PQ and QR , and hence determine whether P , Q and R are collinear:
 (a) $P(-2, 7)$, $Q(1, 1)$, $R(4, -6)$ (b) $P(-5, -4)$, $Q(-2, -2)$, $R(1, 0)$
9. (a) Triangle ABC has vertices $A(-1, 0)$, $B(3, 2)$ and $C(4, 0)$. Calculate the gradient of each side and hence show that $\triangle ABC$ is a right-angled triangle.
 (b) Do likewise for the triangles with the vertices given below. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle:
 (i) $P(2, -1)$, $Q(3, 3)$, $R(-1, 4)$ (ii) $X(-1, -3)$, $Y(2, 4)$, $Z(-3, 2)$
10. Quadrilateral $ABCD$ has vertices $A(-1, 1)$, $B(3, -1)$, $C(5, 3)$ and $D(1, 5)$.
 (a) Show that it has two pairs of parallel sides. (b) Confirm that $AB \perp BC$.
 (c) Show also that $AB = BC$. (d) What type of quadrilateral is $ABCD$?
11. Use gradients to show that each quadrilateral $ABCD$ below is a parallelogram. Then show that it is:
 (a) a rhombus, for the vertices $A(2, 1)$, $B(-1, 3)$, $C(1, 0)$ and $D(4, -2)$,
 (b) a rectangle, for the vertices $A(4, 0)$, $B(-2, 3)$, $C(-3, 1)$ and $D(3, -2)$,
 (c) a square, for the vertices $A(3, 3)$, $B(-1, 2)$, $C(0, -2)$ and $D(4, -1)$.
12. The interval PQ has gradient -3 . A second line passes through $A(-2, 4)$ and $B(1, y)$. Find the value of y if: (a) AB is parallel to PQ , (b) AB is perpendicular to PQ .
13. Find λ for the points $X(-1, 0)$, $Y(1, \lambda)$ and $Z(\lambda, 2)$, if $\angle YXZ = 90^\circ$.
14. For the four points $P(k, 1)$, $Q(-2, -3)$, $R(2, 3)$ and $S(1, k)$, it is known that PQ is parallel to RS . Find the possible values of k .
15. On a number plane, mark the origin O and the points $A(2, 1)$ and $B(3, -1)$.
 (a) Find the gradients of OA and AB and hence show that they are perpendicular.
 (b) Show that $OA = AB$. (c) Find the midpoint D of OB .
 (d) Given that D is the midpoint of AC , find the coordinates of C .
 (e) What shape best describes quadrilateral $OABC$?
16. Answer the following questions for the points $W(2, 3)$, $X(-7, 5)$, $Y(-1, -3)$ and $Z(5, -1)$.
 (a) Show that WZ is parallel to XY .
 (b) Find the lengths WZ and XY . Hence deduce the type of the quadrilateral $WXYZ$.
 (c) Show that the diagonals WY and XZ are perpendicular.
17. Quadrilateral $ABCD$ has vertices $A(1, -4)$, $B(3, 2)$, $C(-5, 6)$ and $D(-1, -2)$.
 (a) Find the midpoints P of AB , Q of BC , R of CD , and S of DA .
 (b) Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.
18. (a) $A(1, 4)$, $B(5, 0)$ and $C(9, 8)$ form the vertices of a triangle. Find the coordinates of P and Q if they divide the sides AB and AC respectively in the ratio $1 : 3$.
 (b) Show that PQ is parallel to BC and is one quarter of its length.

19. Given the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, find and simplify the gradient of PQ .
20. The points $A(4, -2)$, $B(-4, 4)$ and $P(x, y)$ form a right angle at P . Form an equation in x and y , and hence find the equation of the curve on which P lies. Describe this curve.
21. The points $O(0, 0)$, $P(4, 0)$ and $Q(x, y)$ form a right angle at Q and $PQ = 1$ unit.
- Form a pair of equations for x and y .
 - Solve them simultaneously to find the coordinates of the two possible locations of Q .

————— EXTENSION —————

22. (a) The points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x, y)$ are collinear. Use the gradient formula to show that

$$(x - x_1)(y - y_2) = (y - y_1)(x - x_2).$$

- (b) If AB is the diameter of a circle and P another point on the circumference then Euclidean geometry tells us that $\angle APB = 90^\circ$. Use this fact to show that the equation of the circle whose diameter has endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

23. (a) Three points $A_1(a_1, b_1)$, $A_2(a_2, b_2)$, $A_3(a_3, b_3)$ form a triangle. By dropping perpendiculars to the x -axis and taking the areas of the resulting trapeziums, show that the area Δ of the triangle $A_1A_2A_3$ is

$$\Delta = \frac{1}{2}|a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 + a_3b_1 - a_1b_3|,$$

with the expression inside the absolute value sign positive if and only if the vertices A_1 , A_2 and A_3 are in anticlockwise order.

- (b) Use part (a) to generate a test for A_1 , A_2 and A_3 to be collinear.
- (c) Generate the same test by putting gradient $A_1A_2 = \text{gradient } A_2A_3$.
24. Consider the points $P(2p, p^2)$, $Q(-\frac{2}{p}, \frac{1}{p^2})$ and $T(x, -1)$. Find the x -coordinate of T if:
- the three points are collinear,
 - PT and QT are perpendicular.
25. The points $P(p, 1/p)$, $Q(q, 1/q)$, $R(r, 1/r)$ and $S(s, 1/s)$ lie on the curve $xy = 1$.
- If $PQ \parallel RS$, show that $pq = rs$.
 - Show that $PQ \perp RS$ if and only if $pqrs = -1$.
 - Use part (b) to conclude that if a triangle is drawn with its vertices on the rectangular hyperbola $xy = 1$, then the altitudes of the triangle intersect at a common point which also lies on the hyperbola (an altitude of a triangle is the perpendicular from a vertex to the opposite side).

5C Equations of Lines

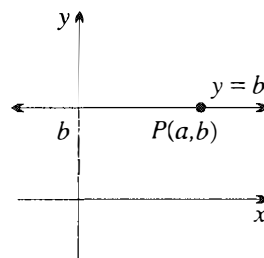
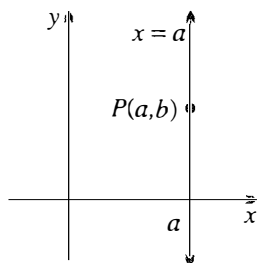
In coordinate geometry, a line in the number plane is represented by an equation in x and y . This section and the next summarise that theory from earlier years, and develop various useful forms for the equation of the line.

Horizontal and Vertical Lines: In a vertical line, all points on the line have the same x -coordinate, but the y -coordinate can take any value.

11 VERTICAL LINES: The vertical line through $P(a, b)$ has equation $x = a$.

In a horizontal line, all points on the line have the same y -coordinate, but the x -coordinate can take any value.

12 HORIZONTAL LINES: The horizontal line through $P(a, b)$ has equation $y = b$.



Gradient-Intercept Form: The problem here is to find a formula for the equation of a line when its gradient and y -intercept are known. Suppose that ℓ has gradient m , and that it has y -intercept b , passing through the point $B(0, b)$. If $Q(x, y)$ is any other point in the plane, then the condition that Q lie on the line ℓ is

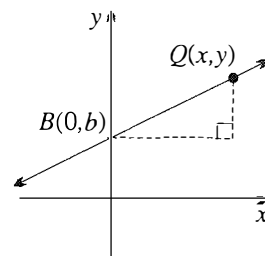
$$\text{gradient of } BQ = m,$$

that is,
$$\frac{y - b}{x} = m,$$

or
$$y - b = mx.$$

Hence
$$y = mx + b,$$

which is the equation of BQ in *gradient-intercept form*.



13 GRADIENT-INTERCEPT FORM: $y = mx + b$

WORKED EXERCISE:

- Write down the gradient and the y -intercept of the line ℓ : $y = 3x - 2$.
- Hence find the equations of the lines through $B(0, 5)$ which are parallel and perpendicular to ℓ .

SOLUTION:

- The line ℓ has gradient 3 and y -intercept -2 .
- The line through B parallel to ℓ has gradient 3 and y -intercept 5, so its equation is $y = 3x + 5$.
The line through B perpendicular to ℓ has gradient $-\frac{1}{3}$ and y -intercept 5, so its equation is $y = -\frac{1}{3}x + 5$.

General Form: It is often useful to have the equation of a line in a standard simplified form, with everything on the LHS. The *general form* of the equation of a line is:

14 GENERAL FORM: $ax + by + c = 0$

When an equation is given in general form, it should be simplified by multiplying out all fractions and dividing out all common factors. It may also be convenient to make the coefficient of x positive.

WORKED EXERCISE:

- (a) Find the gradient and y -intercept of the line ℓ : $2x - 3y + 4 = 0$.
 (b) Find in general form the equations of the lines passing through $B(0, -2)$ and:
 (i) parallel to ℓ , (ii) perpendicular to ℓ , (iii) having angle of inclination 60° .

SOLUTION:

- (a) Solving the line ℓ for y ,

$$3y = 2x + 4$$

$$y = \frac{2}{3}x + \frac{4}{3},$$
 so ℓ has gradient $\frac{2}{3}$ and y -intercept $\frac{4}{3}$.
 (b) (i) The line through B parallel to ℓ has gradient $\frac{2}{3}$ and y -intercept -2 ,
 so its equation is

$$\boxed{\times 3} \quad y = \frac{2}{3}x - 2$$

$$3y = 2x - 6$$

$$2x - 3y - 6 = 0.$$
 (ii) The line through B perpendicular to ℓ has gradient $-\frac{3}{2}$ and y -intercept -2 ,
 so its equation is

$$\boxed{\times 2} \quad y = -\frac{3}{2}x - 2$$

$$2y = -3x - 4$$

$$3x + 2y + 4 = 0.$$
 (iii) The line through B with angle of inclination 60° has gradient $\tan 60^\circ = \sqrt{3}$,
 so its equation is

$$y = x\sqrt{3} - 2$$

$$x\sqrt{3} - y - 2 = 0.$$

Exercise 5C

- Determine by substitution whether the point $A(3, -2)$ lies on the line:
 (a) $y = 4x - 10$ (b) $8x + 10y - 4 = 0$ (c) $x = 3$
- Write down the coordinates of any three points on the line $2x + 3y = 4$.
- Write down the equations of the vertical and horizontal lines through:
 (a) $(1, 2)$ (b) $(-1, 1)$ (c) $(3, -4)$ (d) $(5, 1)$ (e) $(-2, -3)$ (f) $(-4, 1)$
- Write down the gradient and y -intercept of each line:
 (a) $y = 4x - 2$ (b) $y = \frac{1}{5}x - 3$ (c) $y = 2 - x$
- Use the formula $y = mx + b$ to write down the equation of each of the lines specified below, then put that equation into general form:
 (a) with gradient 1 and y -intercept 3 (c) with gradient $\frac{1}{5}$ and y -intercept -1
 (b) with gradient -2 and y -intercept 5 (d) with gradient $-\frac{1}{2}$ and y -intercept 3
- Solve each equation for y and hence write down the gradient and y -intercept:
 (a) $x - y + 3 = 0$ (c) $2x - y = 5$ (e) $3x + 4y = 5$
 (b) $y + x - 2 = 0$ (d) $x - 3y + 6 = 0$ (f) $2y - 3x = -4$
- For each line in question 6, substitute $y = 0$ and $x = 0$ to find the points A and B where the line intersects the x -axis and y -axis respectively, and hence sketch the curve.
- For each line in question 6, use the formula gradient $= \tan \alpha$ to find its angle of inclination, to the nearest minute where appropriate.
- Show by substitution that the line $y = mx + b$ passes through $A(0, b)$ and $B(1, m + b)$. Then show that the gradient of AB , and hence of the line, is m .

DEVELOPMENT

10. Find the gradient of each line below, and hence find in gradient–intercept form the equation of a line passing through $A(0,3)$ and (i) parallel to it, (ii) perpendicular to it:
 (a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $3x + 4y - 5 = 0$
11. In each part below, the angle of inclination α and the y -intercept A of a line are given. Use the formula $\text{gradient} = \tan \alpha$ to find the gradient of each line, then find its equation in general form:
 (a) $\alpha = 45^\circ$, $A = (0,3)$ (c) $\alpha = 30^\circ$, $A = (0,-2)$
 (b) $\alpha = 60^\circ$, $A = (0,-1)$ (d) $\alpha = 145^\circ$, $A = (0,1)$
12. A triangle is formed by the x -axis and the lines $5y = 9x$ and $5y + 9x = 45$. Find (to the nearest degree) the angles of inclination of the two lines, and hence show that the triangle is isosceles.
13. Consider the two lines $\ell_1: 3x - y + 4 = 0$ and $\ell_2: x + ky + \ell = 0$. Find the value of k if:
 (a) ℓ_1 is parallel to ℓ_2 , (b) ℓ_1 is perpendicular to ℓ_2 .
14. [HINT: In each part of this question, draw a diagram of the situation, then use congruent or similar triangles to find the gradient and y -intercept of the line.] Find the equation of the line through $M(-1,2)$ if:
 (a) M is the midpoint of the intercepts on the x -axis and y -axis,
 (b) M divides the intercepts in the ratio $2 : 1$ (x -intercept to y -intercept),
 (c) M divides the intercepts in the ratio $-2 : 5$ (x -intercept to y -intercept).

EXTENSION

15. Find the equations of the four circles which are tangent to the x -axis, the y -axis, and the line $x + y = 2$.
16. (a) Show that the four lines $y = 2x - 1$, $y = 2x + 1$, $y = 3 - \frac{1}{2}x$ and $y = k - \frac{1}{2}x$ enclose a rectangle. (b) Find the possible values of k if they enclose a square.

5D Further Equations of Lines

This section introduces two further standard forms of the equations of lines, namely point–gradient form and the two-intercept form. It also deals with lines through two given points, and the point of intersection of two lines.

Point–Gradient Form: The problem here is to find a formula for the equation of a line ℓ when we know that ℓ has a particular gradient m and passes through a particular point $P(x_1, y_1)$. If $Q(x, y)$ is any other point in the plane, then the condition that Q lie on the line ℓ is

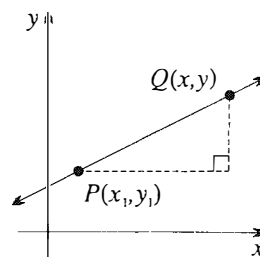
$$\text{gradient of } PQ = m,$$

$$\text{that is, } \frac{y - y_1}{x - x_1} = m,$$

$$\text{or } y - y_1 = m(x - x_1),$$

which is the equation of PQ in *point–gradient form*.

15 **POINT–GRADIENT FORM:** $y - y_1 = m(x - x_1)$



NOTE: Careful readers will realise that the equation $\frac{y - y_1}{x - x_1} = m$ actually describes a line with the point P itself removed, because substituting $P(x_1, y_1)$ into the LHS yields $\frac{0}{0}$. However, when both sides are multiplied by $x - x_1$ to give $y - y_1 = m(x - x_1)$, then the point P is now included, because substitution into either side gives 0.

WORKED EXERCISE:

- (a) Find the equation of a line through $(-2, -5)$ perpendicular to $y = 3x + 2$.
 (b) Express the answer in gradient–intercept form, and hence write down its y -intercept.

SOLUTION:

- (a) The given line has gradient 3, so the perpendicular gradient is $-\frac{1}{3}$.

Hence, using point–gradient form, the required line is

$$y - y_1 = m(x - x_1)$$

$$y + 5 = -\frac{1}{3}(x + 2)$$

$$y = -\frac{1}{3}x - 5\frac{2}{3}.$$

- (b) This is gradient–intercept form, and so its y -intercept is $-5\frac{2}{3}$.

The Line through Two Given Points: Given two distinct points, there is just one line passing through them both. Its equation is best found by a two-step approach.

THE LINE THROUGH TWO GIVEN POINTS:

16

1. Find the gradient of the line.
2. Use point–gradient form to find the equation of the line.

WORKED EXERCISE: Find the equation of the line through $A(1, 5)$ and $B(4, -1)$.

SOLUTION: First, gradient $AB = \frac{-1 - 5}{4 - 1}$
 $= -2.$

Then, using point–gradient form for a line with gradient -2 through $A(1, 5)$, the line AB is

$$y - y_1 = -2(x - x_1)$$

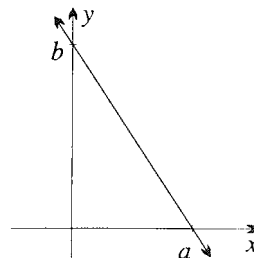
$$y - 5 = -2(x - 1)$$

$$y = -2x + 7.$$

Two-Intercept Form: The problem here is to find the equation of a line ℓ whose x -intercept is a and whose y -intercept is b . This time, the result is very obvious once it is written down:

17

TWO-INTERCEPT FORM: $\frac{x}{a} + \frac{y}{b} = 1$



PROOF: The line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the two points $(a, 0)$ and $(0, b)$, because both points satisfy the equation. Notice that if the line passes through the origin, then $a = b = 0$ and the equation fails. It also fails if the line is vertical (when it has no y -intercept), or horizontal (when it has no x -intercept).

WORKED EXERCISE: Given $A(6, 0)$ and $B(0, 9)$, find in general form the equations of:

- (a) the line AB ,
- (b) the line ℓ_1 perpendicular to AB through A ,
- (c) the line ℓ_2 perpendicular to AB through B .

SOLUTION:

- (a) Using the two-intercept form,

$$AB \text{ is } \frac{x}{6} + \frac{y}{9} = 1$$

$$\boxed{\times 18} \quad 3x + 2y - 18 = 0.$$

- (b) Since AB has gradient $-\frac{3}{2}$,
the gradient perpendicular to AB is $\frac{2}{3}$,

so using point-gradient form,

$$\ell_1 \text{ is } y - 0 = \frac{2}{3}(x - 6)$$

$$2x - 3y - 12 = 0.$$

- (c) Using gradient-intercept form,

$$\ell_2 \text{ is } y = \frac{2}{3}x + 9$$

$$2x - 3y + 27 = 0.$$

Intersection of Lines — Concurrent Lines: The point where two distinct lines intersect can be found using simultaneous equations, as discussed in Chapter One.

Three distinct lines are called *concurrent* if they all pass through the same point. To test whether three given lines are concurrent, the most straightforward method is to find the point where two of them intersect, then test by substitution whether this point lies on the third line.

WORKED EXERCISE: Test whether the three lines $\ell_1: 5x - y - 10 = 0$, $\ell_2: x + y - 8 = 0$ and $\ell_3: 2x - 3y + 9 = 0$ are concurrent.

SOLUTION: First we solve ℓ_1 and ℓ_2 simultaneously.

$$\text{Adding } \ell_1 \text{ and } \ell_2, \quad 6x - 18 = 0$$

$$x = 3$$

$$\text{and substituting into } \ell_2, \quad 3 + y - 8 = 0$$

$$y = 5$$

so ℓ_1 and ℓ_2 intersect at $(3, 5)$.

Then substituting $(3, 5)$ into the third line ℓ_3 ,

$$\text{LHS} = 6 - 15 + 9$$

$$= 0$$

$$= \text{RHS},$$

so the three lines are concurrent, meeting at $(3, 5)$.

Some Consequences: It may be useful to know some of the obvious consequences of all these formulae. Here are a few remarks, chosen from many more that could usefully be made. Most problems, however, are best done without quoting these consequences, which can easily be derived on the spot if needed.

1. The gradient of the line $ax + by + c = 0$ is $-\frac{a}{b}$, its y -intercept is $-\frac{c}{b}$, and its x -intercept is $-\frac{c}{a}$.
2. Any line parallel to $ax + by + c = 0$ has the form $ax + by + k = 0$, for some constant k determined by the particular circumstances.
3. Any line perpendicular to $ax + by + c = 0$ has the form $bx - ay + k = 0$, for some constant k determined by the particular circumstances.

4. Two lines $ax + by + c = 0$ and $Ax + By + C = 0$:
 - (a) intersect in a single point if their gradients $-a/b$ and $-A/B$ are unequal,
 - (b) are parallel if their gradients $-a/b$ and $-A/B$ are equal, but their y -intercepts $-c/b$ and $-C/B$ are unequal,
 - (c) are the same line if their gradients $-a/b$ and $-A/B$ are equal, and their y -intercepts $-c/b$ and $-C/B$ are equal.
5. Two lines $ax + by + c = 0$ and $Ax + By + C = 0$ are perpendicular if and only if the products of their gradients $-a/b$ and $-A/B$ is -1 .

Exercise 5D

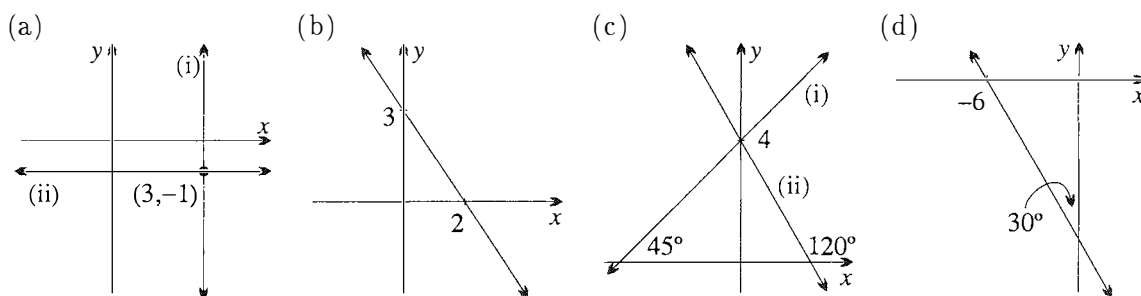
NOTE: Selection of questions from this long exercise will depend on students' previous knowledge.

1. Use point-gradient form $y - y_1 = m(x - x_1)$ to find in general form the equation of the line:
 - (a) through $(1, 1)$ with gradient 2,
 - (b) with gradient -1 through $(3, 1)$,
 - (c) with gradient 3 through $(-5, -7)$,
 - (d) through $(0, 0)$ with gradient -5 ,
 - (e) through $(-1, 3)$ with gradient $-\frac{1}{3}$,
 - (f) with gradient $-\frac{4}{5}$ through $(3, -4)$.
2. Find the gradient of the line through each pair of given points, and hence find its equation:
 - (a) $(3, 4), (5, 8)$
 - (b) $(-1, 3), (1, -1)$
 - (c) $(-4, -1), (6, -6)$
 - (d) $(5, 6), (-1, 4)$
3. Write down the equation of the line with the given intercepts, then rewrite it in general form:
 - (a) $(-1, 0), (0, 2)$
 - (b) $(2, 0), (0, 3)$
 - (c) $(0, -1), (-4, 0)$
 - (d) $(0, -3), (3, 0)$
4. (a) Find the point M of intersection of the lines $\ell_1: x + y = 2$ and $\ell_2: 4x - y = 13$.
 (b) Show that M lies on $\ell_3: 2x - 5y = 11$, and hence that ℓ_1, ℓ_2 and ℓ_3 are concurrent.
 (c) Use the same method to test whether each set of lines is concurrent:
 - (i) $2x + y = -1, x - 2y = -18$ and $x + 3y = 15$
 - (ii) $6x - y = 26, 5x - 4y = 9$ and $x + y = 9$
5. Find the gradient of each line below and hence find, in gradient-intercept form, the equation of a line: (i) parallel to it passing through $A(3, -1)$, (ii) perpendicular to it passing through $B(-2, 5)$.
 - (a) $2x + y + 3 = 0$
 - (b) $5x - 2y - 1 = 0$
 - (c) $4x + 3y - 5 = 0$
6. Given the points $A(1, -2)$ and $B(-3, 4)$, find in general form the equation of:
 - (a) the line AB ,
 - (b) the line through A perpendicular to AB .

DEVELOPMENT

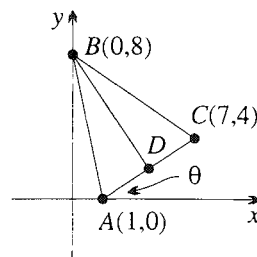
7. The angle of inclination α and a point A on a line are given below. Use the formula $\text{gradient} = \tan \alpha$ to find the gradient of each line, then find its equation in general form:
 - (a) $\alpha = 45^\circ, A = (1, 0)$
 - (b) $\alpha = 120^\circ, A = (-1, 0)$
 - (c) $\alpha = 30^\circ, A = (4, -3)$
 - (d) $\alpha = 150^\circ, A = (-2, -5)$
8. Explain why the four lines $\ell_1: y = x + 1, \ell_2: y = x - 3, \ell_3: y = 3x + 5$ and $\ell_4: y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.
9. Triangle ABC has vertices $A(1, 0), B(6, 5)$ and $C(0, 2)$. Show that it is right-angled, then find the equation of each side.

10. Determine in general form the equation of each straight line sketched:



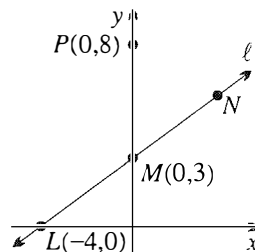
11. The points A , B and C have coordinates $(1,0)$, $(0,8)$ and $(7,4)$, and the angle between AC and the x -axis is θ .

- Find the gradient of the line AC and hence determine θ to the nearest degree.
- Derive the equation of AC .
- Find the coordinates of D , the midpoint of AC .
- Show that AC is perpendicular to BD .
- What type of triangle is ABC ?
- Find the area of this triangle.
- Write down the coordinates of a point E such that $ABCE$ is a rhombus.

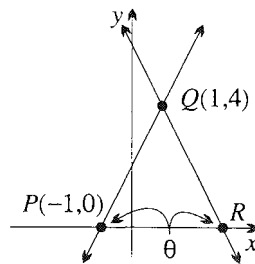


12. (a) On a number plane plot the points $A(4,3)$, $B(0,-3)$ and $C(4,0)$.
 (b) Find the equation of BC . (c) Explain why $OABC$ is a parallelogram.
 (d) Find the area of $OABC$ and the length of the diagonal AB .

13. (a) The line ℓ has intercepts at $L(-4,0)$ and $M(0,3)$. N is a point on ℓ and P has coordinates $(0,8)$.
 (b) Find the equation of ℓ .
 (c) Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
 (d) If M is the midpoint of LN , find the coordinates of N .
 (e) Show that $\angle NPL = 90^\circ$.
 (f) Write down the equation of the circle through N , P and L .



14. The vertices of the triangle are $P(-1,0)$ and $Q(1,4)$ and R , where R lies on the x -axis and $\angle QPR = \angle QRP = \theta$.
 (a) Find the coordinates of the midpoint of PQ .
 (b) Find the gradient of PQ and show that $\tan \theta = 2$.
 (c) Show that PQ has equation $y = 2x + 2$.
 (d) Explain why QR has gradient -2 , and hence find its equation.
 (e) Find the coordinates of R and hence the area of triangle PQR .
 (f) Find the length QR , and hence find the perpendicular distance from P to QR .



15. Find k if the lines $\ell_1: x + 3y + 13 = 0$, $\ell_2: 4x + y - 3 = 0$ and $\ell_3: kx - y - 10 = 0$ are concurrent. [HINT: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3 .]

16. Consider the two lines $\ell_1: 3x + 2y + 4 = 0$ and $\ell_2: 6x + \mu y + \lambda = 0$.

- Write down the value of μ if: (i) ℓ_1 is parallel to ℓ_2 , (ii) ℓ_1 is perpendicular to ℓ_2 .
- Given that ℓ_1 and ℓ_2 intersect at a point, what condition must be placed on μ ?
- Given that ℓ_1 is parallel to ℓ_2 , write down the value of λ if: (i) ℓ_1 is the same line as ℓ_2 , (ii) the distance between the y -intercepts of the two lines is 2.

17. (a) Write down, in general form, the equation of a line parallel to $2x - 3y + 1 = 0$.
 (b) Hence find the equation of the line if it passes through: (i) $(2, 2)$ (ii) $(3, -1)$
18. (a) Write down, in general form, the equation of a line perpendicular to $3x + 4y - 3 = 0$.
 (b) Hence find the equation of the line if it passes through: (i) $(-1, -4)$ (ii) $(-2, 1)$
19. (a) What is the natural domain of the relation $\frac{y}{x-1} = 3$? (b) Graph this relation and indicate how it differs from the graph of the straight line $y = 3x - 3$.
20. Explain how $x + y = 1$ may be transformed into $\frac{x}{a} + \frac{y}{b} = 1$ by stretching.
21. Find the point of intersection of $px + qy = 1$ and $qx + py = 1$, and explain why these lines intersect on the line $y = x$.
22. Determine the equation of the line through $M(4, 3)$ if M is the midpoint of the intercepts on the x -axis and y -axis. [HINT: This time, let the gradient of the line be m , and begin by writing down the equation of the line in point-gradient form.]
23. Write down the equation of the line through $M(-1, 2)$ with gradient m . Hence determine the equation of the line through M if:
 (a) M is the midpoint of the intercepts on the x -axis and y -axis,
 (b) M divides the intercepts in the ratio $2 : 1$ (x -intercept to y -intercept),
 (c) M divides the intercepts in the ratio $-2 : 5$ (x -intercept to y -intercept).
24. Two distinct linear functions $f(x)$ and $g(x)$ have zeroes at $x = a$ and have gradients ℓ and m respectively. Show that $f(x) : g(x) = \ell : m$, for $x \neq a$.

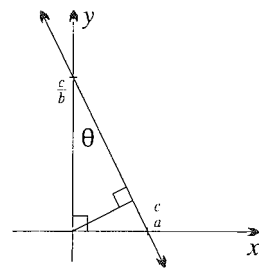
EXTENSION

25. The line passing through $M(a, b)$ intersects the x -axis at A and the y -axis at B . Find the equation of the line if: (a) M bisects AB , (b) M divides AB in the ratio $2 : 1$, (c) M divides AB in the ratio $k : \ell$.
26. The tangent to a circle is perpendicular to the radius at the point of contact. Use this fact to show that the tangent to $x^2 + y^2 = r^2$ at the point (a, b) has equation $ax + by = r^2$.
27. Show that the parametric equations $x = t \cos \alpha + a$ and $y = t \sin \alpha + b$ represent a straight line through (a, b) with gradient $m = \tan \alpha$.
28. [Perpendicular form of a line] Consider the line ℓ with equation $ax + by = c$ where, for the sake of convenience, the values of a , b and c are positive. Suppose this line makes an acute angle θ with the y -axis as shown.

- (a) Show that $\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}$.
 (b) The *perpendicular form* of the line ℓ is

$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{c}{\sqrt{a^2 + b^2}}.$$

Use part (a) to help show that the RHS of this equation is the perpendicular distance from the line to the origin.



- (c) Write these lines in perpendicular form and hence find their perpendicular distances from the origin: (i) $3x + 4y = 5$ (ii) $3x - 2y = 1$

5E Perpendicular Distance

If P is a point and ℓ is a line not passing through P , then the shortest distance from P to ℓ is the perpendicular distance. It is useful to have a formula for this perpendicular distance, rather than having to find the equation of the perpendicular line and the coordinates of the closest point. This formula, and the method developed in the next section, both make use of the general form of the equation of a line.

A Formula for the Distance from the Origin: The first step is to develop a formula for the perpendicular distance p of a given line $\ell: ax + by + c = 0$ from the origin. This can be done very quickly by looking at the triangle OAB formed by the line ℓ and the two axes. We find two different expressions for the area of this triangle OAB and equate them.

The line $ax + by + c = 0$ has x -intercept $-c/a$ and y -intercept $-c/b$, so OA has length $\left|\frac{c}{a}\right|$, and OB has length $\left|\frac{c}{b}\right|$.

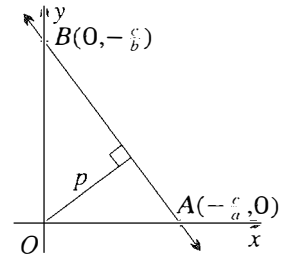
Using OA as the base, area of $\triangle OAB = \frac{1}{2} \times \left|\frac{c}{a}\right| \times \left|\frac{c}{b}\right|$
 $= \frac{1}{2} \left|\frac{c^2}{ab}\right|$.

The length of the hypotenuse AB is $AB^2 = \frac{c^2}{a^2} + \frac{c^2}{b^2}$
 $= \frac{c^2(a^2 + b^2)}{a^2b^2}$,

$$AB = \left|\frac{c}{ab}\right| \sqrt{a^2 + b^2},$$

so, using AB as the base, area of $\triangle OAB = \frac{1}{2}p \left|\frac{c}{ab}\right| \sqrt{a^2 + b^2}$.

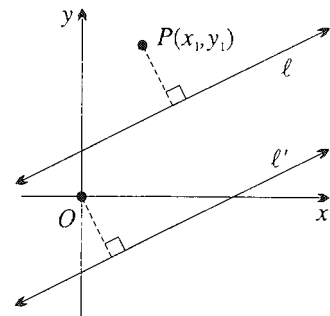
Equating the two expressions for area, $p = \frac{|c|}{\sqrt{a^2 + b^2}}$.



The Perpendicular Distance Formula: Now we can use the shifting procedures of Chapter Two to generalise the result above to generate a formula for the perpendicular distance p of the line $\ell: ax + by + c = 0$ from any given point $P(x_1, y_1)$. The perpendicular distance remains the same if we shift both line and point x_1 units to the left and y_1 units down. This shift moves the point P to the origin, and moves the line ℓ to the new line ℓ' with equation

$$\begin{aligned} a(x + x_1) + b(y + y_1) + c &= 0 \\ ax + by + (ax_1 + by_1 + c) &= 0. \end{aligned}$$

Then, using the formula previously established for the distance from the origin, we obtain:



18 PERPENDICULAR DISTANCE FORMULA: $p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

WORKED EXERCISE: Find the perpendicular distance of $P(-2, 5)$ from $y = 2x - 1$.

SOLUTION: The line in general form is $2x - y - 1 = 0$,

$$\begin{aligned}\text{so distance} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|2 \times (-2) - 1 \times 5 - 1|}{\sqrt{2^2 + (-1)^2}} \\ &= \frac{|-10|}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{10\sqrt{5}}{5} \\ &= 2\sqrt{5}.\end{aligned}$$

Circles and the Perpendicular Distance Formula: A line is a tangent to a circle when its perpendicular distance from the centre is equal to the radius. Lines closer to the centre are secants, and lines more distant miss the circle entirely.

WORKED EXERCISE: Solve these using the perpendicular distance formula.

(a) Show that $\ell: 3x + 4y - 20 = 0$ is a tangent to the circle $C: (x-7)^2 + (y-6)^2 = 25$.

(b) Find the length of the chord of C cut off by the line $m: 3x + 4y - 60 = 0$.

SOLUTION: The circle has centre $(7, 6)$ and radius 5.

(a) The distance p_ℓ from the line ℓ to the centre is

$$\begin{aligned}p_\ell &= \frac{|21 + 24 - 20|}{\sqrt{3^2 + 4^2}} \\ &= \frac{25}{5} \\ &= 5,\end{aligned}$$

so ℓ is a tangent to the circle.

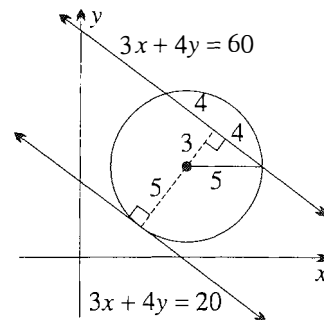
(b) The distance p_m from the line m to the centre is

$$\begin{aligned}p_m &= \frac{|21 + 24 - 60|}{\sqrt{3^2 + 4^2}} \\ &= \frac{|-15|}{5} \\ &= 3.\end{aligned}$$

Using Pythagoras' theorem in the circle on the right,

chord length $= 2 \times 4$

$= 8$ units.



WORKED EXERCISE: For what values of k will the line $5x - 12y + k = 0$ never intersect the circle with centre $P(-3, 1)$ and radius 6?

SOLUTION: The condition is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} > 6$

$$\frac{|-15 - 12 + k|}{\sqrt{5^2 + 12^2}} > 6$$

$$\frac{|-27 + k|}{13} > 6$$

$$|k - 27| > 78$$

$$k < -51 \text{ or } k > 105.$$

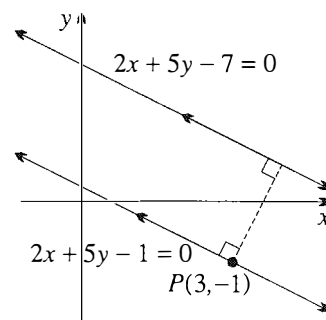
Distance between Parallel Lines: The distance between two parallel lines can be found by choosing any point on one line and finding its perpendicular distance from the second line.

WORKED EXERCISE: Find the perpendicular distance between the two parallel lines $2x + 5y - 1 = 0$ and $2x + 5y - 7 = 0$.

SOLUTION: Choose any point on the first line, say $P(3, -1)$.

The distance between the lines is the perpendicular distance from P to the second line:

$$\begin{aligned} \text{distance} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|6 - 5 - 7|}{\sqrt{2^2 + 5^2}} \\ &= \frac{|-6|}{\sqrt{4 + 25}} \\ &= \frac{6}{\sqrt{29}}. \end{aligned}$$



Exercise 5E

- Find the perpendicular distance from each line to the origin:
 - $x + 3y + 5 = 0$
 - $2x - y + 4 = 0$
 - $2x + 4y - 5 = 0$
- Find the perpendicular distance between each point and line:
 - $(2, 0)$ and $3x + 4y - 1 = 0$
 - $(-3, -2)$ and $x + 3y + 4 = 0$
 - $(-2, 1)$ and $12x - 5y + 3 = 0$
 - $(3, -1)$ and $x + 2y - 1 = 0$
 - $(-3, 2)$ and $4x - y - 3 = 0$
 - $(1, 3)$ and $2x + 4y + 1 = 0$
- Which of the given points is: (a) closest to, (b) furthest from, the line $6x - 8y - 9 = 0$?

$A(1, -1)$	$B(3, 2)$	$C(-4, 1)$	$D(-3, -3)$
- Which of the given lines is: (a) closest to, (b) furthest from, the point $(-1, 5)$?

$\ell_1: 2x + 3y + 4 = 0$	$\ell_2: x - 4y + 7 = 0$	$\ell_3: 3x + y - 8 = 0$

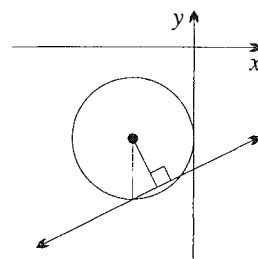
DEVELOPMENT

- The line $y - 2x + \mu = 0$ is $2\sqrt{5}$ units from the point $(1, -3)$. Find the possible values of μ .
 - The line $3x - 4y + 2 = 0$ is $\frac{3}{5}$ units from the point $(-1, \lambda)$. Find the possible values of λ .
- The line $y - x + h = 0$ is more than $\frac{1}{\sqrt{2}}$ units from the point $(2, 7)$. What range of values may h take?
 - The line $x + 2y - 5 = 0$ is at most $\sqrt{5}$ units from the point $(k, 3)$. What range of values may k take?
- Use the perpendicular distance formula to determine how many times each line intersects the given circle:

(a) $3x - 5y + 16 = 0, x^2 + y^2 = 5$	(c) $3x - y - 8 = 0, (x - 1)^2 + (y - 5)^2 = 10$
(b) $7x + y - 10 = 0, x^2 + y^2 = 2$	(d) $x + 2y + 3 = 0, (x + 2)^2 + (y - 1)^2 = 6$
- Use a point on the first line to find the distance between each pair of parallel lines:

(a) $x - 3y + 5 = 0, x - 3y - 2 = 0$	(b) $4x + y - 2 = 0, 4x + y + 8 = 0$

9. The vertices of a triangle are $A(-3, -2)$, $B(3, 1)$ and $C(-1, 4)$.
- Find the equation of the side AB in general form. (b) How far is C from this line?
 - Find the length of AB and hence find the area of this triangle.
 - Similarly find the area of the triangle with vertices $P(1, -1)$, $Q(-1, 5)$ and $R(-3, 1)$.
10. Draw on a number plane the triangle ABC with vertices $A(5, 0)$, $B(8, 4)$ and $C(0, 10)$.
- Show that the line AB has equation $3y = 4x - 20$.
 - Show that the gradient of BC is $-\frac{3}{4}$.
 - Hence show that AB and BC are perpendicular.
 - Show that AB is 5 units.
 - Show that triangles AOC and ABC are congruent.
 - Find the area of quadrilateral $OABC$.
 - Find the distance from the point $D(8, 0)$ to the line AB .
11. (a) Write down the centre and radius of the circle with equation $(x+2)^2 + (y+3)^2 = 4$. Then find the distance from the line $2y - x + 8 = 0$ to the centre.
- (b) Hence determine the length of the chord cut off from the line by the circle.
12. Choose two points in the first quadrant on the line $3x - 5y + 4 = 0$ and find their distances from the line $4x - 5y - 3 = 0$. What can be concluded about the two lines?
13. The point $P(x, y)$ is equidistant from the lines $2x + y - 3 = 0$ and $x - 2y + 1 = 0$, which intersect at A .
- Use the distance formula to show that $|2x + y - 3| = |x - 2y + 1|$.
 - Hence find the equations of the lines that bisect the angles at A .
14. (a) Write down the equation of a line through the origin with gradient m .
- (b) Write down the distance from this line to the point $(3, 1)$.
- (c) If the line is tangent to the circle $(x - 3)^2 + (y - 1)^2 = 4$, show that m satisfies the equation $5m^2 - 6m - 3 = 0$.
- (d) Find the possible values of m and hence the equations of the two tangents.



EXTENSION

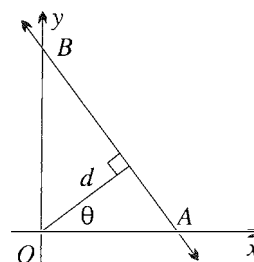
15. Use the perpendicular distance formula to prove that the distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.
16. (a) Find the vertex of $y = x^2 - 2x + 3$, and show that the minimum value of y is 2.
- (b) The point $Q(q, q^2)$ lies on the parabola $\mathcal{P}: y = x^2$. Write down the distance from Q to the line $\ell: 2x - y - 3 = 0$. (c) Hence find the minimum distance from the line ℓ to the parabola $\mathcal{P}$.
17. (a) If the centre of the circle $(x - 4)^2 + (y - 1)^2 = 25$ is moved 3 right and 2 down, what is the equation of the new circle? (b) Write down the distance from the centre of the second circle to the line $y = mx$. (c) Find the values of m if $y = mx$ is tangent to this circle. (d) Hence find the equations of the two tangents from the point $(-3, 2)$ to the circle $(x - 4)^2 + (y - 1)^2 = 25$.

18. [Perpendicular distance — a derivation by trigonometry] Consider the straight line $ax + by = c$ where, for the sake of convenience, the values a , b and c are positive. The perpendicular from the origin to the line has length d and makes an angle θ with the positive x -axis.

(a) Show that $\angle OBA = \theta$.

(b) Show that $\cos \theta = \frac{ad}{c}$ and $\sin \theta = \frac{bd}{c}$.

(c) Use the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$ in order to show that $d = \frac{c}{\sqrt{a^2 + b^2}}$.



19. [Perpendicular distance — a quadratic derivation] Consider the circle $x^2 + y^2 = d^2$ and the line $px + qy + r = 0$.

(a) Show that the coordinates of any point of intersection of the line and the circle must satisfy the quadratic equation $(p^2 + q^2)x^2 + 2prx + r^2 - q^2d^2 = 0$.

(b) If the line is tangent to the circle, then this equation has only one solution and in the quadratic formula $b^2 - 4ac = 0$. Use this result to find the distance to the origin.

5F Lines Through the Intersection of Two Given Lines

This section develops an ingenious way of finding the equations of particular lines through the intersection of two given lines without actually solving the two given lines simultaneously to find their point of intersection. The method is another situation where the general form of the equation of the line is used.

The General Form of Such a Line: When two lines intersect, the set of all the lines through the point M of intersection forms a family of lines through M . The first task is to write the equations of all the lines in the family in the one form.

Suppose that $\ell_1: a_1x + b_1y + c_1 = 0$ and $\ell_2: a_2x + b_2y + c_2 = 0$ are two lines intersecting at a point $M(x_0, y_0)$. Let ℓ be any line of the form

$$\ell: (a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0, \quad (*)$$

where k is a constant. We are going to prove that ℓ also passes through M .

First, ℓ_1 passes through M , and so, substituting the coordinates of M into the equation of ℓ_1 ,

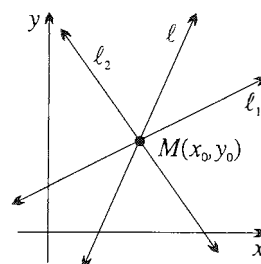
$$a_1x_0 + b_1y_0 + c_1 = 0. \quad (1)$$

Similarly, ℓ_2 passes through M , and so

$$a_2x_0 + b_2y_0 + c_2 = 0. \quad (2)$$

To prove that ℓ passes through M , substitute $M(x_0, y_0)$ into $(*)$:

$$\begin{aligned} \text{LHS} &= (a_1x_0 + b_1y_0 + c_1) + k(a_2x_0 + b_2y_0 + c_2) \\ &= 0 + 0, \text{ by (1) and (2)} \\ &= \text{RHS,} \end{aligned}$$



as required. Hence the equation of every line through the intersection of ℓ_1 and ℓ_2 has the following form:

19

LINE THROUGH THE INTERSECTION OF TWO GIVEN LINES:

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0, \text{ where } k \text{ is a constant.}$$

NOTE: Careful readers will realise that there is one line through M which is not included in this standard form, namely the second line ℓ_2 . When this case occurs, it can be recognised because k becomes infinite. To overcome this problem, it is not good enough to put the k in front of the first expression instead, because then the line ℓ_1 would not be included. The way through is a little more complicated; one must use a *homogeneous form* such as

$$h(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0,$$

in which every line through M corresponds to a unique ratio of h and k or, when the line is ℓ_1 or ℓ_2 , to one of k and h becoming zero.

This same situation arose with gradient. When we replaced the two parameters rise and run by their ratio, a single parameter, the case of vertical gradient was excluded, because when the run is zero the ratio is undefined. There too it would be more correct, though also more complicated, to define gradient as the ordered pair (rise, run), with the qualification that two ordered pairs like (3, 1) and (6, 2) in which one pair is a multiple of another represent the same gradient. This is the easiest way to generalise gradient to three or more dimensions.

WORKED EXERCISE: Write down the equation of a line through the intersection M of the lines

$$\ell_1: x + 2y - 6 = 0 \quad \text{and} \quad \ell_2: 3x - 2y - 6 = 0.$$

Hence, without finding the point of intersection, find the line through M that:

- (a) passes through $P(2, -1)$, (c) is horizontal,
 (b) has gradient 5, (d) is vertical.

SOLUTION: The general form of a line through M is

$$(x + 2y - 6) + k(3x - 2y - 6) = 0, \text{ for some constant } k. \quad (1)$$

$$\begin{aligned} \text{(a) Substituting } P(2, -1) \text{ into (1),} \quad (2 - 2 - 6) + k(6 + 2 - 6) &= 0 \\ 2k &= 6 \\ k &= 3, \end{aligned}$$

and substituting into (1), the required line is

$$\begin{aligned} (x + 2y - 6) + 3(3x - 2y - 6) &= 0 \\ 10x - 4y - 24 &= 0 \\ 5x - 2y - 12 &= 0. \end{aligned}$$

$$\text{(b) Rearranging (1) gives } (1 + 3k)x + (2 - 2k)y + (-6 - 6k) = 0, \quad (2)$$

$$\begin{aligned} \text{which has gradient } \frac{1 + 3k}{2k - 2}. \text{ Hence} \quad \frac{1 + 3k}{2k - 2} &= 5 \\ 1 + 3k &= 10k - 10 \\ k &= \frac{11}{7}, \end{aligned}$$

so substituting into (1), the required line is

$$\begin{aligned}(x + 2y - 6) + \frac{11}{7}(3x - 2y - 6) &= 0 \\ 7(x + 2y - 6) + 11(3x - 2y - 6) &= 0 \\ 40x - 8y - 108 &= 0 \\ 10x - 2y - 27 &= 0.\end{aligned}$$

- (c) The gradient is zero, so the coefficient of x in (2) is zero, $1 + 3k = 0$
 $k = -\frac{1}{3}.$

Substituting into (1), the required line is

$$\begin{aligned}(x + 2y - 6) - \frac{1}{3}(3x - 2y - 6) &= 0 \\ 3(x + 2y - 6) - (3x - 2y - 6) &= 0 \\ 8y - 12 &= 0 \\ y &= 1\frac{1}{2}.\end{aligned}$$

- (d) The line is vertical, so the coefficient of y in (2) is zero, $2 - 2k = 0$
 $k = 1.$

Substituting into (1), the required line is

$$\begin{aligned}(x + 2y - 6) + (3x - 2y - 6) &= 0 \\ 4x - 12 &= 0 \\ x &= 3.\end{aligned}$$

NOTE: Parts (c) and (d) together actually do tell us that the two given lines intersect at $(3, 1\frac{1}{2})$.

Exercise 5F

- (a) Graph the lines $x - y = 0$ and $x + y - 2 = 0$ on grid or graph paper and label them (they intersect at $(1, 1)$).
- (b) Simplify the equation $(x - y) + k(x + y - 2) = 0$ for $k = 2, 1, \frac{1}{2}$ and 0 . Add these lines to your graph, and label each line with its value of k . Observe that each line passes through $(1, 1)$.
- (c) Repeat this process for $k = -\frac{1}{2}, -1$ and -2 , adding these lines to your graph.
- The lines $x + 2y + 9 = 0$ and $2x - y + 3 = 0$ intersect at B .
 - Write down the general equation of a line through B .
 - Hence find the equation of the line ℓ through B and the origin O .
- Find the equation of the line through the intersection of the lines $x - y - 3 = 0$ and $y + 3x - 5 = 0$ and the given point, without finding the point of intersection of the lines:
 - $(0, -2)$
 - $(-1, 5)$
 - $(3, 0)$
- The lines $2x + y - 5 = 0$ and $x - y + 2 = 0$ intersect at A .
 - Write down the general equation of a line through A , and show that it can be written in the form $x(2 + k) + y(1 - k) + (2k - 5) = 0$.
 - Find the value of k that makes the coefficient of x zero, and hence find the equation of the horizontal line through A .
 - Find the value of k that makes the coefficient of y zero, and hence find the equation of the vertical line through A .
 - Hence write down the coordinates of A .

5. (a) Find the point P of intersection of $x + y - 2 = 0$ and $2x - y - 1 = 0$.
 (b) Show that P satisfies the equation $x + y - 2 + k(2x - y - 1) = 0$.
 (c) Find the equation of the line through P and $Q(-2, 2)$:
 (i) using the coordinates of both P and Q ,
 (ii) without using the coordinates of P . Your answers should be the same.

DEVELOPMENT

6. (a) Write down the general form of a line through the point T of intersection of the two lines $2x - 3y + 6 = 0$ and $x + 3y - 15 = 0$.
 (b) Hence find the equation of the line through T and:
 (i) $(3, 8)$ (ii) $(6, 0)$ (iii) $(-3, 3)$ (iv) $(0, 0)$
7. (a) The general form of a line through the intersection M of $x - 2y + 5 = 0$ and $x + y + 2 = 0$ is $\ell: (x - 2y + 5) + k(x + y + 2) = 0$. Show that the gradient of ℓ is $\frac{1+k}{2-k}$.
 (b) Hence find the equation of the lines through M :
 (i) parallel to $3x + 4y = 5$, (iii) perpendicular to $5y - 2x = 4$,
 (ii) perpendicular to $2x - 3y = 6$, (iv) parallel to $x - y - 7 = 0$.
8. (a) Show that every line of the family $(2x - y - 7) + k(x + y - 5) = 0$ passes through a fixed point by finding the coordinates of that point. [HINT: Use the method of question 4 to find the point.]
 (b) Similarly, show that the lines of each family pass through a fixed point:
 (i) $(x + 2y - 8) + k(x - y + 4) = 0$ (ii) $(3x + y + 2) + k(5x + 2y + 1) = 0$
9. Show that the three lines $\ell_1: 2x - 3y + 13 = 0$, $\ell_2: x + y - 1 = 0$ and $\ell_3: 4x + 3y - 1 = 0$ are concurrent by the following method:
 (a) Without finding any points of intersection, find the equation of the line through the intersection of ℓ_1 and ℓ_2 parallel to ℓ_3 . (b) Show that this line is the same line as ℓ_3 .
10. (a) Use the perpendicular distance formula to show that $A(-2, 3)$ is equidistant from the two lines $x - 3y + 1 = 0$ and $3x + y - 7 = 0$.
 (b) Hence find the equation of the line through A that bisects the angle between the two lines, without finding their point of intersection.
11. (a) It is known that $\ell: x + 2y + 10 = 0$ is tangent to the circle $\mathcal{C}: x^2 + y^2 = 20$ at T . Write down the equation of the radius of the circle at T (it will be perpendicular to ℓ).
 (b) Use part (a) to find the equation of the line through $S(1, -3)$ and the point of contact without actually finding the point of contact.
12. The lines $3x - y + 2 = 0$ and $x - 4y - 3 = 0$ form two sides PQ and QR of parallelogram $PQRS$. Given that S has coordinates $(4, 3)$, find the equations of the following lines without finding the coordinates of any other point: (a) RS (b) PS (c) QS (d) PR . [HINT: PR is the line through the intersection of PQ and PS that is parallel to the line through the intersection of QR and RS and has the same y -intercept.]

EXTENSION

13. (a) Show that every circle that passes through the intersections of the circle $x^2 + y^2 = 2$ and the straight line $y = x$ can be written in the form $(x - \mu)^2 + (y + \mu)^2 = 2(1 + \mu^2)$.
 (b) Hence find the equation of such a circle which also passes through $(4, -1)$.

14. [A form of the circle of Apollonius] Consider the expressions $C_1 = (x - 2)^2 + y^2 - 5$ and $C_2 = (x + 2)^2 + y^2 - 5$. (a) Show that the equations $C_1 = 0$ and $C_2 = 0$ represent circles of radius $\sqrt{5}$ which intersect at 1 and -1 on the y -axis. (b) Show that the equation $C_1 + \lambda C_2 = 0$ can be rewritten as $x^2 + 4x \frac{\lambda - 1}{\lambda + 1} + y^2 = 1$, provided $\lambda \neq -1$. (c) Draw graphs of the circles $C_1 + \lambda C_2 = 0$ for $\lambda = 0, \frac{1}{3}, 1, 3$ and note the y -intercepts. (d) Investigate the equation $C_1 + \lambda C_2 = 0$ for negative values of λ . [HINT: Begin with the values $-\frac{1}{3}, -1, -3$.] (e) Find the equation of the circle through $(0, 1)$, $(0, -1)$ and $(5, 1)$.
15. (a) Show that the parabola through the intersection of the parabolas $y = x^2 - 1$ and $y = 1 - x^2$ can be written in the form $y = \frac{1 - k}{1 + k}(x^2 - 1)$, provided $k \neq -1$. (b) Hence find the parabola that also passes through $(2, 12)$. (c) Show that there is no parabola that passes through the intersection of the parabolas $y = 2x - x^2$ and $y = x^2 - 4x$, and the point $(1, -1)$.

5G Coordinate Methods in Geometry

When the French mathematician and philosopher René Descartes introduced coordinate geometry in the 17th century, he intended it to be a system in which all the theorems of Euclidean geometry could be proven in an alternative way. Here are two well-known theorems about centres of triangles proven by this alternative coordinate method. In the first proof, it is convenient to place the figure carefully with important points on the axes or at the origin so that the algebra is simplified. In the second proof, however, using general coordinates for all three vertices displays the symmetry of the situation. Many questions use the words *altitude* and *median*, which should be known.

20

MEDIAN: A *median* of a triangle is the interval from a vertex of the triangle to the midpoint of the opposite side.

ALTITUDE: An *altitude* of a triangle is the perpendicular from a vertex of the triangle to the opposite side (produced if necessary).

Example — The Three Altitudes of a Triangle are Concurrent: This theorem asserts the concurrence of the three altitudes of a triangle, and is most easily proven by placing one side on the x -axis and the opposite vertex on the y -axis.

THEOREM: The three altitudes of a triangle are concurrent. (Their point of intersection is called the *orthocentre* of the triangle.)

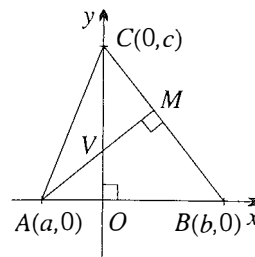
PROOF: Let the side AB of the triangle ABC lie on the x -axis, and the vertex C lie on the positive side of the y -axis. Let the coordinates of the vertices be $A(a, 0)$, $B(b, 0)$ and $C(0, c)$, as in the diagram.

The altitude through C is the interval CO on the y -axis.

Let M be the foot of the altitude through A .

Since BC has gradient $-c/b$, AM has gradient b/c ,

so the equation of AM is $y - 0 = \frac{b}{c}(x - a)$

$$y = \frac{bx}{c} - \frac{ab}{c}$$


and substituting $x = 0$, the altitude AM meets CO at $V(0, -ab/c)$.

Exchanging a and b , the third altitude BN through B similarly has equation

$$y = \frac{ax}{c} - \frac{ab}{c}$$

and again substituting $x = 0$, BN meets CO at the same point $V(0, -ab/c)$.

Hence the three altitudes are concurrent.

Example — The Three Medians of a Triangle are Concurrent: The three medians of any triangle are also concurrent. In contrast to the previous proof, the concurrence of the medians is most clearly proven by allowing the vertices to have general coordinates.

THEOREM: The medians of a triangle are concurrent, and their point of intersection (called the *centroid* of the triangle) divides each median in the ratio 2 : 1.

PROOF: Let the triangle $A_1A_2A_3$ have as coordinates $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ and $A_3(x_3, y_3)$. Let M_1 , M_2 and M_3 be the midpoints of A_2A_3 , A_3A_1 and A_1A_2 respectively.

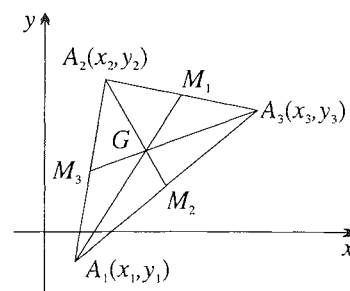
Let G be the point dividing A_1M_1 in the ratio 2 : 1.

By the midpoint formula, the coordinates of M_1 are

$$M_1 = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right),$$

and so, using the ratio division formula,

$$\begin{aligned} G &= \left(\frac{1 \times x_1 + 2 \times \frac{1}{2}(x_2 + x_3)}{1 + 2}, \frac{1 \times y_1 + 2 \times \frac{1}{2}(y_2 + y_3)}{1 + 2} \right) \\ &= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right). \end{aligned}$$



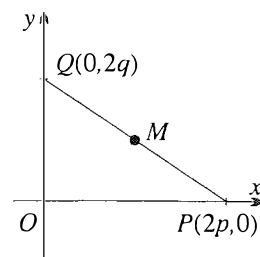
Because this result is symmetric, G must divide A_2M_2 and A_3M_3 in the ratio 2 : 1.

So each median passes through G , which divides it in the ratio 2 : 1.

Exercise 5G

NOTE: Diagrams should be drawn wherever possible.

- (a) The points O , $P(8,0)$ and $Q(0,10)$ form a right-angled triangle, and M is the midpoint of PQ . (i) Find the coordinates of M . (ii) Then find the distance OM , PM and QM , and show that M is equidistant from each of the vertices. (iii) Explain why a circle with centre M can be drawn through the three vertices O , P and Q .

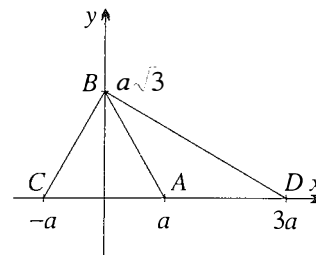


- (b) It is true in general that the midpoint of the hypotenuse of a right triangle is the centre of a circle through all three vertices. Prove that this result is true for any right triangle by placing its vertices at $O(0,0)$, $P(2p,0)$ and $Q(0,2q)$, and repeating the procedures of part (a).
- (a) $PQRS$ is a quadrilateral with vertices on the axes at $P(1,0)$, $Q(0,2)$, $R(-3,0)$ and $S(0,-4)$. Show that $PQ^2 + RS^2 = PS^2 + QR^2$.

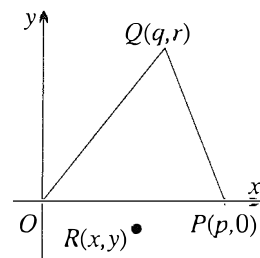
- (b) It is true in general that if the diagonals of a quadrilateral are perpendicular, then the two sums of squares of opposite sides are equal. Prove that this result is true for any quadrilateral by placing the vertices on the axes, giving them coordinates $P(p, 0)$, $Q(0, q)$, $R(-r, 0)$ and $S(0, -s)$, and proceeding as in part (a).
3. (a) A triangle has vertices at $A(1, -3)$, $B(3, 3)$ and $C(-3, 1)$.
 (i) Find the coordinates of P and Q , the midpoints of AB and BC respectively.
 (ii) Show that PQ is parallel to AC and that $PQ = \frac{1}{2}AC$.
- (b) It is true in general that the line joining the midpoints of two sides of a triangle is parallel to the base and half its length. Prove that this is true for any triangle by placing its vertices at $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$, and proceeding as in part (a).
4. Triangle OPA has vertices at the origin O , $A(3, 0)$ and $B(0, 4)$. C is a point on AB such that OC is perpendicular to AB .
 (a) Find the equations of AB and OC and hence find the coordinates of C .
 (b) Find the lengths OA , AB , OC , BC and AC .
 (c) Thus confirm these important corollaries for a right-angled triangle:
 (i) $OC^2 = AC \times BC$ (ii) $OA^2 = AC \times AB$
5. $A(a, 0)$ and $Q(q, 0)$ are points on the positive x -axis, and $B(0, b)$ and $P(0, p)$ are points on the positive y -axis. Show that $AB^2 - AP^2 = QB^2 - QP^2$.

DEVELOPMENT

6. The diagram opposite shows the points A , B , C and D on the number plane.

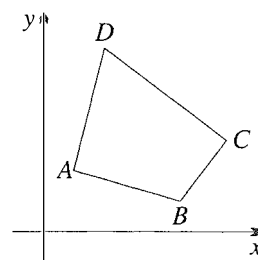


- (a) Show that $\triangle ABC$ is equilateral.
 (b) Show that $\triangle ABD$ is isosceles, with $AB = AD$.
 (c) Show that $AB^2 = \frac{1}{3}BD^2$.
7. (a) Suppose D is the midpoint of AC in triangle ABC . Prove that $AB^2 + BC^2 = 2(CD^2 + BD^2)$. Begin by placing the vertices at $A(a, 0)$, $B(b, c)$ and $C(-a, 0)$. Then find the coordinates of D , then find the squares AB^2 , BC^2 , BD^2 and CD^2 .
 (b) The result proven in part (a) is *Apollonius' theorem*. Give a geometric statement of the result (use the word *median* for the interval joining any vertex to the midpoint of the opposite side).
8. Prove that the diagonals of a parallelogram bisect each other. Begin by showing that the quadrilateral with vertices $W(a, 0)$, $X(b, c)$, $Y(-a, 0)$ and $Z(-b, -c)$, where the constants a , b and c are all positive, is a parallelogram. Then show that the midpoints of both diagonals coincide.
9. [A condition for a point to lie on an altitude of a triangle] In the diagram, $\triangle OPQ$ has its vertices at the origin, $P(p, 0)$ and $Q(q, r)$. Another point $R(x, y)$ satisfies the condition $PQ^2 - OQ^2 = PR^2 - OR^2$. Substitute the coordinates of O , P , Q and R into this condition, and show that R lies on the altitude through Q .



10. The points $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ and $D(d_1, d_2)$ are the vertices of a quadrilateral such as the one in the diagram.

- Find the midpoints P , Q , R and S of AB , BC , CD and DA respectively.
- Show that the diagonals of $PQRS$ bisect each other, by showing that the midpoints of PR and QS coincide.
- What is the most general type of quadrilateral $PQRS$ can be?



11. The previous question proves the general result that the midpoints of any quadrilateral form a parallelogram. Prove this result in an alternative way by finding the gradients of the four sides of $PQRS$ and showing that opposite sides are parallel.
12. The points $A(1, -2)$, $B(5, 6)$ and $C(-3, 2)$ are the vertices of a triangle, and P , Q and R are the midpoints of BC , AC and AB respectively.
- Find the equations of the three medians BQ , CR and AP .
 - Find the intersection of BQ and CR , and show that it lies on the third median AP .
13. [The three medians of a triangle are concurrent — an alternative proof] The previous question was a special case of the general result that the medians of a triangle are concurrent (their point of intersection is called the *centroid*). Prove that the result is true for any triangle by choosing as vertices $A(6a, 6b)$, $B(-6a, -6b)$ and $C(0, 6c)$, and following these steps:
- Find the midpoints P , Q and R of BC , CA and AB respectively. Show that the median through C is $x = 0$, and find the equations of the other two medians.
 - Find where the median through C meets another median, and show that the point lies on the third median.
14. It is true in general that the perpendicular bisectors of the sides of a triangle are concurrent, and that the point of intersection (called the *circumcentre*) is the centre of a circle through all three vertices (called the *circumcircle*). Prove this result in general by placing the vertices at $A(2a, 0)$, $B(-2a, 0)$ and $C(2b, 2c)$, and proceeding as follows:
- Find the gradients of AB , BC and CA , and hence find the equations of the three perpendicular bisectors.
 - Find the intersection M of any two perpendicular bisectors, and show that it lies on the third.
 - Explain why the circumcentre must be equidistant from each vertex.

EXTENSION

15. Triangle ABC is right-angled at A . P is the midpoint of AB and Q is the midpoint of BC . Choose suitable coordinates in order to prove that $BQ^2 - PC^2 = 3(PB^2 - QC^2)$.
16. Prove the corollaries in question 4 for a general right-angled triangle using the vertices $A(a, 0)$, $B(0, b)$ and the origin.
17. The points $P(cp, c/p)$, $Q(cq, c/q)$, $R(cr, c/r)$ and $S(cs, c/s)$ lie on the curve $xy = c^2$.
- If $PQ \parallel RS$, show that $pq = rs$.
 - Show that $PQ \perp RS$ if and only if $pqrs = -1$.
 - Conclude from part (b) that if a triangle is drawn with its vertices on a rectangular hyperbola, then the orthocentre (intersection of the altitudes) will lie on the hyperbola.

CHAPTER SIX

Sequences and Series

Many situations give rise to a sequence of numbers with a simple pattern. For example, the weight of a tray carrying a stack of plates increases steadily as each new plate is added. When cells continually divide into two, then the numbers in successive generations descending from a single cell form the sequence 1, 2, 4, 8, ... of powers of 2. Someone thinking about the half-life of a radioactive substance will need to ask what happens when we add up more and more terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

The highly structured world of mathematics is full of sequences, and, in particular, knowledge of them will be needed to establish important results in calculus in the next chapter.

STUDY NOTES: Sections 6A and 6B are a review of the algebraic work of indices and logarithms at a more demanding level, and could be studied independently. In Sections 6C–6L on sequences and series, computers and graphics calculators could perhaps help to represent examples in alternative graphical forms, to emphasise the linear and exponential functions that lie behind arithmetic and geometric sequences, and to give some interactive experience of the limits of sequences. Section 6M generalises the difference-of-squares and difference-of-cubes identities in preparation for their use in calculus. The final Section 6N on mathematical induction could also be studied at some other time — it is included in this chapter because it involves recursion like APs and GPs. Applications of series, particularly to financial situations, will be covered in the Year 12 text.

6A Indices

We begin with a review of the index laws, which will be needed in calculations later in the chapter. The accompanying exercise is intended to cover a wide variety of arithmetic and algebraic manipulations.

Definition of Indices: An expression of the form a^x is called a *power*. The number a is called the *base* of the power, and the number x is called the *index* (plural *indices*) or *exponent*. The power a^x is defined in different ways for various types of indices.

First, we define $a^0 = 1$. Then for integers $n \geq 1$, we define $a^n = a \times a^{n-1}$, so that $a^1 = a \times a^0$, $a^2 = a \times a^1$, $a^3 = a \times a^2$, ... This is called a *recursive* definition.

1

$$a^0 = 1,$$

$$a^n = a \times a^{n-1}, \text{ for integers } n \geq 1.$$

For positive rational indices m/n , where m and n are positive integers, we define:

2

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m, \text{ where if } n \text{ is even, } \sqrt[n]{a} \text{ means the positive } n\text{th root.}$$

For negative rational indices $-q$, we define:

3

$$a^{-q} = \frac{1}{a^q}.$$

NOTE: Defining powers like 2^π and $3^{\sqrt{2}}$ with irrational indices is quite beyond this course. We will make the quite reasonable assumption that powers with irrational indices can be formed, and that they work as expected. One approach to a power like 2^π is to consider powers with rational indices increasingly close to π , such as 2^3 , $2^{3\frac{1}{7}}$, $2^{3\frac{16}{113}}$ and so on. This point will be taken up later in the chapters on the logarithmic and exponential functions.

WORKED EXERCISE: Simplify the following powers:

(a) 7^{-1} (b) $(\frac{3}{5})^{-1}$ (c) $(1\frac{1}{2})^{-2}$ (d) $121^{\frac{1}{2}}$ (e) $49^{-\frac{1}{2}}$

SOLUTION:

(a) $7^{-1} = \frac{1}{7}$ (c) $(1\frac{1}{2})^{-2} = (\frac{3}{2})^{-2} = (\frac{2}{3})^2 = \frac{4}{9}$ (d) $(121)^{\frac{1}{2}} = 11$
 (b) $(\frac{3}{5})^{-1} = \frac{5}{3}$ (e) $(49)^{-\frac{1}{2}} = (\frac{1}{49})^{\frac{1}{2}} = \frac{1}{7}$

WORKED EXERCISE: Simplify: (a) $(\frac{4}{9})^{\frac{1}{2}}$ (b) $9^{\frac{3}{2}}$ (c) $125^{-\frac{2}{3}}$ (d) $(2\frac{1}{4})^{-\frac{5}{2}}$

SOLUTION:

(a) $(\frac{4}{9})^{\frac{1}{2}} = \frac{2}{3}$ (c) $125^{-\frac{2}{3}} = (\frac{1}{125})^{\frac{2}{3}} = (\frac{1}{5})^2 = \frac{1}{25}$ (d) $(2\frac{1}{4})^{-\frac{5}{2}} = (\frac{4}{9})^{\frac{5}{2}} = (\frac{2}{3})^5 = \frac{32}{243}$
 (b) $9^{\frac{3}{2}} = 3^3 = 27$

Laws for Indices: These rules should be well known from earlier years. Group A restates some of the definitions, Group B involves *compound indices*, and Group C involves *compound bases*.

4

INDEX LAWS:

A. $a^{\frac{1}{2}} = \sqrt{a}$

B. $a^{x+y} = a^x a^y$

C. $(ab)^x = a^x b^x$

$$a^{-1} = \frac{1}{a}$$

$$a^{x-y} = \frac{a^x}{a^y}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$a^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}$$

$$(a^x)^n = a^{xn}$$

The Use of Primes: Bases that are composite numbers are often best factored into primes when calculations are required.

WORKED EXERCISE: Simplify: (a) $12^n \times 18^{-2n}$ (b) $9^{-\frac{4}{3}} \times 81^{\frac{2}{3}}$

SOLUTION:

$$\begin{aligned}
 \text{(a) } 12^n \times 18^{-2n} &= (2^2 \times 3)^n \times (2 \times 3^2)^{-2n} \\
 &= 2^{2n} \times 3^n \times 2^{-2n} \times 3^{-4n} \\
 &= 3^{-3n} \\
 \text{(b) } 9^{-\frac{4}{3}} \times 81^{\frac{2}{3}} &= (3^2)^{-\frac{4}{3}} \times (3^4)^{\frac{2}{3}} \\
 &= 3^{-\frac{8}{3}} \times 3^{\frac{8}{3}} \\
 &= 3^0 \\
 &= 1
 \end{aligned}$$

WORKED EXERCISE: Solve: (a) $16^x = \sqrt{8}$ (b) $27^x = 9^{1-x}$

SOLUTION:

$$\begin{aligned}
 \text{(a) } 16^x &= \sqrt{8} \\
 (2^4)^x &= (2^3)^{\frac{1}{2}} \\
 2^{4x} &= 2^{\frac{3}{2}} \\
 4x &= \frac{3}{2} \\
 x &= \frac{3}{8} \\
 \text{(b) } 27^x &= 9^{1-x} \\
 3^{3x} &= 3^{2-2x} \\
 3x &= 2 - 2x \\
 x &= \frac{2}{5}
 \end{aligned}$$

Negative and Zero Bases: Negative numbers do not have square roots, and so negative bases are impossible if a square root or fourth root or any even root is involved. For example, $(-64)^{\frac{1}{2}}$ is undefined, but $(-64)^{\frac{1}{3}} = -4$.

Powers of zero are only defined for positive indices x , in which case $0^x = 0$ — for example, $0^3 = 0$. If the index x is negative, then 0^x is undefined, being the reciprocal of zero — for example, 0^{-3} is undefined. If the index is zero, then 0^0 is also undefined.

Exercise 6A

NOTE: Do not use the calculator in this exercise at all.

1. Simplify these numerical expressions:

$$\begin{array}{lllll}
 \text{(a) } 3^0 & \text{(c) } \left(\frac{2}{7}\right)^{-1} & \text{(e) } 7^{-2} & \text{(g) } \left(\frac{2}{3}\right)^{-4} & \text{(i) } \left(3\frac{1}{3}\right)^{-3} \\
 \text{(b) } 5^{-1} & \text{(d) } \left(5\frac{1}{4}\right)^{-1} & \text{(f) } \left(\frac{1}{4}\right)^{-3} & \text{(h) } \left(2\frac{1}{2}\right)^{-2} & \text{(j) } (45\frac{3}{7})^0
 \end{array}$$

2. Simplify:

$$\begin{array}{lllll}
 \text{(a) } 25^{\frac{1}{2}} & \text{(c) } 27^{\frac{2}{3}} & \text{(e) } 16^{\frac{3}{4}} & \text{(g) } 27^{\frac{4}{3}} & \text{(i) } \left(\frac{4}{9}\right)^{1\frac{1}{2}} \\
 \text{(b) } 27^{\frac{1}{3}} & \text{(d) } 8^{\frac{2}{3}} & \text{(f) } 9^{\frac{3}{2}} & \text{(h) } \left(\frac{25}{4}\right)^{\frac{3}{2}} & \text{(j) } \left(5\frac{1}{16}\right)^{0.25}
 \end{array}$$

3. Simplify:

$$\begin{array}{lllll}
 \text{(a) } \frac{1}{13^{-2}} & \text{(b) } \frac{2^{-3}}{3^{-2}} & \text{(c) } \frac{4^{-2}}{2^{-3}} & \text{(d) } \frac{5^3}{2^{-5}} & \text{(e) } \frac{2^{-2}}{15^2}
 \end{array}$$

4. Write as single fractions without negative indices:

$$\begin{array}{lllll}
 \text{(a) } 7x^{-1} & \text{(b) } 3x^{-3} & \text{(c) } x^2(y+1)^{-2} & \text{(d) } \frac{s^{-2}}{(t+3)^{-1}} & \text{(e) } \frac{2^{-3}x^2}{3^{-2}y^{-2}}
 \end{array}$$

5. Simplify, giving the answers without negative indices:

$$\begin{array}{lllll}
 \text{(a) } x^{-5}y^3 \times x^3y^{-2} & \text{(c) } (s^2y^{-3})^3 & \text{(e) } 7m^{\frac{4}{3}} \times 3m^{\frac{2}{3}} & \text{(g) } (8x^3y^{-6})^{\frac{1}{3}} \\
 \text{(b) } 3x^{-2} \times 7x & \text{(d) } (5c^{-2}d^3)^{-1} & \text{(f) } (a^{-2}b^4)^{\frac{1}{2}} & \text{(h) } (p^{\frac{1}{5}}q^{-\frac{3}{5}})^{10}
 \end{array}$$

6. Write down the solutions of these index equations:

(a) $17^x = \frac{1}{17}$

(c) $64^x = 4$

(e) $(\frac{1}{25})^x = \frac{1}{5}$

(b) $9^x = \frac{1}{81}$

(d) $4^x = 8$

(f) $49^x = \frac{1}{7}$

————— DEVELOPMENT —————

7. Simplify:

(a) $16^{-\frac{1}{4}}$

(c) $(\frac{8}{125})^{-\frac{2}{3}}$

(e) $(3\frac{3}{8})^{-\frac{4}{3}}$

(g) $(1.2)^{-2}$

(i) $(2.25)^{-0.5}$

(b) $32^{-1.4}$

(d) $(\frac{16}{49})^{-\frac{3}{2}}$

(f) $(0.4)^{-3}$

(h) $(0.25)^{1.5}$

(j) $(0.36)^{-1.5}$

8. Write these expressions using fractional and negative indices:

(a) $\sqrt[3]{7}$

(c) $\frac{\sqrt[3]{x^4}}{5}$

(e) $\frac{7}{\sqrt[5]{y^4}}$

(g) $x^2\sqrt{x}$

(i) $\frac{1}{8x}$

(b) $\sqrt[4]{5^3}$

(d) $\frac{\sqrt{a}}{\sqrt[5]{11}}$

(f) $x\sqrt{x}$

(h) $\frac{2\sqrt{x}}{x}$

(j) $\frac{7}{3x+2}$

9. Given that $x = 16$ and $y = 25$, evaluate:

(a) $x^{\frac{1}{2}} + y^{\frac{1}{2}}$

(b) $x^{\frac{3}{4}} - y^{\frac{1}{2}}$

(c) $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$

(d) $(y-x)^{\frac{1}{2}} \times (4y)^{-\frac{1}{2}}$

10. Expand and simplify, answering without using negative indices:

(a) $(x + 5x^{-1})^2$

(b) $(x^2 - 7x^{-2})^2$

(c) $(3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}})^2$

11. Explain why $8^n = (2^3)^n = 2^{3n}$. Using similar methods, write these expressions with prime bases and simplify:

(a) $2^n \times 8^n$

(c) $3 \times 9^x \times 81^x$

(e) $\frac{1}{7} \times 49^n \times \sqrt{7}$

(b) $\frac{25^n}{5^{n+1}}$

(d) $\frac{121^{-n} \times 11}{11^{3n}}$

(f) $\frac{32^x \times \frac{1}{2}}{4^x \times 16}$

12. Explain why if $3^{3x-1} = 9$, then $3x - 1 = 2$, and so $x = 1$. Similarly, by reducing both sides to powers of the same base, solve:

(a) $125^x = \frac{1}{5}$

(c) $8^x = \frac{1}{4}$

(e) $8^{x+1} = 2 \times 4^{x-1}$

(b) $25^x = \sqrt{5}$

(d) $64^x = \sqrt{32}$

(f) $(\frac{1}{9})^x = 3^4$

13. By taking appropriate powers of both sides, solve:

(a) $b^{\frac{1}{3}} = \frac{1}{7}$

(b) $n^{-2} = 121$

(c) $x^{-\frac{3}{4}} = 27$

14. Solve simultaneously:

(a) $7^{2x-y} = 49$

(b) $8^x \div 4^y = 4$

(c) $13^{x+4y} = 1$

$2^{x+y} = 128$

$11^{y-x} = \frac{1}{11}$

$25^{x+5y} = 5$

15. Write as a single fraction, without negative indices, and simplify:

(a) $a^{-1} - b^{-1}$

(c) $(x^{-2} - y^{-2})^{-1}$

(e) $x^{-2}y^{-2}(x^2y^{-1} - y^2x^{-1})$

(b) $\frac{1 - y^{-1}}{1 - y^{-2}}$

(d) $\frac{a^{-1} + b^{-1}}{a^{-2} - b^{-2}}$

(f) $\frac{(a^2 - 1)^{-1}}{(a - 1)^{-1}}$

16. Explain why $12^n = (2^2 \times 3)^n = 2^{2n} \times 3^n$. Using similar methods, write these expressions with prime bases and simplify:

(a) $2^n \times 4^n \times 8^n$

(c) $6^x \times 4^x \div 3^x$

(e) $100^{2n-1} \times 25^{-1} \times 8^{-1}$

(b) $\frac{9^{n+2} \times 3^{n+1}}{3 \times 27^n}$

(d) $\frac{12^x \times 18^x}{3^x \times 2^x}$

(f) $\frac{24^{x+1} \times 8^{-1}}{6^{2x}}$

17. Explain why $3^n + 3^{n+1} = 3^n(1 + 3) = 4 \times 3^n$. Then use similar methods to simplify:
- (a) $7^{n+2} + 7^n$ (c) $5^n - 5^{n-3}$ (e) $2^{2n+2} - 2^{2n-1}$
 (b) $\frac{3^{n+3} - 3^n}{3^n}$ (d) $\frac{7^n + 7^{n+2}}{7^{n-1} + 7^{n+1}}$ (f) $\frac{2^{2n} - 2^{n-1}}{2^n - 2^{-1}}$
18. Use the methods of the last two questions to simplify:
- (a) $\frac{6^n + 3^n}{2^{n+1} + 2}$ (b) $\frac{12^x + 1}{6^{2x} + 3^x}$ (c) $\frac{12^n - 18^n}{3^n - 2^n}$
19. By taking 6th powers of both sides, show that $11^{\frac{1}{3}} < 5^{\frac{1}{2}}$. Using similar methods (followed perhaps by a check on the calculator), compare:
- (a) $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$ (c) $7^{\frac{3}{2}}$ and 20 (e) $2^{\frac{1}{2}} \times 5^{\frac{1}{3}}$ and $2 \times 3^{\frac{1}{6}}$
 (b) $2^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ (d) $5^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$ (f) $5^{\frac{1}{3}}$ and $2^{\frac{1}{4}} \times 3^{\frac{1}{12}} \times 5^{\frac{1}{6}}$
20. If $a = 2^{\frac{1}{2}} + 2^{-\frac{1}{2}}$ and $b = 2^{\frac{1}{2}} - 2^{-\frac{1}{2}}$, find:
- (a) ab (c) $a^2 + b^2$ (e) $(a + b)a$
 (b) a^2 (d) $a^2 - b^2$ (f) $a^3 + b^3$
21. (a) If $x = 2^{\frac{1}{3}} + 4^{\frac{1}{3}}$, show that $x^3 = 6(1 + x)$.
 (b) If $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, show that $\frac{x^2 + x^{-2}}{x - x^{-1}} = 3$.
 (c) Show that $\frac{pq^{-1} - p^{-1}q}{p^2q^{-2} - p^{-2}q^2} = \frac{pq}{p^2 + q^2}$.
- EXTENSION
22. Find the smallest positive integers m and n for which:
- (a) $12 < 2^{m/n} < 13$ (b) $13 < 2^{m/n} < 14$
23. Find $\lim_{x \rightarrow 0^+} 0^x$ and $\lim_{x \rightarrow 0} x^0$. Explain what these two limits have to do with the remark made in the notes that 0^0 is undefined.

6B Logarithms

The most important thing to learn about logarithms is that the logarithmic function $y = \log_a x$ is simply the inverse function of the exponential function $y = a^x$. The ability to convert between statements about indices and statements about logarithms is a fundamental skill that must be developed.

Definition of Logarithms: Suppose a and x are positive numbers, with $a \neq 1$. Then $\log_a x$ is the index, when x is written as a power of a .

5

DEFINITION: $\log_a x$ is the index, when x is written as a power of a .
 In symbols, $y = \log_a x$ means $x = a^y$.

We read $\log_a x$ as ‘the logarithm of x base a ’, or ‘log x base a ’.

WORKED EXERCISE:

$$\begin{array}{ll} \log_2 8 = 3, \text{ because } 2^3 = 8. & \log_3 1 = 0, \text{ because } 3^0 = 1. \\ \log_5 25 = 2, \text{ because } 5^2 = 25. & \log_{10} \frac{1}{10} = -1, \text{ because } 10^{-1} = \frac{1}{10}. \\ \log_7 7 = 1, \text{ because } 7^1 = 7. & \log_6 \sqrt{6} = \frac{1}{2}, \text{ because } 6^{\frac{1}{2}} = \sqrt{6}. \end{array}$$

WORKED EXERCISE: Rewrite these equations in index form, then solve:

(a) $x = \log_3 \sqrt{3}$ (b) $\log_2 x = 5$ (c) $\log_x 10\,000 = 4$

SOLUTION: When rewriting the equation in index form, remember that ‘the base of the log is the base of the power’, and ‘the log is the index’.

(a) $x = \log_3 \sqrt{3}$ (b) $\log_2 x = 5$ (c) $\log_x 10\,000 = 4$
 $3^x = \sqrt{3}$ $x = 2^5$ $x^4 = 10\,000$
 $x = \frac{1}{2}$ $x = 32$ $x = 10$

Laws for Logarithms: The laws in Group A are four important special cases. Those in Group B are the Group B index laws written in logarithmic form.

6	A. $\log_a 1 = 0$ (because $1 = a^0$)	B. $\log_a xy = \log_a x + \log_a y$
	$\log_a a = 1$ (because $a = a^1$)	$\log_a \frac{x}{y} = \log_a x - \log_a y$
	$\log_a \frac{1}{a} = -1$ (because $\frac{1}{a} = a^{-1}$)	$\log_a x^n = n \log_a x$
	$\log_a \sqrt{a} = \frac{1}{2}$ (because $\sqrt{a} = a^{\frac{1}{2}}$)	

WORKED EXERCISE: Write each of the following in terms of $\log_2 3$:

(a) $\log_2 81$ (b) $\log_2 2\sqrt{3}$ (c) $\log_2 \frac{8}{9}$

SOLUTION: Each number must be written in terms of powers of 2 and 3.

(a) $\log_2 81$ (b) $\log_2 2\sqrt{3}$ (c) $\log_2 \frac{8}{9}$
 $= \log_2 3^4$ $= \log_2 2 + \log_2 3^{\frac{1}{2}}$ $= \log_2 2^3 - \log_2 3^2$
 $= 4 \log_2 3$ $= \log_2 2 + \frac{1}{2} \log_2 3$ $= 3 \log_2 2 - 2 \log_2 3$
 $= 1 + \frac{1}{2} \log_2 3,$ $= 3 - 2 \log_2 3,$
 since $\log_2 2 = 1.$ since $\log_2 2 = 1.$

The Change of Base Law: Conversion of logarithms from one base to another is often needed. For example, the calculator only gives approximations to logarithms to the two bases 10 and e , so the ability to change the base is necessary to approximate logarithms to other bases.

7	CHANGE OF BASE: $\log_b x = \frac{\log_a x}{\log_a b}$
	‘Take the log of the number over the log of the base.’

PROOF: To prove this formula, let $y = \log_b x$.
 Then by the definition of logs, $x = b^y$
 and taking logs base a of both sides, $\log_a x = \log_a b^y$.
 Now by the third law in Group B above, $\log_a x = y \log_a b$
 and rearranging, $y = \frac{\log_a x}{\log_a b}$, as required.

WORKED EXERCISE: $\log_2 7 = \frac{\log_{10} 7}{\log_{10} 2}$
 $\doteq 2.807$ (using the calculator)

WORKED EXERCISE: Solve $2^x = 7$, correct to four significant figures.

SOLUTION:

$$2^x = 7$$

$$\begin{aligned} x &= \log_2 7 \\ &= \frac{\log_{10} 7}{\log_{10} 2} \\ &\doteq 2.807 \end{aligned}$$

Alternatively, taking logs of both sides,

$$\begin{aligned} \log_{10} 2^x &= \log_{10} 7 \\ x \log_{10} 2 &= \log_{10} 7 \\ x &= \frac{\log_{10} 7}{\log_{10} 2} \\ &\doteq 2.807. \end{aligned}$$

WORKED EXERCISE: How many positive integer powers of 7 are less than 10^9 ?

SOLUTION: Put $7^n < 1\,000\,000\,000$.

$$\begin{aligned} \text{Then } n &< \log_7 1\,000\,000\,000 \\ \text{so } n &< \frac{\log_{10} 1\,000\,000\,000}{\log_{10} 7} \\ n &< \frac{9}{\log_{10} 7} \doteq 10.649 \dots \\ n &= 1, 2, \dots, 10, \end{aligned}$$

so 10 powers of 7 are less than $1\,000\,000\,000$.

NOTE: The calculator key marked $\boxed{\log}$ gives logarithms base 10, and should be the only log key used for the moment. The key $\boxed{\ln}$ also gives logarithms, but to a different base $e \doteq 2.7183$ — it will be needed in Chapter 12.

Exponential and Logarithmic Functions: The definition of logarithms means that for any base a , the *logarithmic function* $y = \log_a x$ and the *exponential function* $y = a^x$ are inverse functions, as discussed in Section 2H. This means that when the functions are applied in succession they cancel each other out.

8

POWERS AND LOGARITHMS: $y = \log_a x$ and $y = a^x$ are inverse functions. Applying them in succession, $\log_a a^x = x$ and $a^{\log_a x} = x$.

For example, $\log_2 2^3 = \log_2 8 = 3$, and $2^{\log_2 8} = 2^3 = 8$.

NOTE: The calculator reflects this structure. On most calculators, the function $\boxed{10^x}$ is reached by pressing $\boxed{\text{inv}}$ or $\boxed{\text{shift}}$ followed by $\boxed{\log}$. In Chapter 13, we will need to reach the function $\boxed{e^x}$ by pressing $\boxed{\text{inv}}$ followed by $\boxed{\ln}$.

WORKED EXERCISE:

- (a) Simplify $\log_3 3^{17}$ and $2^{\log_2 7}$. (b) Express 5 and x as powers of 2.

SOLUTION:

- (a) $\log_3 3^{17} = 17$, since $y = \log_3 x$ and $y = 3^x$ are inverse functions.
 $2^{\log_2 7} = 7$, since $y = \log_2 x$ and $y = 2^x$ are inverse functions.

- (b) Similarly, $5 = 2^{\log_2 5}$, and $x = 2^{\log_2 x}$.

Exercise 6B

1. Rewrite each equation in index form, then solve:

(a) $x = \log_3 9$	(c) $x = \log_5 125$	(e) $x = \log_7 \frac{1}{49}$	(g) $x = \log_5 \sqrt{5}$
(b) $x = \log_2 16$	(d) $x = \log_{10} \frac{1}{10}$	(f) $x = \log_{\frac{1}{3}} \frac{1}{81}$	(h) $x = \log_{11} \frac{1}{\sqrt{11}}$

2. Copy and complete the tables of values below, and verify that the functions $y = 2^x$ and $y = \log_2 x$ are inverse functions. Then sketch both curves on the one set of axes, and verify that they are reflections of each other in $y = x$.

x	-3	-2	-1	0	1	2	3
2^x							

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2 x$							

3. Rewrite each equation in index form, then solve:

(a) $\log_4 x = 3$	(c) $\log_9 x = \frac{1}{2}$	(e) $\log_{\frac{1}{16}} x = -\frac{1}{4}$	(g) $\log_{36} x = 1.5$
(b) $\log_{13} x = -1$	(d) $\log_{10} x = -2$	(f) $\log_7 x = -\frac{1}{2}$	(h) $\log_8 x = -\frac{2}{3}$

4. Rewrite each equation in index form, then solve:

(a) $\log_x 27 = 3$	(c) $\log_x 1000 = 3$	(e) $\log_x 25 = -2$	(g) $\log_x 16 = \frac{4}{3}$
(b) $\log_x \frac{1}{7} = -1$	(d) $\log_x 3 = \frac{1}{2}$	(f) $\log_x \frac{4}{9} = 2$	(h) $\log_x 9 = -\frac{1}{2}$

5. Given that a is a positive real number not equal to 1, evaluate:

(a) $\log_a a$	(c) $\log_a a^3$	(e) $\log_a \sqrt{a}$	(g) $\log_a 1$
(b) $\log_a \frac{1}{a}$	(d) $\log_a \frac{1}{a^2}$	(f) $\log_a \frac{1}{\sqrt{a}}$	(h) $\log_a \frac{1}{a\sqrt{a}}$

6. Find which two integers these expressions lie between. Then use the change of base formula and the calculator to find, correct to three significant figures:

(a) $\log_2 11$	(b) $\log_5 127$	(c) $\log_{11} 200$	(d) $\log_{\sqrt{2}} 20$

7. Express in terms of $\log_2 3$ and $\log_2 5$ (remember that $\log_2 2 = 1$):

(a) $\log_2 9$	(b) $\log_2 18$	(c) $\log_2 \frac{1}{6}$	(d) $\log_2 2\frac{1}{2}$

DEVELOPMENT

8. Rewrite in logarithmic form, then solve using the change of base formula. Give your answers in exact form, then to four significant figures:

(a) $2^x = 13$	(c) $7^x > 1000$	(e) $5^x < 0.04$	(g) $(\frac{1}{3})^x > 100$
(b) $3^{x-2} = 20$	(d) $2^{x+1} < 10$	(f) $(\frac{1}{2})^{x+1} = 10$	(h) $(0.06)^x < 0.001$

9. (a) How many positive integer powers of 2 are less than 10^{10} ?

(b) How many positive integer powers of $\frac{1}{5}$ are greater than 10^{-10} ?

10. If $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, simplify:

(a) $\log_a 64$	(c) $\log_a 27a^5$	(e) $\log_a 1.5$	(g) $\log_a 0.04$
(b) $\log_a \frac{1}{30}$	(d) $\log_a \frac{100}{a}$	(f) $\log_a \frac{18}{25a}$	(h) $\log_a \frac{8}{15a^2}$

11. Express in terms of $\log_2 3$ and $\log_2 5$:

(a) $\log_2 \frac{5}{8}$	(b) $\log_2 15\sqrt{3}$	(c) $\log_2 \frac{1}{6}\sqrt{2}$	(d) $\log_2 \frac{3}{25}\sqrt{30}$

12. Use the identities $\log_a a^x = x$ and $a^{\log_a x} = x$ to simplify:

(a) $\log_7 7^5$	(b) $3^{\log_3 7}$	(c) $\log_{12} 12^n$	(d) $6^{\log_6 y}$

13. Using the identity $x = a^{\log_a x}$:
- (a) express 3 as a power of 2,
 - (b) express u as a power of 3,
 - (c) express 7 as a power of a ,
 - (d) express u as a power of v .
14. Simplify these expressions:
- (a) $5^{-\log_5 2}$
 - (b) $12^{2\log_{12} 7}$
 - (c) $2^{\log_2 3 + \log_2 5}$
 - (d) $a^{n \log_a x}$
 - (e) $7^{-\log_7 x}$
 - (f) $5^{x + \log_5 x}$
 - (g) $2^{x \log_2 x}$
 - (h) $3^{\frac{\log_3 x}{x}}$
15. Rewrite these relations in index form (that is, without using logarithms):
- (a) $\log_a (x + y) = \log_a x + \log_a y$
 - (b) $\log_{10} x = 3 + \log_{10} y$
 - (c) $\log_3 x = 4 \log_3 y$
 - (d) $2 \log_2 x + 3 \log_2 y - 4 \log_2 z = 0$
 - (e) $x \log_a 2 = \log_a y$
 - (f) $\log_a x - \log_a y = n \log_a z$
 - (g) $\frac{1}{2} \log_2 x = \frac{1}{3} \log_2 y - 1$
 - (h) $2 \log_3 (2x + 1) = 3 \log_3 (2x - 1)$
16. (a) Prove the identities: (i) $\log_a x = -\log_a \frac{1}{x}$ (ii) $\log_a x = -\log_{\frac{1}{a}} x$
 (b) Check these identities by evaluating $\log_5 25$, $\log_5 \frac{1}{25}$ and $\log_{\frac{1}{5}} 25$.
17. Prove by contradiction that $\log_2 7$ and $\log_7 3$ are irrational (see the relevant worked exercise in Section 2B).

EXTENSION

18. Let $S = \frac{1}{2}(2^x + 2^{-x})$ and $D = \frac{1}{2}(2^x - 2^{-x})$.
- (a) Simplify SD , $S + D$, $S - D$ and $S^2 - D^2$.
 - (b) Rewrite the formulae for S and D as quadratic equations in 2^x . Hence express x in terms of S , and in terms of D , in the case where $x > 1$.
 - (c) Show that $x = \frac{1}{2} \log_2 \frac{1+y}{1-y}$, where $y = DS^{-1}$.

6C Sequences and How to Specify Them

A typical *infinite sequence* is formed by the positive odd integers, arranged in increasing order:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \dots$$

The three dots ... indicate that the sequence goes on forever, with no last term. Notice, however, that the sequence starts abruptly with first term 1, then has second term 3, third term 5, and so on. Using the symbol T_n to stand for the n th term:

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 5, \quad T_4 = 7, \quad T_5 = 9, \quad \dots$$

The two-digit odd numbers arranged in increasing order form a *finite sequence*:

$$11, 13, 15, \dots, 99,$$

where the dots ... here stand for the 41 terms that have been omitted.

There are three different ways to specify a sequence, and it is important to be able to display a given sequence in these different ways.

Write Out the First Few Terms: The easiest way is to write out the first few terms until the pattern is clear. Our example of the positive odd integers could be written

$$1, 3, 5, 7, 9, \dots$$

This sequence clearly continues $\dots, 11, 13, 15, 17, 19, \dots$, and with a few more calculations, it is clear that $T_{11} = 21$, $T_{14} = 27$, and $T_{16} = 31$.

Give a Formula for the n th Term: The formula for the n th term of this sequence is

$$T_n = 2n - 1,$$

because the n th term is always one less than $2n$. Notice that n must be a positive integer. Giving the formula does not rely on the reader recognising a pattern, and any particular term of the sequence can now be calculated quickly:

$$\begin{array}{lll} T_{30} = 60 - 1 & T_{100} = 200 - 1 & T_{244} = 488 - 1 \\ = 59 & = 199 & = 487 \end{array}$$

Say Where to Start and How to Proceed (Recursive Formula): This sequence of odd cardinals starts with 1, then each term is 2 more than the previous one. Thus the sequence is completely specified by writing down these two statements:

$$\begin{aligned} T_1 &= 1, \\ T_n &= T_{n-1} + 2, \text{ for } n \geq 2. \end{aligned}$$

Such a specification is called a *recursive* formula of a sequence, and some important definitions later in this chapter are based on this idea.

WORKED EXERCISE: Give all three specifications of the sequence of positive multiples of seven, arranged in increasing order.

SOLUTION: The sequence is 7, 14, 21, 28, $\dots$

The formula for the n th term is $T_n = 7n$.

The recursive formula is $T_1 = 7$, and $T_n = T_{n-1} + 7$ for $n \geq 2$.

WORKED EXERCISE: Find the first five terms, and the formula for the n th term, of the sequence given by

$$T_1 = 1 \quad \text{and} \quad T_n = \frac{n-1}{n} T_{n-1}, \text{ for } n \geq 2.$$

SOLUTION: Using the formula, the first five terms are $T_1 = 1$,

$$\begin{array}{lllll} T_2 = \frac{1}{2} \times T_1 & T_3 = \frac{2}{3} \times T_2 & T_4 = \frac{3}{4} \times T_3 & T_5 = \frac{4}{5} \times T_4 \\ = \frac{1}{2}, & = \frac{1}{3}, & = \frac{1}{4}, & = \frac{1}{5}. \end{array}$$

From this pattern it is clear that the formula for the n th term is $T_n = \frac{1}{n}$.

WORKED EXERCISE: Find whether 411 and 500 are members of the sequence whose n th term is $T_n = n^2 - 30$.

SOLUTION:

$$\text{Put } T_n = 411.$$

$$\text{Then } n^2 - 30 = 411$$

$$n^2 = 441$$

$$n = 21 \text{ or } -21.$$

But n cannot be negative,
so 411 is the 21st term.

$$\text{Put } T_n = 500.$$

$$\text{Then } n^2 - 30 = 500$$

$$n^2 = 530.$$

But $\sqrt{530}$ is not a positive integer,
so 500 is not a term of the sequence.

WORKED EXERCISE:

- (a) Find how many negative terms there are in the sequence $T_n = 12n - 100$, and find the first positive term (its number and its value).
 (b) How many positive terms are less than 200?

SOLUTION:

(a) Put $T_n < 0$. Then $12n - 100 < 0$ $n < 8\frac{1}{3}$, so there are eight negative terms, and the first positive term is $T_9 = 8$.	(b) Put $0 < T_n < 200$. Then $0 < 12n - 100 < 200$ $8\frac{1}{3} < n < 25$, so the 16 terms from T_9 to T_{24} inclusive are positive and less than 200.

Exercise 6C

- Write down the next four terms of each sequence:

(a) 9, 13, 17, ...	(c) 26, 17, 8, ...	(e) -1, 1, -1, ...	(g) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$
(b) 3, 6, 12, ...	(d) 81, 27, 9, ...	(f) 25, 36, 49, ...	(h) 16, -8, 4, ...
- Write down the first four terms of the sequence whose n th term is:

(a) $T_n = 5n - 2$	(c) $T_n = n^3$	(e) $T_n = 4 \times 3^n$	(g) $T_n = (-1)^n \times n$
(b) $T_n = 5^n$	(d) $T_n = 12 - 7n$	(f) $T_n = 2n(n + 1)$	(h) $T_n = (-3)^n$
- Write down the first four terms of the sequence defined recursively by:

(a) $T_1 = 5,$ $T_n = T_{n-1} + 12$	(c) $T_1 = 1,$ $T_n = nT_{n-1}$	(e) $T_1 = 37,$ $T_n = T_{n-1} - 24$	(g) $T_1 = 5,$ $T_n = T_{n-1} + 5n$
(b) $T_1 = \frac{3}{4},$ $T_n = 2T_{n-1}$	(d) $T_1 = 28,$ $T_n = -\frac{1}{2}T_{n-1}$	(f) $T_1 = 2\sqrt{2},$ $T_n = T_{n-1}\sqrt{2}$	(h) $T_1 = \frac{1}{2},$ $T_n = T_{n-1} + (\frac{1}{2})^n$
- The n th term of a sequence is given by $T_n = 6n + 17$.
 - By forming equations and solving them, find whether each of the numbers 77, 349 and 1577 is a member of the sequence, and if so, which term it is.
 - By forming an inequation and solving it, find how many terms of the sequence are less than 400, and find the value of the first term greater than 400.
- The n th term of a sequence is given by $T_n = 5n^2$.
 - By forming an equation and solving it, find whether each of the numbers 60, 80 and 605 is a member of the sequence, and if so, which term it is.
 - By forming an inequation and solving it, find how many terms of the sequence are less than 1000, and find the value of the first term greater than 1000.

DEVELOPMENT

- Write down the first four terms of these sequences (where a and x are constants):

(a) $T_n = 1 + (-1)^n$	(d) $T_n = 7a - 2an$	(g) $T_n = (-1)^n(4n - 7)$
(b) $T_n = 25 \times (-2)^n$	(e) $T_n = 4a \times 2^{n-1}$	(h) $T_n = (2\sqrt{2})^{n-1}$
(c) $T_n = -36x \times (-\frac{1}{2})^{n-1}$	(f) $T_n = 3^n - 2^n$	(i) $T_n = \frac{3}{4}n^2x$
- Give a recursive formula for the n th term T_n of each sequence in terms of the $(n - 1)$ th term T_{n-1} :

(a) 16, 21, 26, ...	(b) 7, 14, 28, ...	(c) 9, 2, -5, ...	(d) 4, -4, 4, ...

8. (a) Find whether -10 and -15 are members of the sequence $T_n = 48 - 7n$, and if so, what terms they are. (b) How many terms in this sequence are greater than -700 ?
9. (a) Find whether 28 and 70 are members of the sequence $T_n = n^2 - 3n$, and if so, what terms they are. (b) How many terms of this sequence are less than 18 ?
10. (a) Find whether $1\frac{1}{2}$ and 96 are members of the sequence $T_n = \frac{3}{32} \times 2^n$, and if so, what terms they are. (b) Find the first term in this sequence which is greater than 10 .
11. The rigorous definition of a sequence is: 'A *sequence* is a function whose domain is the set of positive integers'. Graph the sequences in question 2, with n on the horizontal axis and T_n on the vertical axis. If there is a simple curve joining the points, draw it and give its equation.
12. Write down the first four terms, then state which terms are zero:
 (a) $T_n = \sin 90n^\circ$ (c) $T_n = \cos 180n^\circ$ (e) $T_1 = -1$ and $T_n = T_{n-1} + \cos 180n^\circ$
 (b) $T_n = \cos 90n^\circ$ (d) $T_n = \sin 180n^\circ$ (f) $T_1 = 1$ and $T_n = T_{n-1} + \sin 90n^\circ$
13. (a) A sequence satisfies $T_n = \frac{1}{2}(T_{n-1} + T_{n+1})$, with $T_1 = 3$ and $T_2 = 7$. Find T_3 and T_4 .
 (b) A sequence satisfies $T_n = \sqrt{T_{n-1} \times T_{n+1}}$, with $T_1 = 1$ and $T_2 = 2$. Find T_3 and T_4 .
14. A sequence is defined by $T_n = \frac{1}{n} - \frac{1}{n+1}$. (a) Find $T_1 + T_2 + T_3 + T_4$.
 (b) Give a formula for $T_1 + T_2 + \cdots + T_n$. (c) Show that $T_n = \frac{1}{n(n+1)}$.
 (d) Which term of the sequence is $\frac{1}{30}$?
15. (a) Which terms of the sequence $T_n = \frac{n-1}{n}$ are 0.9 and 0.99 ? (b) Find $T_{n+1} : T_n$.
 (c) Prove that $\frac{T_n}{T_{n+1}} + \frac{1}{n^2} = 1$. (d) Find $T_2 \times T_3 \times \cdots \times T_n$.
 (e) Prove that $T_{n+1} - T_{n-1} = \frac{2}{n^2 - 1}$.

EXTENSION

16. [The Fibonacci and Lucas sequences] These sequences are defined recursively by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 3,$$

$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 3.$$

- (a) Write out the first 12 terms of each sequence. Explain why every third term of each sequence is even and the rest are odd.
- (b) Write out the sequences $L_1 + F_1, L_2 + F_2, L_3 + F_3, \dots$ and $L_1 - F_1, L_2 - F_2, L_3 - F_3, \dots$. How are these two new sequences related to the Fibonacci sequence, and why?
- (c) Expand and simplify the first four terms of the sequence $T_n = (\frac{1}{2} + \frac{1}{2}\sqrt{5})^n$. Let the two sequences A_n and B_n of rational numbers be defined by $T_n = \frac{1}{2}A_n + \frac{1}{2}B_n\sqrt{5}$. Show that

$$A_{n+2} = A_{n+1} + A_n \quad \text{and} \quad B_{n+2} = B_{n+1} + B_n,$$

and hence that A_n is the Lucas sequence and B_n is the Fibonacci sequence.

- (d) Examine similarly the sequence $U_n = (\frac{1}{2} - \frac{1}{2}\sqrt{5})^n$.

6D Arithmetic Sequences

A very simple type of sequence is an *arithmetic sequence*. This is a sequence like

$$3, 7, 11, 15, 19, 23, 27, 31, 35, 39, \dots,$$

in which the difference between successive terms is constant — in this example each term is 4 more than the previous term. Because the difference is constant, all the terms can be generated from the first term 3 by repeated addition of this common difference 4. Arithmetic sequences are called APs for short, standing for ‘arithmetic progression’, an old name for the same thing.

Definition of an Arithmetic Sequence: An arithmetic sequence is a sequence in which the difference between successive terms is constant.

9

DEFINITION: A sequence T_n is called an *arithmetic sequence* if

$$T_n - T_{n-1} = d, \text{ for } n \geq 2,$$

where d is a constant, called the *common difference*.

This definition is essentially a recursive definition, because if a is the first term, then the terms of the sequence are defined by

$$T_1 = a \quad \text{and} \quad T_n = T_{n-1} + d, \text{ for } n \geq 2.$$

The first few terms of the sequence are

$$T_1 = a, \quad T_2 = a + d, \quad T_3 = a + 2d, \quad T_4 = a + 3d, \quad \dots$$

and from this pattern it is clear that the general formula for the n th term of an AP is:

10

THE n TH TERM OF AN AP: $T_n = a + (n - 1)d$

WORKED EXERCISE: Write out the first five terms, and calculate the 20th term, of the AP with: (a) $a = 2$ and $d = 5$, (b) $a = 20$ and $d = -3$.

SOLUTION:

(a) 2, 7, 12, 17, 22, ...

$$\begin{aligned} T_{20} &= a + 19d \\ &= 2 + 5 \times 19 \\ &= 97 \end{aligned}$$

(b) 20, 17, 14, 11, 8, ...

$$\begin{aligned} T_{20} &= a + 19d \\ &= 20 - 3 \times 19 \\ &= -37 \end{aligned}$$

WORKED EXERCISE: Show that the sequence 200, 193, 186, ... is an AP. Then find a formula for the n th term, and find the first negative term.

SOLUTION:

$$\text{Since } T_2 - T_1 = -7$$

$$\text{and } T_3 - T_2 = -7,$$

it is an AP with $a = 200$ and $d = -7$,

$$\text{so } T_n = 200 - 7(n - 1)$$

$$= 207 - 7n.$$

$$\text{Put } T_n < 0.$$

$$\text{Then } 207 - 7n < 0$$

$$7n > 207$$

$$n > 29\frac{4}{7},$$

so the first negative term is $T_{30} = -3$.

WORKED EXERCISE: Test whether these sequences are APs:

(a) $3, 9, 27, \dots$

(b) $\log_5 6, \log_5 12, \log_5 24, \dots$

SOLUTION:

(a) $T_2 - T_1 = 6$
and $T_3 - T_2 = 18$,
so it is not an AP.

(b) $T_2 - T_1 = \log_5 2$
and $T_3 - T_2 = \log_5 2$,
so it is an AP, and $d = \log_5 2$.

Further Problems: The first example below uses simultaneous equations. The second uses a double inequality to find the number of terms between two given numbers.

WORKED EXERCISE: The third term of an AP is 16, and the 12th term is 79. Find the 41st term.

SOLUTION: Let the first term be a and the common difference be d .

Since $T_3 = 16$, $a + 2d = 16$, (1)

and since $T_{12} = 79$, $a + 11d = 79$. (2)

Subtracting (1) from (2), $9d = 63$

$$d = 7.$$

Substituting into (1) gives $a = 2$, and so $T_{41} = a + 40d$
 $= 282$.

WORKED EXERCISE: Use the fact that the positive multiples of 7 form an AP to find how many multiples of 7 lie between 1000 and 10 000.

SOLUTION: The positive multiples of 7 form an AP $7, 14, 21, \dots$
in which $a = 7$ and $d = 7$.

The n th term of the AP is $T_n = 7 + 7(n - 1)$
 $= 7n$

(or one can simply claim that it's obvious that $T_n = 7n$).

To find the multiples of 7 between 1000 and 10 000, put

$$1000 < T_n < 10\,000$$

$$1000 < 7n < 10\,000$$

$\div 7$

$$142\frac{6}{7} < n < 1428\frac{4}{7},$$

so there are 1428 multiples of 7 less than 10 000, and 142 less than 1000,
leaving $1428 - 142 = 1286$ multiples of 7 between 1000 and 10 000.

Exercise 6D

1. Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If it is, write down the common difference d , find T_{10} , then find a formula for the n th term T_n :

(a) $8, 11, 14, \dots$

(d) $-3, 1, 5, \dots$

(g) $5 + \sqrt{2}, 5, 5 - \sqrt{2}, \dots$

(b) $21, 15, 9, \dots$

(e) $1\frac{3}{4}, 3, 4\frac{1}{4}, \dots$

(h) $1, 4, 9, 16, \dots$

(c) $8, 4, 2, \dots$

(f) $12, -5, -22, \dots$

(i) $-2\frac{1}{2}, 1, 4\frac{1}{2}, \dots$

2. Find T_n for each AP, then find T_{25} and the first negative term:

(a) $82, 79, 76, \dots$

(b) $345, 337, 329, \dots$

(c) $24\frac{1}{2}, 23\frac{1}{4}, 22, \dots$

3. Find x and the common difference if the following numbers form an arithmetic sequence.
[HINT: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find x .]
(a) 14, x , 32 (b) x , 14, 32 (c) $x - 1$, 17, $x + 15$ (d) $2x + 2$, $x - 4$, $5x$

DEVELOPMENT

4. The price of windows in a house is \$500 for the first window, then \$300 for each additional window. (a) Find a formula for the cost of n windows. (b) How much will fifteen windows cost? (c) What is the maximum number of windows whose total cost is less than \$10 000?
5. [Simple interest and APs] A principal of \$2000 is invested at 6% per annum simple interest. Let A_n be the total amount (principal plus interest) at the end of n years. (a) Write out the values of A_1 , A_2 , A_3 and A_4 . (b) Find a formula for A_n , and evaluate A_{12} . (c) How many years will it take before the total amount exceeds \$6000?
6. Use the formula $T_n = a + (n - 1)d$ to find how many terms there are in each sequence:
(a) 2, 5, 8, ..., 2000 (b) 100, 92, 84, ..., -244 (c) $-12, -10\frac{1}{2}, -9, \dots, 108$
7. The n th term of an arithmetic sequence is $T_n = 7 + 4n$.
(a) Write out the first four terms, and hence find the values of a and d .
(b) Find the sum and the difference of the 50th and the 25th terms.
(c) Prove that $5T_1 + 4T_2 = T_{27}$. (d) Which term of the sequence is 815?
(e) Find the last term less than 1000 and the first term greater than 1000.
(f) Find which terms are between 200 and 300, and how many of them there are.
8. (a) Let T_n be the sequence of positive multiples of 8. (i) Find the first term of the sequence greater than 500. (ii) Find the last term of the sequence less than 850. (iii) Hence find the number of positive multiples of 8 between 500 and 850.
(b) Use similar methods to find: (i) the number of multiples of 11 between 1000 and 2000, (ii) the number of multiples of 7 between 800 and 2000.
9. Find the first term and the common difference of the AP $T_n = a + (n - 1)d$ with:
(a) $T_2 = 3$ and $T_{10} = 35$ (c) $T_4 = 6$ and $T_{12} = 34$
(b) $T_5 = 24$ and $T_9 = -12$ (d) $T_7 = \sqrt{5} - 4$ and $T_{13} = 8 - 5\sqrt{5}$
10. (a) The third term of an AP is 7, and the seventh term is 31. Find the eighth term.
(b) The fourth, sixth and eighth terms of an AP add to -6 . Find the sixth term.
11. Find the common difference of each AP, then find x , given that $T_{11} = 36$:
(a) $5x - 9, 5x - 5, 5x - 1, \dots$ (b) $16, 16 + 6x, 16 + 12x, \dots$
(c) $2x + 10, 7 - x, 4 - 4x, \dots$
12. Find the common difference and a formula for the n th term of each AP:
(a) $\log_3 2, \log_3 4, \log_3 8, \dots$ (d) $5 - 6\sqrt{5}, 1 + \sqrt{5}, -3 + 8\sqrt{5}, \dots$
(b) $\log_a 54, \log_a 18, \log_a 6, \dots$ (e) $1.36, -0.52, -2.4, \dots$
(c) $x - 3y, 2x + y, 3x + 5y, \dots$ (f) $\log_a 3x^2, \log_a 3x, \log_a 3, \dots$
13. How many terms of the series 100, 97, 94, ... have squares less than 400?
14. [APs are essentially linear functions.] (a) Show that if $f(x) = mx + b$ is any linear function, then the sequence T_n defined by $T_n = mn + b$ is an AP, and find its first term and common difference. (b) Conversely, if T_n is an AP with first term a and difference d , find the linear function $f(x)$ such that $T_n = f(n)$. (c) Plot on the same axes the points of the AP $T_n = 8 - 3(n - 1)$ and the graph of the continuous linear function $y = 8 - 3(x - 1)$.

EXTENSION

15. [The set of all APs forms a two-dimensional space.] Let $\mathcal{A}(a, d)$ represent the AP whose first term is a and difference is d .
- The sum of the two sequences T_n and U_n is defined to be the sequence whose n th term is $T_n + U_n$. Show that for all constants λ and μ , and for all values of a_1, a_2, d_1 and d_2 , the sequence $\lambda\mathcal{A}(a_1, d_1) + \mu\mathcal{A}(a_2, d_2)$ is an AP, and find its first term and common difference.
 - Write out the sequences $\mathcal{A}(1, 0)$ and $\mathcal{A}(0, 1)$. Show that any AP $\mathcal{A}(a, d)$ with first term a and difference d can be written in the form $\lambda\mathcal{A}(1, 0) + \mu\mathcal{A}(0, 1)$, and find λ and μ .
 - Show more generally that, provided $a_1 : a_2 \neq d_1 : d_2$, any AP $\mathcal{A}(a, d)$ can be written in the form $\lambda\mathcal{A}(a_1, d_1) + \mu\mathcal{A}(a_2, d_2)$, and find expressions for λ and μ .

6E Geometric Sequences

A *geometric sequence* is a sequence like this:

$$6, 18, 54, 162, 486, 1458, 4374, \dots,$$

in which the ratio of successive terms is constant — in this example, each term is 3 times the previous term. This is a very similar situation to the APs of the last section, where the difference of successive terms was constant. Because the ratio is constant, all the terms can be generated from the first term 6 by repeated multiplication by this common ratio 3. The old name was ‘geometric progression’ and so geometric sequences are called GPs for short.

Definition of a Geometric Sequence: A geometric sequence is a sequence in which the ratio of successive terms is constant.

11 **DEFINITION:** A sequence T_n is called a *geometric sequence* if

$$\frac{T_n}{T_{n-1}} = r, \text{ for } n \geq 2,$$

where r is a constant, called the *common ratio*.

This definition, like the definition of an AP, is a recursive definition. If a is the first term, then the terms of the sequence are

$$T_1 = a \quad \text{and} \quad T_n = rT_{n-1}, \text{ for } n \geq 2.$$

The first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad \dots$$

and it follows from this pattern that the general formula for the n th term of a GP is:

12 **THE n TH TERM OF A GP:** $T_n = ar^{n-1}$, for $n \geq 1$.

WORKED EXERCISE: The sequence 6, 18, 54, 162, 486, 1458, 4374, ... at the start of this section is a GP with first term $a = 6$ and common ratio $r = 3$,

$$\begin{aligned}\text{so } T_n &= ar^{n-1} \\ &= 6 \times 3^{n-1}.\end{aligned}$$

For example, $T_6 = 6 \times 3^5 = 1458$,

and $T_{15} = 6 \times 3^{14}$ (the large number is best left factored).

Negative Ratios and Alternating Signs: The sequence 6, -18, 54, -162, ... formed by alternating the signs of the previous sequence is also a GP — its first term is still $a = 6$ but its ratio is $r = -3$. The repeated multiplication by -3 makes the terms alternate in sign.

WORKED EXERCISE: Find T_n , T_6 and T_{15} for 6, -18, 54, -162, ...

SOLUTION:

The sequence is a GP
with $a = 6$ and $r = -3$,

$$\begin{aligned}\text{so } T_n &= ar^{n-1} \\ &= 6 \times (-3)^{n-1}.\end{aligned}$$

$$\begin{aligned}\text{Also, } T_6 &= (-3)^5 \times 6 \\ &= -1458,\end{aligned}$$

$$\begin{aligned}\text{and } T_{15} &= (-3)^{14} \times 6 \\ &= 6 \times 3^{14}.\end{aligned}$$

A Condition for Three Numbers to be in AP or GP: Three numbers T_1 , T_2 , T_3 form an AP when the differences $T_3 - T_2$ and $T_2 - T_1$ are equal. Similarly, they form a GP when the ratios T_3/T_2 and T_2/T_1 are equal.

13

CONDITION FOR AN AP: $T_3 - T_2 = T_2 - T_1$

CONDITION FOR A GP: $\frac{T_3}{T_2} = \frac{T_2}{T_1}$

WORKED EXERCISE: Find the value of x such that 3, $x + 4$ and $x + 10$ form:

(a) an arithmetic sequence,

(b) a geometric sequence.

SOLUTION:

$$\begin{aligned}\text{(a) Put } T_3 - T_2 &= T_2 - T_1 \\ (x + 10) - (x + 4) &= (x + 4) - 3 \\ 6 &= x + 1, \\ \text{so } x &= 5, \\ \text{and the numbers are } 3, 9 &\text{ and } 15.\end{aligned}$$

$$\begin{aligned}\text{(b) Put } \frac{x + 10}{x + 4} &= \frac{x + 4}{3} \\ 3(x + 10) &= (x + 4)^2 \\ x^2 + 5x - 14 &= 0 \\ (x + 7)(x - 2) &= 0, \\ \text{so } x &= 2, \text{ giving } 3, 6 \text{ and } 12, \\ \text{or } x &= -7, \text{ giving } 3, -3 \text{ and } 3.\end{aligned}$$

Further Problems: The first example below uses elimination to solve simultaneous equations, but takes the ratio rather than the difference of the two equations. The second shows the solution of an inequality involving indices.

WORKED EXERCISE: Find the first term a and the common ratio r of a GP in which the fourth term is 30 and the sixth term is 480.

$$\text{SOLUTION: Since } T_4 = 30, ar^3 = 30 \quad (1)$$

$$\text{and since } T_6 = 480, ar^5 = 480 \quad (2)$$

$$\text{Dividing (2) by (1), } r^2 = 16,$$

$$\text{so } r = 4 \text{ and } a = \frac{15}{32}, \text{ or } r = -4 \text{ and } a = -\frac{15}{32}.$$

WORKED EXERCISE: [A harder example] (a) Show that the sequence whose terms are 1000, 400, 160, ... forms a GP, and then find the formula for the n th term.
 (b) Find the first term less than $\frac{1}{1000}$.

SOLUTION:

(a) Since $\frac{400}{1000} = \frac{2}{5}$
 and $\frac{160}{400} = \frac{2}{5}$,
 it is a GP with $a = 1000$ and $r = \frac{2}{5}$,
 so $T_n = ar^{n-1}$
 $= 1000 \times (\frac{2}{5})^{n-1}$.

(b) Put $T_n < \frac{1}{1000}$.
 Using the calculator,
 $T_{16} = 0.00107\dots$
 and $T_{17} = 0.0004\dots$,
 so the first term less than 0.001 is

$$T_{17} = 1000 \times (\frac{2}{5})^{16} \\ \doteq 0.000429.$$

Alternatively, the inequation can be solved analytically using logarithms:

$$\begin{aligned} \text{Put } T_n &< \frac{1}{1000}. \\ \text{Then } 1000 \times (\frac{2}{5})^{n-1} &< \frac{1}{1000} \\ (\frac{2}{5})^{n-1} &< \frac{1}{1\,000\,000} \\ (\frac{5}{2})^{n-1} &> 1\,000\,000 \\ n-1 &> \log_{\frac{5}{2}} 1\,000\,000 \\ n-1 &> \frac{\log_{10} 1\,000\,000}{\log_{10} 2\frac{1}{2}} \\ n-1 &> 15.07\dots \\ n &> 16.07\dots \end{aligned}$$

Hence the first term less than $\frac{1}{1000}$ is
 $T_{17} = 1000 \times (\frac{2}{5})^{16} \doteq 0.000429$.

Exercise 6E

- Find the first four terms, and the formula for the n th term, of the GP with:

(a) $a = 1$ and $r = 3$	(c) $a = 18$ and $r = \frac{1}{3}$	(e) $a = 1$ and $r = \sqrt{2}$
(b) $a = 5$ and $r = -2$	(d) $a = 6$ and $r = -\frac{1}{2}$	(f) $a = -7$ and $r = -1$
- Find $\frac{T_3}{T_2}$ and $\frac{T_2}{T_1}$ to test whether each sequence is a GP. If it is, write down the common ratio, find T_6 , then find a formula for the n th term:

(a) 10, 20, 40, ...	(c) 64, 81, 100, ...	(e) $\frac{3}{4}, 3, 12, \dots$
(b) 180, 60, 20, ...	(d) 35, 50, 65, ...	(f) $-24, -6, -1\frac{1}{2}, \dots$
- Find the common ratio, find a formula for T_n , and find T_6 :

(a) 1, -1, 1, ...	(c) -8, 24, -72, ...	(e) -1024, 512, -256, ...
(b) -2, 4, -8, ...	(d) 60, -30, 15, ...	(f) $\frac{3}{8}, -\frac{9}{2}, 54, \dots$
- Use the formula $T_n = ar^{n-1}$ to find r for a GP where:

(a) $a = 3$ and $T_6 = 96$	(c) $a = 486$ and $T_5 = \frac{2}{27}$	(e) $a = 1000$ and $T_7 = 0.001$
(b) $a = 1$ and $T_5 = 81$	(d) $a = 32$ and $T_6 = -243$	(f) $a = 5$ and $T_7 = 40$

DEVELOPMENT

- Use the formula $T_n = ar^{n-1}$ to find a and r for a GP with:

(a) $T_3 = 1$ and $T_6 = 64$	(c) $T_9 = 24$ and $T_5 = 6$
(b) $T_2 = \frac{1}{3}$ and $T_6 = 27$	(d) $T_7 = 2\sqrt{2}$ and $T_{12} = \frac{1}{16}\sqrt{2}$
- Find the n th term of each GP:

(a) $\sqrt{6}, 2\sqrt{3}, 2\sqrt{6}, \dots$	(b) $ax, a^2x^3, a^3x^5, \dots$	(c) $-x/y, -1, -y/x, \dots$
- The n th term of a geometric sequence is $T_n = 25 \times 2^n$.

(a) Write out the first six terms, and hence find the values of a and r .

- (b) Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.
 (c) Prove that $T_9 \times T_{11} = 25T_{20}$. (d) Which term of the sequence is 6400?
 (e) Verify that $T_{11} = 51\,200$ is the last term less than 100 000, and $T_{12} = 102\,400$ is the first term greater than 100 000.
 (f) Find which terms are between 1000 and 100 000, and how many of them there are.
8. Find x and the common difference or ratio, if these form (i) an AP, (ii) a GP:
 (a) $x, 24, 96$ (b) $24, x, 96$ (c) $x-4, x+1, x+11$ (d) $x-2, x+2, 5x-2$
9. Find T_n for each GP, then find how many terms there are:
 (a) 7, 14, 28, ..., 224 (b) 2, 14, 98, ..., 4802 (c) $\frac{1}{25}, \frac{1}{5}, 1, \dots, 625$
10. Use logs to find how many terms in each of the previous sequences are less than 1 000 000.
11. How many terms are between 1000 and 1 000 000 in the sequences in question 9?
12. [Compound interest and GPs] A principal $\$P$ is invested at 7% per annum compound interest. Let A_n be the total amount at the end of n years.
 (a) Write down A_1, A_2 and A_3 .
 (b) Show that the total amount at the end of n years forms a GP with first term $1.07 \times P$ and ratio 1.07, and find the n th term A_n .
 (c) How many full years does it take for the amount to double, and how many years does it take for it to become ten times the original principal?
13. Find T_n for each GP, then use logs to find how many terms exceed 10^{-6} :
 (a) 98, 14, 2, ... (b) 25, 5, 1, ... (c) 1, 0.9, 0.81, ...
14. [Depreciation and GPs] A car originally costs $\$20\,000$, then at the end of every year, it is worth only 80% of what it was worth a year before. Let W_n be its worth at the end of n years.
 (a) Write down expressions for W_1, W_2 and W_3 , and find a formula for W_n .
 (b) Find how many complete years it takes for the value to fall below $\$2000$.
15. When light passes through one sheet of very thin glass, its intensity is reduced by 3%. What is the minimum number of sheets that will reduce the intensity below 1%?
16. (a) Find a formula for T_n in $2x, 2x^2, 2x^3, \dots$, then find x given that $T_6 = 2$.
 (b) Find a formula for T_n in $x^4, x^2, 1, \dots$, then find x given that $T_6 = 3^6$.
 (c) Find a formula for T_n in $2^{-16}x, 2^{-12}x, 2^{-8}x, \dots$, then find x given that $T_6 = 96$.
17. (a) Find a and b if $a, b, 1$ forms a GP, and $b, a, 10$ forms an AP.
 (b) Find a and b if $a, 1, a+b$ forms a GP, and $b, \frac{1}{2}, a-b$ forms an AP.
 (c) Find the first term of the AP with common difference -7 in which $T_{10} = 3$.
 (d) Find the first term of the GP with common ratio 2 in which $T_6 = 6$.
 (e) Find a and d of the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.
 (f) Find a and r of the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.
18. (a) Show that if the first, second and fourth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the positive integer multiples of the first term.
 (b) Show that if the first, second and fifth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the odd positive integer multiples of the first term.

- (c) Find the common ratio of the GP in which the first, third and fourth terms form an arithmetic sequence. [HINT: $r^3 - 2r^2 + 1 = (r - 1)(r^2 - r - 1)$]
- (d) Find the GP in which each term is one more than the sum of all the previous terms.
19. (a) Show that $2^5, 2^2, 2^{-1}, 2^{-4}, \dots$ is a GP, and find its n th term.
 (b) Show that $\log_2 96, \log_2 24, \log_2 6, \dots$ is an AP, and show that $T_n = 7 - 2n + \log_2 3$.
20. [The relationship between APs and GPs]
 (a) Suppose $T_n = a + (n - 1)d$ is an AP with first term a and difference d . Show that the sequence $U_n = 2^{T_n}$ is a GP, and find its first term and ratio.
 (b) Suppose $T_n = ar^{n-1}$ is a GP with first term a and ratio r . Show that the sequence $U_n = \log_2 T_n$ is an AP, and find its first term and difference.
 (c) Does the base have to be 2?
21. [GPs are essentially exponential functions.]
 (a) Show that if $f(x) = kb^x$ is any exponential function, then the sequence $T_n = kb^n$ is a GP, and find its first term and common ratio.
 (b) Conversely, if T_n is a GP with first term a and ratio r , find the exponential function $f(x)$ such that $T_n = f(n)$.
 (c) Plot on the same axes the points of the GP $T_n = 2^{4-n}$ and the graph of the continuous function $y = 2^{4-x}$.

EXTENSION

22. [Products and sums of GPs] Suppose that $T_n = ar^{n-1}$ and $U_n = AR^{n-1}$ are two GPs.
 (a) Show that the sequence $V_n = T_n U_n$ is a GP, and find its first term and common ratio.
 (b) Show that the sequence $W_n = T_n + U_n$ is a GP if and only if $r = R$. [HINT: The condition for W_n to be a GP is $W_n W_{n+2} = W_{n+1}^2$ — substitute into this condition, and deduce that $(R - r)^2 = 0$.]
23. [The set of all GPs] Let $\mathcal{G}(a, r)$ represent the GP whose first term is a and ratio is r .
 (a) The *product* of two sequences T_n and U_n is defined to be the sequence whose n th term is $T_n U_n$. Show that for all positive constants λ and μ , and for all non-zero a_1, a_2, r_1 and r_2 , the sequence $\mathcal{G}(a_1, r_1)^\lambda \mathcal{G}(a_2, r_2)^\mu$ is a GP, and find its first term and common difference.
 (b) Write out the sequences $\mathcal{G}(2, 1)$ and $\mathcal{G}(1, 2)$. Show that any GP $\mathcal{G}(a, r)$ with first term a and ratio r can be written in the form $\mathcal{G}(2, 1)^\lambda \mathcal{G}(1, 2)^\mu$, and find the values of λ and μ .

6F Arithmetic and Geometric Means

What number x should be placed between 3 and 12 to make a satisfactory pattern 3, x , 12? There are three obvious answers to this question:

$$3, 7\frac{1}{2}, 12 \quad \text{and} \quad 3, 6, 12 \quad \text{and} \quad 3, -6, 12.$$

The number $7\frac{1}{2}$ makes the sequence an AP and is called the *arithmetic mean* of 3 and 12. Notice that $7\frac{1}{2}$ is calculated by taking half the sum of 3 and 12.

The numbers 6 and -6 each make the sequence a GP, with ratio 2 and -2 respectively, and are both called *geometric means* of 3 and 12. Notice that the numbers 6 and -6 can easily be calculated, being the positive and negative square roots of the product 36 of 3 and 12.

Definition — Arithmetic and Geometric Means: Suppose a and b are two numbers.

ARITHMETIC MEAN: The *arithmetic mean* (AM) of a and b is the number x such that a, x, b forms an AP. Then $b - x = x - a$, so $2x = a + b$, giving

$$\text{AM} = \frac{1}{2}(a + b).$$

14 GEOMETRIC MEAN: A *geometric mean* (GM) of a and b is a number x such that a, x, b forms a GP. Then $\frac{b}{x} = \frac{x}{a}$, so $x^2 = ab$, giving

$$\text{GM} = \sqrt{ab} \text{ or } -\sqrt{ab}.$$

Geometric Interpretation of the Means: Arithmetic and geometric means occur often in geometry. The diagram on the right is particularly interesting in that it illustrates both means.

Let a and b be two given lengths. Construct the interval AXB with $AX = a$ and $XB = b$. Construct the midpoint O of AB , and construct the circle with diameter AB . Construct the chord through X perpendicular to AB , meeting the circle at P and Q .

First, the radius AO is the arithmetic mean of a and b , because $AO = \frac{1}{2}(AX + XB)$. Secondly, by circle geometry,

$$AX \times XB = PX \times XQ$$

because the chords AB and PQ intersect at X . So, since $PX = XQ$, it follows that $PX^2 = a \times b$, and hence PX is the geometric mean of a and b .

The semichord PX cannot exceed the radius AO . This gives a geometric proof of the following important inequality (not explicitly part of the course).

THEOREM: The GM of two positive numbers cannot exceed their AM.

Inserting More than One Mean: One can also insert several terms in arithmetic or geometric sequence between two given numbers. This process is called *inserting arithmetic or geometric means*. It should be done by forming an AP or GP with the given numbers as first and last terms.

WORKED EXERCISE: (a) Insert four numbers in arithmetic sequence (that is, insert four arithmetic means) between 10 and 30. (b) Insert three numbers in geometric sequence (geometric means) between 10 and 40.

SOLUTION:

(a) Form an AP with $a = 10$
and $T_6 = 30$.

$$\text{Then } a + 5d = 30$$

$$5d = 20$$

$$d = 4,$$

so the means are 14, 18, 22 and 26.

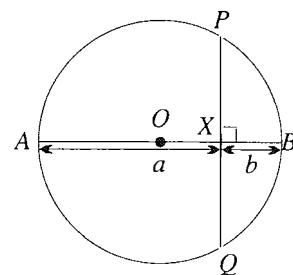
(b) Form a GP with $a = 10$ and $T_5 = 40$.

$$\text{Then } ar^4 = 40$$

$$r^4 = 4$$

$$r = \sqrt{2} \text{ or } -\sqrt{2},$$

so the means are $10\sqrt{2}$, 20 and $20\sqrt{2}$,
or $-10\sqrt{2}$, 20 and $-20\sqrt{2}$.



Exercise 6F

- Find the arithmetic and geometric means of the following pairs of numbers:

(a) 4 and 16	(d) 10 and -40	(g) a and $-a$	(j) 2^4 and 2^7
(b) 16 and 25	(e) $1\frac{1}{2}$ and 6	(h) 1 and a	(k) a^3 and a^5
(c) -5 and -20	(f) a^2 and $49a^2$	(i) 2^4 and 2^6	(l) x^{-3} and x^3
- Find the value of x , then write out the three numbers, if:

(a) 5 is the AM of $x - 3$ and $2x + 7$,	(c) $x - 1$ is the GM of $x - 3$ and $x + 4$,
(b) x is the AM of $3x - 2$ and $x + 10$,	(d) 2 is the GM of $2 - x$ and $5 - x$.
- Insert four numbers in arithmetic sequence between 7 and 42.
 - Insert two numbers in geometric sequence between 27 and 8.
 - Insert nine arithmetic means between 40 and 5.
 - Insert five geometric means between 1 and 1000.
- Find a , b and c such that 3, a , b , c , 48 is: (a) an AP, (b) a GP.

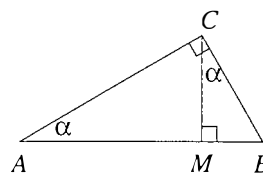
DEVELOPMENT

- Find the arithmetic and geometric means of the following pairs of numbers:

(a) $\sqrt{5} + 1$ and $\sqrt{5} - 1$	(d) $(x - y)^2$ and $(x + y)^2$	(g) $\log_2 3$ and $\log_2 27$
(b) $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{8}}$	(e) $\frac{1}{x - y}$ and $\frac{1}{x + y}$	(h) $\log_b 4$ and $\log_b 256$
(c) $x - y$ and $x + y$	(f) $\log_2 3$ and $\log_2 81$	(i) $\frac{1}{\sqrt{5} + 1}$ and $\frac{1}{\sqrt{5} - 1}$
- Find the arithmetic mean and geometric mean of 0.2 and 0.000 02.
 - Insert three arithmetic means and three geometric means between 0.2 and 0.000 02.
- Suppose x and y are positive numbers.
 - Find the arithmetic mean and the positive geometric mean of $\frac{y}{x}$ and $\frac{x}{y}$.
 - Show that the difference between the two means is $\frac{(x - y)^2}{2xy}$.
 - What is the condition on x and y for the two means to be equal?
- Show that if a and b have opposite signs, then they do not have a geometric mean.
 - If a and b have opposite signs, what determines the sign of the arithmetic mean?
 - Three nonzero numbers form both an AP and a GP. Prove that they are all equal. [HINT: Let the numbers be $x - d$, x and $x + d$, and prove that $d = 0$.]
 - Show that the fourth term of an AP is the arithmetic mean of the first and seventh terms.
 - Show that the fourth term of a GP is a geometric mean of the first and seventh terms.
 - Show that if the fifth term of an AP is a geometric mean of the third and eighth terms, then the seventh term is a geometric mean of the third and fifteenth terms.
- [The relationship between arithmetic means and geometric means]
 - Show that if m is the arithmetic mean of a and b , then 3^m is the geometric mean of 3^a and 3^b .
 - Show that if m is the positive geometric mean of a and b , then $\log_3 m$ is the arithmetic mean of $\log_3 a$ and $\log_3 b$.
- [An algebraic proof of the AM/GM inequality] Suppose a and b are two positive numbers.
 - Expand $(a - b)^2$.
 - Use the fact that $(a - b)^2$ cannot be negative to prove that the arithmetic mean of a and b is never less than the geometric mean.
 - When are the two means equal?

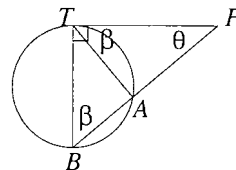
11. [The altitude to the hypotenuse of a right-angled triangle] Let $\triangle ABC$ be right-angled at C . Let CM be the altitude from C to the side AB .

- (a) Show that $\triangle ABC \sim \triangle ACM \sim \triangle CBM$.
 (b) Show that CM is the geometric mean of AM and BM .
 (c) Show that BC is the geometric mean of AB and BM .



12. [The tangent as the GM of two secants] Let PT be a tangent from a point P outside a circle, touching the circle at T . Let PAB be a secant through P meeting the circle at A and B .

- (a) Show that $\triangle PTA \sim \triangle PBT$ (recall the alternate segment theorem that tells us that $\angle TBP = \angle PTA$).
 (b) Use this similarity to show that PT is the geometric mean of PA and PB .



13. [Arithmetic means and midpoints] Suppose A and B are two distinct points, and M is the midpoint of AB .

- (a) Suppose P is any point between A and B . Explain why AM is the arithmetic mean of AP and PB .
 (b) Suppose now that P lies on AB , but beyond B . Where is the point X so that AX is the arithmetic mean of AP and PB ?



14. Let $\triangle ABC$ be right-angled at C , so $a^2 + b^2 = c^2$. Find the ratio $c : a$ if:

- (a) b is the AM of a and c , (b) [The golden mean] b is the GM of a and c .

15. [Geometric means in musical instruments] The pipe length in a modern rank of organ pipes decreases from left to right in such a way that the lengths form a GP, and the thirteenth pipe along is exactly half the length of the first pipe (making an interval called an *octave*).

- (a) Show that the ratio of the GP is $r = (\frac{1}{2})^{\frac{1}{12}}$.
 (b) Show that the 8th pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).
 (c) Show that the 5th pipe along is just under four-fifths the length of the first pipe (a *major third*).
 (d) Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.
 (e) What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

EXTENSION

16. [The golden mean] (a) The point M divides the interval AB in the ratio $1 : \lambda$ in such a way that AM is the geometric mean of BM and BA . Find λ , and draw a diagram.
 (b) The point M divides the interval AB externally in the ratio $1 : \lambda$ in such a way that AB is the geometric mean of AM and BM . Find λ , and draw a diagram.
17. (a) Let $A(a, 2^a)$, $M(m, 2^m)$ and $B(b, 2^b)$ be three points on the curve $y = 2^x$. (i) Show that the x -coordinates form an AP if and only if the y -coordinates form a GP.
 (ii) Sketch $y = 2^x$, then use the fact that the chord AB lies above the curve $y = 2^x$

to show that the geometric mean of two distinct positive numbers is less than their arithmetic mean.

- (b) Let A , M and B be three points on the curve $y = \log_2 x$. Show that the x -coordinates form a GP if and only if the y -coordinates form an AP. Sketch $y = \log_2 x$, then use the fact that the chord AB lies below the curve $y = \log_2 x$ to show that the GM of two distinct positive numbers is less than their AM.
18. Suppose a and b are positive numbers, with $a < b$, and let m and g be the arithmetic and positive geometric means respectively of a and b .
- (a) Show that g is the arithmetic mean of a and m if and only if $b = 9a$.
- (b) Show that g is closer to a than to m if and only if $b > 9a$.
19. (a) Using the fact that the GM of two numbers cannot exceed their AM, prove that if a , b , c and d are any four positive numbers, then $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.
- (b) By letting $d = (abc)^{\frac{1}{3}}$ in part (a), prove that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

6G Sigma Notation

We turn now to the problem of adding up some of the terms of a sequence. For example, we may want to evaluate the sum

$$1 + 4 + 9 + \cdots + 100$$

of the first ten positive square numbers. The purpose of this section is to introduce a concise notation for such sums, called *sigma notation*.

Sigma Notation: The notation for the sum above is

$$\sum_{n=1}^{10} n^2 = 1 + 4 + 9 + \cdots + 81 + 100 = 385,$$

which says ‘evaluate the function n^2 for all the integers from $n = 1$ to $n = 10$, then add up the resulting values’, giving the final answer 385. More generally, if k and ℓ are integers and T_n is defined for all integers from $n = k$ to $n = \ell$, then:

15 DEFINITION: $\sum_{n=k}^{\ell} T_n = T_k + T_{k+1} + T_{k+2} + \cdots + T_{\ell}$

The symbol $\sum$ used here is a large version of the Greek capital letter called ‘sigma’, which is pronounced ‘s’. It stands for the word ‘sum’.

WORKED EXERCISE: Evaluate: (a) $\sum_{n=4}^7 (5n+1)$ (b) $\sum_{n=1}^5 3 \times (-2)^n$.

SOLUTION:

(a) $\sum_{n=4}^7 (5n+1) = 21 + 26 + 31 + 36 = 114$

(b) $\sum_{n=1}^5 3 \times (-2)^n = -6 + 12 - 24 + 48 - 96 = -66$

WORKED EXERCISE: Express the sum $\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12}$ in sigma notation.

SOLUTION: $\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} = \sum_{n=1}^5 \frac{1}{2n+2}, \text{ or } \sum_{n=2}^6 \frac{1}{2n}, \text{ or } \sum_{n=3}^7 \frac{1}{2n-2}.$

There are many answers, depending on the initial value of n .

Exercise 6G

1. Rewrite each sum without sigma notation, and evaluate:

(a) $\sum_{n=1}^6 (3n+2)$	(d) $\sum_{n=1}^5 (n^2 - n)$	(g) $\sum_{k=4}^6 5^{k-4}$	(j) $\sum_{\ell=1}^{31} (-1)^\ell$
(b) $\sum_{n=1}^5 n^2$	(e) $\sum_{n=5}^{105} 4$	(h) $\sum_{n=0}^4 (-1)^n n^2$	(k) $\sum_{\ell=1}^{31} (-1)^{\ell-1}$
(c) $\sum_{n=-2}^2 n(n+1)$	(f) $\sum_{n=-4}^4 n^3$	(i) $\sum_{n=0}^4 (-1)^{n+1} n^2$	(l) $\sum_{a=1}^4 (3^a - 3^{a-1})$

DEVELOPMENT

2. Rewrite each sum in sigma notation, starting each sum at $n = 1$ (do not evaluate):

(a) $1^3 + 2^3 + 3^3 + \cdots + 40^3$	(f) $a + ar + ar^2 + \cdots + ar^{k-1}$
(b) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40}$	(g) $a + (a+d) + (a+2d) + \cdots + (a+(k-1)d)$
(c) $3 + 4 + 5 + \cdots + 22$	(h) $-1 + 2 - 3 + \cdots + 10$
(d) $2 + 2^2 + 2^3 + \cdots + 2^{12}$	(i) $1 - 2 + 3 - \cdots - 10$
(e) $1 + 2 + 2^2 + \cdots + 2^{12}$	(j) $1 - x + x^2 - x^3 + \cdots + x^{2k}$

3. (a) By writing out the terms, show that $\sum_{r=1}^6 r^3 = \sum_{t=5}^{10} (t-4)^3$.

(b) Show similarly by writing out the terms that $\sum_{k=1}^5 \frac{3k-1}{k+2} = \sum_{k=4}^8 \frac{3k-10}{k-1}$.

(c) Write $1 + 4 + 7 + \cdots + 19$ as: (i) $\sum_{n=0}^6 \cdots$ (ii) $\sum_{n=2}^8 \cdots$ (iii) $\sum_{n=7}^{\cdots} \cdots$

4. Write out the terms of $\sum_{r=1}^{10} \left(\frac{1}{r} - \frac{1}{r+1} \right)$, and hence show that the sum is $\frac{10}{11}$.

EXTENSION

5. (a) Show that $\frac{1}{(\sqrt{k} + \sqrt{k+1})(\sqrt[4]{k} + \sqrt[4]{k+1})} = \sqrt[4]{k+1} - \sqrt[4]{k}$.

(b) Hence evaluate $\sum_{k=1}^{255} \frac{1}{(\sqrt{k} + \sqrt{k+1})(\sqrt[4]{k} + \sqrt[4]{k+1})}$.

6. Evaluate: (a) $\frac{1}{2} \sum_{r=1}^4 \left(\frac{1}{2} \sum_{s=1}^4 \left(\frac{1}{2} \sum_{t=1}^4 rst \right) \right)$ (b) $\sum_{r=1}^4 \left(\sum_{s=1}^4 \left(\sum_{t=1}^4 (r-s)(s-t)(t-r) \right) \right)$

(c) $\sum_{n=1}^6 \left(\prod_{k=1}^n k \right)$, where $\prod_{n=k}^{\ell} u_n = u_k \times u_{k+1} \times \cdots \times u_{\ell}$ (d) $\prod_{n=1}^6 \left(\sum_{k=1}^n k \right)$

6H Partial Sums of a Sequence

The n th partial sum S_n of a sequence $T_1, T_2, T_3, T_4, \dots$ is the sum of the first n terms.

16 THE n TH PARTIAL SUM: $S_n = T_1 + T_2 + T_3 + \dots + T_n$

For example, the first ten partial sums of the sequence 1, 2, 4, 8, ... are:

T_n	1	2	4	8	16	32	64	128	256	512	...
S_n	1	3	7	15	31	63	127	255	511	1023	...

There may be a simple formula for the n th partial sum — in this example it should be reasonably clear that

$$S_n = 2^n - 1.$$

For most sequences, however, it will require somewhat more effort than this to arrive at the formula for the n th partial sum. Notice that the partial sums form a second sequence $S_1, S_2, S_3, \dots$, although this will usually not concern us explicitly.

Recovering the Sequence from the Partial Sums: The partial sums S_n have a very simple recursive definition as follows:

$$S_1 = T_1 \quad \text{and} \quad S_n = S_{n-1} + T_n, \text{ for } n \geq 2,$$

because each partial sum is just the previous partial sum plus the next term. Rearranging these equations so that T_1 and T_n are the subjects gives a formula for T_n .

17 RECOVERING THE SEQUENCE: $T_1 = S_1$ and $T_n = S_n - S_{n-1}$, for $n \geq 2$.

The formula $T_n = S_n - S_{n-1}$ should also be understood as a subtraction:

$$T_n = (T_1 + T_2 + \dots + T_n) - (T_1 + T_2 + \dots + T_{n-1}).$$

These equations allow the original sequence to be recovered from the partial sums.

WORKED EXERCISE: Given that $S_n = n^2$, find a formula for the n th term.

SOLUTION: For $n \geq 2$, $T_n = S_n - S_{n-1}$
 $= n^2 - (n-1)^2$
 $= 2n - 1.$

Also $T_1 = S_1 = 1$, so T_1 satisfies this formula.

So $T_n = 2n - 1$, for all $n \geq 1$.

Since $T_n = 2n - 1$ is the formula for the n th odd cardinal, this particular example establishes the following well-known and important result (not an explicit part of our course).

THEOREM: The sum of the first n odd cardinals is n^2 :

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2, \text{ for } n \geq 1.$$

Series: The word *series* is a rather imprecise term, but it always refers to the activity of adding up terms of a sequence. For example,

‘the series $1 + 4 + 9 + \cdots + 81 + 100$ ’

means the expression giving the sum of the first 10 terms of the sequence of positive squares, and the value of this series is 385. One can also speak of

‘the series $1 + 4 + 9 + \cdots$ ’,

which means that one is considering the sequence of positive squares and their successive partial sums. In practice, the words ‘series’ and ‘sequence’ tend to be used interchangeably.

Exercise 6H

1. Copy and complete these tables of a sequence and its partial sums. Then describe each sequence:

(a)

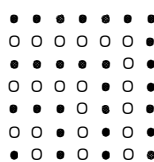
T_n	2	5	8	11	14	17	20
S_n							

(b)

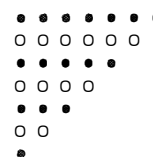
T_n							
S_n	2	6	14	30	62	126	254

2. The maximum numbers of electrons in the successive electron shells of an atom are 2, 8, 18, 32, ... By taking successive differences, make sense of these numbers as the partial sums of a simple series.
3. The n th partial sum of a series is $S_n = n^2 + 2n$. (a) Write out the first five partial sums. (b) Take differences to write out the first five terms of the original sequence. (c) Find S_{n-1} , then use the result $T_n = S_n - S_{n-1}$ to find a formula for T_n .
4. Repeat the steps of the previous question for the sequence whose n th partial sum is:
- (a) $S_n = 4n - n^2$ (b) $S_n = 3n^2 - 5n$ (c) $S_n = 6n - 5n^2$

5. (a) Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2 .



- (b) Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.



DEVELOPMENT

6. The n th partial sum of a series is $S_n = 3^n - 1$. (a) Write out the first five partial sums. (b) Take differences to find the first five terms of the original sequence. (c) Find S_{n-1} , then use the result $T_n = S_n - S_{n-1}$ to find a formula for T_n .
[HINT: This will need the factorisation $3^n - 3^{n-1} = 3^{n-1}(3 - 1) = 2 \times 3^{n-1}$.]
7. Repeat the steps of the previous question for the sequence whose n th partial sum is:
- (a) $S_n = 10(2^n - 1)$ (b) $S_n = 4(5^n - 1)$ (c) $S_n = \frac{1}{4}(4^n - 1)$
[HINT: You will need factorisations such as $2^n - 2^{n-1} = 2^{n-1}(2 - 1)$.]
8. Find the n th term and the first three terms of the sequence for which S_n is:
- (a) $S_n = 3n(n+1)$ (e) $S_n = n^3$ (i) $S_n = \frac{1}{6}n(n+1)(2n+1)$
(b) $S_n = \frac{1}{2}n^2 + \frac{3}{2}n$ (f) $S_n = 1 - 3^{-n}$ (j) $S_n = \frac{1}{4}n^2(n+1)^2$
(c) $S_n = 5n - n^2$ (g) $S_n = (\frac{1}{7})^n - 1$ (k) $S_n = \frac{a(r^n - 1)}{r - 1}$
(d) $S_n = 4n$ (h) $S_n = \frac{1}{2}n(2a + (n-1)d)$

EXTENSION

9. In these sequences, the first term will not necessarily obey the same rule as the succeeding terms, in which case the formula for the sequence will need to be given piecewise:
 (a) $S_n = n^2 + 4n + 3$ (b) $S_n = 7(3^n - 4)$ (c) $S_n = \frac{1}{n}$ (d) $S_n = n^3 + n^2 + n$
 Find T_1 and a formula for T_n for each sequence. How could you have predicted whether or not the general formula would hold for T_1 ?
10. [Fibonacci and Lucas sequences] Examine the sequence of differences between successive terms of the Fibonacci and Lucas sequences.
11. (a) Write down the fourth powers of the positive integers, form the new sequence of differences between successive terms, repeat the process with the new sequence, and continue the process until the resulting sequence is constant. Why is the result 24? What happens when this process is applied to the sequence of some other fixed powers of the integers?
- (b) Apply this same repeated process to the sequence of positive integer powers of 2, or of 3, or of some other base. Examine the situation and justify what you observe.

6I Summing an Arithmetic Series

There is a clever way to add up the terms of an arithmetic series. Here is an example of adding up the first ten terms of the AP with $a = 4$ and $d = 5$:

$$S_{10} = 4 + 9 + 14 + 19 + 24 + 29 + 34 + 39 + 44 + 49.$$

Reversing the sum, $S_{10} = 49 + 44 + 39 + 34 + 29 + 24 + 19 + 14 + 9 + 4$,

and adding the two, $2S_{10} = 53 + 53 + 53 + 53 + 53 + 53 + 53 + 53 + 53 + 53$,

$$2S_{10} = 10 \times 53 \quad (53 \text{ is the sum of } T_1 = 4 \text{ and } T_{10} = 49).$$

Hence

$$S_{10} = \frac{1}{2} \times 10 \times 53 \\ = 265.$$

This process can be done just as well with the general arithmetic series. Let the first term be a , the common difference be d , and the last term T_n be ℓ :

$$S_n = a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.$$

Reversing the sum, $S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a$,

and adding, $2S_n = (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)$

$$2S_n = n(a + \ell) \quad (\text{there are } n \text{ terms}),$$

hence

$$S_n = \frac{1}{2}n(a + \ell).$$

Substituting $\ell = a + (n - 1)d$ gives a second equally useful form of this formula:

$$S_n = \frac{1}{2}n(2a + (n - 1)d).$$

Method for Summing an AP: The two formulae to remember are:

18

PARTIAL SUMS OF APS: $S_n = \frac{1}{2}n(a + \ell)$ (use when $\ell = T_n$ is known) $S_n = \frac{1}{2}n(2a + (n - 1)d)$ (use when d is known)

WORKED EXERCISE: Add up all the integers from 100 to 200 inclusive.

SOLUTION: The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms, in which the first term is $a = 100$ and the last term is $\ell = 200$.

$$\begin{aligned}\text{So } S_{101} &= \frac{1}{2}n(a + \ell) \\ &= \frac{1}{2} \times 101 \times 300 \\ &= 15\,150.\end{aligned}$$

WORKED EXERCISE: (a) Given the AP $40 + 37 + 34 + \cdots$, find S_{10} and an expression for S_n . (b) What is the first negative partial sum?

SOLUTION:

(a) Here $a = 40$ and $d = -3$, so $S_{10} = 5(2a + 9d)$ $= 5 \times (80 - 27)$ $= 265$, and $S_n = \frac{1}{2}n(2a + (n-1)d)$ $= \frac{1}{2}n(80 - 3(n-1))$ $= \frac{1}{2}n(83 - 3n)$.	(b) Put $S_n < 0$. Then $\frac{1}{2}n(83 - 3n) < 0$. Since n must be positive, $83 - 3n < 0$ $n > 27\frac{2}{3}$. S_{28} is the first negative partial sum, and by the formula, $S_{28} = -14$.

WORKED EXERCISE: The sum of the first ten terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION:

First, $S_{10} = 0$ $5(2a + 9d) = 0$ $2a + 9d = 0$. (1) Secondly, $a + (a + d) = 24$ $2a + d = 24$. (2)	(1) - (2) $8d = -24$ $d = -3$, so from (2), $2a - 3 = 24$ $a = 13\frac{1}{2}$. Hence the AP is $13\frac{1}{2} + 10\frac{1}{2} + 7\frac{1}{2} + \cdots$.

Exercise 6I

- Let $S_{10} = 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32$. By reversing the sum and adding in columns, evaluate S_{10} .
- Use the formula $S_n = \frac{1}{2}n(2a + (n-1)d)$ to find these sums:

(a) $2 + 5 + 8 + \cdots$ (12 terms)	(d) $33 + 30 + 27 + \cdots$ (23 terms)
(b) $40 + 33 + 26 + \cdots$ (21 terms)	(e) $-10 - 7\frac{1}{2} - 5 + \cdots$ (13 terms)
(c) $-6 - 2 + 2 + \cdots$ (200 terms)	(f) $10\frac{1}{2} + 10 + 9\frac{1}{2} + \cdots$ (40 terms)
- First use the formula $T_n = a + (n-1)d$ to find the number of terms in each sum. Then find the sum using the formula $S_n = \frac{1}{2}n(a + \ell)$, where ℓ is the last term T_n :

(a) $50 + 51 + 52 + \cdots + 150$	(d) $4 + 7 + 10 + \cdots + 301$
(b) $8 + 15 + 22 + \cdots + 92$	(e) $6\frac{1}{2} + 11 + 15\frac{1}{2} + \cdots + 51\frac{1}{2}$
(c) $-10 - 3 + 4 + \cdots + 60$	(f) $-1\frac{1}{3} + \frac{1}{3} + 2 + \cdots + 13\frac{2}{3}$
- Find these sums by any appropriate method:

(a) $2 + 4 + 6 + \cdots + 1000$	(c) $1 + 5 + 9 + \cdots$ (40 terms)
(b) $1000 + 1001 + \cdots + 3000$	(d) $10 + 30 + 50 + \cdots$ (12 terms)

5. Find the formula for the n th partial sums of the series:
- (a) $3 + 7 + 11 + \dots$ (c) $5 + 4\frac{1}{2} + 4 + \dots$
 (b) $-9 - 4 + 1 + \dots$ (d) $(1 - \sqrt{2}) + 1 + (1 + \sqrt{2}) + \dots$
6. Find formulae for the sums of the first n :
- (a) positive integers, (c) positive integers divisible by 3,
 (b) odd positive integers, (d) odd positive multiples of 100.
7. (a) How many legs are there on 15 fish, 15 ducks, 15 dogs, 15 beetles, 15 spiders, and 15 ten-legged grubs? How many of these creatures have the mean number of legs?
 (b) A school has 1560 pupils, with equal numbers of each age from 6 to 17 years inclusive. It also has 120 teachers and ancillary staff all aged 32 years, and one Principal aged 55 years. What is the total of the ages of everyone in the school?
 (c) A graduate earns \$28 000 per annum in her first year, then each successive year her salary rises by \$1600. What are her total earnings over ten years?

————— DEVELOPMENT —————

8. Find these sums:
- (a) $x + 2x + 3x + \dots + nx$
 (b) $3 + (3 + d) + (3 + 2d) + \dots$ (20 terms)
 (c) $a + (a - 5) + (a - 10) + \dots + (a - 100)$
 (d) $3b + 5b + 7b + \dots$ (200 terms)
 (e) $(1 + \sqrt{2}) + (2 + 3\sqrt{2}) + (3 + 5\sqrt{2}) + \dots$ (12 terms)
 (f) $\sqrt{12} + \sqrt{27} + \sqrt{48} + \dots + 21\sqrt{3}$
9. (a) Show that the n th partial sum of the series $60 + 52 + 44 + 36 + \dots$ is $S_n = 4n(16 - n)$.
 (b) Hence find how many terms must be taken to make the sum: (i) zero, (ii) negative.
 (c) Find the two values of n for which the partial sum S_n is 220.
 (d) Show that $S_n = -144$ has two integer solutions, but that only one has meaning.
 (e) For what values of n does the partial sum S_n exceed 156?
 (f) Prove that no partial sum can exceed 256.
 (g) Write out the first 16 terms and partial sums, and check your results.
10. (a) Prove that the sum of the first n positive integers is $S_n = \frac{1}{2}n(n + 1)$.
 (b) Find n if the sum is: (i) 6 (ii) 55 (iii) 820
 (c) How many terms must be taken for the sum to exceed 210?
 (d) Show that the sum can never be 50.
11. (a) Logs of wood are stacked with 10 on the top row, 11 on the next, and so on. If there are 390 logs, find the number of rows, and the number of logs on the bottom row.
 (b) A stone dropped from the top of a 245 metre cliff falls 5 metres in the first second, 15 metres in the second second, and so on in arithmetic sequence. Find a formula for the distance after n seconds, and find how long the stone takes to fall to the ground.
 (c) A truck spends the day depositing truckloads of gravel from a quarry at equally spaced intervals along a straight road. The first load is deposited 20 km from the quarry, the last is 10 km further along the road. If the truck travels 550 km during the day, how many trips does it make, and how far apart are the deposits?

12. Find the sums of these APs whose terms are logarithms:

- (a) $\log_a 2 + \log_a 4 + \log_a 8 + \cdots + \log_a 1024$
- (b) $\log_5 243 + \log_5 81 + \log_5 27 + \cdots + \log_5 \frac{1}{243}$
- (c) $\log_b 36 + \log_b 18 + \log_b 9 + \cdots + \log_b \frac{9}{8}$
- (d) $\log_x \frac{27}{8} + \log_x \frac{9}{4} + \log_x \frac{3}{2} + \cdots$ (10 terms)

13. (a) Find the common difference, if a series with 8 terms and first term 5 has sum 348.
 (b) Find the last term, if a series with 10 terms and first term -23 has sum -5 .
 (c) Find the first term, if a series with 40 terms and last term $8\frac{1}{2}$ has sum 28.
 (d) Find the first term, if a series with 15 terms and difference $\frac{2}{7}$ has sum -15 .
 (e) The sum of the first and fourth terms of an AP is 16, and the sum of the third and eighth terms is 4. Find the sum of the first ten terms.
 (f) The tenth partial sum of an AP is zero, and the tenth term is -9 . Find the first and second terms.
 (g) The sum to 16 terms of an AP is 96, and the sum of the second and fourth terms is 45. Find the fourth term, and show the sum to four terms is also 96.
14. (a) Prove that if the tenth and twentieth partial sums of an AP are equal, then the thirtieth partial sum must be zero.
 (b) Prove that if the twelfth partial sum of an AP is twice the sixth partial sum, then the sequence is a constant sequence.
 (c) Find the first term and common difference of an AP in which the sum to ten terms is three times the sum to four terms, and the 28th term is -81 .
 (d) Find n , if the sum of the first n terms of the series $48 + 44 + 40 + \cdots$ equals the sum of the first n terms of the series $-1 + 2 + 5 + \cdots$.
15. (a) Insert 9 arithmetic means between 29 and 109, then find their sum.
 (b) Show that the sum of n arithmetic means inserted between a and b is $\frac{1}{2}n(a + b)$.
 (c) Find n , if n arithmetic means inserted between 10 and 82 have sum 506.
 (d) How many arithmetic means must be inserted between 1 and 2 if their sum exceeds 1 000 000?

16. (a) Use the formula $S_n = \frac{1}{2}n(a + \ell)$ to simplify $S_n = \sum_{k=1}^n (44 - 2k)$, and find n if $S_n = 0$.

(b) Solve similarly: (i) $\sum_{k=1}^n (63 - 3k) = 0$ (ii) $\sum_{k=1}^n (39 + 6k) = 153$ (iii) $\sum_{r=1}^n (2 + \frac{1}{2}r) = 22\frac{1}{2}$

17. (a) (i) Find the sum of all positive multiples of 3 less than 300.
 (ii) Find the sum of all the other positive integers less than 300.
 (b) What is the sum of all numbers ending in 5 between 1000 and 2000?
 (c) What is the sum of all numbers ending in 2 or 9 between 1000 and 2000?
 (d) How many multiples of 7 lie between 250 and 2500, and what is their sum?

18. Find the first term and the number of terms if a series has:

- (a) $d = 4$, $\ell = 32$ and $S_n = 0$
- (b) $d = -3$, $\ell = -10$ and $S_n = 55$

19. (a) Find $1 + 2 + \cdots + 24$. (b) Show that $\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n} = \frac{n+1}{2}$.

(c) Hence find the sum of the first 300 terms of $\frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \cdots$.

EXTENSION

20. (a) Show that $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$, for all $r \geq 1$, and hence, by writing out the first few terms, evaluate $\sum_{r=1}^n \frac{1}{r(r+1)}$.

- (b) Use similar methods to evaluate $\sum_{r=1}^n \frac{1}{r(r+2)}$ and $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)}$.

6J Summing a Geometric Series

The method used to find a partial sum of an AP will not work for a GP. There is, however, another equally clever way available. Suppose that a GP has first term a and common ratio r .

$$\text{Let } S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying both sides by the ratio r ,

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$$

$$\text{Taking } (2) - (1), \quad (r-1)S_n = ar^n - a$$

$$\text{and provided } r \neq 1, \quad S_n = \frac{a(r^n - 1)}{r - 1}.$$

Taking opposites of numerator and denominator gives an alternative form:

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Method for Summing a GP: Both both forms of the formula are useful, depending on whether the ratio is greater or less than 1.

19

$$\begin{aligned} \text{PARTIAL SUMS OF GPS: } S_n &= \frac{a(r^n - 1)}{r - 1} \quad (\text{easier when } r > 1) \\ S_n &= \frac{a(1 - r^n)}{1 - r} \quad (\text{easier when } r < 1) \end{aligned}$$

WORKED EXERCISE:

- (a) Find the sum of all the powers of 5 from 5^0 to 5^7 .
 (b) Find the sixth partial sum of the GP $2 - 6 + 18 - \cdots$.

SOLUTION:

- (a) The sum $5^0 + 5^1 + \cdots + 5^7$ is a GP with 8 terms, with $a = 1$ and $r = 5$.
 (b) The series is a GP in which $a = 2$ and $r = -3$.

$$\begin{aligned} \text{So } S_8 &= \frac{a(r^8 - 1)}{r - 1} \quad (\text{here } r > 1) \\ &= \frac{1 \times (5^8 - 1)}{5 - 1} \\ &= 97\,656. \end{aligned}$$

$$\begin{aligned} \text{So } S_6 &= \frac{a(1 - r^6)}{1 - r} \quad (\text{here } r < 1) \\ &= \frac{2 \times (1 - (-3)^6)}{1 - (-3)} \\ &= -364. \end{aligned}$$

WORKED EXERCISE: [A harder example] How many terms of the GP $2+6+18+\dots$ must be taken for the partial sum to exceed one billion?

SOLUTION:

Here $a = 2$ and $r = 3$,
so the n th partial sum is

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{2(3^n - 1)}{3 - 1} \\ &= 3^n - 1. \end{aligned}$$

Put $S_n > 1\,000\,000\,000$.

Then $3^n - 1 > 1\,000\,000\,000$

$$3^n > 1\,000\,000\,001$$

$$n > \frac{\log_{10} 1\,000\,000\,001}{\log_{10} 3}$$

$$n > 18.86\dots,$$

so S_{19} is the first sum over one billion
(or use trial and error).

Two Exceptional Cases: There are two particular values of the common ratio that need special attention, namely 0 and 1. First, according to our definition of a GP, the ratio cannot ever be zero, for then the second and third terms would be zero and the quotient T_3/T_2 would be undefined.

Secondly, if the ratio is 1, then the formula above for S_n doesn't work, because the denominator $r - 1$ would be zero. All the terms, however, are equal to the first term a , and so the formula for the n th partial sum is just $S_n = an$.

Exercise 6J

- Let $S_7 = 2 + 6 + 18 + 54 + 162 + 486 + 1458$. By taking $3S_7$ and subtracting S_7 in columns, evaluate S_7 .
- 'As I was going to St Ives, I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits. Kits, cats, sacks and wives, how many were going to St Ives?' Only the speaker was going to St Ives, but how many were going the other way?
- Use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ (when $r > 1$) or $S_n = \frac{a(1 - r^n)}{1 - r}$ (when $r < 1$) to find these sums, then find a formula for the sum to n terms:

(a) $1 + 2 + 4 + 8 + \dots$ (10 terms)	(g) $9 + 3 + 1 + \dots$ (6 terms)
(b) $1 - 2 + 4 - 8 + \dots$ (10 terms)	(h) $9 - 3 + 1 - \dots$ (6 terms)
(c) $2 + 6 + 18 + \dots$ (5 terms)	(i) $45 + 15 + 5 + \dots$ (6 terms)
(d) $2 - 6 + 18 - \dots$ (5 terms)	(j) $-1 - 10 - 100 - \dots$ (5 terms)
(e) $8 + 4 + 2 + \dots$ (10 terms)	(k) $-1 + 10 - 100 + \dots$ (5 terms)
(f) $8 - 4 + 2 - \dots$ (10 terms)	(l) $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$
- Find an expression for S_n . Hence approximate S_{10} to four significant figures:

(a) $1 + 1.2 + (1.2)^2 + \dots$	(c) $1 + 1.01 + (1.01)^2 + \dots$
(b) $1 + 0.95 + (0.95)^2 + \dots$	(d) $1 + 0.99 + (0.99)^2 + \dots$
- The King takes a chessboard of 64 squares, and places 1 grain of wheat on the first square, 2 on the next, 4 on the next and so on.
 - How many grains are on: (i) the last square (ii) the whole chessboard?
 - Given that 1 litre of wheat contains about 30 000 grains, how many cubic kilometres of wheat are on the chessboard?

DEVELOPMENT

6. Find the sum to n terms of each series, where c , x and y are constants:
- (a) $cx + 3cx^2 + 9cx^3 + \dots$ (c) $cx - 3cx^2 + 9cx^3 - \dots$
 (b) $1 + \frac{1}{x} + \frac{1}{x^2} + \dots$ (d) $1 + \frac{x}{y} + \frac{x^2}{y^2} + \dots$
7. Find S_n and S_{10} , rationalising denominators:
- (a) $1 + \sqrt{2} + 2 + \dots$ (b) $\frac{1}{5} - \frac{1}{\sqrt{5}} + 1 - \dots$
8. (a) Find: (i) $\sum_{n=3}^8 3^{n-4}$ (ii) $\sum_{n=3}^8 \log_a 3^{n-4}$ (iii) $\sum_{n=1}^8 3 \times 2^{3-n}$
 (b) Insert 3 geometric means between $\frac{1}{8}$ and 162, then find their sum.
9. (a) Show that the n th partial sum of the series $7 + 14 + 28 + \dots$ is $S_n = 7(2^n - 1)$.
 (b) For what value of n is the partial sum equal to 1785?
 (c) Show that $T_n = 7 \times 2^{n-1}$, and find how many terms are less than 70 000.
 (d) Use trial and error to find the first partial sum greater than 70 000.
 (e) Prove that the n th partial sum is always 7 less than the $(n + 1)$ th term.
10. The powers of 3 greater than 1 form a GP 3, 9, 27, ...
 (a) Find using logs how many powers of 3 there are between 2 and 10^{20} .
 (b) Show that $S_n = \frac{3}{2}(3^n - 1)$, and find how many terms must be added for the sum to exceed 10^{20} .
11. (a) Each year when a paddock is weeded, only half the previous weight of weed is dug out. In the first year, 6 tonnes of weed is dug out. (i) How much is dug out in the tenth year? (ii) What is the total dug out over the ten years (to four significant figures)?
 (b) Every two hours, half of a particular medical isotope decays. If there was originally 20 g, how much remains after a day (to two significant figures)?
 (c) The price of shoes is increasing with inflation over a ten-year period by 10% per annum, so that the price in each of those ten years is $P, 1.1P, (1.1)^2P, \dots$. I buy one pair of these shoes each year.
 (i) Find an expression for the total amount I pay over the ten years.
 (ii) Hence find the initial price P (to the nearest cent) if the total paid is \$900.
12. The number of people attending the yearly Abletown Show is rising by 5% per annum, and the number attending the yearly Bush Creek Show is falling by 5% per annum. In the first year under consideration, 5000 people attended both shows.
 (a) Find the total number attending each show during the first six years.
 (b) Show that the number attending the Abletown Show first exceeds ten times the number attending the Bush Creek Show in the 25th year.
 (c) What is the ratio (to three significant figures) of the total number attending the Abletown Show over these 25 years to the total attending the Bush Creek Show?
13. Find the n th terms of the sequences:
- (a) $\frac{2}{1}, \frac{2+4}{1+3}, \frac{2+4+6}{1+3+5}, \dots$ (b) $\frac{1}{1}, \frac{1+2}{1+4}, \frac{1+2+4}{1+4+16}, \dots$

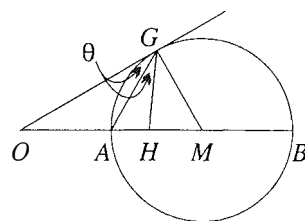
14. (a) Show that the n th partial sum of the series $4 - 12 + 36 - \dots$ is $S_n = (1 - (-3)^n)$.
 (b) For what value of n is the partial sum equal to -728 ?
 (c) What is the last term with absolute value less than 1 000 000?
 (d) Find the first partial sum with absolute value greater than 1 000 000.
15. Show that the formula for the n th partial sum of a GP can also be written independently of n , in terms only of a , r and the last term $\ell = T_n = ar^{n-1}$, as

$$S_n = \frac{r\ell - a}{r - 1} \quad \text{or} \quad S_n = \frac{a - r\ell}{1 - r}.$$

- (a) Hence find: (i) $1 + 2 + 4 + \dots + 1048576$ (ii) $1 + \frac{1}{3} + \frac{1}{9} + \dots + \frac{1}{2187}$
 (b) Find n and r if $a = 1$, $\ell = 64$, $S_n = 85$.
 (c) Find ℓ and n if $a = 5$, $r = -3$, $S_n = -910$.
16. (a) Show that in any GP, $S_{2n} : S_n = (r^n + 1) : 1$. Hence find the common ratio of the GP if $S_{12} : S_6 = 65 : 1$.
 (b) Show that if S_n and Σ_n are the sums to n terms of GPs with ratios r and r^2 respectively, but the same first term, then $\Sigma_n : S_n = (r^n + 1) : (r + 1)$.
 (c) In any GP, let $R_n = T_{n+1} + T_{n+2} + \dots + T_{2n}$. Show that $R_n : S_n = r^n : 1$, and hence find r if $R_8 : S_8 = 1 : 81$.
17. (a) The sequence $T_n = 2 \times 3^n + 3 \times 2^n$ is the sum of two GPs. Find S_n .
 (b) The sequence $T_n = 2n + 3 + 2^n$ is the sum of an AP and a GP. Use a combination of AP and GP formulae to find S_n .
 (c) It is given that the sequence 10, 19, 34, 61, ... has the form $T_n = a + nd + b2^n$, for some values of a , d and b . Find these values, and hence find S_n .

EXTENSION

18. Given a GP in which $T_1 + T_2 + \dots + T_{10} = 2$ and $T_{11} + T_{12} + \dots + T_{30} = 12$, find $T_{31} + T_{32} + \dots + T_{60}$.
19. Show that if n geometric means are inserted between 1 and 2, then their sum is given by $S_n = \frac{1}{2^{\frac{1}{n+1}} - 1} - 1$. Show that $S_n \rightarrow \infty$ as $n \rightarrow \infty$, and find how many means must be inserted for the sum to be at least 1000.
20. [The harmonic mean] The *harmonic mean* of two positive numbers a and b is the number h such that $1/h$ is the arithmetic mean of $1/a$ and $1/b$.
- (a) Show that $h = \frac{2ab}{a+b}$ and $\frac{b-h}{h-a} = \frac{b}{a}$.
- (b) Given a line $OAHB$, show that OH is the harmonic mean of OA and OB if and only if H divides AB internally in the same ratio as O divides AB externally.
- (c) Given a line OAB , construct the circle with diameter AB , construct the centre M , and construct a tangent from O touching the circle at G . Construct H between A and B so that $\angle OGA = \angle HGA$. Show that OM , OG and OH are respectively the arithmetic, geometric and harmonic means of OA and OB . [HINT: Use the sine rule to show that $OG : GH = OA : AH = OB : BH$.]



6K The Limiting Sum of a Geometric Series

There is a sad story of a perishing frog, dying of thirst only 8 metres from the edge of a waterhole. He first jumps 4 metres towards it, his second jump is 2 metres, then each successive jump is half the previous jump. Does the frog perish?

The jumps form a GP, whose terms and partial sums are as follows:

T_n	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...
S_n	4	6	7	$7\frac{1}{2}$	$7\frac{3}{4}$	$7\frac{7}{8}$	$7\frac{15}{16}$	...

The successive jumps have limit zero, meaning they get ‘as close as we like’ to zero. It seems too that the successive partial sums have limit 8, meaning that the frog’s total distance gets ‘as close as we like’ to 8 metres. So provided the frog can stick his tongue out even the merest fraction of a millimetre, eventually he will get some water to drink and be saved.

The General Case: Suppose now that T_n is a GP with first term a and ratio r , so that

$$T_n = ar^{n-1} \quad \text{and} \quad S_n = \frac{a(1-r^n)}{1-r}.$$

- A. When $r > 1$ or $r < -1$, then r^n increases in size without bound. This means that there is no limit for the n th term, and no limit for the n th partial sum. For example, if the ratio is 2 or -2 , then the terms and partial sums are:

For $r = 2$:

T_n	a	$2a$	$4a$	$8a$	$16a$	...
S_n	a	$3a$	$7a$	$15a$	$31a$	...

For $r = -2$:

T_n	a	$-2a$	$4a$	$-8a$	$16a$	...
S_n	a	$-a$	$3a$	$-5a$	$11a$	...

- B. When $r = 1$ the terms are all the same, and when $r = -1$ the terms have the same size but alternate in sign. Again the partial sums do not have a limit:

For $r = 1$:

T_n	a	a	a	a	a	...
S_n	a	$2a$	$3a$	$4a$	$5a$	...

For $r = -1$:

T_n	a	$-a$	a	$-a$	a	...
S_n	a	0	a	0	a	...

- C. When $-1 < r < 1$, however, $r^n \rightarrow 0$ as $n \rightarrow \infty$,
and so $1 - r^n \rightarrow 1$ as $n \rightarrow \infty$.
Hence as n tends to infinity, both the n th term T_n and the n th partial sum S_n tend to a limit, or as we also say, they *converge* to a limit:

$$\begin{aligned} \lim_{n \rightarrow \infty} T_n &= \lim_{n \rightarrow \infty} ar^{n-1} & \text{and} & & \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} \\ &= 0, & & & &= \frac{a}{1-r}. \end{aligned}$$

The new notation $\lim_{n \rightarrow \infty} T_n = 0$ means that $T_n \rightarrow 0$ as $n \rightarrow \infty$.

Similarly, $\lim_{n \rightarrow \infty} S_n = \frac{a}{1-r}$ means that $S_n \rightarrow \frac{a}{1-r}$ as $n \rightarrow \infty$.

For example, if $r = \frac{1}{2}$, then $\lim_{n \rightarrow \infty} S_n = \frac{a}{1 - \frac{1}{2}} = 2a$,

and if $r = -\frac{1}{2}$, then $\lim_{n \rightarrow \infty} S_n = \frac{a}{1 + \frac{1}{2}} = \frac{2}{3}a$:

For $r = \frac{1}{2}$:

$$\begin{array}{l|l} T_n & a \quad \frac{1}{2}a \quad \frac{1}{4}a \quad \frac{1}{8}a \quad \frac{1}{16}a \rightarrow 0 \\ \hline S_n & a \quad \frac{3}{2}a \quad \frac{7}{4}a \quad \frac{15}{8}a \quad \frac{31}{16}a \rightarrow 2a \end{array}$$

For $r = -\frac{1}{2}$:

$$\begin{array}{l|l} T_n & a \quad -\frac{1}{2}a \quad \frac{1}{4}a \quad -\frac{1}{8}a \quad \frac{1}{16}a \rightarrow 0 \\ \hline S_n & a \quad \frac{1}{2}a \quad \frac{3}{4}a \quad \frac{5}{8}a \quad \frac{11}{16}a \rightarrow \frac{2}{3}a \end{array}$$

To summarise all this in a single statement:

20 **LIMITING SUMS OF GEOMETRIC SERIES:**
 The partial sums S_n converge to a limit if and only if $-1 < r < 1$.
 The value of the limit is $S_\infty = \frac{a}{1 - r}$.

WORKED EXERCISE: Explain why these series have limiting sums and find them:

(a) $18 - 6 + 2 - \dots$

(b) $2 + \sqrt{2} + 1 + \dots$

SOLUTION:

(a) Here $a = 18$ and $r = -\frac{1}{3}$.

Since $-1 < r < 1$,

we know that the series converges.

$$\begin{aligned} S_\infty &= \frac{18}{1 + \frac{1}{3}} \\ &= 18 \times \frac{3}{4} \\ &= 13\frac{1}{2} \end{aligned}$$

(b) Here $a = 1$ and $r = \frac{1}{2}\sqrt{2}$.

Since $-1 < r < 1$,

we know that the series converges.

$$\begin{aligned} S_\infty &= \frac{2}{1 - \frac{1}{2}\sqrt{2}} \times \frac{1 + \frac{1}{2}\sqrt{2}}{1 + \frac{1}{2}\sqrt{2}} \\ &= \frac{2 + \sqrt{2}}{1 - \frac{1}{2}} \\ &= 4 + 2\sqrt{2} \end{aligned}$$

WORKED EXERCISE: For what values of x does $1 + (x - 2) + (x - 2)^2 + \dots$ converge, and what is the limiting sum?

SOLUTION:

The GP converges when

$$-1 < r < 1$$

$$-1 < x - 2 < 1$$

$$\boxed{+2} \quad 1 < x < 3.$$

The limiting sum is then

$$\begin{aligned} S_\infty &= \frac{1}{1 - (x - 2)} \\ &= \frac{1}{3 - x}. \end{aligned}$$

The Notation for Infinite Sums: When $-1 < r < 1$ and the GP converges, the limiting sum S_∞ can also be written as an infinite sum, either using sigma notation or using dots, so that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \quad \text{or} \quad a + ar + ar^2 + \dots = \frac{a}{1 - r},$$

and we say that 'the series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$ converges to $\frac{a}{1 - r}$ '.

Exercise 6K

- Copy and complete the table of values for the GP with $a = 18$ and $r = \frac{1}{3}$. Then find the limiting sum S_∞ , and the difference $S_\infty - S_6$.

T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n						
- Test whether these GPs have limiting sums, and find them if they do:
 - $1 + \frac{1}{2} + \frac{1}{4} + \dots$
 - $1 - \frac{1}{2} + \frac{1}{4} - \dots$
 - $12 + 4 + \frac{4}{3} + \dots$
 - $1 - 1 + 1 - \dots$
 - $100 + 90 + 81 + \dots$
 - $-2 + \frac{2}{5} - \frac{2}{25} + \dots$
 - $-\frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \dots$
 - $1 + (1.01) + (1.01)^2 + \dots$
 - $1 - 0.99 + (0.99)^2 - \dots$
 - $1 + (1.01)^{-1} + (1.01)^{-2} + \dots$
 - $0.72 - 0.12 + 0.02 - \dots$
 - $16\sqrt{5} + 4\sqrt{5} + \sqrt{5} + \dots$
- Find the value of x , given the limiting sums of these GPs:
 - $5 + 5x + 5x^2 + \dots = 10$
 - $5 + 5x + 5x^2 + \dots = 3$
 - $5 - 5x + 5x^2 - \dots = 15$
- Find the value of a , given the limiting sums of these GPs:
 - $a + \frac{a}{3} + \frac{a}{9} + \dots = 2$
 - $a - \frac{a}{3} + \frac{a}{9} - \dots = 2$
 - $a + \frac{2}{3}a + \frac{4}{9}a + \dots = 2$
- Find the condition for each GP to have a limiting sum, then find that limiting sum:
 - $1 + (x - 1) + (x - 1)^2 + \dots$
 - $1 + (1 + x) + (1 + x)^2 + \dots$
 - $1 + (3x - 2) + (3x - 2)^2 + \dots$
 - $1 - (3x + 2) + (3x + 2)^2 - \dots$

DEVELOPMENT

- Find the limiting sums if they exist, rationalising denominators:
 - $7 + \sqrt{7} + 1 + \dots$
 - $4 - 2\sqrt{2} + 2 - \dots$
 - $5 - 2\sqrt{5} + 4 - \dots$
 - $9 + 3\sqrt{10} + 10 + \dots$
 - $1 + (1 - \sqrt{3}) + (1 - \sqrt{3})^2 + \dots$
 - $1 + (2 - \sqrt{3}) + (2 - \sqrt{3})^2 + \dots$
 - $(\sqrt{5} + 1) + 2 + (\sqrt{5} - 1) + \dots$
 - $(\sqrt{5} - 1) + 2 + (\sqrt{5} + 1) + \dots$
- When a council offers free reflective house numbers, 30% of residents install them in the first month, the numbers in the second month are only 30% of those in the first month, and so on. What proportion of residents eventually install them?
- A bouncy ball drops from a height of 9 metres and bounces continually, each successive height being $\frac{2}{3}$ of the previous height.
 - Show that the first distance travelled down-and-up is 15 metres, and show that the successive down-and-up distances form a GP.
 - Through what distance does the ball eventually travel?
- Verify the convergence of each of the following series, then find the limit:
 - $\sum_{n=1}^{\infty} 7 \times \left(\frac{1}{2}\right)^{n-1}$
 - $\sum_{n=1}^{\infty} (-1)^n \times \frac{25}{4} \times \left(\frac{4}{25}\right)^n$
 - $\sum_{n=0}^{\infty} 4 \times 5^{-n} + 5 \times 4^{-n}$
- For the GP $(\sqrt{5} + \sqrt{3}) + (\sqrt{5} - \sqrt{3}) + \dots$, verify that $S_\infty = T_1 + \frac{1}{3}\sqrt{3}$.
- Suppose that $T_n = ar^{n-1}$ is a GP with a limiting sum.
 - Find the common ratio r if the limiting sum equals 5 times the first term.
 - Find the first three terms if the second term is 6 and the limiting sum is 27.
 - Find the ratio if the sum of all terms except the first equals 5 times the first term.

- (d) Show that the sum S of all terms from the third on is $\frac{ar^2}{1-r}$. Hence find r if S equals:
- the first term,
 - the second term,
 - the sum of the first and second terms.
- (e) Find the ratio r if the sum of the first three terms equals half the limiting sum.
12. (a) Suppose $a + ar + ar^2 + \dots$ is a GP with limiting sum. Show that the four sequences
- $$a + ar + ar^2 + \dots, \quad a - ar + ar^2 + \dots, \quad a + ar^2 + ar^4 + \dots, \quad ar + ar^3 + ar^5 + \dots,$$
- are all GPs, and that their limiting sums are in the ratio $1 + r : 1 - r : 1 : r$.
- (b) Find the limiting sums of these four GPs, and verify the ratio proven above:
- $48 + 24 + 12 + \dots$
 - $48 - 24 + 12 + \dots$
 - $48 + 12 + \dots$
 - $24 + 6 + \dots$
13. Each spring, fresh flowers are gathered from a patch of bush. Each annual yield, however, is only 90% of the previous year's yield. (a) Find the ratio between the first year's yield and the total yield, if the gathering continues indefinitely into the future. (b) Find also in which year the annual yield will first drop to less than 1% of the first year's yield.
14. A clever new toy comes onto the market, and sells 20 000 units in the first month. Popularity wanes, and each month the sales are only 70% of the sales in the previous month.
- How many units are sold eventually?
 - What proportion are sold in the first 6 months?
 - In which month will monthly sales first drop below 500 per month?
 - What proportion are sold before this month?
15. (a) Show that a GP has a limiting sum if $0 < 1 - r < 2$.
- (b) By calculating the common ratio, show that there is no GP with first term 8 and limiting sum 2.
- (c) A GP has positive first term a , and has a limiting sum S_∞ . Show that $S_\infty > \frac{1}{2}a$.
- (d) Find the range of values of the limiting sum of a GP with:
- $a = 6$
 - $a = -8$
 - $a > 0$
 - $a < 0$
16. Suppose $T_n = ar^{n-1}$ is a GP with limiting sum S_∞ . For any value of n , define the defect D_n to be the difference $D_n = S_\infty - S_n$ between the limiting sum and the n th partial sum.
- Show that $D_n = r^n S_\infty$, show that it is a GP, find its first term and common ratio, and prove that it converges to zero.
 - Find the defect D_n for the GP $18 + 6 + 2 + \dots$, find the defect if 5 terms are taken, and find how many terms must be taken for the defect to be less than $\frac{1}{1\,000\,000}$.
 - How many terms of the sequence $75, 15, 3, \dots$ must be taken for the partial sum of those terms to differ from the limiting sum by less than $\frac{1}{100\,000}$?
17. Find the condition for each GP to have a limiting sum, then find that limiting sum:
- $1 + (x^2 - 1) + (x^2 - 1)^2 + \dots$
 - $1 - (2 - x^2) + (2 - x^2)^2 - \dots$
 - $1 + \frac{1}{5x} + \frac{1}{(5x)^2} + \dots$
 - $1 - \frac{2}{x} + \frac{4}{x^2} - \dots$
 - $1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \dots$
 - $1 - \frac{1}{3-x} + \frac{1}{(3-x)^2} - \dots$
 - $1 + \frac{2x}{1+x^2} + \frac{(2x)^2}{(1+x^2)^2} + \dots$

18. (a) By writing out the terms, show that $\sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right) = 1 - \frac{1}{n+1}$.

(b) Hence explain why $\sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+1} \right) = 1$.

(c) Prove the identity $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$, and hence show that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots = 1.$$

EXTENSION

19. Consider the series $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \cdots$.

(a) Write out the terms of S_n and $2S_n$.

(b) Subtract to get an expression for S_n .

(c) Find the limit as $n \rightarrow \infty$, and hence find S_{∞} .

20. Now consider the general series $a + \frac{a+d}{x} + \frac{a+2d}{x^2} + \frac{a+3d}{x^3} + \cdots$, where a , d and x are constants with $|x| > 1$.

(a) Write out the terms of S_n and xS_n .

(b) Subtract to get an expression for $(x-1)S_n$ and hence for S_n .

(c) Find the limit as $n \rightarrow \infty$, and hence find S_{∞} .

21. The series $4 + 12 + 36 + \cdots$ has no limiting sum because $r > 1$. Nevertheless, substitution into the formula for the limiting sum gives

$$S_{\infty} = \frac{4}{1-3} = -2.$$

Can any meaning be given to this calculation and its result? [HINT: Look at the extension of the series to the *left* of the first term.]

6L Recurring Decimals and Geometric Series

It is now possible to give a precise explanation of recurring decimals. They are infinite GPs, and their value is the limiting sum of that GP.

WORKED EXERCISE: Express the repeating decimals $0.\dot{2}7$ and $2.\dot{6}4\dot{5}$ as infinite GPs, and use the formula for the limiting sum to find their values as fractions reduced to lowest terms.

SOLUTION:

$$0.\dot{2}7 = 0.272727 \dots$$

$$= 0.27 + 0.0027 + 0.000027 + \cdots$$

This is an infinite GP

with $a = 0.27$ and $r = 0.01$,

$$\begin{aligned} \text{so } 0.\dot{2}7 &= \frac{a}{1-r} \\ &= \frac{0.27}{0.99} \\ &= \frac{27}{99} \\ &= \frac{3}{11}. \end{aligned}$$

$$2.\dot{6}4\dot{5} = 2.645454545 \dots$$

$$= 2.6 + (0.045 + 0.00045 + \cdots)$$

This is 2.6 plus an infinite GP

with $a = 0.045$ and $r = 0.01$,

$$\begin{aligned} \text{so } 2.\dot{6}4\dot{5} &= 2.6 + \frac{0.045}{0.99} \\ &= \frac{26}{10} + \frac{45}{990} \\ &= \frac{286}{110} + \frac{5}{110} \\ &= \frac{291}{110}. \end{aligned}$$

Exercise 6L

NOTE: The following prime factorisations will be useful in this exercise:

$$\begin{array}{lll} 9 = 3^2 & 999 = 3^3 \times 37 & 99\,999 = 3^2 \times 41 \times 271 \\ 99 = 3^2 \times 11 & 9999 = 3^2 \times 11 \times 101 & 999\,999 = 3^3 \times 7 \times 11 \times 13 \times 37 \end{array}$$

1. Write each of these recurring decimals as an infinite GP, and hence use the formula for the limiting sum of a GP to express it as a rational number in lowest terms:

$$\begin{array}{llll} \text{(a)} 0.\dot{7} & \text{(c)} 0.\dot{2}\dot{7} & \text{(e)} 0.\dot{4}\dot{5} & \text{(g)} 0.\dot{1}\dot{3}\dot{5} \\ \text{(b)} 0.\dot{6} & \text{(d)} 0.\dot{7}\dot{8} & \text{(f)} 0.\dot{0}\dot{2}\dot{7} & \text{(h)} 0.\dot{1}\dot{8}\dot{5} \end{array}$$

2. Write each recurring decimal as the sum of an integer or terminating decimal and an infinite GP, and hence express it as a fraction in lowest terms:

$$\begin{array}{llll} \text{(a)} 12.\dot{4} & \text{(b)} 7.\dot{8}\dot{1} & \text{(c)} 8.\dot{4}\dot{6} & \text{(d)} 0.2\dot{3}\dot{6} \end{array}$$

3. Apply the earlier method — multiplying by 10^n where n is the cycle length (see Section 2A), then subtracting — to every second recurring decimal in the previous two questions.

DEVELOPMENT

4. (a) Express the repeating decimal $0.\dot{9}$ as an infinite GP, and hence show that it equals 1.
 (b) If $0.\dot{9}$ were not equal to 1, then the difference $1 - 0.\dot{9}$ would be positive. Let $\varepsilon = 1 - 0.\dot{9}$, explain why ε must be less than every positive number, and hence deduce that $\varepsilon = 0$.
 (c) Express $12.47\dot{9}$ as 12.47 plus an infinite GP, and hence show that it equals 12.48 .
 (d) Express 74 and 7.282 as recurring decimals ending in repeated 9s.
5. Use GPs to express these as fractions in lowest terms:

$$\begin{array}{llll} \text{(a)} 0.\dot{0}95\dot{7} & \text{(c)} 0.\dot{2}30\,76\dot{9} & \text{(e)} 0.25\dot{5}\dot{7} & \text{(g)} 0.\dot{0}00\,27\dot{1} \\ \text{(b)} 0.\dot{2}47\dot{5} & \text{(d)} 0.\dot{4}28\,57\dot{1} & \text{(f)} 1.\dot{1}0\dot{3}\dot{7} & \text{(h)} 7.\dot{7}\dot{7}1\,428\dot{5} \end{array}$$

6. The earlier method of handling recurring GPs is a special case of the method of deriving the formula for the n th partial sum of a GP. Compare the proof of that formula (Section 6J) with the earlier method of handling $0.\dot{1}\dot{8}$ (Section 2A), and find the correspondence between them.

EXTENSION

7. (a) Write the base 2 'decimals' 0.01 , 0.1101 and $0.011\,011$ as normal fractions. (b) Express $\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{8}$ and $\frac{11}{16}$ as 'decimals' base 2. (c) By writing them as infinite GPs, express the base 2 'decimals' $0.\dot{1}\dot{0}$, $0.\dot{1}\dot{0}\dot{1}$, $0.\dot{0}01\dot{1}$ and $0.\dot{1}$ as normal fractions. (d) Express $\frac{1}{3}$, $\frac{4}{5}$ and $\frac{1}{7}$ as recurring 'decimals' base 2. (e) Experiment with 'decimals' written to other bases.
8. (a) [The periods of recurring decimals] Let p be any prime other than 2 or 5. Explain why the cycle length of the recurring decimal equal to $1/p$ is n digits, where n is the least power of 10 that has remainder 1 when divided by p .
 (b) Use the factorisations of $10^k - 1$ given at the start of this exercise to predict the periods of the decimal representations of $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{27}$, $\frac{1}{37}$, $\frac{1}{41}$, $\frac{1}{101}$ and $\frac{1}{271}$, then write each as a recurring decimal.
9. [Extension — for further reading] *Fermat's little theorem* says that if p is a prime, and a is not a multiple of p , then a^{p-1} has remainder 1 after division by p . Using this theorem with $a = 10$, deduce that for all primes p except 2 and 5, the period of the decimal representation of $1/p$ is a divisor of $p - 1$.

6M Factoring Sums and Differences of Powers

The well-known difference of squares identity can now be generalised to sums and differences of n th powers. This factorisation will be needed in Section 7C in the proof of one of the fundamental results of the next chapter.

Differences of n th Powers: The polynomial $1 + x + x^2 + \cdots + x^{n-1}$ is a GP with $a = 1$ and $r = x$. So, using the formula for the sum of a GP,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1},$$

and rearranging, this becomes a factorisation of $x^n - 1$:

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

More generally, here is the factorisation of the difference of n th powers.

21	DIFFERENCE OF POWERS: $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$

PROOF: The last identity is easily proven directly by multiplying out the RHS:

$$\begin{aligned} \text{RHS} &= x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} \\ &\quad - x^{n-1}y - x^{n-2}y^2 - x^{n-3}y^3 - \cdots - y^n \\ &= x^n - y^n \end{aligned}$$

WORKED EXERCISE: Here are some examples, beginning with the difference of squares:

$$\begin{aligned} x^2 - 49 &= (x - 7)(x + 7) \\ x^3 - 1 &= (x - 1)(x^2 + x + 1) \\ x^4 - 81y^4 &= (x - 3y)(x^3 + 3x^2y + 9xy^2 + 27y^3) \end{aligned}$$

Sums of Odd Powers: The sum of two squares cannot be factored. The sum of two cubes, however, can easily be converted to the difference of powers, and can then be factored:

$$\begin{aligned} x^3 + y^3 &= x^3 - (-y)^3 \\ &= (x - (-y))(x^2 + x(-y) + (-y)^2) \\ &= (x + y)(x^2 - xy + y^2). \end{aligned}$$

The same device works for all sums of odd powers, so if n is an odd positive integer:

22	SUMS OF ODD POWERS: $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1})$

WORKED EXERCISE: Some further examples of factoring sums of odd powers:

$$\begin{aligned} x^3 + 125 &= (x + 5)(x^2 - 5x + 25) \\ x^5 + 32y^5 &= (x + 2y)(x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4) \\ 1 + a^7 &= (1 + a)(1 - a + a^2 - a^3 + a^4 - a^5 + a^6) \end{aligned}$$

WORKED EXERCISE: Factor $x^6 - 64$ completely.

$$\begin{aligned} \text{SOLUTION: } x^6 - 64 &= (x^3 - 8)(x^3 + 8) \quad (\text{using difference of squares}) \\ &= (x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4) \end{aligned}$$

(Neither quadratic can be factored, since $b^2 - 4ac = -12 < 0$.)

Exercise 6M

1. Factor these expressions, using the difference or sum of n th powers:

(a) $x^2 - 1$	(e) $t^3 + 1$	(i) $x^3 + 8$	(m) $x^5 + 32$
(b) $x^3 - 1$	(f) $t^5 + 1$	(j) $x^5 - 243$	(n) $32t^5 + 1$
(c) $x^5 - 1$	(g) $x^7 + 1$	(k) $x^3 + 125$	(o) $1 - a^7x^7$
(d) $t^7 - 1$	(h) $x^3 - 125$	(l) $x^5 + y^5$	(p) $27t^3 + 8a^3$

2. Factor numerator and denominator, then simplify:

(a) $\frac{x^3 - y^3}{x - y}$	(c) $\frac{x^3 + 1}{x + 1}$	(e) $\frac{x^7 - y^7}{x^5 - y^5}$
(b) $\frac{x^4 - y^4}{x - y}$	(d) $\frac{32x^5 + y^5}{2x + y}$	(f) $\frac{x^7 + y^7}{x^5 + y^5}$

DEVELOPMENT

3. (a) Factor $x^4 - 1$ first as $(x^2)^2 - 1$, then go on.

(b) Factor $x^6 - 1$ first as $(x^3)^2 - 1$, then go on.

(c) Similarly factor: (i) $x^8 - a^8$ (ii) $x^{10} - 1$

4. By expressing x as $(\sqrt{x})^2$ and y as $(\sqrt{y})^2$, factor:

(a) $x - y$	(b) $x^{\frac{3}{2}} - y^{\frac{3}{2}}$	(c) $x^{\frac{3}{2}} + y^{\frac{3}{2}}$

5. By the same method, simplify these algebraic fractions:

(a) $\frac{\sqrt{x} - \sqrt{y}}{x - y}$	(b) $\frac{x - y}{\sqrt{x} + \sqrt{y}}$	(c) $\frac{x^{\frac{3}{2}} - y^{\frac{3}{2}}}{\sqrt{x} - \sqrt{y}}$	(d) $\frac{\sqrt{x} + \sqrt{y}}{x^{\frac{3}{2}} + y^{\frac{3}{2}}}$

6. Simplify, using sums and differences of n th powers or otherwise:

(a) $(n + 1)^2 - n^2$	(c) $(n + 1)^3 - n^3$	(e) $(n + a)^2 - (n - a)^2$
(b) $(n + 1)^2 - (n - 1)^2$	(d) $(n + 1)^3 - (n - 1)^3$	(f) $(6n + a)^2 - (n + 6a)^2$

7. Simplify $\frac{f(u) - f(x)}{u - x}$ when:

(a) $f(x) = x^2$	(c) $f(x) = x^4$	(e) $f(x) = \sqrt{x}$
(b) $f(x) = x^3$	(d) $f(x) = \frac{1}{x}$	(f) $f(x) = \frac{1}{x^2}$

EXTENSION

8. (a) Show by rearranging each LHS as a difference of squares, or by expansion, that:

- (i) $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$
 (ii) $x^4 - x^2 + 1 = (x^2 + x\sqrt{3} + 1)(x^2 - x\sqrt{3} + 1)$
 (iii) $x^4 + 1 = (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$

(b) Hence factor completely:

(i) $x^6 + 1$	(ii) $x^8 - 1$	(iii) $x^{12} - 1$

9. (a) [Mersenne primes] Use the factorisation of differences of powers to show that $M_k = 2^k - 1$ can only be prime if k is a prime number p . Such primes M_p are called *Mersenne primes*. List the first few Mersenne primes, and find the first prime p such that M_p is not prime.

(b) [Fermat primes] Use the factorisation of sums of odd powers to show that $2^k + 1$ can only be prime if k is a power of 2. Such primes are called *Fermat primes*. List the first few Fermat primes, but accept the fact that $2^{32} + 1 = 641 \times 6\,700\,417$ is not prime.

- (c) [Mersenne primes and perfect numbers] Prove that if $M_p = 2^p - 1$ is a Mersenne prime, then $N = 2^{p-1}M_p = 2^{p-1}(2^p - 1)$ is a *perfect number*, meaning that the sum of all factors of N less than N is N itself. Hence list some perfect numbers.
- (d) Let $F_n = 2^{2^n} + 1$, and prove that $F_{n+1} = F_0 F_1 F_2 \dots F_n + 2$. Deduce that F_n and F_m are relatively prime when m and n are distinct. By considering the difference $M_p - M_q$, prove also that M_p and M_q are relatively prime when p and q are distinct primes. [NOTE: Two numbers are called *relatively prime* if their only common factor is 1.]

6N Proof by Mathematical Induction

Mathematical induction is a method of proof quite different from other methods of proof seen so far. It is based on recursion, which is why it belongs with the work on sequences and series, and it is used for proving theorems which claim that a certain statement is true for integer values of some variable.

As far as this course is concerned, proof by mathematical induction can only be applied after a clear statement of the theorem to be proven has already been obtained. So let us examine a typical situation in which a clear pattern is easily generated, but no obvious explanation emerges for why that pattern occurs.

Example 1 — Proving a Formula for the Sum of a Series: Find a formula for the sum of the first n cubes, and prove it by mathematical induction.

SOME CALCULATIONS FOR LOW VALUES OF n : Here is a table of values of the first 10 cubes and their partial sums:

n	1	2	3	4	5	6	7	8	9	10	...
n^3	1	8	27	64	125	216	343	512	729	1000	...
$1^3 + 2^3 + \dots + n^3$	1	9	36	100	225	441	784	1296	2025	3025	...
Form	1^2	3^2	6^2	10^2	15^2	21^2	28^2	36^2	45^2	55^2	...

The surprising thing here is that the last row is the square of the *triangular numbers*, where the n th triangular number is the sum of all the positive integers up to n . Using the formula for the sum of an AP (the number of terms times the average of first and last term), the formula for the n th triangular number is $\frac{1}{2}n(n+1)$. So the sum of the first n cubes seems to be $\frac{1}{4}n^2(n+1)^2$.

Thus we have arrived at a *conjecture*, meaning that we appear to have a true theorem, but we have no clear idea why it is true. We cannot really be sure yet even whether it is true, because showing that a statement is true for the first 10 positive integers is most definitely not a proof that it is true for all integers. The following worked exercise gives a precise statement of the result we want to prove.

WORKED EXERCISE: Prove by mathematical induction that for all integers $n \geq 1$,

$$1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

The proof below is a proof by mathematical induction. Read it carefully, then read the explanation of the proof in the notes below.

PROOF (BY MATHEMATICAL INDUCTION):

$$\begin{aligned} \text{A. When } n = 1, \text{ RHS} &= \frac{1}{4} \times 1 \times 2^2 \\ &= 1 \\ &= \text{LHS.} \end{aligned}$$

So the statement is true for $n = 1$.

B. Suppose that k is a positive integer for which the statement is true.

$$\text{That is, suppose } 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2. \quad (**)$$

We prove the statement for $n = k + 1$.

$$\text{That is, we prove } 1^3 + 2^3 + 3^3 + 4^3 + \cdots + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2.$$

$$\begin{aligned} \text{LHS} &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3, \text{ by the induction hypothesis } (**), \\ &= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \text{RHS.} \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the statement is true for all positive integers n .

Notes on the Proof: First, there are three clear parts. Part A proves the statement for the starting value 1. Part B is the most complicated, and proves that whenever the statement is true for some integer $k \geq 1$, then it is also true for the next integer $k + 1$. Part C simply appeals to the principle of mathematical induction to write a conclusion.

Secondly, any question on proof by mathematical induction is testing your ability to write a coherent account of the proof — you are advised not to deviate from the structure given here. The language of Part B is particularly important. It begins with four sentences, and these four sentences should be repeated strictly in all proofs. The first and second sentences of Part B set up what is assumed about k , writing down the specific statement for $n = k$, a statement later referred to as ‘the induction hypothesis’. The third and fourth sentences set up specifically what it is that we intend to prove.

Statement of the Principle of Mathematical Induction: With this proof as an example, here is a formal statement of the principle of mathematical induction.

23

MATHEMATICAL INDUCTION: Suppose that some statement is to be proven for all integers n greater than or equal to some starting value n_0 . Suppose also that it has been proven that:

1. the statement is true for $n = n_0$,
2. whenever the statement is true for some positive integer $k \geq n_0$, then it is also true for the next integer $k + 1$.

Then the statement must be true for all positive integers $n \geq n_0$.

Example 2 — Proving Divisibility: Find the largest integer that is a divisor of $3^{4n} - 1$, where $n \geq 0$ is any integer, and prove the result by mathematical induction.

SOME CALCULATIONS FOR LOW VALUES OF n : Again, here is a table of values for the first four values of n :

n	0	1	2	3	...
$3^{4n} - 1$	0	80	6560	531440	...

It seems likely from this that 80 is a divisor of all the numbers. Certainly no number bigger than 80 can be a divisor. So we write down the theorem, and try to provide a proof. Various proofs are available, but here is the proof by mathematical induction:

WORKED EXERCISE: Prove by mathematical induction that for all cardinals n ,
 $3^{4n} - 1$ is divisible by 80.

PROOF (BY MATHEMATICAL INDUCTION):

A. When $n = 0$, $3^{4n} - 1 = 0$, which is divisible by 80
 (remember that every number is a divisor of zero).

So the statement is true for $n = 0$.

B. Suppose that k is a cardinal for which the statement is true.

That is, suppose $3^{4k} - 1 = 80m$, for some integer m . (**)

We prove the statement for $n = k + 1$.

That is, we prove $3^{4k+4} - 1$ is divisible by 80.

$$\begin{aligned}
 3^{4k+4} - 1 &= 3^{4k} \times 3^4 - 1 \\
 &= (80m + 1) \times 81 - 1, \text{ by the induction hypothesis (**),} \\
 &= 80 \times 81m + 81 - 1 \\
 &= 80m \times 81 + 80 \\
 &= 80(81m + 1), \text{ which is divisible by 80, as required.}
 \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the statement is true for all cardinals n .

NOTES ON THE PROOF: Notice that the induction hypothesis (**) interprets divisibility by 80 as being $80m$ where m is an integer, whereas the fourth sentence of Part B stating what is to be proven does not interpret divisibility. Proofs of divisibility work more easily this way.

Example 3 — Proving an Inequality: For what integer values of n is 2^n greater than n^2 ?

SOME CALCULATIONS FOR LOW VALUES OF n : Here is a table of values:

n	0	1	2	3	4	5	6	7	8	9	10	...
n^2	0	1	4	9	16	25	36	49	64	81	100	...
2^n	1	2	4	8	16	32	64	128	256	512	1024	...

It seems obvious now that from $n = 5$ onwards, 2^n quickly becomes far larger than n^2 . Reasons for this may seem clearer here, and other proofs are available, but here is the proof by mathematical induction.

WORKED EXERCISE: Prove by mathematical induction that for all integers $n \geq 5$,

$$n^2 \text{ is less than } 2^n.$$

PROOF (BY MATHEMATICAL INDUCTION):

A. When $n = 5$, $n^2 = 25$ and $2^n = 32$. So the statement is true for $n = 5$.

B. Suppose that $k \geq 5$ is an integer for which the statement is true.

$$\text{That is, suppose } k^2 < 2^k. \quad (**)$$

We prove the statement for $n = k + 1$.

$$\text{That is, we prove } (k + 1)^2 < 2^{k+1}.$$

This is best done by proving that $\text{RHS} - \text{LHS} > 0$:

$$\begin{aligned} \text{RHS} - \text{LHS} &= 2 \times 2^k - (k + 1)^2 \\ &> 2k^2 - (k^2 + 2k + 1), \text{ by the induction hypothesis } (**), \\ &= k^2 - 2k - 1 \\ &= (k - 1)^2 - 2, \text{ completing the square,} \\ &> 0, \text{ since } k \geq 5 \text{ and so } (k - 1)^2 \geq 16. \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 5$.

NOTES ON THE PROOF: Proofs of inequalities can be difficult. The most systematic approach is probably the method shown here, 'Prove $\text{RHS} - \text{LHS} > 0$ '.

Further Remarks on Mathematical Induction: Summing a series, divisibility, and inequalities are the major places where proof by mathematical induction is applied. The method is, however, quite general, and the following exercise gives further applications beyond those three situations. In particular, some geometrical situations require proof by mathematical induction. In all cases, the precise structure and words given in these three examples should be followed exactly.

Finally, all the proofs given earlier in the chapter of formulae associated with APs and GPs really require the axiom of mathematical induction for their validity. In fact, if one were to be very strict about logic, any situation where there are dots ... only has meaning because of the axiom of mathematical induction.

Exercise 6N

1. Use mathematical induction to prove that for all positive integers n :

(a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

(b) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

(c) $1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$

(d) $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1)$

(e) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

(f) $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

(g) $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(h) $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

(i) $2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (n+1)2^{n-1} = n \times 2^n$

$$(j) 1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

$$(k) \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

$$(l) a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, \text{ provided } r \neq 1$$

$$(m) a + (a + d) + (a + 2d) \cdots + (a + (n-1)d) = \frac{1}{2}n(2a + (n-1)d)$$

2. Hence find the limiting sums of the series in parts (g), (h) and (k) of the previous question.

DEVELOPMENT

3. Use mathematical induction to prove these divisibility results for all positive integers n :

$$(a) 5^n + 3 \text{ is divisible by } 4$$

$$(d) 5^n + 2 \times 11^n \text{ is a multiple of } 3$$

$$(b) 9^n - 3 \text{ is a multiple of } 6$$

$$(e) 5^{2n} - 1 \text{ is a multiple of } 24$$

$$(c) 11^n - 1 \text{ is divisible by } 10$$

$$(f) x^n - 1 \text{ is divisible by } x - 1$$

4. Prove these divisibility results, advancing in part B of the proof from k to $k+2$:

$$(a) \text{ For even } n: (i) n^3 + 2n \text{ is divisible by } 12 \quad (ii) n^2 + 2n \text{ is a multiple of } 8$$

$$(b) \text{ For odd } n: (i) 3^n + 7^n \text{ is divisible by } 10 \quad (ii) 7^n + 6^n \text{ is divisible by } 13$$

5. Examine the divisors of $n^3 - n$ for low odd values of n , make a judgement about the largest integer divisor, and prove your result by induction.

6. Prove these inequalities by mathematical induction:

$$(a) n^2 > 10n + 7, \text{ for } n \geq 11$$

$$(c) 3^n > n^2, \text{ for } n \geq 2 \text{ (and also for } n = 0 \text{ and } 1)$$

$$(b) 2^n > 3n^2, \text{ for } n \geq 8$$

$$(d) (1 + \alpha)^n \geq 1 + n\alpha, \text{ for } n \geq 1, \text{ where } \alpha > -1$$

7. Examine 2^n and $2n^3$ for low values of n , make a judgement about which is eventually bigger, and prove your result by induction.

$$8. \text{ Prove: (a) } \sum_{r=1}^n \frac{1}{r^2} \leq 2 - \frac{1}{n}, \text{ for } n \geq 1 \quad (b) \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times 2n} \geq \frac{1}{2n}, \text{ for } n \geq 1$$

$$9. (a) \text{ Given that } T_n = 2T_{n-1} + 1 \text{ and } T_1 = 5, \text{ prove that } T_n = 6 \times 2^{n-1} - 1.$$

$$(b) \text{ Given that } T_n = \frac{3T_{n-1} - 1}{4T_{n-1} - 1} \text{ and } T_1 = 1, \text{ prove that } T_n = \frac{n}{2n-1}.$$

10. Prove by induction that the sum of the angles of a polygon with n sides is $n-2$ straight angles. [HINT: Dissect the $(k+1)$ -gon into a k -gon and a triangle.]

11. Prove by induction that n lines in the plane, no two being parallel and no three concurrent, divide the plane into $\frac{1}{2}n(n+1) + 1$ regions. [HINT: The $(k+1)$ th line will cross k lines in k distinct points, and so will add $k+1$ regions.]

12. Prove by mathematical induction that every set with n members has 2^n subsets. [HINT: When a new member is added to a k -member set, then every subset of the resulting $(k+1)$ -member set either contains or does not contain the new member.]

13. Defining $n! = 1 \times 2 \times 3 \times \cdots \times n$ (this is called ' n factorial'), prove that:

$$(a) \sum_{r=1}^n r \times r! = (n+1)! - 1 \quad (b) \sum_{r=1}^n \frac{r-1}{r!} = 1 - \frac{1}{n!}$$

$$14. \text{ Prove: (a) } \sum_{r=1}^n r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \quad (b) \sum_{r=1}^n r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

15. (a) By rationalising the numerator, prove that $\sqrt{n+1} - \sqrt{n} > \frac{1}{2\sqrt{n+1}}$.
 (b) Hence prove by induction that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \sqrt{n}$, for $n \geq 7$.
16. (a) Show that $f(n) = n^2 - n + 17$ is prime for $n = 0, 1, 2, \dots, 16$. Show, however, that $f(17)$ is not prime. Which step of proof by induction does this counterexample show is necessary?
 (b) Begin to show that $f(n) = n^2 + n + 41$ is prime for $n = 0, 1, 2, \dots, 40$, but not for 41.
- NOTE: There is no formula for generating prime numbers — these quadratics are interesting because of the long unbroken sequences of primes they produce.

EXTENSION

17. These proofs require a stronger form of mathematical induction in which ‘the statement’ is assumed true not only for $n = k$, but for any integer from the starting integer up to k .
- (a) Given that $T_{n+2} = 3T_{n+1} - 2T_n$, where $T_1 = 5$ and $T_2 = 7$, prove that $T_n = 3 + 2^n$.
 (b) The Fibonacci series F_n is defined by $F_{n+2} = F_{n+1} + F_n$, where $F_1 = F_2 = 1$. Prove:
 (i) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ (ii) $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ (iii) $F_n \leq \left(\frac{5}{3}\right)^{n-1}$
 (c) Prove that $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$.
 (d) The Lucas series L_n is defined by $L_{n+2} = L_{n+1} + L_n$, where $L_1 = 1$ and $L_2 = 3$. Use the observation that $L_n = F_n + 2F_{n-1}$ to generate a formula for L_n , then prove it by induction.
 (e) Prove that every integer greater than or equal to 2 is the product of prime numbers. (Further reading: Find how to prove that this factorisation into primes is unique.)
18. [A rather difficult proof] (a) Prove by induction on n that the geometric mean of 2^n positive numbers never exceeds their arithmetic mean, that is, for all cardinals n ,

$$\frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \geq (a_1 a_2 \cdots a_{2^n})^{\frac{1}{2^n}}, \text{ for all positive numbers } a_1, a_2, \dots, a_{2^n}.$$

- (b) Induction can work backwards as well as forwards. Suppose that for some integer $k \geq 2$

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq (a_1 a_2 \cdots a_k)^{\frac{1}{k}}, \text{ for all positive numbers } a_1, a_2, \dots, a_k.$$

By substituting $a_k = (a_1 a_2 \cdots a_{k-1})^{\frac{1}{k-1}}$, show that it follows that

$$\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \geq (a_1 a_2 \cdots a_{k-1})^{\frac{1}{k-1}}, \text{ for all positive numbers } a_1, a_2, \dots, a_{k-1}.$$

- (c) Deduce from all this that the geometric mean of any set of positive numbers never exceeds their arithmetic mean.

The Derivative

Our study of functions has now prepared us for some quite new approaches known as *calculus*. Calculus begins with two processes called differentiation and integration, both based on limiting processes:

- *Differentiation* is the examination of the changing steepness of a curve as one moves along it.
- *Integration* is the examination of areas of regions bounded by curves.

These processes were used by the Greeks, for whom tangents and areas were routine parts of their geometry, but it was not until the late seventeenth century that Sir Isaac Newton in England and Gottfried Leibniz in Germany independently gave systematic accounts of them. These were based on the realisation that finding the gradients of tangents and finding areas are inverse processes — a surprising insight so central that it is called ‘the fundamental theorem of calculus’. In this chapter we will be concerned with differentiation, introducing it in the context of functions, geometry and limiting processes.

STUDY NOTES: The derivative is first defined geometrically using tangents in Section 7A, and is then characterised in Section 7B as a limiting process. Sections 7C–7G develop the standard algebraic techniques of differentiation, interlocked in the exercises with the geometry of tangents and normals, particularly to the circle, the parabola and the rectangular hyperbola. Rates of change are included in Section 7H of this introductory chapter, because this is one of the most illuminating interpretations of the derivative and so should occur at the outset. General remarks about limits, continuity and differentiability have been left until Section 7I and 7J, and the study of these two sections could well be delayed until later in the course. The final Section 7K on implicit differentiation is a 4 Unit topic, but the techniques are very useful in the 3 Unit course.

7A The Derivative — Geometric Definition

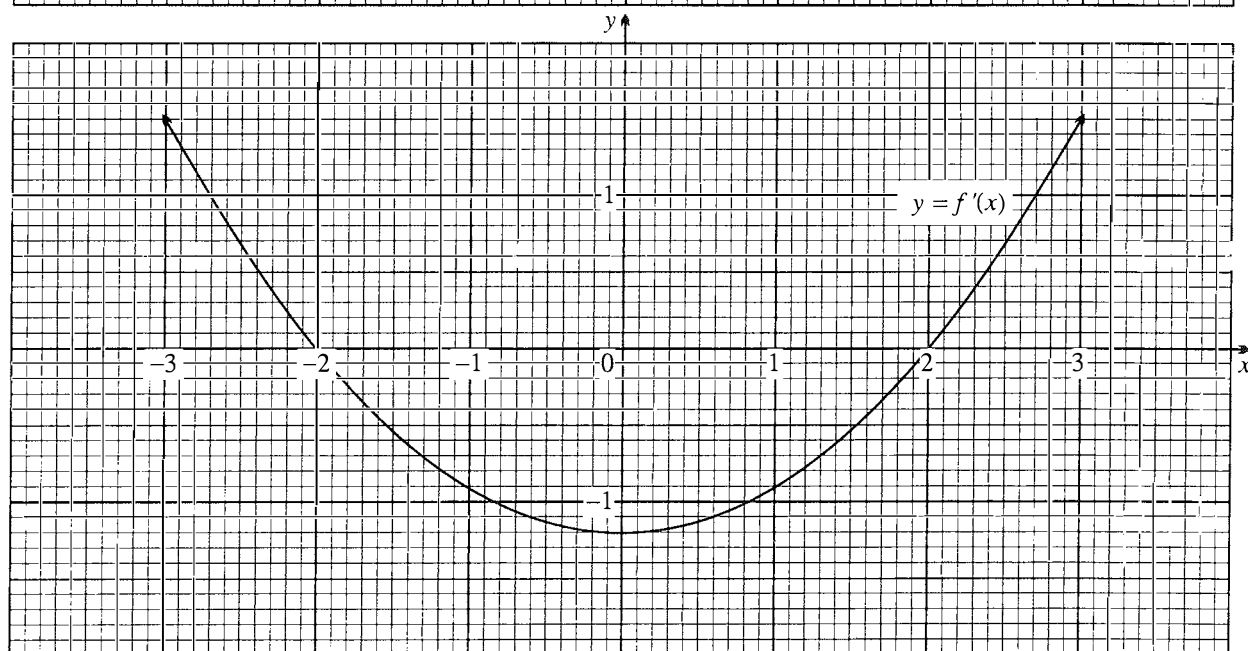
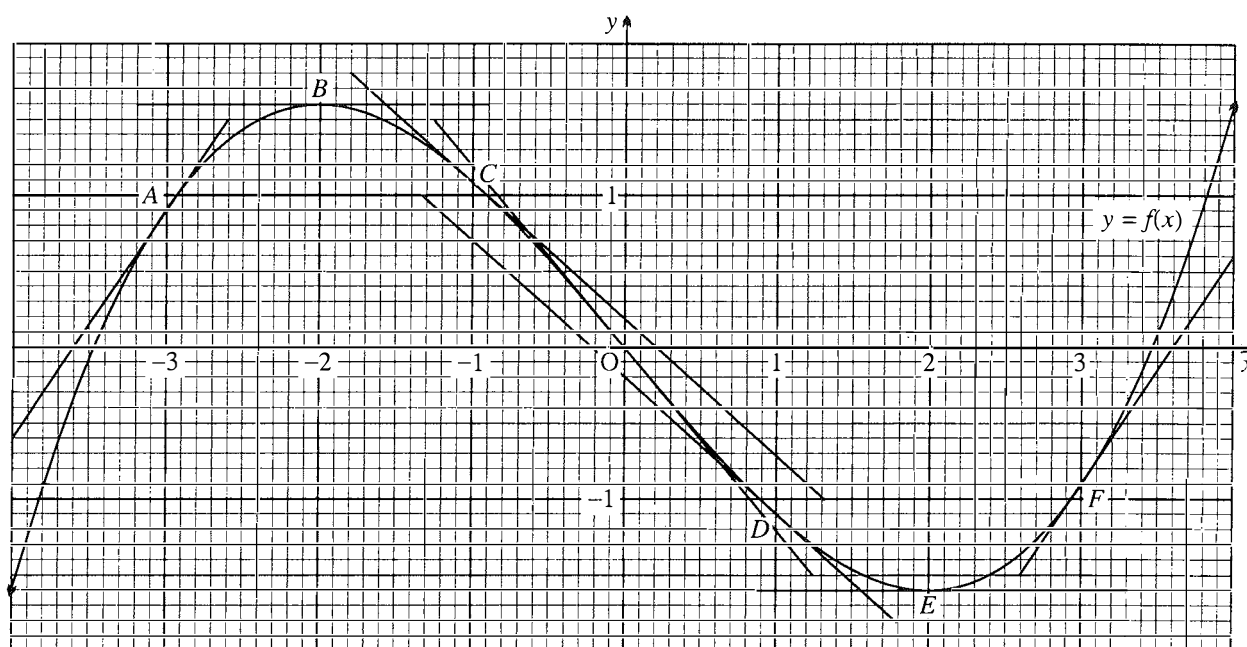
Sketched below on graph paper is the graph of a function $y = f(x)$ — for reasons of convenience the cubic $y = \frac{1}{10}(x^3 - 12x)$ was chosen. Like any curve that is not a straight line, its steepness keeps changing as one moves along the curve. Tangents have been drawn at several points on the curve, because the steepness of the curve at any point is measured by drawing a tangent at the point and measuring the gradient of the tangent.

The gradient of each tangent can easily be found by measuring its rise and run against the grid lines behind it. Counting ten little divisions for the run and measuring the corresponding rise give a natural decimal value for the gradient. Here is the resulting table of values of the gradients:

x	-3	-2	-1	0	1	2	3
gradient of tangent	1.5	0	-0.9	-1.2	-0.9	0	1.5

Notice that the horizontal tangents at B and E have gradient zero. The tangents between B and E have negative gradients, because they slope downwards. Everywhere else the tangents slope upwards and their gradients are positive.

We can get a complete picture of all this by plotting these gradients on a second number plane and joining the points. This gives the second graph below, which shows the gradient at each point on the curve $y = f(x)$. This second graph looks suspiciously like that of a quadratic function, and later we will be able to compute its equation exactly — it is $\frac{1}{10}(3x^2 - 12)$. But for now, it is enough to realise that the resulting function in this second sketch is a new function. This new function is called the *derivative of $f(x)$* , and is written $f'(x)$.



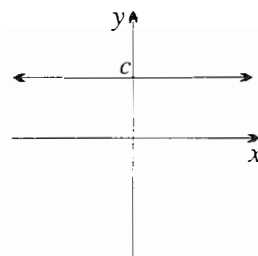
Geometric Definition of the Derivative: Here is the essential definition. Let $f(x)$ be a function. The *derivative* or *derived function* of $f(x)$, written as $f'(x)$, is defined by:

- 1 **DEFINITION:** $f'(x)$ is the gradient of the tangent to $y = f(x)$ at $P(x, f(x))$.

At present, circles are the only curves whose tangents we know much about, so the only functions we can apply our definition to are constant functions, linear functions and semicircular functions.

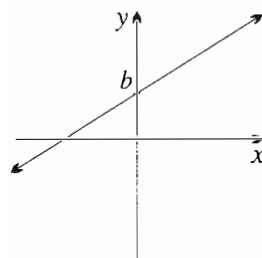
The Derivative of a Constant Function: Let $f(x) = c$ be a constant function. The tangent to the straight line $y = c$ at any point P on the line is of course just the line itself. So every tangent has gradient zero, and $f'(x)$ is the zero function.

- 2 **THEOREM:** The derivative of a constant function $f(x) = c$ is the zero function $f'(x) = 0$.



The Derivative of a Linear Function: Let $f(x) = mx + b$ be a linear function. Again, the tangent to $y = mx + b$ at any point P on the line is just the line itself. So every tangent has gradient m , and $f'(x) = m$ is a constant function.

- 3 **THEOREM:** The derivative of a linear function $f(x) = mx + b$ is the constant function $f'(x) = m$.



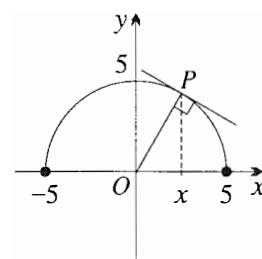
The Derivative of a Semicircle Function: Let $f(x) = \sqrt{25 - x^2}$ be the upper semicircle with centre O and radius 5. We know from geometry that at any point $P(x, \sqrt{25 - x^2})$ on the semicircle, the tangent at P is perpendicular to the radius OP .

Now gradient of radius $OP = \frac{\sqrt{25 - x^2}}{x}$,

so gradient of tangent at $P = -\frac{x}{\sqrt{25 - x^2}}$,

meaning that $f'(x) = -\frac{x}{\sqrt{25 - x^2}}$.

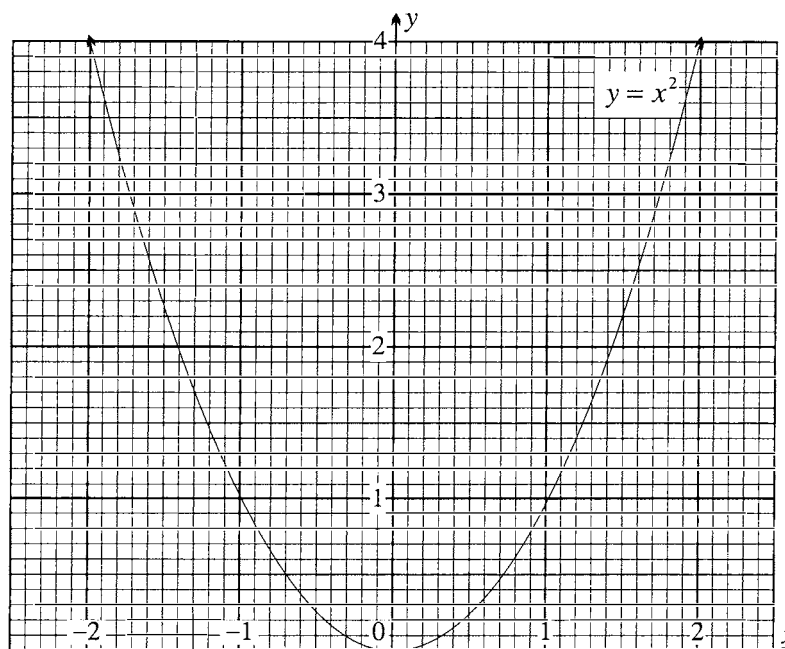
(This result is not to be memorised.)



The General Case: These examples of straight lines and circles are the only functions we can differentiate until we can use the methods developed in the next section. Question 1 in the following exercise continues the curve sketching methods used above, and asks for a reasonably precise construction of the derived function of $f(x) = x^2$, in preparation for Section 7B.

Exercise 7A

1.



- (a) Photocopy the accompanying sketch of $f(x) = x^2$.
- (b) At the point $P(1, 1)$, construct a tangent. Place the pencil point on P , bring the ruler to the pencil, then rotate the ruler about P until it seems reasonably like a tangent.
- (c) Use the definition $\text{gradient} = \frac{\text{rise}}{\text{run}}$ to measure the gradient of the tangent to at most two decimal places. Choose the run to be 10 little divisions, and count how many vertical divisions the tangent rises as it runs across the 10 horizontal divisions.
- (d) Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the nine points on the curve and measuring its gradient.

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x)$									

- (e) On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.
- (f) Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.
2. Write each function in the form $f(x) = mx + b$, and hence write down $f'(x)$:
- (a) $f(x) = 2x + 3$ (d) $f(x) = -4$ (g) $f(x) = \frac{3 - 5x}{4}$
- (b) $f(x) = 5 - 3x$ (e) $f(x) = ax + b$ (h) $f(x) = \frac{5}{2}(7 - \frac{4}{3}x)$
- (c) $f(x) = \frac{1}{2}x - 7$ (f) $f(x) = \frac{2}{3}(x + 4)$ (i) $f(x) = \frac{1}{2} + \frac{1}{3}$

DEVELOPMENT

3. Write each function in the form $f(x) = mx + b$, and hence write down the derived function:
- (a) $f(x) = \frac{1}{2}(3 + 5x) - \frac{1}{2}(5 - 2x)$ (b) $f(x) = (x + 3)^2 - (x - 3)^2$ (c) $f(x) = \frac{k - \ell x}{r} + \frac{k + \ell x}{r}$
4. Sketch the upper semicircle $f(x) = \sqrt{25 - x^2}$, mark the points $(4, 3)$, $(3, 4)$, $(0, 5)$, $(-3, 4)$ and $(-4, 3)$ on it, and sketch tangents and radii at these points. By using the fact that the tangent is perpendicular to the radius at the point of contact, find:
- (a) $f'(4)$ (b) $f'(3)$ (c) $f'(0)$ (d) $f'(-4)$ (e) $f'(-3)$

5. Use the fact that the tangent to a circle is perpendicular to the radius at the point of contact to find the derived functions of the following. Begin with a sketch.

(a) $f(x) = \sqrt{1-x^2}$ (b) $f(x) = -\sqrt{1-x^2}$ (c) $f(x) = \sqrt{4-x^2}$

6. Sketch graphs of these functions, draw tangents at the points where $x = -2, -1, 0, 1, 2$, estimate their gradients, and hence draw a reasonable sketch of the derivative.

(a) $f(x) = 4 - x^2$ (b) $f(x) = \frac{1}{x}$ (c) $f(x) = 2^x$

————— EXTENSION —————

7. Use the radius and tangent theorem to find the derivatives of:

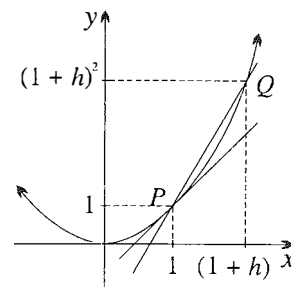
(a) $f(x) = \sqrt{9-x^2} + 4$ (c) $f(x) = \sqrt{36-(x-7)^2}$
 (b) $f(x) = 3 - \sqrt{16-x^2}$ (d) $f(x) = 7 - \sqrt{2x-x^2}$

7B The Derivative as a Limit

The gradient of a line is found by taking two distinct points on it and taking the ratio of rise over run. The difficulty with a tangent is that we only know one point on it — the point of contact — and unless the curve is a straight line, no other points on the curve near the point of contact actually lie on the tangent. The only general way to get at the tangent at some point P on the curve is to use a limiting process involving the family of lines through P .

The Tangent as the Limit of Secants: The diagram opposite shows the graph of $f(x) = x^2$ and the tangent at $P(1,1)$ on the curve. Let $Q(1+h, (1+h)^2)$ be any other point on the curve. Then the straight line through P and Q is a secant whose gradient is the ratio of rise over run:

$$\begin{aligned} \text{gradient } PQ &= \frac{(1+h)^2 - 1}{(1+h) - 1} \\ &= \frac{2h + h^2}{h} \\ &= 2 + h, \text{ since } h \neq 0. \end{aligned}$$



As Q moves along the curve to the right of P (or to the left of P) the secant PQ rotates around P . But the closer Q is to P , the closer the secant PQ is to the tangent at P . In fact, we can make the gradient of the secant PQ 'as close as we like' to the gradient of the tangent by taking Q sufficiently close to P . That means we take the limit as $Q \rightarrow P$:

$$\begin{aligned} \text{gradient (tangent at } P) &= \lim_{Q \rightarrow P} (\text{gradient } PQ) \\ &= \lim_{h \rightarrow 0} (2 + h), \text{ because } h \rightarrow 0 \text{ as } Q \rightarrow P \\ &= 2, \text{ because } 2 + h \rightarrow 2 \text{ as } h \rightarrow 0. \end{aligned}$$

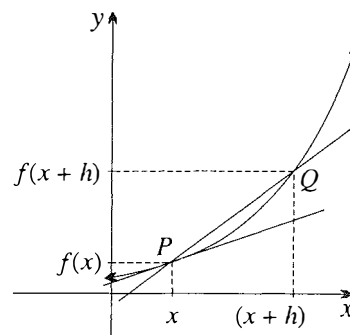
Thus the tangent at P has gradient 2, and so $f'(1) = 2$. Notice that Q cannot actually coincide with P , or both rise and run would be zero, and the calculation would be invalid.

The Derivative as a Limit: This brings us to the formula for the derivative as a limit. The derivative of any function $f(x)$ is the new function $f'(x)$ defined at any value of x by:

4 **DEFINITION:**
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The accompanying diagram explains the definition. The secant PQ joins the point P with coordinates $(x, f(x))$ and the point Q with coordinates $(x+h, f(x+h))$. Using the usual formula for gradient,

$$\text{gradient } PQ = \frac{f(x+h) - f(x)}{h} \quad (\text{rise over run}).$$

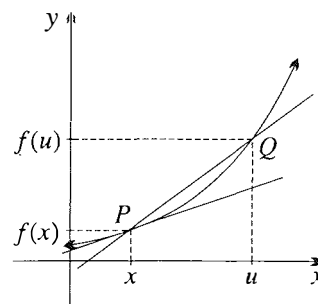


Then the gradient of the tangent is the limit as $Q \rightarrow P$, that is as $h \rightarrow 0$.

NOTE: The diagram shows Q to the right of P . However, Q could as well be on the left, which corresponds algebraically to h being negative.

An Alternative Notation: There is an alternative notation which in some situations is more convenient to use. The diagram is the same, but we let Q have x -coordinate u and y -coordinate $f(u)$. In this case:

5 **DEFINITION:**
$$f'(x) = \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x}$$



WORKED EXERCISE: As an example of this definition of the derivative as a limit, let us calculate the derivative of $f(x) = x^2$, using both notations above. (The graphical work in the first question of the previous exercise should already have obtained the answer $f'(x) = 2x$ for this derivative.) Calculating in this way is called differentiating ‘from first principles’ or ‘from the definition of the derivative’.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2x + h), \text{ since } h \neq 0, \\ &= 2x. \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x} \\ &= \lim_{u \rightarrow x} \frac{u^2 - x^2}{u - x} \\ &= \lim_{u \rightarrow x} \frac{(u-x)(u+x)}{u-x} \\ &= \lim_{u \rightarrow x} (u+x), \text{ since } u \neq x, \\ &= 2x. \end{aligned}$$

What is a Tangent: The careful reader will realise that the word ‘tangent’ was introduced without definition in Section 7A. Whereas tangents to circles are well understood, tangents to more general curves are not so easily defined. It is possible to define a tangent geometrically, but it is far easier to take the formula for the derivative as part of its actual definition. So our strict definition of the tangent at a point $P(x, f(x))$ is that it is the line through P with gradient $f'(x)$.

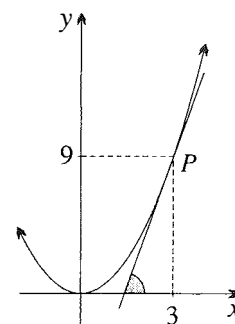
WORKED EXERCISE: Use the fact that the derivative of $f(x) = x^2$ is $f'(x) = 2x$ to find the gradient, the angle of inclination (to the nearest minute), and the equation of the tangent to the curve $y = x^2$ at the point $P(3, 9)$ on the curve.

SOLUTION: Substituting $x = 3$ into $f'(x) = 2x$ gives $f'(3) = 6$, so the tangent at P has gradient 6.

Hence the tangent is $y - 9 = 6(x - 3)$

$$y = 6x - 9.$$

Since the gradient is 6, the angle of inclination is about $80^\circ 32'$ (using the calculator to solve $\tan(\text{angle of inclination}) = 6$).



Exercise 7B

1. Consider the function $f(x) = x^2 - 4x$.

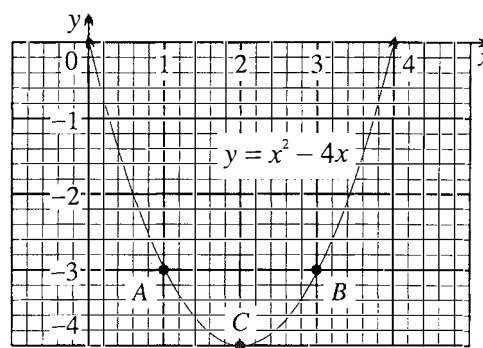
(a) Simplify $\frac{f(x+h) - f(x)}{h}$.

(b) Show that $f'(x) = 2x - 4$, using the definition $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

(c) Substitute $x = 1$ into $f'(x)$ to find the gradient of the tangent at $A(1, -3)$.

(d) Similarly find the gradients of the tangents at $B(3, -3)$ and $C(2, -4)$.

(e) The function $f(x) = x^2 - 4x$ is graphed above. Place your ruler on the curve at A , B and C to check the reasonableness of the results obtained above.



2. For each function below, simplify $\frac{f(x+h) - f(x)}{h}$, then take $\lim_{h \rightarrow 0}$ to find the derivative.

For part (i) you will need the result $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

(a) $f(x) = 5x + 1$	(d) $f(x) = x^2 - 4x$	(g) $f(x) = 9 - 4x^2$
(b) $f(x) = 4 - 3x$	(e) $f(x) = x^2 + 3x + 2$	(h) $f(x) = x^3$
(c) $f(x) = x^2 + 10$	(f) $f(x) = 2x^2 + 3x$	(i) $f(x) = x^4$

3. For each function in question 2: (i) use the derivative to evaluate $f'(2)$, (ii) find the y -coordinate of the point P on $y = f(x)$ where $x = 2$, (iii) find the equation of the tangent at P , (iv) sketch the curve and the tangent.

4. For each of the functions in question 2, find any values of x for which the tangent is horizontal (that is, for which $f'(x) = 0$).

5. Find the derivatives of the functions in question 2 by first simplifying $\frac{f(u) - f(x)}{u - x}$, then taking $\lim_{u \rightarrow x}$ to find the derivative. [HINT: Group the corresponding powers of u and x in the numerator, then factor each pair using difference of powers, then factor the whole numerator using grouping. For example in part (d):

$$u^2 - 4u - x^2 + 4x = (u^2 - x^2) - 4(u - x) = (u + x)(u - x) - 4(u - x) = (u - x)(u + x - 4).]$$

DEVELOPMENT

6. (a) Sketch $f(x) = x^2 + 6x$, then use the $u \rightarrow x$ method to show that the derivative is $2x + 6$.

- (b) Hence find the gradient and angle of inclination of the tangent (nearest minute) at the point where: (i) $x = 0$ (ii) $x = -3$ (iii) $x = -2\frac{1}{2}$ (iv) $x = -3\frac{1}{2}$ (v) $x = -5$
7. (a) Use the $h \rightarrow 0$ method to show that the derivative of $f(x) = x^2 - 5x$ is $f'(x) = 2x - 5$.
 (b) Hence find the points on $y = x^2 - 5x$ where the tangent has the following gradients. Then sketch the curve and the tangents. (i) 1 (ii) -1 (iii) 5 (iv) -5 (v) 0
8. (a) Use the $h \rightarrow 0$ method to show that the derivative of $f(x) = \frac{1}{4}x^2$ is $f'(x) = \frac{1}{2}x$.
 (b) Hence find the x -coordinates of the points on $y = \frac{1}{4}x^2$ where the tangent has angle of inclination:
 (i) 45° (ii) 135° (iii) 60° (iv) 120° (v) 30° (vi) 150° (vii) 37°
9. (a) Use the $h \rightarrow 0$ method to show that the derivative of $f(x) = mx + b$ is $f'(x) = m$.
 (b) Similarly show that the derivative of $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$.
10. (a) Sketch $f(x) = x^2 - 5x + 6$, then use the $u \rightarrow x$ method to show that the derivative is $2x - 5$. (b) Find the gradient at the y -intercept, the equation of the tangent there, and the x -intercept of that tangent.
 (c) Find the gradients at the two x -intercepts $(2, 0)$ and $(3, 0)$ and show that they are opposites. Find the angles of inclination there and show that they are supplementary.
 (d) Find, to the nearest minute, the angles of inclination when $x = 4$ and when $x = 1$.
11. Let $P(1, -3)$ and $Q(1 + h, (1 + h)^2 - 4)$ be two points on the graph of $f(x) = x^2 - 4$.
 (a) Show that the gradient of the chord PQ is $2 + h$ and deduce that $f'(1) = 2$.
 (b) Find the gradient of PQ when:
 (i) $h = 2$ (ii) $h = -3$ (iii) $h = -2$ (iv) $h = 0.01$
 (c) Sketch the curve for $-2 \leq x \leq 3$, using a table of values, and add the chords PQ .
12. Use both the $u \rightarrow x$ method and the $h \rightarrow 0$ method to prove that:
 (a) the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$, (b) the derivative of $\frac{1}{x^2}$ is $-\frac{2}{x^3}$.
13. (a) Prove the identity $u - x = (\sqrt{u} + \sqrt{x})(\sqrt{u} - \sqrt{x})$, for positive values of u and x .
 (b) Hence prove, using the $u \rightarrow x$ method, that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.
14. Use the $u \rightarrow x$ method to show that the derivative of $f(x) = x^2 - ax$ is $f'(x) = 2x - a$, and find the value of a in the following cases: (a) the tangent at the origin has gradient 7, (b) $y = f(x)$ has a horizontal tangent at $x = 3$, (c) the tangent at the point where $x = 1$ has an angle of inclination of 45° , (d) the tangent at the nonzero x -intercept has gradient 5, (e) the tangent at the vertex has y -intercept -9 .

EXTENSION

15. [Algebraic differentiation of x^2] Let $P(a, a^2)$ be any point on the curve $y = x^2$, then the line ℓ through P with gradient m has equation $y - a^2 = m(x - a)$. Show that the x -coordinates of the points where ℓ meets the curve are $x = a$ and $x = m - a$. Find the value of the gradient m for which these two points coincide, and explain why it follows that the derivative of x^2 is $2x$.
16. [An alternative algebraic approach] Find the x -coordinates of the points where the line $\ell: y = mx + b$ meets the curve $y = x^2$, and hence deduce that the derivative of x^2 is $2x$.

7C A Rule for Differentiating Powers of x

It was surely very obvious that the long calculations of the previous exercise had quite simple answers. Fortunately, there is a straightforward rule which allows the derivative of any power of x to be written down in one step.

THEOREM: Suppose $f(x) = x^n$, where n is any real number.
 Then the derivative is $f'(x) = nx^{n-1}$.

6 OR (expressing it as a process)

Take the index as a factor, and reduce the index by 1.

The result will be proven in this section where n is a cardinal number or -1 or $\frac{1}{2}$. The proof will be extended in Section 7E to rational numbers — for the sake of convenience, however, the exercise of this section will use the general result for all real numbers. First, here are four examples of the theorem.

WORKED EXERCISE: Differentiate: (a) x^8 (b) x^{100} (c) x^{-4} (d) $x^{\frac{2}{3}}$

SOLUTION:

$$\begin{array}{llll}
 \text{(a) } f(x) = x^8 & \text{(b) } f(x) = x^{100} & \text{(c) } f(x) = x^{-4} & \text{(d) } f(x) = x^{\frac{2}{3}} \\
 f'(x) = 8x^7 & f'(x) = 100x^{99} & f'(x) = -4x^{-5} & f'(x) = \frac{2}{3}x^{-\frac{1}{3}}
 \end{array}$$

Proof when n is a Cardinal Number: The result was proven in the last section for the cases where n was zero or 1. Suppose then that n is an integer with $n \geq 2$. The proof depends on the factorisation of the difference of n th powers, which was developed from partial sums of GPs in Section 6M of the previous chapter.

$$\begin{aligned}
 \text{Using the definition, } f'(x) &= \lim_{u \rightarrow x} \frac{u^n - x^n}{u - x} \\
 &= \lim_{u \rightarrow x} \frac{(u - x)(u^{n-1} + u^{n-2}x + \cdots + x^{n-1})}{u - x} \\
 &= \lim_{u \rightarrow x} (u^{n-1} + u^{n-2}x + \cdots + x^{n-1}), \text{ since } u \neq x, \\
 &= x^{n-1} + x^{n-1} + \cdots + x^{n-1} \quad (n \text{ terms}) \\
 &= nx^{n-1}.
 \end{aligned}$$

The Derivatives of $1/x$ and $\sqrt{x}$: The derivatives of $1/x$ and $\sqrt{x}$ occur so often that they deserve special attention. Differentiating them from first principles:

A. Suppose that $f(x) = \frac{1}{x}$.

$$\begin{aligned}
 f'(x) &= \lim_{u \rightarrow x} \frac{1/u - 1/x}{u - x} \times \frac{ux}{ux} \\
 &= \lim_{u \rightarrow x} \frac{x - u}{ux(u - x)} \\
 &= \lim_{u \rightarrow x} \frac{-1}{ux}, \text{ since } u \neq x, \\
 &= -\frac{1}{x^2}, \text{ which is } -x^{-2}.
 \end{aligned}$$

B. Suppose that $f(x) = \sqrt{x}$.

$$\begin{aligned}
 f'(x) &= \lim_{u \rightarrow x} \frac{\sqrt{u} - \sqrt{x}}{u - x} \\
 &= \lim_{u \rightarrow x} \frac{\sqrt{u} - \sqrt{x}}{(\sqrt{u} - \sqrt{x})(\sqrt{u} + \sqrt{x})} \\
 &= \lim_{u \rightarrow x} \frac{1}{\sqrt{u} + \sqrt{x}}, \text{ since } u \neq x, \\
 &= \frac{1}{2\sqrt{x}}, \text{ which is } \frac{1}{2}x^{-\frac{1}{2}}.
 \end{aligned}$$

These are the same results as are obtained by applying the formula above for differentiating powers of x :

$$\begin{aligned}\text{A. } f(x) &= \frac{1}{x} \\ &= x^{-1} \\ f'(x) &= -x^{-2}\end{aligned}$$

$$\begin{aligned}\text{B. } f(x) &= \sqrt{x} \\ &= x^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}\end{aligned}$$

7

TWO SPECIAL FORMS: The derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$.
The derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

Linear Combinations of Functions: Compound functions formed by taking sums and multiples of simpler functions are quite straightforward to handle.

8

DERIVATIVE OF A SUM: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
DERIVATIVE OF A MULTIPLE: If $f(x) = kg(x)$, then $f'(x) = kg'(x)$.

PROOF:

For the first:

$$\begin{aligned}f'(x) &= \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x} \\ &= \lim_{u \rightarrow x} \frac{g(u) + h(u) - g(x) - h(x)}{u - x} \\ &= \lim_{u \rightarrow x} \frac{g(u) - g(x)}{u - x} + \lim_{u \rightarrow x} \frac{h(u) - h(x)}{u - x} \\ &= g'(x) + h'(x).\end{aligned}$$

For the second:

$$\begin{aligned}f'(x) &= \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x} \\ &= \lim_{u \rightarrow x} \frac{kg(u) - kg(x)}{u - x} \\ &= k \lim_{u \rightarrow x} \frac{g(u) - g(x)}{u - x} \\ &= kg'(x).\end{aligned}$$

WORKED EXERCISE: Differentiate: (a) $4x^2 - 3x + 2$ (b) $\frac{1}{2}x^6 - \frac{1}{6}x^3$ (c) $(2x - 3)(3x - 2)$

SOLUTION:

$$\begin{aligned}\text{(a) } f(x) &= 4x^2 - 3x + 2 & \text{(b) } f(x) &= \frac{1}{2}x^6 - \frac{1}{6}x^3 & \text{(c) } f(x) &= (2x - 3)(3x - 2) \\ f'(x) &= 8x - 3 & f'(x) &= 3x^5 - \frac{1}{2}x^2 & &= 6x^2 - 13x + 6 \\ & & & & f'(x) &= 12x - 13\end{aligned}$$

WORKED EXERCISE: Differentiate: (a) $\frac{16}{x^3} - \frac{16}{x^2}$ (b) $\sqrt{5x}$ (c) $\frac{5}{12x}$

SOLUTION:

$$\begin{aligned}\text{(a) } f(x) &= \frac{16}{x^3} - \frac{16}{x^2} & \text{(b) } f(x) &= \sqrt{5x} & \text{(c) } f(x) &= \frac{5}{12x} \\ &= 16x^{-3} - 16x^{-2} & &= \sqrt{5} \times \sqrt{x} & &= \frac{5}{12} \times \frac{1}{x} \\ f'(x) &= -48x^{-4} + 32x^{-3} & f'(x) &= \frac{\sqrt{5}}{2\sqrt{x}} & f'(x) &= -\frac{5}{12x^2} \\ &= -\frac{48}{x^4} + \frac{32}{x^3} & & & &\end{aligned}$$

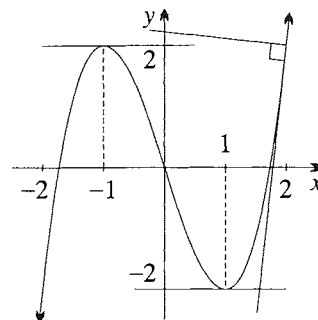
Tangents and Normals to a Curve: Suppose P is a point on a curve $y = f(x)$. The tangent at P is, as we have said, the line through P with gradient equal to the derivative at P . The *normal* at P is defined to be the line through P perpendicular to the tangent at P . Equations of tangents and normals are easily calculated using the derivative.

WORKED EXERCISE: Given that $f(x) = x^3 - 3x$, find the equations of the tangent and normal to the curve $y = f(x)$ at the point $P(2, 2)$ on the curve. Find also the points on the curve where the tangent is horizontal.

SOLUTION: Here $f'(x) = 3x^2 - 3$,
 so at $P(2, 2)$, $f'(2) = 9$,
 so the tangent has gradient 9 and the normal has gradient $-\frac{1}{9}$.
 Hence the tangent is $y - 2 = 9(x - 2)$

$y = 9x - 16$,
 and the normal is $y - 2 = -\frac{1}{9}(x - 2)$
 $y = -\frac{1}{9}x + 2\frac{2}{9}$.

Also, the tangent has gradient zero when $3x^2 - 3 = 0$
 $x = 1$ or -1 ,
 so the tangent is horizontal at $(1, -2)$ and at $(-1, 2)$.



WORKED EXERCISE: Find the points on the graph of $f(x) = x + \frac{1}{x}$ where:

- (a) the tangent is horizontal, (b) the normal has gradient -2 ,
 (c) the tangent has angle of inclination 45° .

SOLUTION: Since $f(x) = x + \frac{1}{x}$, $f'(x) = 1 - \frac{1}{x^2}$.

- (a) Put $f'(x) = 0$, then $\frac{1}{x^2} = 1$

$$x = 1 \text{ or } -1,$$

so the tangent is horizontal at $(1, 2)$ and $(-1, -2)$.

- (b) When the normal has gradient -2 , the tangent has gradient $\frac{1}{2}$,

$$\text{so put } f'(x) = \frac{1}{2}, \text{ then } \frac{1}{x^2} = \frac{1}{2}$$

$$x = \sqrt{2} \text{ or } -\sqrt{2},$$

so the normal has gradient -2 at $(\sqrt{2}, \frac{3}{2}\sqrt{2})$ and at $(-\sqrt{2}, -\frac{3}{2}\sqrt{2})$.

- (c) When the angle of inclination is 45° , the tangent has gradient 1,

$$\text{so put } f'(x) = 1, \text{ then } \frac{1}{x^2} = 0$$

which is impossible, so there is no such point.

Exercise 7C

- Use the rule for differentiating x^n to differentiate (where a , b , c and ℓ are constants):

(a) $f(x) = x^7$	(e) $f(x) = x^4 + x^3 + x^2 + x + 1$	(i) $f(x) = ax^4 - bx^2 + c$
(b) $f(x) = 9x^5$	(f) $f(x) = 2 - 3x - 5x^3$	(j) $f(x) = x^\ell$
(c) $f(x) = \frac{1}{3}x^6$	(g) $f(x) = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2 - 2$	(k) $f(x) = bx^{3b}$
(d) $f(x) = 3x^2 - 5x$	(h) $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1$	(l) $f(x) = x^{5a+1}$
- Find $f'(0)$ and $f'(1)$ for each function in the previous question.
- Differentiate these functions by first expanding the products:

(a) $x(x^2 + 1)$	(c) $(x + 4)(x - 2)$	(e) $(x^2 + 3)^2$	(g) $(x^2 + 3)(x - 5)$
(b) $x^2(3 - 2x - 4x^2)$	(d) $(2x + 1)(2x - 1)$	(f) $x(7 - x)^2$	(h) $(ax - 5)^2$

4. Use the rule for differentiating x^n to differentiate:
- (a) $3x^{-1}$ (b) $5x^{-2}$ (c) $-\frac{4}{3}x^{-3}$ (d) $2x^{-2} + \frac{1}{2}x^{-8}$
5. Write these functions using negative powers of x , then differentiate. Give the final answers in fractional form without negative indices.
- (a) $\frac{1}{x^2}$ (c) $\frac{1}{2x^4}$ (e) $\frac{c}{ax}$ (g) $\frac{a}{x} - \frac{b}{x^2}$
 (b) $\frac{5}{x^3}$ (d) $-\frac{3}{5x^5}$ (f) $\frac{1}{x^6} - \frac{1}{x^8}$ (h) $\frac{7}{2x^n}$
6. Use the fact that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$ to differentiate:
- (a) $\frac{3}{x}$ (b) $\frac{1}{3x}$ (c) $-\frac{7}{3x}$ (d) $\frac{a}{x}$
7. Use the fact that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ to differentiate:
- (a) $3\sqrt{x}$ (b) $10\sqrt{x}$ (c) $\sqrt{49x}$ (d) $\sqrt{7x}$
8. Find the gradients of the tangent and normal at the point on $y = f(x)$ where $x = 3$:
 (a) $f(x) = x^2 - 5x + 2$ (b) $f(x) = x^3 - 3x^2 - 10x$ (c) $f(x) = 2x^2 - 18x$ (d) $f(x) = 2\sqrt{x}$
9. Find the angles of inclination of the tangents and normals in the previous question.
10. Find the equations of the tangent and normal to the graph of $f(x) = x^2 - 8x + 15$ at:
 (a) $A(1, 8)$ (b) $B(6, 3)$ (c) the y -intercept (d) $C(4, -1)$
11. Differentiate $f(x) = x^3$. Hence show that the tangents to $y = x^3$ have positive gradient everywhere except at the origin, and show that the tangent there is horizontal. Explain the situation using a sketch.
12. Find the equation of the tangent to $f(x) = 10x - x^3$ at the point $P(2, 12)$. Then find the points A and B where the tangent meets the x -axis and y -axis respectively, and find the length of AB and the area of $\triangle OAB$.
13. Find any points on the graph of each function where the tangent is parallel to the x -axis:
 (a) $f(x) = 4 + 4x - x^2$ (c) $f(x) = 4ax - x^2$
 (b) $f(x) = x^3 - 12x + 24$ (d) $f(x) = x^4 - 2x^2$

DEVELOPMENT

14. Find the tangent and normal to $f(x) = 12/x$ at: (a) $M(2, 6)$ (b) $N(6, 2)$
15. Show that the line $y = 3$ meets the parabola $y = 4 - x^2$ at $D(1, 3)$ and $E(-1, 3)$. Find the equations of the tangents to $y = 4 - x^2$ at D and E , and find the point where these tangents intersect. Sketch the situation.
16. The tangent and normal to $f(x) = 9 - x^2$ at the point $K(1, 8)$ meet the x -axis at A and B respectively. Sketch the situation, find the equations of the tangent and normal, find the coordinates of A and B , and hence find the length AB and the area of $\triangle AKB$.
17. The tangent and normal to the cubic $f(x) = x^3$ at the point $U(1, 1)$ meet the y -axis at P and Q respectively. Sketch the situation and find the equations of the tangent and normal. Find the coordinates of P and Q , and the area of $\triangle QUP$.
18. Find the derivative of the general quadratic $f(x) = ax^2 + bx + c$, and hence find the coordinates of the point on its graph where the tangent is horizontal.

19. Show that the tangents at the x -intercepts of $f(x) = x^2 - 4x - 45$ have opposite gradients.
20. Find the derivative of the cubic $f(x) = x^3 + ax + b$, and hence find the x -coordinates of the points where the tangent is horizontal. For what values of a and b do such points exist?
21. [Change of pronumeral] (a) Find $G'(3)$, if $G(t) = t^3 - 4t^2 + 6t - 27$.
 (b) Given that $\ell(H) = \frac{1}{H}$, find $\ell'(2)$.
 (c) If $Q(k) = ak^2 - a^2k$, where a is constant, find $Q'(a)$, $Q'(0)$ and $|Q'(0) - Q'(a)|$.
22. Sketch the graph of $f(x) = x^2 - 6x$ and find the gradient of the tangent and normal at the point $A(a, a^2 - 6a)$ on the curve. Hence find the value of a if:
 (a) the tangent has gradient (i) 0, (ii) 2, (iii) $\frac{1}{2}$,
 (b) the normal has gradient (i) 4, (ii) $\frac{1}{4}$, (iii) 0,
 (c) the tangent has angle of inclination 135° ,
 (d) the normal has angle of inclination 30° ,
 (e) the tangent is (i) parallel, (ii) perpendicular, to $2x - 3y + 4 = 0$.
23. (a) The tangent at $T(a, a^2)$ on the graph of $f(x) = x^2$ meets the x -axis at U and the y -axis at V . Find the equation of this tangent, and show that $\triangle OUV$ has area $|\frac{1}{4}a^3|$ square units. (b) Hence find the coordinates of T for which this area will be $31\frac{1}{4}$.
24. (a) Find the equation of the tangent to $y = x^2 + 9$ at the point P with x -coordinate x_0 , and hence show that its x -intercept is $\frac{x_0^2 - 9}{2x_0}$. (b) Hence find the point(s) on the curve whose tangents pass through the origin. Draw a sketch of the situation.
25. Show that the equation of the tangent to $y = 1/x$ at the point $A(a, 1/a)$ is $x + a^2y = 2a$. Hence, with an explanatory sketch, find the point(s) where the tangent:
 (a) has x -intercept 1, (c) passes through $(\frac{3}{2}, \frac{1}{2})$,
 (b) has y -intercept -1 , (d) passes through the origin.
26. (a) Find the equation of the tangent to $y = \sqrt{x} - 1$ at the point where $x = t$.
 (b) Hence find t and the equation of the tangent if the tangent passes through the origin.
 (c) Draw a sketch.
27. Using similar methods, find the points on $y = x^2 + 5$ where a line drawn from the origin can touch the curve (and draw a sketch of the situation).
28. Use the $u \rightarrow x$ method to differentiate $f(x) = x^7$ by first principles.
29. Differentiate $\sqrt{x}$ by the $h \rightarrow 0$ method, using the method of 'rationalising the numerator':

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}.$$

EXTENSION

30. Yet another formula for the derivative is $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$. Draw a diagram to justify this formula, then use it to find the derivatives of x^2 , x^3 , $1/x$ and $\sqrt{x}$.
31. The tangents to $y = x^2$ at two points $A(a, a^2)$ and $B(b, b^2)$ on the curve meet at K . Prove that the x -coordinate of K is the arithmetic mean of the x -coordinates of A and B , and the y -coordinate of K is the geometric mean of the y -coordinates of A and B .

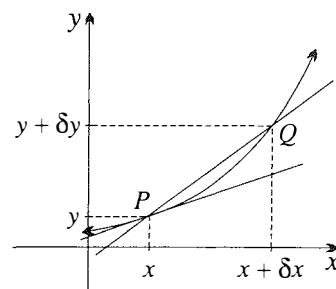
32. (a) Write down the equation of the tangent to the parabola $y = ax^2 + bx + c$ (where $a \neq 0$) at the point P where $x = x_0$, and show that the condition for the tangent at P to pass through the origin is $ax_0^2 - c = 0$. Hence find the condition on a , b and c for such tangents to exist, and the equations of these tangents. (b) Find the points A and B where the tangents from the origin touch the curve, and show that the y -intercept $C(0, c)$ is the midpoint of the interval joining the origin and the midpoint of the chord AB . Show also that the tangent at the y -intercept C is parallel to the chord AB . (c) Hence show that $\triangle OAB$ has four times the area of $\triangle OCA$, and find the area of $\triangle OAB$.

7D The Notation $\frac{dy}{dx}$ for the Derivative

The purpose of this section is to introduce Leibniz's original notation for the derivative, which remains the most widely used and best known notation — it is even said that Dee Why Beach was named after the derivative dy/dx . The notation is extremely flexible, as will soon become evident, and clearly expresses the fact that the derivative is very like a fraction.

Small Changes in x and in y : Let $P(x, y)$ be any point on the graph of a function. Suppose that x changes by a small amount δx to $x + \delta x$, and let y change by a corresponding amount δy to $y + \delta y$. Let the new point be $Q(x + \delta x, y + \delta y)$. Then

$$\text{gradient } PQ = \frac{\delta y}{\delta x} \quad (\text{rise over run}).$$



When δx is small, the secant PQ is almost the same as the tangent at P , and, as before, the derivative is the limit of $\delta y/\delta x$ as $\delta x \rightarrow 0$. This is the basis for Leibniz's notation.

DEFINITION: Let δy be the small change in y resulting from a small change δx in x . Then the derivative dy/dx is

9

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}.$$

The object dx is intuitively understood as an 'infinitesimal change' in x , dy as the corresponding 'infinitesimal change' in y , and the derivative dy/dx as the ratio of these infinitesimal changes. Infinitesimal changes, however, are for the intuition only — the logic of the situation is that:

10 The derivative $\frac{dy}{dx}$ is not a fraction, but is the limit of the fraction $\frac{\delta y}{\delta x}$.

The genius of the notation is that the derivative is a gradient, and the gradient is a fraction, and the notation dy/dx preserves the intuition of fractions. The small differences δx and δy , and the infinitesimal differences dx and dy , are the origins of the word 'differentiation'.

Operator Notation: The derivative $\frac{dy}{dx}$ can also be regarded as the operator $\frac{d}{dx}$ operating on the function y . This operator is also written D_x , giving two further alternative notations for the derivative:

$$\frac{d}{dx}(x^2 + x - 1) = 2x + 1 \quad \text{and} \quad D_x(x^2 + x - 1) = 2x + 1.$$

WORKED EXERCISE: [These are examples of two further techniques used in differentiation — dividing through by the denominator, and using fractional indices.] Differentiate the following functions:

$$(a) \ y = \frac{x^3 + x^2 + x + 1}{x} \quad (b) \ y = 6x\sqrt{x} \quad (c) \ y = \sqrt[3]{x^2} \quad (d) \ y = \frac{10x - 2}{\sqrt{x}}$$

SOLUTION:

$$\begin{aligned} (a) \ y &= \frac{x^3 + x^2 + x + 1}{x} & (b) \ y &= 6x\sqrt{x} & (c) \ y &= \sqrt[3]{x^2} & (d) \ y &= \frac{10x - 2}{\sqrt{x}} \\ &= x^2 + x + 1 + \frac{1}{x} & &= 6x^{1\frac{1}{2}} & &= x^{\frac{2}{3}} & &= 10x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= 2x + 1 - \frac{1}{x^2} & \frac{dy}{dx} &= 6 \times \frac{3}{2}x^{\frac{1}{2}} & \frac{dy}{dx} &= \frac{2}{3}x^{-\frac{1}{3}} & \frac{dy}{dx} &= 5x^{-\frac{1}{2}} + x^{-\frac{3}{2}} \\ & & &= 9\sqrt{x} & & & &= \frac{5x + 1}{x\sqrt{x}} \end{aligned}$$

WORKED EXERCISE: [These two worked exercises show how dy/dx notation is used to perform calculations on the geometry of a curve.] Find the equations of the tangent and normal to the curve $y = 4 - x^2$ at the point $P(1, 3)$ on the curve.

SOLUTION: Here $\frac{dy}{dx} = -2x$, so at $P(1, 3)$, $\frac{dy}{dx} = -2$.

Hence the tangent at P has gradient -2 and the normal has gradient $\frac{1}{2}$, so the tangent is $y - 3 = -2(x - 1)$

$$y = -2x + 5,$$

and the normal is $y - 3 = \frac{1}{2}(x - 1)$

$$y = \frac{1}{2}x + 2\frac{1}{2}.$$

WORKED EXERCISE:

(a) Find the equation of the tangent to $y = x^2 + x + 1$ at $P(a, a^2 + a + 1)$.

(b) Hence find the equations of the tangents passing through the origin.

SOLUTION:

(a) Differentiating, $\frac{dy}{dx} = 2x + 1$,

so at P , $\frac{dy}{dx} = 2a + 1$,

and the tangent is $y - (a^2 + a + 1) = (2a + 1)(x - a)$

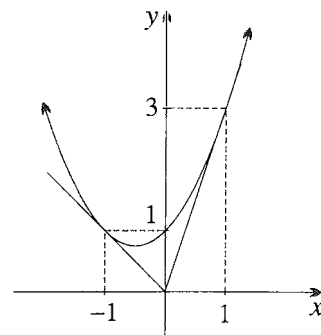
$$y = (2a + 1)x - a^2 + 1.$$

(b) Substituting $(0, 0)$,

$$0 = -a^2 + 1$$

$$a = 1 \text{ or } -1,$$

and the tangents are $y = 3x$ and $y = -x$.



Exercise 7D

- Find the derivative $\frac{dy}{dx}$ of each function, and the value of $\frac{dy}{dx}$ when $x = -1$:
 - $y = x^4 - x^2 + 1$
 - $y = ax^2 + bx + c$
 - $y = (2x - 1)(x - 2)$
 - $y = x^2(ax - c)$
 - $y = \frac{9}{x^3}$
 - $y = 12\sqrt{x}$
 - $y = \frac{a}{x} + \frac{a}{x^2}$
 - $y = \sqrt{121x}$
 - Differentiate each function by first dividing through by the denominator:
 - $\frac{3x^4 - 5x^2}{x}$
 - $\frac{5x^6 + 4x^5}{3x^3}$
 - $\frac{4x^3 - 6}{2x^2}$
 - $\frac{ax^3 - bx^2 + cx - d}{x^2}$
 - Find, in index form, the derivative $\frac{dy}{dx}$ of:
 - $y = x^{2\frac{1}{2}}$
 - $y = x^{-\frac{1}{2}}$
 - $y = 4x^{\frac{3}{4}}$
 - $y = 5x^{-\frac{2}{3}}$
 - $y = -10x^{-0.6}$
 - Differentiate each function by rewriting it using index form:
 - $y = 12x\sqrt{x}$
 - $y = 4x^2\sqrt{x}$
 - $y = \frac{6}{\sqrt{x}}$
 - $y = \frac{5}{x\sqrt{x}}$
 - $y = 15\sqrt[5]{x}$
 - Using $\frac{dy}{dx}$ notation, find the tangent and normal to each curve at the point indicated:
 - $y = x^2 - 6x$ at $O(0, 0)$,
 - $y = \sqrt{x}$ at $K(4, 2)$,
 - $y = x^2 - x^4$ at $J(-1, 0)$,
 - $y = x^3 - 3x + 2$ at $P(1, 0)$.
 - Find any points on each graph where the tangent has gradient -1 :
 - $y = 1/x$
 - $y = \frac{1}{2}x^{-2}$
 - $y = 3 - \frac{1}{3}x^3$
 - $y = x^3 + 1$
 - $y = -\sqrt{x}$
 - Find any points on each curve where the tangent has the given angle of inclination:
 - $y = \frac{1}{3}x^3 - 7$, 45°
 - $y = x^2 + \frac{1}{3}x^3$, 135°
 - $y = x^2 + 1$, 120°
 - $y = 2\sqrt{x}$, 30°
 - Find the x -coordinates of any points on each curve where the normal is vertical:
 - $y = 3 - 2x + x^2$
 - $y = x^4 - 18x^2$
 - $y = x + \frac{1}{x}$
- DEVELOPMENT —————
- Find where $y = -2x$ meets $y = (x + 2)(x - 3)$.
 - Find, to the nearest minute, the angles that the tangents to $y = (x + 2)(x - 3)$ at these points make with the x -axis. Sketch the situation.
 - Find, to four significant figures, the x -coordinate of the point where the tangent has the given angle of inclination:
 - $y = x^2 + 3x$, 22°
 - $y = x^4$, $142^\circ 17'$
 - $y = x^{-1}$, 70°
 - For each curve below: (i) find the equation of the tangent at the point P where $x = a$, (ii) hence find the equations of any tangents passing through the origin.
 - $y = x^2 - 10x + 9$
 - $y = x^2 + 15x + 36$
 - $y = 2x^2 - 7x + 6$
 - Differentiate $y = x^2 + bx + c$, and hence find b and c if:
 - the parabola passes through the origin, and the tangent there has gradient 7,
 - the parabola has y -intercept -3 and has gradient -2 there,
 - the parabola is tangent to the x -axis at the point $(5, 0)$,
 - when $x = 3$ the gradient is 5, and $x = 2$ is a zero,
 - the parabola is tangent to $3x + y - 5 = 0$ at the point $T(3, -1)$,
 - the line $3x + y - 5 = 0$ is a normal at the point $T(3, -1)$.

13. Find the derivative $\frac{dy}{dx}$ of each of the following functions:

$$\begin{array}{lll}
 \text{(a)} \ y = 3x^2\sqrt{x} - 2x\sqrt{x} & \text{(e)} \ y = \frac{3x^2 - 2x + 4}{\sqrt{x}} & \text{(h)} \ y = \sqrt{x^3} \\
 \text{(b)} \ y = 3x\sqrt{x} \times 4x^2\sqrt{x} & \text{(f)} \ y = \frac{x - 2\sqrt{x} + 1}{\sqrt{x}} & \text{(i)} \ y = 4\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \\
 \text{(c)} \ y = 3\sqrt{2x} \times 2\sqrt{18x} & & \text{(j)} \ y = a\left(x^2 - \frac{1}{x^2}\right)^2 \\
 \text{(d)} \ y = \frac{1}{\pi}x^\pi + \pi x^{\frac{1}{\pi}} & \text{(g)} \ y = \left(x + \frac{1}{x}\right)^2 &
 \end{array}$$

14. For each of the following functions, find the value of $\frac{dy}{dx}$ when $x = 1$:

$$\begin{array}{lll}
 \text{(a)} \ y = a^2x - ax^2 & \text{(d)} \ y = 3(4x^{-1} - 2x^{-2}) & \text{(g)} \ y = 1 + x^{-1} + \dots + x^{-6} \\
 \text{(b)} \ y = \frac{a}{x} - \frac{x}{a} & \text{(e)} \ y = n^3x^2\left(nx + \frac{n}{x}\right) & \text{(h)} \ y = \frac{1}{x^3\sqrt{x}} \\
 \text{(c)} \ y = (\sqrt{x} - 3)(\sqrt{x} - 4) & \text{(f)} \ y = x^6 + x^5 + \dots + 1 & \text{(i)} \ y = (2x)^n
 \end{array}$$

15. If $P = tx^2 + 3tu^2 + 3xu + t$, find $\frac{dP}{dx}$, $\frac{dP}{du}$ and $\frac{dP}{dt}$ (assuming that when differentiating with respect to one variable, the other pronumerals are constant).

16. The equation of the path of a ball thrown from the origin is $y = x(12 - x)$, with units in metres (the origin is at ground level). Sketch the curve and find its derivative, keeping in mind that the direction of motion at any point is the direction of the tangent at that point.

- How far from the origin does the ball land if the ground is level?
- Find the x -coordinate of the point H where the direction of motion is horizontal.
- Hence find the maximum height of the ball above the ground.
- Find at what angle the ball was initially thrown (find the gradient at O).
- Show that on level ground, it lands at the same acute angle to the ground.
- At what angle to the ground is the ball moving when it is at the point $P(2, 20)$?
- Show that the gradient of the flight path when $x = a$ is the opposite of the gradient of the flight path when $x = 12 - a$. What does this tell you about the two directions of flight?
- Let ℓ be the line of flight if there were no gravity to deflect the ball. Let A be the point on ℓ directly to the left of the point H , and B be the point on ℓ directly above H . Find the equation of ℓ and the distances HA and HB .

17. Show that the line $x + y + 2 = 0$ is a tangent to $y = x^3 - 4x$, and find the point of contact. [HINT: Find the equations of the tangents parallel to $x + y + 2 = 0$, and show that one of them is this very line.]

18. Find the tangent to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin, and show that this line is also the tangent to the curve at the point $(2, 2)$.

19. Find the points where the line $x + 2y = 4$ cuts the parabola $y = (x - 1)^2$, and show that the line is the normal to the curve at one of these points.

20. Find the equation of the tangent to $y = x^2 + 2x - 8$ at the point K on the curve with x -coordinate a . Hence find the points on the curve where the tangents from $H(2, -1)$ touch the curve.

21. (a) If $y = Ax^n$, show that $x \frac{dy}{dx} = ny$. (b) If $y = \frac{C}{x^n}$, show that $x \frac{dy}{dx} = -ny$.
- (c) (i) If $y = a\sqrt{x}$, show that $y \frac{dy}{dx}$ is a constant. (ii) Conversely, if $y = ax^n$ and $a \neq 0$, find $y \frac{dy}{dx}$ and show that it is constant if and only if $n = \frac{1}{2}$ or 0.
22. Find the equation of the tangent to the parabola $y = (x - 3)^2$ at the point T where $x = a$, find the coordinates of the x -intercept A and y -intercept B of the tangent, and find the midpoint M of AB . For what value of a does M coincide with T ?
23. (a) Find the equation of the tangent to the hyperbola $xy = c$ at the point $T(t, c/t)$, find the points A and B where the tangent meets the x -axis and y -axis respectively, and show that A is independent of c .
- (b) Find the area of $\triangle OAB$ and show that it is constant as T varies.
- (c) Show that T bisects AB and that $OT = AT = BT$.
- (d) Hence explain why the rectangle with diagonal OT has a constant area that is half the area of $\triangle OAB$.
- (e) Find the perpendicular distance from the tangent to O , and the length AB , and hence calculate the area of $\triangle OAB$ by this alternative method. Draw sketches for c and t positive and negative.
24. [For discussion] Sketch the graph of $y = x^3$. Then choose any point in the plane and check by examining the graph that at least one tangent to the curve passes through every point in the plane. What points in the plane have three tangents to the curve passing through them? This problem can also be solved algebraically, but that is considerably harder.

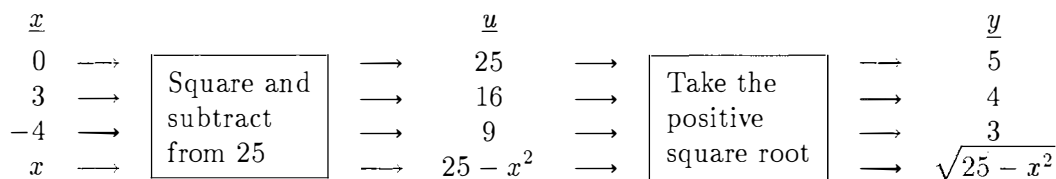
EXTENSION

25. (a) Show that the tangent to $\mathcal{P}: y = ax^2 + bx + c$ with gradient m has y -intercept $c - \frac{(m - b)^2}{4a}$. (b) Hence find the equations of any quadratics that pass through the origin and are tangent to both $y = -2x - 4$ and to $y = 8x - 49$. (c) Find also any quadratics that are tangent to $y = -5x - 10$, to $y = -3x - 7$ and to $y = x - 7$.
26. Suppose $y = ax^3 + bx^2 + cx + d$ is a cubic (so that $a \neq 0$). Show that every point in the plane lies on at least one tangent to this cubic.

7E The Chain Rule

Sections 7E, 7F and 7G will develop three methods that extend the rules for differentiation to cover compound functions of various types.

A Chain of Functions: The semicircle function $y = \sqrt{25 - x^2}$ is the composition of two functions — ‘square and subtract from 25’, followed by ‘take the positive square root’. We can represent the situation by a chain of functions:



The middle column is the output of the first function ‘subtract the square from 25’, and is then the input of the second function ‘take the positive square root’. This decomposition of the original function $y = \sqrt{25 - x^2}$ into the chain of functions may be expressed as follows:

$$\text{‘Let } u = 25 - x^2, \text{ then } y = \sqrt{u}.’$$

The Chain Rule: Suppose then that y is a function of u , where u is a function of x . Using the dy/dx notation for the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right) \quad (\text{multiplying top and bottom by } \delta u) \\ &= \lim_{\delta u \rightarrow 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x} \quad (\text{because } \delta u \rightarrow 0 \text{ as } \delta x \rightarrow 0) \\ &= \frac{dy}{du} \times \frac{du}{dx}. \end{aligned}$$

In practice, although the proof uses limits, the usual attitude to this rule is that ‘the du ’s cancel out’. The chain rule should thus be remembered in the form:

11	THE CHAIN RULE: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

WORKED EXERCISE: Use the chain rule to differentiate the functions:

(a) $(x^2 + 1)^6$ (b) $7(3x + 4)^5$ (c) $(ax + b)^n$ (d) $\sqrt{25 - x^2}$

NOTE: The working in the right-hand column is a recommended way to set out the calculation. The calculation should begin with that working, because the first step is the decomposition of the function into a chain of two functions.

SOLUTION:

(a) Let $y = (x^2 + 1)^6$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 6(x^2 + 1)^5 \times 2x \\ &= 12x(x^2 + 1)^5. \end{aligned}$$

Let $u = x^2 + 1$,

then $y = u^6$.

So $\frac{du}{dx} = 2x$

and $\frac{dy}{du} = 6u^5$.

(b) Let $y = 7(3x + 4)^5$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 35(3x + 4)^4 \times 3 \\ &= 105(3x + 4)^4. \end{aligned}$$

Let $u = 3x + 4$,

then $y = 7u^5$.

So $\frac{du}{dx} = 3$

and $\frac{dy}{du} = 35u^4$.

(c) Let $y = (ax + b)^n$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= n(ax + b)^{n-1} \times a \\ &= an(ax + b)^{n-1}. \end{aligned}$$

Let $u = ax + b$,

then $y = u^n$.

So $\frac{du}{dx} = a$

and $\frac{dy}{du} = nu^{n-1}$.

(d) Let $y = \sqrt{25 - x^2}$.

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2\sqrt{25 - x^2}} \times (-2x) \\ &= -\frac{x}{\sqrt{25 - x^2}},\end{aligned}$$

Let $u = 25 - x^2$,

then $y = \sqrt{u}$.

So $\frac{du}{dx} = -2x$

and $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$.

which agrees with the calculation by geometric methods in Section 7A.

Powers of a Linear Function: Part (c) of the previous exercise should be remembered as a formula for differentiating any linear function of x raised to a power.

12 POWERS OF A LINEAR FUNCTION: $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$

WORKED EXERCISE: Use the standard form above to differentiate:

(a) $(4x - 1)^7$

(b) $\sqrt{5 - 3x}$

(c) $\frac{1}{7 - x}$

SOLUTION:

(a) $\frac{d}{dx}(4x - 1)^7 = 28(4x - 1)^6$ (with $a = 4$, $b = -1$ and $n = 7$).

(b) $\frac{d}{dx}\sqrt{5 - 3x} = \frac{-3}{2\sqrt{5 - 3x}}$ (with $a = -3$, $b = 5$ and $n = \frac{1}{2}$).

(c) $\frac{d}{dx}(7 - x)^{-1} = (7 - x)^{-2}$ (with $a = -1$, $b = 7$ and $n = -1$).

Parametric Differentiation: In many later situations, a curve will be specified by two equations giving x and y in terms of some third variable t , called a *parameter*. For example,

$$x = 2t, \quad y = t^2$$

specifies the parabola $y = \frac{1}{4}x^2$, as can be seen by eliminating t from the two equations. In this situation it is very simple to calculate dy/dx directly using *parametric differentiation*. The formula below is another version of the chain rule, because ‘the dt ’s just cancel out’.

13 PARAMETRIC FUNCTIONS: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

WORKED EXERCISE: In the example above, $\frac{dy}{dx} = \frac{2t}{2} = t$.

Differentiating Inverse Functions: Suppose that y is a function of x , and that the inverse is also a function, so that x is a function of y . Then by the chain rule, $\frac{dy}{dx} \times \frac{dx}{dy} = 1$, and so:

14 INVERSE FUNCTIONS: $\frac{dx}{dy} = \frac{1}{dy/dx}$ (provided neither is zero).

WORKED EXERCISE: Differentiate $y = x^{\frac{1}{3}}$: (a) directly, (b) by first forming the inverse function and then differentiating.

SOLUTION:

(a) Using the usual rule for differentiating powers of x ,

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}.$$

(b) Solving for x ,

Then

$$\begin{aligned} x &= y^3. \\ \frac{dx}{dy} &= 3y^2, \\ \text{and taking reciprocals, } \frac{dy}{dx} &= \frac{1}{3y^2} \\ &= \frac{1}{3}x^{-\frac{2}{3}}. \end{aligned}$$

Completion of Proof that x^n has Derivative nx^{n-1} : The chain rule allows us to complete the proof of the derivative of x^n , at least for rational values of the index n .

THEOREM: $\frac{d}{dx}x^n = nx^{n-1}$, for all rational values of n .

PROOF: The result is already proven when n is a cardinal number, when $n = \frac{1}{2}$, and when $n = -1$.

A. Suppose $y = x^{-m}$, where $m \geq 2$ is an integer.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= \left(-\frac{1}{x^{2m}}\right) \times mx^{m-1}$ $= -mx^{-m-1}, \text{ as required.}$	Let $u = x^m$, then $y = \frac{1}{u}$. So $\frac{du}{dx} = mx^{m-1}$, and $\frac{dy}{du} = -\frac{1}{u^2}$ (proven earlier).

B. Suppose $y = x^{\frac{1}{k}}$, where $k \geq 2$ is an integer.

Then $x = y^k$,

so $\frac{dx}{dy} = ky^{k-1}$, since k is a positive integer,

and $\frac{dy}{dx} = \frac{1}{ky^{k-1}}$, since $\frac{dy}{dx}$ is the reciprocal of $\frac{dx}{dy}$,

$$\begin{aligned} &= \frac{1}{kx^{\frac{k-1}{k}}} \\ &= \frac{1}{k}x^{\frac{1}{k}-1}, \text{ as required.} \end{aligned}$$

C. Suppose $y = x^{\frac{m}{k}}$, where m and k are integers and $k \geq 2$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= mx^{\frac{m-1}{k}} \times \frac{1}{k}x^{\frac{1}{k}-1}$ $= \frac{m}{k}x^{\frac{m}{k}-1}, \text{ as required.}$	Let $u = x^{\frac{1}{k}}$, then $y = u^m$. So $\frac{du}{dx} = \frac{1}{k}x^{\frac{1}{k}-1}$, (by B), and $\frac{dy}{du} = mu^{m-1}$.

NOTE ON IRRATIONAL INDICES: We do not have a precise definition of powers like x^π or $x^{\sqrt{2}}$ with irrational indices, so we can hardly give a rigorous proof that the derivative of x^n is indeed nx^{n-1} for irrational values of n . Nevertheless, since every irrational is 'as close as we like' to a rational number for which the theorem is certainly true, the result is intuitively clear.

Exercise 7E

1. Use the chain rule to differentiate each function. Be careful in each example to identify u as a function of x , and y as a function of u .

(a) $y = (3x + 7)^4$	(e) $y = 8(7 - x^2)^4$	(i) $y = \sqrt{3 - 2x}$
(b) $y = (5 - 4x)^7$	(f) $y = (x^2 + 3x + 1)^9$	(j) $y = 7\sqrt{x^2 + 1}$
(c) $y = (px + q)^8$	(g) $y = -3(x^3 + x + 1)^6$	(k) $y = \sqrt{9 - x^2}$
(d) $y = (x^2 + 1)^{12}$	(h) $y = \sqrt{5x + 4}$	(l) $y = -\sqrt{a^2 - b^2x^2}$

2. Use the standard form $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate:

(a) $y = (5x - 7)^5$	(e) $y = \frac{1}{2 - x}$	(h) $y = \sqrt{x + 4}$
(b) $y = (4 - 3x)^7$	(f) $y = \frac{1}{3 + 5x}$	(i) $y = \sqrt{4 - 3x}$
(c) $y = (2 - 3x)^{-5}$	(g) $y = -\frac{5}{(x + 1)^3}$	(j) $y = \sqrt{mx - b}$
(d) $y = p(q - x)^{-4}$		(k) $y = (5 - x)^{-\frac{1}{2}}$

3. Use parametric differentiation to find dy/dx , then evaluate dy/dx when $t = -1$:

(a) $x = 5t$	(b) $x = ct$	(c) $x = at + b$	(d) $x = 2t^2$
$y = 10t^2$	$y = c/t$	$y = bt + a$	$y = 3t^3$

4. Differentiate each function, and hence find the coordinates of any points where the tangent is horizontal:

(a) $y = (x^2 - 1)^3$	(e) $y = 24 - 7(x - 5)^2$	(i) $y = \frac{1}{1 + x^2}$
(b) $y = (x^2 - 4x)^4$	(f) $y = 4 + (x - 5)^6$	(j) $y = \sqrt{x^2 - 2x + 5}$
(c) $y = (2x + x^2)^5$	(g) $y = a(x - h)^2 + k$	(k) $y = \sqrt{x^2 - 2x}$
(d) $y = \frac{1}{5x + 2}$	(h) $y = \sqrt{3 - 2x}$	

5. Find the equations of the tangent and normal at the point where $x = 1$ to:

(a) $y = (5x - 4)^4$	(b) $y = (x^2 + 1)^3$	(c) $y = (x^2 + 1)^{-1}$	(d) $y = \sqrt{x - 2}$

6. Find the x -coordinates of any points on $y = (4x - 7)^3$ where the tangent is:

(a) parallel to $y = 108x + 7$	(b) perpendicular to $x + 12y + 6 = 0$

DEVELOPMENT

7. Find the tangent to each curve at the point where $t = 3$:

(a) $x = 5t^2, y = 10t$	(b) $x = (t - 1)^2, y = (t - 1)^3$

8. Find the value of a if:

(a) $y = \frac{1}{x + a}$ has gradient -1 when $x = 6$,
(b) $y = (x - a)^3$ has gradient 12 when $x = 6$.

9. (a) Find the equation of the tangent to $y = \frac{1}{x - 4}$ at the point L where $x = b$.

- (b) Hence find the equations of the tangents passing through:

(i) the origin,	(ii) $W(6, 0)$.

10. Differentiate:

$$(a) y = (\sqrt{x} - 3)^{11}$$

$$(b) y = 3\sqrt{4 - \frac{1}{2}x}$$

$$(c) y = \frac{3}{1 - x\sqrt{2}}$$

$$(d) y = (5 - x)^{-\frac{1}{2}}$$

$$(e) y = \frac{-a}{\sqrt{1 + ax}}$$

$$(f) y = \frac{b}{\sqrt{c - \frac{1}{2}x}}$$

$$(g) y = -4 \left(x + \frac{1}{x} \right)^4$$

$$(h) y = \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^6$$

11. Find the values of a and b if the parabola $y = a(x + b)^2 - 8$:

(a) has tangent $y = 2x$ at the point $P(4, 8)$,

(b) has a common tangent with $y = 2 - x^2$ at the point $A(1, 1)$.

12. Use the chain rule to show that: (a) $\frac{d}{dx}(x^2)^3 = 6x^5$ (b) $\frac{d}{dx}(x^k)^\ell = k\ell x^{k\ell-1}$

13. (a) Differentiate the semicircle $y = \sqrt{169 - x^2}$, find the equation of the tangent at $P(12, 5)$, and find the x -intercept and y -intercept of the tangent.

(b) Show that the perpendicular distance from the tangent to the centre equals the radius.

(c) Find the area of the triangle enclosed by the tangent and the two axes.

(d) Find the perimeter of this triangle.

14. (a) Let the point $P(4, 3)$ lie on the semicircle $y = \sqrt{25 - x^2}$, and let $Q(4, \frac{9}{5})$ lie on the curve $y = \frac{3}{5}\sqrt{25 - x^2}$ (which is half an ellipse). Find the equations of the tangents at P and at Q , and show that they intersect on the x -axis.

(b) Find the equation of the tangent at the point P with x -coordinate $x_0 > 0$ on the curve $y = \lambda\sqrt{25 - x^2}$ (again, half an ellipse). Let the tangent meet the x -axis at T , let the ellipse meet the x -axis at $A(5, 0)$, and let the vertical line through P meet the x -axis at M . Show that the point T is independent of λ , and show that OA is the geometric mean of OM and OT .

15. (a) Find the x -coordinates of the points P and Q on $y = (x - 7)^2 + 3$ such that the tangents at P and Q have gradients 1 and -1 respectively.

(b) Show that the square formed by the tangents and normals at P and Q has area $\frac{1}{2}$.

EXTENSION

16. (a) Find the x -coordinates of the points P and Q on $y = (x - h)^2 + k$ such that the tangents at P and Q have gradients m and $-m$ respectively.

(b) Find the area of the quadrilateral formed by the tangents and normals at P and Q .

17. (a) Show that the tangent to $\mathcal{P}$: $y = a(x - h)^2 + k$ at the point T where $x = \alpha$ is $y = 2a(\alpha - h)x + k - a(\alpha^2 - h^2)$.

(b) Hence show that the vertical distance between the vertex $V(h, k)$ and the tangent at T is proportional to the square of the distance between α and the axis of symmetry.

(c) Find the equations of the tangents to $\mathcal{P}$ through the origin, and the x -coordinates of the points of contact.

18. (a) Develop a three-step chain rule for the derivative dy/dx , where y is a function of u , u is a function of v , and v is a function of x . Hence differentiate $y = \frac{1}{1 + \sqrt{1 - x^2}}$.

(b) Generalise the chain rule to n steps.

7F The Product Rule

The product rule extends the methods for differentiation to cover functions that are products of two simpler functions. Suppose then that $y = uv$ is the product of two functions u and v , each of which is a function of x . Then, as we shall prove after the worked exercise:

15 DERIVATIVE OF A PRODUCT: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$ or $y' = vu' + uv'$

The second form uses the convention of the dash ' to represent differentiation with respect to x , so $y' = dy/dx$ and $u' = du/dx$ and $v' = dv/dx$.

NOTE: The product rule can seem difficult to use with the algebraic functions under consideration at present, because the calculations can easily become quite involved. The rule will seem more straightforward later, in the context of exponential and trigonometric functions.

WORKED EXERCISE: Differentiate each function, expressing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

(a) $x(x-10)^4$ (b) $x^2(3x+2)^3$ (c) $x\sqrt{x+3}$

SOLUTION:

(a) Let $y = x(x-10)^4$. Then $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$ $= (x-10)^4 \times 1 + x \times 4(x-10)^3$ $= (x-10)^3(x-10+4x)$ $= 5(x-10)^3(x-2).$	Let $u = x$ and $v = (x-10)^4$. Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 4(x-10)^3$.

So the derivative is zero for $x = 10$ and for $x = 2$.

(b) Let $y = x^2(3x+2)^3$. Then $y' = vu' + uv'$ $= 2x(3x+2)^3 + 9x^2(3x+2)^2$ $= x(3x+2)^2(6x+4+9x)$ $= x(3x+2)^2(15x+4).$	Let $u = x^2$ and $v = (3x+2)^3$. Then $u' = 2x$ and $v' = 9(3x+2)^2$.

So the derivative is zero for $x = 0$, $x = -\frac{2}{3}$ and for $x = -\frac{4}{15}$.

(c) Let $y = x\sqrt{x+3}$. Then $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$ $= \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$ $= \frac{2(x+3) + x}{2\sqrt{x+3}} \quad (\text{common denominator})$ $= \frac{3(x+2)}{2\sqrt{x+3}}, \text{ which is zero for } x = -2.$	Let $u = x$ and $v = \sqrt{x+3}$. Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{2\sqrt{x+3}}$.

Proof of the Product Rule: Suppose that x changes to $x + \delta x$, and that as a result, u changes to $u + \delta u$, v changes to $v + \delta v$, and y changes to $y + \delta y$.

$$\begin{aligned}
&\text{Here} && y = uv, \\
&\text{and} && y + \delta y = (u + \delta u)(v + \delta v) \\
&&& = uv + v\delta u + u\delta v + \delta u\delta v, \\
&\text{so} && \delta y = v\delta u + u\delta v + \delta u\delta v. \\
&\text{Hence, dividing by } \delta x, && \frac{\delta y}{\delta x} = v \frac{\delta u}{\delta x} + u \frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} \times \frac{\delta v}{\delta x} \times \delta x, \\
&\text{and taking limits as } \delta x \rightarrow 0, && \frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} + 0, \text{ as required.}
\end{aligned}$$

Exercise 7F

- Differentiate each function: (i) by expanding the product and differentiating each term, (ii) using the product rule.
 - $y = x^3(x - 2)$
 - $y = (2x + 1)(x - 5)$
 - $y = (x^2 - 3)(x^2 + 3)$
- Differentiate these functions using the product rule, identifying the factors u and v in each example. Express your answers in fully factored form, and state the values of x for which the derivative is zero.
 - $y = x(3 - 2x)^5$
 - $y = x^5(1 - x)^7$
 - $y = 2(x + 1)^3(x + 2)^4$
 - $y = x^3(x + 1)^4$
 - $y = (x - 1)(x - 2)^3$
 - $y = (2x - 3)^4(2x + 3)^5$
- Find the tangents and normals to these curves at the indicated points:
 - $y = x(1 - x)^6$ at the origin
 - $y = (2x - 1)^3(x - 2)^4$ at $A(1, 1)$

DEVELOPMENT

- Differentiate each function using the product rule, giving your answer in fully factored form. At least one of the factors will require the chain rule to differentiate it.
 - $y = x(x^2 + 1)^5$
 - $y = 2\pi x^3(1 - x^2)^4$
 - $y = -2(x^2 + x + 1)^3x$
 - $y = (2 - 3x^2)^4(2 + 3x^2)^5$
- Differentiate $y = (x^2 - 10)^3x^4$, using the chain rule to differentiate the first factor. Hence find the points on the curve where the tangent is horizontal.
- Differentiate each function using the product rule, combining terms using a common denominator and factoring the numerator completely. State the values of x for which the derivative is zero.
 - $y = 6x\sqrt{x + 1}$
 - $y = -4x\sqrt{1 - 2x}$
 - $y = 10x^2\sqrt{2x - 1}$
- What is the domain of $y = x\sqrt{1 - x^2}$?
 - Differentiate $y = x\sqrt{1 - x^2}$, using the chain rule to differentiate the second factor, then combine the terms using a common denominator.
 - Find the points on the curve where the tangent is horizontal.
 - Find the tangent and the normal at the origin.
- Differentiate $y = a(x - \alpha)(x - \beta)$ using the product rule.
 - Show that the tangents at the x -intercepts have opposite gradients and meet at a point M whose x -coordinate is the average of the x -intercepts.
 - Find the point V where the tangent is horizontal. Show that M is vertically above or below V and twice as far from the x -axis. Sketch the situation.

9. Show that if a polynomial $f(x)$ can be written as a product $f(x) = (x - a)^n q(x)$ of the polynomials $(x - a)^n$ and $q(x)$, where $n \geq 2$, then $f'(x)$ can be written as a multiple of $(x - a)^{n-1}$. What does this say about the shape of the curve near $x = a$?
10. Show that the function $y = x^3(1 - x)^5$ has a horizontal tangent at a point P with x -coordinate $\frac{3}{8}$. Show that the y -coordinate of P is $3^3 \times 5^5/8^8$.
11. Prove by mathematical induction that for all positive integers n , $\frac{d}{dx}x^n = nx^{n-1}$. Use only the product rule, and the fact that the derivative of the identity function $f(x) = x$ is $f'(x) = 1$.

EXTENSION

12. (a) Show that the function $y = x^r(1 - x)^s$, where $r, s > 1$, has a horizontal tangent at a point P whose x -coordinate p lies between 0 and 1.
 (b) Show that P divides the interval joining $O(0, 0)$ and $A(1, 0)$ in the ratio $r : s$, and find the y -coordinate of P . What are the coordinates of P if $s = r$?
13. Establish the rule for differentiating a product $y = uvw$, where u, v and w are functions of x . Hence find the derivative of these functions, and the values of x where the tangent is horizontal: (a) $x^5(x - 1)^4(x - 2)^3$ (b) $x(x - 2)^4\sqrt{2x + 1}$
14. Establish the rule for differentiating a product $y = u_1 u_2 \dots u_n$ of n functions of x .

7G The Quotient Rule

The last of these three methods extends the formulae for differentiation to cover functions that are quotients of two simpler functions. Suppose that $y = u/v$ is the quotient of two functions u and v , each of which is a function of x . Then we shall prove, again after the worked exercise:

16

$$\text{DERIVATIVE OF A QUOTIENT: } \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad \text{or} \quad y' = \frac{vu' - uv'}{v^2}$$

WORKED EXERCISE: Differentiate, stating when the derivative is zero:

(a) $\frac{2x + 1}{2x - 1}$ (b) $\frac{\sqrt{x + 1}}{x}$

NOTE: Although these functions could be differentiated using the product rule by expressing them as $(2x + 1)(2x - 1)^{-1}$ and $x^{-1}\sqrt{x + 1}$, the quotient rule makes the work much easier.

SOLUTION:

(a) Let $y = \frac{2x + 1}{2x - 1}$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\ &= \frac{2(2x - 1) - 2(2x + 1)}{(2x - 1)^2} \\ &= \frac{-4}{(2x - 1)^2}, \text{ which is never zero.} \end{aligned}$$

$$\begin{array}{l} \text{Let } u = 2x + 1 \\ \text{and } v = 2x - 1. \\ \text{Then } \frac{du}{dx} = 2 \\ \text{and } \frac{dv}{dx} = 2. \end{array}$$

(b) Let $y = \frac{\sqrt{x+1}}{x}$. Then $y' = \frac{vu' - uv'}{v^2}$ $= \frac{\frac{x}{2\sqrt{x+1}} - \sqrt{x+1}}{x^2} \times \frac{2\sqrt{x+1}}{2\sqrt{x+1}}$ $= \frac{x - 2(x+1)}{2x^2\sqrt{x+1}}$ $= \frac{-x-2}{2x^2\sqrt{x+1}}, \text{ which has no zeroes } (x = -2 \text{ is outside the domain}).$	Let $u = \sqrt{x+1}$ and $v = x$. Then $u' = \frac{1}{2\sqrt{x+1}}$ and $v' = 1$.

Proof of Quotient Rule: We differentiate uv^{-1} using the product rule.

Let $y = uv^{-1}$. Then $\frac{dy}{dx} = V \frac{dU}{dx} + U \frac{dV}{dx}$ $= v^{-1} \frac{du}{dx} - uv^{-2} \frac{dv}{dx}$ $= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ (after multiplying by $\frac{v^2}{v^2}$).	Let $U = u$ and $V = v^{-1}$. Then $\frac{dU}{dx} = \frac{du}{dx}$ and by the chain rule, $\frac{dV}{dx} = \frac{dV}{dv} \times \frac{dv}{dx}$ $= -v^{-2} \frac{dv}{dx}.$

Exercise 7G

1. Differentiate each function using the quotient rule, taking care to identify u and v first. Express your answer in fully factored form, and state any values of x for which the tangent is horizontal.

(a) $y = \frac{x+1}{x-1}$	(c) $y = \frac{3-2x}{x+5}$	(e) $y = \frac{x^2-1}{x^2+1}$	(g) $y = \frac{x^2-a}{x^2-b}$
(b) $y = \frac{2x}{x+2}$	(d) $y = \frac{x^2}{1-x}$	(f) $y = \frac{mx+b}{bx+m}$	(h) $y = \frac{x^n-3}{x^n+3}$

2. Differentiate $y = \frac{1}{3x-2}$:

- (a) by using the chain rule with $u = 3x-2$ and $y = \frac{1}{u}$ (the better method),
- (b) by using the quotient rule with $u = 1$ and $v = 3x-2$.

3. Differentiate $y = \frac{5+2x}{5-2x}$:

- (a) by using the quotient rule (the better method),
- (b) by using the product rule, with the function in the form $y = (5+2x)(5-2x)^{-1}$.

4. For each curve below, find the equations of the tangent and normal and their angles of inclination at the given point:

(a) $y = \frac{x}{5-3x}$ at $K(2, -2)$	(b) $y = \frac{x^2-4}{x-1}$ at $L(4, 4)$

DEVELOPMENT

5. Differentiate, stating any zeroes of the derivative: (a) $\frac{\sqrt{x}+1}{\sqrt{x}+2}$ (b) $\frac{x-3}{\sqrt{x+1}}$
6. (a) Find the value of c if $f'(c) = -3$, where $f(x) = \frac{x^2}{x+1}$.
 (b) Find the value of k if $f'(-3) = 1$, where $f(x) = \frac{x^2+k}{x^2-k}$.
7. (a) Differentiate $y = \frac{x-\alpha}{x-\beta}$.
 (b) Show that for $\alpha > \beta$, all tangents have positive gradient, and for $\alpha < \beta$, all tangents have negative gradient.
 (c) What happens when $\alpha = \beta$?
8. (a) Find the normal to the curve $x = \frac{t}{t+1}$ and $y = \frac{t}{t-1}$ at the point T where $t = 2$.
 (b) Eliminate t from the two equations (by solving the first for t and substituting into the second). Then differentiate this equation to find the gradient of the normal at T .
9. (a) Evaluate $f'(8)$ if $f(x) = \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}}$. (b) Evaluate $g'(5)$ if $g(r) = \frac{(r-6)^3-1}{(r-4)^3+1}$.
10. (a) Sketch the hyperbola $y = \frac{x}{x+1}$, showing the horizontal and vertical asymptotes, and state its domain and range.
 (b) Show that the tangent at the point P where $x = a$ is $x - (a+1)^2y + a^2 = 0$.
 (c) Let the tangent at $A(1, \frac{1}{2})$ meet the x -axis at I , and let G be the point on the x -axis below A . Show that the origin bisects GI .
 (d) Let $T(c, 0)$ be any point on the x -axis. (i) Show that for $c > 0$, no tangents pass through T . (ii) Show that for $c < 0$ and $c \neq -1$, there are two tangents through T whose x -coordinates of their points of contact are opposites of each other. For what values of c are these two points of contact on the same and on different branches of the hyperbola?
11. (a) Suppose $y = \frac{u}{x}$, where u is a function of x . Show that $y + x \frac{dy}{dx} = \frac{du}{dx}$.
 (b) Suppose $y = \frac{x}{u}$, where u is a function of x . Show that $y \frac{du}{dx} + u \frac{dy}{dx} = 1$.

EXTENSION

12. Sketch a point P on a curve $y = f(x)$ where x , $f(x)$ and $f'(x)$ are all positive. Let the tangent, normal and vertical at P meet the x -axis at T , N and M respectively. Let the (acute) angle of inclination of the tangent be $\theta = \angle PTN$, so that $y' = \tan \theta$.
- (a) Using trigonometry, show that:
- | | | |
|------------------|---|-------------------------------|
| (i) $MN = yy'$ | (iii) $\sec \theta = \sqrt{1+y'^2}$ | (v) $PN = y\sqrt{1+y'^2}$ |
| (ii) $TM = y/y'$ | (iv) $\operatorname{cosec} \theta = \sqrt{1+y'^2}/y'$ | (vi) $PT = y\sqrt{1+y'^2}/y'$ |
- (b) Hence find the four lengths when $x = 3$ and: (i) $y = x^2$ (ii) $y = \frac{3x-1}{x+1}$

7H Rates of Change

The derivative has been defined geometrically in this chapter using tangents to the curve, but ever since its introduction the derivative has always been understood also as a rate of change. Let the variable on the horizontal axis be time t , then dy/dt is the ratio of the change in y corresponding to a change in t , when both changes are infinitesimally small. The fractional notation for the derivative as a ratio carries this interpretation of the derivative as a rate:

$$\frac{dy}{dt} = \lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t}.$$

This section deals with rates of change which are not necessarily constant over time.

Using the Chain Rule to Compare Rates: The method is simply to use the chain rule to differentiate with respect to time. This will establish a relation between two rates.

17

RATES: Express one quantity as a function of the other quantity, then differentiate with respect to time using the chain rule.

WORKED EXERCISE: Suppose water is flowing into a large spherical balloon at a constant rate of $50 \text{ cm}^3/\text{s}$.

- At what rate is the radius r increasing when the radius is 7 cm?
- At what rate is the radius increasing when the volume V is $4500\pi \text{ cm}^3$?
- What should the flow rate be changed to so that when the radius is 7 cm, it is increasing at 1 cm/s ?

There are two quantities varying with time here, the volume and the radius. The volume is increasing at a constant rate, but the radius is increasing at a rate that decreases as the balloon expands. The chain rule will allow the two rates of change to be related to each other.

SOLUTION: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Differentiating with respect to time t , $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ (chain rule)

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

- Substituting the known rate $dV/dt = 50$ and the radius $r = 7$,

$$50 = 4\pi \times 49 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{25}{98\pi} \text{ cm/s} \quad (\doteq 0.81 \text{ mm/s}), \text{ the rate of increase of the radius.}$$

- When $V = 4500\pi$, $\frac{4}{3}\pi r^3 = 4500\pi$

$$r^3 = 3375$$

$$r = 15,$$

so substituting again, $50 = 4\pi \times 225 \times \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{18\pi} \quad (\doteq 0.177 \text{ mm/s}).$$

(c) Substituting $r = 7$ and $dr/dt = 1$ gives the rate of change of volume:

$$\begin{aligned}\frac{dV}{dt} &= 4\pi \times 49 \times 1 \\ &= 196\pi \text{ cm}^3/\text{s} \quad (\doteq 616 \text{ cm}^3/\text{s}).\end{aligned}$$

Some Formulae for Solids: These formulae were proven in earlier years.

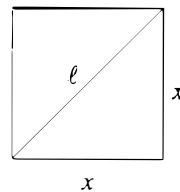
VOLUME AND SURFACE AREA OF SOLIDS:			
18	For a sphere: $V = \frac{4}{3}\pi r^3$ $A = 4\pi r^2$	For a cylinder: $V = \pi r^2 h$ $A = 2\pi r^2 + 2\pi rh$	For a cone: $V = \frac{1}{3}\pi r^2 h$ $A = \pi r^2 + \pi r\ell$ ($\ell = \sqrt{r^2 + h^2}$ is slant height of cone.)
			For a pyramid: $V = \frac{1}{3} \times \text{base} \times \text{height}$ $A = \text{sum of faces}$

Exercise 7H

- Given that $y = x^3 + x$, differentiate with respect to time, using the chain rule.
 - If $dx/dt = 5$, find dy/dt when $x = 2$.
 - If $dy/dt = -6$, find dx/dt when $x = -3$.
- A circular oil stain of radius r and area A is spreading on water. Differentiate the area formula with respect to time to show that $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Hence find:
 - the rate of increase of area when $r = 40$ cm if the radius is increasing at 3 cm/s,
 - the rate of increase of the radius when $r = 60$ cm if the area is increasing at $10 \text{ cm}^2/\text{s}$.
- A spherical bubble of radius r is shrinking so that its volume V is decreasing at a constant rate of $200 \text{ cm}^3/\text{s}$. Show that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.
 - At what rate is its radius decreasing when the radius is 5 cm?
 - What is the radius when the radius is decreasing at $2\pi \text{ cm/s}$?
 - At what rate is the radius decreasing when the volume is $36\pi \text{ cm}^3$?
[HINT: First find the radius when the volume is $36\pi \text{ cm}^3$.]
- The side length x of a square shadow is increasing at 6 cm/s. If the area is A and the length of the diagonal is ℓ , show that

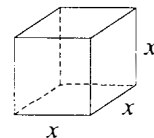
$$\frac{dA}{dt} = 2x \frac{dx}{dt} \quad \text{and} \quad \frac{d\ell}{dt} = \sqrt{2} \frac{dx}{dt}.$$

Hence find the rates of increase of the area and the diagonal when: (a) the side length is 70 cm, (b) the area is 1 m^2 .



- The side length x of a shrinking cube is decreasing at a constant rate of 5 mm per minute. Show that the rates of change of volume V , surface area A , and the total edge length ℓ are

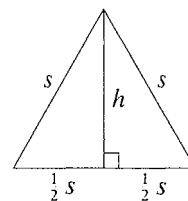
$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \quad \text{and} \quad \frac{dA}{dt} = 12x \frac{dx}{dt} \quad \text{and} \quad \frac{d\ell}{dt} = 12 \frac{dx}{dt}.$$



- Find the rate at which volume, surface area and edge length are decreasing when:
 - the side length is 30 cm,
 - the volume is 8000 cm^3 .
- Find the side length when the volume is decreasing at $300 \text{ cm}^3/\text{min}$.

6. Show that in an equilateral triangle of side length s , the area is given by $A = \frac{1}{4}s^2\sqrt{3}$ and the height by $h = \frac{1}{2}s\sqrt{3}$.

- (a) Find formulae for the rates of change of area and height.
 (b) Hence find the rate at which the area and the height are increasing when the side length is 12 cm and is increasing at 3 mm/s.

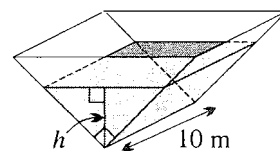


DEVELOPMENT

7. A spherical balloon is to be filled with water so that its surface area increases at a constant rate of $1 \text{ cm}^2/\text{s}$.
 (a) Find, when the radius is 3 cm: (i) the required rate of increase of the radius, (ii) the rate at which the water must be flowing in at that time.
 (b) Find the volume when the volume is increasing at $10 \text{ cm}^3/\text{s}$.

8. A water trough is 10 metres long, with cross section a right isosceles triangle. Show that when the water has depth h cm, its volume is $V = 1000h^2$ and its surface area is $A = 2000h$.

- (a) Find the rates at which depth and surface area are increasing when the depth is 60 cm if the trough is filling at 5 litres per minute (remember that 1 litre is 1000 cm^3).
 (b) Find the rates at which the volume and the surface area must increase when the depth is 40 cm, if the depth is required to increase at a constant rate of $0.1 \text{ cm}/\text{min}$.



9. The equation of the path of a bullet fired into the air is $y = -20x(x - 20)$, where x and y are displacements in metres horizontally and vertically from the origin. The bullet is moving horizontally at a constant rate of $\frac{1}{2} \text{ m/s}$.

- (a) Find the rate at which the bullet is rising:
 (i) when $x = 8$, (ii) when $x = 18$, (iii) when $x = 10$, (iv) when $y = 1500$.
 (b) Find the height when the bullet is: (i) rising at 30 m/s , (ii) falling at 70 m/s .
 (c) Use the gradient function dy/dx to find the angle of flight when the bullet is rising at 10 m/s .
 (d) How high does the bullet go, and how far away does it land?

10. Sand being poured from a conveyor belt forms a cone with height h and semivertical angle 60° . Show that the volume of the pile is $V = \pi h^3$, and differentiate with respect to t .

- (a) Suppose that the sand is being poured at a constant rate of $0.3 \text{ m}^3/\text{min}$, and let A be the area of the base. Find the rate at which the height is increasing:
 (i) when the height is 4 metres, (ii) when the radius is 4 metres.
 (b) Show that $\frac{dA}{dt} = 6\pi h \frac{dh}{dt}$, and find the rate of increase of the base area at these times.
 (c) At what rate must the sand be poured if it is required that the height increase at $8 \text{ cm}/\text{min}$, when the height is 4 metres?

11. An upturned cone of semivertical angle 45° is being filled with water at a constant rate of $20 \text{ cm}^3/\text{s}$. Find the rate at which the height, the area of the water surface, and the area of the cone wetted by the water, are increasing when the height is 50 cm.

12. A square pyramid has height twice its side length s .

- (a) Show that the volume V and the surface area A are $V = \frac{2}{3}s^3$ and $A = (\sqrt{17} + 1)s^2$.
 (b) Hence find the rate at which V and A are decreasing when the side length is 4 metres if the side length is shrinking at 3 mm/s .

13. A ladder 13 metres long rests against a wall, with its base x metres from the wall and its top y metres high. Explain why $x^2 + y^2 = 169$, solve for y , and differentiate with respect to t . Hence find, when the base is 5 metres from the wall:
- the rate at which the top is slipping down when the base is slipping out at 1 cm/s,
 - the rate at which the base is slipping out when the top is slipping down at 5 mm/s.
14. Water is flowing into a hemispherical container of radius 10 cm at a constant rate of 6 cm^3 per second. It is known that the formula for the volume of a solid segment cut off a sphere is $V = \frac{\pi}{3}h^2(3r - h)$, where r is the radius of the sphere and h is the height of the segment.
- Find the rate at which the height of the water is rising when the water height is 2 cm.
 - Use Pythagoras' theorem to find the radius of the circular water surface when the height is h , and hence find the rate of increase of the surface area when the water height is 2 cm.

EXTENSION

15. Show that the volume V of a regular tetrahedron, all of whose side lengths are s , is $V = \frac{1}{12}s^3\sqrt{2}$ (all four faces of a regular tetrahedron are equilateral triangles). Hence find the rate of increase of the surface area when the volume is $144\sqrt{2} \text{ cm}^3$ and is increasing at a rate of $12 \text{ cm}^3/\text{s}$.
16. A large vase has a square base of side length 6 cm, and flat sides sloping outwards at an angle of 120° with the base. Water is flowing in at $12 \text{ cm}^3/\text{s}$. Find, to three significant figures, the rate at which the height of water is rising when the water has been flowing in for 3 seconds.

7I Limits and Continuity

If a tangent can be drawn at a point on a curve, the curve must be smooth at that point without any sharp corner — the technical word is *differentiable*. The curve must also be *continuous* at the point, without any break. The purpose of Sections 7I and 7J is to make a little more precise what is meant by saying that a curve is continuous at a point and what is meant by saying that it is differentiable there. Both these definitions rest firmly on the idea of a limit.

Some Rules for Limits: It is not the intention of this course to provide anything more than an intuitive introduction to limits, and some fairly obvious rules about how to handle them. Here is the informal definition of a limit that we have been using.

19 **DEFINITION:** $\lim_{x \rightarrow a} f(x) = \ell$ means $f(x)$ is 'as close as we like' to ℓ when x is near a .

Here are some of the assumptions we have been making about the behaviour of limits.

20 **LIMIT OF A SUM:** $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x),$
LIMIT OF A MULTIPLE: $\lim_{x \rightarrow a} kf(x) = k \times \lim_{x \rightarrow a} f(x),$
LIMIT OF A PRODUCT: $\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x),$
LIMIT OF A QUOTIENT: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)},$ provided $\lim_{x \rightarrow a} g(x) \neq 0.$

WORKED EXERCISE: Find: (a) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ (b) $\lim_{x \rightarrow 3} \frac{x^2 - 7x + 12}{x^2 + x - 12}$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1), \text{ since } x \neq 1, \\ &= 2 \end{aligned}$$

(The value at $x = 1$ is irrelevant.)

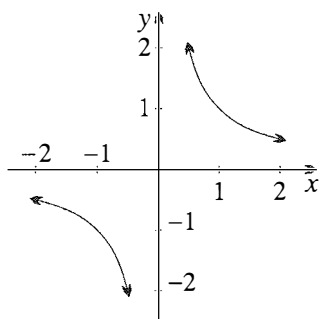
$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 3} \frac{x^2 - 7x + 12}{x^2 + x - 12} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x - 4)}{(x - 3)(x + 4)} \\ &= \lim_{x \rightarrow 3} \frac{x - 4}{x + 4}, \text{ since } x \neq 3, \\ &= -\frac{1}{7} \end{aligned}$$

(The value at $x = 3$ is irrelevant.)

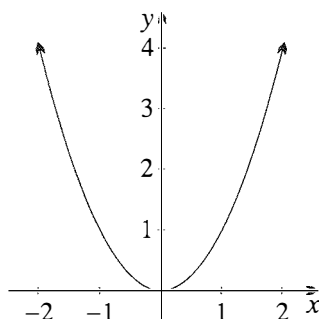
Continuity at a Point — Informal Definition: As discussed already in Chapter 3, continuity at a point means that there is no break in the curve around that point.

21

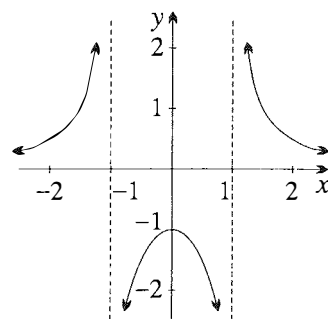
DEFINITION: A function $f(x)$ is called *continuous* at $x = a$ if the graph of $y = f(x)$ can be drawn through the point where $x = a$ without any break. Otherwise we say that there is a *discontinuity* at $x = a$.



EXAMPLE: $y = 1/x$ has a discontinuity at $x = 0$, and is continuous everywhere else.



EXAMPLE: $y = x^2$ is continuous for all values of x .



EXAMPLE: $y = \frac{1}{x^2 - 1}$ has discontinuities at $x = 1$ and at $x = -1$, and is continuous everywhere else.

Piecewise Defined Functions: A function can be *piecewise defined* by giving different definitions in different parts of its domain. For example,

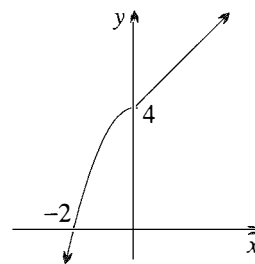
$$f(x) = \begin{cases} 4 - x^2, & \text{for } x \leq 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$$

Clearly the two pieces of this graph join up at the point $(0, 4)$, making the function continuous at $x = 0$.

The more formal way of talking about this involves analysing the behaviour of $f(x)$ on each side of $x = 0$, and taking two limits, first when x is near zero and greater than zero, secondly when x is near zero and less than zero:

$\lim_{x \rightarrow 0^+} f(x)$, meaning ‘the limit as x approaches 0 from the positive side’,

$\lim_{x \rightarrow 0^-} f(x)$, meaning ‘the limit as x approaches 0 from the negative side’.



We now look at these two limits, as well as the value of $f(x)$ at $x = 0$:

$$\begin{array}{lll} \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (4 - x^2) & \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (4 + x) & f(0) = 4 - 0^2 \\ = 4 & = 4 & = 4 \end{array}$$

and the reason why $f(x)$ is continuous at $x = 0$ is that these three values all exist and are all equal.

Continuity at a Point — Formal Definition: Here then is a somewhat stricter definition of continuity at a point, using the machinery of limits.

22

DEFINITION: A function $f(x)$ is called *continuous* at $x = a$ if

$$\lim_{x \rightarrow a^-} f(x) \text{ and } \lim_{x \rightarrow a^+} f(x) \text{ and } f(a)$$

all exist and are all equal.

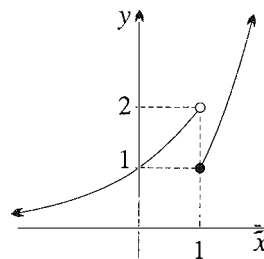
WORKED EXERCISE: Examine for continuity at $x = 1$, then sketch, the function

$$f(x) = \begin{cases} 2^x, & \text{for } x < 1, \\ x^2, & \text{for } x \geq 1. \end{cases}$$

SOLUTION:

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} 2^x = 2, \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 = 1, \\ f(1) &= 1, \end{aligned}$$

so the curve is not continuous at $x = 1$.

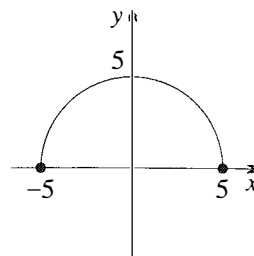


An Assumption of Continuity: It is intuitively obvious that a function like $y = x^2$ is continuous for every value of x . However, it is not possible in this course to give the sort of rigorous proof that mathematicians so enjoy, because the required rigorous treatment of limits is missing. It is therefore necessary to make a general assumption of continuity, loosely stated as follows.

23

ASSUMPTION: The functions in this course are continuous for every value of x in their domain, except where there is an obvious problem.

Continuity in a Closed Interval: The semicircular function with equation $f(x) = \sqrt{25 - x^2}$ presents an interesting test for the definition of continuity. At the right-hand endpoint $x = 5$ the curve is not continuous, because the right-hand limit $\lim_{x \rightarrow 5^+} f(x)$ does not exist. Neither is the curve continuous at the left-hand endpoint $x = -5$, because the left-hand limit $\lim_{x \rightarrow (-5)^-} f(x)$ does not exist.



In fact, no curve is continuous at an endpoint of its domain.

Nevertheless, it will be important later to say that $f(x)$ is continuous in the closed interval $-5 \leq x \leq 5$, and to justify this by the fact that the situation at the left and right-hand sides is

$$\lim_{x \rightarrow 5^-} f(x) = f(5) = 0 \quad \text{and} \quad \lim_{x \rightarrow (-5)^+} f(x) = f(-5) = 0.$$

This leads to the following definition of continuity in a closed interval.

- 24** **DEFINITION:** $f(x)$ is called *continuous in the closed interval* $a \leq x \leq b$ if:
1. $f(x)$ is continuous for every value of x in the open interval $a < x < b$, and
 2. $\lim_{x \rightarrow a^+} f(x)$ and $f(a)$ both exist and are equal, and
 3. $\lim_{x \rightarrow b^-} f(x)$ and $f(b)$ both exist and are equal.

Then by this definition, $y = \sqrt{25 - x^2}$ is indeed continuous in the closed interval $-5 \leq x \leq 5$.

Continuous Functions: A function $f(x)$ is called *continuous* if it is continuous at every point in its domain. This turns out, however, to be a rather unsatisfactory definition for our purposes, because, for example, the function $y = 1/x$ is continuous at every value of x except $x = 0$, which lies outside its domain, and so we have to conclude that $y = 1/x$ is a continuous function with a discontinuity at $x = 0$. Consequently this course will rarely speak of continuous functions, and the emphasis will be on continuity at a point, and less often on continuity in a closed interval.

Exercise 71

NOTE: In every graph, every curve must end with a closed circle if the endpoint is included, an open circle if it is not, or an arrow if it continues forever. After working on these limit questions, one should revise differentiation from first principles in Exercise 7B.

1. (a) Use the rule from Chapter Three, 'Divide top and bottom by the highest power of x in the denominator' to find the behaviour of these functions as $x \rightarrow \infty$:

(i) $y = \frac{x^2 - 4x + 3}{2x^2 - 7x + 6}$ (ii) $y = \frac{2 - 5x}{15x + 11}$ (iii) $y = \frac{x^2 + x + 1}{x^3 + x^2 + x + 1}$ (iv) $y = \frac{x^2 - 5}{1 + \sqrt{x}}$

 (b) Use the same rule to find the behaviour of those functions as $x \rightarrow -\infty$.
2. First factor top and bottom and cancel any common factors, then find:

(a) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ (c) $\lim_{h \rightarrow 0} \frac{h^3 - 9h^2 + h}{h}$ (e) $\lim_{x \rightarrow 0} \frac{x^3 + 6x}{x^2 - 3x}$

(b) $\lim_{u \rightarrow 3} \frac{u^3 - 27}{u - 3}$ (d) $\lim_{h \rightarrow -3} \frac{h^2 - 9}{h^2 + 7h + 12}$ (f) $\lim_{x \rightarrow 0} \frac{x^4 - 4x^2}{x^2 - 2x}$
3. Discuss the behaviour of $y = \frac{x^2 - x - 12}{2x^2 + 7x + 3} = \frac{(x - 4)(x + 3)}{(2x + 1)(x + 3)}$:

(a) as $x \rightarrow \infty$ (c) as $x \rightarrow 0$ (e) as $x \rightarrow 4$ (g) as $x \rightarrow -\infty$

(b) as $x \rightarrow -3$ (d) as $x \rightarrow -\frac{1}{2}$ (f) as $x \rightarrow 1$ (h) as $x \rightarrow -1$
4. For each function below: (i) sketch the curve, (ii) find $\lim_{x \rightarrow 2^-} f(x)$, $\lim_{x \rightarrow 2^+} f(x)$ and $f(2)$, (iii) draw a conclusion about continuity at $x = 2$, (iv) give the domain and range.

(a) $f(x) = \begin{cases} x^3, & \text{for } x \leq 2, \\ 10 - x, & \text{for } x > 2. \end{cases}$ (c) $f(x) = \begin{cases} 1/x, & \text{for } 0 < x < 2, \\ 1 - \frac{1}{4}x, & \text{for } x > 2, \\ \frac{1}{2}, & \text{for } x = 2. \end{cases}$

(b) $f(x) = \begin{cases} 3^x, & \text{for } x < 2, \\ 13 - x^2, & \text{for } x > 2, \\ 4, & \text{for } x = 2. \end{cases}$ (d) $f(x) = \begin{cases} x, & \text{for } x < 2, \\ 2 - x, & \text{for } x > 2, \\ 2, & \text{for } x = 2. \end{cases}$

5. Cancel the algebraic fraction in each function, noting first the value of x for which the function is undefined. Then sketch the curve and state its domain and range:

$$(a) y = \frac{x^2 + 2x + 1}{x + 1} \quad (b) y = \frac{x^4 - x^2}{x^2 - 1} \quad (c) y = \frac{x - 3}{x^2 - 4x + 3} \quad (d) y = \frac{3x + 3}{x + 1}$$

DEVELOPMENT

6. (a) Find the gradient of the secant joining the points $P(x, f(x))$ and $Q(x + h, f(x + h))$ on the curve $y = x^2 - x + 1$, then take $\lim_{h \rightarrow 0} (\text{gradient of } PQ)$.
 (b) Find the gradient of the secant joining the points $P(x, g(x))$ and $Q(u, g(u))$ on the curve $y = x^4 - 3x$, then take $\lim_{u \rightarrow x} (\text{gradient of } PQ)$.

7. (a) Find: (i) $\lim_{x \rightarrow c} \frac{x^4 - c^4}{x^2 - c^2}$ (ii) $\lim_{x \rightarrow c} \frac{x^4 - c^4}{x^3 - c^3}$ (iii) $\lim_{x \rightarrow -c} \frac{x^5 + c^5}{x^3 + c^3}$

(b) Factor the difference of powers $x^n - a^n$ and hence find $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$.

(c) Factor the sum of odd powers $u^{2n+1} + 2^{2n+1}$ and hence find $\lim_{u \rightarrow -2} \frac{u^{2n+1} + 2^{2n+1}}{u + 2}$.

8. For what values of a are these functions continuous:

$$(a) f(x) = \begin{cases} ax^2, & \text{for } x \leq 1, \\ 6 - x, & \text{for } x > 1. \end{cases} \quad (b) g(x) = \begin{cases} \frac{a(x^2 - 9)}{x + 3}, & \text{for } x \neq -3, \\ 12, & \text{for } x = -3. \end{cases}$$

9. Find all zeroes and discontinuities of these functions:

$$(a) y = \frac{x}{x - 3} \quad (c) y = \operatorname{cosec} x^\circ \quad (e) y = \tan x^\circ$$

$$(b) y = \frac{x}{x^2 - 6x - 7} \quad (d) y = \frac{1}{\cos x^\circ - 1} \quad (f) y = \frac{x^3 - x}{x^3 - 9x}$$

10. (a) Simplify $y = \frac{|x|}{x}$, find $\lim_{x \rightarrow 0^+} y$, $\lim_{x \rightarrow 0^-} y$ and $y(0)$, discuss the continuity at $x = 0$, and sketch it.

- (b) Repeat the steps in part (a) for:

$$(i) y = \frac{x^2}{x^2} \quad (ii) y = \frac{x^2}{\sqrt{x^2}} \quad (iii) y = \frac{|x|}{\sqrt{x}} \quad (iv) y = \frac{|x^2 - 2x|}{x}$$

11. (a) Show that the GP $u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \dots + x^{n-1}$ has common ratio $\frac{x}{u}$.

(b) Use the formula for the partial sum of a GP to show that its sum is $\frac{u^n - x^n}{u - x}$.

- (c) Use this identity to find the derivative of $f(x) = x^n$ from first principles.

12. (a) Use the method of 'rationalising the numerator' to find these limits:

$$(i) \lim_{u \rightarrow x} \frac{\sqrt{u} - \sqrt{x}}{u - x} \quad (ii) \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad (iii) \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x-h}}{2h}$$

(b) Explain how each limit can be used to show that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

EXTENSION

13. Which of these functions are continuous in the closed interval $-1 \leq x \leq 1$:

$$(a) \sqrt{x+1} \quad (b) \frac{1}{x^2 - 1} \quad (c) 7\sqrt{1-x^2} \quad (d) \frac{x^2 - 1}{x^2 - 2}$$

14. Find zeroes and discontinuities of: (a) $y = \frac{\cos x^\circ + \sin x^\circ}{\cos x^\circ - \sin x^\circ}$ (b) $y = \frac{\cos x^\circ - \sin x^\circ}{\cos x^\circ + \sin x^\circ}$

15. Find these limits by rationalising the numerator or otherwise:

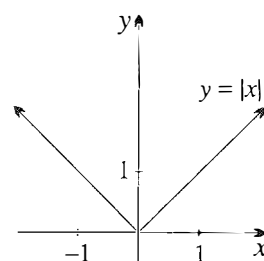
(a) $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x^2}$ (b) $\lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 3} - 2}{x^2 - 1}$ (c) $\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$ (d) $\lim_{x \rightarrow 25} \frac{\frac{1}{\sqrt{x}} - \frac{1}{5}}{x - 25}$

16. Sketch these functions over the whole real line:

(a) $y = \frac{|\sin 180x^\circ|}{\sin 180x^\circ}$ (d) $y = \frac{|x(x^2 - 1)(x^2 - 4)|}{x(x^2 - 1)(x^2 - 4)}$
 (b) $y = \frac{|\cos 180x^\circ|}{\cos 180x^\circ}$ (e) $y = \lim_{n \rightarrow \infty} \frac{\left| x \prod_{k=1}^{2n} (x^2 - k^2) \right|}{x \prod_{k=1}^{2n} (x^2 - k^2)}$
 (c) $y = \frac{|\tan 180x^\circ|}{\tan 180x^\circ}$

7J Differentiability

A tangent can only be drawn at a point P on a curve if the curve is smooth at that point, meaning that the curve can be drawn through P without any sharp change of direction. For example, the curve $y = |x|$ sketched opposite has a sharp point at the origin, where it changes gradient abruptly from -1 to 1 . A tangent cannot be drawn there, and the function has no derivative at $x = 0$. This suggests the following definition.



25 DEFINITION: A function $f(x)$ is called *differentiable* (or *smooth*) at $x = a$ if the derivative $f'(a)$ exists there.

So $y = |x|$ is continuous at $x = 0$, but is not differentiable there.

Clearly a function that is not even continuous at some value $x = a$ cannot be differentiable there, because there is no way of drawing a tangent at a place where there is a break in the curve.

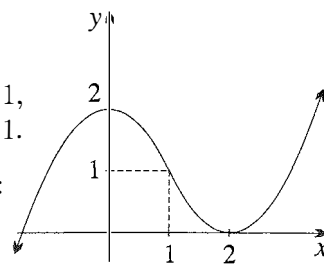
26 If $f(x)$ is not continuous at $x = a$, then it is certainly not differentiable there.

Piecewise Defined Functions: The sketch opposite shows

$$f(x) = \begin{cases} 2 - x^2, & \text{for } x < 1, \\ (x - 2)^2 & \text{for } x \geq 1, \end{cases} \quad \text{so } f'(x) = \begin{cases} -2x, & \text{for } x < 1, \\ 2(x - 2) & \text{for } x > 1. \end{cases}$$

The graph is continuous, because the two pieces join at $P(1, 1)$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1) = 1.$$



But in this case, when the two pieces join, they do so with the same gradient, so that the combined curve is smooth at the point $P(1, 1)$. The reason for this is that the gradients on the left and right of $x = 1$ also converge to the same limit of -2 :

$$\lim_{x \rightarrow 1^-} f'(x) = \lim_{x \rightarrow 1^-} (-2x) = -2, \quad \text{and} \quad \lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^+} 2(x-2) = -2.$$

So the function does have a well-defined derivative of -2 when $x = 1$, and the curve is indeed differentiable there, with a well-defined tangent at the point $P(1, 1)$.

DIFFERENTIABILITY: To test a piecewise defined function for differentiability at a join $x = a$ between pieces:

27

1. Test whether the function is continuous at $x = a$.
2. Test whether $\lim_{x \rightarrow a^-} f'(x)$ and $\lim_{x \rightarrow a^+} f'(x)$ exist and are equal.

WORKED EXERCISE: Test the following functions for continuity and for differentiability at $x = 2$, and if they are differentiable, state the value of the derivative there. Then sketch the curves:

$$(a) f(x) = \begin{cases} x^2 - 1, & \text{for } x \leq 0, \\ x + 1, & \text{for } x > 0. \end{cases} \quad (b) f(x) = \begin{cases} x, & \text{for } x \leq 0, \\ x - x^2, & \text{for } x > 0. \end{cases}$$

SOLUTION:

$$(a) \text{ First, } \lim_{x \rightarrow 0^-} f(x) = -1,$$

$$\text{and } \lim_{x \rightarrow 0^+} f(x) = 1,$$

so the function is not even continuous at $x = 0$.

[Notice, however, that for $x \neq 0$, $f'(x) = 2x$,

$$\text{so } \lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^-} f'(x) = 0,$$

but this is irrelevant since the curve is not continuous at $x = 0$.]

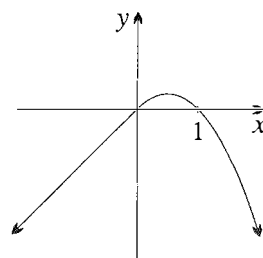
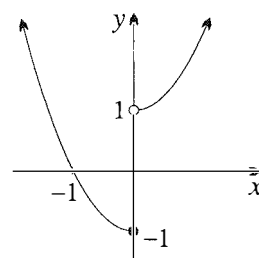
$$(b) \text{ First, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) = 0,$$

so the function is continuous at $x = 0$.

$$\text{Secondly, } f'(x) = \begin{cases} 1, & \text{for } x > 0, \\ 1 - 2x, & \text{for } x < 0, \end{cases}$$

$$\text{so } \lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^-} f'(x) = 1,$$

and the function is differentiable at $x = 0$, with $f'(0) = 1$.

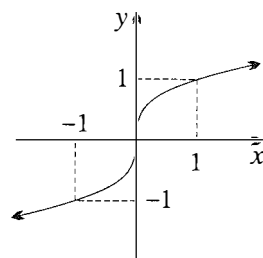


Tangents and Differentiability: It would be natural to think that being differentiable at $x = a$ and having a tangent at $x = a$ were equivalent. Some curves, however, have a point where there is a vertical tangent. But vertical tangents don't have a gradient, so at such a point the derivative is undefined, meaning that the curve is not differentiable there. For example, the curve on the right is

$$f(x) = x^{\frac{1}{3}}, \quad \text{whose derivative is } f'(x) = \frac{1}{3}x^{-\frac{2}{3}}.$$

This function is simply the inverse function of $y = x^3$, so its graph is the graph of $y = x^3$ reflected in the diagonal line $y = x$. There is no problem about the continuity of $f(x)$ at $x = 0$, where the graph passes through the origin. But $f'(0)$ is undefined, and

$$f'(x) \rightarrow \infty \text{ as } x \rightarrow 0^+ \quad \text{and} \quad f'(x) \rightarrow \infty \text{ as } x \rightarrow 0^-.$$



So the gradient of the curve becomes infinitely steep on both sides of the origin, and although the curve is not differentiable there, the y -axis is a vertical tangent. The complete story is:

28

TANGENTS AND DIFFERENTIABILITY: $f(x)$ is differentiable at $x = a$ if and only if there is a tangent there, and the tangent is non-vertical.

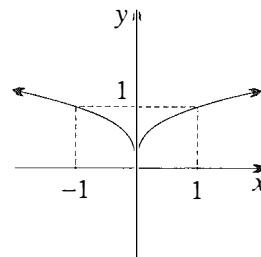
Cusps and Differentiability: A stranger picture is provided by the closely related function

$$f(x) = x^{\frac{2}{3}}, \quad \text{whose derivative is } f'(x) = \frac{2}{3}x^{-\frac{1}{3}}.$$

Again the function is continuous at $x = 0$ where the graph passes through the origin, and again $f'(0)$ is undefined. But this time the function can never be negative, since it is a square, and

$$f'(x) \rightarrow \infty \text{ as } x \rightarrow 0^+ \quad \text{and} \quad f'(x) \rightarrow -\infty \text{ as } x \rightarrow 0^-.$$

So although the gradient of the curve still becomes infinitely steep on both sides of the origin, the two sides are sloping one backwards and one forwards, and this time there is no tangent at $x = 0$. The point $(0, 0)$ is called a *cusp* of the curve.



Exercise 7J

1. Test these functions for continuity at $x = 1$. If the function is continuous there, find $\lim_{x \rightarrow 1^+} f'(x)$ and $\lim_{x \rightarrow 1^-} f'(x)$ to check for differentiability at $x = 1$. Then sketch the graph.

(a) $f(x) = \begin{cases} x^2, & \text{for } x \leq 1, \\ 2x - 1, & \text{for } x > 1. \end{cases}$

(c) $f(x) = \begin{cases} (x + 1)^2, & \text{for } x \leq 1, \\ 4x - 2, & \text{for } x > 1. \end{cases}$

(b) $f(x) = \begin{cases} 3 - 2x, & \text{for } x < 1, \\ 1/x, & \text{for } x \geq 1. \end{cases}$

(d) $f(x) = \begin{cases} (x - 1)^3, & \text{for } x \leq 1, \\ (x - 1)^2, & \text{for } x > 1. \end{cases}$

2. Sketch the graph of the function $y = \begin{cases} x^3 - x, & \text{for } -1 \leq x \leq 1, \\ x^2 - 1, & \text{for } x > 1 \text{ or } x < -1, \end{cases}$ after first checking for any values of x where the curve is not continuous or not differentiable.

DEVELOPMENT

3. (a) Sketch $f(x) = \begin{cases} \sqrt{1 - (x + 1)^2}, & \text{for } -2 \leq x \leq 0, \\ \sqrt{1 - (x - 1)^2}, & \text{for } 0 < x \leq 2. \end{cases}$ Describe the situation at $x = 0$.

(b) Repeat part (a) for $f(x) = \begin{cases} \sqrt{1 - (x + 1)^2}, & \text{for } -2 \leq x \leq 0, \\ -\sqrt{1 - (x - 1)^2}, & \text{for } 0 < x \leq 2. \end{cases}$

4. Sketch each function, giving any values of x where it is not continuous or not differentiable:

(a) $y = |x + 2|$

(d) $y = \frac{1}{|x^2 - 4x + 3|}$

(g) $y = |x^3|$

(b) $y = \frac{1}{|x + 2|}$

(e) $y = |x^2 + 2x + 2|$

(h) $y = |x^2(x - 2)|$

(c) $y = |x^2 - 4x + 3|$

(f) $y = \frac{1}{|x^2 + 2x + 2|}$

(i) $y = |(3 - x)^2|$

(j) $y = \sqrt{(x - 2)^2}$

5. (a) Differentiate $f(x) = x^{\frac{1}{5}}$, and find $\lim_{x \rightarrow 0^+} f'(x)$ and $\lim_{x \rightarrow 0^-} f'(x)$. Sketch the curve and state whether it has a vertical tangent or a cusp at $x = 0$. (b) Repeat for $f(x) = x^{\frac{2}{5}}$.

6. Each example following gives a curve and two points P and Q on the curve. Find the gradient of the chord PQ , and find the x -coordinates of any points M on the curve between P and Q such that the tangent at M is parallel to PQ .

- (a) $y = x^2 - 6x$, $P = (1, -5)$, $Q = (8, 16)$ (f) $y = 1/x$, $P = (-1, -1)$, $Q = (1, 1)$
 (b) $y = x^3 - 9$, $P = (-1, -10)$, $Q = (2, -1)$ (g) $y = |x|$, $P = (-1, 1)$, $Q = (1, 1)$
 (c) $y = x^3$, $P = (-1, -1)$, $Q = (1, 1)$ (h) $y = x^2$, $P = (\alpha, \alpha^2)$, $Q = (-\alpha, \alpha^2)$
 (d) $y = \sqrt{x}$, $P = (1, 1)$, $Q = (4, 2)$ (i) $y = x^2$, $P = (\alpha, \alpha^2)$, $Q = (\beta, \beta^2)$
 (e) $y = 1/x$, $P = (1, 1)$, $Q = (2, \frac{1}{2})$ (j) $y = 1/x$, $P = (\alpha, 1/\alpha)$, $Q = (\beta, 1/\beta)$

NOTE: The existence of at least one such point is guaranteed by a theorem called the *mean value theorem*, provided that the curve is differentiable everywhere between the two points.

EXTENSION

7. Find the equation of the tangent at the point $P(a, ka^{\frac{4}{3}})$ on $y = kx^{\frac{4}{3}}$, where $k > 0$, and the coordinates of the points A and B where it meets the x -axis and y -axis respectively. Sketch the situation, and let the vertical and horizontal lines through P meet the x -axis and y -axis at G and H respectively. Show that A divides OG in the ratio $1 : 3$, and O divides BH in the ratio $1 : 3$. Find the ratio of the areas of the rectangle $OGPH$ and $\triangle OAB$.
8. Consider now the point $P(a, ka^n)$ on the general curve $y = kx^n$, where n is any nonzero real number and $k > 0$. Find the coordinates of the points A , B , G and H defined in the previous question, and show that A divides OG in the ratio $n - 1 : 1$ and that O divides BH in the ratio $n - 1 : 1$. Find the ratio of the areas of the rectangle $OGPH$ and $\triangle OAB$, and find when the rectangle is bigger. For what values of n is the point B above the origin? Can a be negative?
9. Suppose p and q are integers with no common factors and with $q > 0$. Write down the derivative of $f(x) = x^{\frac{p}{q}}$, and hence find the conditions on p and q for which:
- (a) $f(x)$ is defined for $x < 0$, (e) $f(x)$ is continuous for $x \geq 0$,
 (b) $f(x)$ is defined at $x = 0$, (f) $f(x)$ is differentiable at $x = 0$,
 (c) $f(x)$ is defined for $x > 0$, (g) there is a vertical tangent at the origin,
 (d) $f(x)$ is continuous at $x = 0$, (h) there is a cusp at the origin.

7K Extension — Implicit Differentiation

So far we have only been differentiating curves whose equation has the form $y = f(x)$, where $f(x)$ is a function. But solving an equation for y can sometimes be difficult or impossible, and sometimes the curve may not even be a function. The purpose of this rather more difficult section is to extend differentiation to curves like the circle $x^2 + y^2 = 25$, which may not be functions, yet are still defined by an algebraic equation in x and y . This is a 4 Unit topic which is useful, but not necessary, for the 3 Unit course.

Differentiating Expressions in x and y : The first step is using the chain, product and quotient rules to differentiate expressions in x and y where x and y are related. In this situation, neither x nor y is constant, and in particular y must be regarded as a function of x .

WORKED EXERCISE: Differentiate the following expressions with respect to x :

- (a) y^2 (b) x^2y (c) $\frac{x^2}{y^2}$ (d) $(x^2 + y^2)^2$

SOLUTION:

(a) Using the chain rule:

$$\begin{aligned}\frac{d}{dx}(y^2) &= \frac{d}{dy}(y^2) \times \frac{dy}{dx} \\ &= 2y \frac{dy}{dx}.\end{aligned}$$

(b) Using the product rule:

$$\begin{aligned}\frac{d}{dx}(x^2y) &= y \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(y) \\ &= 2xy + x^2 \frac{dy}{dx}.\end{aligned}$$

(c) Using the quotient rule:

$$\begin{aligned}\frac{d}{dx} \left(\frac{x^2}{y^2} \right) &= \frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{y^4} \\ &= \frac{1}{y^4} \left(2xy^2 - 2x^2y \frac{dy}{dx} \right) \\ &= \frac{2x}{y^3} \left(y - x \frac{dy}{dx} \right).\end{aligned}$$

(d) Using the chain rule with $u = x^2 + y^2$:

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2)^2 &= 2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) \\ &= 4(x^2 + y^2) \left(x + y \frac{dy}{dx} \right).\end{aligned}$$

Finding Tangents to Implicitly Defined Curves: When a curve is defined by an algebraic equation in x and y , implicit differentiation will find the derivative as a function of x and y without solving the equation for y . Hence we can find the tangent at any given point on the curve.

WORKED EXERCISE: Use implicit differentiation to find the gradient of the tangent to $x^2 + y^2 = 25$ at the point $P(3,4)$ on the curve.

SOLUTION: Given that $x^2 + y^2 = 25$,

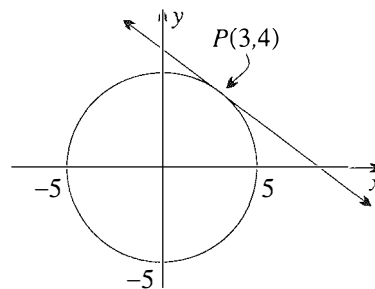
$$\begin{aligned}\text{differentiating implicitly, } 2x + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}.\end{aligned}$$

Hence at $P(3,4)$,

$$\frac{dy}{dx} = -\frac{3}{4},$$

and the tangent is

$$\begin{aligned}y - 4 &= -\frac{3}{4}(x - 3) \\ 3x + 4y &= 25.\end{aligned}$$



NOTE: In this particular case, the geometry of the circle is known independently of differentiation. The tangent is perpendicular to the radius joining the origin and $P(3,4)$, and since the radius has gradient $\frac{4}{3}$, the tangent must have gradient $-\frac{3}{4}$ (this geometric approach to differentiating the circle was used at the beginning of the chapter). This question could also be answered by differentiating the semicircular function, but implicit differentiation is much easier.

Exercise 7K

1. Differentiate the following expressions with respect to x (where x and y are related):

- (a) y^4 (c) $1 - x + y - xy$ (e) $x^3y + y^3x$ (g) $(x + y)^3$ (i) $\sqrt{x + y}$
 (b) xy (d) $3x^2 + 4y^2$ (f) $\frac{x}{y}$ (h) $\frac{x + y}{x - y}$ (j) $\sqrt{x^2 + y^2}$

2. Find dy/dx for the curves defined by these equations:

(a) $x^2 + y^2 = 49$	(d) $x^2 + 3xy + 2y^2 = 5$	(g) $ax^r + by^s = c$
(b) $3x^2 + 2y^2 = 25$	(e) $x^3 + xy^2 = x^2y + y^3$	(h) $\sqrt{x} + \sqrt{y} = 4$
(c) $x^2 - y^2 = 1$	(f) $x^2y^3 = 32$	(i) $\frac{x}{y} + \frac{y}{x} = 1$

DEVELOPMENT

3. (a) Differentiate the circle $x^2 + y^2 = 169$ implicitly, and hence find the tangent and normal at the point $P(-5, 12)$. (Why does the normal pass through the origin?)
 (b) Find the points A and B where the tangent meets the x -axis and the y -axis.
 (c) Find the area of $\triangle AOB$: (i) using OA as the base, (ii) using AB as the base.
4. (a) Differentiate the rectangular hyperbola $xy = 6$ implicitly, and hence find the equations of the tangent and normal at the point $P(2, 3)$.
 (b) Show that P is the midpoint of the interval cut off the tangent by the x -intercept and the y -intercept.
5. (a) Differentiate the curve $y^2 = x$ (which is a parabola whose axis of symmetry is the x -axis), and hence find the equation of the tangent and normal at the point $P(9, 3)$.
 (b) Show that the y -intercept of the tangent at P bisects the interval joining P and its x -intercept.
6. (a) Use parametric differentiation to differentiate the function defined by $x = t + 1/t$ and $y = t - 1/t$, and find the tangent and normal at the point T where $t = 2$.
 (b) Eliminate t from these equations, and use implicit differentiation to find the gradient of the curve at the same point T . [HINT: Square x and y and subtract.]
7. A ladder 8 metres long rests against a wall, with its base x metres from the wall and its top y metres high. Explain why $x^2 + y^2 = 64$, and differentiate the equation implicitly with respect to time. Find, to three significant figures:
 (a) the rate at which the top is slipping down when the base is slipping out at a constant rate of 2 cm/s and is 2 metres from the wall,
 (b) the rate at which the base is slipping out when the top is slipping down at a constant rate of 2 mm/s and is 7 metres high.
8. (a) Show that the volume and surface area of a sphere are related by $S^3 = 36\pi V^2$, and differentiate this equation with respect to time.
 (b) A balloon is to be filled with water so that the rubber expands at a constant rate of 4 cm^2 per second. Use part (a) to find at what rate the water should be flowing in when the radius is 5 cm.
9. (a) Explain why the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ has line symmetry in the x -axis, the y -axis and the lines $y = x$ and $y = -x$.
 (b) Explain why its domain is $-8 \leq x \leq 8$ and its range is $-8 \leq y \leq 8$.
 (c) Show that $dy/dx = -x^{-\frac{1}{3}}y^{\frac{1}{3}}$, and examine the behaviour of the curve as $x \rightarrow 8^-$.
 (d) Hence sketch the curve.
10. Using methods similar to those in the previous question, or otherwise, sketch:
 (a) $x^{\frac{4}{3}} + y^{\frac{4}{3}} = 16$ (b) $x^4 + y^4 = 16$ (c) $|x| + |y| = 2$

EXTENSION

11. (a) Differentiate $x^3 + y^3 = 8$, and hence find the equations of the tangents at the x - and y -intercepts and at the point where the curve meets $y = x$.
- (b) Rewrite the equation as $1 + (y/x)^3 = 8/x^3$ and show that $y/x \rightarrow -1$ as $|x| \rightarrow \infty$. Show that $x + y = \frac{8}{x^2 - xy + y^2}$, and hence that $y = -x$ is an oblique asymptote.
- (c) Sketch the curve.
12. (a) Differentiate $x^3 + y^3 = 3xy$, called the *Folium of Descartes*. Hence find the equation of the tangent at the point where the curve meets $y = x$, and find the points on the curve where the tangents are horizontal and vertical (leave the origin out of consideration at this stage). Sketch the curve after carrying out these steps:
- (b) Show that $\frac{1 + (y/x)^3}{y/x} = \frac{3}{x}$, and hence that $y/x \rightarrow -1$ as $x \rightarrow \infty$.
- (c) Show that $x + y = \frac{3}{x/y - 1 + y/x}$, and hence that $x + y = -1$ is an oblique asymptote.
13. Differentiate $(x^2 + y^2)^2 = 2(x^2 - y^2)$, called the *Lemniscate of Bernoulli*, and find the points where the tangents are horizontal or vertical (ignore the origin). Sketch the curve.
14. Assume that $\frac{d}{dx}x^n = nx^{n-1}$, for $n \in \mathbf{N}$. Implicit differentiation allows a slightly more elegant proof of the successive extensions of this rule to $n \in \mathbf{Z}$ and then to $n \in \mathbf{Q}$.
- (a) Let $y = x^{-n}$, where $n \in \mathbf{N}$. Begin with $yx^n = 1$, and prove that $\frac{dy}{dx} = -nx^{-n-1}$.
- (b) Let $y = x^{\frac{m}{k}}$, where m and k are integers with $k \neq 0$. Begin with $y^k = x^m$, and prove that $\frac{dy}{dx} = \frac{m}{k}x^{\frac{m}{k}-1}$.

CHAPTER EIGHT

The Quadratic Function

The previous chapter on differentiation established that the derivative of any quadratic function is a linear function, for example,

$$\frac{d}{dx}(x^2 - 5x + 6) = 2x - 5.$$

In this sense, quadratics are the next most elementary functions to study after the linear functions of Chapter Five. This relationship between linear and quadratic functions is the underlying reason why quadratics arise in so many applications of mathematics.

STUDY NOTES: Sections 8A–8D review the known theory of quadratics — factoring, completing the square, and the formulae for the roots and the axis of symmetry — presenting them in the more general context of functions and their graphs, and leading to maximisation problems in Section 8E. From this basis, Sections 8F and 8G develop a classification of quadratics based on the discriminant. The final Sections 8H and 8I on the sum and product of roots and quadratic identities will be generalised later to polynomials of higher degree.

8A Factorisation and the Graph

A *quadratic function* is a function that can be written in the form

$$f(x) = ax^2 + bx + c, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

A *quadratic equation* is an equation that can be written in the form

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0,$$

that is, in a form where the LHS is a quadratic function. The requirement that $a \neq 0$ means that the term in x^2 cannot vanish, so that linear functions and equations are not to be regarded as special cases of quadratics. The word ‘quadratic’ comes from the Latin root *quadrat*, meaning ‘square’, and reminds us that quadratics tend to arise as the area of a plane shape, or more generally as the product of two linear functions. In the same way, the terms ‘square of x ’ and ‘cube of x ’ are used for x^2 and x^3 because they are the area and volume respectively of a square and cube of side length x .

Zeroes and Roots: One usually speaks of the solutions of a quadratic equation as *roots* of the equation, and of the x -intercepts of a quadratic function as *zeroes* of the function. However, the distinction between the words ‘roots’ and ‘zeroes’ is not strictly observed, and questions about quadratic functions and their graphs are closely related to questions about quadratic equations.

The Four Questions about the Graph of a Quadratic: Our first task is to review the sketching of the graph of a quadratic function $f(x) = ax^2 + bx + c$. The graph is a parabola, and before attempting any sketch, there are four questions that need to be asked.

FOUR QUESTIONS ABOUT THE GRAPH OF A PARABOLA:

1. Which way up is the curve? Answer: Look at the sign of a .
2. What is the y -intercept? Answer: Put $x = 0$, and then $y = c$.
3. Where are the axis of symmetry and the vertex?
4. Where are the x -intercepts, if there are any?

The first two questions are very straightforward to answer, but the second two questions need close attention. This section and the next two will review in succession the three standard approaches to them: factorisation, completing the square, and using the formulae generated by completing the square.

Factorisation and the Zeroes: Most quadratics cannot easily be factored, but when factorisation is possible, this is usually the quickest approach to sketching the curve. The zeroes are found using the following principle:

2. **FACTORISATION AND THE ZEROES:** If $AB = 0$, then $A = 0$ or $B = 0$, so we find the zeroes by putting each factor equal to zero.

For example, if $y = (2x - 3)(2x - 5)$, then the zeroes are given by

$$2x - 3 = 0 \quad \text{or} \quad 2x - 5 = 0,$$

so they are $x = 1\frac{1}{2}$ and $x = 2\frac{1}{2}$.

Finding the Axis of Symmetry and Vertex from the Zeroes: The axis of symmetry is always midway between the x -intercepts, so it can be found by taking the average of the zeroes.

ZEROES AND THE AXIS OF SYMMETRY AND VERTEX:

3. 1. If a quadratic has zeroes α and β , its axis is the line $x = \frac{1}{2}(\alpha + \beta)$.
2. Substitution into the quadratic gives the y -coordinate of the vertex.

For example, we saw that $y = (2x - 3)(2x - 5)$ has zeroes $x = 1\frac{1}{2}$ and $x = 2\frac{1}{2}$. Averaging these zeroes, the axis of symmetry is the line $x = 2$. Substituting $x = 2$ gives $y = -1$, so the vertex is $(2, -1)$.

WORKED EXERCISE: Sketch the curve $y = x^2 - 2x - 3$.

SOLUTION: Since $a > 0$, the curve is concave up.

When $x = 0$, $y = -3$.

Factoring, $y = (x + 1)(x - 3)$.

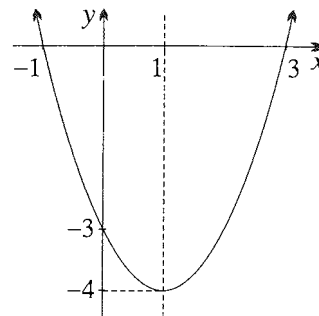
When $y = 0$, $x + 1 = 0$ or $x - 3 = 0$

$$x = -1 \text{ or } x = 3.$$

Then the axis of symmetry is $x = \frac{1}{2}(-1 + 3)$

$$x = 1.$$

When $x = 1$, $y = -4$, so the vertex is $(1, -4)$.



WORKED EXERCISE: Sketch the graph of the function $f(x) = -2x^2 + 9x - 7$.

SOLUTION: Since $a < 0$, the curve is concave down.

When $x = 0$, $f(0) = -7$.

Factoring, $f(x) = -(2x - 7)(x - 1)$.

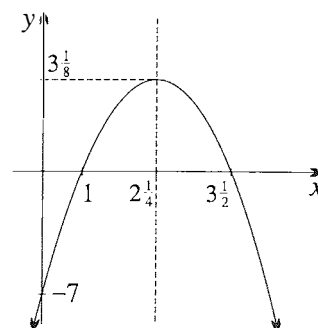
When $f(x) = 0$, $2x - 7 = 0$ or $x - 1 = 0$

$$x = 3\frac{1}{2} \text{ or } x = 1.$$

Then the axis of symmetry is $x = \frac{1}{2}(3\frac{1}{2} + 1)$

$$x = 2\frac{1}{4}.$$

Since $f(2\frac{1}{4}) = 3\frac{1}{8}$, the vertex is $(2\frac{1}{4}, 3\frac{1}{8})$.



Quadratic Inequalities: As discussed in Chapter Three, a quadratic inequation is best solved from a sketch of the quadratic function.

WORKED EXERCISE: From the graphs above, solve:

(a) $x^2 - 2x - 3 \leq 0$

(b) $2x^2 + 7 > 9x$

SOLUTION:

(a) $x^2 - 2x - 3 \leq 0$

From the first graph,

$$-1 \leq x \leq 3.$$

(b) $2x^2 + 7 > 9x$

$$-2x^2 + 9x - 7 < 0$$

From the second graph,

$$x < 1 \text{ or } x > 3\frac{1}{2}.$$

Domain and Range of Quadratic Functions: The natural domain of a quadratic function is the set $\mathbf{R}$ of real numbers, and the graphs above show that its range is clear once the vertex and concavity are established. When the domain is restricted, the range can be read off the graph, taking account of the vertex and endpoints.

WORKED EXERCISE: From the graph of $y = x^2 - 2x - 3$ on the previous page, find the range of the function: (a) with unrestricted domain, (b) with domain $x \geq 4$, (c) with domain $0 \leq x \leq 4$.

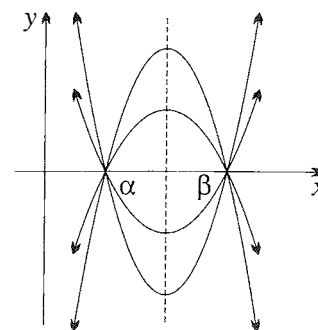
SOLUTION:

(a) With no restriction on the domain, the range is $y \geq -4$.

(b) When $x = 4$, $y = 5$, so the range is $y \geq 5$.

(c) When $x = 0$, $y = -3$, so the range is $-4 \leq y \leq 5$.

Quadratics with Given Zeroes: If it is known that a quadratic $f(x)$ has zeroes α and β , then the quadratic must have the form $f(x) = a(x - \alpha)(x - \beta)$, where a is the coefficient of x^2 . By taking different values of a , this equation forms a family of quadratics all with the same x -intercepts, as sketched opposite. In general:



QUADRATICS WITH GIVEN ZEROES:

4

The quadratics whose zeroes are α and β form a family with equation $y = a(x - \alpha)(x - \beta)$.

WORKED EXERCISE: Write down the family of quadratics with zeroes -2 and 4 , then find the equation of such a quadratic:

(a) with y -intercept 6 ,

(b) with vertex $(1, 21)$.

SOLUTION: The family of quadratics with zeroes -2 and 4 is $y = a(x + 2)(x - 4)$.

- (a) When $x = 0$, $y = -8a$, so $-8a = 6$,
so $a = -\frac{3}{4}$, and the quadratic is $y = -\frac{3}{4}(x + 2)(x - 4)$.
- (b) [Taking the average of the zeroes, the axis of symmetry is indeed $x = 1$.]
When $x = 1$, $y = -9a$, so $-9a = 21$,
so $a = -\frac{7}{3}$, and the quadratic is $y = -\frac{7}{3}(x + 2)(x - 4)$.

Monic Quadratics: Factorisation in quadratics is a little easier to handle when the coefficient of x^2 is 1. Such quadratics are called *monic*.

5 **DEFINITION:** A quadratic is called *monic* if the coefficient of x^2 is 1.

Then $(x - \alpha)(x - \beta)$ is the only monic quadratic whose zeroes are α and β .

Exercise 8A

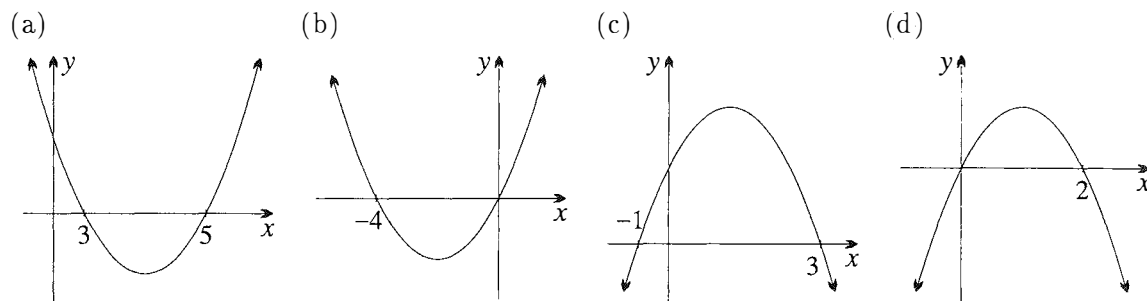
1. Use factorisation where necessary to find the zeroes of these quadratic functions. Use the fact that the axis of symmetry is halfway between the zeroes to find the equation of the axis, then find the vertex. Hence sketch a graph showing all intercepts and the vertex.

- (a) $y = (x - 1)(x + 3)$ (d) $y = (2x - 1)(2x + 5)$ (g) $y = x^2 + 4x + 3$
 (b) $y = x(x - 3)$ (e) $y = x^2 - 9$ (h) $y = 3 + 2x - x^2$
 (c) $y = (5 - x)(x + 1)$ (f) $y = x^2 - 5x + 6$

2. Use the graphs sketched in the question above to solve the following inequations:

- (a) $(x - 1)(x + 3) > 0$ (d) $4x^2 + 8x - 5 < 0$ (g) $x^2 + 4x \leq -3$
 (b) $x(x - 3) \leq 0$ (e) $x^2 \geq 9$ (h) $3 + 2x > x^2$
 (c) $5 + 4x - x^2 \geq 0$ (f) $x^2 < 5x - 6$

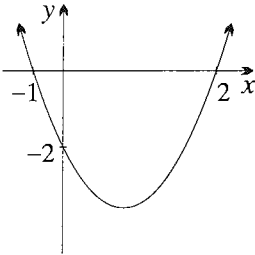
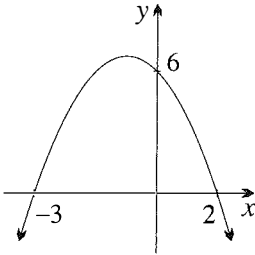
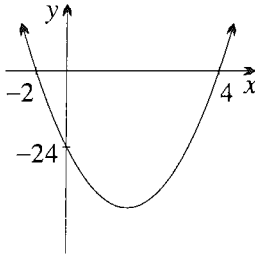
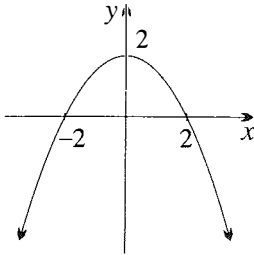
3. Give a possible equation of each quadratic function sketched below:



4. State the axis of symmetry and equation of the monic quadratic with zeroes:
- (a) 4 and 6 (b) 3 and 8 (c) -3 and 5 (d) -6 and -1
5. Use factorisation to sketch each quadratic, showing the intercepts and vertex:
- (a) $y = 2x^2 - 9x - 5$ (c) $y = -3x^2 - 5x + 2$
 (b) $y = 3x^2 - 10x - 8$ (d) $y = 7x - 3 - 4x^2$

DEVELOPMENT

6. The general form of a quadratic with zeroes 2 and 8 is $y = a(x - 2)(x - 8)$. By evaluating a , find the equation if such a quadratic:
- (a) has y -intercept -16 , (c) has constant term -3 , (e) has vertex $(5, -12)$,
 (b) passes through $(3, 10)$, (d) has coefficient of x^2 3, (f) passes through $(1, -20)$.

7. Write down the general form of a monic quadratic for which one of the zeroes is $x = 1$. Then find the equation of such a quadratic in which:
- (a) the curve passes through the origin, (c) the axis of symmetry is $x = -7$,
 (b) there are no other zeroes, (d) the curve passes through $(3, 9)$.
8. Factor and sketch each quadratic. Hence find the range of each function (i) with unrestricted domain, (ii) with domain $x \geq 5$, (iii) with domain $0 \leq x \leq 5$:
- (a) $y = x^2 - 6x + 8$ (b) $y = x^2 - 12x + 27$ (c) $y = -x^2 + 4x - 3$
9. Sketch each of the following regions on a number plane:
- (a) $y \geq x^2$ (c) $-x^2 \leq y \leq x^2$ (e) $x^2 - 1 \leq y \leq 9 - x^2$
 (b) $y \leq -x^2$ (d) $x^2 - 4 \leq y \leq 4 - x^2$ (f) $x^2 + 2x - 3 \leq y \leq 4x - x^2$
10. Explain why $y = ax(x - \alpha)$ is the general form of a quadratic whose graph passes through the origin. Hence find the equation of such a quadratic:
- (a) with another x -intercept at $x = -3$, and monic,
 (b) with vertex $(1, 4)$,
 (c) with no other x -intercepts, and passing through $(2, 6)$,
 (d) with no other x -intercepts, and having gradient 1 at $x = -2$,
 (e) with another x -intercept at $x = 5$, and having gradient 2 at the origin,
 (f) with axis of symmetry $x = -3$, and with -12 as coefficient of x .
11. The general form of a quadratic with zeroes α and β is $y = a(x - \alpha)(x - \beta)$. Find a in terms of α and β if:
- (a) the y -intercept is c , (b) the coefficient of x is b , (c) the curve passes through $(1, 2)$.
12. Find the equations of each quadratic function sketched below:
- (a)  (b)  (c)  (d) 
13. Use factorisation to find the x -intercepts of each graph, and hence find its axis:
- (a) $y = x^2 + bx + cx + bc$ (c) $y = ax^2 - bx - (a + b)$
 (b) $y = x^2 + (1 - a^2)x - a^2$ (d) $y = x^2 - 2cx - 1 + c^2$
14. (a) (i) Sketch the graph of $f(x) = (x - 3)^2$. (ii) Find $f'(x)$, show that $f(3) = f'(3) = 0$, and explain the geometrical significance of this result.
 (b) Show that the derivative of $f(x) = p(x - q)^2$ is $f'(x) = 2p(x - q)$. Hence show that $f(q) = f'(q) = 0$, and explain the geometrical significance of this result.
15. Use the product rule to show that the derivative of $y = a(x - \alpha)(x - \beta)$ is $y' = a(2x - \alpha - \beta)$. Hence show that the vertex is at $x = \frac{1}{2}(\alpha + \beta)$, and that the gradients at the x -intercepts are opposites of each other. Show more generally that the gradients at $x = \frac{1}{2}(\alpha + \beta) + h$ and $x = \frac{1}{2}(\alpha + \beta) - h$ are opposites of each other.

EXTENSION

16. (a) Show that $y = x^4 - 13x^2 + 36$ is an even function, then sketch its graph.
17. (a) If $f(x) = (x-3)(x-1)(x+4)(x+6)$, show that $f(-1) = f(-2)$ and $f(2) = f(-5)$.
 (b) Show that $f(-\frac{3}{2} + a) = f(-\frac{3}{2} - a)$.
 (c) Show that $f'(-\frac{3}{2} - a) = -f'(-\frac{3}{2} + a)$.
 (d) On the same set of axes sketch $f(x) = (x-3)(x-1)(x+4)(x+6)$ and the line $x = -\frac{3}{2}$.
 (e) Sketch a graph of the function $f(x) = (x-a)(x-b)(x-c)(x-d)$ where $b-a = d-c$.
18. Suppose that a quadratic has equation $f(x) = k(x-\alpha)(x-\beta)$. Prove the following identities, and explain how each establishes that $x = \frac{1}{2}(\alpha + \beta)$ is the axis of symmetry.
- (a) $f(\frac{1}{2}(\alpha + \beta) + h) = f(\frac{1}{2}(\alpha + \beta) - h)$ (b) $f(\alpha + \beta - x) = f(x)$

8B Completing the Square and the Graph

Completing the square is the fundamental method of approach to quadratics. It works in every case, in contrast with factoring, which really only works in exceptional cases. Although important formulae for the zeroes and vertex can be developed by completing the square in a general quadratic, the method remains necessary in many situations and needs to be learnt well.

The Method of Completing the Square: For *monic quadratics*, where the coefficient of x^2 is 1, the goal is to express the quadratic $y = x^2 + bx + c$ in the form

$$y = (x - h)^2 + k, \text{ which expands to } y = x^2 - 2hx + h^2 + k.$$

Since then $h = -\frac{1}{2}b$, the method is usually expressed rather concisely as:

6

COMPLETING THE SQUARE: Halve the coefficient of x ,
then add and subtract its square.

For *non-monic quadratics*, where the coefficient of x^2 is not 1, the coefficient should be removed by bracketing before the calculation begins.

WORKED EXERCISE: Complete the square in each of the following quadratics:

(a) $y = x^2 - 4x - 5$

(c) $y = x^2 + x + 1$

(b) $y = 2x^2 - 12x + 16$

(d) $y = -3x^2 - 4x + 2$

SOLUTION:

(a) $y = x^2 - 4x - 5$

$$= (x^2 - 4x + 4) - 4 - 5$$

$$= (x - 2)^2 - 9$$

(c) $y = x^2 + x + 1$

$$= (x^2 + x + \frac{1}{4}) - \frac{1}{4} + 1$$

$$= (x + \frac{1}{2})^2 + \frac{3}{4}$$

(b) $y = 2x^2 - 12x + 16$

$$= 2(x^2 - 6x + 8)$$

$$= 2((x^2 - 6x + 9) - 9 + 8)$$

$$= 2(x - 3)^2 - 2$$

(d) $y = -3x^2 - 4x + 2$

$$= -3(x^2 + \frac{4}{3}x - \frac{2}{3})$$

$$= -3((x^2 + \frac{4}{3}x + \frac{4}{9}) - \frac{4}{9} - \frac{2}{3})$$

$$= -3(x + \frac{2}{3})^2 + \frac{10}{3}$$

Sketching the Graph from the Completed Square: The work in Chapter Two on transformations of graphs tells us that $y = a(x - h)^2 + k$ is just $y = ax^2$ shifted h units to the right and k units upwards. Hence its vertex must be at (h, k) .

7

THE COMPLETED SQUARE AND TRANSLATION: The curve $y = a(x - h)^2 + k$ is the translation of $y = ax^2$ to a parabola with vertex at (h, k) .

From the completed square form, the zeroes can then be calculated by the usual method of setting $y = 0$. If the zeroes exist, the quadratic can then be written in factored form.

WORKED EXERCISE: Use the previous completed squares to sketch the graphs of:

(a) $y = x^2 - 4x - 5$

(c) $y = x^2 + x + 1$

(b) $y = 2x^2 - 12x + 16$

(d) $y = -3x^2 - 4x + 2$

If possible, express each quadratic in the form $y = a(x - \alpha)(x - \beta)$.

NOTE: When the square is completed, the axis of symmetry and vertex can be read off directly — the zeroes of the quadratic can then be found with a subsequent calculation. This contrasts with factoring, where the zeroes are found first and the vertex then follows.

SOLUTION:

(a) $y = x^2 - 4x - 5$ is concave up with y -intercept -5 .

Since $y = (x - 2)^2 - 9$, the vertex is $(2, -9)$.

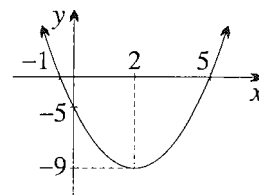
Put $y = 0$, then $(x - 2)^2 = 9$

$$x - 2 = 3 \text{ or } x - 2 = -3$$

$$x = 5 \text{ or } -1.$$

Hence also

$$y = (x - 5)(x + 1).$$



(b) $y = 2x^2 - 12x + 16$ is concave up with y -intercept 16 .

Since $y = 2(x - 3)^2 - 2$, the vertex is $(3, -2)$.

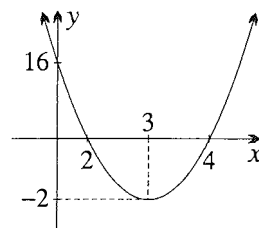
Put $y = 0$, then $2(x - 3)^2 = 2$

$$x - 3 = 1 \text{ or } x - 3 = -1$$

$$x = 4 \text{ or } 2.$$

Hence also

$$y = 2(x - 4)(x - 2).$$

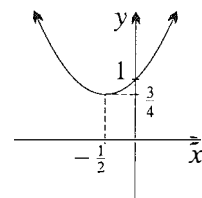


(c) $y = x^2 + x + 1$ is concave up with y -intercept 1 .

Since $y = (x + \frac{1}{2})^2 + \frac{3}{4}$, the vertex is $(-\frac{1}{2}, \frac{3}{4})$.

Put $y = 0$, then $(x + \frac{1}{2})^2 = -\frac{3}{4}$,

and since this has no solutions, there are no x -intercepts.



(d) $y = -3x^2 - 4x + 2$ is concave down with y -intercept 2 .

Since $y = -3(x + \frac{2}{3})^2 + \frac{10}{3}$, the vertex is $(-\frac{2}{3}, \frac{10}{3})$.

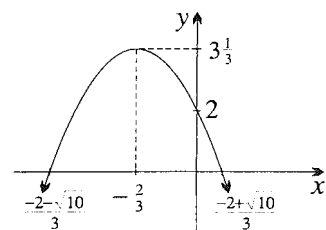
Put $y = 0$, then $3(x + \frac{2}{3})^2 = \frac{10}{3}$

$$(x + \frac{2}{3})^2 = \frac{10}{9}$$

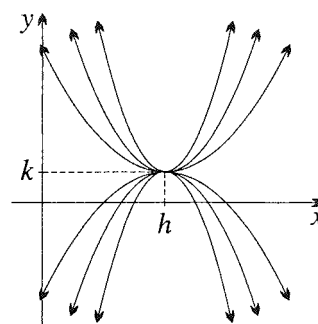
$$x + \frac{2}{3} = \frac{1}{3}\sqrt{10} \text{ or } x + \frac{2}{3} = -\frac{1}{3}\sqrt{10}$$

$$x = -\frac{2}{3} + \frac{1}{3}\sqrt{10} \text{ or } -\frac{2}{3} - \frac{1}{3}\sqrt{10}.$$

Hence also $y = -3 \left(x + \frac{2}{3} - \frac{1}{3}\sqrt{10} \right) \left(x + \frac{2}{3} + \frac{1}{3}\sqrt{10} \right).$



The Family of Quadratics with a Common Vertex: If a quadratic is known to have its vertex at (h, k) , then by the theory above, its equation must have the form $y = a(x - h)^2 + k$, for some value of a . This equation gives a family of quadratics with vertex (h, k) , as different values of a are taken, as in the sketch opposite. The general case is:



8

QUADRATICS WITH A COMMON VERTEX: The quadratics with vertex (h, k) form a family of curves with equation $y = a(x - h)^2 + k$.

WORKED EXERCISE: Write down the family of quadratics with vertex $(-3, 2)$, then find the equation of such a quadratic:

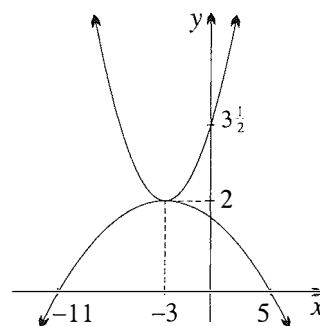
- (a) if $x = 5$ is one of its zeroes, (b) if the coefficient of x is equal to 1.

SOLUTION: The family of quadratics with vertex $(-3, 2)$ is

$$y = a(x + 3)^2 + 2.$$

- (a) Substituting $(5, 0)$ gives $0 = a \times 64 + 2$,
so $a = -\frac{1}{32}$, and the quadratic is $y = -\frac{1}{32}(x + 3)^2 + 2$.

- (b) Expanding, $y = ax^2 + 6ax + (9a + 2)$,
so $6a = 1$,
so $a = \frac{1}{6}$ and the quadratic is $y = \frac{1}{6}(x + 3)^2 + 2$.

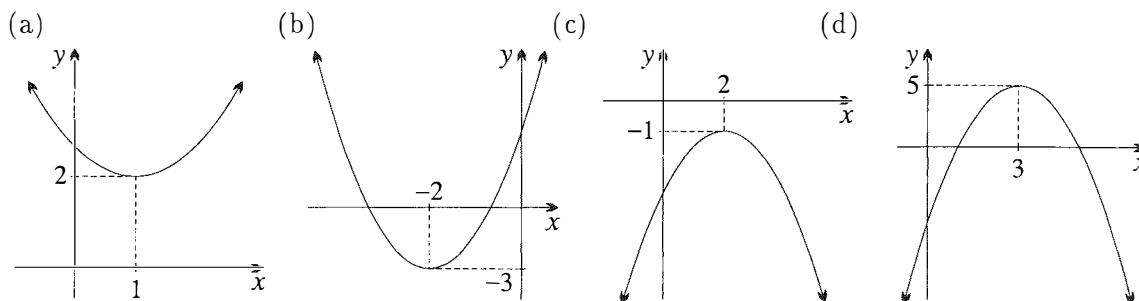


Exercise 8B

1. Complete the square where necessary in each quadratic, expressing it in the form $y = (x - h)^2 + k$. Hence find the axis of symmetry, vertex and any intercepts, giving irrational zeroes in exact form. Sketch their graphs, showing vertex and intercepts.

- (a) $y = (x - 3)^2 - 9$ (d) $y = (x - 5)^2 - 2$ (g) $y = x^2 - 2x + 5$
 (b) $y = (x + 2)^2 - 1$ (e) $y = x^2 - 2x$ (h) $y = x^2 + x + 1$
 (c) $y = (x + 1)^2 + 3$ (f) $y = x^2 - 4x + 3$ (i) $y = x^2 - 3x + 1$

2. Give a possible equation for each of the quadratic functions sketched below:



3. Write down the equation of the monic quadratics with the following vertices. Also find the axis of symmetry and y -intercept of each.

- (a) $(2, 5)$ (b) $(0, -3)$ (c) $(-1, 7)$ (d) $(3, -11)$

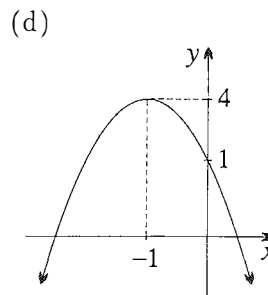
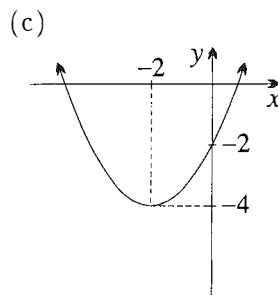
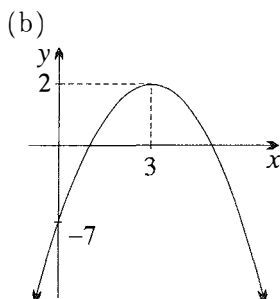
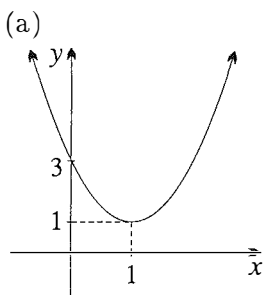
4. Explain why any quadratic with vertex $(0, 1)$ has equation $y = ax^2 + 1$, for some value of a . Hence find the equation of such a quadratic passing through:

- (a) $(1, 3)$ (b) $(-2, -11)$ (c) $(9, 28)$ (d) $(\frac{1}{2}, \frac{15}{16})$

5. Explain why $y = a(x+4)^2 + 2$ is the general form of a quadratic with vertex $(-4, 2)$. Then find the equation of such a quadratic for which:
- (a) the quadratic is monic,
 - (b) the coefficient of x^2 is 3,
 - (c) one of the zeroes is $x = 3$,
 - (d) the y -intercept is 16,
 - (e) the curve passes through the origin,
 - (f) the curve passes through $(1, 20)$.

DEVELOPMENT

6. Express each quadratic function in the form $y = a(x - h)^2 + k$ (notice that $a \neq 1$ in each example). Find any x -intercepts, the y -intercept and the vertex. Write down the equation of the axis of symmetry. Then sketch the curve, showing this information.
- (a) $y = -x^2 - 2x$
 - (b) $y = -x^2 + 4x + 1$
 - (c) $y = -x^2 + 5x - 6$
 - (d) $y = 2x^2 - 4x + 3$
 - (e) $y = 4x^2 - 16x + 13$
 - (f) $y = -3x^2 + 6x + 3$
 - (g) $y = -5x^2 - 20x - 23$
 - (h) $y = 2x^2 + 5x - 12$
 - (i) $y = 3x^2 + 2x - 8$
7. Sketch the graph of each function, showing the intercepts and vertex. From the graph, find the range of each function with: (i) unrestricted domain, (ii) domain $x \leq -1$, (iii) domain $-1 \leq x \leq 2$.
- (a) $y = (x - 1)^2 + 2$
 - (b) $y = 2(x - 3)^2 + 1$
 - (c) $y = -(x + 2)^2 + 5$
8. Complete the square to find the vertex of the function $y = x^2 - 6x + c$. Hence find the values of c for which the graph of the function: (a) touches the x -axis, (b) cuts the x -axis in two places, (c) does not intersect the x -axis.
9. Write down the general form of a monic quadratic whose axis of symmetry is $x = -2$. Hence find the equation of such a quadratic:
- (a) passing through the origin,
 - (b) passing through $(5, 1)$,
 - (c) with $x = 1$ as one zero,
 - (d) with y -intercept -6 ,
 - (e) touching the line $y = -2$,
 - (f) with range $y \geq 7$,
 - (g) whose tangent at $x = 1$ passes through $(0, 0)$,
 - (h) which is tangent to $y = -x^2$.
10. Find the equations of the quadratic functions sketched below.



11. Complete the square to find the roots α and β , and show that $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.
- (a) $x^2 - x - 6 = 0$
 - (b) $x^2 - 4x + 1 = 0$
 - (c) $2x^2 + 3x - 5 = 0$
 - (d) $5x^2 - 15x + 11 = 0$
12. (a) Complete the square to find the vertex of each quadratic function. Then sketch all five functions on the same number plane. (i) $y = x^2 - 4x + 4$ (ii) $y = x^2 - 2x + 4$ (iii) $y = x^2 + 4$ (iv) $y = x^2 + 2x + 4$ (v) $y = x^2 + 4x + 4$
- (b) What is the effect of varying b on the graph of $y = x^2 + bx + 4$?
13. Complete the square to find the vertex and x -intercepts of the function $y = x^2 + ax + b$. Then sketch a possible graph of the function if:
- (a) $a > 0$ and $a^2 > 4b$,
 - (b) $a > 0$ and $a^2 < 4b$,
 - (c) $a > 0$ and $a^2 = 4b$,
 - (d) $a < 0$ and $a^2 > 4b$,
 - (e) $a < 0$ and $a^2 < 4b$,
 - (f) $a < 0$ and $a^2 = 4b$.

14. Complete the square in $y = ax^2 + bx + c$. Hence write down the vertex and find the zeroes.
15. Expand $y = a(x - \alpha)(x - \beta)$, complete the square, and hence find the vertex.
16. If $f(x) = (x - h)^2 + k$, show that $f'(h + r) = -f'(h - r)$. Give a geometric interpretation.
17. Write down the general form of a quadratic with vertex (h, k) . Find an expression for the coefficient of x^2 if:
- the y -intercept is c ,
 - the curve passes through $(1, 2)$,
 - the coefficient of x is b ,
 - one of the zeroes is α .
18. (a) Find the zeroes of the monic quadratic $y = (x + d)^2 - e$, where $e > 0$.
 (b) Find an expression for the difference between the two zeroes.
 (c) Hence find the condition for the difference between the two zeroes to be 2, and describe geometrically the family of quadratics with this property.
19. The monic quadratics $y = (x - h_1)^2 + k_1$ and $y = (x - h_2)^2 + k_2$ do not intersect at all. Find the corresponding condition on the constants h_1, h_2, k_1 and k_2 , and describe geometrically the relationship between the two curves.

EXTENSION

20. (a) Complete the square to find the vertex and x -intercepts of $y = x^2 + 2x - 3$, and sketch the curve. (b) Hence sketch $y = |x^2 + 2x - 3|$. (c) Complete the square to solve the equations $x^2 + 2x - 3 = 1$ and $x^2 + 2x - 3 = -1$. (d) Hence sketch $y = \frac{1}{x^2 + 2x - 3}$.
21. Consider the quadratic $f(x) = a(x - h)^2 + k$ with vertex (h, k) . Prove the following identities and hence establish that $x = h$ is the axis of symmetry.
- $f(h + t) = f(h - t)$
 - $f(2h - x) = f(x)$.
22. Show that the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, cannot have more than two distinct roots. [HINT: Assume that the equation can have three distinct roots α, β and γ . Substitute α, β and γ into the equation and conclude that $a = b = c = 0$.]

8C The Quadratic Formulae and the Graph

Completing the square in a general quadratic yields formulae for the axis of symmetry and for the zeroes of a quadratic function. These formulae are extremely useful, and will allow the theory of quadratics to be advanced considerably. The previous exercise asked for these formulae to be generated, but in view of their importance, they are derived again here.

Completing the Square in the General Quadratic: Here then is the completion of the square in the general quadratic $y = ax^2 + bx + c$:

$$\begin{aligned} y &= a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) \\ &= a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} \right), \text{ since half the coefficient of } x \text{ is } \frac{b}{2a}, \\ &= a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}. \end{aligned}$$

Hence the axis of symmetry is $x = -\frac{b}{2a}$, and the vertex is $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a} \right)$.

Remember the formula for the axis, and find the y -coordinate of the vertex by substitution.

9 THE AXIS OF SYMMETRY: The axis of symmetry is the line $x = -\frac{b}{2a}$.

To find the formula for the zeroes, put $y = 0$ into the completed square:

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

The quantity $b^2 - 4ac$ is called the *discriminant* and is given the symbol Δ (Greek capital delta). When calculating the zeroes, the discriminant should always be found first, so the formula for the zeroes should be remembered in the form:

10 THE ZEROES: $x = \frac{-b + \sqrt{\Delta}}{2a}$ or $\frac{-b - \sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$.

The discriminant $\Delta = b^2 - 4ac$ will become important theoretically as the chapter develops. For now it will be enough to notice two things. First, if the discriminant is negative, then the quadratic has no zeroes, because negative numbers don't have square roots. Secondly, if the discriminant is zero, then the quadratic has only one zero, because the only square root of zero is zero itself.

NOTE: The vertex (found above) can be written in the form $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$. Some people prefer memorising this formula for the coordinates of the vertex rather than substituting the axis of symmetry to find the y -coordinate.

WORKED EXERCISE: Use the quadratic formulae to sketch the following quadratics. Give any irrational zeroes first in simplified surd form, then approximated to four significant figures. If possible, write each quadratic in factored form.

(a) $y = -x^2 + 6x + 1$

(b) $y = 3x^2 - 6x + 4$

SOLUTION:

(a) The curve $y = -x^2 + 6x + 1$ is concave down with y -intercept 1.

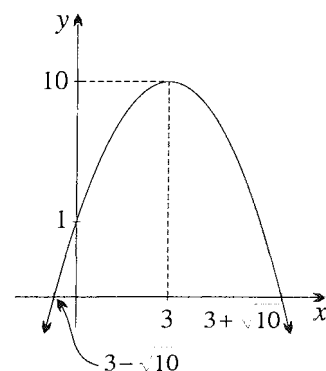
The formulae are applied with $a = -1$, $b = 6$ and $c = 1$.

The axis is $x = -\frac{b}{2a}$
 $x = 3$.

When $x = 3$, $y = 10$, so the vertex is $(3, 10)$.

Also $\Delta = b^2 - 4ac$
 $= 40 = 4 \times 10$,

so $y = 0$ when $x = \frac{-b + \sqrt{\Delta}}{2a}$ or $\frac{-b - \sqrt{\Delta}}{2a}$



$$= 3 - \sqrt{10} \text{ or } 3 + \sqrt{10}$$

$$\doteq -0.1623 \text{ or } 6.162.$$

Hence also $y = (x - 3 + \sqrt{10})(x - 3 - \sqrt{10})$.

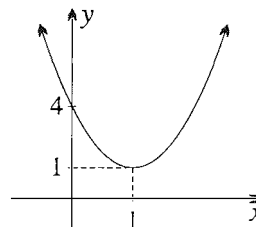
- (b) The curve $y = 3x^2 - 6x + 4$ is concave up,
and its y -intercept is 4.

Using the formulae with $a = 3$, $b = -6$ and $c = 4$,
the axis is $x = 1$,

and substituting $x = 1$, the vertex is $(1, 1)$.

Also $\Delta = 36 - 48$

which is negative, so there are no zeroes.



Exercise 8C

- Find the discriminant $\Delta = b^2 - 4ac$, and hence the zeroes, of these quadratics. Give irrational zeroes in surd form, then approximated to four significant figures.

(a) $y = x^2 + 6x + 5$	(d) $y = -x^2 + 2x + 1$	(g) $y = -5x^2 + 7x + 3$
(b) $y = x^2 + 4x + 4$	(e) $y = x^2 + 4x - 1$	(h) $y = 4x^2 - 3x - 3$
(c) $y = -x^2 + 2x + 24$	(f) $y = 2x^2 + 2x - 1$	(i) $y = 4x^2 - 9$
- For each quadratic in the previous question, find the equation of the axis of symmetry using the formula $x = -b/2a$. Substitute into the function to find the vertex, then sketch the curve, showing the vertex and all intercepts.
- Use the graphs in parts (a)–(d) of the previous question to solve:

(a) $x^2 + 6x + 5 < 0$	(b) $x^2 + 4x > -4$	(c) $2x + 24 \leq x^2$	(d) $x^2 \leq 2x + 1$
- By substituting the axis of symmetry $x = -b/2a$ into the equation of the general quadratic $y = ax^2 + bx + c$, show that the vertex has y -coordinate $-\Delta/4a$. Use this formula to check the vertices that you obtained in question 2 above.
- Evaluate the discriminant $\Delta = b^2 - 4ac$ for each quadratic, and hence establish how many times each function will intersect the x -axis:

(a) $y = x^2 + 2x - 3$	(b) $y = x^2 + 3x + 1$	(c) $y = 9x^2 - 6x + 1$	(d) $y = -2x^2 + 5x - 7$
- Use the quadratic formula to find the roots α and β of each quadratic, and show that $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

(a) $3x^2 - 10x - 8 = 0$	(b) $x^2 - 2x - 4 = 0$	(c) $x^2 - 6x + 1 = 0$	(d) $-3x^2 + 5x + 2 = 0$

DEVELOPMENT

- Use the quadratic formula to find the zeroes of each of the following quadratic functions. Hence write each function in the form $y = a(x - \alpha)(x - \beta)$.

(a) $y = x^2 - 6x + 4$	(b) $y = 3x^2 + 6x + 2$	(c) $y = -x^2 + 3x + 1$	(d) $y = -2x^2 - x + 1$
- Solve the following pairs of equations simultaneously. Hence state how many times the parabola and the line intersect.

(a) $y = x^2 - 4x + 3$ and $y = x + 3$,	(c) $y = -x^2 + x - 3$ and $y = 2x + 1$,
(b) $y = 2x^2 + 7x - 4$ and $y = 3x - 6$,	(d) $y = -2x^2 + 5x - 1$ and $y = 3 - x$.
- The interval PQ has length p , and the point A lies between the points P and Q . Find PA when $PQ \times QA = PA^2$.

10. Find the derivative $f'(x)$ of the general quadratic $f(x) = ax^2 + bx + c$, and hence show that the derivative is zero when $x = -b/2a$. Explain how this relates to the axis of symmetry.
11. (a) Expand $f(x) = (x-h)^2 + k$ and show that $\Delta = -4k$. Then use the quadratic formulae to show that the axis is $x = h$ (as expected) and to find an expression for the zeroes.
 (b) Expand $f(x) = (x-\alpha)(x-\beta)$ and show that $\Delta = (\alpha-\beta)^2$. Hence show that the zeroes are $x = \alpha$ and $x = \beta$ (as expected) and that the vertex is $\left(\frac{\alpha+\beta}{2}, -\left(\frac{\alpha-\beta}{2}\right)^2\right)$.
12. (a) Find the axis of symmetry and the vertex of $y = x^2 + bx + c$. (b) Find the zeroes, and then find the difference between them. (c) What condition on the constants b and c must be satisfied for the difference to be exactly 1? (d) Hence show that the family of such quadratics is the family of parabolas whose vertices are on the line $y = -\frac{1}{4}$.

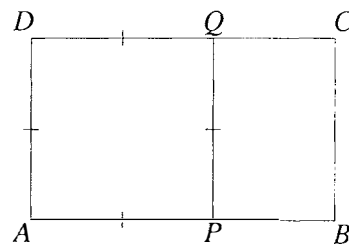
EXTENSION

13. [The golden mean] Sketch $y = x^2 - x - 1$, showing the vertex and all intercepts.

(a) If $\alpha = \frac{1}{2}(\sqrt{5} + 1)$, show that: (i) $\alpha^2 = \alpha + 1$

(ii) $\frac{1}{\alpha} = \alpha - 1$ (iii) $\alpha^6 = 8\alpha + 5$

- (b) $ABCD$ is a rectangle whose length and breadth are in the ratio $\alpha : 1$. It is divided into a square $APQD$ and a second rectangle $PBCQ$, as shown. Show that the length and breadth of rectangle $PBCQ$ are also in the ratio $\alpha : 1$.



8D Equations Reducible to Quadratics

There are many equations, including many trigonometric equations, which can be solved by using substitutions that reduce them to quadratic equations. For example, the degree 4 equation $x^4 - 13x^2 + 36 = 0$ becomes a quadratic equation with the substitution $u = x^2$. Substitution can also help to determine the graph if the function is reducible to a quadratic.

WORKED EXERCISE: By making substitutions that will reduce them to quadratic equations, solve:

- (a) $x^4 - 13x^2 + 36 = 0$ (b) $2\cos^2 x - 3\cos x + 1 = 0$, for $0^\circ \leq x \leq 360^\circ$

SOLUTION:

- (a) Let $u = x^2$.

$$\text{Then } u^2 - 13u + 36 = 0$$

$$(u-9)(u-4) = 0$$

$$u = 9 \text{ or } u = 4.$$

$$\text{So } x^2 = 9 \text{ or } x^2 = 4$$

$$x = 3, -3, 2 \text{ or } -2.$$

- (b) Let $u = \cos x$.

$$\text{Then } 2u^2 - 3u + 1 = 0$$

$$(2u-1)(u-1) = 0$$

$$u = \frac{1}{2} \text{ or } u = 1.$$

$$\text{So } \cos x = \frac{1}{2} \text{ or } \cos x = 1$$

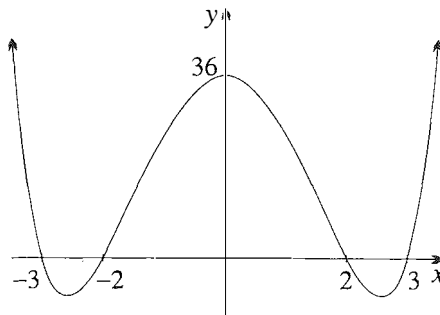
$$x = 0^\circ, 60^\circ, 300^\circ \text{ or } 360^\circ.$$

WORKED EXERCISE: Using only factorisation, sketch $y = x^4 - 13x^2 + 36$.

SOLUTION: From the previous example, the zeroes are 3, -3, 2 and -2,

$$\text{so } y = (x-3)(x+3)(x-2)(x+2).$$

The y -intercept is 36.



Exercise 8D

1. Solve the following equations for real values of x by reducing them to quadratic equations:

(a) $x^4 - 10x^2 + 9 = 0$

(f) $16x^2 + 16x^{-2} = 257$

(b) $x^4 + 100 = 29x^2$

(g) $(x^2 - x)^2 - 18(x^2 - x) + 72 = 0$

(c) $3x^4 - 10x^2 + 8 = 0$

(h) $(x^2 - 4x) + 8 = \frac{48}{x^2 - 4x}$

(d) $x^6 - 9x^3 + 8 = 0$

(i) $3^{2x} - 12 \times 3^x + 27 = 0$

(e) $x^3 + \frac{27}{x^3} = 28$

(j) $4^x - 12 \times 2^x + 32 = 0$

DEVELOPMENT

2. By making suitable substitutions, solve the following for $0^\circ \leq x \leq 360^\circ$:

(a) $2\sin^2 x - 3\sin x + 1 = 0$

(c) $\sec^2 x + 2\tan x = 0$

(b) $2\sin^2 x = 3(\cos x + 1)$

(d) $\cot^2 x = \operatorname{cosec} x + 1$

3. Solve the following simultaneous equations:

(a) $x^2 + y^2 = 10$ and $x + 2y = 7$

(b) $x^2 + y^2 - 2y = 7$ and $x - y = 3$

(c) $x^2 + y^2 - 2x + 6y - 35 = 0$ and $2x + 3y = 5$

4. Solve the following equations for real values of x by reducing them to quadratic equations:

(a) $2\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 15 = 0$

(b) $x(x+1)(x+2)(x+3) = 35$

[HINT: Expand $(x+1)(x+2)$ and $x(x+3)$ and let $u = x^2 + 3x$.]

(c) $(x+1)(x+2)(x+3)(x+4) = 18 + 5(x^2 + 5x)$

5. Solve for x . Each solution must be checked in the original equation.

(a) $3x - \sqrt{x} = 2$

(c) $\sqrt{5+x} + \sqrt{x} = 5$

(b) $x + 2\sqrt{x+1} = 7$

(d) $\sqrt{x+5} + \sqrt{x-2} = \sqrt{5x-6}$

6. Solve for x (the second will need the change of base law):

(a) $\frac{x+5}{x-5} - \frac{x-6}{x+6} = \frac{x+4}{x-4} - \frac{x-7}{x+7}$

(b) $2\log_5 x - 9\log_x 5 = 3$

7. (a) Solve for x and y simultaneously $\frac{p^2}{x} + \frac{q^2}{y} = \frac{(p+q)^2}{r}$ and $x+y=r$.

(b) Hence solve simultaneously $\frac{9}{x} + \frac{16}{y} = \frac{49}{2}$ and $x+y=2$.

8. (a) Sketch the following functions, clearly indicating all x - and y -intercepts:

(i) $y = x^4 - 10x^2 + 9$ (ii) $y = 2x^4 - 11x^2 + 12$ (iii) $y = (x^2 - 4x)^2 - (x^2 - 4x) - 6$

(b) Use the graphs drawn in part (a) to solve the following inequations:

(i) $x^4 - 10x^2 + 9 \geq 0$ (ii) $2(x^4 + 6) \leq 11x^2$ (iii) $x^4 - 8x^3 + 15x^2 + 4x - 6 > 0$

EXTENSION

9. Quartic equations of the form $Ax^4 + Bx^3 + Cx^2 + Bx + A = 0$, $A \neq 0$, are also reducible to quadratics using the substitution $u = x + 1/x$ and grouping terms appropriately.

(a) Copy and complete: ‘ $x^4 - 5x^3 + 8x^2 - 5x + 1 = 0$

$$x^2 \left(x^2 - 5x + 8 - \frac{5}{x} + \frac{1}{x^2} \right) = 0$$

$$x^2 \left(x^2 + 2 + \frac{1}{x^2} - 5x - \frac{5}{x} + 6 \right) = 0.$$

Let $u = x + \frac{1}{x}$, then $x^2(u^2 - 5u + 6) = 0$.

(b) Solve for x : (i) $x^4 + 3x^3 - 8x^2 + 3x + 1 = 0$ (ii) $3x^4 - 10x^3 + 13x^2 - 10x + 3 = 0$

8E Problems on Maximisation and Minimisation

We come now to an entirely new type of problem, which involves finding the maximum or minimum value of a function, and the value of x for which it occurs. This section will only be able to deal with quadratic functions, but in the next chapter, the calculus will be used to deal with far more general functions.

There are, as usual, three approaches to maximising a quadratic — completing the square, using the formula for the axis of symmetry, and factorisation. While completing the square may sometimes seem a little complicated, it is worth repeating that this approach is the real foundation of work on quadratics, and will repay study.

Finding the Maximum or Minimum By Completing the Square: A square like $(x - 6)^2$ can never be negative, and it reaches its minimum value of zero when $x = 6$. This is the key observation that allows us to deal with any quadratic whose square has been completed. Consider, for example,

$$(x - 6)^2 + 5 \quad \text{and} \quad -(x - 6)^2 + 7.$$

The first has a minimum of 5 when $x = 6$, and the second has a maximum of 7 when $x = 6$. Hence the general method of approaching the maximum or minimum values of a quadratic is:

11 **MAXIMISATION AND MINIMISATION BY COMPLETING THE SQUARE:** Complete the square, then use the fact that a square can never be negative to read off the maximum or minimum and the value of x for which it occurs.

WORKED EXERCISE: Find the maximum or minimum values of these quadratic functions, and the values of x for which they occur:

(a) $y = x^2 - 4x + 7$

(b) $y = 3 - 8x - x^2$

SOLUTION:

(a) Completing the square, $y = x^2 - 4x + 7$

$$y = (x^2 - 4x + 4) - 4 + 7$$

$$y = (x - 2)^2 + 3.$$

Now $(x - 2)^2$ can never be negative, and $(x - 2)^2$ is zero when $x = 2$, so y has a minimum of 3 when $x = 2$.

(b) Completing the square, $y = 3 - 8x - x^2$

$$y = -\left((x^2 + 8x + 16) - 16 - 3\right)$$

$$y = -(x + 4)^2 + 19.$$

Now $-(x + 4)^2$ can never be positive, and $-(x + 4)^2$ is zero when $x = -4$, so y has a maximum of 19 when $x = -4$.

Maximisation and Minimisation Using the Axis of Symmetry: The maximum or minimum of a quadratic must occur at the vertex. When $a > 0$, the graph is concave up and so must have a minimum, while if $a < 0$, it is concave down and so must have a maximum. This gives an alternative approach using the formula for the axis of symmetry.

MAXIMISATION AND MINIMISATION USING THE AXIS OF SYMMETRY:

12

1. Find the axis of symmetry and substitute it to find the vertex.
2. The sign of a distinguishes between maximum and minimum.

WORKED EXERCISE: Repeat the previous worked example using the formula for the axis of symmetry.

SOLUTION:

- (a) For $y = x^2 - 4x + 7$, the axis of symmetry is $x = 2$.

When $x = 2$, $y = 4 - 8 + 7 = 3$.

Since $a > 0$, the curve is concave up,

so there is a minimum of 3 when $x = 2$.

- (b) For $y = 3 - 8x - x^2$, the axis of symmetry is $x = -4$.

When $x = -4$, $y = 3 + 32 - 16 = 19$.

Since $a < 0$, the curve is concave down,

so there is a maximum of 19 when $x = -4$.

Maximisation and Minimisation Using Factorisation: The axis of symmetry is the arithmetic mean of the zeroes, so if the quadratic can be factored (or is already factored), the axis of symmetry is easily found and substituted. As before, the sign of a will distinguish between maximum and minimum. An example of this approach is given in the problem below.

Solving Problems on Maxima and Minima: When a maximisation problem is presented in words rather than symbols, great care needs to be taken when setting up the function to be maximised. Two variables will need to be introduced — one variable (usually called y) will be the quantity to be maximised, the other (usually called x) will be the quantity that can be changed.

PROBLEMS ON MAXIMA AND MINIMA: After drawing a picture:

13

1. If no variables have been named, introduce two variables:
'Let y (or whatever) be the variable to be maximised or minimised.
Let x (or whatever) be the variable that can be changed.'
2. Express y as a function of x .
3. Use an acceptable method to find the maximum or minimum value of y , and the value of x for which it occurs.
4. Write a careful conclusion.

WORKED EXERCISE: Farmer Brown builds a rectangular chookyard using an existing wall as one fence. If she has 20 metres of fencing, find the maximum area of the chookyard and the length of the fence parallel to the wall.

SOLUTION: Let x be the length in metres perpendicular to the wall.

Let A be the area of the chookyard.

The length parallel to the wall is $20 - 2x$ metres,

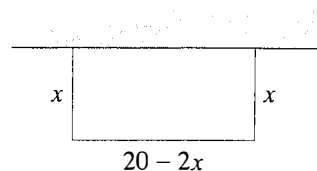
so $A = x(20 - 2x)$.

Since the zeroes are 0 and 10, the axis is $x = 5$,

and when $x = 5$, $A = 50$.

Hence the maximum area is 50 square metres,

and occurs when the fence parallel to the wall is 10 metres long.



WORKED EXERCISE: [A subtle choice of variables to prove a significant result]

The point P lies on the hypotenuse AB of a right triangle $\triangle OAB$. The points X and Y are the feet of the perpendiculars from P to the sides OA and OB respectively. Show that the rectangle $OXPY$ has a maximum area equal to half the area of the triangle OAB when P is the midpoint of the hypotenuse AB .

SOLUTION: Let P divide AB in the ratio $\lambda : (1 - \lambda)$. Let $a = OA$ and $b = OB$.

Then by similar triangles, X divides AO in the ratio $\lambda : (1 - \lambda)$,

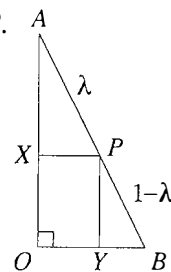
and Y divides OB in the ratio $\lambda : (1 - \lambda)$,

so $XO = (1 - \lambda)a$ and $OY = \lambda b$, and area of $OXPY = ab\lambda(1 - \lambda)$.

The zeroes of this quadratic are $\lambda = 0$ and $\lambda = 1$, and it is upside down,

so the area is maximum when $\lambda = \frac{1}{2}$ and P is the midpoint of AB ,

and then $OXPY$ has area $\frac{1}{4}ab$, which is half the area of $\triangle OAB$.



Exercise 8E

- (a) Complete the square to find the minimum value of each quadratic function:

(i) $y = x^2 - 4$ (ii) $y = x^2 - 10x + 16$ (iii) $y = x^2 - 5x + 6$ (iv) $y = 2x^2 + 5x - 3$

(b) Using the formula for the axis of symmetry, $x = -\frac{b}{2a}$, find the minimum value of each of the following quadratic functions:

(i) $y = x^2 - 2x + 5$ (ii) $y = 2x^2 + 4x + 5$ (iii) $y = x^2 - 6x + 7$ (iv) $y = 4x^2 - 2x + 3$

(c) Factor each of the following quadratic functions in order to find its zeroes. Sketch a graph of each function, clearly indicating its minimum value.

(i) $y = x^2 - 2x - 35$ (ii) $y = 6x^2 - 13x + 6$
- (a) Complete the square in each function to find the maximum value:

(i) $y = 9 - x^2$ (ii) $y = 6 + x - x^2$ (iii) $y = 8 - 2x - x^2$ (iv) $y = 5x - 2x^2 - 3$

(b) Using the formula for the axis of symmetry, $x = -\frac{b}{2a}$, find the maximum value of each of the following quadratic functions:

(i) $y = -x^2 - 4x - 5$ (ii) $y = -3x^2 + 3x - 2$ (iii) $y = 4 + x - x^2$ (iv) $y = 3x - 2x^2 + 1$

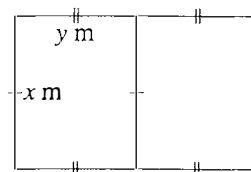
(c) Factor each of the following quadratic functions in order to find its zeroes. Sketch a graph of each function, clearly indicating its maximum value.

(i) $y = 3 + 2x - x^2$ (ii) $y = 13x - 10x^2 - 4$

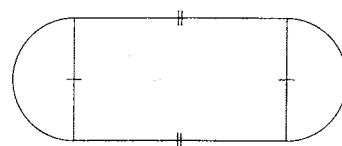
3. Two numbers have a sum of 3.
 - (a) Let the numbers be x and $3 - x$, and show that their product is $P = 3x - x^2$.
 - (b) Find the value of x for which P will be a maximum, and hence find the maximum value of P .
4. Two numbers have a sum of 30. Using the method of the previous question, find the numbers if their product is a maximum.
5. Two numbers have a sum of 6.
 - (a) Let the two numbers be x and $6 - x$, and show that the sum of the squares of the two numbers is $S = 2x^2 - 12x + 36$.
 - (b) Find the value of x for which S is a minimum, and hence find the least value of S .
6. A rectangle has a perimeter of 16 metres. Let x be the length of one side, and find a formula for the area A in terms of x . Hence find the maximum value of A .
7. A stone is thrown upwards so that at any time t seconds after throwing, the height of the stone is $h = 100 + 10t - 5t^2$ metres. Find the maximum height reached.
8. A manufacturer finds that the cost $C(x)$, in thousands of dollars, of manufacturing his product is given by $C(x) = 2x^2 - 8x + 15$, where x is the number of machines operating. Find how many machines he should operate in order to minimise the cost of production, and hence the minimum cost of production.

DEVELOPMENT

9. (a) A rectangle has a perimeter of 64 cm. If the length of the rectangle is x and its width is y , find an expression for the square of the length of the diagonal in terms of x .
- (b) Find the dimensions of the rectangle if the square of the length of the diagonal is a minimum.
10. $PQRS$ is a square of side length 5 cm. A and B are points on the sides PQ and SP of the square respectively such that $PA = BP = x$.
 - (a) Show that the area of the quadrilateral $BARS$ is given by $\frac{1}{2}(25 + 5x - x^2)$.
 - (b) Hence find the maximum area of the quadrilateral $BARS$.
11. A dairy farmer has 4 km of fencing to enclose a rectangular paddock. There is to be a gate of length 15 metres on each of the shortest sides of the paddock. The gates require no fencing.
 - (a) If she uses x metres of fencing on each of the longer sides, and y metres of fencing on each of the shorter sides, find an expression for the area enclosed in terms of x only.
 - (b) Hence find the maximum area that the dairy farmer can enclose.
12. A piece of wire of length 80 cm is to be cut into two sections. One section is to be bent into a square, and the other into a rectangle 4 times as long as it is wide.
 - (a) Let x be the side length of the square and y be the width of the rectangle. Write a formula connecting x and y and show that if A is the sum of the areas of the square and rectangle, then $A = \frac{41}{4}y^2 - 100y + 400$.
 - (b) Find the lengths of both sections of wire if A is to be a minimum.
13. 1600 metres of fencing is to be used to enclose a rectangular area and to divide it into two equal areas as shown.
 - (a) Using the pronumerals given, show that the combined enclosed area A is given by $A = 800x - \frac{3}{2}x^2$.
 - (b) Hence find the values of x and y for which the area enclosed is greatest.



14. A Tasmanian orchardist notices that an apple tree will produce 300 apples per year if 16 trees are planted in every standard-sized field. For every additional tree planted in the standard-sized field, she finds that the yield per tree decreases by 10 apples per year.
- If she plants an additional x trees in every standard-sized field, show that the total number of apples produced will be $N = -10x^2 + 140x + 4800$.
 - How many trees should be planted in each field in order to maximise the number of apples that are produced?
15. A string 72 cm long is to be cut into two pieces. One piece is used to form a circle and the other a square. What should be the perimeter of the square in order to minimise the sum of the two areas.
16. A farmer with m dollars to spend is constructing a rectangular paddock $PQRS$. The side PQ runs along a river and costs n dollars per metre to fence. The remaining three sides of the paddock cost r dollars per metre to fence. Find in terms of m , n , and r the lengths of the sides of the paddock in order to maximise its area.
17. The total cost of producing x items per day is $\frac{1}{3}x^2 + 45x + 27$ dollars, and the price per item at which each may be sold is $60 - \frac{1}{2}x$ dollars. Find an expression for the daily profit, and hence find the maximum possible profit.
18. Suppose that the cost of producing x items per hour is given by $C(x)$ where $C(x) = x^2 + 10$, and the number of items sold per hour at a price of p dollars per item is $x = 16 - p$.
- Find in terms of x the revenue gained from the sales.
 - Hence show that the profit achieved per hour is given by $-2x^2 + 16x - 10$.
 - Find the number of items that should be produced each hour in order to maximise the profit.
 - Find the maximum profit.
19. (a) Find where the graphs of the functions $y = x(x - 4)$ and $y = x(5 - x)$ intersect, and carefully draw graphs of both functions on the same number plane.
- (b) P is a point on the the graph of the function $y = x(x - 4)$ and Q is a point on the graph of the function $y = x(5 - x)$. P and Q have the same x -coordinate, where $0 \leq x \leq \frac{9}{2}$. Find an expression for the length of PQ and hence the maximum length of PQ .
20. A running track is 1000 metres long. It is designed using two sides of a rectangle and the perimeter of two semicircles as shown. The shaded rectangular section is to be used for field events. Find the dimensions of this section so as to maximise its area.



21. Highway A and Highway B intersect at right angles. A car on Highway A is presently 80 km from the intersection and is travelling towards the intersection at 50 km per hour. A car on Highway B is presently 70 km from the intersection and is travelling towards the intersection at 45 km per hour.
- Find an expression for the square of the distance between the two cars if they continue in this manner for h hours.
 - If the cars can continue through the intersection and remain on the same highways, in how many minutes will the distance between them be a minimum?
22. The point $P(x, y)$ lies on the curve $y = 3x^2$. Find the coordinates of P so that the distance from P to the line $y = 2x - 1$ is a minimum.
23. A piece of string of length ℓ is bent to form the sector of a circle of radius r . Show that the area of the sector is maximised when $r = \frac{1}{4}\ell$.
24. Prove that the rectangle of greatest area that can be inscribed in a circle is a square. [HINT: Recall that the maximum of A occurs when the maximum of A^2 occurs.]

25. The sum of the radii of two circles remains constant. Prove that the sum of the areas of the circles is least when the circles are congruent. [HINT: Let the radii be r and $k - r$, where k is a constant.]
26. OAB is a triangle in which $OA \perp OB$. OA and OB have lengths of 60 cm and 80 cm respectively. A rectangle is inscribed inside the triangle so that one of its sides lies along the base OA of the triangle. (a) By using similar triangles find the size of the rectangle of maximum area that may be inscribed in the triangle. (b) Repeat the question using a method similar to that in the second worked example.
27. A rectangle is inscribed in an isosceles triangle with one of the sides of the rectangle on the base of the triangle. Prove that the rectangle of greatest area occupies half the area of the triangle.

EXTENSION

28. Give a complete proof that the largest triangle that can be inscribed in a circle is an equilateral triangle.

8F The Theory of the Discriminant

In Section 8C, we established that the zeroes of the general quadratic function $y = ax^2 + bx + c$ are

$$x = \frac{-b + \sqrt{\Delta}}{2a} \text{ or } \frac{-b - \sqrt{\Delta}}{2a}, \text{ where } \Delta = b^2 - 4ac.$$

In this section, we develop the theory of the discriminant Δ a little further, because it is one of the keys to understanding the behaviour of a quadratic function.

The Discriminant Discriminates: At first glance, one would expect the formula above to mean that every quadratic has two zeroes. The square root in the formula, however, makes the situation more complicated, because negative numbers have no square roots, zero has just one square root, and only positive numbers have two square roots. This means that the number of zeroes depends on the sign of the discriminant.

THE DISCRIMINANT AND THE NUMBER OF ZEROES:

14

If $\Delta > 0$, there are two zeroes, $x = \frac{-b + \sqrt{\Delta}}{2a}$ and $x = \frac{-b - \sqrt{\Delta}}{2a}$.

If $\Delta = 0$, there is only one zero, $x = -\frac{b}{2a}$.

If $\Delta < 0$, there are no zeroes.

Unreal Zeroes and Double Zeroes: When $\Delta < 0$, we will sometimes say that *there are two unreal zeroes* (meaning two zeroes whose values are not real numbers), rather than saying that there are no zeroes. When $\Delta = 0$, it's often appropriate to think of the situation as two zeroes coinciding, and we say that *there are two equal zeroes*, or that the zero is a *double zero*. This adjustment of the language allows us to say that every quadratic has two zeroes, and the question then is whether those zeroes are real or unreal, and whether they are equal or distinct.

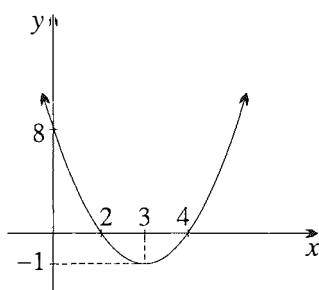
THE LANGUAGE OF DOUBLE ZEROES AND UNREAL ZEROES:

15

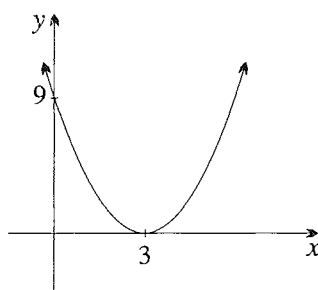
If $\Delta > 0$, there are two distinct real zeroes.

If $\Delta = 0$, there is one real double zero (or two equal zeroes).

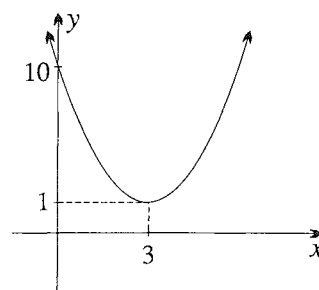
If $\Delta < 0$, there are no zeroes (or two distinct unreal zeroes).



$$\begin{aligned} y &= x^2 - 6x + 8 \\ &= (x - 3)^2 - 1, \\ \Delta &= 6^2 - 4 \times 8 = 4 \end{aligned}$$



$$\begin{aligned} y &= x^2 - 6x + 9 \\ &= (x - 3)^2, \\ \Delta &= 6^2 - 4 \times 9 = 0 \end{aligned}$$



$$\begin{aligned} y &= x^2 - 6x + 10 \\ &= (x - 3)^2 + 1, \\ \Delta &= 6^2 - 4 \times 10 = -4 \end{aligned}$$

The three graphs above all have axis of symmetry $x = 3$ and differ only in their constant terms. The first has two real and distinct zeroes, 2 and 4. In the second, the parabola has risen so that the two zeroes coincide to give the one double zero, $x = 3$. In the third, the parabola has risen further so that there are no longer any zeroes (or as we shall sometimes say, there are two unreal zeroes).

Quadratics that are Perfect Squares: The middle graph above is an example of a quadratic that is a *perfect square*:

$$x^2 - 6x + 9 = (x - 3)^2 \quad \text{and} \quad \Delta = 6^2 - 4 \times 9 = 0.$$

In general, when $\Delta = 0$ and the two zeroes coincide, the quadratic meets the x -axis in only one point, where it is tangent, and the quadratic can be expressed as a multiple of a perfect square.

THE DISCRIMINANT AND PERFECT SQUARES:

16

When $\Delta = 0$, the quadratic is a multiple of a perfect square, $y = a(x - \alpha)^2$, and the x -axis is a tangent to the parabola at the double zero.

Are the Zeroes Rational or Irrational: Suppose now that all the three coefficients in $y = ax^2 + bx + c$ are rational numbers. Then because we need to take the square root of Δ , the zeroes will also be rational numbers if Δ is square, otherwise the zeroes will involve a surd and be irrational. So the discriminant allows another distinction to be made about the zeroes:

THE DISCRIMINANT AND RATIONAL ZEROES: Suppose a , b and c are rational.

17

If Δ is a square, then the zeroes are rational.

If Δ is positive but not a square, then the zeroes are irrational.

WORKED EXERCISE: Use the discriminant to describe the zeroes of:

(a) $y = 5x^2 - 2x - 3$ (b) $y = 3x^2 - 12x + 12$ (c) $y = 8 + 3x - 2x^2$

If the quadratic is a multiple of a perfect square, express it in this form.

SOLUTION:

(a) For $y = 5x^2 - 2x - 3$, $\Delta = 4 + 4 \times 15$
 $= 64$,

so there are two real zeroes, and they are rational.

(b) For $y = 3x^2 - 12x + 12$, $\Delta = 144 - 4 \times 36$
 $= 0$,

so there is one rational zero.

Also $y = 3(x - 2)^2$.

(c) For $y = 8 + 3x - 2x^2$, $\Delta = 9 + 4 \times 16$
 $= 73$,

so there are two real zeroes, and they are irrational.

WORKED EXERCISE: For what values of λ does $x^2 - (\lambda + 5)x + 9 = 0$ have:

(a) equal roots,

(b) no roots?

SOLUTION: Here $\Delta = (\lambda + 5)^2 - 36$.

(a) $\Delta = 0$ when $(\lambda + 5)^2 = 36$

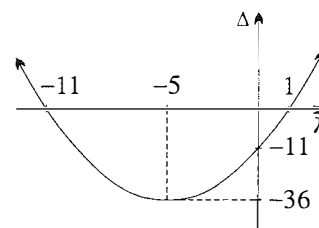
$$\lambda + 5 = 6 \text{ or } \lambda + 5 = -6$$

$$\lambda = 1 \text{ or } -11,$$

so there are equal roots when $\lambda = 1$ and when $\lambda = -11$.

(b) There are no roots when Δ is negative,

so from the graph of Δ as a function of λ , there are no roots for $-11 < \lambda < 1$.



WORKED EXERCISE: [A harder example] Use the discriminant to find the equations of the lines through $A(3, -3)$ which are tangent to the rectangular hyperbola $y = 3/x$.

SOLUTION: The family of lines through $A(3, -3)$ is

$$y + 3 = m(x - 3), \text{ where } m \text{ is the gradient,}$$

$$y = mx - (3m + 3).$$

Solving this line simultaneously with the hyperbola,

$$mx - (3m + 3) = \frac{3}{x}$$

$$mx^2 - 3(m + 1)x - 3 = 0.$$

For the line to be a tangent, there must be a double zero.

So putting $\Delta = 0$, $9(m + 1)^2 + 12m = 0$

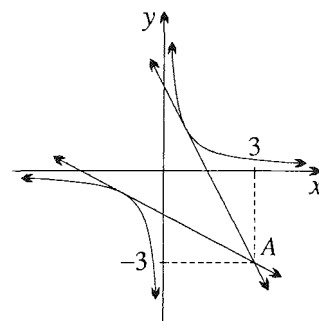
$\boxed{\div 3}$ $3m^2 + 6m + 3 + 4m = 0$

$$3m^2 + 10m + 3 = 0$$

$$(m + 3)(3m + 1) = 0$$

$$m = -3 \text{ or } -\frac{1}{3}.$$

So the lines are $y = -3x + 6$ and $y = -\frac{1}{3}x - 2$.



Exercise 8F

1. Describe the roots of quadratic equations with rational coefficients that have the following discriminants. If the roots are real, state whether they are equal or unequal, rational or irrational.

(a) $\Delta = 7$

(c) $\Delta = 0$

(e) $\Delta = \frac{4}{9}$

(b) $\Delta = -9$

(d) $\Delta = 64$

(f) $\Delta = -0.3$

2. Find the discriminant Δ of each equation. Hence state how many roots there are, and whether or not they are rational.

(a) $x^2 - 4x + 3 = 0$

(d) $x^2 + 2x - 7 = 0$

(b) $2x^2 - 3x + 5 = 0$

(e) $6x^2 + 11x - 10 = 0$

(c) $x^2 - 6x + 9 = 0$

(f) $9x^2 - 1 = 0$

NOTE: In questions 3–7, first find the discriminant Δ , then answer the question. If it is necessary to solve a quadratic inequality, this should be done by sketching a graph.

3. Find g , if the following quadratic functions have exactly one distinct zero:

(a) $y = x^2 + 10x + g$

(e) $y = gx^2 - gx + 1$

(b) $y = gx^2 - 4x + 1$

(f) $y = gx^2 + 7x + g$

(c) $y = 2x^2 - 3x + (g + 1)$

(g) $y = 4x^2 + 4gx + (6g + 7)$

(d) $y = (g - 2)x^2 + 6x + 1$

(h) $y = 9x^2 - 2(g + 1)x + 1$

4. Find the values of k for which the roots of the following equations are real numbers:

(a) $x^2 + 2x + k = 0$

(e) $x^2 + kx + 4 = 0$

(b) $kx^2 - 8x + 2 = 0$

(f) $x^2 - 3kx + 9 = 0$

(c) $3x^2 - 4x + (k + 1) = 0$

(g) $4x^2 - (6 + k)x + 1 = 0$

(d) $(2k - 1)x^2 - 5x + 2 = 0$

(h) $9x^2 + (k - 6)x + 1 = 0$

5. Find the values of ℓ for which the following quadratic functions have no real zeroes:

(a) $y = x^2 + \ell x + 4$

(d) $y = 9x^2 - 4(\ell - 1)x - \ell$

(b) $y = \ell x^2 + 6x + \ell$

(e) $y = \ell x^2 - 4\ell x - (\ell - 5)$

(c) $y = x^2 + (\ell + 1)x + 4$

(f) $y = (\ell - 3)x^2 + 2\ell x + (\ell + 2)$

6. (a) Show that the x -coordinates of the points of intersection of the circle $x^2 + y^2 = 4$ and the line $y = x + 1$ satisfy the equation $2x^2 + 2x - 3 = 0$.

- (b) Evaluate the discriminant Δ and explain why this shows that there are two points of intersection.

7. Using the method outlined in the previous question, determine how many times the line and circle intersect in each case:

(a) $x^2 + y^2 = 9, y = 2 - x$

(c) $x^2 + y^2 = 5, y = -2x + 5$

(b) $x^2 + y^2 = 1, y = x + 2$

(d) $(x - 3)^2 + y^2 = 4, y = x - 4$

DEVELOPMENT

8. Find Δ for each equation. By writing Δ as a perfect square, show that each equation has rational roots for all rational values of m and n :

(a) $4x^2 + (m - 4)x - m = 0$

(d) $2mx^2 - (4m + 1)x + 2 = 0$

(b) $(m - 1)x^2 + mx + 1 = 0$

(e) $2(m - 2)x^2 + (6 - 7m)x + 6m = 0$

(c) $mx^2 + (2m + n)x + 2n = 0$

(f) $(4m + 1)x^2 - 2(m + 1)x + (1 - 2m) = 0$

9. Prove that the roots of the following equations are real and distinct for all real values of λ .
[HINT: Find Δ and write it in such a way that it is obviously positive.]

(a) $x^2 + \lambda x - 1 = 0$

(c) $\lambda x^2 - (\lambda + 4)x + 2 = 0$

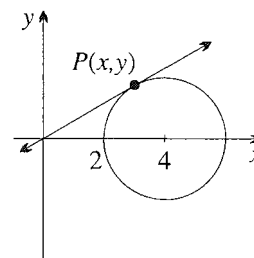
(b) $3x^2 + 2\lambda x - 4 = 0$

(d) $x^2 + (\lambda + 1)x + (\lambda - 2) = 0$

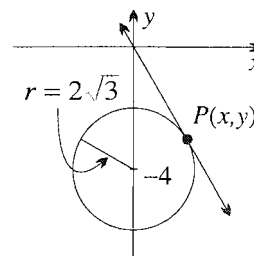
NOTE: In the following questions you may need to rearrange the quadratic equation into the form $ax^2 + bx + c = 0$ before finding Δ .

10. Find the values of m for which the roots of the quadratic equation:
- $1 - 3x - mx^2 = 0$ are real and distinct,
 - $2x^2 + 4x + 5 = 3x + m$ are real and equal,
 - $x(x - 2m) = m - 2x - 3$ are unreal,
 - $12m(x^2 - 2x) + 12(2x^2 + x) = 38m + 11$ are real.
11. Show that the roots of $(x - a)(x - b) = c^2$ are always real, where a , b and c are real.
12. (a) For what values of b is the line $y = x + b$ a tangent to the curve $y = 2x^2 - 7x + 4$?
 (b) The line $2x + y + b = 0$ is a tangent to $y = 2x^2 + 3x + 1$. Find the value of b .
 (c) The line $y = mx + 4$ is a tangent to $y = 3x^2 + 5x + 7$. Find the value of m .
13. Find the equation of the tangent to the parabola $y = x^2 - 5x - 3$ that is parallel to the line $3x - y - 7 = 0$.
14. Find the gradients of the lines that pass through the point $(1, 7)$ and are tangent to the parabola $y = (2 - x)(1 + 3x)$.
15. The line $y = 4x - 7$ is tangent to a parabola that has a y -intercept of -3 and the line $x = \frac{1}{2}$ as its axis of symmetry. Find the equation of the parabola.
16. How many horizontal tangents may be drawn to each of the following cubic functions?
 [HINT: You will need to differentiate and set the derivative equal to zero, then use the discriminant to find how many solutions this equation has.]
- $y = x^3 + 5x^2 - 8x + 7$
 - $y = 3x^3 - 3x^2 + x - 1$
 - $y = \frac{1}{3}x^3 + x^2 + 5x + 11$
17. If in $y = ax^2 + bx + c$ we find $ac < 0$, explain why the graph of the parabola must have two x -intercepts.

18. (a) Write down the equation of the circle in the diagram.
 (b) Write down the equation of the line through the origin with gradient m .
 (c) By solving the circle and the line simultaneously, show that the x -coordinate of the point P in the diagram satisfies the equation $(m^2 + 1)x^2 - 8x + 12 = 0$.
 (d) Use the theory of the discriminant to find the value of m .
 (e) Hence or otherwise find the coordinates of P .



19. Use the method outlined in the previous question to find the gradient of the line in the diagram and hence the coordinates of the point P .
20. Use the discriminant to find the gradients of the lines that pass through the point $(7, 1)$ and are tangent to the circle $x^2 + y^2 = 25$.



21. (a) Find a if $y = 3(a + 2)x^2 + 6ax + (4 - 3a)$ has no zeroes.
 (b) Find b if $y = (2b - 3)x^2 + (5b - 1)x + (3b + 2)$ has two distinct zeroes.
 (c) Find g if $y = (g + 1)x^2 - (3 - 5g)x - (g - 12)$ has one zero.
 (d) Find k if $(3k - 2)x^2 + 2(k + 6)x + (k - 4) = 0$ has two distinct roots.
22. (a) Show that the quadratic equation $(a^2 + b^2)x^2 + 2b(a + c)x + (b^2 + c^2) = 0$, where a , b and c are real constants, has real roots when $(b^2 - ac)^2 \leq 0$.
 (b) State this condition in a simpler form.

_____ EXTENSION _____

23. Show that $\frac{x^2 - mn}{2x - m - n}$ takes no real values between m and n . [HINT: Put $\frac{x^2 - mn}{2x - m - n} = \lambda$ and find the discriminant.]
24. Show that the equation $(x - m)(x - n) + (x - n)(x - \ell) + (x - \ell)(x - m) = 0$ cannot have equal roots unless $m = n = \ell$.

8G Definite and Indefinite Quadratics

The last section showed how the discriminant discriminated between quadratics with two, one or no zeroes. The distinction between quadratics with no zeroes and those with one zero or two is sufficiently important for special words to be used to describe them.

Definition: Suppose $f(x) = ax^2 + bx + c$ is a quadratic.

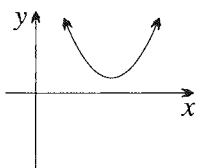
- 18** **POSITIVE DEFINITE, NEGATIVE DEFINITE AND INDEFINITE QUADRATICS:**
 $f(x)$ is called *positive definite* if $f(x)$ is positive for all values of x ,
 and *negative definite* if $f(x)$ is negative for all values of x .
 $f(x)$ is called *definite* if it is positive definite or negative definite.
 $f(x)$ is called *indefinite* if it is not definite.

These definitions may be clearer when expressed in terms of zeroes:

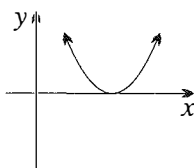
- 19** **DEFINITE AND INDEFINITE AND ZEROES:**
 A quadratic is *definite* if it has no zeroes,
 being *positive definite* if it is always positive,
 and *negative definite* if it is always negative.
 A quadratic is *indefinite* if it has at least one zero.

The word ‘definite’ means ‘we can be definite about the sign of $f(x)$ whatever the value of x ’. An ‘indefinite’ quadratic takes different signs (or is zero at least once) for different values of x .

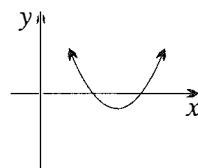
The Six Cases: There are three possibilities for Δ — negative, zero and positive — and two possibilities for a — positive and negative. This makes six possible cases altogether, and these cases are graphed below.



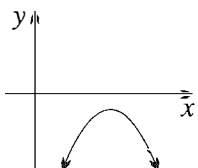
$\Delta < 0$ and $a > 0$,
positive definite



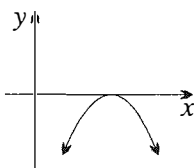
$\Delta = 0$ and $a > 0$,
indefinite



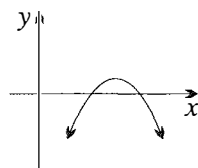
$\Delta > 0$ and $a > 0$,
indefinite



$\Delta < 0$ and $a < 0$,
negative definite



$\Delta = 0$ and $a < 0$,
indefinite



$\Delta > 0$ and $a < 0$,
indefinite

Definite Quadratics and Factorisation: If a quadratic is indefinite, then it can be factored, either as $a(x - \alpha)(x - \beta)$ if it has two zeroes α and β , or as $a(x - \alpha)^2$ if it has one double zero α . A definite quadratic, however, cannot be factored, because otherwise it would have zeroes.

20

INDEFINITE QUADRATICS AND FACTORING: A quadratic can be factored into real factors if and only if it is indefinite.

WORKED EXERCISE: For what values of a is $f(x) = ax^2 + 8x + a$ positive definite, negative definite and indefinite?

SOLUTION: Here $\Delta = 64 - 4a^2$

$$= 4(16 - a^2),$$

so $\Delta \geq 0$ when $-4 \leq a \leq 4$,

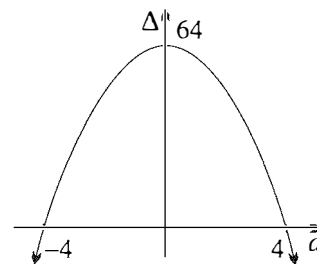
and $\Delta < 0$ when $a < -4$ and when $a > 4$.

Hence $f(x)$ is indefinite for $-4 \leq a \leq 4$

(but $a \neq 0$, because when $a = 0$ it is not a quadratic),

and $f(x)$ is positive definite for $a > 4$,

and $f(x)$ is negative definite for $a < -4$.



Exercise 8G

1. Use a graph to solve the following quadratic inequations:

(a) $x^2 \geq x$

(d) $(x + 1)^2 \leq 34$

(g) $-5 > 4x(2 - x)$

(b) $7 - x^2 > 0$

(e) $3x^2 + 5x - 2 \leq 0$

(h) $x^2 + 4x + 5 \leq 0$

(c) $x^2 + 9 > 6x$

(f) $-2x^2 + 13x \geq 15$

(i) $-2x^2 - 3x - 3 < 0$

2. Evaluate the discriminant and look carefully at the coefficient of x^2 to determine whether the following functions are positive definite, negative definite or indefinite:

(a) $y = 2x^2 - 5x + 7$

(d) $y = -x^2 + 7x - 3$

(b) $y = x^2 - 4x + 4$

(e) $y = 25 - 20x + 4x^2$

(c) $y = 5x - x^2 - 9$

(f) $y = 3x + 2x^2 + 11$

3. Find the discriminant as a function of k , and hence find the values of k for which the following expressions are: (i) positive definite, (ii) indefinite.
- (a) $2x^2 - 5x + 4k$ (c) $3x^2 + (12 - k)x + 12$
 (b) $2x^2 - kx + 8$ (d) $x^2 - 2(k - 3)x + (k - 1)$
4. Find the discriminant as a function of m , and hence find the values of m for which these expressions are: (i) negative definite, (ii) indefinite.
- (a) $-x^2 + mx - 4$ (c) $-x^2 + (m - 2)x - 25$
 (b) $-2x^2 + 3x - m$ (d) $-4x^2 + 4(m + 1)x - (4m + 1)$
5. Find the values of ℓ that will make each quadratic below a perfect square:
- (a) $x^2 - 2\ell x + 16$ (c) $(5\ell - 1)x^2 - 4x + (2\ell - 1)$
 (b) $2\ell x^2 + 2\ell x + 1$ (d) $(4\ell + 1)x^2 - 6\ell x + 4$

DEVELOPMENT

6. Show that the discriminant of $(3k - 5)x^2 + 2(4 - k)x + 4 = 0$ is $\Delta = 4(k^2 - 20k + 36)$, and hence find the values of k for which $(3k - 5)x^2 + 2(4 - k)x + 4 = 0$ is:
- (a) positive definite, (b) negative definite, (c) indefinite.
7. (a) Sketch a graph of the function $y = \frac{(x - 1)(x + 2)}{x(x + 4)}$.
 (b) Prove that the roots of the equation $kx(x + 4) = (x - 1)(x + 2)$ are always real.
 (c) How can you establish this result from the graph you have drawn?
8. Find the domain of each of the following expressions:
- (a) $\sqrt{x^2 - 5x + 1}$ (c) $\sqrt{2x^2 - 9x + 4}$ (e) $\sqrt{6 + 5x - 4x^2}$
 (b) $\sqrt{2x - 3 - x^2}$ (d) $\frac{x + 2}{\sqrt{2x - x^2}}$ (f) $\frac{(x - 1)(2x + 3)}{\sqrt{x^2 - 9}}$
9. State in terms of a , b and c the conditions necessary for $ax^2 + 2bx + 3c$ to be:
- (a) positive definite, (b) negative definite, (c) indefinite.
10. Sketch a possible graph of the quadratic function $y = ax^2 + bx + c$ if:
- (a) $a > 0$, $b > 0$, $c > 0$ and $b^2 - 4ac > 0$ (c) $a > 0$, $b < 0$, $c > 0$ and $b^2 = 4ac$
 (b) $a < 0$, $c < 0$ and $b = 0$ (d) $a > 0$, $b < 0$ and $b^2 - 4ac < 0$
11. State in terms of b and c the condition for the roots of $x^2 + 2bx + 3c = 0$ to be:
- (a) equal, (c) unreal, (e) distinct and positive,
 (b) real and distinct, (d) opposite in sign, (f) distinct and negative.
12. The expression $x^2 - xy - 2y^2 + x + 7y - 5$ can be treated as a function in x with y as an arbitrary constant. Show that it is positive definite when $1 < y < \frac{7}{3}$.
13. Find the range of values of x for which the equation in y , $2x^2 - 3xy + y^2 - 5x + 11 = 0$ will have real roots. Find also for what values of y the equation in x will have real roots.
14. The expression $3x^2 + 2xy - 8y^2 - 8x + 14y - 3$ can be treated as a function in x with y as an arbitrary constant or as a function in y with x as an arbitrary constant. Show that in either case the expression is indefinite. Factor the expression.
15. Find the values of λ for which $4a^2 - 10ab + 10b^2 + \lambda(3a^2 - 10ab + 3b^2)$ is a perfect square.

EXTENSION

16. The equation $2x^2 + ax + (b + 3) = 0$ has real roots. Find the minimum value of $a^2 + b^2$.
17. (a) Show that $f(x) = (x - 5)^2 + (x + 2)^2$ is positive definite, first by expanding and finding the discriminant and secondly by explaining directly why $f(x)$ must be positive for all values of x .
- (b) Express $2x^2 + 4x + 10$ in the form $(x - r)^2 + (x - s)^2$.
- (c) Find the discriminant of $f(x) = 2x^2 + 2bx + c$. Hence show that $f(x)$ can be expressed as a sum $(x - r)^2 + (x - s)^2$ of two distinct squares (that is, with $r \neq s$) if and only if $f(x)$ is positive definite.

8H Sum and Product of Roots

Many problems on quadratics depend on the sum and product of the roots rather than on the roots themselves. For example, the axis of symmetry is found by taking the average of the zeroes, which is half the sum of the zeroes. The formulae for the sum and product of the roots are very straightforward, and do not involve the surds that often appear in the roots themselves.

Forming a Quadratic Equation with Given Roots: Suppose we are asked to form a quadratic equation with roots α and β . The simplest such equation is

$$(x - \alpha)(x - \beta) = 0.$$

Expanding this out, $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

21

A QUADRATIC WITH GIVEN ROOTS α AND β : $(x - \alpha)(x - \beta) = 0$
 OR $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$.

WORKED EXERCISE: Form quadratic equations with integer coefficients and roots:

(a) $3\frac{1}{2}$ and $-2\frac{1}{3}$,

(b) $2 + \sqrt{7}$ and $2 - \sqrt{7}$.

SOLUTION:

(a) Such an equation is

$$(x - 3\frac{1}{2})(x + 2\frac{1}{3}) = 0$$

OR

Since $\alpha + \beta = \frac{7}{6}$ and $\alpha\beta = -\frac{49}{6}$,

such an equation is

$$\begin{array}{l} \boxed{\times 6} \quad (2x - 7)(3x + 7) = 0 \\ \quad \quad \quad 6x^2 - 7x - 49 = 0. \end{array}$$

$$\begin{array}{l} x^2 - \frac{7}{6}x - \frac{49}{6} = 0 \\ \boxed{\times 6} \quad 6x^2 - 7x - 49 = 0. \end{array}$$

(b) Taking the sum,

$$\alpha + \beta = 4,$$

and taking the product,

$$\alpha\beta = 4 - 7 = -3 \quad (\text{difference of squares}),$$

so such an equation is $x^2 - 4x - 3 = 0$.

Formulae for the Sum and Product of Roots: Suppose $ax^2 + bx + c = 0$ has roots α and β . Dividing through by a gives $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, so by the previous result:

22

SUM AND PRODUCT OF ROOTS: $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

These formulae can also be proven directly using the general equation for the roots — see the first question in the second group of the exercise below.

WORKED EXERCISE: If α and β are the roots of the equation $2x^2 - 6x - 1 = 0$, find:

(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2\beta + \alpha\beta^2$ (d) $1/\alpha + 1/\beta$ (e) $\alpha^2 + \beta^2$

SOLUTION: Since $a = 2$, $b = -6$ and $c = -1$:

(a) $\alpha + \beta = 3$

(b) $\alpha\beta = -\frac{1}{2}$

(c) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$
 $= -1\frac{1}{2}$

(d) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$
 $= -6$

(e) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
 $= 9 + 1$
 $= 10$

Expressions Symmetric in α and β : Expressions in α and β like those in the worked exercise above are called *symmetric* in α and β , because if α and β are exchanged, the expression remains the same. With enough ingenuity, it should be possible to evaluate any expression symmetric in α and β by the sort of methods used here.

WORKED EXERCISE: Prove that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$. Then use this identity to find the difference $|\alpha - \beta|$ of the roots of the equation $x^2 - 9x + 2 = 0$.

SOLUTION: $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$
 $= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta$
 $= (\alpha + \beta)^2 - 4\alpha\beta$

In the given equation, $\alpha + \beta = 9$ and $\alpha\beta = 2$, so $(\alpha - \beta)^2 = 9^2 - 4 \times 2$
 $= 73,$

so the difference of the roots is $|\alpha - \beta| = \sqrt{73}.$

WORKED EXERCISE: [A problem where a relation between the roots is known]
 Find m , given that one of the roots of $x^2 + mx + 18 = 0$ is twice the other.

SOLUTION: Let the roots be α and 2α . [NOTE: This is the essential step here.]
 Then using the product of roots, $\alpha \times 2\alpha = 18$

$$\alpha = 3 \text{ or } -3.$$

Now using the sum of roots, $\alpha + 2\alpha = -m$

$$m = -3\alpha$$

$$m = 9 \text{ or } -9.$$

Unreal Roots: A quadratic equation like $x^2 - 4x + 8 = 0$ has no roots, because its discriminant is $\Delta = -16$ which has no square roots. Nevertheless, the formulae for the sum and product of the roots give answers as usual:

$$\alpha + \beta = 4 \quad \text{and} \quad \alpha\beta = 8,$$

and the question is, what meaning do these answers have? Now blind use of the formula for the roots of a quadratic would give the following expressions for them:

$$\alpha = 2 + \sqrt{-4} \quad \text{and} \quad \beta = 2 - \sqrt{-4}.$$

If we calculate $\alpha + \beta$ and $\alpha\beta$ ignoring the fact that these expressions are meaningless:

$$\alpha + \beta = 2 + 2 = 4 \quad \text{and} \quad \alpha\beta = 2^2 - (-4) = 8 \quad (\text{difference of squares})$$

which are the values we obtained above. Considerations like these lead mathematicians to take seriously objects such as $\sqrt{-4}$. Since $\sqrt{-4}$ can't be a real number, such arithmetic requires an extension of the real number system. Square roots of negative numbers are called *imaginary numbers*, and sums like $2 + \sqrt{-4}$ of real and imaginary numbers are called *complex numbers*.

Arithmetic and Geometric Means: If α and β are the roots of a given quadratic equation, then their arithmetic and geometric means are easily found, because they are simply half the sum of the roots and the square root of the product of the roots. This means that various problems in Euclidean and coordinate geometry can be solved far more easily.

WORKED EXERCISE: The line $y = 2x + b$ intersects the circle $x^2 + y^2 = 25$ at P and Q . Use the sum and product of roots to find the coordinates of the midpoint $M(X, Y)$ of PQ . Hence find the locus of M as b varies, and describe it geometrically.

SOLUTION: Solving the line and the circle simultaneously,

$$x^2 + (4x^2 + 4bx + b^2) = 25$$

$$5x^2 + 4bx + (b^2 - 25) = 0.$$

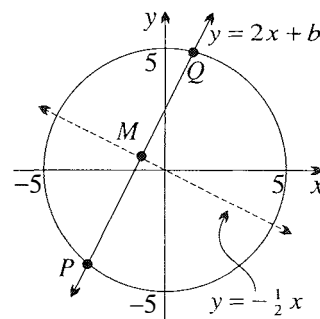
The x -coordinate of M is the arithmetic mean of the roots:

$$\begin{aligned} X &= \frac{1}{2}(\alpha + \beta) \\ &= \frac{1}{2} \times \left(-\frac{4}{5}b\right) \\ &= -\frac{2}{5}b, \end{aligned}$$

and substituting into the line, $M = \left(-\frac{2}{5}b, \frac{1}{5}b\right)$.

This point lies on the line $y = -\frac{1}{2}x$,

which is the diameter perpendicular to the family of lines $y = 2x + b$.



Exercise 8H

- Use the formulae $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$ to write down the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots of $x^2 + 7x + 10 = 0$. Then solve $x^2 + 7x + 10 = 0$ by factoring, and check your results.
- Repeat the previous question for these equations. In parts (b) and (c), use the formula to find the roots, and the difference of squares identity to find their product.
 - $3x^2 - 10x + 3 = 0$
 - $x^2 + 4x + 1 = 0$
 - $x^2 - x - 1 = 0$
- If α and β are the roots of the following quadratic equations, write down the values of $\alpha + \beta$ and $\alpha\beta$ without solving the equations.
 - $x^2 - 2x + 5 = 0$
 - $x^2 + x - 6 = 0$
 - $x^2 + x = 0$
 - $2x^2 + 3x - 1 = 0$
 - $4 + 5x^2 = -5x$
 - $3x^2 + 2x = 4(x + 1)$
 - $x^2 - mx + n = 0$
 - $px^2 + qx - 3r = 0$
 - $ax(x - 1) = 3 - 4x$
- A quadratic equation with roots α and β has the form $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. Form a quadratic equation, with integral coefficients, whose roots are:
 - 1 and 3
 - 2 and 6
 - 1 and -4
 - $\frac{1}{2}$ and $\frac{3}{2}$
 - $2 + \sqrt{3}$ and $2 - \sqrt{3}$
 - $-1 - \sqrt{5}$ and $-1 + \sqrt{5}$
- If α and β are the zeroes of $y = x^2 - 3x + 2$, without finding the zeroes, find the values of:
 - $\alpha + \beta$
 - $\alpha\beta$
 - $7\alpha + 7\beta$
 - $\alpha^2\beta + \alpha\beta^2$
 - $(\alpha + 3)(\beta + 3)$
 - $\frac{1}{\alpha} + \frac{1}{\beta}$
 - $\alpha^2 + \beta^2$
 - $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

6. If α and β are the roots of $2x^2 - 5x + 1 = 0$, without solving the equation, find the values of:

(a) $\alpha + \beta$	(c) $(\alpha - 1)(\beta - 1)$	(e) $\alpha^3\beta^2 + \alpha^2\beta^3$	(g) $\alpha^2 + \beta^2$
(b) $\alpha\beta$	(d) $\alpha^{-1} + \beta^{-1}$	(f) $\frac{2}{\alpha} + \frac{2}{\beta}$	(h) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

7. (a) Show that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.
 (b) If α and β are the roots of the following equations, without solving the equation, find the values of $\alpha + \beta$, $\alpha\beta$ and hence $(\alpha - \beta)^2$.

(i) $x^2 - 3x + 1 = 0$	(ii) $x^2 + 5x - 7 = 0$	(iii) $3x^2 - 7x + 2 = 0$

- (c) Hence find the difference $|\alpha - \beta|$ for each equation.

8. If α and β are the roots of the equation $ax^2 + bx + c = 0$, show that $(\alpha - \beta)^2 = \frac{\Delta}{a^2}$.

DEVELOPMENT

9. Write down the formulae for the two zeroes α and β of $f(x) = ax^2 + bx + c$, assuming that $\alpha < \beta$. Hence prove directly that $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$ (you will need to use the difference of squares identity).
10. Without solving, find the arithmetic and geometric means of the roots of $3x^2 - 5x + 4 = 0$.
11. If α and β are the roots of the equation $3x^2 + 2x + 7 = 0$, find $\alpha + \beta$ and $\alpha\beta$. Hence form the equation with integer coefficients having roots:
- | | | | |
|----------------------------|--|---|------------------------------|
| (a) 2α and 2β | (b) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ | (c) $\alpha + 2\beta$ and $\beta + 2\alpha$ | (d) α^2 and β^2 |
|----------------------------|--|---|------------------------------|
12. Find the values of g for which the function $y = 2x^2 - (3g - 1)x + (2g - 5)$ has:
- one zero equal to 0 (let the zeroes be 0 and α),
 - the sum of the zeroes equal to their product (put $\alpha + \beta = \alpha\beta$),
 - the zeroes as reciprocals of one another (let the zeroes be α and $1/\alpha$),
 - the zeroes equal in magnitude but opposite in sign (let the zeroes be α and $-\alpha$).
13. Given that α and β are the roots of $(2m - 1)x^2 + (1 + m)x + 1 = 0$, find m if:
- | | | | |
|-----------------------|--------------------------------|------------------|-------------------------------------|
| (a) $\alpha = -\beta$ | (b) $\alpha = \frac{1}{\beta}$ | (c) $\alpha = 2$ | (d) $\alpha + \beta = 2\alpha\beta$ |
|-----------------------|--------------------------------|------------------|-------------------------------------|
14. Given that the roots of the quadratic equation $ax^2 + bx + c = 0$ are real, explain by inspection of the coefficients how one can determine:
- whether the roots have opposite signs,
 - the sign of the roots if they both have the same sign,
 - the sign of the numerically greater root if they have opposite signs.
15. If α and β are the zeroes of the function $y = 3x^2 - 5x - 4$, find the value of $\alpha^3 + \beta^3$.
16. Find the value of m if one root of the equation $x^2 + 6x + m = 0$ is double the other.
17. Find the value of ℓ if one zero of the function $y = x^2 - 2\ell x + (\ell + 3)$ is three times the other.
18. If the roots of the quadratic equation $x^2 - mx + n = 0$ differ by 1, without solving the equation, prove that $m^2 = 4n + 1$.
19. If one zero of the function $y = x^2 + mx + n$ is the square of the other, without finding the zeroes, prove that $m^3 = n(3m - n - 1)$.

20. The line $y = 6x + 9$ crosses the parabola $y = x^2 + 1$ at A and B .
 (a) Show that the x -coordinates of A and B satisfy the equation $x^2 - 6x - 8 = 0$.
 (b) Without solving the equation, find the sum of the roots of the quadratic.
 (c) Hence find the coordinates of the midpoint M of AB .
21. Using a method similar to that in the previous question, find the midpoints of the chords formed by:
 (a) the line $x + y - 1 = 0$ and the circle $x^2 + y^2 = 13$,
 (b) the line $y = x + 3$ and the parabola $y = (x - 2)^2$,
 (c) the line $x + y + 4 = 0$ and the rectangular hyperbola $y = \frac{1}{x}$.
22. The line $3x - y + b = 0$ cuts the circle $x^2 + y^2 = 16$ at two points P and Q . Find the coordinates $M(X, Y)$ of the midpoint of PQ . Hence find the locus of M as b varies, and describe it geometrically.
23. (a) Expand $(\sqrt{\alpha} + \sqrt{\beta})^2$. (b) If α and β are the zeroes of the function $y = x^2 - 6x + 16$, without finding the zeroes, evaluate $\sqrt{\alpha} + \sqrt{\beta}$.
24. The line $x + y - 1 = 0$ intersects the circle $x^2 + y^2 = 13$ at $A(\alpha_1, \alpha_2)$ and $B(\beta_1, \beta_2)$. Without finding the coordinates of A and B , find the length of the chord AB . [HINT: Form a quadratic equation in x and evaluate $|\alpha_1 - \beta_1|$, and similarly find $|\alpha_2 - \beta_2|$.]
25. For what values of m are the roots of $x^2 + 2x + 3 = m(2x + 1)$ real and positive?

————— EXTENSION —————

26. If the equations $mx^2 + 2x + 1 = 0$ and $x^2 + 2x + m = 0$ have a common root, find the possible values of m and the value of the common root in each case.
27. If α and β are the roots of $x^2 = 5x - 8$, find $\sqrt[3]{\alpha} + \sqrt[3]{\beta}$ without finding the roots.
28. The distinct positive numbers α and β are roots of the quadratic equation $ax^2 + bx + c = 0$. Use the fact that the discriminant is positive to prove that their arithmetic mean is greater than their geometric mean.

8I Quadratic Identities

This final section has three purposes. The first purpose is to prove some theorems about the coefficients of quadratics that have been tacitly assumed throughout the chapter. The second purpose is to develop some elegant methods of proving quadratic identities and finding coefficients in quadratics. The third purpose is to establish some very general geometrical ideas lying behind the algebra of quadratics.

Theorem — Three Values Determine a Quadratic: We shall prove that if two quadratics agree for at least three distinct values of x , then they agree for all values of x and their coefficients are equal.

THEOREM: Suppose that the two expressions

$$f(x) = ax^2 + bx + c \quad \text{and} \quad g(x) = a_1x^2 + b_1x + c_1$$

23

take the same value for at least three distinct values of x . Then $f(x)$ and $g(x)$ are equal for all values of x , and the coefficients a_1 , b_1 and c_1 are respectively equal to the coefficients a , b and c .

PROOF:

A. First we prove the following particular case:

‘Suppose that the expression $Ax^2 + Bx + C$ is zero for the three distinct values $x = \alpha$, $x = \beta$ and $x = \gamma$. Then $A = B = C = 0$.’

$$\text{Substituting,} \quad A\alpha^2 + B\alpha + C = 0 \quad (1)$$

$$A\beta^2 + B\beta + C = 0 \quad (2)$$

$$A\gamma^2 + B\gamma + C = 0. \quad (3)$$

Subtracting (2) from (1), $A(\alpha^2 - \beta^2) + B(\alpha - \beta) = 0$,
and since $\alpha \neq \beta$, we can divide through by $\alpha - \beta$ so that

$$A(\alpha + \beta) + B = 0. \quad (4)$$

$$\text{Similarly from (2) and (3),} \quad A(\beta + \gamma) + B = 0. \quad (5)$$

$$\text{Now subtracting (5) from (4),} \quad A(\alpha - \gamma) = 0,$$

$$\text{and since } \alpha \neq \gamma, \quad A = 0.$$

Then from (4), $B = 0$ and from (1), $C = 0$.

B. To prove the main theorem, let $h(x) = f(x) - g(x)$

$$= (a - a_1)x^2 + (b - b_1)x + (c - c_1).$$

Then since $f(x)$ and $g(x)$ agree at three distinct values of x ,

it follows that $h(x)$ is zero for these three values of x ,

so, by the result in part A, $a - a_1 = b - b_1 = c - c_1 = 0$, as required.

NOTE: We have of course been equating coefficients of two identically equal quadratic functions since the first section of this chapter, but the theorem now justifies that practice. The theorem is a special case of the result, to be proven later, that if two polynomials of degree n are equal for $n + 1$ values of x , then they are equal for all values of x and their coefficients are the same.

A Notation for Identically Equal: If two functions $f(x)$ and $g(x)$ are equal for all values of x , they are called *identically equal*. The notation for this is $f(x) \equiv g(x)$. There is an essential distinction between

$$f(x) = g(x) \quad \text{and} \quad f(x) \equiv g(x),$$

in that the first is an *equation*, which will be true for some set of values of x , and the second is an *identity*, which is true for all values of x .

Application of the Theorem to Identities: An identity involving quadratics can now be proven by proving it true for just three values of x , conveniently chosen to simplify the calculations.

WORKED EXERCISE: Given that a , b and c are distinct constants, prove that

$$(x - a)(x - b) + (x - b)(x - c) \equiv (x - b)(2x - a - c).$$

SOLUTION: It will be sufficient to prove the result when $x = a$, $x = b$ and $x = c$.

When $x = a$, LHS = $0 + (a - b)(a - c)$ and RHS = $(a - b)(a - c)$,

when $x = b$, LHS = $0 + 0$ and RHS = 0 ,

when $x = c$, LHS = $(c - a)(c - b)$ and RHS = $(c - b)(c - a)$,

so, being true for three distinct values of x , the identity holds for all x .

Application of the Theorem to Finding Coefficients: If two quadratic expressions are identically equal, there are two ways of generating equations for finding unknown coefficients.

TWO METHODS OF GENERATING EQUATIONS FOR FINDING COEFFICIENTS:

24

1. Equate coefficients of like terms.
2. Substitute carefully chosen values of x .

WORKED EXERCISE: Express n^2 as a quadratic in $n - 3$.

SOLUTION: Let $n^2 \equiv a(n - 3)^2 + b(n - 3) + c$.
 Equating coefficients of n^2 , $1 = a$.
 Put $n = 3$, then since $n - 3 = 0$, $9 = c$.
 Put $n = 0$, then $0 = 9a - 3b + c$,
 and since $a = 1$ and $c = 9$, $b = 6$.
 So $n^2 \equiv (n - 3)^2 + 6(n - 3) + 9$.

Geometrical Implications of the Theorem: Here are some of the geometrical versions of the theorem, given in the language of coordinate geometry.

GEOMETRICAL IMPLICATIONS:

25

1. The graph of a quadratic function is completely determined by any three points on the curve.
2. The graphs of two distinct quadratic functions cannot intersect in more than two points.
3. A line cannot intersect a parabola in more than two points.

Algebraically, a quadratic is said to have *degree 2* because it has only a term in x^2 , a term in x and a constant term. But geometrically, a parabola is said to have *degree 2* because some lines intersect it in two points, but no line intersects it in more than two points. The third statement above shows that these two contrasting ideas of degree coincide. Hence the theorem at the start of this section provides one of the fundamental links between the algebra of the quadratic function and the geometry of the parabola. This theorem will later be generalised to polynomial functions of any degree, and so will link the algebra of polynomials to the geometry of their graphs.

Exercise 8I

1. Show that the following quadratic identity holds when $x = a$, $x = b$ and $x = 0$, where a and b are distinct and nonzero. Explain why it follows that it is true for all values of x .

$$x(x - b) + x(x - a) \equiv x(2x - a - b)$$

2. Show that $x = 0$, $x = a$ and $x = b$ are all solutions of the quadratic equation

$$\frac{x(x - a)}{b - a} + \frac{x(x - b)}{a - b} = x,$$

where a and b are distinct and nonzero. For what values of x does it now follow that this equation is true?

3. Similarly prove the following identities in x , where a , b and c are distinct:

$$(a) \frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$$

$$(b) (b-c)(x-a) + (c-a)(x-b) + (a-b)(x-c) \equiv 0$$

NOTE: The last identity is only linear, and so only two values of x need be tested.

4. (a) If $x^2 + x + 1 = a(x-1)^2 + b(x-1) + c$ for all values of x , form three equations by substituting $x = 0$ and $x = 1$, and equating coefficients of x^2 . Hence find a , b and c .
 (b) Find a , b and c if $n^2 - n \equiv a(n-4)^2 + b(n-4) + c$. [HINT: Substitute $n = 4$, substitute $n = 0$, and equate coefficients of n^2 .]
 5. (a) Express $2x^2 + 3x - 6$ in the form $a(x+1)^2 + b(x+1) + c$.
 (b) Find a , b and c if $2x^2 + 4x + 5 \equiv a(x-3)^2 + b(x-3) + c$.
 (c) Express $2x^2 - 5x + 3$ in the form $a(x-2)^2 + b(x-2) + c$.
 6. (a) Express x^2 in the form $a(x+1)^2 + b(x+1) + c$.
 (b) Express n^2 as a quadratic in $(n-4)$.
 (c) Express x^2 as a quadratic in $(x+2)$.

DEVELOPMENT

7. Prove the following identities, where p , q and r are distinct:
- $$(a) (p+q+x)(pq+px+qx) - pqx \equiv (p+q)(p+x)(q+x)$$
- $$(b) \frac{p(x-q)(x-r)}{(p-q)(p-r)} + \frac{q(x-p)(x-r)}{(q-r)(q-p)} + \frac{r(x-p)(x-q)}{(r-p)(r-q)} \equiv x$$
- $$(c) \frac{(p-x)(q-x)}{(p-r)(q-r)} + \frac{(q-x)(r-x)}{(q-p)(r-p)} + \frac{(r-x)(p-x)}{(r-q)(p-q)} \equiv 1$$
8. (a) If $2x^2 - x - 1 = a(x-b)(x-c)$ for all real values of x , find a , b and c .
 (b) Express m^2 in the form $a(m-1)^2 + b(m-2)^2 + c(m-3)^2$.
 9. (a) Express $2x - 5$ in the form $A(2x+1) + B(x+1)$.
 (b) Express $\frac{2x+3}{(x+1)(x+2)}$ in the form $\frac{A}{x+1} + \frac{B}{x+2}$. [HINT: You will need to multiply both sides of $\frac{2x+3}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$ by $(x+1)(x+2)$.]
 (c) Express $\frac{x+1}{x(x+2)}$ in the form $\frac{A}{x} + \frac{B}{x+2}$.
 10. (a) Find a , b and c if the graph of the quadratic function $f(x) = ax^2 + bx + c$ passes through $O(0,0)$, $A(4,0)$ and $B(5,5)$. Then check by substitution whether the point $D(-2,10)$ lies on the curve.
 (b) Use a similar method to find whether the points $P(0,6)$, $Q(2,0)$, $R(4,2)$ and $S(6,12)$ lie on the graph of a quadratic function.
 11. (a) Show that $m^2 - (m-1)^2 = 2m - 1$.
 (b) Hence find the sum of the series $1 + 3 + 5 + \cdots + 61$.
 (c) Check your answer using the formula for the sum of an arithmetic series.
 (d) Find the sum of the series $1 + 3 + 5 + \cdots$ to n terms, using each of these two methods.

EXTENSION

12. (a) Express $\frac{x^2}{(x-m)(x-n)}$ in the form $A + \frac{B}{x-m} + \frac{C}{x-n}$. Hence or otherwise, find the value of $\frac{m^2}{(m-n)(m-r)} + \frac{n^2}{(n-m)(n-r)} + \frac{r^2}{(r-m)(r-n)}$, where m , n and r are distinct.
- (b) Similarly, given that the theorems proven in this section may be extended to cubics, find the value of $\frac{m^3}{(m-n)(m-r)(m-p)} + \frac{n^3}{(n-m)(n-r)(n-p)} + \frac{r^3}{(r-m)(r-n)(r-p)} + \frac{p^3}{(p-m)(p-n)(p-r)}$, where m , n , r and p are distinct.
13. Show that the curve $ax^2 + by^2 + cxy + dx + ey + f = 0$ has degree at most 2 in the geometric sense, by showing that no line can intersect it in more than two points. [HINT: Substitute the general form for the equation of a straight line into the curve.]

The Geometry of the Parabola

In the previous chapter, parabolic graphs of quadratic functions were used to help solve problems that were mostly algebraic. All that is reversed in this chapter, where algebraic methods will merely be the tool used to establish the basic geometry of the parabola. In keeping with this geometric approach, the parabola will now be defined not as the graph of a quadratic, but will have a purely geometric definition in which the curve is completely determined by a point called the *focus* and a line called the *directrix*.

Although the motivation here is geometric, the methods used are mostly algebraic, and they make an interesting contrast with the Euclidean methods used to study the geometry of triangles, quadrilaterals and circles. The various configurations are placed almost at once on the coordinate plane so that the methods of functions, algebra and calculus can be applied to them. There will be particular emphasis on a new method of describing curves called *parametrisation*.

STUDY NOTES: Sections 9A–9C complete the 2 Unit material on loci and on the parabola in particular — they form a convenient unit in themselves, the work on locus possibly having been covered previously. The remainder of the chapter presents the 3 Unit theory of the parabola and could be studied later, since it requires maturity in the use of algebra and coordinate geometry. Section 9D introduces parameters, then in Sections 9E–9H the theory of chords, tangents and normals is developed using parametric and non-parametric methods. Section 9I is a collection of general theorems about the parabola, and Section 9J deals with the solution of various locus problems generated by parabolas.

9A A Locus and its Equation

A *locus* is a set of points. The word usually implies that the set has been described in terms of some geometric constraint. Usually a locus is some sort of curve, and can be thought of as the path traced out by a moving point. Our tasks in this section are to find the algebraic equation of a locus that has been geometrically specified, and to supply a geometric description of a locus specified algebraically.

Simple Loci — Sketch and Write Down the Equation: Some simple loci require no more than a sketch, after which the equation can be easily written down.

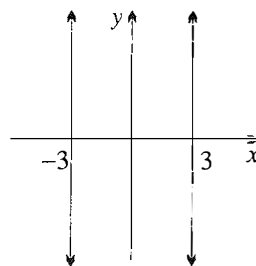
- 1 **SIMPLE LOCI:** Sketch the locus, then write down its equation.

WORKED EXERCISE: Sketch the locus of a point whose distance from the y -axis is 3 units, then write down its equation.

SOLUTION: From the sketch, the equation of the locus is

$$x = 3 \text{ or } x = -3.$$

Notice that this locus is not a function. It can also be written down algebraically as a single equation, $x^2 = 9$, or as $|x| = 3$.



Finding the Locus Algebraically: Generally, however, questions will require the use of algebraic methods to find the equation of a locus. Some sort of sketch should be made, with a general point $P(x, y)$ placed on the coordinate plane, then the formal algebraic work should begin as follows:

- 2 **HARDER LOCI:** 'Let $P(x, y)$ be any point in the plane.
The condition that P lie on the locus is ...'.

We start with any point in the plane, because we seek an algebraic equation such that the point lies on the locus if and only if its coordinates satisfy the equation.

NOTE: When the distance formula is used, it is best to square the geometric condition first.

Example—Finding the Equation of a Circle: A *circle* is defined geometrically as a locus in terms of its centre and radius.

- 3 **DEFINITION:** A *circle* is the locus of all points in the plane that are a fixed distance (called the *radius*) from a given point (called the *centre*).

The well-known equation for a circle can easily be established from this definition.

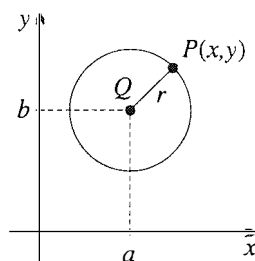
WORKED EXERCISE: Use the definition of a circle to find the equation of the circle with centre $Q(a, b)$ and radius r .

SOLUTION: Let $P(x, y)$ be any point in the plane.

The condition that P lie on the locus is $PQ = r$.

Squaring both sides, $PQ^2 = r^2$.

Using the distance formula, $(x - a)^2 + (y - b)^2 = r^2$.



WORKED EXERCISE: Find the equation of the locus of a point which moves so that its distance from the point $A(2, 1)$ is twice its distance from the point $B(-4, -5)$. Describe the locus geometrically.

SOLUTION: Let $P(x, y)$ be any point in the plane.

The condition that P lie on the locus is

$$PA = 2 \times PB$$

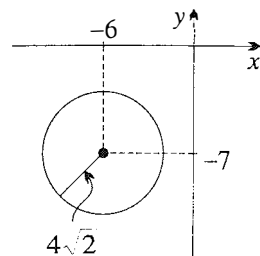
square

$$PA^2 = 4 \times PB^2$$

$$(x - 2)^2 + (y - 1)^2 = 4(x + 4)^2 + 4(y + 5)^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 4x^2 + 32x + 64 + 4y^2 + 40y + 100$$

$$3x^2 + 36x + 3y^2 + 42y = -159.$$



Then dividing by 3 and completing both squares,

$$\begin{aligned}x^2 + 12x + 36 + y^2 + 14y + 49 &= -53 + 36 + 49 \\(x + 6)^2 + (y + 7)^2 &= 32,\end{aligned}$$

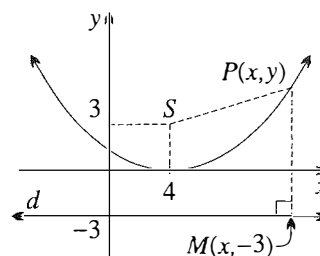
so the locus is a circle with centre $(-6, -7)$ and radius $4\sqrt{2}$.

Example — Finding the Equation of a Parabola: The locus in the worked example below is a parabola, whose geometric definition is the subject of the next section.

WORKED EXERCISE: Find the equation of the locus of all points equidistant from the point $S(4, 3)$ and the line $d: y = -3$.

SOLUTION: Let $P(x, y)$ be any point in the plane, and let $M(x, -3)$ be the foot of the perpendicular from P to d . The condition that P lie on the locus is

$$\begin{aligned}PS &= PM \\ \boxed{\text{square}} \quad PS^2 &= PM^2 \\ (x - 4)^2 + (y - 3)^2 &= (y + 3)^2 \\ (x - 4)^2 &= 12y.\end{aligned}$$



Exercise 9A

NOTE: Solve question 1 by sketching the points $P(x, y)$ on the number plane. All other questions should be solved by starting with the phrase: ‘The condition that $P(x, y)$ lie on the locus is ...’.

- Sketch each locus of the point $P(x, y)$, and hence write down its equation.
 - P is two units below the x -axis.
 - P is one unit to the left of the y -axis.
 - P is equidistant from the lines $y = -1$ and $y = 5$.
 - The distance of P from the x -axis is three times its distance from the y -axis.
 - P is three units from the origin.
 - P is equidistant from the lines with equations $y = x + 3$ and $y = x + 7$.
 - P is 3 units from the point $A(-3, 1)$.
- Derive the equation of the locus of the point $P(x, y)$ which moves so that it is always a distance of 4 units from the point $A(3, 1)$.
- Use the distance formula to find the equation of the locus of the point $P(x, y)$ which moves so that it is equidistant from the points $R(-2, 4)$ and $S(1, 2)$.
 - Find the equation of the line through the midpoint of RS and perpendicular to RS . Comment on the result.
- Given the points $A(4, 0)$, $B(-2, 0)$ and $P(x, y)$, find the gradients of AP and BP .
 - Hence show that the equation of the locus of the point $P(x, y)$ which moves so that $\angle APB$ is a right angle is $x^2 + y^2 - 2x - 8 = 0$.
 - Complete the square in x , and hence describe the locus geometrically.
- Given the points $A(1, 4)$ and $B(-3, 2)$, find the equation of each locus of the point $P(x, y)$, and describe each locus geometrically.
 - P is equidistant from A and B .
 - $\angle APB$ is a right angle.
 - P is equidistant from A and the x -axis.

6. A point $P(x, y)$ moves so that it is equidistant from the point $K(-1, 3)$ and the line $y = 0$. Show that the locus is a parabola and find its vertex.

DEVELOPMENT

7. (a) Use the distance formula to find the square of the distance between the point $P(x, y)$ and each of the points $A(0, 4)$, $B(0, -4)$ and $C(6, 3)$.
 (b) A point $P(x, y)$ moves so that the sum of the squares of its distances from the points A , B and C is 77. Show that the locus is a circle and find its centre and radius.
8. Find, and describe geometrically, the equation of the locus of the point $P(x, y)$ which moves so that the sum of squares of its distances from the points $A(1, 1)$, $B(-1, 1)$, $C(1, -1)$ and $D(-1, -1)$ is 12.
9. (a) Find the locus of the point $P(x, y)$ which moves so that its distance from the point $A(4, 0)$ is always twice its distance from the point $B(1, 0)$.
 (b) Find the locus of the point $P(x, y)$ which moves so that its distance from the point $A(2, 5)$ is always twice its distance from the point $B(4, -1)$.
10. A point $P(x, y)$ moves so that its distance from the point $K(2, 5)$ is twice its distance from the line $x = -1$. Draw a diagram, and find the equation of the locus of P .
11. (a) By using the perpendicular distance formula, find the locus of a point $P(x, y)$ which is equidistant from the lines $3x + 4y = 36$ and $4x + 3y = 24$.
 (b) Show that the locus consists of two perpendicular lines, and sketch all four lines on the same number plane.
12. Find the locus of a point $P(x, y)$ which lies above the x -axis so that the sum of its distances to the origin and the x -axis is 2.
13. If $P(x, y)$ is any point on the line $y = 4x + 3$, show that the midpoint M of OP has coordinates $(\frac{1}{2}x, \frac{1}{2}(4x + 3))$. Hence find the locus of M .
14. $P(x, y)$ is a variable point on the line $2x - 3y + 6 = 0$, and the point Q divides OP in the ratio $3 : 2$. Show that Q has coordinates $(\frac{3}{5}x, \frac{2}{5}(x + 3))$. Hence find the locus of Q .
15. (a) Show that the points $A(0, 2)$, $B(-\sqrt{3}, -1)$ and $C(\sqrt{3}, -1)$ form an equilateral triangle.
 (b) Find the equations of the lines that form the sides AB , BC and CA .
 (c) Find the locus of the point $P(x, y)$ which moves so that the sum of the squares of the distances from the sides of the triangle is 9.
 (d) Describe the locus geometrically and state its relation to the triangle.

EXTENSION

16. (a) Show that if $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three-dimensional space, then $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$.
 (b) $P(x, y, z)$ is a variable point. The sum of the squares of the distances from P to the points $A(a, 0, 0)$ and $B(-a, 0, 0)$ is $4a^2$, where a is a constant. Find the equation of the locus of P and describe it geometrically.
 (c) Consider the eight corners of a cube $A(1, 1, 1)$, $B(-1, 1, 1)$, $C(1, -1, 1)$, $D(-1, -1, 1)$, $E(1, 1, -1)$, $F(-1, 1, -1)$, $G(1, -1, -1)$ and $H(-1, -1, -1)$. A point $P(x, y, z)$ moves so that $PA^2 + PB^2 + PC^2 + PD^2 + PE^2 + PF^2 + PG^2 + PH^2 = 30$. Show that the locus of P is a sphere and find its centre and radius.
17. Let $P(x, y)$ be a point and L_1 and L_2 be two lines in the number plane. Let $\mathcal{C}$ be the set of all points P such that the sum of the squares of the distances of P from L_1 and L_2 is r^2 . Prove that $\mathcal{C}$ is a circle if and only if L_1 and L_2 are perpendicular.

9B The Geometric Definition of the Parabola

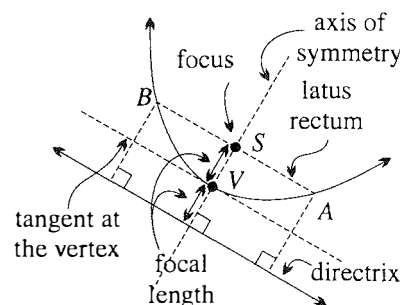
Until now, we have used the word ‘parabola’ to refer to any curve whose equation is a quadratic. The parabola, however, is a geometric object, and to be understood in its true form, needs to be defined geometrically. This section will define the parabola as a locus, similar to the locus definition of a circle, and the general algebraic equation will then be derived from this geometric definition. The definition is surprising in that the whole curve is entirely determined simply by taking any point in the plane and any line not through the point.

4

DEFINITION: A *parabola* is the locus of all points that are equidistant from a given point (called the *focus*) and given line (called the *directrix*) not passing through the focus.

Vertex, Axis of Symmetry, Latus Rectum and Focal Length: The diagram below begins to unpackage the definition and to define some special points and lines associated with a parabola. Suppose that the focus S and the directrix d have been given.

1. The *vertex* V is the point midway between the focus and the directrix.
2. The line through the vertex V parallel to the directrix is the tangent to the parabola at the vertex, because every point on it other than V is closer to the directrix than to S .
3. The line SV through the focus and vertex, perpendicular to the directrix, is called the *axis of symmetry* (or just *axis*), because the whole diagram is symmetric in that line.
4. Let A and B be the two points where the parabola meets the line through S parallel to the directrix. Being on the parabola, their distances from the focus equal the distance from the focus to the directrix.
5. The interval joining the points A and B passes through the focus and is parallel to the directrix. It is called the *latus rectum*.



Focal Length, Chords and Focal Chords: The *focal length* is the distance between the vertex and the focus (or between the vertex and the directrix) and is usually assigned the pronumeral a . Since it is a distance, $a > 0$. Then other important distances can be expressed in terms of the focal length, using the two squares formed by the axis, the directrix and the latus rectum.

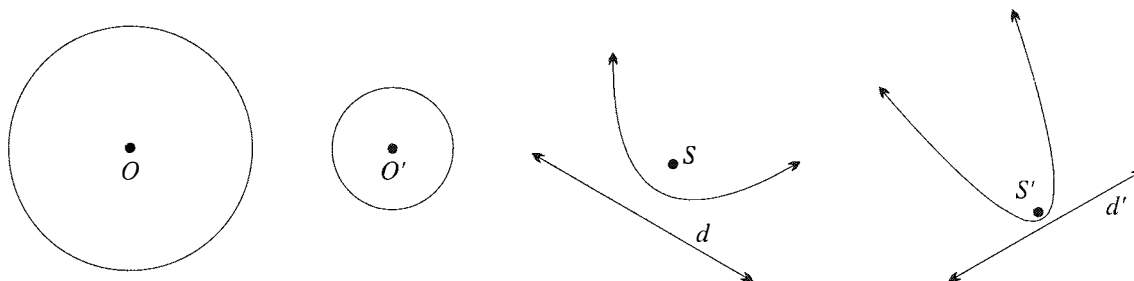
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THE FOCAL LENGTH:

Let distance from focus to vertex $= a$ (the focal length).
 Then distance from focus to directrix $= 2a$ (twice focal length),
 and length of latus rectum $= 4a$ (four times focal length).

By analogy with the circle, any interval joining two points on the parabola is called a *chord*, and the line through the two points is called a *secant*. A chord passing through the focus is called a *focal chord*. The latus rectum is then distinguished from all other chords because it is the focal chord parallel to the directrix.

The focus of a parabola has some analogy with the centre of a circle, and focal chords are distinguished from other chords in a manner similar to the way that a diameter is a chord passing through a circle's centre. Either the length of the latus rectum, or the focal length, gives a measure of how opened out the arms of the parabola are, just as the diameter of a circle (or the radius) is the measure of a circle's size.



It is obvious from the circle's definition that any two circles of the same radius are congruent, and that any two circles are similar. In the same way, any two parabolas with the same focal length can be mapped to each other by congruence transformations — translate the second focus onto the first, then rotate the second directrix until it coincides with the first. Any two parabolas must then be similar, because an enlargement can be used to change the focal length.

SIMILARITY AND CONGRUENCE OF CIRCLES AND PARABOLAS:

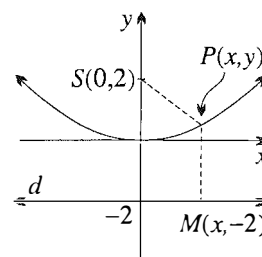
- 6 Any two circles with the same radius are congruent.
 Any two parabolas with the same focal length are congruent.
 Any two circles are similar.
 Any two parabolas are similar.

Using the Definition of a Parabola to Find its Equation: You must be able, by locus methods, to use the definition of a parabola to find its equation.

WORKED EXERCISE: [The locus method] Use the definition of the parabola to find the equation of the parabola with focus $S(0,2)$ and directrix $d: y = -2$. What are the vertex, focal length, and length of the latus rectum?

SOLUTION: Let $P(x, y)$ be any point in the plane, and let $M(x, -2)$ be the foot of the perpendicular from P to d . The condition that P lie on the parabola is

$$\begin{aligned} PS &= PM \\ \boxed{\text{square}} \quad PS^2 &= PM^2 \\ x^2 + (y-2)^2 &= (y+2)^2 \\ x^2 + y^2 - 4y + 4 &= y^2 + 4y + 4 \\ x^2 &= 8y. \end{aligned}$$



The diagram makes it clear that the vertex is $(0,0)$ and the focal length is 2. Hence the length of the latus rectum is 8.

The Four Standard Positions of the Parabola: The intention of the rest of this chapter is to study the parabola using the methods of coordinate geometry. Although the parabola can be placed anywhere on the plane, in any orientation, the equation

will obviously be simpler if the directrix is parallel to one of the axes, and even simpler if the vertex is at the origin. This gives four standard positions for a parabola with focal length a — facing up, facing down, facing right, and facing left. The four diagrams below show these four positions.

7

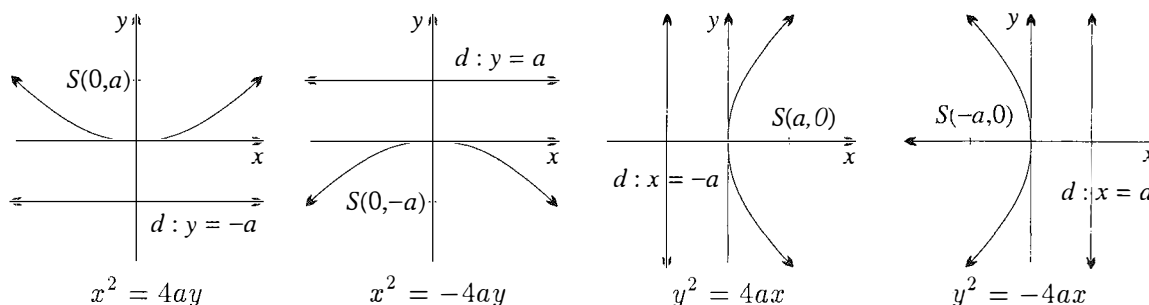
THE FOUR STANDARD POSITIONS OF THE PARABOLA: The equation of every parabola whose vertex is at the origin and whose axis is vertical or horizontal can be put into exactly one of the four forms

$$x^2 = 4ay$$

$$x^2 = -4ay$$

$$y^2 = 4ax$$

$$y^2 = -4ax.$$



The first position — vertex at the origin and facing upwards — is the most usual. We will prove that its equation is $x^2 = 4ay$; the other three equations then follow using reflections in the x -axis and in the line $y = x$.

PROOF: The parabola with vertex at the origin, with focal length a and facing upwards will have focus $S(0, a)$ and directrix $d: y = -a$.

Let $P(x, y)$ be any point in the plane,

and let $M(x, -a)$ be the foot of the perpendicular from P to d .

The condition that P lie on the parabola is

$$PS = PM$$

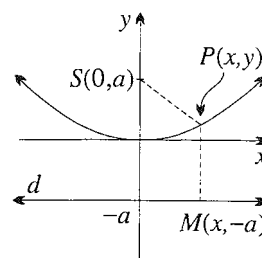
square

$$PS^2 = PM^2$$

$$x^2 + (y - a)^2 = (y + a)^2$$

$$x^2 = (y + a)^2 - (y - a)^2$$

$$x^2 = 4ay.$$



Using the Four Standard Equations of a Parabola: Most of the time, there is no need to go back to the definition of a parabola. We can simply use the standard equations of the parabola established above. You need to be able to describe the parabola geometrically given its equation, and you need to be able to write down the equation of a parabola described geometrically.

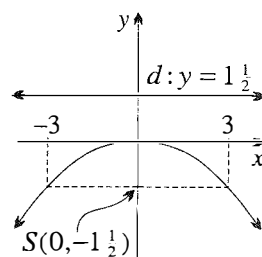
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FIND a FIRST: Establish the orientation, then find the values of $4a$ and of a .

WORKED EXERCISE: [Geometric description of an equation]

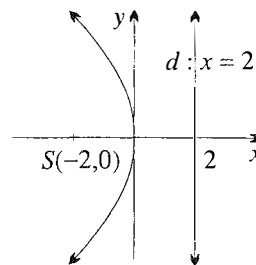
Sketch the parabola $x^2 = -6y$, showing focus, directrix, and the endpoints of the latus rectum.

SOLUTION: The parabola faces down, with $4a = 6$ and $a = 1\frac{1}{2}$. So the focus is $S(0, -1\frac{1}{2})$, the directrix is $y = 1\frac{1}{2}$, and the latus rectum has endpoints $(3, -1\frac{1}{2})$ and $(-3, -1\frac{1}{2})$.



WORKED EXERCISE: [Writing down the equation] Write down the equation of the parabola with vertex at the origin and directrix $x = 2$.

SOLUTION: The parabola is facing left, with $a = 2$ and $4a = 8$.
So its equation is $y^2 = -8x$.



Exercise 9B

NOTE: When a question asks 'Use the definition of a parabola to find its equation', the solution should begin 'Let $P(x, y)$ be any point in the plane. The condition that P lie on the parabola is ...' Otherwise the four standard forms may be used.

- [Construction using ruler and compasses] On a fresh piece of lined paper, rule a directrix d along a line about six lines from the bottom of the page, then mark a focus S two lines above d and horizontally centred on the page. With compasses, construct a set of concentric circles with centre S and radii 1, 2, 3, ... times the distance between the lines on the page. Now use the definition of the parabola to mark points equidistant from the focus and directrix, then join them up by hand to form a parabola.
- [An approach through the tangents to the parabola] On a blank piece of paper, mark a directrix d and a focus S about 4 cm apart. Fold the paper so that the focus S is positioned exactly on the directrix d . Make about twenty such folds, positioning S at different places along d . The set of folds will form an envelope of tangents to the parabola.
- The variable point $P(x, y)$ moves so that it is equidistant from the point $S(0, 3)$ and the line $y + 3 = 0$. Draw a diagram, and let L be the point $(x, -3)$.
 - Show that $PS^2 = x^2 + (y - 3)^2$ and $PL^2 = (y + 3)^2$.
 - By setting $PS^2 = PL^2$, derive the equation of the locus of P .
- Applying the method outlined in the previous question, use the definition of a parabola to derive the equations of:
 - the parabola with focus $(0, 5)$ and directrix $y + 5 = 0$,
 - the parabola with focus $(0, -1)$ and directrix $y - 1 = 0$,
 - the parabola with focus $(2, 0)$ and directrix $x + 2 = 0$,
 - the parabola with focus $(-\frac{3}{2}, 0)$ and directrix $x - \frac{3}{2} = 0$.
- Derive the equation of the locus of the point $P(x, y)$ which moves so that:
 - it is equidistant from the point $S(0, -a)$ and the line $y - a = 0$,
 - it is equidistant from the point $S(a, 0)$ and the line $x + a = 0$.
- For each of the following parabolas, find: (i) the focal length a , (ii) the coordinates of the vertex, (iii) the coordinates of the focus, (iv) the equation of the axis, (v) the equation of the directrix, (vi) the length of the latus rectum. Then sketch a graph of each parabola showing these features.

(a) $x^2 = 4y$	(e) $x^2 = -8y$	(i) $y^2 = 4x$	(m) $y^2 = -8x$
(b) $x^2 = 8y$	(f) $x^2 = -12y$	(j) $y^2 = x$	(n) $y^2 = -12x$
(c) $x^2 = y$	(g) $x^2 = -2y$	(k) $y^2 = 6x$	(o) $y^2 = -x$
(d) $x^2 = \frac{4}{3}y$	(h) $x^2 = -0.4y$	(l) $y^2 = \frac{1}{2}x$	(p) $y^2 = -1.2x$
- Rearrange each equation into the form $x^2 = 4ay$, $x^2 = -4ay$, $y^2 = 4ax$ or $y^2 = -4ax$. Hence sketch a graph of each parabola, indicating the vertex, focus and directrix.

(a) $2x^2 = y$	(b) $4y + x^2 = 0$	(c) $9y^2 = 4x$	(d) $y^2 + 10x = 0$

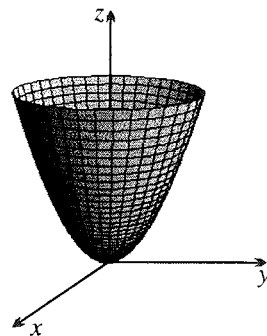
8. Use the four standard forms to find the equation of the parabola with vertex at the origin, axis vertical and:
- (a) focus at $(0, 5)$, (c) directrix $y = -2$, (e) equation of latus rectum $y = 1$,
 (b) focus at $(0, -3)$, (d) directrix $y = \frac{1}{2}$, (f) equation of latus rectum $y = -\frac{1}{8}$.
9. Use the four standard forms to find the equation of the parabola with vertex at the origin, axis horizontal and:
- (a) focus at $(\frac{1}{2}, 0)$, (c) directrix $x = -4$, (e) equation of latus rectum $x = 3$,
 (b) focus at $(-1, 0)$, (d) directrix $x = 2$, (f) equation of latus rectum $x = -\frac{3}{2}$.
10. Use the four standard forms to find the equation of the parabola with vertex at the origin and with the following properties:
- (a) axis vertical, passing through $(4, 1)$,
 (b) axis vertical, passing through $(-2, 8)$,
 (c) axis horizontal, passing through $(2, -2)$,
 (d) axis horizontal, passing through $(-1, 1)$.

DEVELOPMENT

11. Find the equation of the parabola with vertex at the origin and:
- (a) axis vertical, latus rectum 8 units in length (2 parabolas),
 (b) focal length 3 units, and axis horizontal or vertical (4 parabolas),
 (c) passing through $(1, 1)$, and axis horizontal or vertical (2 parabolas),
 (d) axis horizontal, focal length $\frac{1}{2}$ (2 parabolas).
12. (a) The equation of a parabola is of the form $y = kx^2$. If the line $8x - y - 4 = 0$ is a tangent to the parabola, find the value of k .
 (b) A parabola with vertical axis has its vertex at the origin. Find the equation of the parabola if the line $12x - 4y + 3 = 0$ is a tangent.
13. Use the definition of a parabola and perpendicular distance formula to find:
- (a) the equation of the parabola with focus $S(-1, 1)$ and directrix $y = x - 2$,
 (b) the equation of the parabola with focus $S(3, -3)$ and directrix $x - y + 6 = 0$.
- Explain, without referring to its equation, why each parabola passes through the origin.

EXTENSION

14. The variable point $P(x, y, z)$ moves so that it is equidistant from the point $S(0, 0, a)$ and the plane $z = -a$ (such a surface is called a *paraboloid*). Show that the equation of the locus of P is $x^2 + y^2 = 4az$.
15. Find the equations of the following paraboloids:
- (a) focus $(0, 3, 0)$, directrix $y = -3$,
 (b) focus $(1, 0, 0)$, directrix $x = -1$,
 (c) focus $(0, 0, -2)$, directrix $z = 2$,
 (d) focus $(0, -\frac{3}{2}, 0)$, directrix $y - \frac{3}{2} = 0$.
16. Find the vertex and directrix of each of the following paraboloids:
- (a) $x^2 + y^2 + 8z = 0$ (b) $y^2 + z^2 - 2x = 0$ (c) $x^2 + y + z^2 = 0$



9C Translations of the Parabola

When the vertex of a parabola is not at the origin, the normal rules for shifting curves around the plane apply — to move the vertex from $(0, 0)$ to (h, k) , replace x by $x - h$ and y by $y - k$. As with a parabola whose vertex is at the origin, there are two tasks to learn. First, one must be able to write down the equation of a parabola given its geometric description, and conversely, one must be able to describe a parabola geometrically given its equation.

THE FOUR SHIFTED STANDARD FORMS OF THE PARABOLA: Every parabola whose axis is vertical or horizontal has an equation that can be put into exactly one of the four forms

9

$$(x - h)^2 = 4a(y - k) \qquad (x - h)^2 = -4a(y - k)$$

$$(y - k)^2 = 4a(x - h) \qquad (y - k)^2 = -4a(x - h)$$

where $a > 0$ is the focal length, and (h, k) is the vertex.

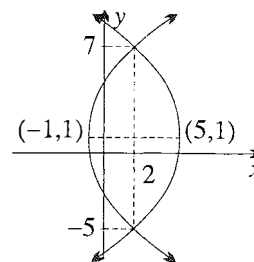
Writing Down the Equation of a Given Parabola: A sketch is essential before anything else. Writing down the equation requires the focal length a , the vertex (h, k) and the orientation of the parabola.

WORKED EXERCISE: Write down the equations of the parabolas with focal length 3, focus $(2, 1)$ and axis parallel to the x -axis. Sketch them, and find and describe their points of intersection.

SOLUTION: The parabola facing right has vertex $(-1, 1)$, so its equation is $(y - 1)^2 = 12(x + 1)$.

The parabola facing left has vertex $(5, 1)$, so its equation is $(y - 1)^2 = -12(x - 5)$.

The two parabolas meet at $(2, 7)$ and $(2, -5)$, which are the endpoints of their common latus rectum.



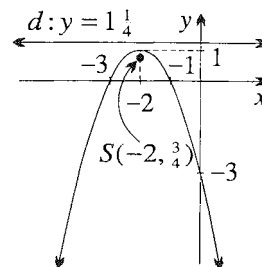
Describing a Parabola Given its Equation: If the equation of a parabola is given, the parabola should be forced into the appropriate standard form by completing the square. As always, find the focal length a . Then a sketch is essential.

WORKED EXERCISE: Find the focus, directrix, focal length and endpoints of the latus rectum of the parabola $y = -3 - 4x - x^2$.

SOLUTION: Completing the square, $x^2 + 4x = -y - 3$
 $x^2 + 4x + 4 = -y - 3 + 4$
 $(x + 2)^2 = -(y - 1).$

So $4a = 1$ and $a = \frac{1}{4}$, the vertex is $(-2, 1)$, and the parabola is concave down.

Thus the focus is $(-2, \frac{3}{4})$ and the directrix is $y = 1\frac{1}{4}$, and the endpoints of the latus rectum are $(-2\frac{1}{2}, \frac{3}{4})$ and $(-1\frac{1}{2}, \frac{3}{4})$.



Exercise 9C

NOTE: When a question asks ‘Use the definition of a parabola to find its equation’, the solution should begin ‘Let $P(x, y)$ be any point in the plane. The condition that P lie on the parabola is’ Otherwise the four standard forms may be used.

- The variable point $P(x, y)$ moves so that it is equidistant from the point $S(3, 3)$ and the line $y + 1 = 0$. Let L be the point $(x, -1)$.
 - Show that $PS^2 = (x - 3)^2 + (y - 3)^2$ and $PL^2 = (y + 1)^2$.
 - By setting $PS^2 = PL^2$, derive the equation of the locus of P .
- Applying the method outlined in the previous question, use the definition of a parabola to derive the equations of the following parabolas:
 - focus $(-7, -2)$, directrix $y + 8 = 0$
 - focus $(0, 2)$, directrix $x + 2 = 0$
- Sketch each of the following parabolas, clearly indicating the coordinates of the vertex and focus, and the equations of the axis and directrix:

(a) $x^2 = 4(y + 1)$	(e) $x^2 = -2(y + 3)$	(i) $(y + 7)^2 = 12(x - 5)$
(b) $(x + 2)^2 = 4y$	(f) $(x + 5)^2 = -4(y - 3)$	(j) $(y + 8)^2 = -4(x - 3)$
(c) $(x - 3)^2 = 8(y + 5)$	(g) $y^2 = 6(x + 2)$	(k) $y^2 = -10(x + 6)$
(d) $(x - 4)^2 = -8y$	(h) $(y - 1)^2 = 16x$	(l) $(y - 3)^2 = -2x$
- Using the four standard forms, find the equation of the parabola with focus and vertex:

(a) $(-2, 6), (-2, 4)$	(d) $(0, 0), (1, 0)$	(g) $(8, -10), (8, -7)$
(b) $(5, 1), (1, 1)$	(e) $(-5, 4), (-5, 2)$	(h) $(-3, -3), (-1, -3)$
(c) $(2, -1), (2, 2)$	(f) $(-3, -2), (-7, -2)$	(i) $(6, 0), (6, -3)$
- Use the standard forms to find the equation of the parabola with vertex and directrix:

(a) $(2, -1), y = -3$	(d) $(2, 5), x = 5$	(g) $(0, -\frac{3}{2}), y = \frac{1}{2}$
(b) $(1, 0), x = 0$	(e) $(3, 1), y = -1$	(h) $(-1, -4), x = 2$
(c) $(-3, 4), y - 6 = 0$	(f) $(-4, 2), x = -7$	(i) $(-7, -5), y = -5\frac{1}{2}$
- Use the standard forms to find the equation of the parabola with focus and directrix:

(a) $(0, 4), y = 0$	(d) $(-4, 0), x = 0$	(g) $(-1, 4), y = 5$
(b) $(6, 0), x = 0$	(e) $(1, 7), y = 3$	(h) $(3, \frac{1}{2}), x = 5$
(c) $(0, -2), y = 0$	(f) $(3, -2), x = 1$	(i) $(5, -4), y = -9$

DEVELOPMENT

- Express the equation of each of the following parabolas in the form $(x - h)^2 = 4a(y - k)$ or $(x - h)^2 = -4a(y - k)$. Sketch a graph, clearly indicating the focus, vertex and directrix.

(a) $y = x^2 + 6x + 5$	(e) $y = (x + 8)(x - 2)$
(b) $x^2 = 1 - y$	(f) $(x + 3)(x + 5) = 8y - 25$
(c) $6y = x^2 - 12x$	(g) $x^2 - 6x + 2y + 12 = 0$
(d) $x^2 = 2(1 + 2y)$	(h) $x^2 - 8x + 12y + 4 = 0$
- Express the equation of each of the following parabolas in the form $(y - k)^2 = 4a(x - h)$ or $(y - k)^2 = -4a(x - h)$. Sketch a graph, clearly indicating the focus, vertex and directrix.

(a) $y^2 - 4x = 0$	(e) $y(y - 4) = 8x$
(b) $y^2 = 6 - 2x$	(f) $y^2 - 6y - 2x + 7 = 0$
(c) $6x = y^2 + 18$	(g) $y^2 + 4y + 6x - 26 = 0$
(d) $y^2 - 2y = 4x - 5$	(h) $(y - 4)(y - 6) = 12x + 11$

9. By using the general form $y = Ax^2 + Bx + C$ or $x = Ay^2 + By + C$, find the equation of the parabola with:
- axis parallel to the y -axis, passing through $(1, 0)$, $(-1, -6)$ and $(2, 9)$;
 - axis parallel to the y -axis, passing through $(1, -5)$, $(-1, 5)$ and $(0, 1)$;
 - axis parallel to the x -axis, passing through $(0, 1)$, $(8, -1)$ and $(-1, 2)$;
 - axis parallel to the x -axis, passing through $(-4, 1)$, $(-6, -1)$ and $(-3, 0)$.
10. Find the equation of each of the following parabolas:
- vertex at $(1, 4)$, axis parallel to the y -axis, passing through $(3, 5)$;
 - vertex at $(-2, 3)$, axis parallel to the y -axis, y -intercept at -1 ;
 - vertex at $(-3, -2)$, axis parallel to the x -axis, passing through $(-1, 0)$;
 - vertex at $(2, 5)$, axis parallel to the x -axis, passing through $(0, 4)$.
11. Find all possible equations of the parabolas with the following constraints, assuming that the axis is parallel to one of the coordinate axes:
- vertex at $(3, -1)$, focal length 2 units (4 parabolas);
 - latus rectum has endpoints $(1, 3)$ and $(1, -5)$ (2 parabolas);
 - focus at $(-2, 4)$, endpoint of latus rectum at $(0, 4)$ (2 parabolas);
 - axis $y - 2 = 0$, vertex at $(3, 2)$, latus rectum has length 6 units (2 parabolas);
 - focus $(6, -3)$, vertex on the line $y = x - 4$ (2 parabolas).
12. Find the equation of each of the following parabolas:
- vertex at $(3, -1)$, axis parallel to the y -axis and the line $4x + y - 7 = 0$ is a tangent;
 - vertex at $(-4, 2)$, axis parallel to the x -axis and the line $x = 6 - 4y$ is a tangent.
13. Use the perpendicular distance formula to find the equation of the parabola with:
- focus $(-1, 4)$ and directrix $x - y - 1 = 0$;
 - focus $(1, 2)$ and directrix $4x + 3y - 2 = 0$.

EXTENSION

14. Derive the equations of the following paraboloids:
- focus $(1, 2, 3)$, directrix $z + 1 = 0$,
 - focus $(-1, 2, -1)$, directrix $x = 1$,
 - focus $(0, -5, 3)$, directrix $y = 2$,
 - focus $(4, -3, 7)$, directrix $z - 4 = 0$.
15. Find the focus, vertex and directrix of the following paraboloids:
- $x^2 + y^2 - 2x - 4y - 4z + 1 = 0$
 - $y^2 + z^2 + 6y + 8z + 1 = 0$

9D Parametric Equations of Curves

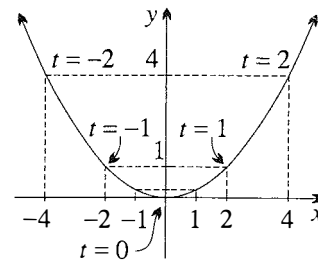
This section introduces an ingenious way of handling curves by making each coordinate a function of a single variable, called a *parameter*. In this way, each point on the curve is specified by a single number, rather than by a pair of coordinates. The main purpose here is to investigate further the geometry of the parabola, but the method is general, and some other curves will be considered, particularly circles and rectangular hyperbolas.

An Example of Parametrisation: The parabola $x^2 = 4y$ with focal length 1 can be *parametrised* by the pair of equations

$$x = 2t \quad \text{and} \quad y = t^2$$

because by simple algebra, elimination of t gives $x^2 = 4y$. The variable point $(2t, t^2)$ now runs along the whole curve as the parameter t takes different values:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-4	-2	-1	0	1	2	4
y	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4



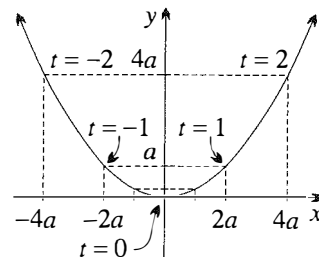
The sketch shows the curve with the seven plotted points labelled by their parameter. In effect, the curve becomes a ‘bent and stretched’ number line. The original equation in x and y is called the *Cartesian equation* of the curve to distinguish it from the *parametric equations* of the curve.

The Standard Parametrisation of the Parabola: The parabola $x^2 = 4ay$ has a parametrisation that is so convenient that it is taken as the *standard parametrisation*:

$$x = 2at \quad \text{and} \quad y = at^2.$$

The parametrisation should be checked by elimination of t — notice that the previous example was a special case with $a = 1$. A short table of values shows how the parabola is divided neatly into four parts by this parametrisation:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	$-4a$	$-2a$	$-a$	0	a	$2a$	$4a$
y	$4a$	a	$\frac{1}{4}a$	0	$\frac{1}{4}a$	a	$4a$



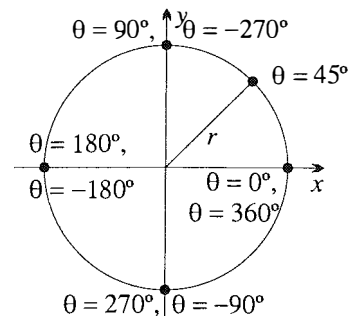
The vertex has parameter $t = 0$, points to the right have positive parameter, and points to the left have negative parameter. The endpoints of the latus rectum have parameters $t = -1$ and $t = 1$, points on the curve between these endpoints have parameter with $|t| < 1$, and points above the latus rectum have parameters with $|t| > 1$. So the three chief points on the parabola are paired with the three most important numbers, 0, 1 and -1 .

A Parametrisation of the Circle: The circle $x^2 + y^2 = r^2$ can be parametrised using trigonometric functions by

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

As we saw in Chapter Four, this parametrisation works because of the Pythagorean identity $r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$. Notice from the table of values below how in this case, each parameter corresponds to just one point, but each point corresponds to infinitely many different values of the parameter, all differing by multiples of 360° :

θ	-360°	-270°	-180°	-90°	0	45°	90°	180°	270°	360°
x	r	0	$-r$	0	r	$\frac{1}{2}r\sqrt{2}$	0	$-r$	0	r
y	0	r	0	$-r$	0	$\frac{1}{2}r\sqrt{2}$	r	0	$-r$	0

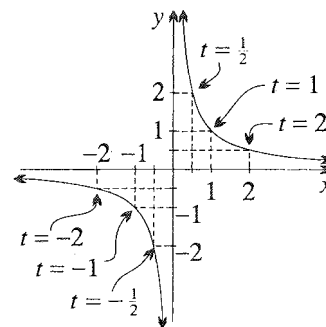


A Parametrisation of the Rectangular Hyperbola: The rectangular hyperbola $xy = 1$ can be parametrised algebraically by

$$x = t \quad \text{and} \quad y = \frac{1}{t}.$$

This time there is a one-to-one correspondence between the points on the curve and the real numbers, with the one exception that $t = 0$ does not correspond to any point:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	*	2	1	$\frac{1}{2}$



WORKED EXERCISE: Find the Cartesian equations of the curves defined by the parametric equations: (a) $x = 4t$, $y = t^2 + 1$ (b) $x = \sec \theta$, $y = \sin \theta$
Describe part (a) geometrically.

SOLUTION:

(a) From the first, $t = \frac{1}{4}x$,
and substituting into the second,

$$y = \frac{1}{16}x^2 + 1$$

$$x^2 = 16(y - 1),$$

which is a parabola with vertex $(0, 1)$,
concave up, with focal length 4.

(b) Squaring, $x^2 = \sec^2 \theta$,
and $y^2 = \sin^2 \theta$
 $= 1 - \cos^2 \theta$,
so $y^2 = 1 - \frac{1}{x^2}$
 $x^2(1 - y^2) = 1$.

Exercise 9D

1. (a) Complete the table below for the curve $x = 2t$, $y = t^2$ and sketch its graph:

t	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
x									
y									

- (b) Eliminate the parameter to find the Cartesian equation of the curve.
(c) State the coordinates of the vertex and focus of the parabola.
(d) What value of t gives the coordinates of the vertex?
(e) What are the coordinates of the endpoints of the latus rectum and what values of t give these coordinates?
2. Repeat the previous question for the curves: (a) $x = 4t$, $y = 2t^2$ (b) $x = t$, $y = \frac{1}{2}t^2$
3. (a) Show that the point $\left(ct, \frac{c}{t}\right)$ lies on the curve $xy = c^2$.

- (b) Complete the table of values below for the curve $x = 2t$, $y = \frac{2}{t}$ and sketch its graph.

t	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
x										
y										

- (c) Explain what happens as $t \rightarrow \infty$, $t \rightarrow -\infty$, $t \rightarrow 0^+$ and $t \rightarrow 0^-$.

4. (a) Show that the point $(a \cos \theta, b \sin \theta)$ lies on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
 (b) Complete a table of values for the curve $x = 4 \cos \theta$, $y = 3 \sin \theta$, where $0^\circ \leq \theta \leq 360^\circ$.
 (c) Sketch the curve and state its Cartesian equation.

DEVELOPMENT

5. Eliminate the parameter and hence find the Cartesian equation of the curve.
 (a) $x = 3 - t$, $y = 2t + 1$ (c) $x = t + \frac{1}{t}$, $y = t^2 + \frac{1}{t^2}$
 (b) $x = 1 + 2 \tan \theta$, $y = 3 \sec \theta - 4$ (d) $x = \cos \theta + \sin \theta$, $y = \cos \theta - \sin \theta$
6. (a) Show that the point $(a \sec \theta, b \tan \theta)$ lies on the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
 (b) Complete a table of values for the curve $x = 4 \sec \theta$, $y = 3 \tan \theta$, where $0^\circ \leq \theta \leq 360^\circ$.
 What happens when $\theta = 90^\circ$ and $\theta = 270^\circ$?
 (c) Sketch the curve (it has two asymptotes) and state its Cartesian equation.
7. (a) Show that $x = a + r \cos \theta$, $y = b + r \sin \theta$ defines a circle with centre (a, b) and radius r .
 (b) Hence sketch a graph of the curve $x = 1 + 2 \cos \theta$, $y = -3 + 2 \sin \theta$.
8. Different parametric representations may result in the same Cartesian equation. The graphical representation, however, may be different.
 (a) Find the Cartesian equation of the curve $x = 2 - t$, $y = t - 1$ and sketch its graph.
 (b) Find the Cartesian equation of the curve $(\sin^2 t, \cos^2 t)$. Explain why $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and sketch a graph of the curve.
 (c) Find the Cartesian equation of the curve $x = 4 - t^2$, $y = t^2 - 3$. Explain why $x \leq 4$ and $y \geq -3$ and sketch a graph of the curve.
9. Find the Cartesian equation of the curve $x = 3 + r \cos \theta$, $y = -2 + r \sin \theta$, and describe it geometrically if: (a) r is constant and θ is variable, (b) θ is constant and r is variable.

EXTENSION

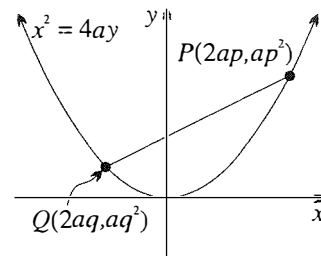
10. P_1 and P_2 are the points (x_1, y_1) and (x_2, y_2) respectively.
 (a) Let $P(x, y)$ divide the interval P_1P_2 in the ratio $\lambda : 1$. Show that $x = \frac{x_1 + \lambda x_2}{1 + \lambda}$,
 $y = \frac{y_1 + \lambda y_2}{1 + \lambda}$ is a parametrisation of the line P_1P_2 .
 (b) What parameters do the points P_1 and P_2 have, and what happens near $\lambda = -1$?
 (c) The line joining $P_1(1, 5)$ and $P_2(4, 9)$ meets the line $3x + y - 11 = 0$ at the point P .
 Find the ratio $\frac{P_1P}{PP_2}$, without finding the coordinates of P .
 (d) (i) The line P_1P_2 intersects the parabola $x^2 = 4ay$. Obtain an equation whose roots are the values of λ corresponding to the points of intersection. (ii) Find a necessary and sufficient condition for the line P_1P_2 to be a tangent to the parabola.

9E Chords of a Parabola

In the remaining sections, the geometry of chords, tangents and normals is developed using parametric as well as using Cartesian methods. Although the more important equations in these sections are boxed as usual, it is not intended that they be learnt and applied — examination questions will either ask for them to be derived, or give the formulae in the question.

The Parametric Equation of the Chord: Suppose that $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two distinct points on the parabola $x^2 = 4ay$. We can find the equation of the chord PQ by finding the gradient of the chord and then using point-gradient form.

$$\begin{aligned}\text{Gradient of chord} &= \frac{ap^2 - aq^2}{2ap - 2aq} \\ &= \frac{a(p - q)(p + q)}{2a(p - q)} \\ &= \frac{1}{2}(p + q), \\ \text{so the chord is } y - ap^2 &= \frac{1}{2}(p + q)(x - 2ap) \\ y - ap^2 &= \frac{1}{2}(p + q)x - ap^2 - apq \\ y &= \frac{1}{2}(p + q)x - apq.\end{aligned}$$



10 THE PARAMETRIC EQUATION OF THE CHORD: $y = \frac{1}{2}(p + q)x - apq$

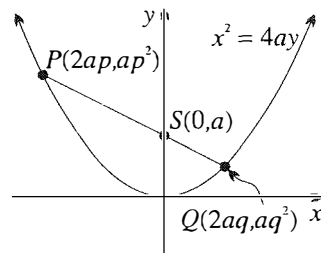
NOTE: If the parameters p and q are exchanged, then the formulae for the gradient of the chord and the equation of the chord remain the same. Geometrically, this is because the chord PQ is the same line as the chord QP . Such expressions are called *symmetric in p and q* , and this can often be a good check that the calculations have been carried out accurately.

Parameters and Focal Chords: As defined in Section 9B, a chord that passes through the focus of a parabola is called a *focal chord*. Substituting the focus $(0, a)$ into the equation of the chord above gives

$$a = 0 - apq,$$

and dividing by a , $pq = -1$ (note that $a \neq 0$).

This is a condition for PQ to be a focal chord.



11 FOCAL CHORDS: PQ is a focal chord if and only if $pq = -1$.

WORKED EXERCISE: Two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ lie on the parabola $x^2 = 4ay$.

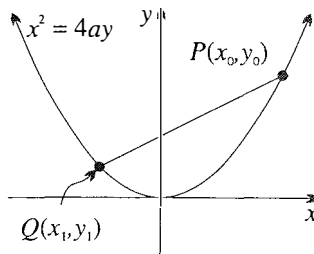
- Show that $y_0 - y_1 = \frac{x_0^2 - x_1^2}{4a}$.
- Show that the chord PQ has equation $4ay = x(x_0 + x_1) - x_0x_1$.
- Show that PQ is a focal chord if and only if $x_0x_1 = -4a^2$.
- Use part (a) to show that the chord joining the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on $x^2 = 4ay$ has equation $y = \frac{1}{2}(p + q)x - apq$.

SOLUTION:

- (a) Since P and Q lie on $x^2 = 4ay$, $x_0^2 = 4ay_0$ and $x_1^2 = 4ay_1$,

$$\text{hence} \quad y_0 - y_1 = \frac{x_0^2 - x_1^2}{4a}.$$

$$\begin{aligned}\text{(b)} \quad \text{Gradient } PQ &= \frac{y_0 - y_1}{x_0 - x_1} \\ &= \frac{x_0^2 - x_1^2}{4a(x_0 - x_1)} \\ &= \frac{x_0 + x_1}{4a},\end{aligned}$$



so the chord is $y - y_1 = \frac{x_0 + x_1}{4a}(x - x_1)$

$$4ay - 4ay_1 = x(x_0 + x_1) - x_0x_1 - x_1^2.$$

Since $4ay_1 = x_1^2$, $4ay = x(x_0 + x_1) - x_0x_1$.

(c) Substituting $(0, a)$ gives $4a^2 = 0 - x_0x_1$, as required.

(d) Substituting $x_0 = 2ap$ and $x_1 = 2aq$,

$$4ay = x(2ap + 2aq) - 4a^2pq$$

$$\boxed{\div 4a} \quad y = \frac{1}{2}x(p + q) - apq.$$

NOTE: This worked exercise gives an alternative, but less elegant, derivation of the parametric equation of a chord.

Exercise 9E

1. Find the equation of the chord of the parabola joining the points with parameters:

(a) 1 and -3 on $x = 2t$, $y = t^2$

(c) -1 and -2 on $x = t$, $y = \frac{1}{2}t^2$

(b) $\frac{1}{2}$ and 2 on $x = 4t$, $y = 2t^2$

(d) -2 and 4 on $x = \frac{1}{2}t$, $y = \frac{1}{4}t^2$

Then use the formula $y = \frac{1}{2}(p + q)x - apq$ to obtain the chords in parts (a)–(d).

2. (a) Find the chord joining the points with parameters 2 and $-\frac{1}{2}$ on $x = 6t$, $y = 3t^2$.

(b) Find the Cartesian equation of the parabola and the coordinates of the focus.

(c) Show by substitution that the chord in part (a) is a focal chord.

DEVELOPMENT

3. The points P and Q on the curve $x = 2at$, $y = at^2$ have parameters p and q respectively.

(a) Show that the chord PQ has gradient $\frac{1}{2}(p + q)$.

(b) Hence show that the equation of the chord is $y - \frac{1}{2}(p + q)x + apq = 0$.

(c) Show that PQ intersects the directrix at $\left(\frac{2a(pq - 1)}{p + q}, -a\right)$.

(d) State the coordinates of the focus of the parabola.

(e) Show that if PQ is a focal chord, then $pq = -1$.

(f) Hence find the point of intersection of a focal chord and the directrix.

4. P and Q are the points with parameters p and q on the parabola $x = 2at$, $y = at^2$.

(a) State the coordinates of P , Q and the focus S .

(b) Use the distance formula to find an expression for the length of PS .

(c) Similarly find an expression for the length of QS .

(d) Hence show that $PS + QS = a(p^2 + q^2 + 2)$.

(e) If PQ is a focal chord (and hence $pq = -1$), show that $PQ = a(p + 1/p)^2$.

5. P and Q are the points with parameters p and q on the parabola $x = at^2$, $y = 2at$.

(a) Show that the chord PQ is $2x - (p + q)y + 2apq = 0$.

(b) If $OP \perp OQ$, show that the x -intercept of PQ is independent of p and q .

6. (a) The line $x + 2y - 8 = 0$ intersects the parabola $x = 4t$, $y = 2t^2$. By forming a quadratic in t , find the parameters at the points of intersection.

(b) Find the parameters of the points where $y = 3 - x$ intersects $x = 2t$, $y = t^2$.

7. P and Q are the points with parameters p and q on the parabola $x = 2at$, $y = at^2$.
- Show that the chord PQ is $y - \frac{1}{2}(p+q)x + apq = 0$.
 - If the chord when extended passes through the point $(0, -a)$, show that $pq = 1$.
 - Hence, if S is the focus of the parabola, show that $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$.
8. (a) The line $y = 2x + 10$ is a chord of the parabola $x = 4t$, $y = 2t^2$. By putting the line in the form $y = \frac{1}{2}(p+q)x - apq$, find $p+q$ and pq , and hence find the coordinates of the endpoints of the chord.
- Show, using a similar method, that $y = 2x - 8$ is a tangent to this parabola.
 - Show, using a similar method, that $y = 2x - 10$ does not meet this parabola.
9. Using the equation $y = \frac{1}{2}(p+q)x - apq$, show that the midpoint of a chord of the parabola $x = 2at$, $y = at^2$ lies on the vertical line $x = k$ if and only if the chord has gradient $k/2a$.
10. The points P , Q , R and S lie on $x = 2at$, $y = at^2$ and have parameters p , q , r and s respectively. If the chords PQ and RS intersect on the axis, show that $p:r = s:q$.

————— EXTENSION —————

11. The parameters of the points P , Q and R on the parabola $x = 2at$, $y = at^2$ form a geometric sequence. Show that the y -intercepts of the chords PQ , PR and QR also form a geometric sequence.
12. A focal chord AB of a parabola meets the directrix at D . Prove that if the focus S divides AB internally in the ratio $k:1$, then D divides AB externally in the ratio $k:1$.

9F Tangents and Normals: Parametric Approach

The easiest way to finding the equation of a tangent is to make an appeal to calculus. We can differentiate in order to find its gradient, and then use point-gradient form to find its equation. Alternatively, we can take the limit of the equation of a chord as its endpoints move together.

The Gradient of the Tangent: Suppose that $P(2ap, ap^2)$ is any point on the parabola with equation $x^2 = 4ay$.

$$\begin{aligned} \text{Differentiating parametrically, } \frac{dy}{dx} &= \frac{dy/dp}{dx/dp} \\ &= \frac{2ap}{2a} \\ &= p. \end{aligned}$$

12

THE PARAMETRIC GRADIENT OF THE TANGENT:

The gradient of the tangent at $P(2ap, ap^2)$ is p .

NOTE: The simplicity of this result is the essential reason why the standard parametrisation $x = 2at$ and $y = at^2$ is the most convenient parametrisation of the parabola $x^2 = 4ay$.

The Parametric Equation of the Tangent: Now the equation of the tangent can be found using the point–gradient form:

$$y - ap^2 = p(x - 2ap)$$

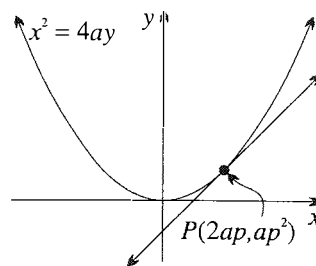
$$y - ap^2 = px - 2ap^2$$

$$y = px - ap^2.$$

13

PARAMETRIC EQUATION OF THE TANGENT:

The tangent at $P(2ap, ap^2)$ is $y = px - ap^2$.



The Tangent as the Limit of the Chord: The equation of the tangent can be developed in a completely different way by starting from the equation of the chord PQ . As the point Q moves closer to P , the line PQ becomes closer and closer to the tangent at P . In fact, the tangent is the limit of the chord PQ as $Q \rightarrow P$.

Algebraically, we can move Q towards P by taking the limit as $q \rightarrow p$.

The chord is $y = \frac{1}{2}(p + q)x - apq$,

and taking the limit as $q \rightarrow p$ gives $y = px - ap^2$, as before.

This process is identical to first principles differentiation, in that a tangent is characterised as the limit of the chord when the endpoints approach each other.

Tangents from an External Point: The parametric form of the tangent gives a straightforward way to find the equations of the two tangents from an external point.

WORKED EXERCISE:

(a) Show that the tangent to $x^2 = 12y$ at the point $P(6p, 3p^2)$ has equation $y = px - 3p^2$.

(b) By substituting the point $A(2, -1)$ into this equation of the tangent, find the points of contact, and the equations, of the tangents to the parabola from A .

SOLUTION:

(a) Differentiating, $\frac{dy}{dx} = \frac{6p}{6}$

$$= p,$$

so the tangent is $y - 3p^2 = p(x - 6p)$

$$y = px - 3p^2$$

(this is the boxed equation above with $a = 3$).

(b) Substituting $A(2, -1)$ gives $-1 = 2p - 3p^2$

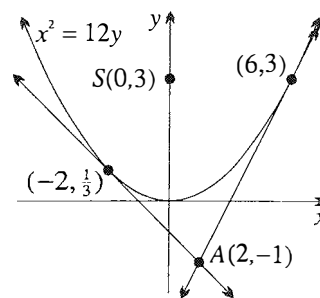
$$3p^2 - 2p - 1 = 0$$

$$(3p + 1)(p - 1) = 0$$

$$p = 1 \text{ or } -\frac{1}{3},$$

so the points of contact are $(6, 3)$ and $(-2, \frac{1}{3})$,

and the corresponding tangents are $y = x - 3$ and $y = -\frac{1}{3}(x + 1)$.



The Intersection of Two Tangents: Simultaneous equations can give us the point T of intersection of the tangents at two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on a parabola. Notice at the outset that the coordinates of T in the solution must be symmetric in p and q .

The tangents are $y = px - ap^2$ (1)

and $y = qx - aq^2$. (2)

Subtracting these, $(p - q)x = a(p^2 - q^2)$

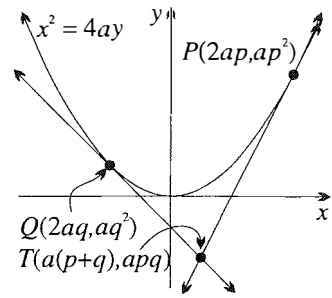
$$\div (p - q)$$

$$x = a(p + q), \text{ since } p \neq q.$$

Substituting into (1),

$$y = ap^2 + apq - ap^2$$

$$y = apq.$$



14

INTERSECTION OF TANGENTS:

The tangents at P and Q meet at $(a(p + q), apq)$.

The Parametric Equation of the Normal: Proceeding as usual,

$$\text{gradient of normal} = -\frac{1}{p},$$

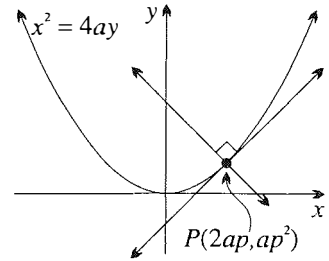
so its equation is

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$\times p$$

$$py - ap^3 = -x + 2ap$$

$$x + py = 2ap + ap^3.$$



15

PARAMETRIC EQUATION OF THE NORMAL:

The normal at $P(2ap, ap^2)$ has gradient $-\frac{1}{p}$ and equation $x + py = 2ap + ap^3$.

WORKED EXERCISE:

- Show that the normal to $x^2 = 20y$ at the point $P(10p, 5p^2)$ has equation $x + py = 10p + 5p^3$.
- Hence find the equations of the normals to $x^2 = 20y$ from $A(0, 30)$.
- Show that a normal passes through $K(0, k)$ if and only if $k > 10$ (apart from the normal at the vertex).

SOLUTION:

$$(a) \text{ Differentiating, } \frac{dy}{dx} = \frac{10p}{10} = p,$$

so the normal has gradient $-\frac{1}{p}$ and its equation is

$$y - 5p^2 = -\frac{1}{p}(x - 10p)$$

$$x + py = 10p + 5p^3$$

(this is the boxed equation above with $a = 5$).

$$(b) \text{ Substituting } A(0, 30) \text{ gives } 0 + 30p = 10p + 5p^3$$

$$\div 5$$

$$5p(p^2 - 4) = 0$$

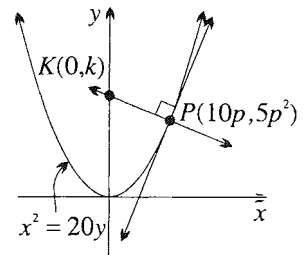
$$p = 0, 2 \text{ or } -2.$$

So the normals from A are $x = 0$, $x + 2y = 60$ and $x - 2y = -60$.

$$(c) \text{ Substituting } K(0, k) \text{ gives } 0 + pk = 10p + 5p^3$$

$$p(5p^2 + 10 - k) = 0,$$

which has nonzero solutions if and only if $k > 10$.



Exercise 9F

- Use the derivative to find the equations of the following tangents:
 - at the point $t = 1$ on $x = 2t$, $y = t^2$
 - at the point $t = \frac{1}{2}$ on $x = 4t$, $y = 2t^2$
 - at the point $t = -3$ on $x = t$, $y = \frac{1}{2}t^2$
 - at the point $t = q$ on $x = 6t$, $y = 3t^2$
 Then use the formula $y = px - ap^2$ to obtain the tangents in parts (a)–(d).
- Use the derivative to find the equations of the following normals:
 - at the point $t = 2$ on $x = 2t$, $y = t^2$
 - at $t = -\frac{1}{2}$ on $x = 4t$, $y = 2t^2$
 - at the point $t = m$ on $x = 6t$, $y = 3t^2$
 - at the point $t = q$ on $x = 2at$, $y = at^2$
 Then use the formula $x + py = 2ap + ap^3$ to obtain the normals in parts (a)–(d).
- Find the equation of the tangent to the parabola $x = 2at$, $y = at^2$ at the point $t = p$.
 - Find the coordinates of the points where the tangent intersects the coordinate axes.
 - Find the area of the triangle formed by the two intercepts and the origin.
- Find the equation of the normal to $x = 2at$, $y = at^2$ at the point $t = p$.
 - Find the points where the normal intersects the coordinate axes.
 - Find the area of the triangle formed by the two intercepts and the origin.
- Show that the endpoints L and R of the latus rectum of the parabola $x = 2at$, $y = at^2$ have parameters 1 and -1 respectively. Then use these parameters to write down the equations of the tangents and normals at L and R .
 - Show that these tangents and normals form a square, and find its vertices and its area.
 - Sketch the parabola, showing the square, the focus and the directrix.

DEVELOPMENT

- Find the equation of the tangent to $x^2 = 4y$ at the point $(2t, t^2)$.
 - If the tangent passes through the point $(2, -3)$, find the values of t .
 - Hence state the equations of the tangents to $x^2 = 4y$ passing through $(2, -3)$.
- Using methods similar to the those in previous question, find the parameters of the points of contact of the tangents to $x = 10t$, $y = 5t^2$ from the point $P(24, -5)$.
 - Hence find the gradients and points of contact of the tangents, and show that they are perpendicular. Show also that P lies on the directrix.
- Sketch the parabola $x = \frac{1}{2}t$, $y = \frac{1}{4}t^2$, and mark the points A and B with parameters $t = -2$ and $t = 4$ respectively.
 - Find the tangents at A and B , and show that they intersect at $C(\frac{1}{2}, -2)$.
 - Find the midpoint M of BC , then use the methods of the previous two questions to find the point D on the parabola between A and B such that the tangent at D passes through M . Show that the tangent at D is parallel to AB .
- Substitute the parabola $x = 6t$, $y = 3t^2$ into the line $x - y - 3 = 0$ to form a quadratic equation in t . Then use the discriminant to show that the line is a tangent, and find its point of contact.
 - Similarly, show that $x - 2y - 1 = 0$ is a tangent to $x = 4t$, $y = 2t^2$, and find its perpendicular distance from the focus.
 - By substituting the parabola $x = 2t$, $y = t^2$ into the line $x + y + a = 0$, show that the line is a tangent if and only if $a = 1$.
 - Similarly, find the value of k if $y = kx - 12$ is a tangent to $x = 6t$, $y = 3t^2$.

10. (a) Show that the endpoints of the latus rectum of the parabola $x = 2at$, $y = at^2$ have parameters $t = 1$ and $t = -1$.
 (b) Hence find the normals to $x^2 = 4ay$ at the endpoints of the latus rectum.
 (c) Show that the normals intersect the curve again when $x = 6a$ and $x = -6a$, and hence that the interval between these points of intersection has length $12a$.
11. (a) Show that the normal to $x^2 = 16y$ at the point $P(8p, 4p^2)$ on the parabola has equation $x + py = 8x + 4p^3$.
 (b) By substituting $A(0, 44)$ into the normal, show that the normals at three points on the parabola pass through A , and find their coordinates.
12. P and Q are the points $t = p$ and $t = q$ on the parabola $x = 2at$, $y = at^2$.
 (a) Find the equations of the normals to the curve at P and Q .
 (b) Prove that $p^3 - q^3 = (p - q)(p^2 + pq + q^2)$.
 (c) Show that the normals intersect at the point $(-apq(p + q), a(p^2 + q^2 + pq + 2))$.
 (d) If $pq = 2$, show that the normals intersect on the parabola.
13. P and Q are the points $t = p$ and $t = q$ on the parabola $x = 2at$, $y = at^2$.
 (a) Find the equations of the tangents to the curve at P and Q and show that they intersect at $R(a(p + q), apq)$.
 (b) Find the equations of the normals at P and Q , and show that they intersect at $U(-apq(p + q), a(p^2 + q^2 + pq + 2))$.
 (c) If PQ is a focal chord, show that the interval RU is parallel to the axis of the parabola.
14. (a) Find the equation of the tangent to $x^2 = 4ay$ at $P(2ap, ap^2)$, and find the point A where the tangent intersects the y -axis.
 (b) Find the equation of the normal at P , and the point B where the normal intersects the y -axis.
 (c) If S is the focus and C is the foot of the perpendicular from P to the axis of the parabola, show that: (i) $AS = SB$ (ii) $CB = 2a$ (iii) $AO = CO$
15. A line is drawn parallel to the axis of the parabola $x^2 = 4ay$, cutting the parabola at $P(2ap, ap^2)$ and the directrix at R . (a) State the coordinates of R . (b) Show that the normal at P is parallel to RS , where S is the focus.
16. The points P and Q on the parabola $x = 2at$, $y = at^2$ have parameters p and q respectively.
 (a) Find the midpoint M of the chord PQ , the point T on the parabola where the tangent is parallel to PQ , and the point I where the tangents at P and Q intersect.
 (b) Show that M , T and I lie in a vertical line, with T the midpoint of MI .
 (c) Show that the tangent at T bisects the tangents PI and QI .
 (d) What is the ratio of the areas of $\triangle PQI$ and $\triangle PQT$?
17. It was proven in the notes above that the tangents to the parabola $x^2 = 4ay$ at two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola intersect at the point $M(a(p + q), apq)$. Explain why this result can be restated as follows: 'The tangents at two points on the parabola $x^2 = 4ay$ meet at a point whose x -coordinate is the arithmetic mean of the x -coordinates of the points, and whose y -coordinate is one of the geometric means of the y -coordinates of the points.' Which geometric mean is it?

EXTENSION

18. Show that the common chord of any two circles having focal chords of $x^2 = 4ay$ as diameters passes through the vertex of the parabola. (You may assume that tangents at the extremities of a focal chord are perpendicular.)

19. (a) The tangents to $x^2 = 4ay$ at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meet at M . Show that the product of the distance PQ and the perpendicular distance from M to PQ is $a^2|p-q|^3$.
 (b) Hence find the area of $\triangle MPQ$.
 (c) Let T be the point on the parabola where the tangent is parallel to the chord PQ . Show that $\triangle TPQ$ has half the area of $\triangle MPQ$.

9G Tangents and Normals: Cartesian Approach

The equations of tangents and normals can also be approached without reference to parameters, by simply using the standard theory, developed in Chapter Seven, of finding equations of tangents through differentiation of the equation.

The Cartesian Equation of the Tangent: Suppose then that $P(x_1, y_1)$ is any point on the parabola $x^2 = 4ay$.

Solving for y ,

$$y = \frac{x^2}{4a}$$

and differentiating,

$$\frac{dy}{dx} = \frac{x}{2a}$$

so gradient at $P = \frac{x_1}{2a}$.

Hence the tangent is

$$y - y_1 = \frac{x_1}{2a}(x - x_1)$$

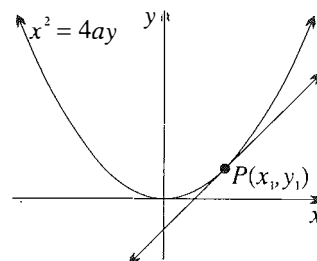
$$2ay - 2ay_1 = xx_1 - x_1^2.$$

Since P lies on the parabola, $x_1^2 = 4ay_1$ (this is a subtle point),

and so

$$2ay - 2ay_1 = xx_1 - 4ay_1$$

$$2a(y + y_1) = xx_1.$$



16

CARTESIAN EQUATION OF THE TANGENT:

The tangent at $P(x_1, y_1)$ is $xx_1 = 2a(y + y_1)$.

NOTE: The relationship between this formula and the equation $x^2 = 4ay$ is quite striking. Notice how the degree 2 term x^2 has been split multiplicatively into $x \times x_1$, and the degree 1 term $4ay$ has been split additively into $2ay + 2ay_1$. It is a general result that the Cartesian equation of the tangent to any second-degree curve can be written down following this procedure. See the last question in Exercise 9G for a clear statement and proof.

The Cartesian Equation of the Normal: Suppose again that the point $P(x_1, y_1)$ lies on the parabola $x^2 = 4ay$. Using the formula for perpendicular gradients,

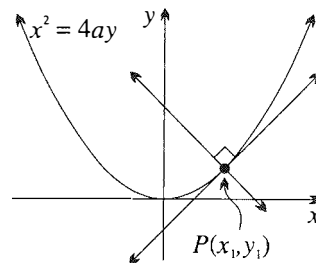
gradient of normal at $P = -\frac{2a}{x_1}$,

so the normal is

$$y - y_1 = -\frac{2a}{x_1}(x - x_1)$$

$$x_1y - x_1y_1 = -2ax + 2ax_1$$

$$x_1y + 2ax = x_1y_1 + 2ax_1.$$



17

CARTESIAN EQUATION OF THE NORMAL:The normal at $P(x_1, y_1)$ is $x_1y + 2ax = x_1y_1 + 2ax_1$.

Algebraic Approaches to the Tangents: Calculus is not necessary for parabolas, and the following worked example shows how to find tangents using the discriminant.

WORKED EXERCISE: Write down the general form of a line with x -intercept 2, and hence use the discriminant to find the tangents to $x^2 = -6y$ with x -intercept 2.

SOLUTION: A line with gradient m and x -intercept 2 has equation $y = m(x - 2)$.

Substituting into $x^2 = -6y$, $x^2 = -6m(x - 2)$

$$x^2 + 6mx - 12m = 0$$

$$\Delta = 36m^2 + 48m$$

$$= 12m(3m + 4).$$

So $\Delta = 0$ when $m = 0$ or $m = -\frac{4}{3}$, and the required tangents are

$$y = 0 \text{ and } y = -\frac{4}{3}(x - 2).$$

Exercise 9G

- Use the derivative to find the equations of the following tangents:
 - at the point $(2, 1)$ on $x^2 = 4y$
 - at the point $(-1, 8)$ on $y = x^2 - 2x + 5$
 - at the point $(3, 3)$ on $x^2 = 3y$
 - at the point $(2, 1)$ on $y = 2x^2 - 4x + 1$
 Then use the formula $xx_1 = 2a(y + y_1)$ to obtain the tangents in parts (a) and (b).
 - Use the derivative to find the equations of the following normals:
 - at the point $(1, 1)$ on $x^2 = y$
 - at the point $(-3, -7)$ on $y = x^2 + 3x - 7$
 - at the point $(-6, 9)$ on $x^2 = 4y$
 - at the point $(0, 3)$ on $y = (2x + 1)(x + 3)$
 Then use the formula $x_1y + 2ax = x_1(2a + y_1)$ to obtain the normals in parts (a) and (b).
 - Show that $y = 3x - 9$ is a tangent to the parabola $x^2 = 4y$ by solving the two equations simultaneously and showing that there is exactly one solution. What is the point of contact?
 - Use a similar method to show that $8x + 4y - 27 = 0$ is a tangent to the parabola $y = x^2 - 3x + 7$, and find the point of contact.
 - Find the equations of the normals to $x^2 = 4y$ at the points where $x = 2$ and $x = 4$.
 - Find the point of intersection of the normals.
 - Show that the endpoints of the latus rectum of the parabola $x^2 = 4ay$ are $A(-2a, a)$ and $B(2a, a)$.
 - Find the equations of the tangents and normals to $x^2 = 4ay$ at these endpoints.
 - Show that these tangents and normals form a square, and find its area.
- DEVELOPMENT**
- Find the equation of the parabola which is symmetrical about the y -axis and passes through the points $(1, 1)$ and $(-2, 2)$. [HINT: It will have the form $y = ax^2 + c$.]
 - Find the tangent and the normal at the point $(1, 1)$.
 - Find where the line $y = 3x + 4$ intersects the parabola $2y = 5x^2$.
 - Find the equations of the tangents to the parabola at the points of intersection.
 - Find the point of intersection of the tangents.

8. (a) Using the formula $xx_0 = 2a(y + y_0)$ for the equation of the tangent to $x^2 = 4ay$ at the point (x_0, y_0) on the curve, show that the tangents at $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the curve intersect at $M\left(\frac{2a(y_2 - y_1)}{x_2 - x_1}, \frac{x_1y_2 - x_2y_1}{x_2 - x_1}\right)$.
- (b) Use the identities $x_1^2 = 4ay_1$ and $x_2^2 = 4ay_2$ to show that M is $\left(\frac{x_1 + x_2}{2}, \frac{x_1x_2}{4a}\right)$.
- (c) Check that the answer to the previous question agrees with this result.
- (d) By substituting $x_1 = 2ap$ and $x_2 = 2aq$, deduce that the tangents to $x^2 = 4ay$ at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meet at $(a(p + q), apq)$.

NOTE: Although approaches using calculus are usually more straightforward, tangents to parabolas can be found using purely algebraic methods based on the discriminant. The remaining questions in the exercise use these methods.

9. (a) Substitute the line $y = mx - 2$ into the parabola $x^2 = 2y$, and show that the resulting quadratic in x has discriminant $\Delta = 4m^2 - 16$.
- (b) Hence find the two tangents to $x^2 = 2y$ with y -intercept -2 .
- (c) Use a similar method to find the two tangents to $x^2 = 8y$ with y -intercept -2 , and show that they are perpendicular.
10. (a) Find the value of b for which $y = -2x + b$ will be a tangent to $x^2 = 6y$.
- (b) Hence write down the tangent to $x^2 = 6y$ with gradient -2 .
- (c) Using a similar method, find the equations of the tangents:
- (i) to $x^2 = 4y$ parallel to $y = 1 - 2x$, (ii) to $x^2 = 9y$ perpendicular to $y = 1 - 2x$.
- (d) Repeat part (c) using the derivative to find the x -coordinate of the point of contact.
11. If $y = 1 - 2x$ is a tangent to $x^2 = 4ay$, find a and the point of contact.
12. (a) Use the discriminant to show that $y = mx + b$ is a tangent to the parabola $\mathcal{P}: x^2 = 4ay$ when $16a(am^2 + b) = 0$. Hence show that $y = mx - am^2$ is tangent to $\mathcal{P}$ for all m .
- (b) Hence write down the tangent to $x^2 = 12y$ parallel to $y = 7x$.
13. (a) Use the discriminant to show that $mx - y + m^2 = 0$ touches the parabola $x^2 = -4y$, for all values of m .
- (b) Hence find the equations of the tangents to $x^2 = -4y$ through the point $A(1, 2)$.
14. Let $\ell: y + 2 = m(x - 6)$ be a line with gradient m through $A(6, -2)$.
- (a) Show that ℓ is a tangent to the parabola $\mathcal{P}: x^2 = 8y$ when $m^2 - 3m - 1 = 0$.
- (b) Without solving this quadratic in m , show that there are two tangents from A to the parabola, and that they are perpendicular.
15. (a) Show that $\ell: ax + by = 1$ is a tangent to $\mathcal{P}: x^2 = 12y$ when $3a^2 + b = 0$.
- (b) Hence find the tangents to $\mathcal{P}$ with y -intercept -27 .
- (c) Show that if ℓ passes through $U(4, 1)$, then $4a + b = 1$. Hence find the tangents to $\mathcal{P}$ through U .
16. [Using the discriminant to derive the general equation of the tangent] Suppose that $P(2ap, ap^2)$ is any point on the parabola $\mathcal{P}: x^2 = 4ay$. Let $\ell: y - ap^2 = m(x - 2ap)$ be a line with any gradient m through P .
- (a) Show that solving the line ℓ and the parabola $\mathcal{P}: x^2 = 4ay$ simultaneously yields the quadratic equation $x^2 - 4amx + (8a^2mp - 4a^2p^2) = 0$.
- (b) Show that the discriminant of this quadratic is $\Delta = 16a^2(m - p)^2$.
- (c) Hence show that the line and the parabola touch when $m = p$, and that the equation of the tangent at P is $y = px - ap^2$.

17. [An alternative algebraic approach] Using the pronumerals of the previous question:
- Show that substituting the parabola $x = 2at$, $y = at^2$ into the line ℓ yields the quadratic equation $t^2 - 2mt + (2mp - p^2) = 0$ in t .
 - Show that the discriminant of this quadratic is $\Delta = 4(m - p)^2$, and hence that the line and the parabola touch when $m = p$.
18. [An algebraic approach without parameters] Suppose that $P(x_0, y_0)$ is any point on the parabola $\mathcal{P}: x^2 = 4ay$. Let $\ell: y - y_0 = m(x - x_0)$ be a line with gradient m through P .
- Show that solving the line ℓ and the parabola $\mathcal{P}: x^2 = 4ay$ simultaneously, and substituting $y_0 = \frac{x_0^2}{4a}$, yields the equation $x^2 - 4amx + (4amx_0 - x_0^2) = 0$.
 - Show that the discriminant of this quadratic is $\Delta = 4(x_0 - 2am)^2$, and hence show that the equation of the tangent to the parabola at P is $xx_0 = 2a(y + y_0)$.

EXTENSION

19. Show that the tangent at the point $P(x_0, y_0)$ on the general degree 2 curve

$$ax^2 + by^2 + 2cxy + 2dx + 2ey + f = 0$$

$$\text{is } axx_0 + byy_0 + c(x_0y + xy_0) + d(x + x_0) + e(y + y_0) + f = 0.$$

9H The Chord of Contact

Establishing the equation of the chord of contact is the principal reason why the Cartesian equation of the tangent was introduced in the last section. The resulting relationships between tangents and chords go to the heart of the study of second-degree curves.

The Chord of Contact: Suppose that $P(x_0, y_0)$ is a point that lies outside the parabola $x^2 = 4ay$. A glance at the graph below will make it clear that there are two tangents to the parabola from P . These two tangents touch the curve at two points of contact, which we call here A and B . The chord AB joining these two points of contact is called the *chord of contact from P* .

It is a remarkable fact that this chord of contact has equation $xx_0 = 2a(y + y_0)$, exactly the same equation as the tangent, except that here the point P does not lie on the curve.

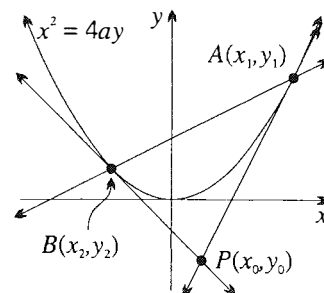
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THE CHORD OF CONTACT: The chord of contact from $P(x_0, y_0)$ is $xx_0 = 2a(y + y_0)$.

PROOF: The proof is very elegant indeed, and involves no calculation whatsoever. Let the points of contact be $A(x_1, y_1)$ and $B(x_2, y_2)$. Then the tangent at A is $xx_1 = 2a(y + y_1)$, and the tangent at B is $xx_2 = 2a(y + y_2)$. Since $P(x_0, y_0)$ lies on the tangent at A ,

$$x_0x_1 = 2a(y_0 + y_1),$$
and since P lies on the tangent at B ,

$$x_0x_2 = 2a(y_0 + y_2).$$



But the first identity shows that $A(x_1, y_1)$ lies on $xx_0 = 2a(y + y_0)$,
 and the second identity shows that $B(x_2, y_2)$ lies on $xx_0 = 2a(y + y_0)$.
 Since both A and B lie on $xx_0 = 2a(y + y_0)$, this equation must be the line AB .

Using the Chord of Contact to Find the Points of Contact: Given a point $P(x_1, y_1)$ outside a parabola, the two points of contact of the tangents from P can be found by finding the chord of contact from P , and then solving simultaneously the parabola and the chord of contact from P .

WORKED EXERCISE:

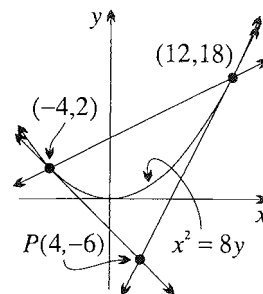
- (a) Given the parabola $x^2 = 8y$, find the chord of contact from $P(4, -6)$.
 (b) Hence find the two points of contact of the tangents from $P(4, -6)$.

SOLUTION:

- (a) Here $4a = 8$, so $a = 2$,
 so the chord of contact is $xx_0 = 4(y + y_0)$
 $4x = 4(y - 6)$
 $y = x + 6$.

- (b) Solving simultaneously with the parabola $x^2 = 8y$,
 $x^2 = 8(x + 6)$
 $x^2 - 8x - 48 = 0$
 $(x - 12)(x + 4) = 0$
 $x = 12$ or $x = -4$.

So the points of contact are $(12, 18)$ and $(-4, 2)$.



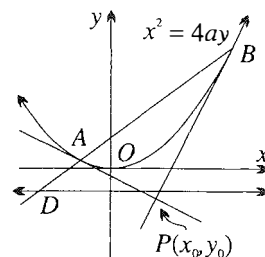
Exercise 9H

- Find the equation of the chord of contact of $x^2 = 4y$ from each point:
 - $(0, -2)$
 - $(3, 0)$
 - $(-2, -1)$
 - $(4, -6)$
- Find the equation of the chord of contact of $x = 6t, y = 3t^2$ from each point:
 - $(0, -3)$
 - $(-2, 0)$
 - $(6, 1)$
 - $(-5, -4)$
- The point $P(2, 0)$ lies outside the parabola $x^2 = 8y$.
 - Find the equation of the chord of contact from P .
 - Find the points of intersection of the chord of contact and the parabola.
 - Find the equations of the two tangents.
- Each point $A(1, -2)$, $B(3, -2)$ and $C(-4, -2)$ lies on the directrix of the parabola $x^2 = 8y$.
 - Write down the coordinates of the focus of the parabola.
 - Find the equations of the chords of contact from A , B and C , and show that each chord is a focal chord.
- Write down the equation of the chord of contact of the parabola $y^2 = 4ax$ from the external point (x_0, y_0) .
 - Show by substitution that the chord of contact from the point $(-5, 2)$ to the parabola $y^2 = 20x$ is a focal chord. Why is it so?

6. Tangents are drawn from the point $T(2, -1)$ to the parabola $x^2 = 4y$. P and Q are the points of contact of the tangents.
- Find the equation of the chord PQ .
 - Show that the x -coordinates of P and Q are the roots of the quadratic $x^2 - 4x - 4 = 0$.
 - Find the sum of the roots of the equation in part (b).
 - Hence find the midpoint M of the chord PQ , and show that TM is parallel to the axis of the parabola.

DEVELOPMENT

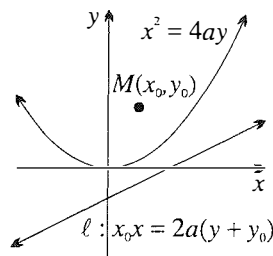
7. $P(x_1, -a)$ is any point on the directrix of the parabola $x^2 = 4ay$.
- Show that the chord of contact from P has equation $x_1x = 2a(y - a)$.
 - Hence show that the chord of contact passes through the focus of the parabola.
8. (a) Find the equation of the chord of contact of the parabola $x^2 = 8y$ from:
- $P(2, 0)$
 - $Q(1, -1)$
- (b) If the chords in part (a) intersect at R , show that the line PQ is a tangent to the parabola and that its point of contact is R .
9. (a) Write down the equation of the chord of contact of the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$.
- Suppose that the points of contact of the tangents are A and B . Find a quadratic equation whose roots are the x -coordinates of A and B .
 - Hence find the coordinates of M , the midpoint of the chord AB .
 - Show that PM is parallel to the axis of the parabola.
 - Show that the midpoint N of PM lies on the parabola.
10. AB is the chord of contact of the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$. The line AB meets the directrix of the parabola at D .
- Write down the equation of AB .
 - Show that D has coordinates $\left(\frac{2a(y_0 - a)}{x_0}, -a\right)$.
 - Prove that PD subtends a right angle at the focus.
11. (a) Write down the equation of the chord of contact of the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$, then write it in gradient-intercept form $y = mx + b$.
- Let this chord meet the axis of the parabola at T , and let the line through P parallel to the axis meet the parabola at N . Use part (a) to show that:
 - the points P and T are equidistant from the tangent at the origin,
 - the chord is parallel to the tangent to the parabola at N .
12. Tangents are drawn to the parabola $y = x^2$ from the point $T(1, -1)$. These tangents touch the parabola at P and Q .
- Obtain a quadratic equation whose roots are the x -coordinates of P and Q , and write down the sum and the product of these roots.
 - Find a quadratic equation whose roots are the y -coordinates of P and Q , and write down the sum and the product of these roots.
 - Prove the identity $(p - q)^2 = (p + q)^2 - 4pq$.
 - Use the distance formula and this identity to find the length of the chord PQ .



13. Repeat the previous question for the parabola $x^2 = 2y$ and $T(2, -3)$.
14. [An alternative derivation of the equation of the chord of contact] Let $T(2at, at^2)$ be any point on the parabola $x^2 = 4ay$.
- Show that the tangent at T has equation $y = tx - at^2$.
 - If this tangent passes through the point $P(x_0, y_0)$, show that $at^2 - x_0t + y_0 = 0$.
 - What is the condition for this quadratic equation in t to have two real roots? Interpret this result geometrically.
 - Suppose that t_1 and t_2 are the roots of the quadratic equation, and let T_1 and T_2 be the points on the parabola corresponding to $t = t_1$ and $t = t_2$ respectively.
 - Show that the chord T_1T_2 has equation $(t_1 + t_2)x = 2y + 2at_1t_2$.
 - Show that $t_1 + t_2 = \frac{x_0}{a}$ and $t_1t_2 = \frac{y_0}{a}$.
 - Hence show that the chord T_1T_2 has equation $x_0x = 2a(y + y_0)$.
15. $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie outside the parabola $x^2 = 4ay$.
- Write down the equation of the chord of contact from P_1 .
 - If the line containing the chord of contact from P_1 passes through P_2 , show that the line containing the chord of contact from P_2 passes through P_1 .
16. Find the condition that must be satisfied if the chord of contact of the parabola $x^2 = 4ay$ from the point (x_0, y_0) is parallel to the line $y = mx + b$.

EXTENSION

17. The parabola $\mathcal{P}$ has equation $x^2 = 4ay$. The point $M(x_0, y_0)$ lies inside $\mathcal{P}$, so that $x_0^2 < 4ay_0$. The line ℓ has equation $x_0x = 2a(y + y_0)$.
- Show that $x_0x_1 \leq \frac{x_0^2 + x_1^2}{2}$, for all real x_0 .
 - Prove that the line ℓ lies entirely outside $\mathcal{P}$. That is, show that if $P(x_1, y_1)$ is any point on ℓ , then $x_1^2 > 4ay_1$. (Use the result in part (a).)
 - The chord of contact from any point $Q(x_2, y_2)$ outside $\mathcal{P}$ has equation $x_2x = 2a(y + y_2)$. Prove that M lies on the chord of contact from any point on ℓ .
18. (a) Use implicit differentiation to show that the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $P(x_1, y_1)$ on the circle is $xx_1 + yy_1 = a^2$.
- (b) Use the methods of this section to prove that if $P(x_0, y_0)$ is a point outside the circle, then $xx_0 + yy_0 = a^2$ is the equation of the chord of contact from P .
- (c) Hence prove that the product of the distances from the centre O to the point P and to the chord of the contact is the square of the radius.
- (d) Find the equation of the chord of contact of the circle $x^2 + y^2 = 25$ from the external point $P(4, 5)$. Then solve the circle and the chord simultaneously to find the points of contact of the tangents from P .
19. (a) Using similar methods, find the equation of the chord of contact to the hyperbola $xy = c^2$ from a point $P(x_0, y_0)$.
- (b) Show that the product of the distances from O to the point P and to the chord of contact is the constant $2c^2$.
- (c) Find the equation of the chord of contact of $xy = 25$ from the point $P(2, 8)$. Then solve the curve and the chord simultaneously to find the points of contact of the tangents from P .



9I Geometrical Theorems about the Parabola

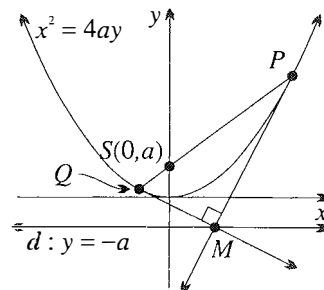
This section is concerned with establishing purely geometric properties of the parabola, that is, properties of the parabola that do not depend on the particular way in which the parabola has been tied to the coordinate system of the plane. The machinery of coordinate geometry becomes here only a convenient set of tools for proving the theorems, but the whole process illustrates well how coordinate geometry has been able to create a unity between geometry and algebra.

Two Geometric Characterisations of Focal Chords: We have already shown that a chord PQ is a focal chord if and only if $pq = -1$. But p and q are the gradients of the tangents at P and Q , so the tangents at P and Q are perpendicular if and only if $pq = -1$, hence:

- 19 FOCAL CHORDS AND PERPENDICULAR TANGENTS:** A chord that joins two points on a parabola is a focal chord if and only if the tangents at the endpoints of the chord are perpendicular.

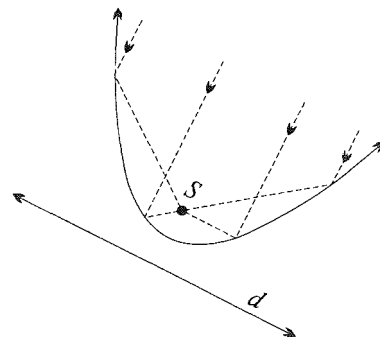
Secondly, the directrix is the line $y = -a$. Now we have seen that the tangents at P and Q meet at the point $M(a(p+q), apq)$, so the condition $pq = -1$ is the same as saying that the intersection M lies on the directrix. Putting all this together:

- 20 FOCAL CHORDS AND INTERSECTION OF TANGENTS:** A chord joining two points on a parabola is a focal chord if and only if the tangents at the endpoints of the chord meet on the directrix.



NOTE: It should be stressed here that these two theorems are purely geometric. Although five pronumerals p , q , a , x and y were used in the proof, they have no place in the final statement of either theorem.

The Reflection Property of the Parabola: Parabolic bowls that are silvered on the inside have a most useful function in focusing light. When light travelling parallel to the axis falls on the bowl, the mirrored surface focuses it at the focus — hence the name focus for that point. Conversely, if a source of light is placed at the focus of the parabola, then it will be reflected from the bowl in a direction parallel to the axis.



Proving this requires the fairly obvious fact from physics that light is reflected from a surface so that the angle between the incident ray and the tangent at the point equals the angle between the reflected ray and the tangent (in physics one usually measures the angles with the normal, but the angle with the tangent is easier to handle in our case). Writing all this geometrically, the necessary theorem is as follows:

- 21 THE REFLECTION PROPERTY:** The interval joining a point on a parabola to the focus, and the line through the point parallel to the axis, are equally inclined to the tangent at the point.

PROOF:

A. Label the diagram as shown, with the tangent at P meeting axis at K ,
and let $\theta = \angle SKP$.

Then $\angle QPB = \theta$ (corresponding angles, $KS \parallel PQ$).

It will suffice to prove that the intervals SK and SP are equal,

because then $\angle SPK = \theta$ (base angles of isosceles $\triangle SPK$),

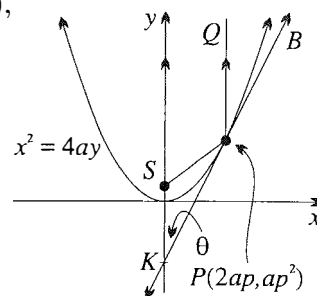
and so $\angle SPK = \angle SKP$, as required.

$$\begin{aligned} \text{B. By the distance formula, } SP^2 &= (2ap - 0)^2 + (ap^2 - a)^2 \\ &= a^2(4p^2 + (p^2 - 1)^2) \\ &= a^2(p^2 + 1)^2. \end{aligned}$$

Also, the tangent $y = px - ap^2$ has y -intercept $-ap^2$,

$$\begin{aligned} \text{so } K &= (0, -ap^2) \text{ and } SK^2 = (a + ap^2)^2 \\ &= a^2(1 + p^2)^2. \end{aligned}$$

Hence $SP = SK$, and the result is proven.



Exercise 9I

NOTE: This exercise and the next are the culmination of the work on the parabola and its properties. The large number of questions is intended to be sufficient for later revision.

1. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two variable points on the parabola $x^2 = 4ay$. M is the midpoint of the chord PQ , and T is the point of intersection of the tangents at P and Q .

 - (a) Show that the tangent at P has equation $y = px - ap^2$ and write down the equation of the tangent at Q .
 - (b) Show that M has coordinates $(a(p+q), \frac{1}{2}a(p^2+q^2))$.
 - (c) Show that T has coordinates $(a(p+q), apq)$.
 - (d) Show that MT is parallel to the axis of the parabola.
 - (e) Find the coordinates of the midpoint N of MT , and show that it lies on the parabola.
2. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on the parabola $x^2 = 4ay$, and PQ is a focal chord. The tangents at P and Q meet at T .

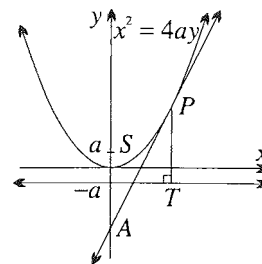
 - (a) Show that the chord PQ has equation $y = \frac{1}{2}(p+q)x - apq$. (b) Show that $pq = -1$.
 - (c) Show that the tangent at P has gradient p , and state the gradient of the tangent at Q .
 - (d) Show that T is the point $(a(p+q), apq)$. (e) Show that the tangents at P and Q are perpendicular and that they meet on the directrix.
3. $P(2at, at^2)$, where $t \neq 0$, is a variable point on the parabola $x^2 = 4ay$. The normal at P meets the axis of the parabola at N , and PB is the perpendicular from P to the axis of the parabola. The interval BN is called the *subnormal* corresponding to P .

 - (a) Show that the normal at P has equation $x + ty = 2at + at^3$.
 - (b) Write down the coordinates of B and N . (c) Hence prove that the length of the subnormal is constant, that is, independent of where P is on the parabola.
4. $P(at^2, 2at)$ is an arbitrary point on the parabola $y^2 = 4ax$ with focus $S(a, 0)$.

 - (a) Show that the tangent at P has equation $x = ty - at^2$.
 - (b) Show that the tangent at P meets the directrix at the point $Q(-a, a(t - \frac{1}{t}))$.
 - (c) Hence prove that $\angle PSQ = 90^\circ$.

5. $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. S is the focus, T is the foot of the perpendicular from P to the directrix, and A is the point where the tangent at P meets the y -axis.

- Write down the coordinates of T .
- Show that A is the point $(0, -at^2)$.
- Show that PA and ST bisect each other, by finding their midpoints.
- Show that PA and ST are perpendicular to each other.
- What type of quadrilateral is $SPTA$?



6. $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P meets the x -axis at A and the y -axis at B .

- Find the coordinates of A and B .
- If $C(c, d)$ is the fourth vertex of the rectangle $BOAC$, where O is the origin, show that $c = td$.

DEVELOPMENT

7. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ vary on the parabola $x^2 = 4ay$. The tangents at P and Q meet at right angles at T .

- Show that $pq = -1$. What does this result tell us about the chord PQ ?
- Show that the tangent at P has equation $y = px - ap^2$, and write down the equation of the tangent at Q .
- Show that T has coordinates $(a(p + q), apq)$.
- Find the gradient of the chord PQ , and hence show that the line through T perpendicular to the chord PQ has equation $y = -\frac{2}{p+q}x + a$.
- Show that the line in part (d) meets the chord PQ at the focus of the parabola.

8. $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $T(2at, at^2)$ are variable points on the parabola $x^2 = 4ay$.

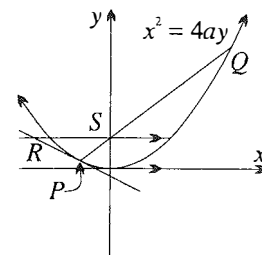
- Show that the chord PQ has equation $y - \frac{1}{2}(p + q)x + apq = 0$.
- Find the equation of the tangent at T .
- The tangent at T cuts the axis of the parabola at R . Find the coordinates of R . If the chord PQ , when extended, passes through R , show that p , t and q form a geometric progression.

9. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $\mathcal{P}$ whose equation is $x^2 = 4ay$.

- Find the point of intersection A of the tangents to $\mathcal{P}$ at P and Q . (You may use the fact that the tangent to $\mathcal{P}$ at any point $T(2at, at^2)$ on $\mathcal{P}$ has equation $y = tx - at^2$.)
- Suppose further that A lies on the line containing the latus rectum of $\mathcal{P}$.
 - Show that $pq = 1$.
 - Show that the chord PQ intersects the axis of symmetry of $\mathcal{P}$ on the directrix.

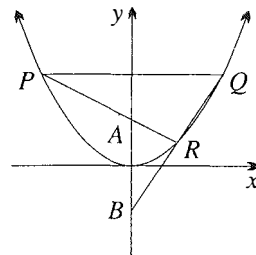
10. P and Q are the points $t = p$ and $t = q$ respectively on the parabola $x = 2at$, $y = at^2$ with focus S . PQ is a focal chord and the tangent at P meets the latus rectum produced at R .

- Show that $pq = -1$. (b) Show that $SP = a(p^2 + 1)$.
- Show that R has coordinates $\left(\frac{a}{p}(p^2 + 1), a\right)$.
- Hence show that $SR^2 = SP \times SQ$.



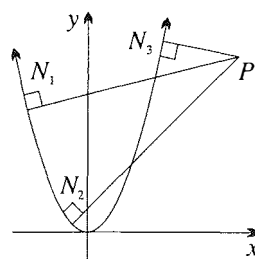
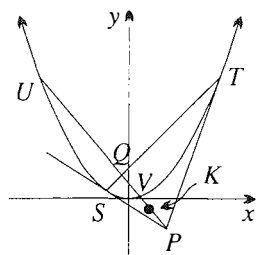
11. [In this question we prove the reflection property of the parabola.] $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, and $t \neq 0$. S is the focus and T is the point where the tangent to the parabola at P meets the axis of the parabola.
- Show that the tangent at P has equation $y = tx - at^2$.
 - Show that $SP = ST$.
 - Hence show that $\angle SPT$ is equal to the acute angle between the tangent and the line through P parallel to the axis of the parabola.
12. [An alternative proof of the reflection property] P is the variable point (x_1, y_1) on the parabola $x^2 = 4ay$, and S is the focus. The tangent at P meets the tangent at the vertex of the parabola at Q and it meets the axis of the parabola at R .
- Explain why $x_1^2 = 4ay_1$. (b) Show that the tangent at P is $x_1x = 2a(y + y_1)$.
 - Find the coordinates of Q and R .
 - Show that $SQ \perp PQ$, and that the tangent at the vertex bisects PR .
 - Hence, using congruent triangles, show that the tangent at P is equally inclined to the axis of the parabola and the focal chord through P .
13. P is a point on a parabola, and ℓ is the axis of symmetry of the parabola. The tangent and normal to the parabola at P meet ℓ at T and N respectively. Prove that P , T and N all lie on a circle whose centre is at the focus of the parabola.
14. The normal at the point $P(2at, at^2)$ on the parabola $x^2 = 4ay$ intersects the y -axis at Q . A is the point $(0, -a)$ and S is the focus. The midpoint of QS is R .
- Show that R has coordinates $(0, \frac{1}{2}a(t^2 + 3))$. (b) Show that $AR^2 - RP^2 = 4a^2$.
15. $P(2ap, ap^2)$ is any point on the parabola $x^2 = 4ay$ other than its vertex. The normal at P meets the parabola again at Q .
- Show that the normal at P cannot pass through the focus of the parabola.
 - Show that the x -coordinate of Q is one of the roots of the quadratic equation $px^2 + 4ax - 4a^2p(2 + p^2) = 0$. Then find the coordinates of Q .
16. $P(2p, p^2)$ is a variable point on the parabola $x^2 = 4y$, whose focus is S . The normal at P meets the y -axis at N and M is the midpoint of PN .
- Find the coordinates of M , and show that SM is parallel to the tangent at P .
 - Suppose that $\triangle SNP$ is equilateral. Find the coordinates of P .
17. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on the parabola $x^2 = 4ay$. R is the intersection of the tangent at P and the line through Q parallel to the axis of the parabola, while U is the intersection of the tangent at Q and the line through P parallel to the axis.
- Show that $PQUR$ is a parallelogram.
 - If $p > q$, show that parallelogram $PQUR$ has area $2a^2(p - q)^3$ units².
18. The points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x = 2t, y = t^2$. S is the focus.
- Show that $PS = p^2 + 1$, by using the fact that any point on a parabola is equidistant from the focus and the directrix.
 - Hence find an expression for $PS + QS$ in terms of p and q .
 - If PQ is a focal chord, show that $PQ = \left(p + \frac{1}{p}\right)^2$.

19. P is a variable point on the parabola $x^2 = 4y$. The normal at P meets the parabola again at Q . The tangents at P and Q meet at T . S is the focus and $QS = 2PS$.
- (a) Prove that $\angle PSQ = 90^\circ$.
- (b) Prove that $PQ = PT$.
20. The points $P(p, \frac{1}{2}p^2)$ and $Q(q, \frac{1}{2}q^2)$ vary on the parabola $2y = x^2$ with focus S . The line PQ passes through the point where the directrix meets the axis of symmetry.
- (a) Show that $pq = 1$.
- (b) Show that $PS \times QS = \frac{1}{2}(PS + QS)$.
21. The diagram shows a parabola. PQ is any chord parallel to the directrix. R is a third point on the parabola, and the lines RP and RQ cut the axis of the parabola at A and B respectively. Show that the interval AB is bisected by the vertex of the parabola.
22. $P(2p, p^2)$ and $Q(2q, q^2)$, where $p \neq q$, are variable points on the parabola $x^2 = 4y$. You may assume that the chord PQ has equation $(p+q)x - 2y - 2pq = 0$, and that the tangents at P and Q meet at the point $T(p+q, pq)$.
- (a) Show that, for each non-zero value of p , there are two values of q for which T lies on the parabola $x^2 = -4y$, and find these values in terms of p .
- (b) For each value of q , show that PQ produced is a tangent to the parabola $x^2 = -4y$.



EXTENSION

23. In the diagram the point $P(x_0, y_0)$ lies outside the parabola $x^2 = 4ay$ (which means that $x_0^2 > 4ay_0$.) The two tangents to the parabola from P touch the parabola at S and T .
- (a) Suppose that $Q(x_1, y_1)$ is any point between S and T on the chord of contact ST . Suppose that k is any real number, and let K be the point on PQ dividing the interval PQ in the ratio $k : 1$. Write down the coordinates of K in terms of x_0, x_1, y_0, y_1 and k .
- (b) Find the condition that K should lie on the parabola, and rearrange this condition as a quadratic equation in k .
- (c) Hence show that the two points U and V where the line PQ meets the parabola divide the interval PQ internally and externally in the same ratio.
24. The diagram shows the parabola $y = x^2$. P is a point that lies on three distinct normals (PN_1, PN_2 and PN_3) to the parabola.
- (a) Show that the equation of the normal to $y = x^2$ at the variable point (t, t^2) is $t^3 + \left(\frac{1-2y}{2}\right)t - \frac{x}{2} = 0$.
- (b) If the function $f(t) = t^3 + ct + d$ has three distinct real zeroes, prove that $27d^2 + 4c^3 < 0$. [HINT: The graph of $f(t)$ must have two points where the tangent is horizontal, one above the t -axis and one below.]
- (c) Suppose that the normals at three distinct points N_1, N_2 and N_3 on the parabola $y = x^2$ all pass through $P(x_0, y_0)$. Use part (b) to show that $y_0 > 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}$.



9J Locus Problems

In many situations a variable point P on a parabola will determine another point M , so that M moves as the point P moves. The problem then is to find the equation of the path or locus of M , and if possible to describe that locus in geometrical terms.

One-parameter Locus Problems: The usual method is to give the point P its parametric coordinates, and then find the coordinates of M in terms of the parameter p . The formulae for the coordinates of M then form two simultaneous equations, and the parameter p can be eliminated from them.

22

LOCUS PROBLEMS: Write the coordinates of the moving point as two simultaneous equations, then eliminate the parameter.

WORKED EXERCISE: Let A be the endpoint $(2, 1)$ of the latus rectum of the parabola $x^2 = 4y$, and let $P(2p, p^2)$ be any point on the parabola. Find and describe the locus of the midpoint M of PA .

SOLUTION: The coordinates of M are

$$x = p + 1 \quad (1)$$

$$y = \frac{1}{2}(p^2 + 1). \quad (2)$$

From (1),

$$p = x - 1,$$

and substituting into (2), $y = \frac{1}{2}(x^2 - 2x + 1 + 1)$

$$2y = x^2 - 2x + 2.$$

Completing the square, $2y - 1 = x^2 - 2x + 1$

$$(x - 1)^2 = 2(y - \frac{1}{2}),$$

so the locus is a parabola with vertex $(1, \frac{1}{2})$ and focal length $\frac{1}{2}$.

WORKED EXERCISE: The tangent at a point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the x -axis at A and the y -axis at B . Find and describe the locus of the midpoint M of AB .

SOLUTION: We assume the tangent is $y = px - ap^2$,

so putting $x = 0$, the point B is $B(0, -ap^2)$,

and putting $y = 0$, the point A is $A(ap, 0)$,

so the coordinates of the midpoint M are

$$x = \frac{1}{2}ap \quad (1)$$

$$y = -\frac{1}{2}ap^2. \quad (2)$$

Squaring (1), $x^2 = \frac{1}{4}a^2p^2$

$$x^2 = -\frac{1}{2}ap^2 \times (-\frac{1}{2}a),$$

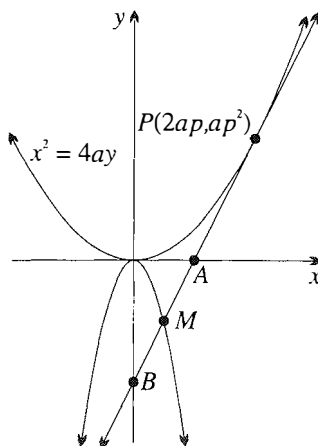
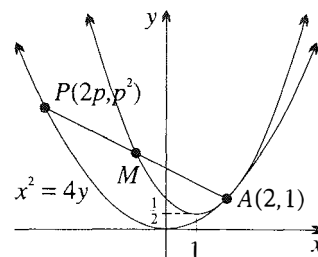
and using (2), $x^2 = -\frac{1}{2}ay$.

This is a parabola facing downwards

with vertex $(0, 0)$ and focal length $\frac{1}{8}a$,

so the focus is $(0, -\frac{1}{8}a)$ and the directrix is $y = \frac{1}{8}a$.

WORKED EXERCISE: Find the locus of the midpoint of the x -intercept and y -intercept of the normal at a variable point on the parabola $y^2 = 4ax$.



SOLUTION: We assume that the normal at $P(ap^2, 2ap)$ is $y + px = 2ap + ap^3$,

so putting $x = 0$, the y -intercept is $(0, 2ap + ap^3)$,

and putting $y = 0$, the x -intercept is $(2a + ap^2, 0)$,

so the coordinates of the midpoint M are

$$x = a + \frac{1}{2}ap^2 \quad (1)$$

$$y = ap + \frac{1}{2}ap^3. \quad (2)$$

$$\text{From (1) and (2), } y = px. \quad (3)$$

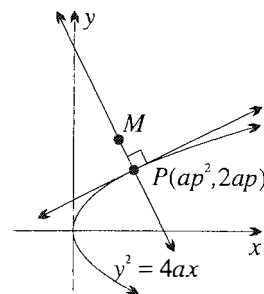
$$\text{From (1), } \frac{1}{2}ap^2 = x - a$$

$$p^2 = \frac{2(x - a)}{a},$$

and substituting this into the square of (3),

$$y^2 = \frac{2x^2(x - a)}{a}$$

$$ay^2 - 2x^3 + 2x^2a = 0.$$



NOTE: This last locus is not a parabola because it involves a term in the cube of x . This is common with locus problems involving the normal, where the algebra of the elimination of the parameters is often more complicated.

Two-parameter Locus Problems: In a more difficult type of problem, the variable point depends on two points with parameters say p and q , but there is a relation between the two parameters. This case produces three simultaneous equations — expressing the coordinates of the variable point in terms of p and q gives two equations, and the relation between p and q is a third equation. From these three equations, both parameters p and q must be eliminated.

23

TWO-PARAMETER LOCUS PROBLEMS: Write the coordinates of the moving point and the relation between the two parameters as three simultaneous equations, then eliminate both parameters.

WORKED EXERCISE: Suppose that PQ is a focal chord of the parabola $x^2 = 4ay$.

- Find and describe the locus of the midpoint M of PQ .
- Find and describe also the locus of the intersection T of the tangents at P and Q , and show that MT is always parallel to the axis.

SOLUTION:

- Let the endpoints of the chord be $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

Then the coordinates of the midpoint M are

$$x = \frac{1}{2}(2ap + 2aq)$$

$$y = \frac{1}{2}(ap^2 + aq^2),$$

$$\text{that is, } x = a(p + q) \quad (1)$$

$$y = \frac{1}{2}a(p^2 + q^2). \quad (2)$$

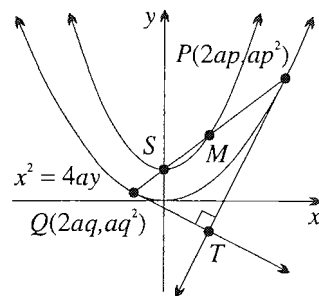
But the parameters p and q are related by

$$pq = -1. \quad (3)$$

$$\text{Squaring (1), } x^2 = a^2(p^2 + q^2 + 2pq),$$

$$\text{and using (2) and (3), } x^2 = a(2y - 2a)$$

$$x^2 = 2a(y - a),$$



so the locus is a parabola with vertex $(0, a)$ and focal length $\frac{1}{2}a$.

Hence the focus is $(0, \frac{3}{2}a)$ and the directrix is $y = \frac{1}{2}a$.

(b) We assume that $(a(p+q), apq)$ is the intersection of the tangents,

$$\text{so } T \text{ has coordinates } x = a(p+q) \quad (4)$$

$$y = apq, \quad (5)$$

$$\text{and again, } pq = -1. \quad (6)$$

Substituting (6) into (5), $y = -a$ (notice that (4) was irrelevant),

so the locus of T is the line $y = -a$, which is the directrix of the parabola.

Since the x -coordinates of M and T are equal, MT is always vertical.

NOTE: We proved in Section 9F that the tangents at the endpoints of a focal chord intersect on the directrix. Part (b) of this locus question has simply proven the same result a different way.

Using Sum and Product of Roots: The previous work in Chapter Eight on sum and product of roots can be very useful in locus questions.

WORKED EXERCISE: Find and describe the locus of the midpoints of the chords cut off a parabola $x^2 = 4y$ by lines parallel to $y = x$. Make clear any restrictions on the locus.

SOLUTION: The family of chords parallel to $y = x$

has equation $y = x + b$, where b can vary.

Substituting the line $y = x + b$ into the parabola $x^2 = 4y$,

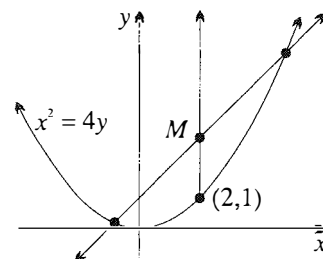
$$x^2 = 4x + 4b$$

$$x^2 - 4x - 4b = 0,$$

so average of roots = 2.

Hence the locus of the midpoints is the vertical line $x = 2$.

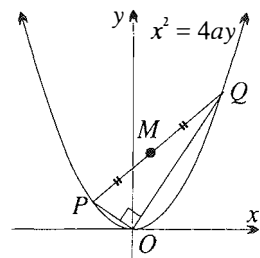
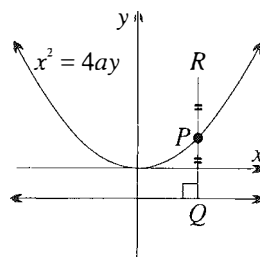
The tangent at the endpoint $(2, 1)$ of the latus rectum is the farthest right a line $y = x + b$ can be, yet still touch the curve, so more carefully stated, the locus is $x = 2$, where $y \geq 1$.



Exercise 9J

- Let $P(2at, at^2)$ be a variable point on the parabola $x^2 = 4ay$. Suppose that M is the midpoint of the interval OP , where O is the origin.
 - Show that M has coordinates $(at, \frac{1}{2}at^2)$.
 - Write down a pair of parametric equations representing the locus of M .
 - Hence show that the locus of M has Cartesian equation $x^2 = 2ay$.
 - Give a geometrical description of this locus.
- The tangent at $P(2at, at^2)$ on the parabola $x^2 = 4ay$ meets the x -axis at T .
 - Show that the tangent has equation $y = tx - at^2$.
 - Find the coordinates of T .
 - Find the coordinates of the midpoint M of PT .
 - Show that, as t varies, the locus of M is the parabola $2x^2 = 9ay$.

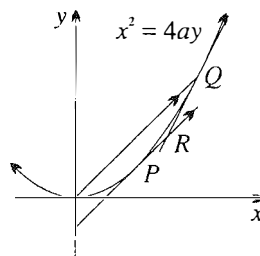
3. P is the variable point $(4t, 2t^2)$ on the parabola $x^2 = 8y$. The normal at P cuts the y -axis at A and R is the midpoint of AP .
- Show that the normal at P has equation $x + ty = 4t + 2t^3$.
 - Show that R has coordinates $(2t, 2t^2 + 2)$.
 - Show that the locus of R is a parabola, and show that the vertex of this parabola is the focus of the original parabola.
4. $P(2at, at^2)$ moves on the curve $x^2 = 4ay$. The tangent at P meets the x -axis at T and the normal at P meets the y -axis at N .
- Show that T and N are the points $(at, 0)$ and $(0, 2a + at^2)$ respectively.
 - Find the coordinates of M , the midpoint of TN .
 - Show that the locus of M is the parabola $x^2 = \frac{1}{2}a(y - a)$.
5. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on $x^2 = 4ay$. $S(0, a)$ is the focus, and M is the midpoint of the chord PQ . The chord PQ passes through S as p and q vary.
- Show that the chord PQ has equation $y = \frac{1}{2}(p + q)x - apq$.
 - Use the fact that the chord PQ passes through S to show that $pq = -1$.
 - Show that $x = a(p + q)$, $y = \frac{1}{2}a(p^2 + q^2)$ are parametric equations of the locus of M .
 - Use the identity $(p + q)^2 = p^2 + q^2 + 2pq$ to show that the Cartesian equation of the locus of M is $x^2 = 2a(y - a)$.
6. $P(2at, at^2)$ lies on the parabola $x^2 = 4ay$. Q is the foot of the perpendicular from P to the directrix of the parabola and the interval QP is extended to R so that $RP = PQ$.
- Write down the coordinates of Q .
 - Show that R has coordinates $(2at, a(2t^2 + 1))$.
 - Hence find the Cartesian equation of the locus of R as t varies, then describe this locus geometrically.
7. $P(2ap, ap^2)$ is any point on the parabola $x^2 = 4ay$.
- Find the equation of the line ℓ through the focus parallel to the tangent at P .
 - Show that the point T where ℓ meets the x -axis has coordinates $\left(-\frac{a}{p}, 0\right)$.
 - Find the Cartesian equation of the locus of M , the midpoint of ST .
 - Show that ℓ meets the normal at P to the parabola at the point $N(ap, a(p^2 + 1))$.
 - Find the Cartesian equation of the locus of N .
8. P is the variable point $(at^2, 2at)$ on the parabola $y^2 = 4ax$. The perpendiculars from P to the y - and x -axis meet them at A and B respectively. M is the midpoint of PB .
- Find the coordinates the midpoint N of MA .
 - Show that as P varies on the parabola $y^2 = 4ax$, N moves on the parabola $2y^2 = 9ax$.
9. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on $x^2 = 4ay$, and the chord PQ subtends a right angle at the vertex O .
- Show that $pq = -4$.
 - Show that the coordinates of the midpoint M of PQ are $(a(p + q), \frac{1}{2}a(p^2 + q^2))$.
 - Use the identity $p^2 + q^2 = (p + q)^2 - 2pq$ to show that the Cartesian equation of the locus of M is $x^2 = 2a(y - 4a)$.
 - Give a geometrical description of this locus.



10. $P(2at, at^2)$ varies on the parabola $x^2 = 4ay$. $S(0, a)$ is the focus.
- Show that the tangent at P has equation $y = tx - at^2$.
 - A line drawn from the focus perpendicular to the tangent at P meets it at T . Find the coordinates of T .
 - What is the locus of T as P varies on the parabola?

DEVELOPMENT

11. P and Q are the points where $t = t_1$ and $t = t_2$ respectively on the parabola $x = 2at$, $y = at^2$. The chord PQ cuts the axis of the parabola at $(0, 3a)$.
- Show that $t_1 t_2 = -3$.
 - Show that as t_1 and t_2 vary, the midpoint of the chord PQ moves on the parabola $x^2 = 2a(y - 3a)$.
12. The variable point $P(2ap, ap^2)$ lies on $x^2 = 4ay$, and the chord OQ is drawn parallel to the tangent at P . The tangents at P and Q meet at R .
- Derive the equation of the tangent at P .
 - Write down the equation of the chord OQ .
 - Show that the coordinates of Q are $(4ap, 4ap^2)$.
 - Find the equation of the tangent at Q .
 - Show that R has coordinates $(3ap, 2ap^2)$.
 - Find the Cartesian equation of the locus of R as P varies on the parabola.
13. Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, where $p > q$, move along the parabola $x^2 = 4ay$. At all times the x -coordinates of P and Q differ by $2a$.
- Find the midpoint M of the chord PQ , and the Cartesian equation of its locus.
 - Give a geometrical description of this locus.
14. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ vary on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the vertex. The tangents at P and Q meet at T , while the normals at P and Q meet at N .
- Show that $pq = -4$.
 - Show that T has coordinates $(a(p + q), apq)$.
 - Find the Cartesian equation of the locus of T .
 - Show that N has coordinates $(-apq(p + q), a(p^2 + pq + q^2 + 2))$.
 - Find the Cartesian equation of the locus of N .
15. A parabola is defined by $x = 2at$, $y = at^2$. The points P and Q lie on the parabola and have parameters $t = p$ and $t = \frac{1}{p}$ respectively. The tangents at P and Q intersect at T .
- Find the equations of the tangents at P and Q .
 - Prove that the locus of T is part of a line parallel to the directrix of the parabola.
16. $P(8p, 4p^2)$ and $Q(8q, 4q^2)$ are variable points on the parabola $x^2 = 16y$. The chord PQ produced passes through the fixed point $(4, 0)$. The tangents at P and Q meet at R .
- Show that the chord PQ has equation $y = \frac{1}{2}(p + q)x - 4pq$.
 - Show that $p + q = 2pq$.
 - Find the coordinates of R .
 - Show that R moves on the line $x = 2y$.

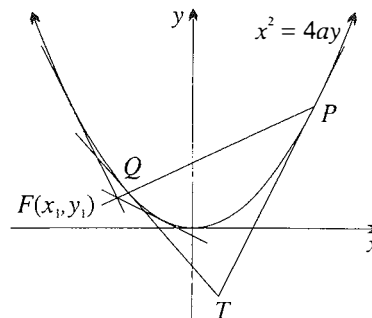


17. $A(2at_1, at_1^2)$ and $B(2at_2, at_2^2)$ are two variable points on the parabola $x^2 = 4ay$. The normals to the parabola at A and B meet at right angles at N .
- Show that $t_1 t_2 = -1$.
 - Show that N has coordinates $(a(t_1 + t_2), a(t_1^2 + t_2^2 + 1))$.
 - Hence find the Cartesian equation of the locus of N .
18. $P(2p, p^2)$ and $Q(2q, q^2)$ are variable points on the parabola $x^2 = 4y$. PQ is a focal chord of gradient m , and the normals at P and Q intersect at N .
- Derive the equation of the normal at P .
 - Show that N is the point $(-pq(p+q), p^2 + pq + q^2 + 2)$.
 - Show that $p + q = 2m$ and that $pq = -1$.
 - Write the coordinates of N in terms of m .
 - Hence find the Cartesian equation of the locus of N as m varies.
19. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. The tangents at P and Q meet at R , and R lies on the parabola $x^2 = -4ay$.
- Show that R has coordinates $(a(p+q), apq)$.
 - Show that $p^2 + q^2 + 6pq = 0$.
 - As P and Q vary, show that the locus of the midpoint of the chord PQ is the parabola $3x^2 = 4ay$.
20. P is the point with parameter $t = p$ on the parabola $x = 6t, y = 3t^2$.
- Show that the tangent to the parabola at P has equation $y = px - 3p^2$.
 - If Q is the point on the parabola where $t = 1 - p$, and P and Q are distinct, show that the tangents at P and Q meet at the point $T(3, 3p - 3p^2)$.
 - Specify algebraically the locus of T .
 - Comment on the points P, Q and T in the case where $p = \frac{1}{2}$.
21. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on the parabola $x^2 = 4ay$. It is given that the tangents at P and Q meet at $T(a(p+q), apq)$, and that the line PQ is a tangent to the parabola $x^2 = 2ay$.
- Show that the line PQ has equation $y = \frac{1}{2}(p+q)x - apq$.
 - Show that $(p+q)^2 = 8pq$.
 - Find the Cartesian equation of the locus of T .
22. $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P cuts the y -axis at Q , and R divides the interval PQ externally in the ratio $2 : 1$.
- Show that R has coordinates $(-2at, 4a + at^2)$.
 - Find the Cartesian equation of the locus of R .
 - Show that if the normal at P passes through the fixed point (h, k) , then the parameter t satisfies $at^3 + (2a - k)t - h = 0$.
 - What is the greatest number of normals to the parabola $x^2 = 4ay$ that can be drawn from any point in the number plane? Give a reason for your answer.
23. Tangents from the point $P(x_1, y_1)$ touch the parabola $x^2 = 8y$ at the points A and B .
- Show that the x -coordinates of A and B are the roots of the quadratic equation $x^2 - 2x_1x + 8y_1 = 0$.
 - Hence show that the midpoint M of the chord AB has coordinates $(x_1, \frac{1}{4}x_1^2 - y_1)$.
 - Suppose that P varies on the line $x - y = 2$. Find the Cartesian equation of the locus of M .

24. The parabola $x^2 = 4ay$ and the line $y = mx + k$ intersect at distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.
- Show that the normal at P has equation $x + py = 2ap + ap^3$, and that the normals at P and Q intersect at the point $N(-apq(p+q), a(p^2 + q^2 + pq + 2))$.
 - Show that: (i) $p + q = 2m$, (ii) $pq = -\frac{k}{a}$.
 - Hence show that $p^2 + q^2 = \frac{2k}{a} + 4m^2$.
 - Express the coordinates of N in terms of a , m and k .
 - If the chord PQ has constant gradient m , show that the locus of N is a straight line, and show that this line is a normal to the parabola.

EXTENSION

25. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ vary on the parabola $x^2 = 4ay$ in such a way that the line PQ always passes through the fixed point $F(x_1, y_1)$, which lies outside the parabola. The tangents at P and Q meet at T .



- Show that $(p+q)x_1 = 2apq + 2y_1$.
 - Hence show that the locus of T is part of a straight line. Show also that the other part of this straight line is the chord of contact of the tangents to the parabola from F .
26. $P_1(2at_1, at_1^2)$, $P_2(2at_2, at_2^2)$ and $P_3(2at_3, at_3^2)$ are variable points on $x^2 = 4ay$. Suppose that T is the point of intersection of the tangents to the parabola at P_2 and P_3 .
- Show that T has coordinates $(a(t_2 + t_3), at_2t_3)$.
 - Show that the line through T perpendicular to the tangent at P_1 meets the directrix at the point $D(a(t_1 + t_2 + t_3 + t_1t_2t_3), -a)$.
 - Hence find the locus of the orthocentre of the triangle formed by the three tangents to the parabola drawn at P_1 , P_2 and P_3 . (The *orthocentre* of a triangle is the point of intersection of its three altitudes.)

The Geometry of the Derivative

Working out the shape of a curve from its equation is a fundamental concern of this course. Now that we have the derivative, the systematic approach to sketching unfamiliar curves, begun in Chapter Three, can be extended by two further questions:

1. Where is the curve sloping upwards, where is it sloping downwards, and where does it have any maximum or minimum values?
2. Where is the curve concave up, where is it concave down, and where does it change from one concavity to the other?

These will become standard procedures for investigating unfamiliar curves (in this text they will be Steps 5 and 6 of a curve sketching menu). In particular, the algorithm for finding maximum and minimum values of a function can be applied to all sorts of practical and theoretical questions.

STUDY NOTES: Sections 10A–10F develop the standard procedures for dealing with the questions raised above about the shape of a curve. This is an important place where algebraic procedures should be freely supplemented by curve sketching software, so that a number of curves similar to those given here can be quickly drawn to demonstrate the effect of changing a constant or the form of an equation in various ways. Sections 10G–10I apply curve sketching methods to maximisation and minimisation problems, particularly in practical and geometric contexts. The final Section 10J begins to reverse the process of differentiation in preparation for the definite integral in Chapter Eleven.

10A Increasing, Decreasing and Stationary at a Point

At a point where a curve is sloping upwards, the tangent has positive gradient, and y is increasing as x increases. At a point where it is sloping downwards, the tangent has negative gradient, and y is decreasing as x increases. That is, for a function $f(x)$ defined at $x = a$:

INCREASING, DECREASING AND STATIONARY AT A POINT:

1

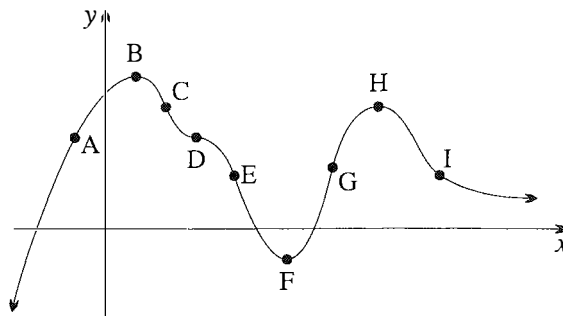
If $f'(a) > 0$, then $f(x)$ is called *increasing* at $x = a$.

If $f'(a) < 0$, then $f(x)$ is called *decreasing* at $x = a$.

If $f'(a) = 0$, then $f(x)$ is called *stationary* at $x = a$.

For example, the curve in the diagram to the right is:

- increasing at A and G ,
- decreasing at C , E and I ,
- stationary at B , D , F and H .



NOTE: These definitions of increasing, decreasing and stationary are *pointwise* definitions, because they concern the behaviour of the function at a point rather than over an interval. In later work on inverse functions and inverse trigonometric functions, we will be considering functions that are increasing or decreasing over an interval rather than at a point.

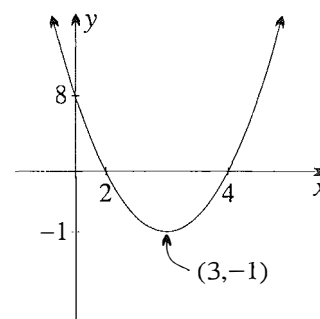
WORKED EXERCISE: Use the derivative to show that the graph of $f(x) = (x-2)(x-4)$ is stationary at the vertex $V(3, -1)$, decreasing to the left of V , and increasing to the right of V .

SOLUTION: Expanding, $f(x) = x^2 - 6x + 8$,
so $f'(x) = 2x - 6$
 $= 2(x - 3)$.

Since $f'(3) = 0$, the curve is stationary at $x = 3$.

Since $f'(x) > 0$ for $x > 3$, the curve is increasing for $x > 3$.

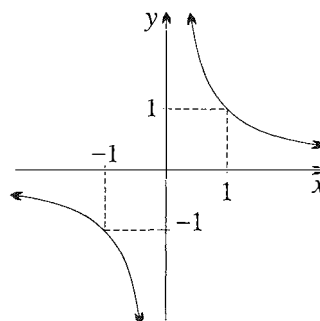
Since $f'(x) < 0$ for $x < 3$, the curve is decreasing for $x < 3$.



WORKED EXERCISE: Use the derivative to show that the graph of $y = \frac{1}{x}$ has no stationary points, and is decreasing for all values of x in its domain.

SOLUTION: Differentiating, $\frac{dy}{dx} = -\frac{1}{x^2}$.

The domain is $x \neq 0$, so for all x in the domain, x^2 is positive, the derivative is negative, and the curve is decreasing.



NOTE: The value $y = \frac{1}{2}$ at $x = 2$ is greater than the value $y = -\frac{1}{2}$ at $x = -2$, despite the fact that the curve is decreasing for all x in the domain. This sort of thing can of course only happen because of the break in the curve at $x = 0$.

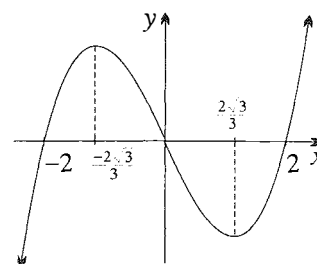
WORKED EXERCISE: Find where $y = x^3 - 4x$ is decreasing.

SOLUTION: $y' = 3x^2 - 4$,

so y' has zeroes at $x = \frac{2}{3}\sqrt{3}$ and at $x = -\frac{2}{3}\sqrt{3}$,
and is negative between the two zeroes.

So the curve is decreasing for $-\frac{2}{3}\sqrt{3} < x < \frac{2}{3}\sqrt{3}$.

[To sketch the curve, notice also that the function is odd, with zeroes at $x = 0$, $x = 2$ and $x = -2$.]



WORKED EXERCISE: Show that $f(x) = x^3 + x - 1$ is always increasing. Find $f(0)$ and $f(1)$, and explain why the curve has exactly one x -intercept. (A diagram is not actually needed in this exercise, although a sketch always helps.)

SOLUTION: Differentiating, $f'(x) = 3x^2 + 1$.

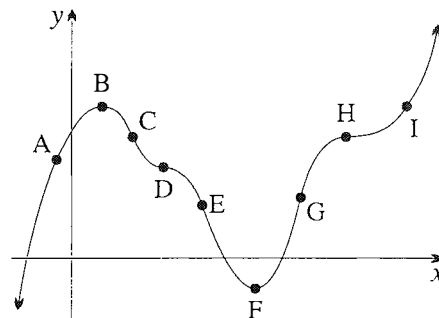
Since squares can never be negative, $f'(x)$ can never be less than 1, so the function is increasing for every value of x .

Because $f(0) = -1$ is negative and $f(1) = 1$ is positive and $f(x)$ is continuous, the curve must cross the x -axis somewhere between 0 and 1, and because the function is increasing for every value of x , it can never go back and cross the x -axis at a second point.

Exercise 10A

1. In the diagram to the right, name the points where:

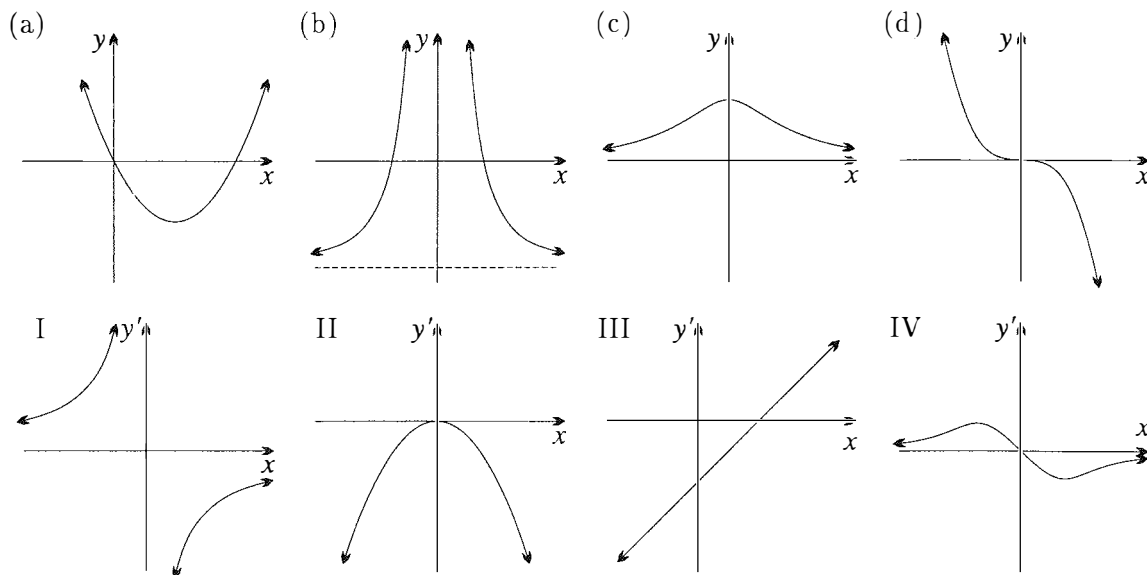
- (a) $f'(x) > 0$
- (b) $f'(x) < 0$
- (c) $f'(x) = 0$



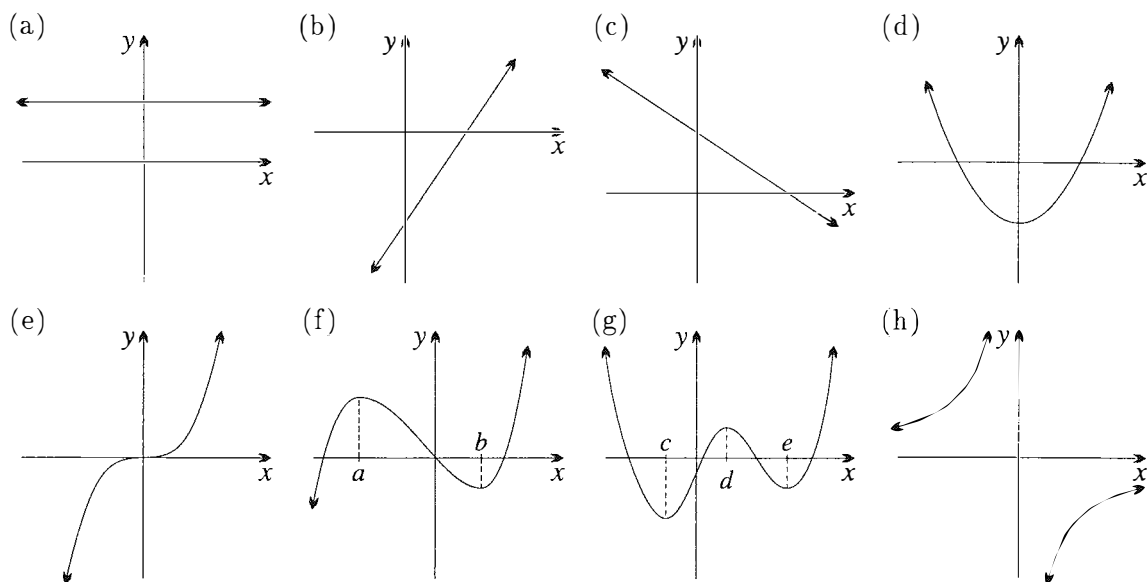
2. (a) Show that $y = -5x + 2$ is decreasing for all x .
 (b) Show that $y = x + 7$ is increasing for all x .
 (c) Show that $y = x^3$ is increasing for all values of x , apart from $x = 0$ where it is stationary.
 (d) Show that $y = 3x^2$ is increasing for $x > 0$ and decreasing for $x < 0$. What happens at $x = 0$?
 (e) Show that the function $y = \sqrt{x}$ is increasing for all values of $x > 0$.
 (f) Show that $y = \frac{1}{x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$.
3. (a) Find $f'(x)$ for the function $f(x) = 4x - x^2$.
 (b) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?
 (c) Find $f(2)$, then, by interpreting these results geometrically, sketch a graph of $f(x)$.
4. (a) Find $f'(x)$ for the function $f(x) = x^3 - 3x^2 + 5$.
 (b) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?
 (c) Evaluate $f(0)$ and $f(2)$, then, by interpreting these results geometrically, sketch a graph of $y = f(x)$.
5. (a) Differentiate $f(x) = -\frac{3}{x}$, and hence prove that $f(x)$ increases for all x in its domain.
 (b) Explain why $f(-1) > f(2)$ despite this fact.
6. Find the derivative of each of the following functions. By solving $dy/dx > 0$, find the values of x for which the function is increasing.
 (a) $y = x^2 - 4x + 1$ (b) $y = 7 - 6x - x^2$ (c) $y = 2x^3 - 6x$ (d) $y = x^3 - 3x^2 + 7$
7. (a) Find the values of x for which $y = x^3 + 2x^2 + x + 7$ is an increasing function.
 (b) Find the values of x for which $y = x^4 - 8x^2 + 7$ is a decreasing function.

DEVELOPMENT

8. The graphs of four functions (a), (b), (c) and (d) are shown below. The graphs of the derivatives of these functions, in scrambled order, are shown in I, II, III and IV. Match the graph of each function with the graph of its derivative.



9. Look carefully at each of the functions drawn below to establish where they are increasing, decreasing and stationary. Hence draw a graph of the derivative of each of the functions.



10. By finding $f'(x)$ show that:

(a) $f(x) = \frac{2x}{x-3}$ is decreasing for all $x \neq 3$,

(b) $f(x) = \frac{x^3}{x^2+1}$ is increasing for all x , apart from $x = 0$ where it is stationary.

11. (a) Find $f'(x)$ for the function $f(x) = \frac{1}{3}x^3 + x^2 + 5x + 7$.

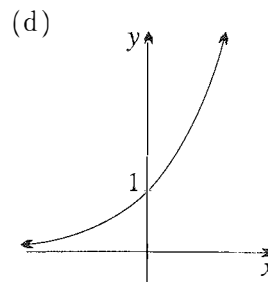
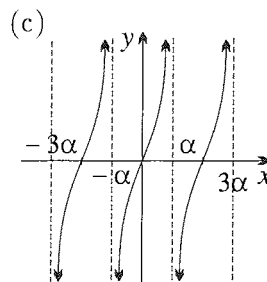
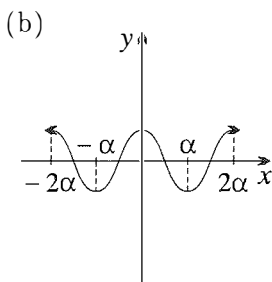
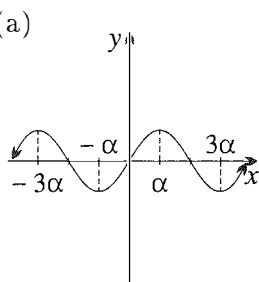
(b) By completing the square, show that $f'(x)$ is always positive, and hence that $f(x)$ is increasing for all x .

12. (a) Prove that $f(x) = 2x^3 - 3x^2 + 5x + 1$ has no stationary points.

(b) Show that $f'(x) > 0$ for all values of x , and hence that the function is always increasing.

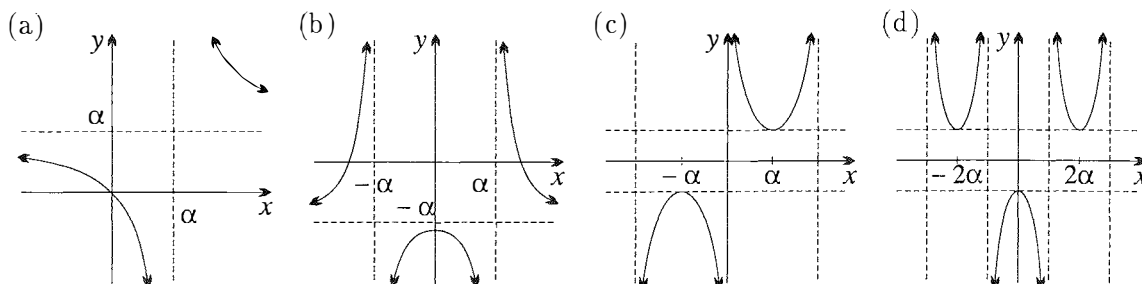
(c) Deduce that the equation $f(x) = 0$ has only one real root.

13. (a) Prove that $y = -x^3 + 2x^2 - 5x + 7$ is decreasing for all values of x .
 (b) Hence deduce the number of solutions to the equation $7 - 5x + 2x^2 - x^3 = 0$.
14. (a) Show that $f(x) = \frac{x^2 + 1}{x}$ is an odd function. (b) Find $f'(x)$.
 (c) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?
 (d) Evaluate $f(1)$ and $f(-1)$. (e) State the equations of any vertical asymptotes.
 (f) By interpreting these results geometrically, sketch a graph of the function.
15. Sketch graphs of continuous curves suggested by the properties below:
- (a) $f(1) = f(-3) = 0$,
 $f'(-1) = 0$,
 $f'(x) > 0$ when $x < -1$,
 $f'(x) < 0$ when $x > -1$.
- (d) $f(x) > 0$ for all x ,
 $f'(0) = 0$,
 $f'(x) < 0$ for $x < 0$,
 $f'(x) > 0$ for $x > 0$.
- (b) $f(2) = f'(2) = 0$,
 $f'(x) > 0$ for all $x \neq 2$.
- (e) $f(0) = 0$,
 $f'(x) < 0$ for all $x < 0$,
 $|f'(x_1)| < |f'(x_2)|$ for $x_1 < x_2 < 0$,
 $f'(x) > 0$ for all $x > 0$,
 $|f'(x_1)| < |f'(x_2)|$ for $x_1 > x_2 > 0$.
- (c) $f(x)$ is odd,
 $f(3) = 0$ and $f'(1) = 0$,
 $f'(x) > 0$ for $x > 1$,
 $f'(x) < 0$ for $0 \leq x < 1$.
16. A function $f(x)$ has derivative $f'(x) = -x(x+2)(x-1)$.
 (a) Draw a graph of $y = f'(x)$, and hence establish where $f(x)$ is increasing, decreasing and stationary.
 (b) Draw a possible graph of $y = f(x)$, given that $f(0) = 2$.
17. (a) If $f(x) = \frac{x^2 - 4}{x^2 - 1}$, find $f'(x)$.
 (b) Establish that $f'(x) < 0$ when $x < 0$ ($x \neq -1$), and $f'(x) > 0$ when $x > 0$ ($x \neq 1$).
 (c) State the equations of any horizontal or vertical asymptotes.
 (d) Hence sketch a graph of $y = f(x)$.
18. For what values of x is $y = \frac{x^2}{2x^2 + x + 1}$ decreasing?
19. (a) For $f(x) = \frac{1 - x^2}{x^2 + 1}$: (i) find $f'(x)$, (ii) evaluate $f(0)$, (iii) show that $f(x)$ is even.
 (b) Hence explain why $f(x) \leq 1$ for all x .
20. Look carefully at each of the functions drawn below to establish where they are increasing, decreasing and stationary. Hence draw a graph of the derivative of each of the functions.



EXTENSION

21. Draw a graph of the derivative of each function graphed below.



22. [This question proves that a differentiable function that is zero at its endpoints must be horizontal somewhere in between.] Suppose that $f(x)$ is continuous in the interval $a \leq x \leq b$ and differentiable for $a < x < b$, and suppose that $f(a) = f(b) = 0$.

(a) Suppose first that $f(x) > 0$ for some x in $a < x < b$, and choose $x = c$ so that $f(c)$ is the maximum value of $f(x)$ in the interval $a < x < b$.

(i) Explain why $\frac{f(x) - f(c)}{x - c} \geq 0$ for $a \leq x < c$, and $\frac{f(x) - f(c)}{x - c} \leq 0$ for $c \leq x < b$.

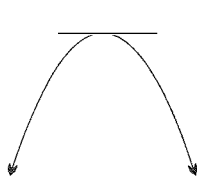
(ii) Hence explain why $f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ must be zero.

(b) Complete the proof by considering the other two possible cases:

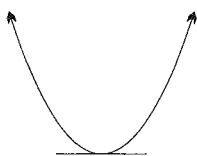
(i) $f(x) < 0$ for some x in $a \leq x \leq b$, (ii) $f(x) = 0$ for $a \leq x \leq b$.

10B Stationary Points and Turning Points

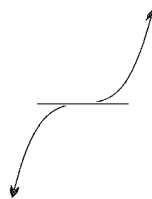
Stationary points can be classified into four different types, according to whether the curve turns upwards or downwards from the tangent to the left and to the right of the stationary point:



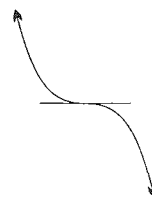
Maximum turning point



Minimum turning point



Stationary point of inflexion



Stationary point of inflexion

The words used in describing this classification need to be properly defined.

Local or Relative Maximum and Minimum: The words *maximum* and *minimum* are usually used for *local* or *relative* maxima and minima, that is, they relate only to the curve in the immediate neighbourhood of the point being considered. Suppose now that $A(a, f(a))$ is a point on a curve $y = f(x)$. Then:

LOCAL MAXIMUM: The point A is called a *local* or *relative maximum* if

$$f(x) \leq f(a), \text{ for all } x \text{ in some small interval around } a.$$

2 LOCAL MINIMUM: Similarly, A is called a *local* or *relative minimum* if

$$f(x) \geq f(a), \text{ for all } x \text{ in some small interval around } a.$$

EXTREMUM: Any local maximum or minimum is called an *extremum*.

Turning Points: A turning point is a stationary point where the curve smoothly turns over from increasing to decreasing or from decreasing to increasing, as in the first and second diagrams above. Such a situation results in a local maximum or minimum.

3 TURNING POINTS: A stationary point is called a *turning point* if the derivative changes sign around the point.

In other words, a turning point is a stationary point that is a local maximum or minimum.

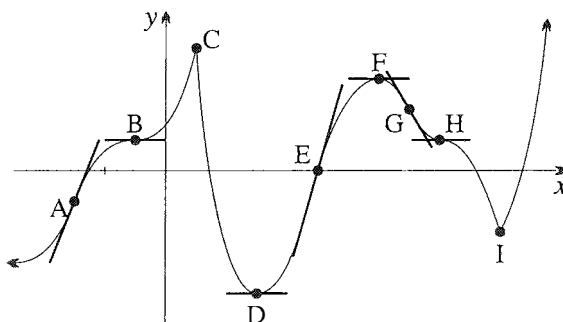
Stationary Points of Inflexion: In the last two diagrams above, there is no turning point, because in the third diagram the curve is increasing on both sides of the stationary point, and in the fourth diagram the curve is decreasing on both sides of the stationary point. Because of the presence of the stationary point, the curve flexes around the stationary point, changing concavity from downwards to upwards, or from upwards to downwards, with the surprising effect that the tangent at the stationary point actually crosses the curve.

4 POINTS OF INFLEXION: A *point of inflexion* is a point on the curve where the tangent crosses the curve. That is, it is a point where the concavity changes from upwards to downwards or from downwards to upwards.

STATIONARY POINTS OF INFLEXION: A *stationary point of inflexion* is a point of inflexion with horizontal tangent. That is, it is both a point of inflexion and a stationary point.

The diagram below demonstrates the various phenomena described in these definitions:

WORKED EXERCISE: Classify the points labelled A – I in the diagram below.



SOLUTION: C and F are local maxima, with F being a maximum turning point.

D and I are local minima, with D being a minimum turning point.

B and H are stationary points of inflexion.

A , E and G are points of inflexion, but are not stationary points.

Analysing Stationary Points and Slope: We now appeal to the theorem, discussed in Chapter Three, to the effect that a function can only change sign at a zero or at a discontinuity. But we apply this theorem now not to the function $f(x)$ but to its derivative $f'(x)$.

- 5** **CHANGES BETWEEN INCREASING AND DECREASING:** A function can only change from increasing to decreasing, or from decreasing to increasing, at a zero or a discontinuity of the derivative.

Here then is the method for analysing the stationary points, and also for gaining an overall picture of the whole slope of the function.

- 6** **USING THE DERIVATIVE $f'(x)$ TO ANALYSE STATIONARY POINTS AND SLOPE:**
1. Find the zeroes and discontinuities of the derivative $f'(x)$.
 2. Draw up a table of test points of the derivative $f'(x)$ around its zeroes and discontinuities, followed by a table of slopes, to see where its sign changes. The table will show not only the nature of each stationary point, but also where the function is increasing and where it is decreasing across its whole domain.

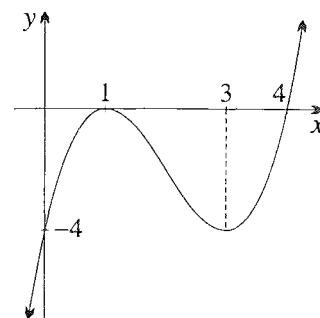
The table of slopes in the third row of the table gives an outline picture of the shape of the curve, and is a good preparation for a proper sketch.

WORKED EXERCISE: Find the stationary points of the cubic $y = x^3 - 6x^2 + 9x - 4$, determine their nature, and sketch the curve.

SOLUTION: $y' = 3x^2 - 12x + 9$
 $= 3(x-1)(x-3),$

so y' has zeroes at $x = 1$ and 3 , and no discontinuities:

x	0	1	2	3	4
y'	9	0	-3	0	9
	/	—	\	—	/



When $x = 1$, $y = 0$, and when $x = 3$, $y = -4$ (from the original equation), so $(1, 0)$ is a maximum turning point, and $(3, -4)$ is a minimum turning point. [In fact, the function factors as $y = (x-1)^2(x-4)$.]

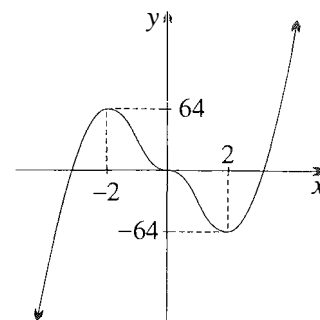
NOTE: Only the signs of y' are relevant. But if the actual values of y' are not calculated, some other argument should be given as to how the signs were obtained.

WORKED EXERCISE: Find the stationary points of the quintic $f(x) = 3x^5 - 20x^3$, determine their nature, and sketch the curve.

SOLUTION: $f'(x) = 15x^4 - 60x^2$
 $= 15x^2(x-2)(x+2),$

so $f'(x)$ has zeroes at $x = -2$, $x = 0$ and $x = 2$, and has no discontinuities:

x	-3	-2	-1	0	1	2	3
$f'(x)$	675	0	-45	0	-45	0	675
	/	—	\	—	\	—	/



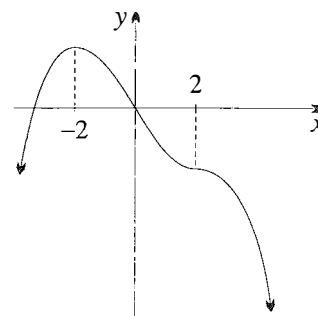
When $x = 0$, $y = 0$, when $x = 2$, $y = -64$, and when $x = -2$, $y = 64$,
so $(-2, 64)$ is a maximum turning point, $(2, -64)$ is a minimum turning point,
and $(0, 0)$ is a stationary point of inflexion.

NOTE: This function $f(x) = 3x^5 - 20x^3$ is an odd function, and it has as its derivative $f'(x) = 15x^4 - 60x^2$, which is even. In general, the derivative of an even function is odd, and the derivative of an odd function is even — this provides a useful check of the working. The result is obvious for polynomials because the indices reduce by 1, but see the last question in Exercise 10E for a general proof.

WORKED EXERCISE: Given the function sketched on the right, write down a possible equation for the derivative of the function, and a table of values to justify it.

SOLUTION: A possibility is $f'(x) = -(x+2)(x-2)^2$. As its table of values shows, this function is zero at $x = 2$ and negative on both sides of it, and it changes sign around $x = -2$.

x	-3	-2	0	2	3
$f'(x)$	25	0	-8	0	-5
	/	—	\	—	\



WORKED EXERCISE: The graph of the cubic $f(x) = x^3 + ax^2 + bx + c$ passes through the origin and has a stationary point at $A(2, 2)$. Find a , b and c .

SOLUTION: To find the three unknown constants, we need three independent equations.

Since $f(0) = 0$, $0 = 0 + 0 + 0 + c$ and so $c = 0$. (1)

Since $f(2) = 2$, $2 = 8 + 4a + 2b + c$
and since $c = 0$, $2a + b = -3$. (2)

Differentiating, $f'(x) = 3x^2 + 2ax + b$
and since $f'(2) = 0$, $0 = 12 + 4a + b$
 $4a + b = -12$. (3)

Subtracting (2) from (3), $2a = -9$
 $a = -4\frac{1}{2}$,

and substituting into (2), $-9 + b = -3$
 $b = 6$.

Exercise 10B

1. Find the derivative of each function and complete the given table to determine the nature of the stationary point. Sketch each graph, indicating all important features.

(a) $y = x^2 - 4x + 3$:	<table> <tr><td>x</td><td>1</td><td>2</td><td>3</td></tr> <tr><td>y'</td><td></td><td></td><td></td></tr> </table>	x	1	2	3	y'				(c) $y = 3x^2 + 11x - 4$:	<table> <tr><td>x</td><td>-2</td><td>$-\frac{11}{6}$</td><td>-1</td></tr> <tr><td>y'</td><td></td><td></td><td></td></tr> </table>	x	-2	$-\frac{11}{6}$	-1	y'			
x	1	2	3																
y'																			
x	-2	$-\frac{11}{6}$	-1																
y'																			
(b) $y = 12 + 4x - x^2$:	<table> <tr><td>x</td><td>1</td><td>2</td><td>3</td></tr> <tr><td>y'</td><td></td><td></td><td></td></tr> </table>	x	1	2	3	y'				(d) $y = 3 + 5x - 2x^2$:	<table> <tr><td>x</td><td>1</td><td>$\frac{5}{4}$</td><td>2</td></tr> <tr><td>y'</td><td></td><td></td><td></td></tr> </table>	x	1	$\frac{5}{4}$	2	y'			
x	1	2	3																
y'																			
x	1	$\frac{5}{4}$	2																
y'																			

2. Find the stationary point(s) of each function and use a table of values of dy/dx to determine its nature. Sketch each graph, indicating all intercepts with the axes.
- (a) $y = 2x^2 - 3x + 5$ (b) $y = 5 - 4x - x^2$ (c) $y = x^3 - 3x^2$ (d) $y = 12x - x^3$
3. Find the stationary points of each of the following functions and use a table of values of dy/dx to determine their nature. Sketch each graph (do not find the x -intercepts).
- (a) $y = 2x^3 + 3x^2 - 36x + 15$ (c) $y = 16 + 4x^3 - x^4$
 (b) $y = x^3 + 4x^2 + 4x$ (d) $y = 3x^4 - 16x^3 + 24x^2 + 11$

DEVELOPMENT

4. (a) Use the product rule to show that if $y = x(x - 2)^3$, then $y' = 2(2x - 1)(x - 2)^2$.
 (b) Find any stationary points and use a table of values of y' to analyse them.
 (c) Sketch a graph of the function, indicating all important features.
5. (a) Expand and simplify $(x + 2)(x - 3)^2$.
 (b) If $f(x) = 3x^4 - 16x^3 - 18x^2 + 216x + 40$, find $f'(x)$ in factored form.
 (c) Hence find all stationary points and analyse them.
 (d) Sketch a graph of $y = f(x)$.
6. (a) If $f(x) = (x - 2)^2(x + 4)^3$, show that $f'(x) = (x - 2)(x + 4)^2(5x + 2)$.
 (b) Find all stationary points and analyse them.
 (c) Sketch $y = f'(x)$ and hence determine where $y = f(x)$ is increasing and decreasing.
 (d) Sketch a graph of $y = f(x)$, indicating all important features.
7. Using the method outlined in the previous question, sketch graphs of the these functions:
- (a) $y = x^2(3 - x)^2$ (c) $y = (x - 5)^2(2x + 1)$
 (b) $y = (1 - x)^3(x + 2)^2$ (d) $y = (3x - 2)^2(2x - 3)^3$
8. (a) If $f(x) = \frac{3x}{x^2 + 1}$, show that $f'(x) = \frac{3(1 - x)(1 + x)}{(x^2 + 1)^2}$.
 (b) Hence find any stationary points and analyse them.
 (c) Sketch a graph of $y = f(x)$, indicating all important features.
 (d) Hence state how many roots the equation $\frac{3x}{x^2 + 1} = c$ has for:
- (i) $c > \frac{3}{2}$ (ii) $c = \frac{3}{2}$ (iii) $0 < c < \frac{3}{2}$ (iv) $c = 0$
9. The tangent to the curve $y = x^2 + ax - 15$ is horizontal at the point where $x = 4$. Find the value of a .
10. The curve $y = ax^2 + bx + c$ passes through the points $(1, 4)$ and $(-1, 6)$ and obtains its maximum value when $x = -\frac{1}{2}$. Find the values of a , b and c .
11. The curve $y = ax^2 + bx + c$ touches the line $y = 2x$ at the origin and has a maximum point when $x = 1$. Find the values of a , b and c .
12. The function $y = ax^3 + bx^2 + cx + d$ has a relative maximum at the point $(-2, 27)$ and a relative minimum at the point $(1, 0)$. Find the values of a , b , c and d .
13. (a) Sketch graphs of the following functions, clearly indicating any stationary points (but leave the y -coordinates in factored form):
 (i) $y = x^4(1 - x)^6$ (ii) $y = x^4(1 - x)^7$ (iii) $y = x^5(1 - x)^6$ (iv) $y = x^5(1 - x)^7$
 (b) Show that $y = x^a(1 - x)^b$ has a turning point whose x -coordinate divides the interval between the points $(0, 0)$ and $(1, 0)$ in the ratio $a : b$.

EXTENSION

14. Let $f(x) = x^3 + 3bx^2 + 3cx + d$.

- (a) Show that $y = f(x)$ has two distinct turning points if and only if $b^2 > c$.
 (b) If $b^2 > c$, show that the vertical distance between the turning points is $4(b^2 - c)^{\frac{3}{2}}$.
 [HINT: Use the sum and product of the roots of the derived function.]

10C Critical Values

As discussed in the previous section, the derivative of a function can change sign at a zero or a discontinuity of the derived function. Such values are called *critical values*. The examples so far have mostly avoided functions whose derivative has a discontinuity, and this section will deal with them more systematically.

CRITICAL VALUES: A zero or discontinuity of the derivative is called a *critical value* of the function.

7 THE TABLE OF TEST POINTS OF $f'(x)$: Because these critical values are the only places where the derivative can change sign, a table of test points of $f'(x)$ around them will be sufficient to analyse the stationary points and to show where the function is increasing and where it is decreasing.

WORKED EXERCISE:

- (a) Find the critical values of $y = \frac{1}{x(x-4)}$, then use a table of test points of $\frac{dy}{dx}$ to analyse stationary points and find where the function is increasing and decreasing.
 (b) Analyse the sign of the function in its domain, find any vertical and horizontal asymptotes, then sketch the curve.

SOLUTION:

- (a) The domain of the function is $x \neq 0$ and $x \neq 4$.

Differentiating using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{x^2(x-4)^2} \times (2x-4) \\ &= \frac{2(2-x)}{x^2(x-4)^2},\end{aligned}$$

so $\frac{dy}{dx}$ has a zero at $x = 2$

and discontinuities at $x = 0$ and $x = 4$:

x	-1	0	1	2	3	4	5
$\frac{dy}{dx}$	$\frac{6}{25}$	*	$\frac{2}{9}$	0	$-\frac{2}{9}$	*	$-\frac{6}{25}$
	/	*	/	—	\	*	\

$$\begin{aligned}\text{Let } u &= x^2 - 4x, \\ \text{then } y &= \frac{1}{u}. \\ \text{So } \frac{du}{dx} &= 2x - 4 \\ \text{and } \frac{dy}{du} &= -\frac{1}{u^2}.\end{aligned}$$

So the function has a maximum turning point at $(2, -\frac{1}{4})$, it is increasing for $x < 2$ (except at $x = 0$), and it is decreasing for $x > 2$ (except at $x = 4$).

(b) The function itself is never zero,

and it has discontinuities at $x = 0$ and $x = 4$:

x	-1	0	2	4	5
y	$\frac{1}{5}$	*	$-\frac{1}{4}$	*	$\frac{1}{5}$

so $y > 0$ for $x < 0$ or $x > 4$, and $y < 0$ for $0 < x < 4$.

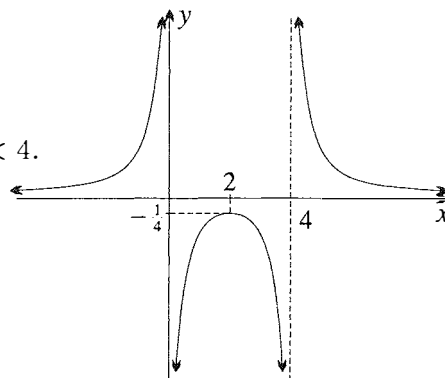
As $x \rightarrow 4^+$, $y \rightarrow \infty$, and as $x \rightarrow 4^-$, $y \rightarrow -\infty$,

as $x \rightarrow 0^+$, $y \rightarrow -\infty$, and as $x \rightarrow 0^-$, $y \rightarrow \infty$,

so $x = 0$ and $x = 4$ are vertical asymptotes.

Also, $y \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$,

so the x -axis is a horizontal asymptote.



Rational Functions: A rational function is a function like $y = \frac{x-2}{x^2-4x}$, which is the ratio of two polynomials. These functions can be very complicated to sketch. Besides the difficult algebra of the quotient rule, there may be asymptotes, zeroes, turning points and inflexions. The curve $y = \frac{1}{x(x-4)}$ above is a rational function, but was a little simpler to handle because of the constant numerator.

Taking the Limit of $f'(x)$ near Critical Values and for Large Values of x : In the previous worked example, there were asymptotes at the two values $x = 0$ and $x = 4$ where $f'(x)$ was undefined, so no further analysis was needed. In other situations, however, the shape of the curve may not be clear near a value $x = a$ where $f'(x)$ is undefined. It may then be necessary to examine the behaviour of $f'(x)$ as $x \rightarrow a^+$ and as $x \rightarrow a^-$. Furthermore, the shape of the curve as $x \rightarrow \infty$ and as $x \rightarrow -\infty$ may need examination of the behaviour of both $f'(x)$ and $f(x)$.

BEHAVIOUR NEAR DISCONTINUITIES OF THE DERIVATIVE AND FOR LARGE x :

8

1. For each discontinuity $x = a$ of $f'(x)$, it may be necessary to examine the behaviour of $f'(x)$ as $x \rightarrow a^+$ and as $x \rightarrow a^-$.
2. It may also be necessary to examine $f'(x)$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

WORKED EXERCISE: [Vertical tangents] Analyse the critical values of $y = x^{\frac{1}{3}}$, then sketch the curve. This curve and the next were discussed in Section 7J on differentiability, but the methods of these sections are well suited to them, provided that the behaviour of the derivative is properly analysed near its discontinuities.

SOLUTION: $y = x^{\frac{1}{3}}$ is an odd function, defined everywhere.

It is zero at $x = 0$, positive for $x > 0$, and negative for $x < 0$.

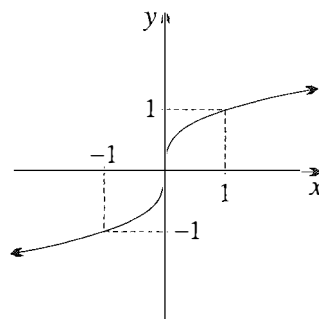
Differentiating, $y' = \frac{1}{3}x^{-\frac{2}{3}}$,

so y' has no zeroes, and has a discontinuity at $x = 0$:

x	-1	0	1
y'	$\frac{1}{3}$	*	$\frac{1}{3}$
	/	*	/

Since $y' \rightarrow \infty$ as $x \rightarrow 0^+$ and as $x \rightarrow 0^-$, there is a vertical tangent at the origin.

Also $y' \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, so the curve flattens out away from the origin, but $y \rightarrow \infty$ as $x \rightarrow \infty$, so there is no horizontal asymptote.



WORKED EXERCISE: [Cusps] Analyse the critical values of $y = x^{\frac{2}{3}}$, then sketch it.

SOLUTION: $y = x^{\frac{2}{3}}$ is an even function, defined everywhere.

It is zero at $x = 0$ and positive elsewhere.

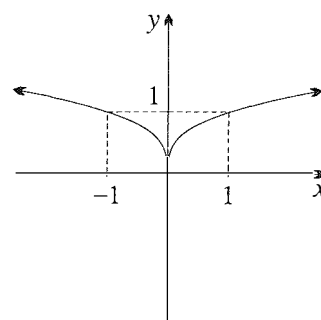
Differentiating, $y' = \frac{2}{3}x^{-\frac{1}{3}}$,

so y' has no zeroes, and has a discontinuity at $x = 0$:

x	-1	0	1
y'	$-\frac{2}{3}$	*	$\frac{2}{3}$
	\	*	/

As $x \rightarrow 0^+$, $y' \rightarrow \infty$, and as $x \rightarrow 0^-$, $y' \rightarrow -\infty$,
so there is a cusp at the origin.

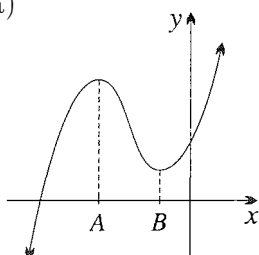
Again $y' \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, so the curve flattens out away from the origin,
and again $y \rightarrow \infty$ as $x \rightarrow \infty$, so there is no horizontal asymptote.



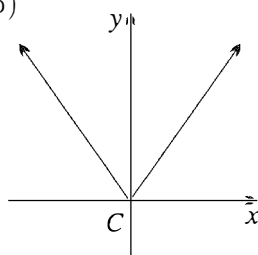
Exercise 10C

1. All the critical points have been labelled on the graphs of the functions drawn below. State which of these are relative maxima or minima or horizontal points of inflexion.

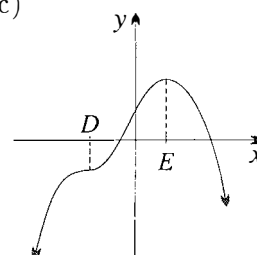
(a)



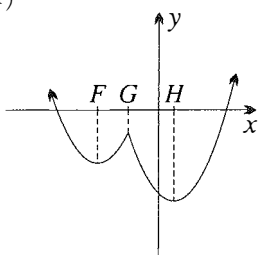
(b)



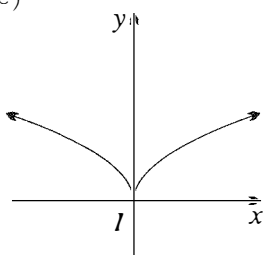
(c)



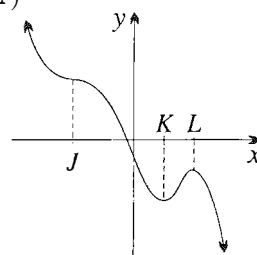
(d)



(e)



(f)



2. The derivatives of various functions have been given below. Find the critical values. Use a table of test points of dy/dx to find which critical values give turning points or horizontal points of inflexion:

(a) $\frac{dy}{dx} = x - 1$

(e) $\frac{dy}{dx} = \frac{x}{x-1}$

(i) $\frac{dy}{dx} = x^{-\frac{1}{3}}$

(b) $\frac{dy}{dx} = (x-3)(2x+1)$

(f) $\frac{dy}{dx} = \frac{x^2}{x-1}$

(j) $\frac{dy}{dx} = x - \frac{1}{x}$

(c) $\frac{dy}{dx} = x(x-3)^2$

(g) $\frac{dy}{dx} = \frac{x}{(x-1)^2}$

(k) $\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$

(d) $\frac{dy}{dx} = (x+2)^3(x-4)$

(h) $\frac{dy}{dx} = \frac{x^2}{(x-1)^3}$

(l) $\frac{dy}{dx} = \frac{2-x}{\sqrt{2+x}(1-x)^3}$

DEVELOPMENT

3. (a) State the domain of the function $y = x + \frac{1}{x}$.
- (b) Show that $\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$, and write down any critical values.
- (c) Find and analyse any stationary points, using a table of values of $\frac{dy}{dx}$.
- (d) Describe what happens to y as $x \rightarrow \infty$ and $x \rightarrow -\infty$ (and find the oblique asymptote).
- (e) Sketch a graph of the function, indicating all important features.
4. Differentiate these functions using the quotient rule. Use a table of values of y' to analyse any stationary points. Find any asymptotes, then sketch each function:
- (a) $y = \frac{x}{x^2 - 1}$ (b) $y = \frac{x^2}{1 + x^2}$ (c) $y = \frac{x^2 - 4}{x^2 - 1}$ (d) $y = \frac{x^2 + 1}{(x - 1)^2}$
5. (a) State the domain of the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$.
- (b) Show that $f'(x) = \frac{x - 1}{2x\sqrt{x}}$ and write down any critical values.
- (c) Find the stationary point, and use a table of values of dy/dx to determine its nature.
- (d) Describe what happens to $f(x)$ and to $f'(x)$ as $x \rightarrow \infty$.
- (e) Sketch a graph of the function, indicating all important features.
6. Use the steps outlined in the previous question to graph the following functions:
- (a) $y = x - \frac{1}{x}$ (b) $y = x^2 + \frac{1}{x^2}$
7. (a) Find the domain and any asymptotes of $y = \frac{\sqrt{x}}{\sqrt{9 + x^2}}$.
- (b) Show that its derivative is $\frac{dy}{dx} = \frac{(3 - x)(3 + x)}{2(9 + x^2)^{\frac{3}{2}}\sqrt{x}}$.
- (c) Find the critical values, and analyse them with a table of test points.
- (d) By examining the limit of the derivative as $x \rightarrow 0^+$, determine the shape of the curve near the origin, then sketch the curve.
8. Consider the function $y = |x| + 3$. (a) Find $\frac{dy}{dx}$ when $x < 0$ and when $x > 0$.
- (b) Hence find any critical values, and sketch a graph of the function.
9. Consider the function $y = |x - 2|$. (a) Find $\frac{dy}{dx}$ when $x > 2$ and when $x < 2$.
- (b) Hence find any critical values, and sketch a graph of the function.
10. (a) Differentiate $f(x) = (x - 2)^{\frac{1}{5}}$.
- (b) Show that there are no stationary points, but that a critical value occurs at $x = 2$.
- (c) By considering the sign of $f'(x)$, sketch a graph of $y = f(x)$.
11. (a) Differentiate $f(x) = (x - 1)^{\frac{2}{3}}$.
- (b) Show that the critical point $(1, 0)$ is not a stationary point.
- (c) By considering the sign of $f'(x)$ when $x < 1$ and when $x > 1$, sketch $y = f(x)$.
- (d) Hence sketch: (i) $y = 3 + (x - 1)^{\frac{2}{3}}$ (ii) $y = 1 - (x - 1)^{\frac{2}{3}}$

EXTENSION

12. (a) Differentiate $y = x^{\frac{1}{2}} - x^{\frac{3}{2}}$. (b) Find those values of x for which $y' = 0$, and hence determine the coordinates of any critical points.
 (c) Hence sketch a graph of $y^2 = x(1 - x)^2$.
13. Sketch graphs of the following functions, indicating all critical points:
 (a) $y = |(x - 1)(x - 3)|$ (b) $y = |x - 2| + |x + 1|$ (c) $y = x^2 + |x|$
14. (a) State the domain and range of the function $\sqrt{x} + \sqrt{y} = \sqrt{c}$, where c is a constant.
 (b) Use implicit differentiation to show that $y' = -\sqrt{y/x}$.
 (c) By considering the behaviour of y' as $x \rightarrow 0^+$ and $y \rightarrow 0^+$, sketch a graph of the curve, labelling all critical points.

10D Second and Higher Derivatives

The derivative of the derivative of a function is called the *second derivative* of the function. As for the derivative, there is a variety of notations, including

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f^{(2)}(x) \quad \text{or} \quad f''(x) \quad \text{or} \quad y''.$$

This section is concerned with the algebraic manipulation of the second derivative — the geometric implications are left until the next section.

WORKED EXERCISE: Find the successive derivatives of $y = x^4 + x^3 + x^2 + x + 1$.

$$\begin{array}{lll} \text{SOLUTION: } y = x^4 + x^3 + x^2 + x + 1 & \frac{d^2y}{dx^2} = 12x^2 + 6x + 2 & \frac{d^4y}{dx^4} = 24 \\ & \frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1 & \frac{d^3y}{dx^3} = 24x + 6 \\ & & \frac{d^5y}{dx^5} = 0 \end{array}$$

NOTE: The degree of the polynomial goes down by one with each differentiation, so that the fifth and all higher derivatives vanish. In general, the $(n + 1)$ th and higher derivatives of a polynomial of degree n vanish, but the n th derivative does not.

The eventual vanishing of the higher derivatives of polynomials is actually a characteristic property of polynomials, because the converse is also true. If the $(n + 1)$ th derivative of a polynomial vanishes but the n th does not, then the function is a polynomial of degree n . The result seems clear, but as yet we lack the machinery for a formal proof.

Exercise 10D

1. Find the first, second and third derivatives of the following:

$$\begin{array}{lllll} \text{(a) } x^{10} & \text{(c) } 4 - 3x & \text{(e) } 4x^3 - x^2 & \text{(g) } x^{-1} & \text{(i) } 5x^{-3} \\ \text{(b) } 3x^5 & \text{(d) } x^2 - 3x & \text{(f) } x^{0.3} & \text{(h) } \frac{1}{x^2} & \text{(j) } x^2 + \frac{1}{x} \end{array}$$

2. Use the chain rule to find the first and second derivatives of the following:

$$\text{(a) } (x + 1)^2 \quad \text{(b) } (3x - 5)^3 \quad \text{(c) } (1 - 4x)^2 \quad \text{(d) } (8 - x)^{11}$$

3. By writing them with negative indices, find the first and second derivatives of the following:

(a) $\frac{1}{x+2}$ (b) $\frac{1}{(3-x)^2}$ (c) $\frac{1}{(5x+4)^3}$ (d) $\frac{2}{(4-3x)^2}$

4. By writing them with fractional indices, find the first and second derivatives of the following functions:

(a) $\sqrt{x}$ (b) $\sqrt[3]{x}$ (c) $x\sqrt{x}$ (d) $\frac{1}{\sqrt{x}}$ (e) $\sqrt{x+2}$ (f) $\sqrt{1-4x}$

5. (a) If $f(x) = 3x + \frac{1}{x^3}$, find: (i) $f'(2)$ (ii) $f''(2)$

(b) If $f(x) = (2x-3)^4$, find: (i) $f'(1)$ (ii) $f''(1)$ (iii) $f'''(1)$ (iv) $f''''(1)$

6. If $f(x) = ax^2 + bx + c$ and $f(1) = 5$, $f'(1) = 2$ and $f''(1) = -1$, find a , b and c .

DEVELOPMENT

7. Find the first and second derivatives of the following:

(a) $\frac{x}{x+1}$ (b) $\frac{x-1}{2x+5}$ (c) $\frac{x}{1+x^2}$

8. If $f(x) = x(x-1)^4$, use the product rule to find $f'(x)$ and $f''(x)$.

9. Find the values of x for which $y'' = 0$ if:

(a) $y = x^4 - 6x^2 + 11$ (b) $y = x^3 + x^2 - 5x + 7$ (c) $y = x\sqrt{x+1}$

10. (a) If $y = 3x^2 + 7x + 5$, prove that $\frac{d}{dx} \left(x \frac{dy}{dx} \right) = x \frac{d^2y}{dx^2} + \frac{dy}{dx}$.

(b) If $y = (2x-1)^4$, prove that $\frac{d}{dx} \left(y \frac{dy}{dx} \right) = y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2$.

(c) If $y = 2x^2 - \frac{3}{\sqrt{x}}$, prove that $2x^2 \frac{d^2y}{dx^2} = x \frac{dy}{dx} + 2y$.

11. (a) Find the first, second and third derivatives of x^n .

(b) Find the n th and $(n+1)$ th derivatives of x^n .

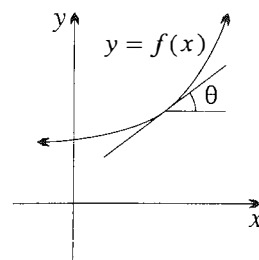
12. Find y' and y'' in terms of t in each of the following cases:

(a) $x = 3t + 1$, $y = 2t$ (c) $x = 1 - 5t$, $y = t^3$ (e) $x = (t-2)^2$, $y = 3t$
 (b) $x = 2t$, $y = \frac{1}{t}$ (d) $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$ (f) $x = \frac{1+t}{1-t}$, $y = \frac{1-t}{1+t}$

13. Find positive integers a and b such that $x^2y'' + 2xy' = 12y$, where $y = x^a + x^{-b}$.

EXTENSION

14. The curvature C of the graph $y = f(x)$ is defined as the absolute value of the rate of change of the angle θ with respect to arc length. That is, $C = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{\frac{3}{2}}}$. Compute the curvature of a circle of radius r .



10E Concavity and Points of Inflexion

Sketched on the right are a cubic function and its first and second derivatives. These sketches are intended to show how the concavity of the original graph is determined by the sign of the second derivative.

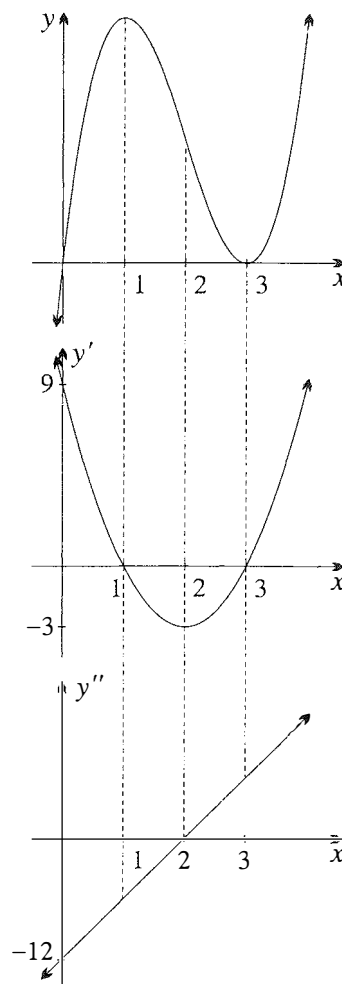
$$\begin{aligned} y &= x^3 - 6x^2 + 9x = x(x-3)^2 \\ y' &= 3x^2 - 12x + 9 = 3(x-1)(x-3) \\ y'' &= 6x - 12 = 6(x-2) \end{aligned}$$

The sign of each derivative tells whether the function above it is increasing or decreasing. So the second graph describes the gradient of the first, and the third graph describes the gradient of the second.

To the right of $x = 2$, the top graph is concave up. This means that as one moves along the curve from left to right, the tangent rotates, with its gradient steadily increasing. Thus for $x > 2$, the gradient function y' is increasing as x increases, as can be seen in the middle graph. The bottom graph is the gradient of the middle graph, and accordingly y'' is positive for $x > 2$.

Similarly, to the left of $x = 2$ the top graph is concave down. This means that its gradient function y' is steadily decreasing as x increases. The bottom graph is the derivative of the middle graph, so y'' is negative for $x < 2$.

This example demonstrates that the concavity of a graph $y = f(x)$ at any value $x = a$ is determined by the sign of its second derivative at $x = a$.



9

CONCAVITY AND THE SECOND DERIVATIVE:

If $f''(a)$ is negative, the curve is concave down at $x = a$.

If $f''(a)$ is positive, the curve is concave up at $x = a$.

Points of Inflexion: A point of inflexion is a point where the tangent crosses the curve, as was defined in Section 10B. This means that the curve must curl away from the tangent on opposite sides of the tangent, so the concavity must change around the point. The three diagrams above show how the point of inflexion at $x = 2$ results in a minimum turning point at $x = 2$ in the middle graph of y' . Hence the bottom graph of y'' has a zero at $x = 2$, and changes sign around $x = 2$.

This discussion gives the full method for analysing concavity and finding points of inflexion. Once again, the method uses the fact that y'' can only change sign at a zero or a discontinuity of y'' .

10

USING $f''(x)$ TO ANALYSE CONCAVITY AND FIND POINTS OF INFLEXION:

1. Find the zeroes and discontinuities of the second derivative $f''(x)$.
2. Use a table of test points of the second derivative $f''(x)$ around its zeroes and discontinuities, followed by a table of concavities, to see where its sign changes. The table will show not only any points of inflexion, but also the concavity of the graph across its whole domain.

Inflexional Tangents: It is often useful in sketching to find the gradient of any *inflexional tangents* (tangents at the point of inflexion). A question will often ask for this before requiring the sketch.

WORKED EXERCISE: Find any points of inflexion of $f(x) = x^5 - 5x^4$ and the gradients of the inflexional tangents, and describe the concavity. Find also any turning points, and sketch the curve.

SOLUTION: Here $f(x) = x^5 - 5x^4 = x^4(x - 5)$
 $f'(x) = 5x^4 - 20x^3 = 5x^3(x - 4)$
 $f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3).$

Hence $f'(x)$ has zeroes at $x = 0$ and $x = 4$, and no discontinuities:

x	-1	0	1	4	5
$f'(x)$	25	0	-15	0	625
	/	—	\	—	/

so $(0, 0)$ is a maximum turning point, and $(4, -256)$ is a minimum turning point.

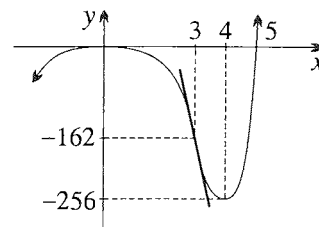
Also, $f''(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities:

x	-1	0	1	3	4
$f''(x)$	-80	0	-40	0	320
	∩	.	∩	.	∪

so $(3, -162)$ is a point of inflexion (but $(0, 0)$ is not).

Since $f'(3) = -135$, the inflexional tangent has gradient -135 .

The graph is concave down for $x < 0$ and $0 < x < 3$, and concave up for $x > 3$.



NOTE: The example given above is intended to show that $f''(x) = 0$ is NOT a sufficient condition for a point of inflexion — the sign of $f''(x)$ must also change around the point.

WORKED EXERCISE: [Finding pronumerals in a function] For what values of b is the graph of the quartic $f(x) = x^4 - bx^3 + 5x^2 + 6x - 8$ concave down at the point where $x = 2$?

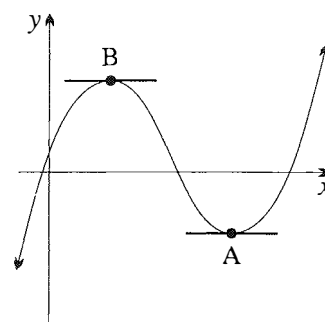
SOLUTION: Differentiating, $f'(x) = 4x^3 - 3bx^2 + 10x + 6$
 $f''(x) = 12x^2 - 6bx + 10.$

Put $f''(2) < 0$, then $48 - 12b + 10 < 0$

$$12b > 58$$

$$b > 4\frac{5}{6}.$$

Using the Second Derivative to Test Stationary Points: If the curve is concave up at a stationary point, then the point must be a minimum turning point, as in the point A on the diagram to the right. Similarly the curve is concave down at B , which must therefore be a maximum turning point.



This gives an alternative test of the nature of a stationary point. Suppose that $x = a$ is a stationary point of a function $f(x)$. Then:

USING THE SECOND DERIVATIVE TO TEST A STATIONARY POINT:

- 11**
- If $f''(a) > 0$, the curve is concave up at $x = a$, and there is a minimum turning point there.
 - If $f''(a) < 0$, the curve is concave down at $x = a$, and there is a maximum turning point there.
 - If $f''(a) = 0$, more work is needed. Go back to the table of values of $f'(x)$, or else use a table of values of $f''(x)$.

The third point is most important — all four cases are possible for the shape of the curve at $x = a$ when the second derivative vanishes there, and without further work, no conclusion can be made at all. The previous example of $y = x^5 - 5x^4$ shows that such a point can be a turning point. The following worked exercise is an example where such a point turns out to be a point of inflexion.

WORKED EXERCISE: Use the second derivative, if possible, to determine the nature of the stationary points of the graph of $f(x) = x^4 - 4x^3$. Find also any points of inflexion, examine the concavity over the whole domain, and sketch the curve.

SOLUTION: Here $f(x) = x^4 - 4x^3 = x^3(x - 4)$
 $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$
 $f''(x) = 12x^2 - 24x = 12x(x - 2),$

so $f'(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities.

Since $f''(3) = 36$ is positive, $(3, -27)$ is a minimum turning point, but $f''(0) = 0$, so no conclusion can be drawn about $x = 0$.

x	-1	0	1	3	4
$f'(x)$	-16	0	-8	0	64
	\	—	\	—	/

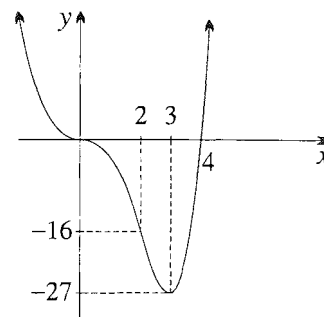
so $(0, 0)$ is a stationary point of inflexion.

$f''(x)$ has zeroes at $x = 0$ and $x = 2$, and no discontinuities:

x	-1	0	1	2	3
$f''(x)$	36	0	-12	0	36
	∪	.	∩	.	∪

so, besides the horizontal inflexion at $(0, 0)$, there is an inflexion at $(2, -16)$, and the inflexional tangent at $(2, -16)$ has gradient -16 .

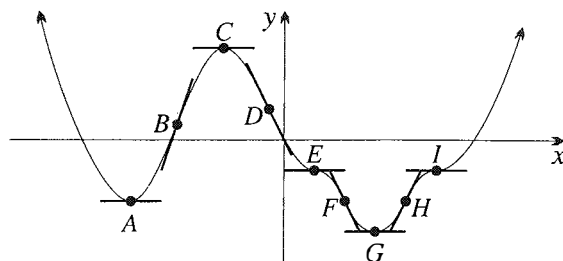
The graph is concave down for $0 < x < 2$, and concave up for $x < 0$ and for $x > 2$.



Exercise 10E

1. Complete the table below for the function to the right, stating at each given point whether the first and second derivative would be positive, negative or zero:

Point	A	B	C	D	E	F	G	H	I
y'									
y''									

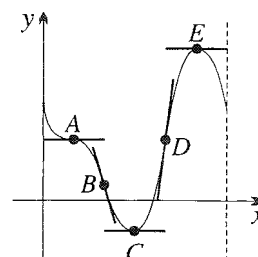


2. (a) Show that $y = x^2 - 3x + 7$ is concave up for all values of x .
 (b) Show that $y = -3x^2 + 2x - 4$ is concave down for all values of x .
 (c) Show that $y = \frac{1}{x}$ is concave up for $x > 0$ and concave down for $x < 0$.
 (d) Show that $y = x^3 - 3x^2 - 5x + 2$ is concave up for $x > 1$ and concave down for $x < 1$.
3. (a) If $f(x) = x^3 - 3x$, find $f'(x)$ and $f''(x)$. (b) Hence find any stationary points and, by examining the sign of $f''(x)$, determine their nature.
 (c) Find the coordinates of any points of inflexion.
 (d) Sketch a graph of the function, indicating all important features.
4. (a) If $f(x) = x^3 - 6x^2 - 15x + 1$, find $f'(x)$ and $f''(x)$. (b) Hence find any stationary points and, by examining the sign of $f''(x)$, determine their nature.
 (c) Find the coordinates of any points of inflexion.
 (d) Sketch a graph of the function, indicating all important features.

DEVELOPMENT

5. Find the range of values of x for which the curve $y = 2x^3 - 3x^2 - 12x + 8$ is:
 (a) increasing, (b) decreasing, (c) concave up, (d) concave down.
6. Find the x -coordinates of any points of inflexion of the following curves:
 (a) $\frac{d^2y}{dx^2} = x + 5$ (b) $\frac{d^2y}{dx^2} = (x + 5)^2$ (c) $\frac{d^2y}{dx^2} = (x - 3)(x + 2)$ (d) $\frac{d^2y}{dx^2} = (x - 3)^2(x + 2)$
7. Sketch a small section of the graph of the continuous function f about $x = a$ if:
 (a) $f'(a) > 0$ and $f''(a) > 0$ (c) $f'(a) < 0$ and $f''(a) > 0$
 (b) $f'(a) > 0$ and $f''(a) < 0$ (d) $f'(a) < 0$ and $f''(a) < 0$
8. Sketch possible graphs of continuous functions with these properties:
 (a) $f(-5) = f(0) = f(5) = 0$, and $f'(3) = f'(-3) = 0$, and $f''(x) > 0$ for $x < 0$, and $f''(x) < 0$ for $x > 0$
 (b) $f'(2) = f''(2) = 0$, and $f''(1) > 0$, and $f''(3) < 0$
9. By finding the second derivative, explain why the curve $y = ax^2 + bx + c$, $a \neq 0$:
 (a) is concave up if $a > 0$, (b) is concave down if $a < 0$, (c) has no points of inflexion.
10. (a) If $f(x) = x^4 - 12x^2$, find $f'(x)$ and $f''(x)$.
 (b) Find the coordinates of any stationary points, and use $f''(x)$ to determine their nature.
 (c) Find the coordinates of any points of inflexion.
 (d) Find the gradient of the curve at the two points of inflexion.
 (e) Sketch a graph of the function, showing all important features.

11. (a) If $f(x) = 7 + 5x - x^2 - x^3$, find $f'(x)$ and $f''(x)$.
 (b) Find any stationary points and distinguish between them.
 (c) Find the coordinates of any points of inflexion.
 (d) Sketch a graph, showing all important features.
 (e) Find the gradient of the curve at the point of inflexion.
 (f) Hence show that the inflexional tangent has equation $144x - 27y + 190 = 0$.
12. (a) If $y = x^3 + 3x^2 - 72x + 14$, find y' and y'' .
 (b) Show that the curve has a point of inflexion at $(-1, 88)$.
 (c) Show that the gradient of the tangent at the point of inflexion is -75 .
 (d) Hence find the equation of the tangent at the point of inflexion.
13. (a) If $f(x) = x^3$ and $g(x) = x^4$, find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$.
 (b) Both $f(x)$ and $g(x)$ have a stationary point at $(0, 0)$. Evaluate $f''(x)$ and $g''(x)$ when $x = 0$. Can you determine the nature of the stationary points from this calculation?
 (c) Use a table of values of $f'(x)$ and $g'(x)$ to determine the nature of the stationary points.
14. A curve has equation $y = ax^3 + bx^2 + cx + d$, a turning point at $(0, 5)$, a point of inflexion when $x = \frac{1}{2}$, and crosses the x -axis at $x = -1$. Find the values of a , b , c and d .
15. (a) Show that if $y = \frac{x+2}{x-3}$, then $y'' = \frac{10}{(x-3)^3}$.
 (b) By examining the sign of $(x-3)^3$, determine when the curve is concave up, and when it is concave down.
 (c) Hence sketch a graph of $y = \frac{x+2}{x-3}$.
16. Given the graph of $y = f(x)$ drawn to the right, on separate axes sketch graphs of:
 (a) $y = f'(x)$
 (b) $y = f''(x)$
17. (a) Show that if $y = x(x-1)^3$, then $f'(x) = (x-1)^2(4x-1)$ and $f''(x) = 6(x-1)(2x-1)$.
 (b) Sketch $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ on the same axes and compare them.
18. (a) Find $f'(x)$ and $f''(x)$ for the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$.
 (b) By completing the square, show that $f'(x) > 0$ for all x , and hence that $f(x)$ is an increasing function.
 (c) Find the coordinates of any points of inflexion.
 (d) Hence sketch a graph of the function.
19. (a) Find the values of x for which the function $y = x^{\frac{2}{3}}$ is:
 (i) increasing, (ii) decreasing, (iii) concave up, (iv) concave down.
 (b) Hence sketch a graph of the function, indicating all critical points.



EXTENSION

20. (a) Use the definition of the derivative to show that the derivative of an even function is odd. [HINT: If $f(x)$ is an even function, then $f(-x+h) - f(-x) = f(x-h) - f(x)$.]
 (b) Show similarly that the derivative of an odd function is even.

10F Curve Sketching using Calculus

Two quite distinct ways have been used so far to sketch a curve from its equation. Familiar equations can often be sketched using the method of ‘known curves and their transformations’ introduced at the end of Chapter Two. For unfamiliar equations, there is the ‘curve sketching menu’ introduced at the end of Chapter Three. This section will complete that curve sketching menu by adding Steps 5 and 6, which apply the calculus.

12

A CURVE SKETCHING MENU:

0. **PREPARATION:** Combine any fractions using a common denominator, and then factor top and bottom as far as possible.
1. **DOMAIN:** Find the domain of $f(x)$ (*always* do this first).
2. **SYMMETRY:** Find whether the function is odd or even, or neither.
3. **A. INTERCEPTS:** Find the y -intercept and all x -intercepts (zeroes).
B. SIGN: Use a table of test points of $f(x)$ to find where the function is positive, and where it is negative.
4. **A. VERTICAL ASYMPTOTES:** Examine the behaviour near any discontinuities, noting any vertical asymptotes.
B. HORIZONTAL ASYMPTOTES: Examine behaviour as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, noting any horizontal asymptotes (and possibly any oblique asymptotes).
5. **THE FIRST DERIVATIVE:**
 - A. Find the zeroes and discontinuities of $f'(x)$.
 - B. Use a table of test points of $f'(x)$ to determine the nature of the stationary points, and the slope of the function throughout its domain.
 - C. It may be necessary to take limits of $f'(x)$ near discontinuities of $f'(x)$ and as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.
6. **THE SECOND DERIVATIVE:**
 - A. Find the zeroes and discontinuities of $f''(x)$.
 - B. Use a table of test points of $f''(x)$ to find any points of inflexion (it may be useful to find the gradients of the inflexional tangents), and the concavity of the function throughout its domain.
7. **ANY OTHER FEATURES.**

The final Step 7 is a routine warning that the variety of functions can never be contained in any simple menu. In particular, symmetries about points other than the origin and lines other than the y -axis have not been considered here, yet every parabola has an axis of symmetry, and every cubic has point symmetry in its point of inflexion. Neither has periodicity been included, despite the fact that the trigonometric functions repeat periodically.

It is rare for all these steps to be entirely relevant for any particular function. The domain should always be considered first in every example, and oddness and evenness are always significant, but beyond this, only experience can be a guide as to which steps are essential to bring out the characteristic shape of the curve. The menu should be a checklist and not a rigid prescription. Examination questions usually give a guide as to what is to be done, and the particular arrangement of the menu above belongs to this text and not to the Syllabus.

Known Curves and their Transformations: It's important to keep in mind that familiar curves can be sketched much more quickly by recognising that the curve is some transformation of a curve that is already well known. Calculus is then quite unnecessary. This approach was discussed in detail in the final section of Chapter Two.

Exercise 10F

- Use the steps of the curve sketching menu to sketch the graphs of the following polynomial functions. Indicate the coordinates of any stationary points, points of inflexion, and intercepts with the axes. Do not attempt to find the intercepts with the x -axis in parts (c), (e) and (f):

(a) $y = 2x^3 - 3x^2 + 5$	(c) $y = 12x^3 - 3x^4 + 11$	(e) $y = x^3 - 3x^2 - 24x + 5$
(b) $y = 9x - x^3$	(d) $y = x(x - 6)^2$	(f) $y = x^4 - 16x^3 + 72x^2 + 10$
- If $f(x) = \frac{x^2}{1+x^2}$, show that $f'(x) = \frac{2x}{(1+x^2)^2}$ and $f''(x) = \frac{2-6x^2}{(1+x^2)^3}$.
 - Hence find the coordinates of any stationary points and determine their nature.
 - Find the coordinates of any points of inflexion.
 - State the equation of the horizontal asymptote.
 - Sketch a graph of the function, indicating all important features.
- If $f(x) = \frac{4x}{x^2+9}$, show that $f'(x) = \frac{36-4x^2}{(x^2+9)^2}$ and $f''(x) = \frac{8x^3-216x}{(x^2+9)^3}$.
 - Hence find the coordinates of any stationary points and determine their nature.
 - Find the coordinates of any points of inflexion.
 - State the equation of the horizontal asymptote.
 - Sketch a graph of the function, indicating all important features.

DEVELOPMENT

- Sketch graphs of the following rational functions, indicating all stationary points, points of inflexion and intercepts with the axes. For each question solve the equation $y = 1$ to see where the graph cuts the horizontal asymptote:

(a) $y = \frac{x^2 - x - 2}{x^2}$	(b) $y = \frac{x^2 - 2x}{(x+2)^2}$
- Without finding inflexions, sketch the graphs of the following functions. Indicate any asymptotes, stationary points and intercepts with the axes (some of them also have oblique asymptotes):

(a) $y = \frac{x}{x-1}$	(d) $y = \frac{x-x^2-1}{1+x+x^2}$	(g) $y = x^2 + \frac{1}{x^2}$
(b) $y = \frac{x^2-1}{x^2-4}$	(e) $y = \frac{x}{x^2+1}$	(h) $y = \frac{x^2+5}{x-2}$
(c) $y = \frac{1}{(x-2)(x+1)}$	(f) $y = x + \frac{1}{x}$	(i) $y = \frac{(x+1)^3}{x}$
- Write down the domain of each of the following functions and sketch a graph, clearly indicating any stationary points and intercepts with the axes:

(a) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$	(b) $y = x\sqrt{3-x}$	(c) $y = \frac{\sqrt{x}}{2+x}$	(d) $y = \frac{x}{\sqrt{1+x}}$

7. By carefully noting their critical points, sketch the graphs of the following functions:
- (a) $y = x^{\frac{2}{7}}$ (b) $y = (x - 2)^{\frac{2}{3}} + 4$ (c) $y = (1 - x)^{\frac{1}{4}} - 2$
8. Using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, find any stationary points and sketch the graphs of the functions:
- (a) $x = 6t, y = 3t^2$ (b) $x = \frac{1}{t}, y = \frac{1}{t^2}$ (c) $x = t + 1, y = t^3 - 3t$

EXTENSION

9. (a) Sketch $f(x) = x(4 - x^2)$, clearly indicating all stationary points and intercepts.
 (b) What is the relationship between the x -coordinates of the stationary points of the function $y = (f(x))^2$ and the information found in part (a)?
 (c) Hence sketch $y = (f(x))^2$ and $y = (f(x))^3$.
10. (a) Sketch $f(x) = (x + 5)(x - 1)$, clearly indicating the turning point.
 (b) Where do the graphs of the functions $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect?
 (c) Differentiate $y = \frac{1}{f(x)}$, and explain why $\frac{1}{f(x)}$ increases as $f(x)$ decreases and vice versa.
 (d) Using part (a), find and analyse the stationary point of $y = \frac{1}{(x + 5)(x - 1)}$.
 (e) Hence sketch a graph of the reciprocal function on the same diagram as part (a).
11. (a) Use implicit differentiation to find dy/dx if $x^3 + y^3 = a^3$ where a is constant.
 (b) Hence sketch the graph of $x^3 + y^3 = a^3$, showing all critical points.
12. (a) Sketch a graph of the function $y = |2x - 1| + |x + 3|$, showing all critical points.
 (b) Hence solve the inequality $|2x - 1| > 4 - |x + 3|$.

10G Global Maximum and Minimum

Australia has many high mountain peaks, each of which is a *local* or *relative maximum*, because each is the highest point relative to other peaks in its immediate locality. Mount Kosciuszko is the highest of these, but it is still not a *global* or *absolute maximum*, because there are higher peaks on other continents of the globe. Mount Everest in Asia is the global maximum over the whole world.

Suppose now that $f(x)$ is a function defined on some domain, not necessarily the natural domain of the function, and that $A(a, f(a))$ is a point on the curve $y = f(x)$ within the domain. Then:

13

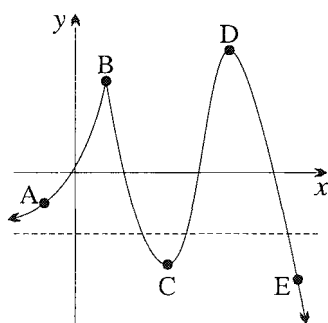
GLOBAL MAXIMUM: The point A is called a *global* or *absolute maximum* if

$$f(x) \leq f(a), \text{ for all } x \text{ in the domain.}$$

GLOBAL MINIMUM: Similarly, A is called a *global* or *absolute minimum* if

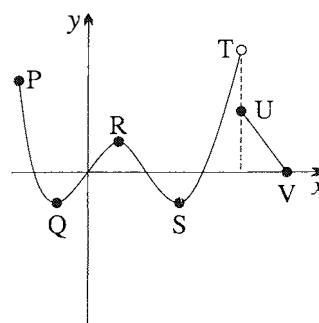
$$f(x) \geq f(a), \text{ for all } x \text{ in the domain.}$$

The following diagrams illustrate what has to be considered in the general case.



The domain of $f(x)$ is the whole real line.

1. There are local maxima at the point B , where $f'(x)$ is undefined, and at the turning point D . This point D is also the global maximum.
2. There is a local minimum at the turning point C , which is lower than all points on the curve to the left past A . But there is no global minimum, because the curve goes infinitely far downwards to the right of E .



The domain of $f(x)$ is the closed interval on the x -axis from P to V .

1. There are local maxima at the turning point R and at the end-point P . But there is no global maximum, because the point T has been omitted from the curve.
2. There are local minima at the two turning points Q and S , and at the endpoint V . These points Q and S have equal heights and are thus both global minima.

Testing for Global Maximum and Minimum: These examples show that there are three types of points that must be considered and compared when finding the global maximum and minimum of a function $f(x)$ defined on some domain.

14

TESTING FOR GLOBAL MAXIMUM AND MINIMUM: Examine and compare:

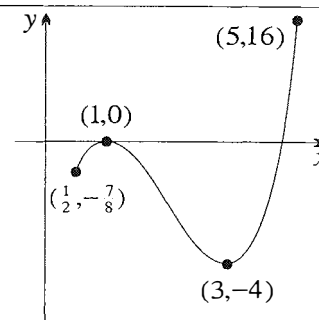
1. turning points,
2. boundaries of the domain (which may involve behaviour for large x),
3. discontinuities of $f'(x)$ (because they may be local extrema).

More simply, examine and compare the critical values and the boundary values.

WORKED EXERCISE: Find the global maximum and minimum of $f(x) = x^3 - 6x^2 + 9x - 4$, where $\frac{1}{2} \leq x \leq 5$.

SOLUTION: The unrestricted curve is sketched in Section 10B, and substituting the boundaries, $f(\frac{1}{2}) = -\frac{7}{8}$ and $f(5) = 16$ gives the diagram on the right.

So the global maximum is 16 when $x = 5$, and the global minimum is -4 when $x = 3$.



WORKED EXERCISE: Find the global maximum and minimum of $y = \frac{x+1}{x^2+3}$ for $x \geq 0$.

$$\begin{aligned} \text{SOLUTION: } y' &= \frac{(x^2+3) - 2x(x+1)}{(x^2+3)^2} \\ &= \frac{-(x^2+2x-3)}{(x^2+3)^2} \end{aligned}$$

$$= \frac{-(x+3)(x-1)}{(x^2+3)^2},$$

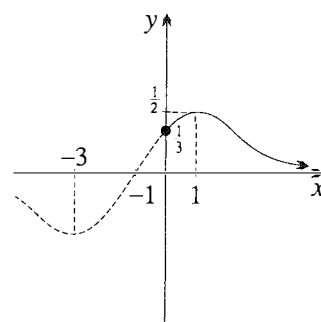
so y' has a zero at $x = 1$ and no discontinuities:

x	0	1	2
y'	$\frac{1}{3}$	0	$-\frac{5}{49}$
	/	—	\

So $(1, \frac{1}{2})$ is a maximum turning point.

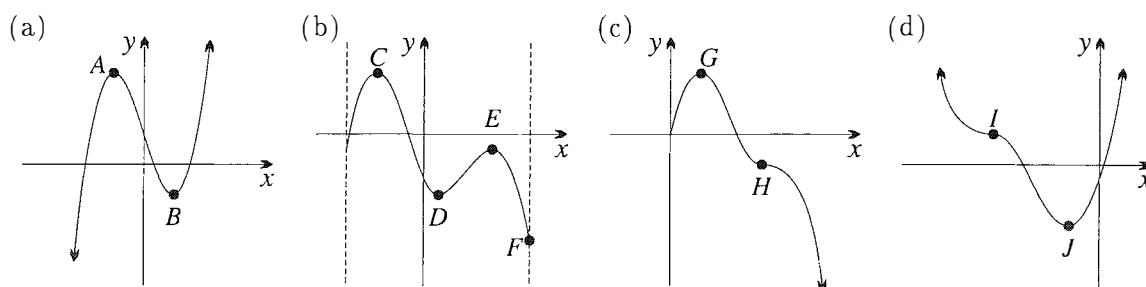
When $x = 0$, $y = \frac{1}{3}$, and $y \rightarrow 0$ as $x \rightarrow \infty$,

so $(1, \frac{1}{2})$ is a global maximum, but there is no global minimum.



Exercise 10G

1. In each diagram below, name the points that are: (i) absolute maxima, (ii) absolute minima, (iii) relative maxima, (iv) relative minima, (v) horizontal points of inflexion.



2. Sketch each of the given functions and state the global minimum and global maximum of each function in the specified domain:

- (a) $y = x^2$, $-2 \leq x \leq 2$ (f) $y = \frac{1}{x}$, $-4 \leq x \leq -1$
 (b) $y = 5 - x$, $0 \leq x \leq 3$ (g) $y = x^{\frac{1}{3}}$, $1 \leq x \leq 8$
 (c) $y = \sqrt{16 - x^2}$, $-4 \leq x \leq 4$ (h) $y = \begin{cases} -1, & \text{for } x < -2, \\ x + 1, & \text{for } -2 \leq x < 1, \\ 2, & \text{for } x \geq 1. \end{cases}$
 (d) $y = |x|$, $-5 \leq x \leq 1$
 (e) $y = \sqrt{x}$, $0 \leq x \leq 8$

DEVELOPMENT

3. Sketch graphs of the following functions, indicating any stationary points. Determine the absolute minimum and maximum points for each function in the specified domain.

- (a) $y = 7$, $-1 \leq x \leq 4$ (d) $y = x^3 - 3x^2 + 5$, $-3 \leq x \leq 2$
 (b) $y = \frac{1}{x^2}$, $3 \leq x \leq 4$ (e) $y = 3x^3 - x + 2$, $-1 \leq x \leq 1$
 (c) $y = x^2 - 4x + 3$, $0 \leq x \leq 5$ (f) $y = x^3 - 6x^2 + 12x$, $0 \leq x \leq 3$

4. Find (i) the local maxima or minima and (ii) the global maximum and minimum of the function $y = x^4 - 8x^2 + 11$ on each of the following domains:

- (a) $1 \leq x \leq 3$ (b) $-4 \leq x \leq 1$ (c) $-1 \leq x \leq 0$

5. Use the complete curve sketching menu to sketch the following functions. State the absolute minimum and maximum values of each function on the domain $-2 \leq x \leq 1$.

- (a) $y = x^2 + 1$ (b) $y = \sqrt{x^2 + 1}$ (c) $y = \frac{1}{\sqrt{x^2 + 1}}$ (d) $y = \frac{x}{\sqrt{x^2 + 1}}$

EXTENSION

6. We have assumed without comment that a function that is continuous on a closed interval has a global maximum in that interval, and that the function reaches that global maximum at some value in the interval (and similarly for minima). Proving this obvious-looking result is beyond the course, but draw sketches, with and without asymptotes, to show that the result is false when either the function is not continuous or the interval is not closed.
7. Consider the even function $y = \sin \frac{360^\circ}{x}$ (and try graphing it on a machine).
- (a) How many zeroes are there in the closed intervals $1 \leq x \leq 10$, $0.1 \leq x \leq 1$ and $0.01 \leq x \leq 0.1$? (b) How many zeroes are there in the open interval $0 < x < 1$?
 (c) Does the function have a limit as $x \rightarrow 0^+$ or as $x \rightarrow \infty$?
 (d) Does the function have a global maximum or minimum?

10H Applications of Maximisation and Minimisation

The practical applications of maximisation and minimisation should be obvious — for example, maximise the volume of a box built from a rectangular sheet of cardboard, minimise the fuel used in a flight, maximise the profits from manufacturing and selling an article, minimise the metal used in a can of soft drink.

Maximisation and Minimisation Using Calculus: Many of these problems involve only quadratic functions, and so can be solved by the methods of Chapter Eight without any appeal to calculus. The use of the derivative to find the global maximum and minimum, however, applies to any differentiable function (and may be more convenient for some quadratics).

MAXIMISATION AND MINIMISATION PROBLEMS: After drawing a diagram:

- 15
1. Introduce the two variables from which the function is to be formed.
 ‘Let y (or whatever) be the quantity that is to be maximised, and let x (or whatever) be the quantity that can be varied.’
 2. Form an equation in the two variables, noting any restrictions.
 3. Find the global maximum or minimum.
 4. Write a careful conclusion.

NOTE: A claim that a stationary point is a maximum or minimum must be justified by a proper analysis of the nature of the stationary point.

WORKED EXERCISE: An open rectangular box is to be made by cutting square corners out of a square piece of cardboard $60 \text{ cm} \times 60 \text{ cm}$ and folding up the sides. What is the maximum volume of the box, and what are its dimensions then? What dimensions give the minimum volume?

SOLUTION: Let V be the volume of the box,
 and let x be the side lengths of the squares.

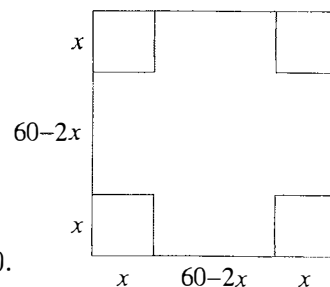
Then the box is $x \text{ cm}$ high,

with base a square of side length $60 - 2x$,

so
$$V = x(60 - 2x)^2,$$

$$= 3600x - 240x^2 + 4x^3, \text{ where } 0 < x < 30.$$

Differentiating, $V' = 3600 - 480x + 12x^2$



$$= 12(x - 30)(x - 10),$$

so V' has zeroes at $x = 10$ and $x = 30$, and no discontinuities.

Furthermore, $V'' = -480 + 24x$,

so $V''(10) = -240 < 0$ and $V''(30) = 240 > 0$.

Hence $(10, 16\,000)$ is a maximum turning point,

and the maximum volume is $16\,000\text{ cm}^3$ when the box is $10\text{ cm} \times 40\text{ cm} \times 40\text{ cm}$.

Also, $V = 0$ at both boundaries $x = 0$ and $x = 30$, so the minimum value is zero when the dimensions are $60\text{ cm} \times 60\text{ cm} \times 0\text{ cm}$ or $0\text{ cm} \times 0\text{ cm} \times 30\text{ cm}$.

Introducing Extra Pronumerals: Some problems require other variable or constant pronumerals to be added during the working, but these other pronumerals must not be confused with the two variables from which the function is to be formed.

WORKED EXERCISE: The cost C , in dollars per hour, of running a boat depends on the speed v km/hr of the boat according to the formula $C = 500 + 40v + 5v^2$. On a trip from Port A to Port B, what speed will minimise the total cost of the trip?

SOLUTION: [NOTE: The introduction of the constant D is the key step here.]

Let T be the total cost of the trip. We seek T as a function of v .

Let D be the distance between the two ports,

then since $\text{time} = \frac{\text{distance}}{\text{speed}}$, the time for the trip is $\frac{D}{v}$,

so the total cost is $T = (\text{cost per hour}) \times \text{time for the trip}$

$$= C \times \frac{D}{v}$$

$$T = \frac{500D}{v} + 40D + 5Dv, \text{ where } v > 0.$$

Differentiating, $\frac{dT}{dv} = -\frac{500D}{v^2} + 5D$, since D is a constant,

$$= \frac{5D}{v^2}(-100 + v^2)$$

$$= \frac{5D}{v^2}(v - 10)(v + 10),$$

so $\frac{dT}{dv}$ has a zero at $v = 10$, and no discontinuities for $v > 0$.

Differentiating again, $\frac{d^2T}{dv^2} = \frac{1000D}{v^3}$, which is positive for $v = 10$,

so $v = 10$ gives a minimum turning point.

When $v = 10$, $C = 500 + 400 + 500 = 1400$ dollars per hour,

so a speed of 10 km/hr will minimise the cost of the trip.

Exercise 10H

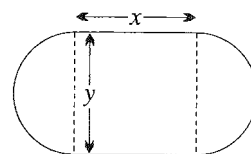
NOTE: You must always prove that any stationary point is a maximum or minimum, either by creating a table of values of the derivative, or by substituting into the second derivative, or perhaps in some other way. It is never acceptable to assume this from the wording of a question.

1. (a) If $x + y = 8$, express $P = x^2 + y^2$ in terms of x only.
 (b) Find $\frac{dP}{dx}$ and hence find the value of x for which P obtains its minimum value.
 (c) Calculate the minimum value of P (and prove that it is a minimum).
2. (a) If $2x + y = 11$, express $P = xy$ in terms of x only.
 (b) Find $\frac{dP}{dx}$ and hence the value of x for which P obtains its maximum value.
 (c) Calculate the maximum value of P (and prove that it is a maximum).
3. At time t seconds, a particle has height $h = 3 + t - 2t^2$ metres.
 (a) Find dh/dt and show that the maximum height occurs after 0.25 seconds.
 (b) Find the maximum height.
4. (a) A rectangle has a constant perimeter of 20 cm. If the length of the rectangle is x cm, show that it must have width $(10 - x)$ cm, and hence that its area is $A = 10x - x^2$.
 (b) Find dA/dx , and hence find the value of x for which A is maximum.
 (c) Hence find the maximum area.
5. A landscaper is constructing a rectangular garden bed. Three of the sides are to be fenced using 40 metres of fencing, while an existing wall will form the fourth side of the rectangle.
 (a) If x is the length of the side opposite the wall, show that the remaining two sides each have length $(20 - \frac{1}{2}x)$ m.
 (b) Show that the area is $A = 20x - \frac{1}{2}x^2$.
 (c) Find dA/dx and hence the value of x for which A obtains its maximum value.
 (d) Find the maximum area of the garden bed.
6. The total cost of producing x telescopes per day is given by $C = (\frac{1}{5}x^2 + 15x + 10)$ dollars, and each telescope is sold for a price of $(47 - \frac{1}{3}x)$ dollars.
 (a) Find an expression for the revenue R raised from the sale of x telescopes.
 (b) Find an expression for the daily profit P made if x telescopes are sold ($P = R - C$).
 (c) How many telescopes should be made daily in order to maximise the profit?

————— DEVELOPMENT —————

7. The sum of two positive numbers is 40. [HINT: Let the numbers be x and $40 - x$.] Find the numbers if:
 - (a) their product is a maximum,
 - (b) the sum of their squares is a minimum, [HINT: Let $S = x^2 + (40 - x)^2$.]
 - (c) the product of the cube of one number and the square of the other number is a maximum. [HINT: Let $P = x^3(40 - x)^2$, and show that $P' = 5x^2(x - 24)(x - 40)$.]
8. A rectangle has a constant area of 36 cm^2 .
 - (a) If the length of the rectangle is x , show that its perimeter is $P = 2x + \frac{72}{x}$.
 - (b) Show that $\frac{dP}{dx} = \frac{2(x - 6)(x + 6)}{x^2}$ and hence find the minimum possible perimeter.
9. A piece of wire of length 10 metres is cut into two pieces and used to form two squares.
 - (a) If one piece of wire has length x metres, find the side length of each square.
 - (b) Show that the combined area of the squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.
 - (c) Find dA/dx and hence find the value of x that makes A a minimum.
 - (d) Find the least possible value of the combined areas.

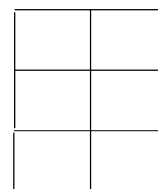
10. A running track of length 400 metres is designed using two sides of a rectangle and two semicircles, as shown. The rectangle has length x metres and the semicircles each have diameter y metres.



- Show that $x = \frac{1}{2}(400 - \pi y)$.
 - The region inside the track will be used for field events. Show that its area is $A = \frac{1}{4}(800y - \pi y^2)$.
 - Hence find the maximum area that may be enclosed.
11. A box has a square base and no lid. Let the square base have length x cm and the box have height h cm.
- Show that the surface area of the box is given by $S = x^2 + 4xh$.
 - If the box has a volume of 32 cm^3 , show that $h = \frac{32}{x^2}$ and hence that $S = x^2 + \frac{128}{x}$.
 - Find dS/dx , and hence find the dimensions of the box that minimise its surface area.

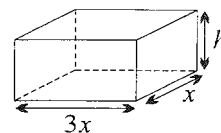
12. A window frame consisting of 6 equal rectangles is shown on the right.

- Let the entire frame have height h metres and width y metres. If 12 metres of frame is available, show that $y = \frac{1}{4}(12 - 3h)$.



- Show that the area of the window is $A = 3h - \frac{3}{4}h^2$.
- Find $\frac{dA}{dh}$ and hence find the dimensions of the frame so that the area of the window is maximum.

13. The steel frame of a rectangular prism, as illustrated in the diagram, is three times as long as it is wide. The prism has a volume of 4374 m^3 . Find the dimensions of the frame so that the minimum amount of steel is used.



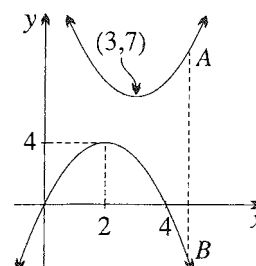
14. A transport company runs a truck from Hobart to Launceston, a distance of 250 km, at a constant speed of v km/hr. For a given speed v , the cost per hour is $6400 + v^2$ cents.

- If the trip costs C cents, show that $C = 250 \left(\frac{6400}{v} + v \right)$.
- Find the speed for which the cost of the journey is minimised.
- Find the minimum cost of the journey.

15. A cardboard box is to have square ends and a volume of 768 cm^3 . It is to be sealed using two pieces of tape, one passing entirely around the length and width of the box and the other passing entirely around the height and width of the box. Find the dimensions of the box so that the least amount of tape is used.

16. Two sides of a rectangle lie along the x and y axes. The vertex opposite the origin is in the first quadrant and lies on $3x + 2y = 6$. What is the maximum area of the rectangle?

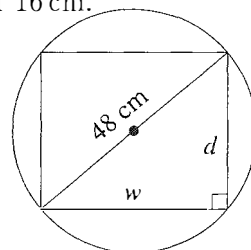
17. The diagram to the right shows a point A on the curve $y = (x - 3)^2 + 7$, and a point B with the same x -coordinate as A on the curve $y = x(4 - x)$. Find an expression for the length of AB , and determine its minimum length.



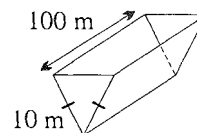
18. A point A lies on the curve $y = x(5 - x)$, and a point B with the same x -coordinate as A lies on the curve $y = x(x - 3)$. Show this information on a diagram, then find an expression for the length of AB , and determine the maximum length if $0 \leq x \leq 4$.
19. (a) Sketch the parabola $y = 4 - x^2$ and the tangent at $P(a, 4 - a^2)$ in the first quadrant. (b) Find the equation of the tangent at P . (c) Hence find the value of a for which the area of the triangle formed by the tangent and the coordinate axes will be minimum.
20. The point $P(x, y)$ lies on the parabola $y = x^2$.
 (a) Show that the distance from P to the line $x - y - 1 = 0$ is $\frac{1}{2}\sqrt{2}(x^2 - x + 1)$.
 (b) Hence find P for which the distance is minimum.

21. Find the maximum area of a right triangle with hypotenuse of length 16 cm.

22. Engineers have determined that the strength s of a rectangular beam varies as the product of the width w and the square of the depth d of the beam, that is, $s = kwd^2$ for some constant k . Find the dimensions of the strongest rectangular beam that can be cut from a cylindrical log with diameter 48 cm.



23. (a) Draw a diagram showing the region enclosed between the parabola $y^2 = 4ax$ and its latus rectum $x = a$.
 (b) Find the dimensions of the rectangle of maximum area that can be inscribed in this region.
24. The point A lies on the positive half of the x -axis, the point B lies on the positive half of the y -axis, and the interval AB passes through the point $P(5, 3)$. Find the coordinates of A and B so that $\triangle AOB$ has minimum area.
25. The ends of a 100 metre long trough are isosceles triangles whose equal sides have length 10 metres. Find the height of the trough in order that its volume is maximised.



26. A man in a rowing boat is presently 6 km from the nearest point A on the shore. He wants to reach as soon as possible a point B that is a further 20 km down the shore from A . If he can row at 8 km/hr and run at 10 km/hr, how far from A should he land?
27. A page of a book is to have 80 cm^2 of printed material. There is to be a 2 cm margin at the top and bottom and a 1 cm margin on each side of the page. What should be the dimensions of the page in order to use the least amount of paper?
28. (a) An open rectangular box is to be formed by cutting squares of side length x cm from the corners of a rectangular sheet of metal that has length 40 cm and width 15 cm. Find the value of x in order to maximise the volume of the box.
 (b) An open rectangular box is to be formed by cutting equal squares from a sheet of tin which has dimensions a metres by b metres. Find the area of the squares to be removed if the box is to have maximum volume. Check your answer to part (a).

EXTENSION

29. If an object is placed u cm in front of a lens of focal length f cm, then the image appears v cm behind the lens, where $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$. Show that the minimum distance between the image and the object is $4f$ cm.

30. The method of least squares can be used to find the line of best fit for a series of points obtained experimentally. The line of best fit through the points (x_1, y_1) , (x_2, y_2) , $\dots$, (x_n, y_n) is obtained by selecting the linear function $f(x)$ that minimises the sum $(y_1 - f(x_1))^2 + (y_2 - f(x_2))^2 + \dots + (y_n - f(x_n))^2$. Find the value of m for which the equation $y = mx + 2$ best fits the points $(1, 6)$, $(2, 7)$, $(3, 13)$ and $(5, 15)$.
31. Snell's law states that light travels through a homogeneous medium in a straight line at a constant velocity dependent upon the medium. Let the velocity of light in air be v_1 and the velocity of light in water be v_2 . Show that light will travel from a point in air to a point in water in the shortest possible time if $\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$, where the light ray makes angles of θ_1 and θ_2 in air and water respectively with the normal to the surface. NOTE: This question can be redone after Chapter Fourteen is completed.

10I Maximisation and Minimisation in Geometry

Geometrical problems provided the classic situations where maximisation and minimisation problems were first clearly stated and solved. In many of these problems, the answer and the solution both have considerable elegance and clarity, and they make the effectiveness of calculus very obvious.

WORKED EXERCISE: [A difficult example] A square pyramid is inscribed in a sphere (the word 'inscribed' means that all five vertices of the pyramid lie on the surface of the sphere). What is the maximum ratio of the volumes of the pyramid and the sphere, and what are the corresponding proportions of the pyramid?

SOLUTION: Let the volume of the pyramid be V .

Let the height MT of the pyramid be $r + h$, where $-r \leq h \leq r$, r being the constant radius of the sphere.

Then from the diagram, $MA^2 = r^2 - h^2$ (Pythagoras' theorem)

and since $MB = MA$, $AB^2 = 2(r^2 - h^2)$.

So
$$V = \frac{1}{3} \times AB^2 \times MT$$

$$= \frac{2}{3}(r^2 - h^2)(r + h)$$

and the required function is $V = \frac{2}{3}(r^3 + r^2h - rh^2 - h^3)$.

Differentiating,
$$\frac{dV}{dh} = \frac{2}{3}(r^2 - 2rh - 3h^2)$$

$$= \frac{2}{3}(r - 3h)(r + h),$$

so dV/dh has zeroes at $h = \frac{1}{3}r$ or $-r$, and no discontinuities.

Also,
$$\frac{d^2V}{dh^2} = \frac{2}{3}(-2r - 6h),$$

which is negative for $h = \frac{1}{3}r$, giving a maximum (it is positive for $h = -r$).

So the volume is maximum when $h = \frac{1}{3}r$, that is, when $MT = \frac{4}{3}r$,

and then
$$AB^2 = 2(r^2 - \frac{1}{9}r^2)$$

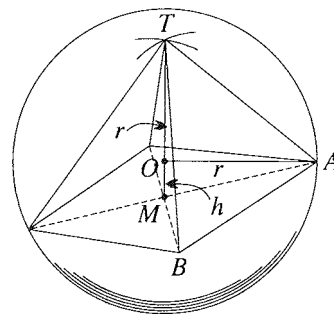
$$= \frac{16}{9}r^2.$$

This means that $AB = MT = \frac{4}{3}r$,

so that the height and base side length of the pyramid are equal.

Then
$$\text{ratio of volumes} = \frac{1}{3} \times \frac{16}{9}r^2 \times \frac{4}{3}r : \frac{4}{3}\pi r^3$$

$$= 16 : 27\pi.$$

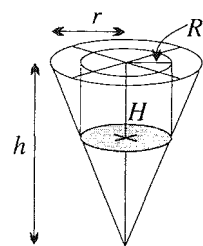
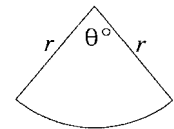


Exercise 10I

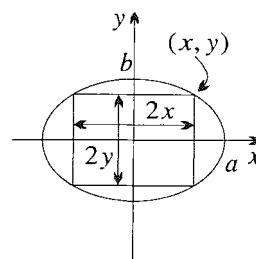
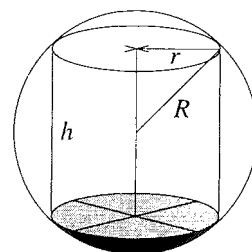
- The sum of the height h of a cylinder and the circumference of its base is 10 metres.
 - If r is the radius of the cylinder, show that the volume is $V = \pi r^2(10 - 2\pi r)$.
 - Find the maximum volume of the cylinder.
- A closed cylindrical can is to have a surface area of $60\pi \text{ cm}^2$.
 - If the cylinder has height h and radius r , show that $h = \frac{30 - r^2}{r}$.
 - Show that the volume of the can is given by $V = \pi r(30 - r^2)$.
 - Find the maximum possible volume of the can in terms of π .
- A rectangle has an area of 121 m^2 .
 - If the length is x metres and the perimeter is P metres, show that $P = 2\left(x + \frac{121}{x}\right)$.
 - Hence show that the perimeter is a minimum when the rectangle is a square.
- A right circular cone is to have a fixed slant height of s cm.
 - Let the cone have height h and radius r . Explain why $s^2 = r^2 + h^2$.
 - Show that the volume of the cone is given by $V = \frac{1}{3}\pi h(s^2 - h^2)$.
 - Show that the volume of the cone is maximised when $h = \frac{1}{3}\sqrt{3}s$.
 - Hence find the maximum volume of the cone.

DEVELOPMENT

- A piece of wire of length ℓ is bent to form a sector of a circle of radius r .
 - If the sector subtends an angle of θ° at the centre, show that $\ell = 2r + \frac{\pi r \theta}{180}$, and find an expression for θ in terms of ℓ and r .
 - Show that the area of the sector is $A = \frac{1}{2}r(\ell - 2r)$.
 - Show that the area of the sector is maximised when $r = \frac{1}{4}\ell$.
- A cylinder of height h metres is inscribed in a sphere of constant radius R metres.
 - If the cylinder has radius r metres, show that $r^2 = R^2 - \frac{1}{4}h^2$.
 - Show that the volume of the cylinder is given by $V = \frac{\pi}{4}h(4R^2 - h^2)$.
 - Show that the volume of the cylinder is maximised when $h = \frac{2}{3}\sqrt{3}R$.
 - Hence show that the ratio of the volume of sphere to the maximum volume of the cylinder is $\sqrt{3} : 1$.
- The sum of the radii of two circles is constant, so that $r_1 + r_2 = k$, where k is constant.
 - Find an expression for the sum of the areas of the circles in terms of one variable only.
 - Hence show that the sum of the areas is least when the circles are congruent.
- A cylinder of height H and radius R is inscribed in a cone of constant height h and constant radius r .
 - Use similar triangles to show that $H = \frac{h(r - R)}{r}$.
 - Find an expression for the volume of the cylinder in terms of the variable R .
 - Find the maximum possible volume of the cylinder in terms of h and r .



9. A cylinder is inscribed in a cone of height h and base radius r . Using a method similar to the previous question, find the maximum possible value of the curved surface area of the cylinder.
10. A rectangle is inscribed in a quadrant of a circle of radius r so that two of its sides are along the bounding radii of the quadrant.
- If the rectangle has length x and width y , show that $x^2 + y^2 = r^2$, and hence that the area of the rectangle is given by $A = y\sqrt{r^2 - y^2}$.
 - Use the product rule and chain rule to find dA/dy .
 - Show that the maximum area of the rectangle is $\frac{1}{2}r^2$.
11. A cylindrical can open at one end is to have a fixed outside surface area S .
- Show that if the can has height h and radius r , then $h = \frac{S - \pi r^2}{2\pi r}$.
 - Find an expression for the volume of the cylinder in terms of r , and hence show that the maximum possible volume is attained when the height of the can equals its radius.
12. A right triangle has base 60 cm and height 80 cm. A rectangle is inscribed in the triangle, so that one of its sides lies along the base of the triangle.
- Let the rectangle have length y cm and height x cm. Then, by using similar triangles, show that $y = \frac{3}{4}(80 - x)$.
 - Hence find the dimensions of the rectangle of maximum area.
13. A cylinder, open at both ends, is inscribed in a sphere of constant radius R .
- Let the cylinder have height h and radius r as illustrated in the diagram. Show that $h = 2\sqrt{R^2 - r^2}$.
 - Show that the surface area of the cylinder is given by $S = 4\pi r\sqrt{R^2 - r^2}$.
 - Hence find the maximum surface area of the cylinder in terms of R .
14. (a) An ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Make y^2 the subject, and hence write down the equation of the upper half of the ellipse.
- (b) A rectangle with vertical side of length $2y$ and horizontal side of length $2x$ is inscribed in the ellipse as shown. Find, in terms of x , an expression for the area of the rectangle.
- (c) Hence find maximum area of the rectangle, and the value of x for which this occurs.
15. Show that a rectangle with fixed perimeter has its shortest diagonal when it is a square.
16. Show that a closed cylindrical can of fixed volume will have minimum surface area when its height is equal in length to its diameter.
17. A cone of height h cm is inscribed in a sphere of constant radius R cm. Find the ratio $h : R$ when the volume of the cone is maximised.
18. (a) The perimeter of an isosceles triangle is 12 cm. Find its maximum area.
- (b) Prove the general result that for all isosceles triangles of constant perimeter, the one with maximum area is equilateral.



19. Find the maximum area of an isosceles triangle in which the equal sides have a fixed length of a units. NOTE: The equilateral triangle is not the triangle of maximum area.
20. A parallelogram is inscribed in a triangle so that they have one vertex in common. The other vertices of the parallelogram lie on the three different sides of the triangle. Show that the maximum area of the parallelogram is half that of the triangle.
21. An isosceles triangle is to circumscribe a circle of constant radius r . Find the minimum area of such a triangle.
22. A cone with semi-vertical angle 45° is to circumscribe a sphere of radius 12 cm. Is there any maximisation problem here, and what can be said about the height and radius of the cone?

EXTENSION

23. A square of side $2x$ and a circle of variable radius r , where $r < x$, have a common centre. In each corner of the square, a circle is constructed so that it touches two sides of the square and the centre circle. Find the value of r that minimises the sum of the areas of the five circles.

10J Primitive Functions

This section reverses the process of differentiation, and asks what we can say about a function if we know its derivative. The discussion of these questions here will be needed for the methods of integration established in the next chapter.

Functions with the Same Derivative: A great many different functions can all have the same derivative. For example, all these functions have the same derivative $2x$:

$$x^2, \quad x^2 + 3, \quad x^2 - 2, \quad x^2 + 4\frac{1}{2}, \quad x^2 - \pi.$$

These functions are all the same apart from a constant term. This is true generally — any two functions with the same derivative differ only by a constant.

THEOREM:

16

- (a) If a function $f(x)$ has derivative zero in an interval $a < x < b$, then $f(x)$ is a constant function in $a < x < b$.
- (b) If $f'(x) = g'(x)$ for all x in an interval $a < x < b$, then $f(x)$ and $g(x)$ differ by a constant in $a < x < b$.

PROOF:

- (a) Because the derivative is zero, the gradient of the curve must be zero throughout the interval. The curve must therefore be a horizontal straight line, and $f(x)$ is a constant function.
- (b) Take the difference between $f(x)$ and $g(x)$ and apply part (a).
 Let $h(x) = f(x) - g(x)$.
 Then $h'(x) = f'(x) - g'(x)$
 $= 0$, for all x in the interval $a < x < b$.
 Hence by part (a), $h(x) = C$, where C is a constant,
 so $f(x) - g(x) = C$, as required.

The Family of Curves with the Same Derivative: Continuing with the very first example, the various functions whose derivatives are $2x$ must all be of the form

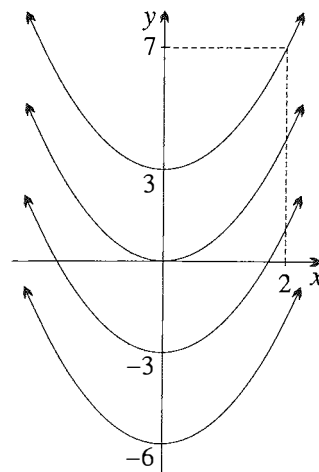
$$f(x) = x^2 + C, \text{ where } C \text{ is a constant.}$$

By taking different values of the constant C , these functions form an infinite family of curves, each consisting of the parabola $y = x^2$ translated upwards or downwards.

Boundary Conditions: If we know also that the curve must pass through a particular point, say $(2, 7)$, then we can evaluate the constant C by substituting the point into $f(x) = x^2 + C$:

$$7 = 4 + C.$$

So $C = 3$ and hence $f(x) = x^2 + 3$ — in place of the infinite family of functions, there is now a single function. Such an extra condition is called a *boundary condition*. (It is also called an *initial condition* if it involves the value of y when $x = 0$, particularly when x is time.)



Primitives: We need a suitable name for the result of this reverse process:

17

DEFINITION: A function $F(x)$ is called a *primitive* of $f(x)$ if the derivative of $F(x)$ is $f(x)$, that is, if $F'(x) = f(x)$.

A function always has infinitely many different primitives, because if $F(x)$ is any primitive of $f(x)$, then $F(x) + C$ is also a primitive of $f(x)$. To give another example, these functions are all primitives of $x^2 + 1$:

$$\frac{1}{3}x^3 + x, \quad \frac{1}{3}x^3 + x + 7, \quad \frac{1}{3}x^3 + x - 13, \quad \frac{1}{3}x^3 + x + 4\pi,$$

and the general primitive of $x^2 + 1$ is $\frac{1}{3}x^3 + x + C$, where C is a constant.

A Rule for Finding Primitives: We have seen that a primitive of x is $\frac{1}{2}x^2$, and a primitive of x^2 is $\frac{1}{3}x^3$. Reversing the formula $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$ gives the general rule:

18

FINDING PRIMITIVES: Suppose $n \neq -1$.

If $\frac{dy}{dx} = x^n$, then $y = \frac{x^{n+1}}{n+1} + C$, for some constant C .

'Increase the index by 1 and divide by the new index.'

WORKED EXERCISE: Find primitives of: (a) $x^3 + x^2 + x + 1$ (b) $5x^3 + 7$

SOLUTION:

$$(a) \quad \frac{dy}{dx} = x^3 + x^2 + x + 1,$$

$$(b) \quad f'(x) = 5x^3 + 7,$$

$$y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C,$$

where C is a constant.

$$f(x) = \frac{5}{4}x^4 + 7x + C,$$

where C is a constant.

WORKED EXERCISE: Find primitives of: (a) $\frac{1}{x^2}$ (b) $\sqrt{x}$

SOLUTION:

$$(a) \quad f'(x) = \frac{1}{x^2}$$

$$= x^{-2},$$

$$f(x) = -x^{-1} + C,$$

where C is a constant,

$$= -\frac{1}{x} + C.$$

$$(b) \quad \frac{dy}{dx} = \sqrt{x}$$

$$= x^{\frac{1}{2}},$$

$$y = \frac{2}{3} x^{\frac{3}{2}} + C,$$

where C is a constant.

Linear Extension: Reversing the formula $\frac{d}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n$ gives:

EXTENSION TO POWERS OF LINEAR FUNCTIONS: Suppose $n \neq -1$.

19 If $\frac{dy}{dx} = (ax+b)^n$, then $y = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, for some constant C .

'Increase the index by 1 and divide by the new index and by the coefficient of x .'

WORKED EXERCISE: Find primitives of:

$$(a) \quad (3x+1)^4$$

$$(b) \quad (1-3x)^6$$

$$(c) \quad \frac{1}{(x+1)^2}$$

$$(d) \quad \sqrt{x+1}$$

SOLUTION:

$$(a) \quad \frac{dy}{dx} = (3x+1)^4,$$

$$y = \frac{1}{15}(3x+1)^5 + C,$$

where C is a constant.

$$(c) \quad \frac{dy}{dx} = (x+1)^{-2},$$

$$y = -(x+1)^{-1} + C,$$

where C is a constant,

$$(b) \quad \frac{dy}{dx} = (1-3x)^6,$$

$$y = -\frac{1}{21}(1-3x)^7 + C,$$

where C is a constant.

$$(d) \quad \frac{dy}{dx} = (x+1)^{\frac{1}{2}},$$

$$y = \frac{2}{3}(x+1)^{\frac{3}{2}} + C,$$

where C is a constant.

Finding the Primitive, Given a Boundary Condition: Often the derivative and a particular value of a function are known. In this case, first find the general primitive, then substitute the known value of the function to work out the constant.

WORKED EXERCISE: Given that $\frac{dy}{dx} = 6x^2 + 1$, and that $y = 12$ when $x = 2$, find y as a function of x .

SOLUTION: Since

$$\frac{dy}{dx} = 6x^2 + 1,$$

$$y = 2x^3 + x + C, \text{ for some constant } C.$$

Substituting $x = 2$ and $y = 12$, $12 = 16 + 2 + C$,

so $C = -6$, and hence

$$y = 2x^3 + x - 6.$$

WORKED EXERCISE: Given that $f''(x) = (2x - 1)^2$, and $f(0) = f(1) = 0$, find $f(2)$.

SOLUTION: Since $f''(x) = (2x - 1)^2$,
 $f'(x) = \frac{1}{6}(2x - 1)^3 + C$, for some constant C
 and $f(x) = \frac{1}{48}(2x - 1)^4 + Cx + D$, for some constant D .
 Since $f(0) = 0$, $0 = \frac{1}{48} + D$
 $D = -\frac{1}{48}$.
 Since $f(1) = 0$, $0 = \frac{1}{48} + C - \frac{1}{48}$.
 So $C = 0$, and hence $f(2) = \frac{81}{48} - \frac{1}{48}$
 $= \frac{5}{3}$.

Primitives of Discontinuous Functions: [This is a subtle point about the primitives of discontinuous functions, which is really beyond the standard course.] When a function is not continuous at some point, the constant may take different values in different parts of the domain. For example, the function $1/x^2$ is not defined at $x = 0$, and its domain disconnects into the two pieces $x > 0$ and $x < 0$. Some primitives of $1/x^2$ are

$$-\frac{1}{x} \quad \text{and} \quad -\frac{1}{x} + 5 \quad \text{and} \quad f(x) = \begin{cases} -\frac{1}{x} + 4, & \text{for } x < 0, \\ -\frac{1}{x} - 7, & \text{for } x > 0. \end{cases}$$

and the general primitive is $F(x) = \begin{cases} -\frac{1}{x} + A, & \text{for } x < 0, \\ -\frac{1}{x} + B, & \text{for } x > 0. \end{cases}$ where A and B

are unrelated constants. In most applications, however, only one branch of the function has any physical significance.

Exercise 10J

1. Find primitives of each of the following (where a and b are constants):

(a) x^6	(d) $5x^9$	(g) 0	(j) $ax^3 + bx^2$
(b) $3x$	(e) $21x^6$	(h) $2x^2 + 5x^7$	(k) x^a
(c) 5	(f) $\frac{1}{3}x^{12}$	(i) $3x^2 - 4x^3 - 5x^4$	(l) $ax^a + bx^b$

2. Find primitive functions of the following by first expanding the products:

(a) $x(x - 3)$	(c) $(3x - 1)(x + 4)$	(e) $(2x^2 + 1)^2$
(b) $(x^2 + 1)^2$	(d) $x^2(5x^3 - 4x)$	(f) $x(ax - 3)^2$

3. Write these functions using negative powers of x . Find the primitive functions, giving your answers in fractional form without negative indices.

(a) $\frac{1}{x^2}$	(c) $\frac{1}{3x^2}$	(e) $\frac{1}{x^2} - \frac{1}{x^3}$	(g) $\frac{1}{x^a}$
(b) $\frac{2}{x^3}$	(d) $-\frac{2}{5x^4}$	(f) $\frac{a}{bx^2}$	(h) $\frac{x^a + x^b}{x^a}$

4. Write these functions with fractional indices, and hence find the primitive functions:

(a) $\sqrt{x}$	(b) $\frac{1}{\sqrt{x}}$	(c) $\sqrt[3]{x}$	(d) $\frac{2}{3\sqrt{x}}$	(e) $\sqrt[5]{x^3}$

5. Find y as a function of x if:

- (a) $\frac{dy}{dx} = 2x + 3$, and $y = 8$ when $x = 1$,
 (b) $\frac{dy}{dx} = 9x^2 + 4$, and $y = 1$ when $x = 0$,
 (c) $\frac{dy}{dx} = \sqrt{x}$, and $y = 2$ when $x = 9$.

6. Box 18 of the text states the rule that the primitive of x^n is $\frac{x^{n+1}}{n+1}$, provided that $n \neq -1$. Why can't this rule be used when $n = -1$?

————— DEVELOPMENT —————

7. Find each family of curves whose gradient function is given below. Then sketch the family, and find the member of the family passing through $A(1, 2)$.

- (a) $\frac{dy}{dx} = -4x$ (b) $\frac{dy}{dx} = 3$ (c) $\frac{dy}{dx} = 3x^2$ (d) $\frac{dy}{dx} = -\frac{1}{x^2}$

8. Find primitive functions of each of the following by recalling that if $y' = (ax + b)^n$, then $y = \frac{(ax + b)^{n+1}}{a(n+1)} + C$.

- (a) $(x + 1)^3$ (c) $(3x - 4)^6$ (e) $(ax - b)^5$
 (b) $\frac{1}{(x - 2)^4}$ (d) $(1 - 7x)^3$ (f) $\frac{2}{(1 - 9x)^{10}}$

9. Find primitive functions of each of the following:

- (a) $\sqrt{x + 1}$ (c) $\sqrt{2x - 7}$ (e) $\sqrt{ax + b}$
 (b) $\sqrt{1 - x}$ (d) $\frac{1}{\sqrt{2 - 3x}}$ (f) $\frac{3}{2\sqrt{4x - 1}}$

10. (a) Find y if $y' = (2x + 1)^3$, and $y = -1$ when $x = 0$.

(b) Find y if $y'' = 6x + 4$, and when $x = 1$, $y' = 2$ and $y = 4$.

(c) Find y if $y'' = \sqrt{3 - x}$, and when $x = -1$, $y' = \frac{16}{3}$ and $y = 14\frac{8}{15}$.

11. Find the primitive functions of each of the following:

- (a) $x^a x^b$ (c) x^{ab} (e) $x\sqrt{x}$ (g) $\sqrt{x}(x + 1)$
 (b) $\frac{x^a}{x^b}$ (d) $ax^b + bx^a$ (f) $\frac{1}{\sqrt{x}} + \sqrt{x}$ (h) $\frac{\sqrt{x} - 1}{2\sqrt{x}}$

12. (a) Find the equation of the curve through the origin whose gradient is $\frac{dy}{dx} = 3x^4 - x^3 + 1$.

(b) Find the curve passing through $(2, 6)$ with gradient function $\frac{dy}{dx} = 2 + 3x^2 - x^3$.

(c) Find the curve through the point $(\frac{1}{5}, 1)$ with gradient function $y' = (2 - 5x)^3$.

(d) A curve with gradient function $f'(x) = cx + d$ has a turning point at $(2, 0)$ and crosses the y -axis when $y = 4$. Find c and d , and hence find the equation of the curve.

13. Given that $\frac{dy}{dt} = 8t^3 - 6t^2 + 5$, and $y = 4$ when $t = 0$, find y when $t = 2$.

14. (a) At any point on a curve, $\frac{d^2y}{dx^2} = 2x - 10$. The curve passes through the point $(3, -34)$, and at this point the tangent to the curve has a gradient of 20. Find the y -intercept.
 (b) Find the curve through the points $(1, 6)$ and $(-1, 8)$ with $y'' = 8 - 6x$.
15. Water is leaking out through a hole in the bottom of a bucket at a rate $\frac{dv}{dt} = -8(25 - t)$, where $v \text{ cm}^3$ is the volume of liquid in the bucket at time t seconds. Initially there was $2\frac{1}{2}$ litres of water in the bucket. Find v in terms of t , and hence find the time taken for the bucket to be emptied.
16. The price of an item on sale is being reduced at a rate given by $\frac{dp}{dt} = -\frac{30}{t^4}$, where p is the price in dollars and t is the number of days the item has been on sale. After two days the item was retailing for \$5.25. Find the price of the item after 5 days. Why will the item always retail for a price above \$4?
17. (a) The velocity $\frac{dx}{dt}$ (rate of change of position x at time t) of a particle is given by $\frac{dx}{dt} = t^2 - 3t$ cm/sec. If the particle starts 1 cm to the right of the origin, find its position after 3 seconds.
 (b) The acceleration at time t of a particle travelling on the x -axis is given by $\frac{d^2x}{dt^2} = 2t - 5$. If the particle is initially at rest at the origin, find its position after 4 seconds.
18. A stone is thrown upwards according to the equation $\frac{dv}{dt} = -10$, where v is the velocity in metres per second and t is the time in seconds. Given that the stone is thrown from the top of a building 30 metres high with an initial velocity of 5 m/s, find how long it takes for the stone to hit the ground.

EXTENSION

19. The gradient function of a curve is given by $f'(x) = -\frac{1}{x^2}$. Find the equation of the curve, given that $f(1) = f(-1) = 2$. Sketch a graph of the function.
20. (a) Prove that for a polynomial of degree n , the $(n + 1)$ th and higher derivatives vanish, but the n th does not.
 (b) Prove that if the $(n + 1)$ th derivative of a polynomial vanishes but the n th does not, then the polynomial has degree n .

Integration

The calculation of areas has so far been restricted to regions bounded by straight lines or parts of circles. *Integration* is the second of the two basic processes of calculus (the first being differentiation), and it extends the study of areas to regions bounded by more general curves — for example we will be able to calculate the area bounded by the parabola $x^2 = 4ay$ and its latus rectum. We will also be able to find the volume of the solid generated by rotating a region in the coordinate plane about one of the axes.

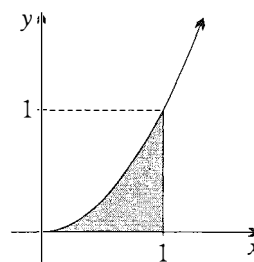
The surprising result at the centre of this section is, as mentioned before, that finding tangents and finding areas are inverse processes, so that integration is the inverse process of differentiation. This result is called the *fundamental theorem of calculus* because it is the basis of the whole theory of differentiation and integration. It will greatly simplify the calculations required.

STUDY NOTES: In Section 11A some simple areas are calculated by a limiting process involving infinite dissections, and the definite integral is defined for functions with positive values. The fundamental theorem is proven and applied in Section 11B. In Section 11C the definite integral is extended to functions with negative values, and some simple theorems on the definite integral are established by dissection. This allows the standard methods of integration to be developed and applied to areas and volumes in Sections 11D–11G. Approximation methods are left until Sections 11I and 11J at the end of the chapter after the exact theory has been developed. The reverse chain rule is developed in Section 11H — this work may prove a little too hard for a first treatment of integration, and could be left until later. Computers or graphics calculators could be used to reinforce the definition of the definite integral in terms of areas, and they are of course particularly suited to the approximation methods. Computer graphics of volumes of rotation, or models constructed on a lathe, would be an effective way of making these solids a little more visible.

11A Finding Areas by a Limiting Process

All work on areas must rest eventually on the basic definition of area, which is that the area of a rectangle is length times breadth. Any region bounded by straight lines, such as a triangle or a trapezium, can be rearranged into rectangles with a few well chosen cuttings and pastings, but any dissection of a curved region into rectangles must involve an infinite number of rectangles, and so must be a limiting process, like differentiation.

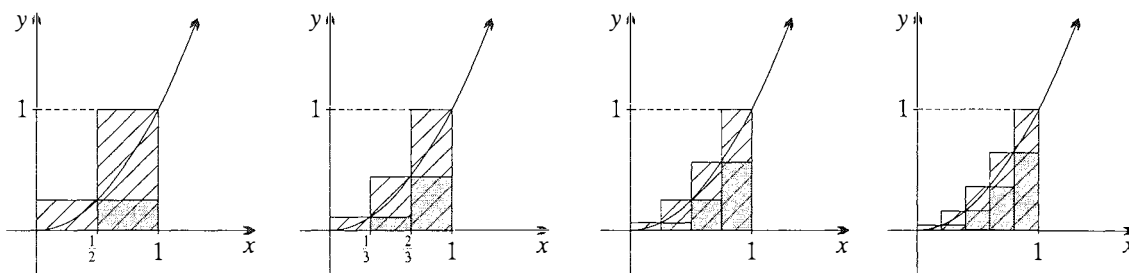
Example — The Area Under the Parabola $y = x^2$: These limiting calculations are much more elaborate than first-principles differentiation, and the fundamental theorem of calculus will soon make them unnecessary. However, it is advisable to carry out a very few such calculations in order to understand what is being done. The example below illustrates the technique, which in different forms was already highly developed by the Greeks independently of our present ideas about functions and graphs. We shall find the shaded area A of the region ‘under the curve’ $y = x^2$ from $x = 0$ to $x = 1$.



To begin the process, notice that the area is completely contained within the unit square, so we know that

$$0 < A < 1.$$

In the four pictures below, the region has been sliced successively into two, three, four and five strips, then ‘upper’ and ‘lower’ rectangles have been constructed so that the region is trapped, or sandwiched, between the upper and lower rectangles. Calculating the areas of these upper and lower rectangles provides tighter and tighter bounds on the area A .



In the first picture,

$$\frac{1}{2} \times \left(\frac{1}{2}\right)^2 < A < \frac{1}{2} \times \left(\frac{1}{2}\right)^2 + \frac{1}{2} \times 1^2$$

$$\frac{1}{8} < A < \frac{5}{8}.$$

In the second picture,

$$\frac{1}{3} \times \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \left(\frac{2}{3}\right)^2 < A < \frac{1}{3} \times \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \left(\frac{2}{3}\right)^2 + \frac{1}{3} \times 1^2$$

$$\frac{5}{27} < A < \frac{14}{27}.$$

In the third picture,

$$\frac{1}{4} \left(\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 \right) < A < \frac{1}{4} \left(\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2 \right)$$

$$\frac{14}{64} < A < \frac{30}{64}.$$

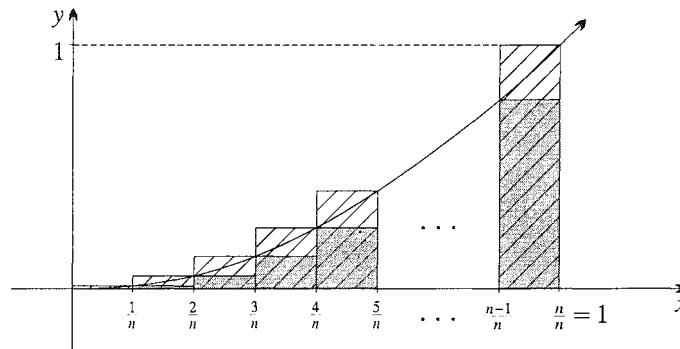
In the fourth picture,

$$\frac{1}{5} \left(\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 \right) < A < \frac{1}{5} \left(\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{5}{5}\right)^2 \right)$$

$$\frac{30}{125} < A < \frac{55}{125}.$$

The Limiting Process: The bounds on the area are getting tighter, but the exact value of the area can only be obtained if this sandwiching process can be turned into a limiting process. The calculations below will need the formula for the sum of the first n squares proven in Exercise 6N by induction:

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$



- A. Divide the interval $0 \leq x \leq 1$ into n subintervals, each of width $\frac{1}{n}$.

On each subinterval form the upper rectangle and the lower rectangle.

Then the required region is entirely contained within the upper rectangles, and, in turn, the lower rectangles are entirely contained within the required region. So however many strips the region has been dissected into,

$$(\text{sum of lower rectangles}) \leq A \leq (\text{sum of upper rectangles}).$$

- B. The heights of the successive upper rectangles are $\frac{1^2}{n^2}, \frac{2^2}{n^2}, \dots, \frac{n^2}{n^2}$, and so, using the formula quoted above,

$$\begin{aligned} \text{sum of upper rectangles} &= \frac{1}{n} \left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \dots + \frac{n^2}{n^2} \right) \\ &= \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2) \\ &= \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6} \\ &= \frac{1}{3} \times \frac{n}{n} \times \frac{n+1}{n} \times \frac{2n+1}{2n} \\ &= \frac{1}{3} \times \left(1 + \frac{1}{n} \right) \times \left(1 + \frac{1}{2n} \right), \end{aligned}$$

hence the sum of the upper rectangles has limit $\frac{1}{3}$ as $n \rightarrow \infty$.

- C. The heights of the successive lower rectangles are $0, \frac{1^2}{n^2}, \frac{2^2}{n^2}, \dots, \frac{(n-1)^2}{n^2}$, so substituting $n-1$ for n into the quoted formula,

$$\begin{aligned} \text{sum of lower rectangles} &= \frac{1}{n} \left(0 + \frac{1^2}{n^2} + \frac{2^2}{n^2} + \dots + \frac{(n-1)^2}{n^2} \right) \\ &= \frac{1}{n^3} (1^2 + 2^2 + \dots + (n-1)^2) \\ &= \frac{1}{n^3} \times \frac{(n-1)n(2n-1)}{6} \\ &= \frac{1}{3} \times \frac{n}{n} \times \frac{n-1}{n} \times \frac{2n-1}{2n} \\ &= \frac{1}{3} \times \left(1 - \frac{1}{n} \right) \times \left(1 - \frac{1}{2n} \right), \end{aligned}$$

hence the sum of the lower rectangles also has limit $\frac{1}{3}$ as $n \rightarrow \infty$.

- D. Finally, since $(\text{sum of lower rectangles}) \leq A \leq (\text{sum of upper rectangles})$, and since both these sums have the same limit $\frac{1}{3}$, it follows that $A = \frac{1}{3}$.

The Definite Integral: More generally, suppose that $f(x)$ is a function that is continuous in a closed interval $a \leq x \leq b$. For the moment (that is, in Sections 11A and 11B), suppose that $f(x)$ is never negative anywhere within this interval. Then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is called the *definite integral of $f(x)$ from $x = a$ to $x = b$* , and is given the symbol $\int_a^b f(x) dx$. The function $f(x)$ is called the *integrand*, and the numbers a and b are called the *upper and lower bounds* of the integral.

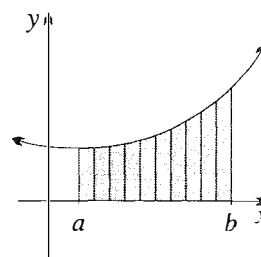
With this notation, the calculations above mean that

$$\int_0^1 x^2 dx = \frac{1}{3},$$

and the process which was carried out can be described as follows:

1 INTEGRATION BY FIRST PRINCIPLES: To find a definite integral by first principles, dissect the interval into n equal subintervals, construct upper and lower rectangles on each subinterval, and find the sums of the upper and lower rectangles. Then their common limit will be the value of the integral.

Leibniz's striking notation for the definite integral arises from the intuitive understanding of the previous limiting process. Dissect the region $a \leq x \leq b$ into infinitely many slices, each of infinitesimal width dx . Each slice, being infinitesimally thin, is essentially a rectangle of width dx and height $f(x)$, so the area of each slice is $f(x) dx$. Now sum these slices from $x = a$ to $x = b$. The symbol $\int$ is an old form of the letter S , and thus $\int_a^b f(x) dx$ becomes the symbol for the sum of all the areas.



Contrast the 'smooth sum' of the integral with the 'jagged sum' of the sigma notation $\sum_{n=a}^b u_n$ introduced in Chapter Six, where the symbol $\sum$ is the Greek letter capital sigma, also meaning S and also standing for sum.

The name 'integration' suggests putting parts together to make a whole, and the approach and the notation both arise from building up a region in the plane out of an infinitely large number of infinitesimally thin strips. So integration is indeed 'making a whole' of these thin slices. Notice that the definite integral has been defined geometrically in terms of areas associated with the graph of a function, and that the language of functions has now been brought into the study of areas.

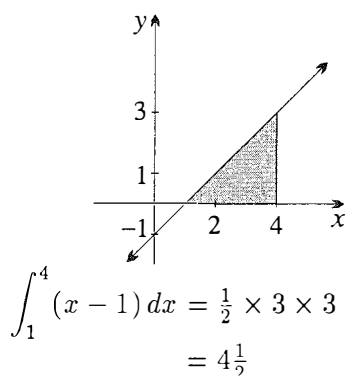
Further Examples of Definite Integrals: When the function is linear, the integral can be calculated using area formulae from mensuration, as in the first two examples (but a quicker method will be developed later). The last example involves a circular function.

WORKED EXERCISE: Evaluate using a graph and area formulae:

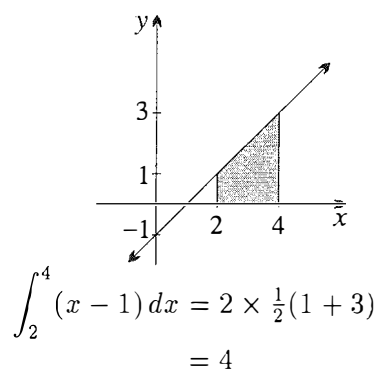
(a) $\int_1^4 (x-1) dx$ (b) $\int_2^4 (x-1) dx$ (c) $\int_{-a}^a |x| dx$ (d) $\int_{-5}^5 \sqrt{25-x^2} dx$

SOLUTION:

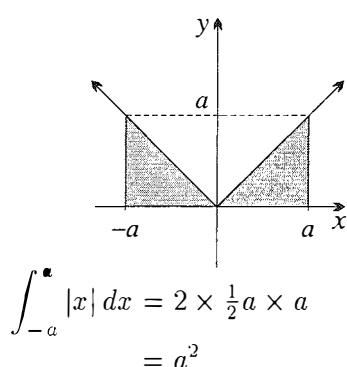
(a)



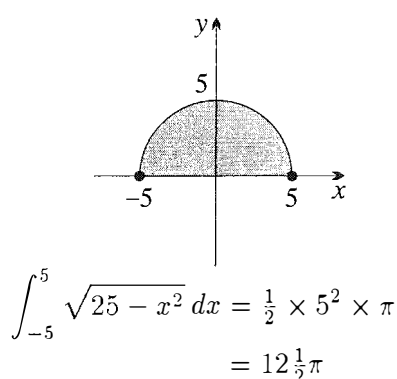
(b)



(c)



(d)



NOTE: Three area formulae from earlier mensuration have been used here:

2

FOR A TRIANGLE:

$$A = \frac{1}{2}bh$$

FOR A TRAPEZIUM:

$$A = \frac{1}{2}h(a + b)$$

FOR A CIRCLE:

$$A = \pi r^2$$

Exercise 11A

1. Sketch a graph of each integral, then use area formulae to evaluate it:

(a) $\int_4^8 (x - 4) dx$

(c) $\int_{-4}^4 \sqrt{16 - x^2} dx$

(e) $\int_3^6 (2x + 1) dx$

(b) $\int_{-1}^3 (x + 5) dx$

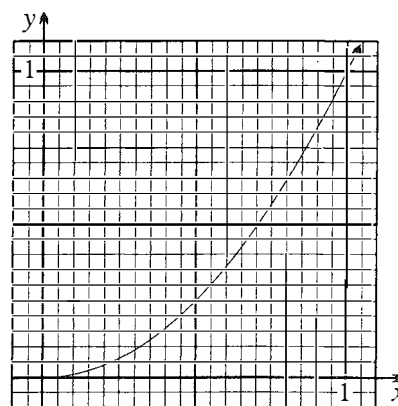
(d) $\int_{-5}^0 \sqrt{25 - x^2} dx$

(f) $\int_0^3 5 dx$

2. The notes above this exercise used arguments involving limits to prove that $\int_0^1 x^2 dx = \frac{1}{3}$. The diagram

on the right shows the graph of $y = x^2$ from $x = 0$ to $x = 1$, drawn with a scale of 20 little divisions to 1 unit. This means that 400 little squares make up 1 square unit.

(a) Count how many little squares there are under the graph from $x = 0$ to $x = 1$ (keeping reasonable track of fragments of squares), then divide by 400 to check how close this result is to $\int_0^1 x^2 dx = \frac{1}{3}$.



- (b) Question 4(a) below establishes the more general result $\int_0^a x^2 dx = \frac{1}{3}a^3$. By counting the appropriate squares, check this result for $a = \frac{1}{4}$, $a = \frac{1}{2}$, $a = \frac{3}{5}$, $a = \frac{3}{4}$ and $a = \frac{4}{5}$.

DEVELOPMENT

3. Using exactly the same setting out as in the example in the notes above this exercise, show by first principles that $\int_0^1 x^3 dx = \frac{1}{4}$, using upper and lower rectangles and taking the limit as the number of rectangular strips becomes infinite.

NOTE: This calculation will need the formula $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$ for the sum of the first n cubes, proven in Section 6N using mathematical induction.

4. Prove the following two definite integrals by first principles, using upper and lower rectangles and taking the limit as the number of rectangular strips becomes infinite:

(a) $\int_0^a x^2 dx = \frac{a^3}{3}$

(b) $\int_0^a x^3 dx = \frac{a^4}{4}$

EXTENSION

5. Draw a large sketch of $y = x^2$ for $0 \leq x \leq 1$, and let U be the point $(1,0)$. For some positive integer n , let $P_0 (= O)$, P_1 , P_2 , ..., P_n be the points on the curve with x -coordinates $x = 0$, $x = \frac{1}{n}$, $x = \frac{2}{n}$, ..., $x = \frac{n}{n} = 1$. Join the chords P_0P_1 , P_1P_2 , ... and $P_{n-1}P_n$, and join P_nU .

(a) Use the area formula for a trapezium to find the area of the polygon $P_0P_1P_2 \dots P_nU$.

(b) Explain geometrically why this area is always greater than $\int_0^1 x^2 dx$.

(c) Show that its limit as $n \rightarrow \infty$ is equal to $\frac{1}{3}$, that is, equal to $\int_0^1 x^2 dx$.

11B The Fundamental Theorem of Calculus

The fundamental theorem will allow us to evaluate definite integrals quickly using a quite straightforward algorithm based on primitives. Because the details of the proof are rather demanding, the algorithm is presented first, by means of some worked examples, and the proof is left to the end of the section.

Statement of the Fundamental Theorem (Integral Form): The integral form of the fundamental theorem, stated below, is essentially a formula for evaluating a definite integral.

3

THE FUNDAMENTAL THEOREM (INTEGRAL FORM): Suppose $f(x)$ is a function continuous in the closed interval $a \leq x \leq b$, and suppose $F(x)$ is a primitive of $f(x)$. Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

This means that a definite integral can be evaluated by writing down any primitive $F(x)$ of $f(x)$, then substituting the upper and lower bounds into it and subtracting.

Using the Fundamental Theorem to Evaluate an Integral: The conventional way to set out these calculations is to enclose the primitive in square brackets, writing the upper and lower bounds as superscript and subscript respectively.

WORKED EXERCISE:

$$\begin{aligned} \text{(a)} \quad \int_0^1 x^2 dx &= \left[\frac{1}{3}x^3 \right]_0^1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3} \end{aligned}$$

(as calculated in Section 11A)

$$\begin{aligned} \text{(b)} \quad \int_2^4 (x-1) dx &= \left[\frac{1}{2}x^2 - x \right]_2^4 \\ &= (8-4) - (2-2) \\ &= 4 \end{aligned}$$

(as calculated in Section 11A)

$$\begin{aligned} \text{(c)} \quad \int_{-2}^2 (x^3 + 8) dx &= \left[\frac{1}{4}x^4 + 8x \right]_{-2}^2 \\ &= (4+16) - (4-16) \\ &= 32 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int_1^2 x^{-2} dx &= \left[-x^{-1} \right]_1^2 \\ &= -\frac{1}{2} + 1 \\ &= \frac{1}{2} \end{aligned}$$

Change of Pronumeral: The rest of this section is an exposition of the proof of the fundamental theorem. First, notice that the pronumeral in the definite integral notation is a *dummy variable*, meaning that it can be replaced by any other pronumeral. For example, the four integrals

$$\int_0^1 x^2 dx = \int_0^1 t^2 dt = \int_0^1 y^2 dy = \int_0^1 \lambda^2 d\lambda$$

all have the same value $\frac{1}{3}$ — the letter used for the variable has changed, but the function remains the same and so the area involved remains the same. Similarly the pronumeral in sigma notation is a dummy variable. For example,

$$\sum_{n=1}^4 n^2 = \sum_{r=1}^4 r^2 = \sum_{x=1}^4 x^2 = \sum_{\lambda=1}^4 \lambda^2$$

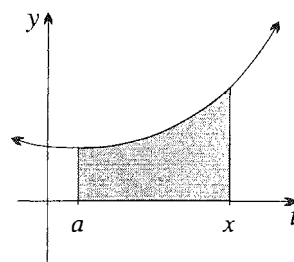
all have the same value $1 + 4 + 9 + 16 = 30$.

The Definite Integral as a Function of its Upper Bound: The value

of the definite integral $\int_a^b f(x) dx$ changes when the value of a or b is changed. This means that it is a function both of its upper bound b and of its lower bound a . In order to suggest the functional relationship with the upper bound b more closely, we shall replace the letter b by the letter x , which is conventionally the variable of a function. The original letter x needs to be replaced in turn by some other letter — a suitable choice is t , which is also conventionally a variable. Then we can speak clearly about the definite integral

$$A(x) = \int_a^x f(t) dt$$

being a function of its upper bound x . This integral is represented in the sketch above.



The Fundamental Theorem of Calculus — Differential Form: There are two forms of the theorem, the *integral form* stated above, and the following *differential form* which will be proven first. The differential form claims that this definite integral $A(x) = \int_a^x f(t) dt$ is a primitive of $f(x)$. It has been stated so as to make clear that the two processes of differentiation and integration are inverse processes and cancel each other out.

4

THE FUNDAMENTAL THEOREM (DIFFERENTIAL FORM): Suppose $f(x)$ is a function that is continuous in the closed interval $a \leq x \leq b$. Then the definite integral $\int_a^x f(t) dt$, regarded as a function of its upper bound x , is a primitive of $f(x)$:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x), \text{ for } a < x < b.$$

Examples of the Fundamental Theorem: Before proving the theorem, let us look at two examples which should make the statement of the theorem clear.

First, we have shown in Exercise 11A that $\int_0^a x^2 dx = \frac{1}{3}a^3$.

Changing the variables as described above, $\int_0^x t^2 dt = \frac{1}{3}x^3$,

and differentiating, $\frac{d}{dx} \int_0^x t^2 dt = x^2$, as the theorem says.

As a second example, we showed also that $\int_0^a x^3 dx = \frac{1}{4}a^4$.

Changing variables and differentiating, $\frac{d}{dx} \int_0^x t^3 dt = x^3$, as the theorem says.

Proof of the Differential Form: The proof is based on the sandwiching technique. Notice that in constructing this proof, we are still working under the assumption that $f(x)$ is never negative anywhere within the interval $a \leq x \leq b$.

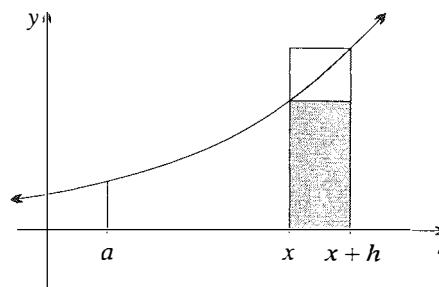
Let $A(x) = \int_a^x f(t) dt$, then we must prove that $A'(x) = f(x)$.

Recall the definition of the derivative as a limit:

$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h}.$$

$$\begin{aligned} \text{Now } A(x+h) - A(x) &= \int_a^{x+h} f(t) dt - \int_a^x f(t) dt \\ &= \int_x^{x+h} f(t) dt, \end{aligned}$$

$$\text{so } A'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt.$$



Suppose $f(t)$ is increasing in the closed interval $x \leq t \leq x+h$, as in the diagram above.

Then the lower rectangle on the interval $x \leq t \leq x+h$ has height $f(x)$,

and the upper rectangle on the interval $x \leq t \leq x+h$ has height $f(x+h)$,

so, using areas, $h \times f(x) \leq \int_x^{x+h} f(t) dt \leq h \times f(x+h)$

$$\boxed{\div h} \quad f(x) \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq f(x+h).$$

Since $f(x)$ is continuous, $f(x+h) \rightarrow f(x)$ as $h \rightarrow 0$,

and so $\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt = f(x)$, meaning that $A'(x) = f(x)$, as required.

If $f(x)$ is decreasing in $x \leq t \leq x+h$, the argument applies with inequalities reversed.

Proof of the Integral Form: It is given that $F(x)$ is a primitive of $f(x)$,

and the fundamental theorem says that $\int_a^x f(t) dt$ is also a primitive of $f(x)$,

so the primitives $\int_a^x f(t) dt$ and $F(x)$ must differ only by a constant,

that is, $\int_a^x f(t) dt = F(x) + C$, for some constant C .

Substituting $x = a$, $\int_a^a f(t) dt = F(a) + C$,

but $\int_a^a f(t) dt = 0$, because the area in this definite integral has zero width,

so $0 = F(a) + C$.

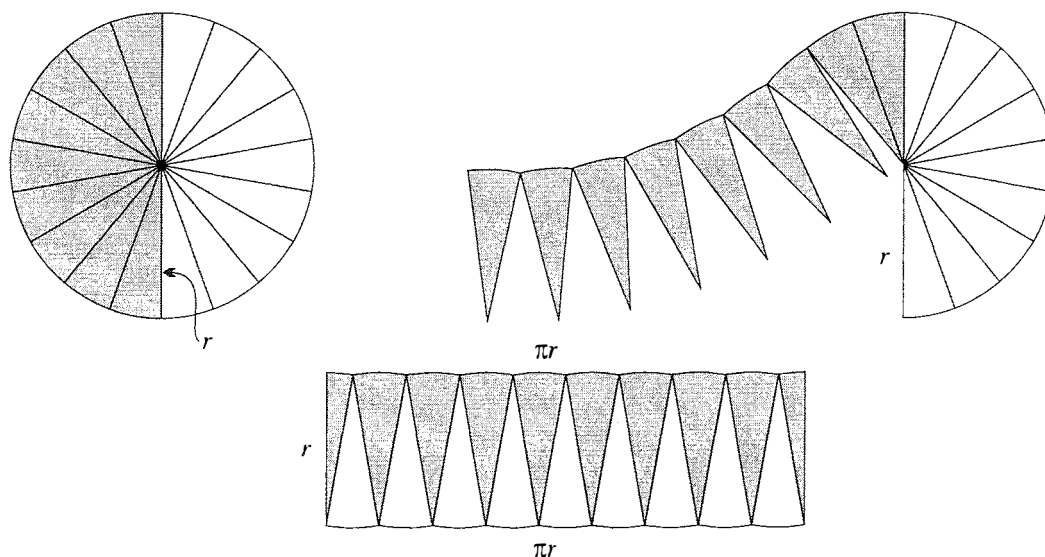
Hence $C = -F(a)$, and $\int_a^x f(t) dt = F(x) - F(a)$.

Changing letters, $\int_a^b f(x) dx = F(b) - F(a)$.

NOTE: Some readers who may have noticed a lack of rigour in the preceding arguments could benefit from the treatment in a more advanced text. In particular, we assumed in the proof of the differential form that $f(t)$ was either increasing or decreasing in the closed interval $x < t < x + h$, and we also assumed that h was positive. Secondly, the proof of the integral form actually requires that $A(x)$ be differentiable in the closed interval $a \leq x \leq b$, which in turn requires constructing a definition of *one-sided derivatives* at the endpoints, in a manner similar to the definition of continuity in a closed interval.

More fundamentally, the definite integral was defined in terms of area, but it is not at all clear that a region bounded by curves actually has an area, since area had previously only been defined for rectangles and then by dissection for regions bounded by straight lines. More rigorous treatments turn a generalisation of the 'first principles' calculation of integrals into the definition of the definite integral, and then define area in terms of the definite integral.

The Area of a Circle:



In earlier years, the formula $A = \pi r^2$ for the area of a circle was proven. Because the boundary is a curve, some limiting process had to be used in that proof. For comparison with the arguments used above about the definite integral, here is the most common version of that argument — a little rough in its logic, but very quick. It involves dissecting the circle into infinitesimally thin sectors, and then rearranging them into a rectangle.

The height of the rectangle in the lower diagram is r . Since the circumference $2\pi r$ is divided equally between the top and bottom sides, the length of the rectangle is πr . So the rectangle has area πr^2 , which is therefore the area of the circle.

Exercise 11B

1. Evaluate the following definite integrals using the fundamental theorem:

(a) $\int_1^4 2x \, dx$	(d) $\int_1^2 10x^4 \, dx$	(g) $\int_{-1}^1 (4x^3 + 3x^2 + 1) \, dx$
(b) $\int_0^5 x^2 \, dx$	(e) $\int_3^6 (2x + 1) \, dx$	(h) $\int_0^3 (x + x^2 + x^3) \, dx$
(c) $\int_2^3 3x^2 \, dx$	(f) $\int_4^7 dx$	(i) $\int_{-3}^1 (2x^2 - 7x + 5) \, dx$

2. (a) Evaluate the following definite integrals:

(i) $\int_5^{10} x^{-2} \, dx$	(ii) $\int_2^3 2x^{-3} \, dx$	(iii) $\int_{\frac{1}{2}}^1 x^{-5} \, dx$

- (b) By writing them with negative indices, evaluate the following definite integrals:

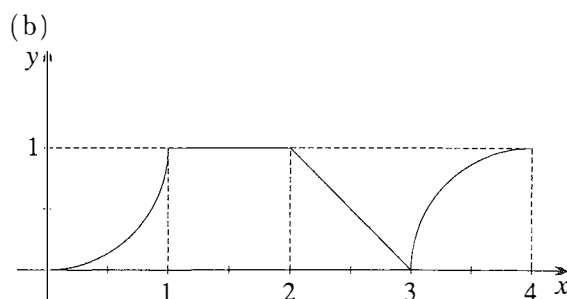
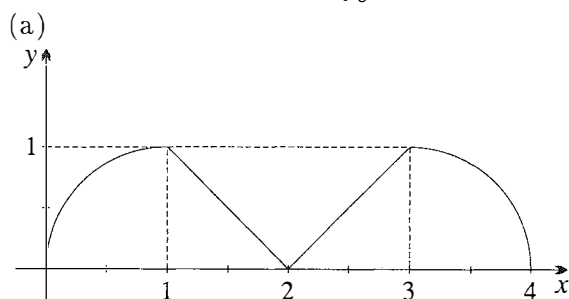
(i) $\int_1^2 \frac{dx}{x^2}$	(ii) $\int_1^4 \frac{dx}{x^3}$	(iii) $\int_{\frac{1}{2}}^1 \frac{3}{x^4} \, dx$

3. By expanding the brackets where necessary, evaluate the following definite integrals:

(a) $\int_0^1 3x(2 + x) \, dx$	(c) $\int_1^2 \frac{1 + x^2}{x^2} \, dx$	(e) $\int_{-1}^0 (1 - x^2)^2 \, dx$
(b) $\int_1^3 (x + 2)^2 \, dx$	(d) $\int_1^3 \left(x + \frac{1}{x}\right)^2 \, dx$	(f) $\int_4^9 (\sqrt{x} + 1)(\sqrt{x} - 1) \, dx$

DEVELOPMENT

4. Use area formulae to find $\int_0^4 f(x) \, dx$ in each sketch of $f(x)$:



5. Find the value of k if:

(a) $\int_2^k \frac{3}{x^2} \, dx = \frac{9}{10}$	(b) $\int_0^3 kx^2 \, dx = 4$	(c) $\int_{-1}^2 (3x^2 + 4x + k) \, dx = 30$

6. Sketch the integrand and explain why this calculation is invalid:

$$\int_{-1}^1 \frac{dx}{x^2} = \left[-\frac{1}{x} \right]_{-1}^1 = -1 - 1 = -2.$$

EXTENSION

7. (a) Find $\frac{d}{dx} \int_3^x (4t^3 - 3t^2 + t - 1) dt$.

(b) Find $\frac{d}{dx} \int_x^2 (7 - 6t)^4 dt$.

- (c) The derivative of the function $U(x)$ is $u(x)$.

(i) Find $V'(x)$, where $V(x) = (a - x)U(x) + \int_0^x U(t) dt$ and a is a constant.

(ii) Hence prove that $\int_0^a U(x) dx = aU(0) + \int_0^a (a - x)u(x) dx$.

8. Is it possible to have a non-negative function $f(x)$ defined on the interval $0 \leq x \leq 1$ such that $f(c) > 0$ for some c such that $0 < c < 1$, but $\int_0^1 f(x) dx = 0$?

11C The Definite Integral and its Properties

This section will first extend the theory to functions with negative values. Then some properties of the definite integral will be established using fairly obvious arguments about the dissection of the area associated with the integral.

Integrating Functions with Negative Values: When a function has negative values, its graph is below the x -axis, so the ‘heights’ of the little rectangles in the dissection are negative numbers. This means that areas below the x -axis should contribute negative values to the final integral. The fundamental theorem will then allow these integrals to be evaluated in the usual way.

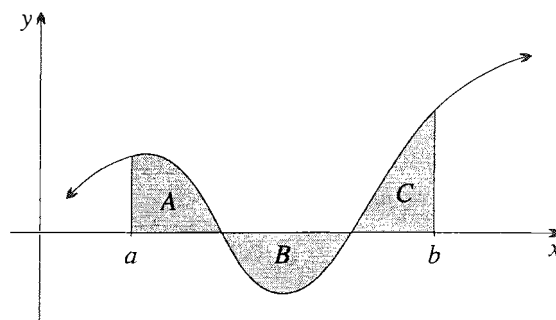
5

DEFINITION: Suppose $f(x)$ is a function which is continuous in some closed interval $a \leq x \leq b$. The *definite integral* $\int_a^b f(x) dx$ is the area between the curve $y = f(x)$ and the x -axis from $x = a$ to $x = b$, with areas above the x -axis counted as positive and areas below the x -axis counted as negative.

In the diagram to the right, the region B is below the x -axis, and so is counted as negative in the definite integral:

$$\int_a^b f(x) dx = \text{area } A - \text{area } B + \text{area } C.$$

Because areas under the x -axis are counted as negative, the definite integral is sometimes referred to as the *signed area* under the curve, to distinguish it from area, which is always positive.

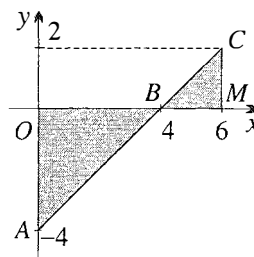


WORKED EXERCISE: Evaluate and sketch: (a) $\int_0^4 (x - 4) dx$ (b) $\int_0^6 (x - 4) dx$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \int_0^4 (x - 4) dx &= \left[\frac{1}{2}x^2 - 4x \right]_0^4 \\ &= (8 - 16) - (0 - 0) \\ &= -8 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_0^6 (x - 4) dx &= \left[\frac{1}{2}x^2 - 4x \right]_0^6 \\ &= (18 - 24) - (0 - 0) \\ &= -6 \end{aligned}$$



Notice that the area of the shaded triangle OAB below the x -axis has area 8, and accordingly the first integral is -8 . The shaded triangle BCM above the x -axis has area 2, and accordingly the second integral is $-8 + 2 = -6$.

Odd and Even Functions: The first example below shows the graph of $y = x^3 - 4x$. Because the function is odd, the area of each shaded hump is the same, so the whole integral from $x = -2$ to $x = 2$ is zero, because the equal humps above and below the x -axis cancel out. But if the function is even, as in the second example, there is a doubling instead of a cancelling:

6

ODD FUNCTIONS: If $f(x)$ is odd, then $\int_{-a}^a f(x) dx = 0$.
EVEN FUNCTIONS: If $f(x)$ is even, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

WORKED EXERCISE: Sketch, then evaluate using symmetry:

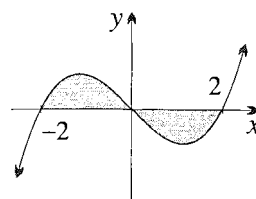
$$\text{(a)} \quad \int_{-2}^2 (x^3 - 4x) dx = 0 \qquad \text{(b)} \quad \int_{-2}^2 (x^2 + 1) dx$$

SOLUTION:

$$\text{(a)} \quad \int_{-2}^2 (x^3 - 4x) dx = 0, \text{ since the integrand is odd.}$$

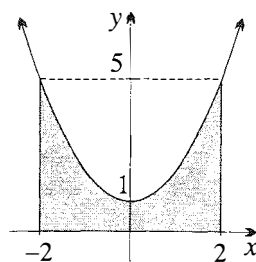
(Without this simplification the calculation is

$$\begin{aligned} \int_{-2}^2 (x^3 - 4x) dx &= \left[\frac{1}{4}x^4 - 2x^2 \right]_{-2}^2 \\ &= (4 - 8) - (4 - 8) \\ &= 0, \text{ as before.} \end{aligned}$$



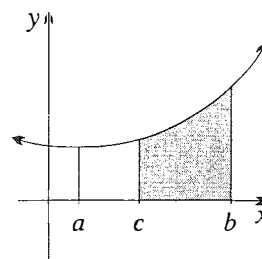
(b) Since the integrand is even,

$$\begin{aligned} \int_{-2}^2 (x^2 + 1) dx &= 2 \int_0^2 (x^2 + 1) dx \\ &= 2 \left[\frac{1}{3}x^3 + x \right]_0^2 \\ &= 2 \left(2\frac{2}{3} + 2 \right) - (0 + 0) \\ &= 9\frac{1}{3}. \end{aligned}$$



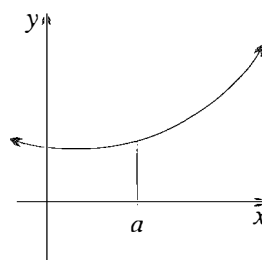
Dissection of the Interval: In the work so far, we have routinely dissected the region by dissecting the interval over which the integration is being performed. If $f(x)$ is continuous in the interval $a \leq x \leq b$ and the number c lies in this interval, then:

7 **DISSECTION:** $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$



Intervals of Zero Width: In the course of the proof in Section 11B of the fundamental theorem, the trivial remark was made that if the interval has width zero, then the integral is zero. Provided that a function $f(x)$ is defined at $x = a$, then:

8 **INTERVALS OF ZERO WIDTH:** $\int_a^a f(x) dx = 0$



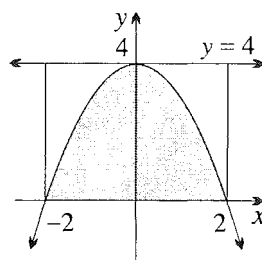
Inequalities with Definite Integrals: If $f(x)$ and $g(x)$ are two functions continuous in the interval $a \leq x \leq b$, with $f(x) \leq g(x)$ throughout the interval, then the integral of $f(x)$ is less than the integral of $g(x)$.

9 **INEQUALITY:** If $f(x) \leq g(x)$ in the interval $a \leq x \leq b$, then
$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

When both functions are positive, then the region under the curve $y = f(x)$ is contained within the region under the curve $y = g(x)$. If one or both functions become negative, then the inequality still holds because of the qualification that areas under the x -axis are counted as negative.

WORKED EXERCISE: Sketch the graph of $f(x) = 4 - x^2$ for $-2 \leq x \leq 2$, and explain why $0 \leq \int_{-2}^2 (4 - x^2) dx \leq 16$.

SOLUTION: Since $0 \leq 4 - x^2 \leq 4$ in the interval $-2 \leq x \leq 2$, it follows that the region associated with the integral is inside the square of side length 4 in the diagram opposite.



Running an Integral Backwards: A further small qualification must be made to the definition of the definite integral. Suppose that the function $f(x)$ is defined in the closed interval $a \leq x \leq b$. Then:

10 **REVERSING THE INTERVAL:** $\int_b^a f(x) dx = - \int_a^b f(x) dx$

So if the integral 'runs backwards' over the interval, then the integral reverses in sign. This agrees perfectly with the fundamental theorem, because

$$F(a) - F(b) = -(F(b) - F(a)).$$

WORKED EXERCISE: By this definition,

$$\begin{aligned}\int_2^{-2} (x^2 + 1) dx &= - \int_{-2}^2 (x^2 + 1) dx \\ &= -9\frac{1}{3} \quad (\text{the last integral was calculated above}).\end{aligned}$$

Alternatively, calculating the integral directly,

$$\begin{aligned}\int_2^{-2} (x^2 + 1) dx &= \left[\frac{1}{3}x^3 + x \right]_2^{-2} \\ &= (-2\frac{2}{3} - 2) - (2\frac{2}{3} + 2) \\ &= -9\frac{1}{3}.\end{aligned}$$

Linear Combinations of Functions: When two functions are added, the two regions are piled on top of each other, so that:

11 INTEGRAL OF A SUM: $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

Similarly, when a function is multiplied by a constant, the region is expanded vertically by that constant, so that:

12 INTEGRAL OF A MULTIPLE: $\int_a^b k f(x) dx = k \int_a^b f(x) dx$

EXAMPLE: Using these rules about linear combinations,

$$\begin{aligned}\int_0^1 (3x^2 + 2x) dx &= \int_0^1 3x^2 dx + \int_0^1 2x dx \\ &= 3 \int_0^1 x^2 dx + 2 \int_0^1 x dx \\ &= 3 \times \frac{1}{3} + 2 \times \frac{1}{2}, \quad \text{since } \int_0^1 x^3 dx = \frac{1}{3} \text{ and } \int_0^1 x^2 dx = \frac{1}{2}, \\ &= 2, \text{ as before.}\end{aligned}$$

This result should be checked by the simpler direct evaluation of the integral.

Exercise 11C

1. Calculate each definite integral using a graph and area formulae. In computing the integral, recall that areas above the x -axis are counted as positive, and areas below the x -axis are counted as negative.

(a) $\int_0^1 \sqrt{1-x^2} dx$

(c) $\int_2^4 (3-2x) dx$

(e) $\int_{-3}^0 (x+2) dx$

(b) $\int_{-3}^3 (-\sqrt{9-x^2}) dx$

(d) $\int_0^4 (-2) dx$

(f) $\int_2^{10} (5-x) dx$

2. Evaluate the following definite integrals using the fundamental theorem:

(a) $\int_{-1}^2 x^3 dx$

(c) $\int_0^6 (x^2 - 6x) dx$

(e) $\int_0^2 (4x^3 - 2x) dx$

(b) $\int_{-1}^1 6x^5 dx$

(d) $\int_{-1}^1 (x^3 - x) dx$

(f) $\int_{-2}^{10} (12 - 3x) dx$

3. (a) Evaluate the following definite integrals:

$$(i) \int_0^1 x^{\frac{1}{2}} dx$$

$$(ii) \int_1^4 x^{-\frac{1}{2}} dx$$

$$(iii) \int_1^8 x^{\frac{1}{3}} dx$$

(b) By writing them with fractional indices, evaluate the following definite integrals:

$$(i) \int_1^4 \sqrt{x} dx$$

$$(ii) \int_1^9 x\sqrt{x} dx$$

$$(iii) \int_1^9 \frac{dx}{\sqrt{x}}$$

4. By expanding the brackets where necessary, evaluate the following definite integrals:

$$(a) \int_0^2 x(1-x) dx$$

$$(c) \int_1^4 \frac{x^3-3}{x^3} dx$$

$$(e) \int_1^4 \left(2\sqrt{x} - \frac{3}{\sqrt{x}} \right) dx$$

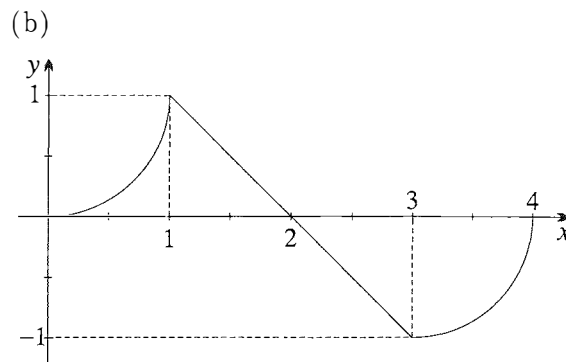
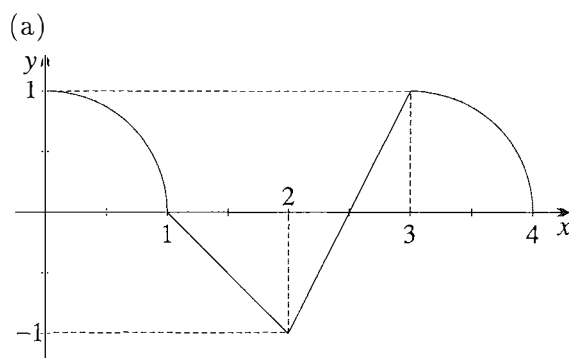
$$(b) \int_{-2}^2 (2-x)(1+x) dx$$

$$(d) \int_0^5 x(x+1)(x-1) dx$$

$$(f) \int_1^8 (\sqrt[3]{x} + x) dx$$

DEVELOPMENT

5. Use area formulae to find $\int_0^4 f(x) dx$ in each sketch of $f(x)$:



6. Find the value of k if:

$$(a) \int_1^k (x+1) dx = 6$$

$$(b) \int_1^k (k+3x) dx = \frac{13}{2}$$

$$(c) \int_1^9 \frac{k}{\sqrt{x}} dx = 4$$

7. Use the properties of the definite integral to evaluate, stating reasons:

$$(a) \int_3^3 \sqrt{9-x^2} dx$$

$$(c) \int_{-1}^1 x^3 dx$$

$$(e) \int_{-3.47}^{3.47} \frac{1}{5x^3-7x} dx$$

$$(b) \int_4^4 (x^3-3x^2+5x-7) dx$$

$$(d) \int_{-5}^5 (x^3-25x) dx$$

$$(f) \int_{-2}^2 \frac{x}{1+x^2} dx$$

8. Using the properties of the definite integral, explain why: (a) $\int_{-\alpha}^{\alpha} (ax^5+cx^3+ex) dx = 0$

$$(b) \int_{-\alpha}^{\alpha} (ax^5+bx^4+cx^3+dx^2+ex+f) dx = 2 \int_0^{\alpha} (bx^4+dx^2+f) dx$$

9. (a) On one set of axes, sketch $y = x^2$ and $y = \sqrt{x}$, showing the point of intersection.

$$(b) \text{Hence explain why } 0 < \int_0^1 x^2 dx < \int_0^1 \sqrt{x} dx < 1.$$

10. (a) Calculate using a graph and area formulae: (i) $\int_0^5 1 dx$ (ii) $\int_0^5 x dx$

(b) Using these results, and the properties of integrals of sums and multiples, evaluate:

$$(i) \int_0^5 2x dx$$

$$(ii) \int_0^5 (x+1) dx$$

$$(iii) \int_0^5 (3x-2) dx$$

11. (a) Calculate the following definite integrals using a graph and area formulae:

$$(i) \int_{-\frac{1}{4}}^{\frac{1}{4}} (1 - 4x) dx \quad (ii) \int_{-5}^1 |x + 5| dx \quad (iii) \int_0^2 (|x| + 3) dx$$

- (b) Hence write down the values of the following definite integrals:

$$(i) \int_{\frac{1}{4}}^{-\frac{1}{4}} (1 - 4x) dx \quad (ii) \int_1^{-5} |x + 5| dx \quad (iii) \int_2^0 (|x| + 3) dx$$

12. (a) (i) Show that $\int_3^4 dx = \int_2^3 dx = \int_1^2 dx = 1$.

$$(ii) \text{ Show that } \frac{2}{7} \int_3^4 x dx = \frac{2}{5} \int_2^3 x dx = \frac{2}{3} \int_1^2 x dx = 1.$$

$$(iii) \text{ Show that } \frac{3}{37} \int_3^4 x^2 dx = \frac{3}{19} \int_2^3 x^2 dx = \frac{3}{7} \int_1^2 x^2 dx = 1.$$

- (b) Using the above results and the theorems on the definite integral in Section 11C only, calculate the following:

$$\begin{array}{lll} (i) \int_1^4 dx & (iii) \int_2^1 x^2 dx & (v) \int_1^3 7x^2 dx \\ (ii) \int_1^3 x dx & (iv) \int_1^2 (x^2 + 1) dx & (vi) \int_1^4 (3x^2 - 6x + 5) dx \end{array}$$

EXTENSION

13. State with reasons whether the following statements are true or false:

$$\begin{array}{ll} (a) \int_{-90}^{90} \sin^3 x^\circ dx = 0 & (d) \int_0^1 2^x dx < \int_0^1 3^x dx \\ (b) \int_{-30}^{30} \sin 4x^\circ \cos 2x^\circ dx = 0 & (e) \int_{-1}^0 2^x dx < \int_{-1}^0 3^x dx \\ (c) \int_{-1}^1 2^{-x^2} dx = 0 & (f) \int_0^1 \frac{dt}{1+t^n} \leq \int_0^1 \frac{dt}{1+t^{n+1}}, \text{ where } n = 1, 2, 3, \dots \end{array}$$

14. First evaluate the integrals, then establish the following results. Give a sketch of each situation.

$$\begin{array}{ll} (a) \int_1^N \frac{dx}{x^2} \text{ converges to } 1 \text{ as } N \rightarrow \infty. & (c) \int_1^N \frac{dx}{\sqrt{x}} \text{ diverges to } \infty \text{ as } N \rightarrow \infty. \\ (b) \int_\varepsilon^1 \frac{dx}{x^2} \text{ diverges to } \infty \text{ as } \varepsilon \rightarrow 0^+. & (d) \int_\varepsilon^1 \frac{dx}{\sqrt{x}} \text{ converges to } 2 \text{ as } \varepsilon \rightarrow 0^+. \end{array}$$

11D The Indefinite Integral

Having established primitives as the key to calculating definite integrals, we now turn again to the calculation of primitives.

The Indefinite Integral: Because of the close connection established by the fundamental theorem between primitives and definite integrals, the term *indefinite integral* is often used for the primitive, and the usual notation for the primitive of a function $f(x)$ is an integral sign without any limits of integration. For example, the primitive or indefinite integral of $x^2 + 1$ is

$$\int (x^2 + 1) dx = \frac{1}{3}x^3 + x + C, \text{ for some constant } C.$$

The word ‘indefinite’ implies that the integral cannot be evaluated further because no limits of integration have yet been specified. Whereas a definite integral is a pure number whose pronumeral is a ‘dummy variable’, an indefinite integral is a function of x whose pronumeral is carried across to the answer. The constant is an important part of the answer, despite being a nuisance to write every time, and it must always be written down. In most problems, it will not be zero.

Standard Forms for Integration: The previous rules in Section 10 J for finding primitives can now be restated a little more elegantly in this new notation.

13

$$\begin{aligned} \text{STANDARD FORMS: (a)} \quad \int x^n dx &= \frac{x^{n+1}}{n+1} + C, \text{ for some constant } C. \\ \text{(b)} \quad \int (ax + b)^n dx &= \frac{(ax + b)^{n+1}}{a(n+1)} + C, \text{ for some constant } C. \end{aligned}$$

Integrals are usually found using the fundamental theorem, so the word ‘integration’ is commonly used to refer both to the finding of a primitive and to the evaluating of a definite integral.

WORKED EXERCISE: Here are some examples of integration techniques.

$$\begin{aligned} \text{(a)} \quad \int (5 - 2x)^2 dx &= \frac{(5 - 2x)^3}{(-2) \times 3} + C \\ &= -\frac{1}{6}(5 - 2x)^3 + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \frac{dx}{\sqrt{9 - 2x}} &= \int (9 - 2x)^{-\frac{1}{2}} dx \\ &= -\frac{1}{2} \times \frac{2}{1} \times (9 - 2x)^{\frac{1}{2}} + C \\ &= -\sqrt{9 - 2x} + C \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int \left(x - \frac{1}{x}\right)^2 dx &= \int (x^2 - 2 + x^{-2}) dx \\ &= \frac{1}{3}x^3 - 2x - \frac{1}{x} + C \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int \frac{x^2 - 1}{\sqrt{x}} dx &= \int (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) dx \\ &= \frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + C \end{aligned}$$

Exercise 11D

1. Find the indefinite integral of each of the following:

(a) 4

(d) 0

(g) $7x^{13} + 3x^8$

(j) $ax^2 + bx$

(b) $2x$

(e) $4x^5$

(h) $4 - 3x$

(k) $x^a, a \neq -1$

(c) $3x^2$

(f) $x^{0.4}$

(i) $3x^2 - 8x^3 + 7x^4$

(l) $ax^a + bx^b, a, b \neq -1$

2. Write these functions using negative indices, then find the indefinite integrals, giving your answers in fractional form:

(a) $\frac{1}{x^2}$

(c) $-\frac{1}{5x^3}$

(e) $\frac{1}{x^a}, a \neq 1$

(g) $\frac{ax^a}{x}, a \neq 0$

(b) $\frac{3}{x^4}$

(d) $\frac{1}{x^2} - \frac{1}{x^5}$

(f) $\frac{x^a}{x^b}, a - b \neq -1$

(h) $\frac{x^a + x^b}{x^a}, b - a \neq -1$

3. Write these functions with fractional indices and find the indefinite integrals:

(a) $\sqrt{x}$ (b) $\sqrt[3]{x}$ (c) $\frac{1}{\sqrt{x}}$ (d) $\sqrt[3]{x^2}$

4. By expanding the brackets where necessary, perform the following integrations:

(a) $\int x^2(5-3x) dx$ (d) $\int \sqrt{x}(3\sqrt{x}-x) dx$ (g) $\int \frac{x^7+x^4}{x^6} dx$
 (b) $\int (2x+1)^2 dx$ (e) $\int (\sqrt{x}-2)(\sqrt{x}+2) dx$ (h) $\int \frac{2x^3-x^4}{4x} dx$
 (c) $\int (1-x^2)^2 dx$ (f) $\int (2\sqrt{x}-1)^2 dx$ (i) $\int \frac{x^2+2x}{\sqrt{x}} dx$

DEVELOPMENT

5. By using the formula $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, find:

(a) $\int (x+1)^5 dx$ (d) $\int (3x+1)^4 dx$ (g) $\int 2(2x-1)^{10} dx$
 (b) $\int (x+2)^3 dx$ (e) $\int \left(1-\frac{x}{5}\right)^3 dx$ (h) $\int \frac{8}{(4x+1)^5} dx$
 (c) $\int (4-x)^4 dx$ (f) $\int \frac{1}{(x+1)^3} dx$ (i) $\int \frac{4}{5(1-4x)^2} dx$

6. Find each of the following indefinite integrals:

(a) $\int \sqrt{x+1} dx$ (d) $\int \sqrt[3]{4x-1} dx$ (g) $\int \left(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x+2}}\right) dx$
 (b) $\int \sqrt{2x-1} dx$ (e) $\int \frac{1}{\sqrt{3x+5}} dx$ (h) $\int \left(\sqrt{4-x} + \frac{1}{\sqrt{4-x}}\right) dx$
 (c) $\int \sqrt{7-4x} dx$ (f) $\int \sqrt{1-\frac{1}{2}x} dx$ (i) $\int \left(\sqrt{ax} + \frac{1}{\sqrt{ax}}\right) dx$

7. Evaluate the following:

(a) $\int_0^2 (x+1)^4 dx$ (e) $\int_0^1 (ax+b)^2 dx$ (i) $\int_1^2 (2x+7(3x-4)^6) dx$
 (b) $\int_2^3 (2x-5)^3 dx$ (f) $\int_0^1 \sqrt{9-8x} dx$ (j) $\int_{-1}^2 \left(\frac{1}{\sqrt{x+2}} + \sqrt{x+2}\right) dx$
 (c) $\int_{-2}^2 (1-x)^5 dx$ (g) $\int_2^7 \frac{dx}{\sqrt{x+2}}$ (k) $\int_1^5 \sqrt{3x+1} dx$
 (d) $\int_0^5 \left(1-\frac{x}{5}\right)^4 dx$ (h) $\int_0^7 \sqrt[3]{x+1} dx$ (l) $\int_{-3}^0 \sqrt{1-5x} dx$

EXTENSION

8. (a) If u and v are differentiable functions in x , prove that $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$.

(b) Hence find $\int x\sqrt{1+x} dx$.

9. A question in Exercise 11B asked for an explanation, with a sketch, why this calculation is invalid:

$$\int_{-1}^1 \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^1 = -1 - 1 = -2.$$

Given that it is invalid, can any meaning nevertheless be given to the final result?

11E Finding Area by Integration

The aim of this section and the next is to use definite integrals to find the areas of regions bounded by curves, lines and axes.

Integral and Area: A definite integral is a pure number, which can be positive or negative because areas of regions below the x -axis are counted as negative. An area has units (called 'square units' or u^2 in the absence of any physical interpretation), and cannot be negative. Any problem on areas requires some care when finding the correct integral or combination of integrals required. Some particular techniques are listed below, but the general rule is to draw a picture first to see which bits need to be added or subtracted.

14 FINDING AN AREA: When using integrals to find the area of a region, first draw a sketch of the curves. The sketch will usually need to show any intercepts and any points of intersection.

Area Above and Below the x -axis: When a curve crosses the x -axis, the area of the region between the curve and the x -axis cannot usually be found by a single integral, because the integral represents areas of regions under the x -axis as negative.

WORKED EXERCISE: Find the area between the x -axis and $y = (x + 1)x(x - 2)$.

SOLUTION: Expanding,

$$y = x^3 - x^2 - 2x.$$

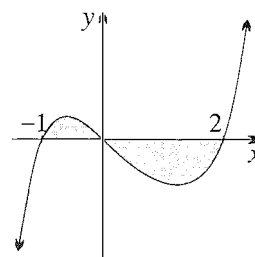
For the region above the x -axis, area = $+\int_{-1}^0 (x^3 - x^2 - 2x) dx$

$$\begin{aligned} &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_{-1}^0 \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \\ &= \frac{5}{12} \text{ square units.} \end{aligned}$$

For the region below the x -axis, area = $-\int_0^2 (x^3 - x^2 - 2x) dx$

$$\begin{aligned} &= -\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_0^2 \\ &= -(4 - 2\frac{2}{3} - 4) + (0 - 0 - 0) \\ &= 2\frac{2}{3} \text{ square units.} \end{aligned}$$

Adding these, total area = $3\frac{1}{12}$ square units.



Using Symmetry: As always, symmetry should be exploited whenever possible, but some explanation of what has been done needs to be given.

WORKED EXERCISE: Find the area between the curve $y = x^3 - x$ and the x -axis.

SOLUTION: Factoring,

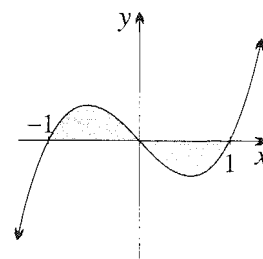
$$y = x(x - 1)(x + 1),$$

and the two regions have equal area, since the function is odd.

Now area below the x -axis = $-\int_0^1 (x^3 - x) dx$

$$\begin{aligned} &= -\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_0^1 \\ &= -\left(\frac{1}{4} - \frac{1}{2} \right) + (0 - 0) \\ &= \frac{1}{4} \text{ square units.} \end{aligned}$$

Doubling, total area = $\frac{1}{2}$ square units.



Area Between a Graph and the y -axis: When x is a function of y , the definite integral with respect to y will find the area of the region between the curve and the y -axis, with areas of regions to the left of the y -axis being counted as negative. The limits of integration are then values of y rather than of x . This technique can often avoid a subtraction.

15 DEFINITION: Suppose x is a function of y in some closed interval $a \leq y \leq b$. Then $\int_a^b x \, dy$ is the area of the region between the curve and the y -axis from $y = a$ to $y = b$, with areas right of the y -axis counted as positive and areas left of the y -axis counted as negative.

WORKED EXERCISE: Use integration with respect to y to find the area of the region between the cubic $y = x^3 + 1$, the y -axis, the x -axis and the line $y = 9$.

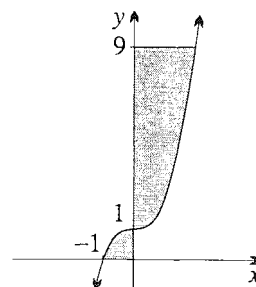
SOLUTION: The cubic crosses the y -axis at $(0, 1)$.

Solving for x , the equation of the cubic is $x = (y - 1)^{\frac{1}{3}}$.

For the region left of the y -axis, area $= - \int_0^1 (y - 1)^{\frac{1}{3}} \, dy$
 $= -\frac{3}{4} \left[(y - 1)^{\frac{4}{3}} \right]_0^1$
 $= -\frac{3}{4}(0 - 1)$
 $= \frac{3}{4}$ square units.

For the region right of the y -axis, area $= + \int_1^9 (y - 1)^{\frac{1}{3}} \, dy$
 $= \frac{3}{4} \left[(y - 1)^{\frac{4}{3}} \right]_1^9$
 $= \frac{3}{4}(16 - 0)$
 $= 12$ square units.

Adding these, total area $= 12\frac{3}{4}$ square units.



Exercise 11E

1. (a) In the figure to the right, find the area of triangle AOD :

(i) by using the formula for the area of a triangle,

(ii) by evaluating $\int_{-2}^0 (x + 2) \, dx$.

- (b) Find the area of the trapezium $OBCD$:

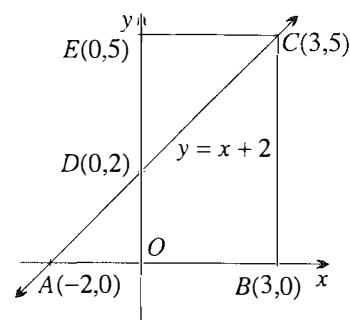
(i) by using the formula for the area of a trapezium,

(ii) by evaluating $\int_0^3 (x + 2) \, dx$.

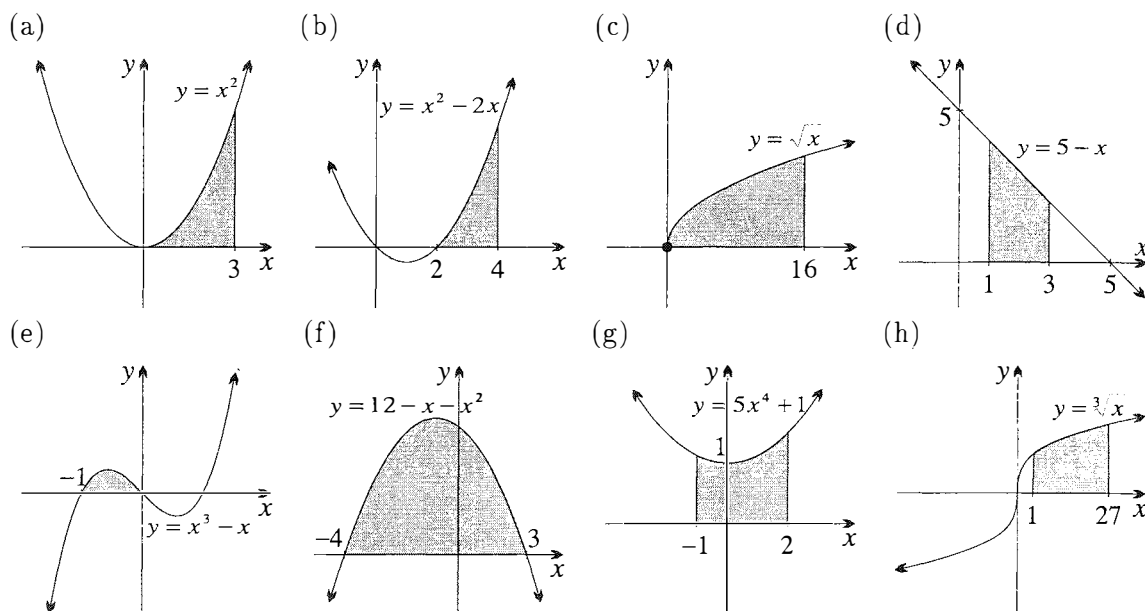
- (c) Find the area of triangle CDE :

(i) by using the formula for the area of a triangle,

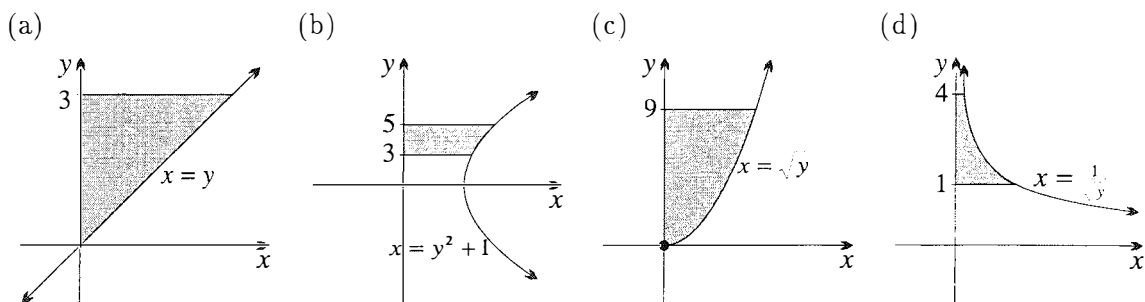
(ii) by evaluating $\int_2^5 (y - 2) \, dy$.



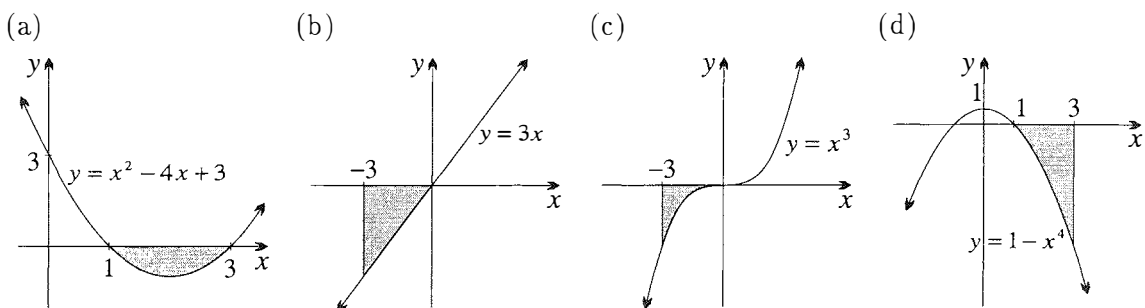
2. Find the area of each shaded region below by evaluating the appropriate integral:



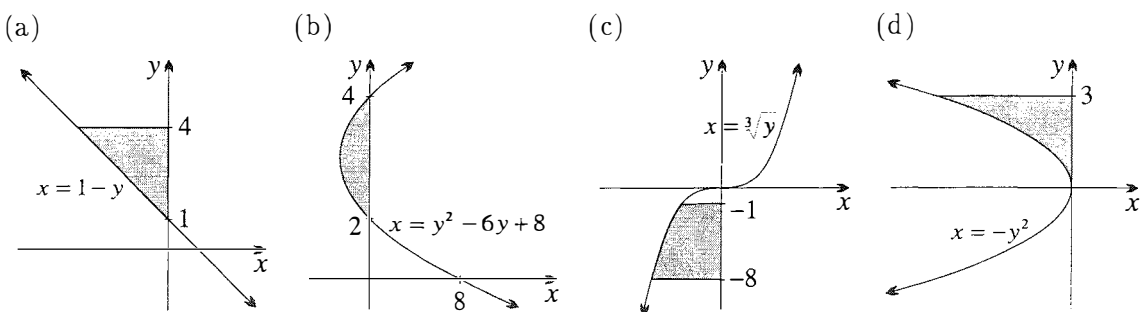
3. Find the area of each shaded region below by evaluating the appropriate integral:



4. Find the area of each shaded region below by evaluating the appropriate integral:

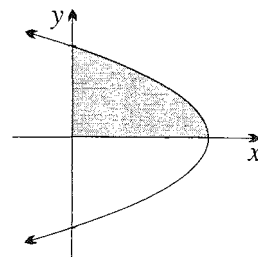


5. Find the area of each shaded region below by evaluating the appropriate integral:



DEVELOPMENT

6. Find the area bounded by the curve $y = |x + 2|$ and the x -axis for $-2 \leq x \leq 2$.
7. Find the area bounded by the graph of the given function and the x -axis between the specified values. You should draw a diagram for each question and check to see whether the area is above or below the x -axis.
- (a) $y = x^2$, $x = -3$ and $x = 2$ (f) $y = x(3 - x)^2$, $x = 0$ and $x = 3$
 (b) $y = 2x^3$, $x = -4$ and $x = 1$ (g) $y = x(x - 1)(x + 3)$, $x = 1$ and $x = -3$
 (c) $y = 3x(x - 2)$, $x = 0$ and $x = 2$ (h) $y = (x - 1)(x + 3)(x - 2)$, $x = -3$ and $x = 2$
 (d) $y = x - 3$, $x = -1$ and $x = 4$ (i) $y = x(1 - x)$, $x = -2$ and $x = 2$
 (e) $y = 4 - x$, $x = 0$ and $x = 8$ (j) $y = x^4 - 4x^2$, $x = 0$ and $x = 3$
8. Find the area bounded by the graph of the given function and the y -axis between the specified values. You should draw a diagram for each question and check to see whether the area is to the right or left of the y -axis.
- (a) $x = y - 5$, $y = 0$ and $y = 6$ (c) $x = y^2$, $y = -1$ and $y = 3$
 (b) $x = 3 - y$, $y = 2$ and $y = 5$ (d) $x = (y - 1)(y + 1)$, $y = -2$ and $y = 0$
9. In these questions you should draw a graph and look carefully for any symmetries that will simplify the calculation.
- (a) Find the area bounded by the curve and the x -axis: (i) $y = x^7$, $-2 \leq x \leq 2$
 (ii) $y = x^3 - 16x$, $-4 \leq x \leq 4$ (iii) $y = x^4 - 3x^2$, $-\sqrt{3} \leq x \leq \sqrt{3}$
 (b) Find the area bounded by the curve and the y -axis: (i) $x = 2y$, $-5 \leq y \leq 5$
 (ii) $x = y^2$, $-3 \leq y \leq 3$ (iii) $x = 4 - y^2$, $-2 \leq y \leq 2$
10. Find the area bounded by the graph of the given function and the coordinate axes:
- (a) $y = (x + 2)^3$ (c) $y = \sqrt{x + 3}$
 (b) $y = (3x - 4)^4$ (d) $y = \sqrt{5 - 2x}$
11. The diagram shows a graph of $y^2 = 16(2 - x)$.
- (a) Find the x and y intercepts.
 (b) Find the magnitude of the shaded area by considering:
 (i) the area between the curve and the x -axis,
 (ii) the area between the curve and the y -axis.
12. From the point $A(2, 2)$ on the curve $y = \frac{1}{4}x^3$, a line is drawn parallel to the y -axis to meet the x -axis at B and a line is drawn parallel to the x -axis to meet the y -axis at C . Show that the curve divides the resultant rectangle in the ratio 1 : 3.
13. (a) The gradient of a curve is given by $f'(x) = x^2 - 2x - 3$, and the curve passes through the origin. Find its equation and sketch its graph, indicating all stationary points.
 (b) Find the area enclosed between the curve and the x -axis between the turning points.
14. Sketch $y = x^2$ and mark the points $A(a, a^2)$, $B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.
- (a) Show that $\int_0^a x^2 dx = \frac{2}{3}$ (area of $\triangle OAP$).
 (b) Show that $\int_{-a}^a x^2 dx = \frac{4}{3}$ (area of rectangle $ABQP$).



15. Given positive real numbers a and n , let A , P and Q be the points (a, a^n) , $(a, 0)$ and $(0, a^n)$ respectively. Find the ratios:
- (a) $\int_0^a x^n dx : (\text{area of } \triangle AOP)$ (b) $\int_0^a x^n dx : (\text{area of rectangle } OPAQ)$
16. (a) Show that $x^4 - 2x^3 + x = x(x-1)(x^2 - x - 1)$. Then sketch a graph of the function $y = x^4 - 2x^3 + x$ and shade the three regions bounded by the graph and the x -axis.
- (b) If $a = \frac{1}{2}(1 + \sqrt{5})$, evaluate a^2 , a^4 and a^5 .
- (c) Show that the area of one shaded region equals the sum of the areas of the other two.
17. (a) Find the area bounded by the curve $y = ax^2 + bx + c$ and the x -axis between $x = h$ and $x = -h$, where $y > 0$ for $-h \leq x \leq h$.
- (b) [Simpson's rule — see Section 11J] Hence show that if $y = y_0$, y_1 and y_2 when $x = -h$, 0 and h respectively, then the area is given by $\frac{1}{3}h(y_0 + 4y_1 + y_2)$.

————— EXTENSION —————

18. Consider the function $G(x) = \int_0^x g(u) du$, where $g(u) = \begin{cases} 4 - \frac{4}{3}u, & \text{for } 0 \leq u < 6, \\ u - 10, & \text{for } 6 \leq u \leq 12. \end{cases}$
- (a) Sketch a graph of $g(u)$.
- (b) Find the stationary points of the function $y = G(x)$ and determine their nature.
- (c) Find those values of x for which $G(x) = 0$.
- (d) Sketch the curve $y = G(x)$, indicating all important features.
- (e) Find the area bounded by the curve $y = G(x)$ and the x -axis for $0 \leq x \leq 6$.
19. (a) Show that for $n < -1$, $\int_1^N x^n dx$ converges as $N \rightarrow \infty$, and find the limit.
- (b) Show that for $n > -1$, $\int_\varepsilon^1 x^n dx$ converges as $\varepsilon \rightarrow 0^+$, and find the limit.
- (c) Interpret these two results as areas.

11F Area of a Compound Region

When a region is bounded by two or more different curves, some dissection process may need to be employed before its area can be calculated using definite integrals. A preliminary sketch of the region therefore becomes all the more important.

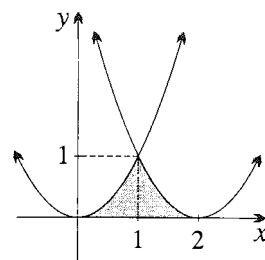
Area Under a Combination of Curves: Sometimes a region is bounded by different curves in different parts of the x -axis.

WORKED EXERCISE: Find the area bounded by $y = x^2$, $y = (x-2)^2$ and the x -axis.

SOLUTION: The two curves intersect at $(1, 1)$.

$$\begin{aligned} \text{First, } \int_0^1 x^2 dx &= \left[\frac{1}{3}x^3 \right]_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

$$\begin{aligned} \text{Secondly, } \int_1^2 (x-2)^2 dx &= \left[\frac{1}{3}(x-2)^3 \right]_1^2 \\ &= 0 - \left(-\frac{1}{3}\right) \\ &= \frac{1}{3}. \end{aligned}$$



Combining these, $\text{area} = \frac{1}{3} + \frac{1}{3}$
 $= \frac{2}{3}$ square units.

Area Between Curves: Suppose $y = f(x)$ and $y = g(x)$ are two curves with $f(x) \leq g(x)$ in the interval $a \leq x \leq b$, so that $y = f(x)$ is never above $y = g(x)$. Then the area of the region contained between the curves can be found by subtraction.

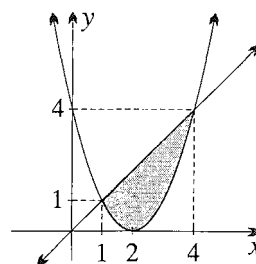
16 AREA BETWEEN CURVES: If $f(x) \leq g(x)$ in the interval $a \leq x \leq b$,
 area between the curves $= \int_a^b (g(x) - f(x)) dx$.
 That is, take the integral of the top curve minus the bottom curve.

The assumption that $f(x) \leq g(x)$ is important. If the curves cross each other, then separate integrals will need to be taken or else the areas where different curves are on top will begin to cancel each other out.

WORKED EXERCISE: Find the area between the two curves $y = (x - 2)^2$ and $y = x$.

SOLUTION: The two curves intersect at $(1, 1)$ and $(4, 4)$, and in the shaded region, the line is above the parabola.

$$\begin{aligned} \text{Area} &= \int_1^4 (x - (x - 2)^2) dx \\ &= \int_1^4 (-x^2 + 5x - 4) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x \right]_1^4 \\ &= \left(-21\frac{1}{3} + 40 - 16 \right) - \left(-\frac{1}{3} + 2\frac{1}{2} - 4 \right) \\ &= 4\frac{1}{2} \text{ square units.} \end{aligned}$$

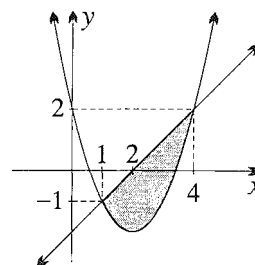


NOTE: This formula for the area of the region between two curves holds even if the region crosses the x -axis. To illustrate this, the next example is the previous example pulled down 2 units so that the region between the line and the parabola crosses the x -axis. The area of course remains the same — and notice how the formula still gives the correct answer.

WORKED EXERCISE: Find the area between $y = x^2 - 4x + 2$ and $y = x - 2$.

SOLUTION: The two curves intersect at $(1, -1)$ and $(4, 2)$.

$$\begin{aligned} \text{Area} &= \int_1^4 ((x - 2) - (x^2 - 4x + 2)) dx \\ &= \int_1^4 (-x^2 + 5x - 4) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x \right]_1^4 \\ &= \left(-21\frac{1}{3} + 40 - 16 \right) - \left(-\frac{1}{3} + 2\frac{1}{2} - 4 \right) \\ &= 4\frac{1}{2} \text{ square units.} \end{aligned}$$



Area Between Curves that Cross: Now suppose that one curve $y = f(x)$ is sometimes above and sometimes below another curve $y = g(x)$ in the region where areas are being calculated. In this case, separate integrals will need to be taken.

WORKED EXERCISE: The diagram below shows the curves $y = -x^2 + 4x - 4$ and $y = x^2 - 8x + 12$ meeting at the points $(2, 0)$ and $(4, -4)$. Find the shaded area.

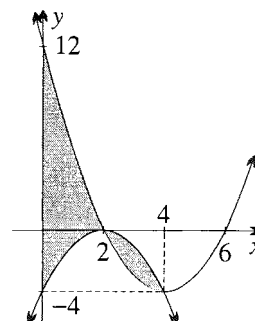
SOLUTION: In the left-hand region, the second curve is above the first,

$$\begin{aligned} \text{so area} &= \int_0^2 ((x^2 - 8x + 12) - (-x^2 + 4x - 4)) dx \\ &= \int_0^2 (2x^2 - 12x + 16) dx \\ &= \left[\frac{2}{3}x^3 - 6x^2 + 16x \right]_0^2 \\ &= 5\frac{1}{3} - 24 + 32 \\ &= 13\frac{1}{3} \text{ square units.} \end{aligned}$$

In the right-hand region, the first curve is above the second,

$$\begin{aligned} \text{so area} &= \int_2^4 ((-x^2 + 4x - 4) - (x^2 - 8x + 12)) dx \\ &= \int_2^4 (-2x^2 + 12x - 16) dx \\ &= \left[-\frac{2}{3}x^3 + 6x^2 - 16x \right]_2^4 \\ &= (-42\frac{2}{3} + 96 - 64) - (-5\frac{1}{3} + 24 - 32) \\ &= 2\frac{2}{3} \text{ square units.} \end{aligned}$$

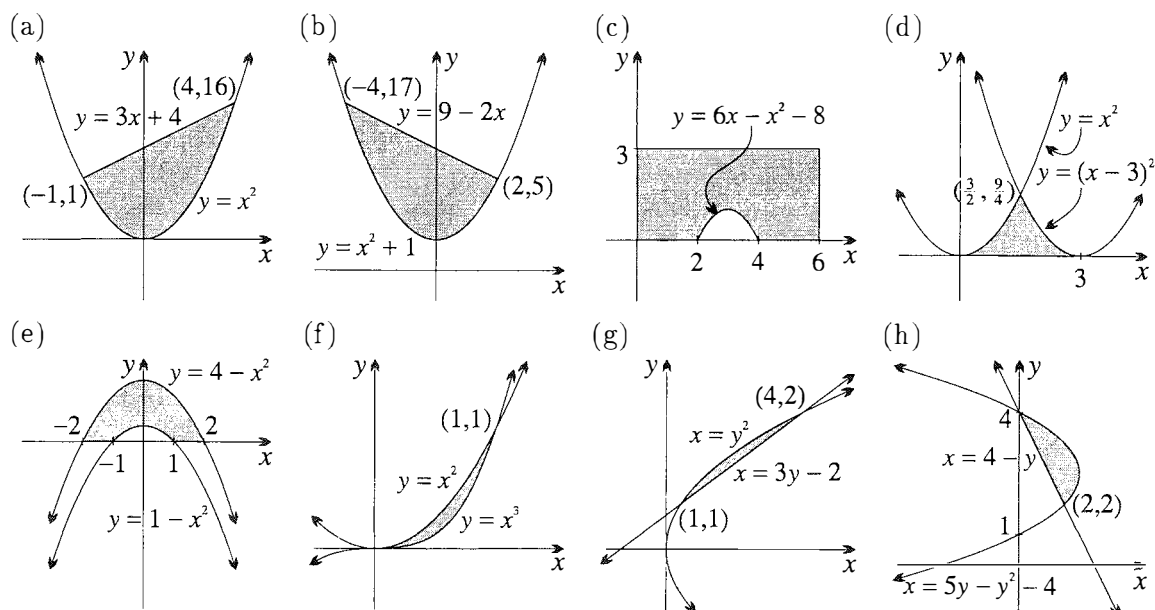
So total area = 16 square units.



NOTE: When one of the curves is the x -axis, then the region involved is the region between the other curve and $y = 0$. So the calculations of areas between a curve and the x -axis in the previous section were special cases of the areas between curves in this section.

Exercise 11F

1. Find the magnitude of each shaded area in the diagrams below:



2. (a) By solving the equations simultaneously, show that the curves $y = x^2 + 4$ and $y = x + 6$ intersect at the points $(-1, 5)$ and $(2, 8)$. Then sketch the curves on the same number plane and shade the area enclosed between them.

(b) Show that this area is equal to $\int_{-1}^2 (x - x^2 + 2) dx$ and evaluate the integral.

3. (a) Find the points of intersection of the curves $y = (x - 3)^2$ and $y = 14 - 2x$, then sketch the curves on the same number plane and shade the area enclosed between them.

(b) Show that this area is equal to $\int_{-1}^5 (4x + 5 - x^2) dx$ and evaluate the integral.

DEVELOPMENT

4. Sketch graphs of each pair of functions, showing their points of intersection. By evaluating the appropriate integral in each case, find the area enclosed between the two curves.

(a) $y = 9 - x^2$ and $y = 3 - x$

(c) $y = x^4$ and $y = x^2$

(b) $y = x + 10$ and $y = (x - 3)^2 + 1$

(d) $y = 3x^2$ and $y = 6x^3$

5. (a) By solving the equations simultaneously, show that the curves $y = x^2 + 2x - 8$ and $y = 2x + 1$ intersect when $x = 3$ and $x = -3$. Then sketch both curves on the same number plane and shade the region enclosed between them.

(b) Despite the fact that it crosses the x -axis, the area of the region between the curves is given by $\int_{-3}^3 ((2x + 1) - (x^2 + 2x - 8)) dx$. Evaluate the integral and hence find the magnitude of the area enclosed between the curves.

6. Find the area bounded by the lines $y = \frac{1}{4}x$ and $y = -\frac{1}{2}x$ between $x = 1$ and $x = 4$.

7. Sketch graphs of each pair of functions, clearly indicating their points of intersection, then find the area enclosed between the two curves in each case:

(a) $y = x^2 - 6x + 5$ and $y = x - 5$

(c) $y = -3x$ and $y = 4 - x^2$

(b) $y = x$ and $y = x^3$

(d) $y = x^2 - 1$ and $y = 7 - x^2$

8. (a) On the same number plane sketch graphs of the functions $y = x^2$ and $x = y^2$, clearly indicating the points of intersection.

(b) Find the magnitude of the area bounded by the two curves.

9. Consider the function $x^2 = 8y$. Tangents are drawn at the points $A(4, 2)$ and $B(-4, 2)$ and intersect on the y -axis. Find the area bounded by the curve and the tangents.

10. (a) Show that the equation of the tangent to the curve $y = x^3$ at the point where $x = 2$ is $y - 12x + 16 = 0$.

(b) Show that the tangent and the curve meet again at the point $(-4, -64)$.

(c) Find the magnitude of the area enclosed between the curve and the tangent.

11. Sketch graphs of each pair of functions, showing their points of intersection, then find the area enclosed between the two curves in each case.

(a) $y = \sqrt{x + 2}$ and $5y = x + 6$

(b) $y = \sqrt{3 - x}$ and $2x + 3y - 6 = 0$

12. (a) Given the two functions $f(x) = (x + 1)(x - 1)(x - 3)$ and $g(x) = (x - 1)(x + 1)$, for what values of x is $f(x) > g(x)$?

(b) Sketch a graph of the two functions on the same number plane and find the magnitude of the area between them.

13. Find the area bounded by the curves $y = x^2(1 - x)$ and $y = x(1 - x)^2$.

14. (a) Sketch a graph of the function $y = 12x - 32 - x^2$, clearly indicating the x -intercepts.
 (b) Find the equation of the tangent to the curve at the point A where $x = 5$.
 (c) If the tangent meets the x -axis at B , and C is the x -intercept of the parabola closer to the origin, find the area of the region bounded by AB , BC and the arc CA .

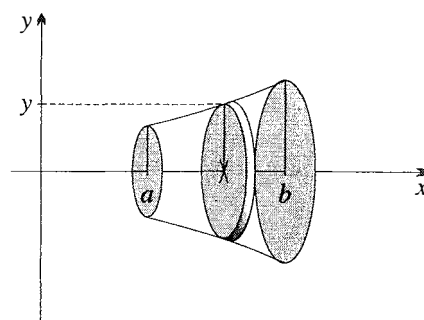
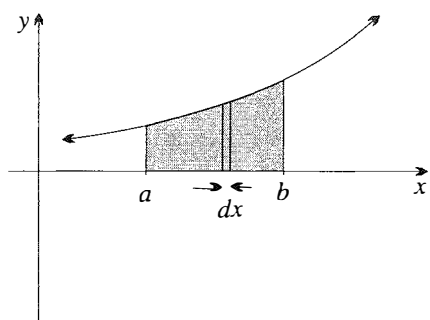
EXTENSION

15. Find the value of k for which the line $y = kx$ bisects the area enclosed by the curve $4y = 4x - x^2$ and the x -axis.
 16. The average value of a continuous function $f(x)$ over an interval $a \leq x \leq b$ is defined to be $\frac{1}{b-a} \int_a^b f(x) dx$ if $a \neq b$, or $f(a)$ if $a = b$. If k is the average value of $f(x)$ on the interval $a \leq x \leq b$, show that the area of the region bounded by $f(x)$ above the line $y = k$ is equal to the area of the region bounded by $f(x)$ below the line $y = k$.

11G Volumes of Solids of Revolution

If a region in the coordinate plane is rotated about either the x -axis or the y -axis, a solid region in three dimensions is generated, called the *solid of revolution*. The process is similar to shaping a piece of wood on a lathe, or making pottery on a wheel, because such shapes have rotational symmetry and circular cross sections. The volumes of such solids can be found using a simple integration formula. The well-known formulae for the volumes of cones and spheres can finally be proven by this method.

Rotating a Region about the x -axis: The first diagram below shows the region under the curve $y = f(x)$ in the interval $a \leq x \leq b$, and the second shows the solid generated when this region is rotated about the x -axis.



Imagine the solid sliced like salami perpendicular to the x -axis into infinitely many circular slices, each of width dx . One of the slices is shown below, and the vertical strip on the first diagram above is what generates this slice when it is rotated about the x -axis.

Now radius of circular slice = y , the height of the strip,

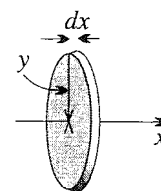
so area of circular slice = πy^2 , because it is a circle.

But the slice is essentially a very thin cylinder of thickness dx ,

so volume of circular slice = $\pi y^2 dx$ (area $\times$ thickness).

To get the total volume, we simply sum all the slices from $x = a$ to $x = b$,

so volume of solid = $\int_a^b \pi y^2 dx$.



17

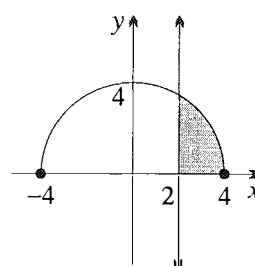
VOLUMES OF REVOLUTION ABOUT THE x -AXIS: The volume generated when the region between a curve and the x -axis from $x = a$ to $x = b$ is rotated about the x -axis is $\int_a^b \pi y^2 dx$ cubic units.

Notice that if the curve is below the x -axis, so that y is negative, the volume calculated is still positive, because it is y^2 rather than y which occurs in the formula. Unless other units are specified, 'cubic units' (u^3), should be used, by analogy with the areas of regions discussed in Section 11E.

WORKED EXERCISE: The shaded region cut off the semicircle $y = \sqrt{16 - x^2}$ by the line $x = 2$ is rotated about the x -axis. Find the volume generated.

SOLUTION: Squaring, $y^2 = 16 - x^2$,

$$\begin{aligned} \text{so} \quad \text{volume} &= \int_2^4 \pi y^2 dx \\ &= \pi \int_2^4 (16 - x^2) dx \\ &= \pi \left[16x - \frac{1}{3}x^3 \right]_2^4 \\ &= \pi \left(64 - \frac{64}{3} - 32 + \frac{8}{3} \right) \\ &= \frac{40\pi}{3} \text{ cubic units.} \end{aligned}$$



Volumes of Revolution about the y -axis: Calculating the volume when some region is rotated about the y -axis is simply a matter of exchanging x and y .

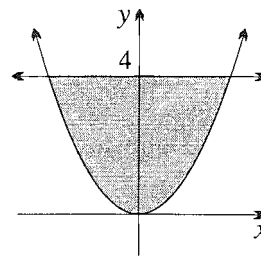
18

VOLUMES OF REVOLUTION ABOUT THE y -AXIS: The volume generated when the region between a curve and the y -axis from $y = a$ to $y = b$ is rotated about the y -axis is $\int_a^b \pi x^2 dy$.

When y is given as a function of x , the equation will need to be written with x^2 as the subject.

WORKED EXERCISE: Find the volume of the solid formed by rotating the region between $y = x^2$ and the line $y = 4$ about the y -axis.

$$\begin{aligned} \text{SOLUTION: Since } x^2 = y, \text{ volume} &= \int_0^4 \pi x^2 dy \\ &= \pi \int_0^4 y dy \\ &= \pi \left[\frac{1}{2}y^2 \right]_0^4 \\ &= 8\pi \text{ cubic units.} \end{aligned}$$



Volume by Subtraction: When rotating the region between two curves lying above the x -axis, the two integrals need to be subtracted, as if the outer volume has first been formed, and the inner volume cut away from it. The two volumes can always be calculated separately and subtracted, but if the two integrals have the same limits of integration, it may be more convenient to combine them.

19

ROTATING THE REGION BETWEEN CURVES: The volume generated when the region between two curves from $x = a$ to $x = b$ is rotated about the x -axis is

$$\int_a^b \pi(y_2^2 - y_1^2) dx \quad (\text{where } y_2 > y_1 > 0).$$

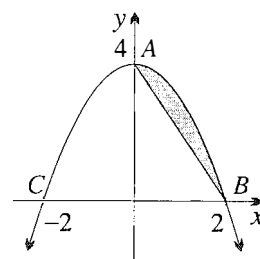
Similarly, $\int_a^b \pi(x_2^2 - x_1^2) dy$ is the volume of the solid generated when the region between two curves from $y = a$ to $y = b$ is rotated about the y -axis (where $x_2 > x_1 > 0$).

WORKED EXERCISE: The curve $y = 4 - x^2$ meets the y -axis at $A(0, 4)$ and the x -axis at $B(2, 0)$ and $C(-2, 0)$. Find the volume generated when the region between the curve and the line AB is rotated: (a) about the x -axis, (b) about the y -axis.

SOLUTION:

(a) The parabola is $y_2 = 4 - x^2$, and the line AB is $y_1 = 4 - 2x$,

$$\begin{aligned} \text{so volume} &= \int_0^2 \pi(y_2^2 - y_1^2) dx \\ &= \pi \int_0^2 \left((16 - 8x^2 + x^4) - (16 - 16x + 4x^2) \right) dx \\ &= \pi \int_0^2 (x^4 - 12x^2 + 16x) dx \\ &= \pi \left[\frac{1}{5}x^5 - 4x^3 + 8x^2 \right]_0^2 \\ &= \pi \left(\frac{32}{5} - 32 + 32 \right) \\ &= \frac{32\pi}{5} \text{ cubic units.} \end{aligned}$$



(b) The parabola is $x_2^2 = 4 - y$, and the line is $x_1 = 2 - \frac{1}{2}y$,

$$\begin{aligned} \text{so volume} &= \int_0^4 \pi(x_2^2 - x_1^2) dy \\ &= \pi \int_0^4 \left((4 - y) - (4 - 2y + \frac{1}{4}y^2) \right) dy \\ &= \pi \int_0^4 \left(-\frac{1}{4}y^2 + y \right) dy \\ &= \pi \left[-\frac{1}{12}y^3 + \frac{1}{2}y^2 \right]_0^4 \\ &= \pi \left(-\frac{16}{3} + 8 \right) \\ &= \frac{8\pi}{3} \text{ cubic units.} \end{aligned}$$

NOTE: One would not normally expect the volumes of revolution about the two different axes to be equal. This is because an element of area will generate a larger element of volume if it is moved further away from the axis of rotation.

Cones and Spheres: The formulae for the volumes of cones and spheres may have been learnt earlier, but they cannot be proven without arguments involving integration. The proofs of both results are developed in the following exercise, and these questions should be carefully worked.

20

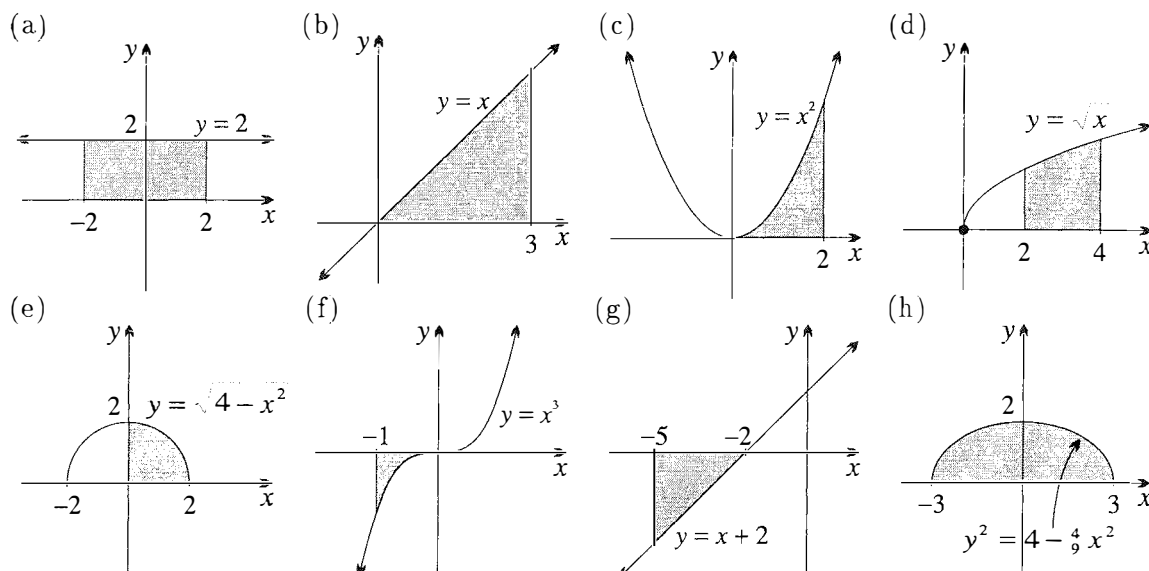
VOLUME OF A CONE: $V = \frac{1}{3}\pi r^2 h$

VOLUME OF A SPHERE: $V = \frac{4}{3}\pi r^3$

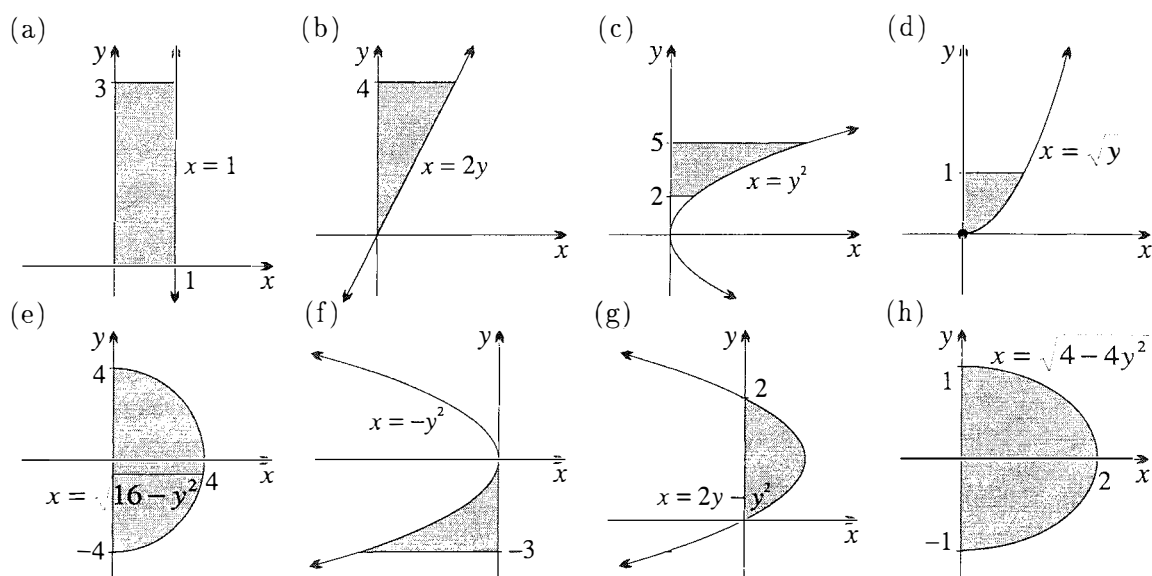
Exercise 11G

- Sketch the region bounded by the line $y = 3x$ and the x -axis between $x = 0$ and $x = 3$.
 - When this region is rotated about the x -axis, a right circular cone will be formed. Find the radius and height of the cone and hence find its volume.
 - Evaluate $\int_0^3 \pi y^2 dx = \pi \int_0^3 9x^2 dx$ in order to check your answer.
- Sketch a graph of the region bounded by the curve $y = \sqrt{9 - x^2}$ and the x -axis between $x = -3$ and $x = 3$.
 - When this region is rotated about the x -axis, a sphere will be formed. Find the radius of the sphere and hence find its volume.
 - Evaluate $\int_{-3}^3 \pi y^2 dx = \pi \int_{-3}^3 (9 - x^2) dx$ in order to check your answer.

3. Calculate the volume generated when each shaded region is rotated about the x -axis:



4. Calculate the volume generated when each shaded region is rotated about the y -axis:



DEVELOPMENT

5. Find the volume of the solid generated by rotating the region bounded by the following curves about the x -axis. Sketch a graph of each region and when finding y^2 , recall that $(a + b)^2 = a^2 + 2ab + b^2$.

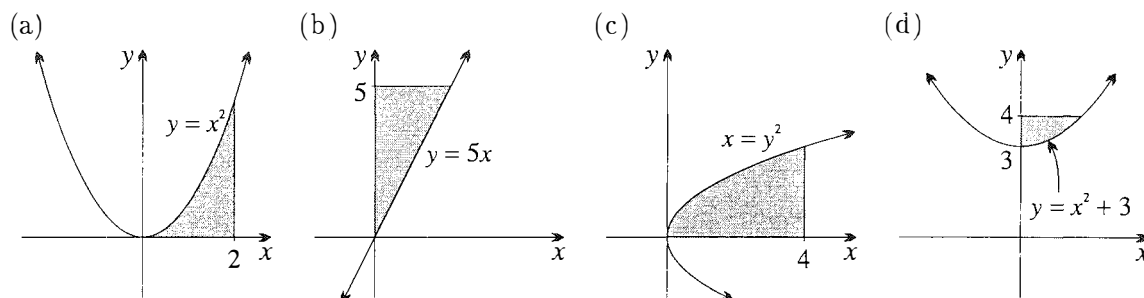
(a) $y = x + 3$, $x = 3$, $x = 5$ and $y = 0$ (c) $y = 5x - x^2$ and $y = 0$
 (b) $y = 1 + \sqrt{x}$, $x = 1$, $x = 4$ and $y = 0$ (d) $y = x^3 - x$ and $y = 0$

6. Find the volume of the solid generated by rotating the region bounded by the following curves about the y -axis. Preliminary sketches will be needed.

(a) $x = y - 2$, $y = 1$ and $x = 0$ (c) $x = y(y - 3)$ and $x = 0$
 (b) $x = y^2 + 1$, $y = 0$, $y = 1$ and $x = 0$ (d) $y = 1 - x^2$ and $y = 0$

7. (a) Sketch the region bounded by $y = x^2$ and the x -axis between $x = 0$ and $x = 4$.
 (b) Find the volume of the solid generated when this region is rotated about the x -axis.
 (c) Find the volume V_1 of the cylinder formed when the line $x = 4$ between $y = 0$ and $y = 16$ is rotated about the y -axis.
 (d) Evaluate $V_2 = \int_0^{16} \pi x^2 dy = \pi \int_0^{16} y dy$.
 (e) Hence evaluate the volume $V = V_1 - V_2$ when the region defined in part (a) is rotated about the y -axis.

8. Find the volume of the solid generated by rotating each region about: (i) the x -axis, (ii) the y -axis. [HINT: In some cases a subtraction of volumes will be necessary.]



9. By evaluating the appropriate integral, find the volume of the sphere generated when the region inside the circle $x^2 + y^2 = 64$ is rotated about the x -axis.
10. A vase is formed by rotating the portion of the curve $y^2 = x - 6$ between $y = 6$ and $y = -6$ about the y -axis. Find the volume of the vase.
11. The region bounded by the curve $y = 3 - x^2$ and the lines $y = 3$ and $x = \sqrt{3}$ is rotated about the y -axis. Sketch this region on a number plane and calculate the volume of the solid that is generated.
12. (a) By solving the equations by substitution, show that the curves $x^2 = 16y$ and $4y^2 = x$ intersect at the point $(4, 1)$.
 (b) Calculate the area of the region bounded by the two curves.
 (c) Find the volume of the solid formed when this region is rotated about:
 (i) the x -axis, (ii) the y -axis.
13. (a) Sketch a graph of the region bounded by the curves $y = x^2$ and $y = x^3$.
 (b) Find the volume of the solid generated when this region is rotated about:
 (i) the x -axis, (ii) the y -axis.

14. (a) On the same number plane sketch graphs of the functions $xy = 5$ and $x + y = 6$, clearly indicating their points of intersection.
 (b) Find the volume of the solid generated when the region bounded by the two curves is rotated about the x -axis.
15. In this question a number of standard volume formulae will be proven.
 (a) (i) A right circular cone of height h and radius r is generated by rotating the straight line $y = \frac{r}{h}x$ between $x = 0$ and $x = h$ about the x -axis. Show that the volume of the cone is given by $\frac{1}{3}\pi r^2 h$.
 (ii) Find the volume of the frustum of the cone in the interval $a \leq x \leq b$.
 (b) A cylinder of height h and radius r is generated by rotating the line $y = r$ between $x = 0$ and $x = h$ about the x -axis. Show that the volume of the cylinder is $\pi r^2 h$.
 (c) (i) A sphere of radius r is generated by rotating the semicircle $y = \sqrt{r^2 - x^2}$ about the x -axis. Show that the volume of the sphere is given by $\frac{4}{3}\pi r^3$.
 (ii) A spherical cap of radius r and height h is formed by rotating the semicircle $y = \sqrt{r^2 - x^2}$ between $x = r$ and $x = r - h$ about the x -axis. Show that the volume of the cap is given by $\frac{1}{3}\pi h^2(3r - h)$.
16. Find the volume of the solid generated when:
 (a) the region between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the x -axis is rotated about the x -axis,
 (b) the region between the parabola $x^2 = 4ay$ and the y -axis between $y = 0$ and $y = h$ is rotated about the y -axis.
17. Sketch $x^2 = 4ay$, and shade the region bounded by the curve and the latus rectum. Find the volume of the solid generated when this region is rotated about:
 (a) the axis of symmetry, (b) the tangent at the origin.
18. (a) State the domain and range of the function $y = \sqrt{9 - x}$.
 (b) Sketch a graph of the function.
 (c) Calculate the area bounded by the curve and the coordinate axes in the first quadrant.
 (d) Calculate the volume of the solid generated when this region is rotated about:
 (i) the x -axis, (ii) the y -axis.
19. (a) Find the equation of the tangent to the curve $y = x^3 + 2$ at the point where $x = 1$.
 (b) Draw a diagram showing the region bounded by the curve, the tangent and the y -axis.
 (c) Calculate the volume of the solid generated when this region is rotated about:
 (i) the x -axis, (ii) the y -axis.
20. (a) On the same set of axes sketch the curves $y = \sqrt{9 - x^2}$ and $y = 18 - 2x^2$.
 (b) Find the area bounded by the two curves.
 (c) Find the volume of the solid formed when this region is rotated about the x -axis.
21. (a) Find all stationary points of the function $y = x + \frac{1}{x}$ and sketch its graph.
 (b) Show that the line $y = \frac{5}{2}$ intersects the curve when $x = \frac{1}{2}$ or $x = 2$.
 (c) Find the volume of the solid generated when the region between the curve and the line $y = \frac{5}{2}$ is rotated about the x -axis.

22. Consider the curves $f(x) = x^n$ and $f(x) = x^{n+1}$, where n is a positive integer.
- Find the points of intersection of the curves.
 - Show that the volume V_n of the solid generated when the region bounded by the two curves is rotated around the x -axis is given by $\pi \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right)$ cubic units.
 - By means of a diagram, show that the solid whose volume is given by $V_1 + V_2 + V_3 + \dots$ will be the cone formed by rotating the line $y = x$ between $x = 0$ and $x = 1$ about the x -axis.
 - Find the volume of the cone in part (c).
 - Hence show that $\frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{7 \times 9} + \dots = \frac{1}{6}$.

EXTENSION

23. (a) Sketch a graph of the function $y = \sqrt{x}(1-x)$, clearly indicating any stationary points and intercepts with the axes.
- Hence sketch a graph of the function $y^2 = x(1-x)^2$.
 - Find the area contained in the loop between $x = 0$ and $x = 1$.
 - Find the volume of the solid formed when this region is rotated about the x -axis.
24. (a) Sketch a graph of the region bounded by $y = -x^2 + 6x - 8$ and the x -axis for $2 \leq x \leq 4$.
- Use the quadratic formula to show that the equation of the curve is $x = 3 + \sqrt{1-y}$ for $3 \leq x \leq 4$, and $x = 3 - \sqrt{1-y}$ for $2 \leq x \leq 3$.
 - The region in part (a) is rotated about the y -axis. Show that the volume of the solid formed is given by $V = \int_0^1 12\pi\sqrt{1-y} \, dy$ and evaluate the integral.
25. (a) Sketch a graph of the circle $(x-3)^2 + y^2 = 1$.
- The region inside the circle is rotated about the y -axis. Show that the volume of the solid formed is given by $V = 24\pi \int_0^1 \sqrt{1-y^2} \, dy$ and evaluate the integral.

11H The Reverse Chain Rule

The result of a chain-rule differentiation is a product, so some products should be able to be integrated using a reverse form of the chain rule.

Running a Chain Rule Derivative Backwards: Once a derivative has been calculated using the chain rule, the result can be reversed to give an integral. There may then need to be some juggling with constants.

WORKED EXERCISE: Differentiate $(x^2 + 1)^4$, and hence integrate $x(x^2 + 1)^3$.

SOLUTION: Let $y = (x^2 + 1)^4$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= 2x \times 4(x^2 + 1)^3 \\ &= 8x(x^2 + 1)^3. \end{aligned}$$

Let $u = x^2 + 1$,

then $y = u^4$.

So $\frac{du}{dx} = 2x$

and $\frac{dy}{du} = 4u^3$.

Hence $\frac{d}{dx}(x^2 + 1)^4 = 8x(x^2 + 1)^3$, and writing this as an indefinite integral,

$$\int 8x(x^2 + 1)^3 dx = (x^2 + 1)^4 + C, \text{ for some constant } C,$$

$$\boxed{\div 8} \quad \int x(x^2 + 1)^3 dx = \frac{1}{8}(x^2 + 1)^4 + D, \text{ where } D \text{ is a constant.}$$

The Reverse Chain Rule: Obtaining this same result without having first been given a derivative to calculate is a little more sophisticated. The chain rule says that if $y = u^{n+1}$ is a power of u , where u is a function of x , then $\frac{dy}{dx} = (n+1)u^n \times \frac{du}{dx}$. Turning this around gives:

21 THE REVERSE CHAIN RULE: $\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C$, for some constant C .

Or, using function notation, $\int f'(x) (f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + C$.

Let us illustrate this by finding the previous integral directly.

WORKED EXERCISE: Find $\int x(x^2 + 1)^3 dx$, using the substitution $u = x^2 + 1$.

NOTE: The key to the whole process is to identify clearly the intermediate function u (or $f(x)$ in function notation).

SOLUTION:

$$\begin{aligned} \int x(x^2 + 1)^3 dx &= \frac{1}{2} \int 2x(x^2 + 1)^3 dx \\ &= \frac{1}{8}(x^2 + 1)^4 + C, \end{aligned}$$

as calculated in the last worked exercise.

Or, using function notation,

$$\begin{aligned} \int x(x^2 + 1)^3 dx &= \frac{1}{2} \int 2x(x^2 + 1)^3 dx \\ &= \frac{1}{8}(x^2 + 1)^4 + C. \end{aligned}$$

$$\text{Let } u = x^2 + 1.$$

$$\text{Then } \frac{du}{dx} = 2x.$$

$$\int u^3 \frac{du}{dx} dx = \frac{1}{4}u^4$$

$$\text{Let } f(x) = x^2 + 1.$$

$$\text{Then } f'(x) = 2x.$$

$$\int f'(x) (f(x))^3 dx = \frac{1}{4}(f(x))^4$$

Exercise 11H

1. (a) Find $\frac{d}{dx}(x^2 + 3)^4$. (b) Hence find: (i) $\int 8x(x^2 + 3)^3 dx$ (ii) $\int x(x^2 + 3)^3 dx$

2. (a) Find $\frac{d}{dx}(x^3 + 3x^2 + 5)^4$.

(b) Hence find: (i) $\int 12(x^2 + 2x)(x^3 + 3x^2 + 5)^3 dx$ (ii) $\int (x^2 + 2x)(x^3 + 3x^2 + 5)^3 dx$

3. (a) Find $\frac{d}{dx}(5 - x^2 - x)^7$.

(b) Hence find: (i) $\int (-14x - 7)(5 - x^2 - x)^6 dx$ (ii) $\int (2x + 1)(5 - x^2 - x)^6 dx$

4. (a) Find $\frac{d}{dx}(x^3 - 1)^5$. (b) Hence find: (i) $\int 15x^2(x^3 - 1)^4 dx$ (ii) $\int 3x^2(x^3 - 1)^4 dx$

5. (a) Find $\frac{d}{dx}\sqrt{2x^2+3}$. (b) Hence find: (i) $\int \frac{2x}{\sqrt{2x^2+3}} dx$ (ii) $\int \frac{x}{\sqrt{2x^2+3}} dx$
6. (a) Find $\frac{d}{dx}(\sqrt{x}+1)^3$. (b) Hence find: (i) $\int \frac{3(\sqrt{x}+1)^2}{2\sqrt{x}} dx$ (ii) $\int \frac{(\sqrt{x}+1)^2}{\sqrt{x}} dx$

DEVELOPMENT

7. Find the following indefinite integrals using the reverse chain rule. Some hints are given for the substitution.

- (a) $\int 10x(5x^2+3)^2 dx$ (Let $u = 5x^2+3$.) (h) $\int x\sqrt{5x^2+1} dx$ (Let $u = 5x^2+1$.)
- (b) $\int 2x(x^2+1)^3 dx$ (Let $u = x^2+1$.) (i) $\int \frac{x}{\sqrt{x^2+3}} dx$ (Let $u = x^2+3$.)
- (c) $\int 12x^2(1+4x^3)^5 dx$ (Let $u = 1+4x^3$.) (j) $\int \frac{x+1}{\sqrt{4x^2+8x+1}} dx$
- (d) $\int x(1+3x^2)^4 dx$ (Let $u = 1+3x^2$.) (k) $\int \frac{x}{(x^2+5)^3} dx$
- (e) $\int (x-2)(x^2-4x-5)^3 dx$ (l) $\int \frac{(\sqrt{x}-3)^4}{\sqrt{x}} dx$ (Let $u = \sqrt{x}-3$.)
- (f) $\int x^3(1-x^4)^7 dx$ (m) $\int px(qx^2-3)^3 dx$
- (g) $\int 3x^2\sqrt{x^3-1} dx$ (Let $u = x^3-1$.) (n) $\int rx^2(px^3+q)^4 dx$

8. Calculate the following definite integrals using the reverse chain rule:

- (a) $\int_{-1}^1 x^2(x^3+1)^4 dx$ (d) $\int_{-3}^{-1} (x+5)(x^2+10x+3)^2 dx$
- (b) $\int_0^1 \frac{x}{(5x^2-1)^3} dx$ (e) $\int_0^a \frac{x}{(ax^2+1)^2} dx$
- (c) $\int_0^{\frac{1}{2}} x\sqrt{1-4x^2} dx$ (f) $\int_{b^2}^{4b^2} \frac{(\sqrt{x}+b)^2}{\sqrt{x}} dx$, where $b > 0$

9. (a) What is the domain of the function $f(x) = x\sqrt{x^2-1}$?
 (b) Find $f'(x)$ and hence show that the function has no stationary points in its domain.
 (c) Show that the function is odd, and hence sketch its graph.
 (d) By evaluating the appropriate integral, find the area enclosed by the curve and the x -axis between $x=1$ and $x=3$.
10. (a) Sketch $y = x(7-x^2)^3$, indicating all stationary points and intercepts with the axes.
 (b) Find the area enclosed between the curve and the x -axis.

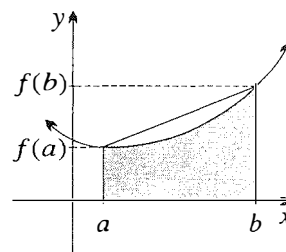
EXTENSION

11. (a) Explain why the graphs of $y = f(x)$ and $y = f(a-x)$ are reflections of each other in the vertical line $x = \frac{1}{2}a$. (b) Deduce $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, and hence evaluate:
- (i) $\int_0^4 x(4-x)^4 dx$ (ii) $\int_0^1 x^2\sqrt{1-x} dx$

11I The Trapezoidal Rule

Approximation methods for definite integrals become necessary when exact calculation through the primitive is not possible. This can happen for two major reasons. First, the primitives of many important functions cannot be written down in a form suitable for calculation. Secondly, some values of the function may be known from experiments, but the function itself may still be unknown.

The Trapezoidal Rule: The most obvious way to approximate an integral is to replace the curve by a straight line. The resulting region is then a trapezium, and so the approximation method is called the *trapezoidal rule*. The width of the trapezium is $b - a$, and the average of the sides is $\frac{1}{2}(f(a) + f(b))$. Hence, using the area formula for a trapezium:



22 **TRAPEZOIDAL RULE:** $\int_a^b f(x) dx \doteq \frac{b-a}{2} (f(a) + f(b))$,
with equality when the function is linear.

WORKED EXERCISE: Find approximations for $\int_1^5 \frac{1}{x} dx$ using the trapezoidal rule with
(a) one application, (b) four applications.

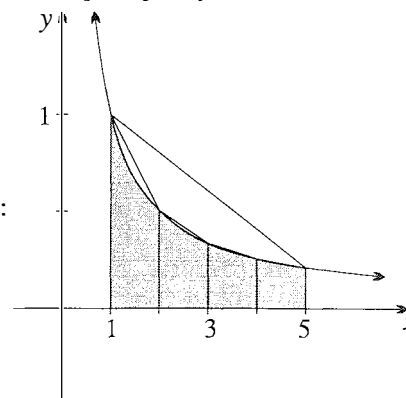
SOLUTION: Constructing a table of values:

x	1	2	3	4	5
$1/x$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

$$\begin{aligned} \text{(a)} \quad \int_1^5 \frac{1}{x} dx &\doteq \frac{5-1}{2} (f(1) + f(5)) \\ &\doteq 2 \times \left(1 + \frac{1}{5}\right) \\ &\doteq 2\frac{2}{5} \end{aligned}$$

(b) Dividing the interval $1 \leq x \leq 5$ into four subintervals:

$$\begin{aligned} \int_1^5 \frac{1}{x} dx &= \int_1^2 \frac{1}{x} dx + \int_2^3 \frac{1}{x} dx + \int_3^4 \frac{1}{x} dx + \int_4^5 \frac{1}{x} dx \\ &\doteq \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2}\right) + \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4}\right) + \frac{1}{2} \left(\frac{1}{4} + \frac{1}{5}\right) \\ &\doteq 1\frac{41}{60} \end{aligned}$$



NOTE: Because the curve is concave up, every approximation found using the trapezoidal rule is greater than the value of the integral. Similarly if a curve is concave down, every trapezoidal rule approximation is less than the integral. For linear functions the rule gives the exact value.

Exercise 11I

1. Show, by means of a diagram, that the trapezoidal rule will:

(a) overestimate $\int_a^b f(x) dx$ if $f''(x) > 0$ for $a \leq x \leq b$,

(b) underestimate $\int_a^b f(x) dx$ if $f''(x) < 0$ for $a \leq x \leq b$.

2. (a) Complete this table for the function $y = x(4 - x)$:
- | | | | | | |
|-----|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 |
| y | | | | | |
- (b) Using the the trapezoidal rule with these five function values, estimate $\int_0^4 x(4 - x) dx$.
- (c) What is the exact value of $\int_0^4 x(4 - x) dx$, and why does it exceed the approximation?
- (d) Calculate the percentage error in the approximation (that is, divide the error by the correct answer and convert to a percentage).

3. (a) Complete this table for the function $y = \frac{6}{x}$:
- | | | | | | |
|-----|---|---|---|---|---|
| x | 1 | 2 | 3 | 4 | 5 |
| y | | | | | |
- (b) Use the trapezoidal rule with the five function values above to estimate $\int_1^5 \frac{6}{x} dx$.
- (c) Find the second derivative of $y = \frac{6}{x}$ and use it to explain why the estimate will exceed the exact value of the integral.

4. (a) Complete this table for the function $y = \sqrt{x}$:
- | | | | | | | | | |
|-----|---|----|----|----|----|----|----|----|
| x | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| y | | | | | | | | |
- (b) Use the trapezoidal rule with the eight function values above to estimate $\int_9^{16} \sqrt{x} dx$. Give your answer correct to three significant figures.
- (c) What is the exact value of $\int_9^{16} \sqrt{x} dx$? Find the second derivative of $y = x^{\frac{1}{2}}$ and use it to explain why the estimate will be less than the exact value of the integral.

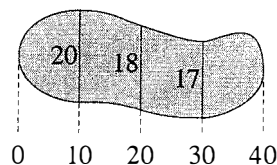
————— DEVELOPMENT —————

5. (a) Show that the function $y = \frac{1}{1 + x^2}$ has a stationary point at $(0, 1)$.
- (b) Sketch a graph of the function, showing all important features.
- (c) Use the trapezoidal rule with five function values to estimate $\int_0^2 \frac{1}{1 + x^2} dx$.
6. Use the trapezoidal rule with three function values to approximate each of these integrals. Answer correct to three decimal places. (a) $\int_0^1 2^{-x} dx$ (b) $\int_1^3 \sqrt[3]{9 - 2x} dx$.
7. (a) Use the trapezoidal rule with five function values to estimate $\int_0^1 \sqrt{1 - x^2} dx$ to four decimal places.
- (b) Use part (a) and the fact that $y = \sqrt{1 - x^2}$ is a semicircle to estimate π . Give your answer to three decimal places, and explain why your estimate is less than π .

8. An object is moving along the x -axis with values of the velocity v in m/s at time t given in the table on the right. Given that the distance travelled may be found by calculating the area under a velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the particle in the first five seconds.

t	0	1	2	3	4	5
v	1.5	1.3	1.4	2.0	2.4	2.7

9. The diagram on the right shows the width of a lake at 10 metre intervals. Use the trapezoidal rule to estimate the surface area of the water.



10. (a) A cone is generated by rotating the line $y = 2x$ about the x -axis between $x = 0$ and $x = 3$. Using the trapezoidal rule with four function values, estimate the volume of the cone.
 (b) Calculate the exact volume of the cone and hence find the percentage error in the approximation.
11. The region under the graph $y = 2^{x+1}$ between $x = 1$ and $x = 3$ is rotated about the x -axis. Using the trapezoidal rule with five function values, estimate the volume of the solid formed.

EXTENSION

12. (a) Show that the function $y = \sqrt{x}$ is increasing for all $x > 0$.
 (b) By dividing the area under the curve $y = \sqrt{x}$ into n equal subintervals, show that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \geq \int_0^n \sqrt{x} \, dx = \frac{2n\sqrt{n}}{3}.$$

- (c) Use mathematical induction to prove that for all integers $n \geq 1$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leq \frac{\sqrt{n}(4n+3)}{6}.$$

- (d) Give an alternative proof of part (c) using the trapezoidal rule.
 (e) Hence estimate $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{12000}$, correct to the nearest hundred.

11J Simpson's Rule

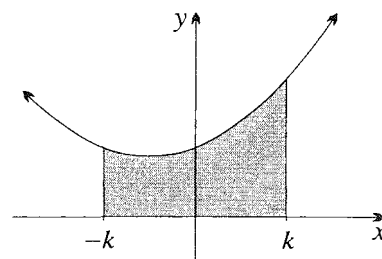
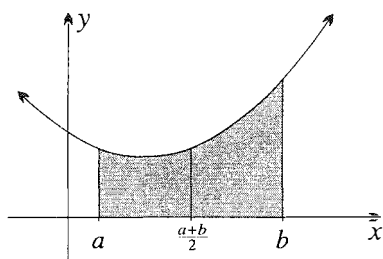
The trapezoidal rule approximates the function by a linear function, which is a polynomial of degree 1. The next most obvious method is to approximate the function by a polynomial of degree 2, that is, by a quadratic function. This is called *Simpson's rule*, and geometrically, it approximates the curve with a parabola.

Simpson's Rule: To approximate a definite integral using Simpson's rule, the value at the midpoint as well as the values at the endpoints must be known.

23

SIMPSON'S RULE: $\int_a^b f(x) \, dx \doteq \frac{1}{6}(b-a) \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right),$
 with equality holding when the function is quadratic.

PROOF:



Shifting the origin to the midpoint of the interval $a \leq x \leq b$ does not change the value of the integral, so we need only deal with the case where the interval is $-k \leq x \leq k$. We must therefore prove that if $f(x) = Ax^2 + Bx + C$ is any quadratic function, then

$$\int_{-k}^k f(x) dx = \frac{1}{3}k \left(f(-k) + 4f(0) + f(k) \right).$$

$$\begin{aligned} \text{LHS} &= \left[\frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx \right]_{-k}^k \\ &= \frac{2}{3}Ak^3 + 2Ck \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{1}{3}k \left((Ak^2 - Bk + C) + 4C + (Ak^2 + Bk + C) \right) \\ &= \frac{2}{3}Ak^3 + 2Ck, \text{ as required.} \end{aligned}$$

NOTE: Simpson's rule also gives the exact answer for cubic functions. This can be seen from the proof above, if one imagines a term Dx^3 being added to the quadratic. Being an odd function, Dx^3 would not affect the value of the integral on the LHS, and would also cancel out of the RHS when k and $-k$ were substituted.

WORKED EXERCISE: Use Simpson's rule to find an approximation to $\int_1^5 f(x) dx$, given the following table of values:

x	1	2	3	4	5
$f(x)$	2.31	4.56	5.34	3.02	0.22

SOLUTION: The best use of the data is to apply Simpson's rule on each of the intervals $1 \leq x \leq 3$ and $3 \leq x \leq 5$, and then add the results.

$$\text{First, } \int_1^3 f(x) dx \doteq \frac{1}{6} \times (3-1) \times (2.31 + 4 \times 4.56 + 5.34) \doteq 8.63.$$

$$\text{Secondly, } \int_3^5 f(x) dx \doteq \frac{1}{6} \times (5-3) \times (5.34 + 4 \times 3.02 + 0.22) \doteq 5.88.$$

Combining these gives

$$\int_1^5 f(x) dx \doteq 14.51.$$

Exercise 11J

1. (a) Complete this table for the function $y = \frac{2}{x}$:
- | x | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 |
|-----|---|----------------|---|----------------|---|
| y | | | | | |

- (b) Use Simpson's rule with three function values to estimate $\int_1^2 \frac{2}{x} dx$.

(c) Use Simpson's rule with three function values to estimate $\int_2^3 \frac{2}{x} dx$.

(d) Hence use Simpson's rule with five function values to estimate $\int_1^3 \frac{2}{x} dx$.

2. (a) Complete this table for the function $y = \sqrt{x+5}$:

x	-4	-3	-2
y			

(b) Use Simpson's rule to estimate $\int_{-4}^{-2} \sqrt{x+5} dx$ correct to three significant figures.

3. (a) Sketch a graph of the function $y = \sqrt{9-x^2}$.

(b) Hence evaluate $\int_{-3}^3 \sqrt{9-x^2} dx$ correct to three decimal places.

(c) Complete this table for $y = \sqrt{9-x^2}$:

x	-3	-1.5	0	1.5	3
y					

(d) Using five function values, estimate $\int_{-3}^3 \sqrt{9-x^2} dx$ correct to three decimal places:

(i) by the trapezoidal rule,

(ii) by Simpson's rule.

4. (a) Complete this table for $y = 3 - 2x - x^2$:

x	-3	-2	-1	0	1
y					

(b) Use Simpson's rule with five function values to approximate $\int_{-3}^1 (3 - 2x - x^2) dx$.

(c) Evaluate $\int_{-3}^1 (3 - 2x - x^2) dx$. How does this compare with the answer obtained in part (b)? Why is this the case?

DEVELOPMENT

5. Use Simpson's rule with three function values to approximate:

(a) $\int_1^3 \frac{dx}{x^2+1}$

(b) $\int_{-1}^1 3^{-x} dx$

6. Use Simpson's rule with five function values to approximate the following integrals. (Give the approximation correct to four significant figures where necessary.)

(a) $\int_3^5 \sqrt{x^2-1} dx$

(b) $\int_{-1}^1 \log_{10}(x+3) dx$

(c) $\int_2^6 3^x dx$

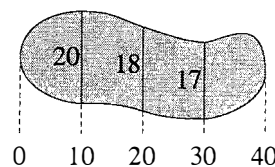
7. (a) Use Simpson's rule with five function values to estimate $\int_0^1 \sqrt{1-x^2} dx$. Give your answer correct to four decimal places.

(b) Use part (a) and the fact that $y = \sqrt{1-x^2}$ is a semicircle to estimate π . Give your answer correct to three decimal places.

8. An object is moving along the x -axis with values of the velocity v in m/s at time t given in the table on the right. Given that the distance travelled may be found by calculating the area under a velocity/time graph, use Simpson's rule to estimate the distance travelled by the particle in the first four seconds.

t	0	1	2	3	4
v	1.5	1.3	1.4	2.0	2.4

9. The diagram on the right shows the width of a lake at 10 metre intervals. Use Simpson's rule to estimate the surface area of the water.



10. The region bounded by the curve $y = 3^{x-1}$ and the x -axis between $x = 1$ and $x = 3$ is rotated about the x -axis. Use Simpson's rule with five function values to approximate the volume of the solid that is formed. Give your answer correct to two decimal places.
11. Consider the function $f(x) = \sqrt{x}(2 - x)$.
- Find $f'(x)$, and hence find the coordinates of any stationary points.
 - Sketch a graph of the function, indicating all important features.
 - Use Simpson's rule with five function values to approximate the area enclosed by the curve and the x -axis between its two zeroes. Answer correct to two decimal places.
 - Use Simpson's rule with five function values to approximate the volume of the solid formed when the region in part (c) is rotated about the x -axis. Answer correct to two decimal places.
 - Check your answers to parts (c) and (d) by evaluating the appropriate integrals.
12. $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$ is the length of the arc of a curve $y = f(x)$ between $x = a$ and $x = b$. Using Simpson's rule with five function values, estimate the length of the graph $y = x^2$ between $x = 0$ and $x = 2$. Answer correct to two decimal places.

EXTENSION

13. Machine computation of approximations to an integral $\int_a^b f(x) dx$ is more efficient with extended forms of the trapezoidal rule and Simpson's rule. Divide the interval $a \leq x \leq b$ into n subintervals, each of width h , so that $h = \frac{b-a}{n}$. Define $y_0, y_1, \dots, y_n$ by

$$y_0 = f(a), \quad y_1 = f(a+h), \quad y_2 = f(a+2h), \quad \dots, \quad y_n = f(b).$$

- (a) Explain why the two rules, using all these function values, can be written as

$$\text{TRAPEZOIDAL RULE: } \int_a^b f(x) dx = \frac{1}{2}h(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$$

$$\text{SIMPSON'S RULE: } \int_a^b f(x) dx = \frac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 4y_{n-1} + y_n)$$

- (b) Apply these formulae with various values of n to find estimates of $\int_1^2 \frac{1}{x} dx$ and

$\int_0^1 2^x dx$. Use a computer or programmable calculator if these are available. To test the accuracy of your calculations, you need to know that according to the methods developed in the next two chapters, the values of these integrals are reciprocals of each other, being 0.693 147 and 1.442 695 respectively, correct to six decimal places.

The Logarithmic Function

So far the calculus has been developed only for algebraic functions like

$$x^2 - 2x + 4, \quad \sqrt{1 - x^2}, \quad \text{and} \quad x^2 + \frac{1}{x^2},$$

which can be written using powers, roots and reciprocals. The first step in extending calculus to non-algebraic or *transcendental functions* is to study the logarithmic and exponential functions, which are so important in dealing with applications of calculus to the natural world.

Direct first principles differentiation of logarithmic and exponential functions, however, is blocked by some intractable limits, and by the fact that the most natural base to use for logarithmic and exponential functions is an irrational real number which is given the symbol e , and which makes its first appearance in this chapter. The most satisfying account of the theory begins not with any particular logarithmic or exponential function, but with the problem of integrating the reciprocal function $y = 1/x$. It is quite surprising that such an indirect approach makes the theory so clear.

STUDY NOTES: The natural logarithmic function $y = \log_e x$ is the central subject of this chapter, but it is vital that logarithmic and exponential functions to more familiar bases like 2 and 10 be well understood before the arguments involving calculus begin. Particularly important are their graphs, their domains and ranges, their asymptotic behaviour, and above all the fact that they are inverses of each other. The short account in Section 12A reviews the earlier discussion of the graphs and the algebra. The theorem on the derivative of the logarithmic functions in Section 12B is the fundamental step in the theory, and it may be best on first reading to be convinced simply by the graphical argument of Steps 1–4 of the explanation, leaving until later the tricky Steps 5 and 6. As always, computer, calculator and graph-paper work with these unfamiliar functions should be used to help establish an intuitive understanding of their behaviour. In particular, integration of $y = 1/x$ by counting squares on graph paper should be used to establish approximations of both the function $\log x$ and the value of e .

12A Review of Logarithmic and Exponential Functions

The algebra of the logarithmic and exponential functions was reviewed in Sections A and B of Chapter Six before the discussion of sequences and series. What is required for this chapter is a clear picture of the graphs, and a clear understanding that the logarithmic and exponential functions to a given base b are mutually inverse (the base b must be positive and different from 1).

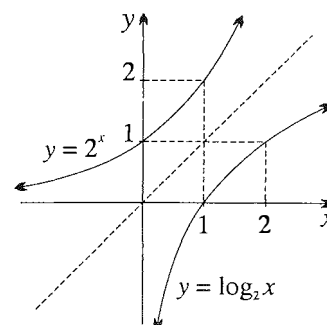
1

DEFINITION OF LOGARITHMS: The *logarithm* base b of a positive number x is the index, when the number x is expressed as a power of the base b :

$$y = \log_b x \quad \text{means that} \quad x = b^y.$$

This means that the functions $y = b^x$ and $y = \log_b x$ are inverse functions. The tables of values of the two functions will be the same except with the rows reversed. Taking the particular case $b = 2$,

$y = 2^x$							
x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$							
x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-3	-2	-1	0	1	2	3



On the right are the resulting graphs of the exponential function $y = 2^x$ and the logarithmic function $y = \log_2 x$, drawn on the one set of axes. They are reflections of each other in the diagonal line $y = x$.

Because the two function are mutually inverse, if they are applied one after the other to a number, then the number remains the same. For example, applying them in turn to 8,

$$\log_2 2^8 = \log_2 256 = 8 \quad \text{and} \quad 2^{\log_2 8} = 2^3 = 8.$$

The general statement of this is:

2

THE FUNCTIONS $y = b^x$ AND $y = \log_b x$ ARE MUTUALLY INVERSE:

$$\log_b b^x = x, \text{ for all real } x \quad \text{and} \quad b^{\log_b x} = x, \text{ for all } x > 0.$$

Here once again is a list of the index laws and the corresponding log laws:

3

THE LOG AND INDEX LAWS BASE b :	
$b^u \times b^v = b^{u+v}$	$\log_b xy = \log_b x + \log_b y$
$b^u \div b^v = b^{u-v}$	$\log_b \frac{x}{y} = \log_b x - \log_b y$
$(b^u)^v = b^{uv}$	$\log_b x^v = v \log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
$b^{-1} = \frac{1}{b}$	$\log_b \frac{1}{b} = -1$
$b^{-u} = \frac{1}{b^u}$	$\log_b \frac{1}{x} = -\log_b x$

Lastly there is the change of base formula. If c is some other base, then:

4

$$\text{CHANGE OF BASE: } \log_c x = \frac{\log_b x}{\log_b c} \quad (\text{log of the number over log of the base}).$$

Exercise 12A

1. (a) Copy and complete these tables of values of the functions
- $y = 3^x$
- and
- $y = \log_3 x$
- :

x	-2	-1	0	1	2
3^x					

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$					

- (b) Sketch both curves on the one set of axes, choosing appropriate scales on the axes.
- (c) Add the line $y = x$ to your graph. What transformation transforms the graph of $y = 3^x$ into the graph of $y = \log_3 x$?
- (d) What are the domain and range of $y = 3^x$ and $y = \log_3 x$?
2. (a) Use your calculator to copy and complete these tables of values of $y = 10^x$ and $y = \log_{10} x$:

x	-2	-1	-0.75	-0.5	0	0.25	0.5	0.75	1
10^x									

x	0.01	0.1	0.5	0.8	1	2	3	5	7	10
$\log_{10} x$										

- (b) Sketch both curves on the one set of axes, choosing appropriate scales on the axes.
- (c) Add the line $y = x$ to your graph. What transformation transforms the graph of $y = 10^x$ into the graph of $y = \log_{10} x$?
- (d) What are the domain and range of $y = 10^x$ and $y = \log_{10} x$?
3. First rewrite each equation in index form and then solve it:
- (a) $x = \log_2 16$ (c) $x = \log_5 25$ (e) $x = \log_{\frac{1}{7}} 49$ (g) $x = \log_{\frac{2}{3}} \left(\frac{8}{27}\right)$
 (b) $x = \log_3 \frac{1}{3}$ (d) $x = \log_{10} 0.01$ (f) $x = \log_{\frac{1}{3}} 27$ (h) $x = \log_{\frac{5}{2}} \left(\frac{4}{25}\right)$
4. First rewrite each equation in index form and then solve it:
- (a) $\log_x 14 = 1$ (c) $\log_x 36 = 2$ (e) $\log_x 25 = -2$ (g) $\log_x 27 = \frac{3}{2}$
 (b) $\log_x 64 = 3$ (d) $\log_x 1000 = 3$ (f) $\log_x \frac{1}{8} = -3$ (h) $\log_x 7 = \frac{1}{2}$
5. Use the identities $\log_b b^x = x$ and $b^{\log_b x} = x$ to simplify:
- (a) $\log_2 2^3$ (c) $\log_7 7^{-2\frac{1}{2}}$ (e) $\theta^{\log_\theta 3.5}$ (g) $t^{\log_t 2\pi}$
 (b) $3^{\log_3 5}$ (d) $2.71^{\log_{2.71} 10}$ (f) $\log_n n^{-2}$ (h) $\log_{2c}(2c)^y$
6. Evaluate without the use of a calculator:
- (a) $\log_2 32 - \log_2 128$ (c) $\log_{10} \sqrt{10} + \log_{10} \sqrt[3]{10}$
 (b) $\log_3 \frac{1}{9} - \log_3 \frac{1}{3} + \log_3 1 - \log_3 3$ (d) $\log_5 125 - \log_5 \frac{1}{25} - \log_5 \sqrt{5}$
7. Simplify:
- (a) $\log_6 3 + \log_6 2$ (c) $\log_2 2 - \log_2 3 + \log_2 6$,
 (b) $\log_5 100 - \log_5 4$ (d) $\log_3 54 - \log_3 10 + \log_3 5$.
8. Solve each equation by converting both sides to a common base:
- (a) $2^x = 2^{2x+3}$ (d) $25^{\frac{x}{3}} = 5^{x+1}$ (g) $4^{2x+3} = 8^{x+5}$
 (b) $2^{2x} = 16^{x-8}$ (e) $3 \times 2^x = 48$ (h) $9^{4x-3} = 3^{x+7}$
 (c) $3^{\frac{x}{2}+1} = 1$ (f) $\frac{6^x}{12} = 18$ (i) $5^{3x-4} = 25^{x-2}$

9. Use the change of base formula and a calculator to evaluate to three significant figures:

- (a) $\log_2 3$ (c) $\log_5 16$ (e) $\log_\pi \left(\frac{22}{7}\right)$
 (b) $\log_{\sqrt{2}} 3$ (d) $\log_3 8$ (f) $\log_{\frac{2}{3}} 5$

DEVELOPMENT

10. Solve each pair of simultaneous equations by converting both sides to a common base:

- (a) $2^{2x-y} = 32$ (b) $5^{x+y} = \frac{1}{5}$ (c) $3^{x+y} = 81$ (d) $7^{x+y} = 49$
 $2^{4x+y} = 128$ $5^{3x+2y} = 1$ $81^{x-y} = 3$ $49^{x-y} = 7$

11. Use the formula to change the base of each logarithm to the base in the brackets, then evaluate:

- (a) $\log_{36} 216$, (6) (c) $\log_8 32$, (2) (e) $\log_{\frac{1}{4}} 32$, (2) (g) $\log_{\sqrt{2}} 4$, (2)
 (b) $\log_{125} 25$, (5) (d) $\log_{27} 81$, (3) (f) $\log_{\frac{1}{9}} 27$, (3) (h) $\log_{\sqrt[3]{3}} \left(3^{\frac{2}{5}}\right)$, (3)

12. Prove the first three logarithm results of Box 3 by putting $x = b^u$ and $y = b^v$.

13. Prove the last four logarithm results of Box 3 by rewriting them in index form.

14. (a) Use the change of base result to show that $\log_a b = \frac{1}{\log_b a}$.

(b) Hence evaluate, without the use of a calculator: (i) $\log_8 2$ (ii) $\log_{\sqrt{125}} 5$

(c) Using the change of base formula, prove:

$$(i) \log_a b \times \log_b a = 1 \quad (ii) \log_p q \times \log_q r \times \log_r p = 1$$

15. (a) Use the change of base result to show that $\log_{a^x} b = \frac{\log_a b}{x}$.

(b) Hence evaluate, without the use of a calculator: (i) $\log_8 128$ (ii) $\log_{\sqrt{27}} 81$

(c) Solve for x the equation $\log_{\sqrt{a}}(x+2) - \log_{\sqrt{a}} 2 = \log_a x + \log_a 2$.

16. (a) Use the change of base result to show that $\log_{a^x}(b^x) = \log_a b$.

(b) Hence simplify: (i) $\log_{16} 81$ (ii) $\log_{\sqrt{3}} \sqrt{2}$ (iii) $\log_{\sqrt{27}} \sqrt{125}$

EXTENSION

17. (a) Show that $\log_{ab} x = \frac{\log_a x}{1 + \log_a b}$. (b) Hence show that $\log_2 5 = \frac{1 - \log_{10} 2}{\log_{10} 2}$.

(c) Use this result and a calculator to evaluate $\log_2 5$ to four significant figures.

18. Solve for x : (a) $\log_{2x} 216 = x$ (b) $\log_{5x} 3375 = x$ [HINT: Rewrite each as an equation in index form and then consider the prime factorisation of 216 and of 3375.]

12B The Logarithmic Function and its Derivative

The principal purpose of this section is to prove that the derivative of the logarithmic function $y = \log_e x$ is the reciprocal function $y = \frac{1}{x}$, where the base e is an irrational number with approximate value $e \doteq 2.7183$.

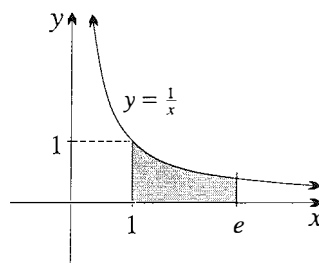
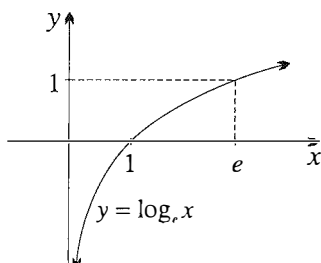
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THE DERIVATIVE OF THE LOGARITHMIC FUNCTION: $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

The number e will be defined by a definite integral as follows:

6 THE DEFINITION OF e : $\int_1^e \frac{1}{x} dx = 1$ (and $e \doteq 2.7183$)

Graphed below on the left is the logarithmic function $y = \log_e x$. On the right is the reciprocal function $y = 1/x$, with the definite integral used to define e shaded.



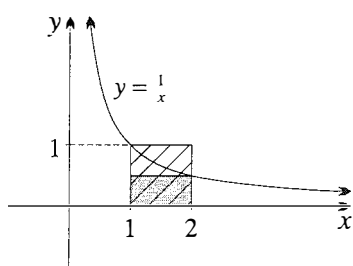
The six-step explanation that follows is rather indirect in that there is no direct attempt to differentiate any logarithmic function $y = \log x$, but instead the fundamental theorem of calculus is used to investigate the integral of the reciprocal function $y = 1/x$.

Step 1 — What is the Primitive of $1/x$: The reciprocal function $y = 1/x$ is obviously an important function which is required whenever two quantities are inversely proportional to each other. If we were to use the rule for integrating powers of x , however, we would get nonsense:

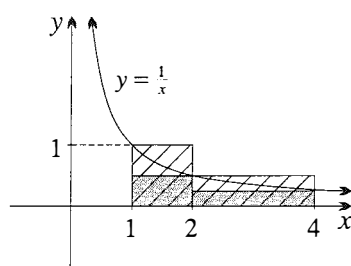
$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad \text{with } n = -1 \text{ gives } \int x^{-1} dx = \frac{x^0}{0},$$

which is undefined because of the division by zero. Yet definite integrals involving the reciprocal function are clearly well defined, provided that the integral does not cross the discontinuity at $x = 0$.

For example, here are diagrams of $\int_1^2 \frac{1}{x} dx$ and $\int_1^4 \frac{1}{x} dx$. Some upper and lower rectangles have been drawn to establish rough bounds for these integrals.



$$\frac{1}{2} < \int_1^2 \frac{1}{x} dx < 1$$



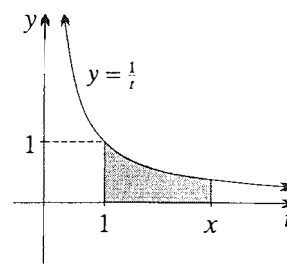
$$1 < \int_1^4 \frac{1}{x} dx < 2$$

So definite integrals of $y = 1/x$ clearly exist, but we don't yet know how to find their values. A useful procedure when this sort of thing happens is to give a name to the object we want to study, and then examine some of its properties.

Step 2 — A Function Defined by a Definite Integral: Define the function $L(x)$, for all $x > 0$, by the formula

$$L(x) = \int_1^x \frac{1}{t} dt.$$

Notice that x is the variable in the function $L(x)$, and so has to be used as a bound of the definite integral. Consequently, as was done in the proof of the fundamental theorem of calculus in Chapter 11, the variable in the integrand has been changed to t (the variable is called a *dummy variable*, because it disappears before any final result). Notice also that the variable x must be positive, because it is not possible to integrate across the asymptote at $x = 0$.



Step 3 — Three Properties of the Function $L(x)$: Three properties follow quickly from the definition of $L(x)$. These will allow us to make a preliminary sketch of $L(x)$.

FIRST PROPERTY: Since the curve $y = \frac{1}{t}$ is always above the x -axis for $t > 0$,

$$\begin{cases} L(x) > 0, & \text{for } x > 1, \\ L(x) = 0, & \text{for } x = 1, \\ L(x) < 0, & \text{for } 0 < x < 1. \end{cases}$$

SECOND PROPERTY: The fundamental theorem of calculus says that for any function $f(x)$, $\frac{d}{dx} \int_a^x f(t) dt = f(x)$. Applying this to the function $L(x)$ gives the result

$$L'(x) = \frac{1}{x}, \text{ for all } x > 0.$$

THIRD PROPERTY: From the two diagrams in Step 1,

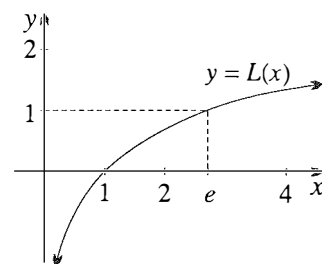
$$\frac{1}{2} < L(2) < 1 \quad \text{and} \quad 1 < L(4) < 2.$$

A PRELIMINARY SKETCH OF $L(x)$: On the right the function $y = L(x)$ is sketched using the information we have gained so far.

First, the curve is above the x -axis for $x > 0$, and below the x -axis for $0 < x < 1$.

Secondly, the gradient of $y = L(x)$ is $\frac{dy}{dx} = \frac{1}{x}$, which means that the gradient is always positive, but becomes ever flatter as $x \rightarrow \infty$, and ever steeper as $x \rightarrow 0^+$.

Thirdly, $L(x)$ reaches the value 1 between $x = 2$ and $x = 4$. Accordingly, we shall define a new number e by $L(e) = 1$.



Step 4 — The Definition of e : Already it is clear that the graph of $y = L(x)$ looks very like a logarithmic function. The next step is to establish its base, and the key to finding the base is the fact that with any logarithmic function, the log of the base is exactly 1:

$$\log_b b = 1, \text{ for all positive bases } b \neq 1,$$

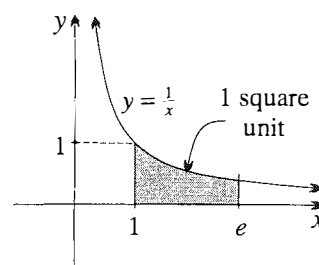
so if $L(x)$ is going to be a log function, its base has to be the number e such that $L(e) = 1$, as is indicated on the previous graph.

DEFINITION: Define e to be the positive real number such that

$$\int_1^e \frac{1}{x} dx = 1.$$

This definition works because $L(x)$ reaches the value 1 somewhere between 2 and 4. Also, $L(x)$ is always increasing, and hence it cannot take the value 1 at more than one place.

Hence e is a well-defined real number between 2 and 4. Some later, more sophisticated calculations will justify the approximation given by the calculator, $e \doteq 2.718\,281\,28$. It is also possible to prove that e is an irrational number (see for example question 7(b) of the 1993 HSC 4 Unit paper).



Step 5 — A Characteristic Property of Logarithmic Functions: This step and the next are difficult, but they establish that $L(x)$ is indeed the function $y = \log_e x$. First, we show that $L(x)$ satisfies a characteristic property of logarithmic functions.

FOURTH PROPERTY: $L(x^a) = aL(x)$, for all real a and for all $x > 0$.

PROOF: In part A we prove that LHS and RHS have the same derivative and so must differ by a constant. In part B we prove that this constant is zero, so that LHS and RHS must be equal.

$$\begin{aligned} \text{A. } \frac{d}{dx}(\text{RHS}) &= aL'(x) \\ &= a \times \frac{1}{x}, \text{ by the second property.} \end{aligned}$$

Using the chain rule:

$$\begin{aligned} \text{LHS} &= L(x^a) \\ \frac{d}{dx}(\text{LHS}) &= \frac{1}{x^a} \times ax^{a-1} \\ &= a \times \frac{1}{x}. \end{aligned}$$

$$\begin{array}{l|l} \text{Let} & u = x^a, \\ \text{then} & \text{LHS} = L(u). \\ \text{So} & \frac{du}{dx} = ax^{a-1} \\ \text{and} & \frac{d}{du}(\text{LHS}) = \frac{1}{u}. \end{array}$$

B. Because RHS and LHS have the same derivative,

$$L(x^a) = aL(x) + C, \text{ for some constant } C.$$

Substituting $x = 1$, $L(1) = aL(1) + C$,

and since we already know that $L(1) = 0$, it follows that $C = 0$, as required.

Step 6 — $L(x)$ is the Logarithmic Function with Base e : We can now prove the main theorem of this section. The proof is short, but it relies on the fourth property above. It also relies on log and exponential functions being inverse functions, and in particular uses the crucial idea that every positive number x can be expressed as a power of e :

$$x = e^{\log_e x}, \text{ for all } x > 0.$$

$$\text{THEOREM: } \int_1^x \frac{1}{t} dt = \log_e x$$

PROOF: We have to prove that $L(x) = \log_e x$.

$$\begin{aligned} L(x) &= L(e^{\log_e x}), \text{ using the remark above,} \\ &= (\log_e x)L(e), \text{ using the fourth property above,} \\ &= \log_e x, \text{ since } L(e) = 1 \text{ by the definition of } e. \end{aligned}$$

COROLLARY: Since $L'(x) = \frac{1}{x}$, it follows now that $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.

NOTE: Careful readers may see difficulties with this presentation, and particularly with the final proof, in that powers have only really been properly defined for rational indices rather than for real indices, and therefore a serious question remains about whether it is possible to take logarithms of arbitrary real numbers.

A standard way around this problem in more advanced treatments is to *define* the logarithmic function to be the function $L(x)$, and then prove the various log laws with the techniques used to prove the fourth property. Powers of e can then be defined as the inverse function of the logarithmic function, and one then has to prove that for rational indices this definition of powers agrees with the earlier definition in terms of roots. In this way the existence, the continuity and the differentiability of the logarithmic function are all placed beyond question.

The Logarithmic Function: The function $y = \log_e x$ is called *the logarithmic function*, in contrast with logarithmic functions with other bases like 2, 3 or 10. As far as calculus is concerned, it is the basic log function, and is often written simply as $\log x$, so that if no base is given, base e will from now on be implied. It is also written as $\ln x$, the 'n' standing for 'natural' logarithms, or for 'Napierian' logarithms in honour of the Scottish mathematician John Napier (1550–1617), who invented tables of logarithms base e for calculations (first published in 1614). The graph of $y = \log x$ was sketched at the start of this section — it should be regarded as one of most important curves in the course.

Be careful of the different convention used on calculators, where $\log x$ stands for $\log_{10} x$.

7

NOTATION: $\log_e x$, $\log x$ and $\ln x$ all mean the same thing, except on calculators, where $\log x$ usually means $\log_{10} x$.

Differentiating Functions Involving the Logarithmic Function: The basic standard form is $\frac{d}{dx}(\log x) = \frac{1}{x}$. The following examples use this in combination with the logarithmic laws and the chain, product and quotient rules.

WORKED EXERCISE: Differentiate: (a) $3 \log x$ (b) $\log 7x^2$ (c) $\log(ax + b)$

SOLUTION:

(a) Since $\frac{d}{dx}(\log x) = \frac{1}{x}$, it follows that $\frac{d}{dx}(3 \log x) = \frac{3}{x}$.

(b) Let $y = \log 7x^2$.

Using the log laws, $y = \log 7 + 2 \log x$,

so $\frac{dy}{dx} = \frac{2}{x}$ (notice that $\log 7$ is a constant).

(c) For $y = \log(ax + b)$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}, \text{ by the chain rule,} \\ &= \frac{a}{ax + b}. \end{aligned}$$

$$\begin{array}{l} \text{Let } u = ax + b, \\ \text{then } y = \log u. \\ \text{So } \frac{du}{dx} = a \\ \text{and } \frac{dy}{du} = \frac{1}{u}. \end{array}$$

WORKED EXERCISE: Using the chain and product rules, differentiate:

(a) $\log \log x$

(b) $x^3 \log x$

SOLUTION:

(a) Let $y = \log \log x$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du}, \text{ by the chain rule,} \\ &= \frac{1}{x} \times \frac{1}{\log x} \\ &= \frac{1}{x \log x}. \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \log x, \\ \text{then } y &= \log u. \\ \text{So } \frac{du}{dx} &= \frac{1}{x} \\ \text{and } \frac{dy}{du} &= \frac{1}{u}. \end{aligned}$$

(b) Let $y = x^3 \log x$. Then, by the product rule with $u = x^3$ and $v = \log x$,

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \log x + x^3 \times \frac{1}{x} \\ &= x^2(1 + 3 \log x). \end{aligned}$$

Standard Forms for Differentiation: It is convenient to write down two further standard forms for differentiation. The second form below was proven in part (c) of the first worked exercise above. The third is the general chain rule extension.

STANDARD FORMS FOR DIFFERENTIATION:

8

A. $\frac{d}{dx} \log_e x = \frac{1}{x}$

B. $\frac{d}{dx} \log_e(ax + b) = \frac{a}{ax + b}$

C. $\frac{d}{dx} \log_e u = \frac{1}{u} \times \frac{du}{dx}$ OR $\frac{d}{dx} \log_e f(x) = \frac{f'(x)}{f(x)}$

WORKED EXERCISE: Differentiate: (a) $\log(4x - 9)$ (b) $\log(4 + x^2)$

SOLUTION:

(a) $\frac{d}{dx} \log(4x - 9) = \frac{4}{4x - 9}$ (second standard form with $ax + b = 4x - 9$).

(b) $\frac{d}{dx} \log(4 + x^2) = \frac{2x}{4 + x^2}$ (third standard form with $u = 4 + x^2$).

Using the Log Laws to Make Differentiation Easier: The following example shows the use of the log laws to avoid a combination of the chain and quotient rules.

WORKED EXERCISE: Differentiate $\log \frac{(1+x)^2}{(1-x)^2}$.

SOLUTION: Let $y = \log \frac{(1+x)^2}{(1-x)^2}$.

Then $y = \log(1+x)^2 - \log(1-x)^2$
 $= 2\log(1+x) - 2\log(1-x),$

so $\frac{dy}{dx} = \frac{2}{1+x} + \frac{2}{1-x}$
 $= \frac{4}{1-x^2}.$

Logarithmic Functions to Other Bases: All other logarithmic functions can be expressed in terms of *the logarithmic function* by the change of base formula, for example,

$$\log_2 x = \frac{\log x}{\log 2},$$

and so every other logarithmic function is just a constant multiple of $\log x$. This allows any other logarithmic function to be differentiated easily.

9

DIFFERENTIATING A LOGARITHMIC FUNCTIONS WITH ANOTHER BASE:

Use the change of base formula to write it as a multiple of $\log x$.

WORKED EXERCISE: Find the derivative of $y = \log_b x$.

SOLUTION: Let $y = \log_b x$.

Then $y = \frac{\log_e x}{\log_e b}$

so $\frac{dy}{dx} = \frac{1}{x \log_e b}$.

A Characterisation of the Logarithmic Function: Since the derivative of $f(x) = \log x$ is $f'(x) = 1/x$, substitution of $x = 1$ shows that the tangent at the x -intercept has gradient exactly 1. This property characterises the logarithmic function amongst all other logarithmic functions.

10

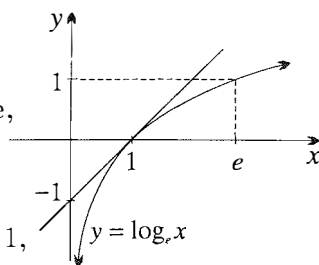
THE GRADIENT AT THE x -INTERCEPT: The function $y = \log x$ is the only logarithmic function whose gradient at the x -intercept is exactly 1.

PROOF: Let $f(x) = \log_b x$ be any other logarithmic function, then $f(1) = 0$, and so the x -intercept is at $x = 1$.

Also $f'(x) = \frac{1}{x \log_e b}$, by the previous worked exercise,

and so $f'(1) = \frac{1}{\log_e b}$.

Hence the gradient at the x -intercept is 1 if and only if $\log_e b = 1$, that is, if and only if the base b is equal to e .



Extension — The Log Laws and Implicit Differentiation: The log laws and implicit differentiation can be combined to differentiate complicated algebraic functions.

WORKED EXERCISE: Use implicit differentiation to differentiate $y = \sqrt{\frac{x-1}{x+1}}$.

SOLUTION: Taking logs of both sides, $\log y = \frac{1}{2} \log(x-1) - \frac{1}{2} \log(x+1)$.

Differentiating with respect to x , $\frac{1}{y} \frac{dy}{dx} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

$$= \frac{1}{(x-1)(x+1)}.$$

$$\frac{dy}{dx} = \frac{1}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}.$$

$\times y$

Exercise 12B

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\log x$ means $\log_{10} x$).

1. Differentiate, using the log laws:

(a) $\log 3x$	(c) $2 \log x$	(e) $\log x + \pi$	(g) $3 \log 5x$
(b) $\log_e 7x$	(d) $x + 4 \log x$	(f) $\ln \frac{x}{2}$	(h) $4x^3 - \ln \frac{4}{3}x$

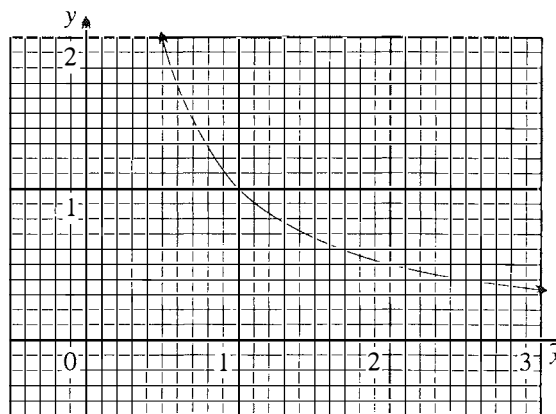
2. Differentiate, using the standard form $\frac{d}{dx} \log_e(ax + b) = \frac{a}{ax + b}$:

(a) $\log(2x + 5)$	(d) $\log(4 - x)$	(g) $x - \ln(1 - x)$	(j) $\ln(\pi x + 1)$
(b) $\ln(3x - 7)$	(e) $\log_e(4 + 7x)$	(h) $\log_e(-ex + 3e)$	(k) $\log(1 - \frac{\pi}{2}x)$
(c) $\log(3 + 2x)$	(f) $\log(2 - 5x)$	(i) $\log(ax - b)$	(l) $\log_e(a - \frac{2x}{b})$

3. In Step 4 of the development of the logarithmic function, e was defined by

$$\int_1^e \frac{1}{x} dx = 1.$$

This question uses this definition to estimate e from a graph of $y = 1/x$. The diagram to the right shows the graph of $y = 1/x$ from $x = 0$ to $x = 3$, drawn with a scale of 10 little divisions to 1 unit, so that 100 little squares make 1 square unit. Count the squares in the column from $x = 1.0$ to 1.1 , then the squares in the column from $x = 1.1$ to 1.2 , and so on. Continue until the number of squares equals 100 — the x -value at this point will be an estimate of e .



4. The notes established the further result that $\log x = \int_1^x \frac{1}{t} dt$. By counting squares, find estimates of $\log x$ for the values of x in the table below, then sketch the graph of $y = \log x$. [HINT: For values of x less than 1, the integral runs backwards, and so will be negative.]

x	0.5	0.6	0.8	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3

5. Simplify these expressions involving logarithms to the base e :

(a) $e \log e$	(d) $\ln \sqrt{e}$	(g) $\log e^e$
(b) $\frac{1}{e} \ln \frac{1}{e}$	(e) $e \log e^3 - e \log e$	(h) $\log(\log e^e)$
(c) $3 \log_e e^2$	(f) $\log_e e + \log_e \frac{1}{e}$	(i) $\log(\log(\log e^e))$

6. Solve: (a) $\ln(x^2 + 5x) = 2 \ln(x + 1)$ (b) $\log(7x - 12) = 2 \log x$.

7. Use the chain rule to differentiate:

(a) $\log(x^2 + 1)$	(b) $\log(x^2 + 3x + 2)$	(c) $\ln(2 - x^2)$	(d) $\log_e(1 + 2\sqrt{x})$

8. Use the logarithm laws to help differentiate:

(a) $\log 7x^2$	(c) $\log \sqrt{x}$	(e) $\log_e \sqrt{2 + x}$	(g) $\log \sqrt[3]{x + 1}$
(b) $\log 5x^3$	(d) $\log \frac{3}{x}$	(f) $\ln \left(\frac{1 + x}{1 - x} \right)$	(h) $\log(x\sqrt{x + 1})$

9. Use the change of base formula to express these to base e , then differentiate them:
 (a) $\log_2 x$ (b) $\log_{10} x$ (c) $\log_2 5x$ (d) $5 \log_3 7x$
10. Differentiate these functions using the product rule:
 (a) $x \log x$ (b) $x \log(2x + 1)$ (c) $(2x + 1) \log x$ (d) $\sqrt{x} \log x$
11. Find the equation of the tangent to $y = \log x$ at the point where $x = e^2$.
12. Find the equation of the normal to $y = \log x$ at the point where $x = \frac{1}{e}$. What is its x -intercept?

DEVELOPMENT

13. Use the logarithm laws to simplify the following, then differentiate them:
 (a) $\ln \left(\frac{\sqrt{x-1}}{x^2+1} \right)$ (b) $\log \left((x^2 - 2x)\sqrt{x} \right)$ (c) $\log_e \pi^x$
14. Differentiate the following, using the chain, product and quotient rules and the logarithm laws:
 (a) $x^2 \log x$ (d) $(\log x)^4$ (g) $(2 \log x - 3)^4$ (j) $\log_x 3$
 (b) $\frac{\log x}{x}$ (e) $\frac{1}{1 + \log x}$ (h) $\frac{1}{\log x}$ (k) $\frac{x}{\log x}$
 (c) $(\log x)^2$ (f) $\sqrt{\log x}$ (i) $\log(\log x)$ (l) $\log_x 3^x$
15. Find the point(s) where the tangent to each of these curves is horizontal:
 (a) $y = x \log x$ (b) $y = \frac{1}{x} + \log x$ (c) $y = x^2 \log \frac{1}{x}$
16. (a) Show that the gradient of $y = \log x$ at $x = 1$ is equal to 1.
 (b) Find the value of the derivative of $y = \log x$ at $x = 1$ by first principles, using the formula $f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$, and hence show that $\lim_{x \rightarrow 1} \frac{\log x}{x - 1} = 1$.
17. For what value of x does the tangent to $y = \log_{10} x$ have gradient 1?
18. Differentiate the following using any appropriate technique. Use the logarithm laws whenever possible.
 (a) $\log(2x^2 - 3x)$ (c) $\log(1 + \log x)$ (e) $(x^2 + 2x) \log \sqrt{x-2}$
 (b) $\log \left(\frac{x-1-2x^2}{5} \right)$ (d) $\log(x^{\frac{1}{2}} + \log x)$ (f) $\log \left(\frac{(x-3)^4 \sqrt{x}}{x+1} \right)$
19. (a) Show that $y = \frac{x}{\log x}$ is a solution of the equation $\frac{dy}{dx} = \left(\frac{y}{x} \right) - \left(\frac{y}{x} \right)^2$.
 (b) Show that $y = \log(\log x)$ is a solution of the equation $x \frac{d^2 y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 + \frac{dy}{dx} = 0$.

EXTENSION

20. Use logarithms to help find the derivatives of:
 (a) $y = \frac{(x+1)\sqrt{x-1}}{x+2}$ (c) $y = \frac{x^2 \sqrt{x+1}}{\sqrt{x-1}}$ (e) $y = \left(\frac{x^2+1}{x} \right)^{\frac{1}{\pi}}$
 (b) $y = \frac{(x-1)^3(x+2)^2}{(x-3)^4}$ (d) $y = \frac{\sqrt{x}(x-1)^2}{x+1}$ (f) $y = \sqrt{x} \sqrt{x+1} \sqrt{x+2}$
21. Take logarithms of both sides and use the log laws to differentiate:
 (a) $y = x^x$ (b) $y = x^{\log x}$ (c) $y = x^{\frac{1}{x}}$

22. It can be shown (with some considerable difficulty) that the continued fraction on the right approaches the value $e-1$. With the help of a calculator, use this continued fraction to find a rational approximation for e that is accurate to four significant figures.

$$1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \dots}}}}}}}}$$

23. (a) If $y = \log x$, use differentiation by first principles to show that $y' = \lim_{h \rightarrow 0} \log \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$.
- (b) Use the fact that $y' = \frac{1}{x}$ to show that $\lim_{h \rightarrow 0} \log \left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \frac{1}{x}$.
- (c) Substitute $n = \frac{1}{h}$ and $u = \frac{1}{x}$ to prove these two important limits:
- (i) $\lim_{n \rightarrow \infty} \left(1 + \frac{u}{n}\right)^n = e^u$ (ii) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$
- (d) Investigate how quickly $\left(1 + \frac{1}{n}\right)^n$ converges to e by using your calculator with the following values of n : (i) 1 (ii) 10 (iii) 100 (iv) 1000 (v) 10 000

12C Applications of Differentiation

Differentiation can now be used in the usual way to study tangents, turning points and inflexions of functions involving the logarithmic function. Systematic sketching of such curves, however, will need the special limits developed after the first worked example.

The Geometry of Tangents and Normals: The following example illustrates how to investigate the geometry of tangents and normals to a curve.

WORKED EXERCISE: (a) Find the point $T(a, \log a)$ on $y = \log x$ where the tangent passes through the origin. (b) Find the area of the triangle formed by the y -axis and the tangent and the normal at T .

SOLUTION:

(a) Differentiating, $\frac{dy}{dx} = \frac{1}{x}$,

so the tangent at $T(a, \log a)$ has gradient $\frac{1}{a}$,

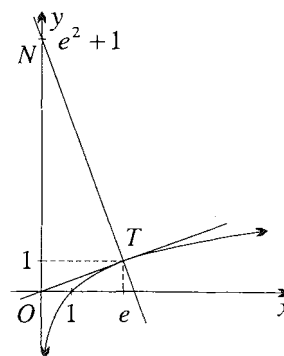
and the tangent is $y - \log a = \frac{1}{a}(x - a)$.

The tangent passes through $(0, 0)$ if and only if

$$\begin{aligned} -\log a &= \frac{1}{a} \times (-a), \\ a &= e, \end{aligned}$$

so T is the point $(e, 1)$.

- (b) The normal at T has gradient $-e$, so its equation is $y - 1 = -e(x - e)$, and its y -intercept N is therefore $(0, e^2 + 1)$. So the base ON of $\triangle ONT$ is $(e^2 + 1)$, and its altitude is e . Hence the triangle has area $\frac{1}{2}e(e^2 + 1)$ square units.



Two Special Limits — x Dominates $\log x$: Curve sketching involving $\log x$ requires knowledge of some special limits that arise from clashes between x and $\log x$. For example, we need to know what happens to $x \log x$ as $x \rightarrow 0^+$. On the one hand, x is approaching zero, but on the other hand, $\log x$ is diverging to $-\infty$, so what does their product do?

The answer is that x *dominates* $\log x$. To put it more colourfully, ‘in a battle between x and $\log x$, x always wins’.

11 THE FUNCTION x DOMINATES THE FUNCTION $\log x$:

$$\lim_{x \rightarrow \infty} \frac{\log x}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} x \log x = 0$$

PROOF: These proofs are not easy. The first limit is proven below by an interesting geometric method, then the second limit follows using a substitution.

A. Consider the definite integral $\int_1^{\sqrt{x}} \frac{1}{t} dt$ sketched below, where $x > 1$.

It is clear from the diagram that

$$0 < \int_1^{\sqrt{x}} \frac{1}{t} dt < \text{area } ABCO,$$

$$0 < [\log t]_1^{\sqrt{x}} < \sqrt{x}$$

$$0 < \log \sqrt{x} - \log 1 < \sqrt{x}$$

$$0 < \frac{1}{2} \log x < \sqrt{x}$$

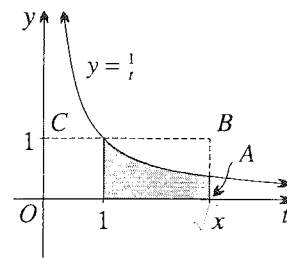
$$\left[\times \frac{2}{x} \right] 0 < \frac{\log x}{x} < \frac{2}{\sqrt{x}}.$$

But $\frac{2}{\sqrt{x}} \rightarrow 0$ as $x \rightarrow \infty$, so by the sandwiching principle, $\lim_{x \rightarrow \infty} \frac{\log x}{x} = 0$.

B. Substitute $u = \frac{1}{x}$ into the limit proven in part (a).

Since $\log u = -\log x$, and $u \rightarrow 0^+$ as $x \rightarrow \infty$, $\lim_{u \rightarrow 0^+} (-u \log u) = 0$.

Then replacing u by x , and taking opposites, $\lim_{x \rightarrow 0^+} x \log x = 0$.



Two More General Limits: A more general version of this result is that x^k dominates $\log x$ for all $k > 0$ (proven in the Extension of the following Exercise 12C).

12 THE FUNCTION x^k DOMINATES THE FUNCTION $\log x$ FOR $k > 0$:

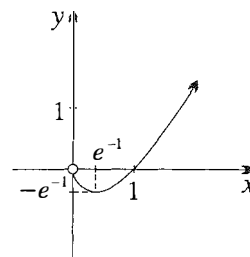
$$\lim_{x \rightarrow \infty} \frac{\log x}{x^k} = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} x^k \log x = 0.$$

An Example of Curve Sketching: Here are the six steps of the curve sketching menu applied to $y = x \log x$. Notice in Step 5 the use of the derivative not only to find the turning point, but also to analyse the gradient of the curve near the boundary of the domain.

1. The domain is $x > 0$, because $\log x$ is undefined for $x \leq 0$.
2. The domain is unsymmetric, so the function is neither even nor odd.

3. The only zero is at $x = 1$, and the curve is continuous for $x > 0$:

x	0	$1/e$	1	e
y	*	$-1/e$	0	e
sign	*	-	0	+



4. Since x dominates $\log x$, $y \rightarrow 0$ as $x \rightarrow 0^+$.

Also $y \rightarrow \infty$ as $x \rightarrow \infty$.

5. Differentiating by the product rule, $f'(x) = \log x + 1$,

$$f''(x) = \frac{1}{x},$$

so $f'(x) = 0$ when $x = 1/e$, and $f''(1/e) = e > 0$,

hence $(1/e, -1/e)$ is a minimum turning point.

Also, $f'(x) \rightarrow -\infty$ as $x \rightarrow 0^+$, so the curve becomes vertical near the origin.

6. Since $f''(x)$ is always positive, there are no inflexions, and the curve is always concave up.

Exercise 12C

- Use your knowledge of transformations to help sketch the graphs of the given functions:
 - $y = \log(x + 1)$
 - $y = \log(-x)$
 - $y = -\log 3x$
 - $y = \log(x - 2)$
 - $y = \log\left(\frac{x}{2}\right)$
 - $y = \log|x|$
- Write down the domain of $y = \log(1 + x^2)$.
 - Show that $y' = \frac{2x}{1 + x^2}$ and $y'' = \frac{2(1 - x^2)}{(1 + x^2)^2}$.
 - Hence show that $y = \log(1 + x^2)$ has one stationary point, and determine its nature.
 - Find the coordinates of the two points of inflexion.
 - Hence sketch the curve, and then write down its range.
 - Hence sketch $y = \log\left(\frac{1 + x^2}{2}\right)$. [HINT: You will need to use the logarithm laws.]
- Find the domain of $y = (\log x)^2$.
 - Find y' and y'' , and hence show that the curve has as an inflexion at $x = e$.
 - Classify the stationary point at $x = 1$, sketch the curve, and write down the range.
- Determine the first two derivatives of $y = x - \log x$.
 - Deduce that the curve is concave up for all values of x in its natural domain.
 - Find the minimum turning point. (d) Sketch the curve and write down its range.
- Write down the domain of $y = \frac{1}{x} + \log x$. (b) Find the first and second derivatives.
 - Show that the curve has a minimum at $(1, 1)$ and an inflexion at $(2, \frac{1}{2} + \log 2)$.
 - Sketch the graph and write down its range.
- Write down the domain of $y = \frac{\log x}{x}$, then find any horizontal or vertical asymptotes.
 - Find y' and y'' . (c) Find any stationary points and determine their nature.
 - Find the exact coordinates of the lone point of inflexion.
 - Sketch the curve, and write down its range.

DEVELOPMENT

7. (a) Write down the equation of the tangent to $y = 2 \log x$ at the point where $x = c$. Hence find any values of c for which the tangent passes through the origin.
 (b) Repeat part (a) for these curves: (i) $y = (\log x)^2$ (ii) $y = x^2 \log x$
8. (a) Show that the equation of the tangent to $y = x \log x$ at the point $x = e$ is $y = 2x - e$.
 (b) Find the distance from this tangent to the origin.
9. (a) Find the equation of the tangent to $y = (\log x)^2$ at the point where $x = t$.
 (b) Find the area of the triangle cut off by this tangent and the coordinate axes when $t = e$.
10. Investigate the curve $y = -x \log x$ as follows, then sketch it and write down its range.
 (a) Find its domain and any intercepts.
 (b) Find and classify any stationary points.
 (c) Examine the behaviour of y and y' as $x \rightarrow 0^+$.
11. (a) Find and classify the lone stationary point of $y = x^2 \log x$ in its natural domain.
 (b) Show that there is an inflexion at $x = e^{-\frac{3}{2}}$.
 (c) Examine the behaviour of y and y' as $x \rightarrow 0^+$.
 (d) Hence sketch the graph of this function, then write down its range.
12. Carefully classify the critical points of $y = \frac{x}{\log x}$ and show that there is an inflexion at $(e^2, \frac{1}{2}e^2)$. Examine the behaviour of y and y' as $x \rightarrow 0^+$ and as $x \rightarrow \infty$, then sketch the curve and write down its range.
13. (a) Write down the domain of $y = \log \left(\frac{x^2}{x+1} \right)$.
 (b) Show that $y' = \frac{x+2}{x(x+1)}$.
 (c) Show that $y' = 0$ at $x = -2$. Explain why there is no stationary point there.
 (d) How many inflexion points does this curve have? (e) Sketch the graph.
14. (a) What is the natural domain of $y = \log(\log x)$? (b) What is the x -intercept?
 (c) Find y' and y'' , and explain why there are no stationary points.
 (d) Confirm that $y''(\frac{1}{e}) = 0$, and explain why there is no inflexion there.
 (e) Sketch the curve.
15. (a) Given $\log ax = \log x + C$, what is the value of C ?
 (b) Hence show that the gradient of $y = \log x$ is everywhere the same as $y = \log ax$ and explain this in terms of enlargements and translations.
 (c) Do likewise for $y = \log_b ax$, where a , b and x are all positive.
16. (a) Find the gradient of the tangent to $y = \log(1+x^2)$ at the point where $x = c$.
 (b) The tangent at another point (x, y) is perpendicular to the one found in part (a).
 Show that $x^2 + \frac{4cx}{1+c^2} + 1 = 0$.
 (c) Show that $\Delta = -4 \left(\frac{1-c^2}{1+c^2} \right)^2$ for this quadratic, and hence that the only tangents to $y = \log(1+x^2)$ which are mutually perpendicular are those at $x = -1$ and $x = 1$.

EXTENSION

17. Use the results $\lim_{u \rightarrow \infty} \frac{\log u}{u} = 0$ and $\lim_{u \rightarrow 0^+} u \log u = 0$, and the substitution $u = x^k$ where $k > 0$, to prove the two further limits in Box 12 of the notes above.
18. Show that $y = x^{\frac{1}{\log x}}$ is a constant function and find the value of this constant. What is the natural domain of this function? Sketch its graph.
19. (a) Differentiate $y = x^x$ by taking logs of both sides. Then examine the behaviour of $y = x^x$ near $x = 0$, and show that the curve becomes vertical as $x \rightarrow 0^+$.
 (b) Locate and classify any stationary points, and where the curve has gradient 1.
 (c) Sketch the function.
20. (a) Find the limits of $y = x^{\frac{1}{x}}$ as $x \rightarrow 0^+$ and as $x \rightarrow \infty$.
 (b) Show that there is a maximum turning point when $x = e$.
 (c) Show that $y = x^x$ and $y = x^{\frac{1}{x}}$ have a common tangent at $x = 1$.
 (d) Sketch the graph of the function.

12D Integration of the Reciprocal Function

Integration of the Reciprocal Function: Since $\log x$ has derivative $1/x$, it follows that $\log x$ is a primitive of $1/x$, provided that x remains positive so that $\log x$ is defined. This gives a new standard form for integration, with the following three versions (omitting constants of integration).

STANDARD FORMS FOR INTEGRATION:

- 13 A. $\int \frac{1}{x} dx = \log x$, provided that $x > 0$
- B. $\int \frac{1}{ax+b} dx = \frac{1}{a} \log(ax+b)$, provided that $ax+b > 0$
- C. $\int \frac{1}{u} \frac{du}{dx} dx = \log u$, provided that $u > 0$ OR $\int \frac{f'(x)}{f(x)} dx = \log f(x)$

WORKED EXERCISE: Evaluate: (a) $\int_e^{e^2} \frac{1}{x} dx$ (b) $\int_0^1 \frac{1}{2x+1} dx$ (c) $\int_0^{\frac{1}{2}} \frac{x}{1-x^2} dx$

SOLUTION:

$$\begin{aligned} \text{(a)} \quad \int_e^{e^2} \frac{1}{x} dx &= [\log x]_e^{e^2} \\ &= \log e^2 - \log e \\ &= 1 \end{aligned} \qquad \begin{aligned} \text{(b)} \quad \int_0^1 \frac{1}{2x+1} dx &= \frac{1}{2} [\log(2x+1)]_0^1 \\ &= \frac{1}{2} (\log 3 - \log 1) \\ &= \frac{1}{2} \log 3 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_0^{\frac{1}{2}} \frac{x}{1-x^2} dx &= -\frac{1}{2} \int_0^{\frac{1}{2}} \frac{-2x}{1-x^2} dx \\ &= -\frac{1}{2} [\log(1-x^2)]_0^{\frac{1}{2}} \\ &= -\frac{1}{2} (\log \frac{3}{4} - \log 1) \\ &= \log 2 - \frac{1}{2} \log 3 \end{aligned}$$

$$\begin{aligned} &\text{Let } u = 1 - x^2. \\ &\text{Then } \frac{du}{dx} = -2x. \\ &\int \frac{1}{u} \frac{du}{dx} dx = \log u \end{aligned}$$

WORKED EXERCISE: Find $f(x)$, if $f'(x) = \frac{3}{4x-1}$, and $f(1) = 0$.

SOLUTION: Integrating, $f(x) = \frac{3}{4} \log(4x-1) + C$, provided that $x > \frac{1}{4}$.

Substituting $x = 1$, $0 = \frac{3}{4} \log 3 + C$,

$$\text{so } f(x) = \frac{3}{4} \log(4x-1) - \frac{3}{4} \log 3$$

$$f(x) = \frac{3}{4} \log \frac{1}{3}(4x-1).$$

Given a Derivative, Find an Integral: Our theory so far has not yielded a primitive of $\log x$, but the following exercise shows how the primitive of $\log x$ can be obtained.

WORKED EXERCISE: Differentiate $x \log x$, and hence find:

(a) $\int_1^e \log x \, dx$

(b) $\int_1^2 \log_2 x \, dx$

SOLUTION: First, $\frac{d}{dx}(x \log x) = \log x + x \times \frac{1}{x}$, by the product rule,

that is, $\frac{d}{dx}(x \log x) = 1 + \log x$.

Reversing this, $\int (1 + \log x) \, dx = x \log x + C$, for some constant C ,

$$\int \log x \, dx = x \log x - x + C.$$

(a) Hence $\int_1^e \log x \, dx = [x \log x - x]_1^e$

$$= (e \log e - e) - (1 \log 1 - 1)$$

$$= (e - e) - (0 - 1)$$

$$= 1.$$

(b) Also, $\int_1^2 \log_2 x \, dx = \int_1^2 \frac{\log x}{\log 2} \, dx$

$$= \frac{1}{\log 2} [x \log x - x]_1^2$$

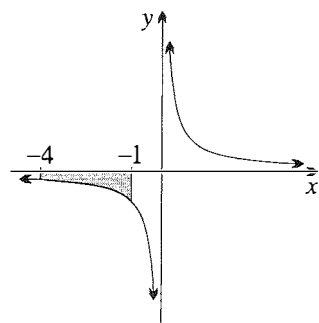
$$= \frac{1}{\log 2} (2 \log 2 - 2 - 1 \log 1 + 1)$$

$$= \frac{2 \log 2 - 1}{\log 2}.$$

Extension — The Primitive of $y = 1/x$ on Both Sides of the Origin: The graph of the function $y = 1/x$ is a hyperbola, with two disconnected branches separated by the discontinuity at $x = 0$.

Clearly we can take definite integrals of $1/x$ provided only that the interval of integration does not cross the asymptote at $x = 0$, and there is no reason why we should not integrate over an interval like $-4 \leq x \leq -1$ on the negative side of the origin. If x is negative, then $\log(-x)$ is well defined, and using the chain rule:

$$\frac{d}{dx} \log(-x) = -\frac{1}{-x} = \frac{1}{x}.$$



So $\log(-x)$ is a primitive of $1/x$ when x is negative. Combining these results, $\log|x|$ is a primitive of $1/x$ for all $x \neq 0$. We now have the following three versions of the more general standard form (omitting constants of integration).

FURTHER STANDARD FORMS FOR INTEGRATION (EXTENSION):

14

$$\text{A. } \int \frac{1}{x} dx = \log|x|$$

$$\text{B. } \int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b|$$

$$\text{C. } \int \frac{1}{u} \frac{du}{dx} dx = \log|u| \quad \text{OR} \quad \int \frac{f'(x)}{f(x)} dx = \log|f(x)|$$

NOTE: Careful readers will notice that because $y = 1/x$ has two disconnected branches, there can be different constants of integration in the two branches. So the general primitive of $1/x$ is

$$\int \frac{1}{x} dx = \begin{cases} \log x + A, & \text{for } x > 0, \\ \log(-x) + B, & \text{for } x < 0, \end{cases} \quad \text{where } A \text{ and } B \text{ are constants.}$$

If a boundary condition is given in one region, this has no implication at all for the constant of integration in the other region. In any physical interpretation, however, the function would normally have meaning in only one of the two branches.

Exercise 12D

1. Determine the following indefinite integrals:

(a) $\int \frac{1}{x} dx$	(e) $\int \frac{5}{3+2x} dx$	(i) $\int \frac{2}{2x-1} dx$	(m) $\int \frac{dx}{2-e^x}$
(b) $\int \frac{2}{x} dx$	(f) $\int \frac{dx}{4x-1}$	(j) $\int \frac{dx}{3-5x}$	(n) $\int \frac{\sqrt{2}}{3x-\pi} dx$
(c) $\int \frac{1}{3x} dx$	(g) $\int \frac{dx}{2x-1}$	(k) $\int \frac{2}{5-7x} dx$	(o) $\int \frac{dx}{b-ax}$
(d) $\int \frac{1}{5x+4} dx$	(h) $\int \frac{3}{2x+1} dx$	(l) $\int \frac{e}{\pi x+1} dx$	(p) $\int \frac{a}{b-cx} dx$

2. Evaluate the following definite integrals:

(a) $\int_1^e \frac{dx}{x}$	(d) $\int_3^9 \frac{1}{x} dx$	(g) $\int_1^3 \frac{dx}{3x-1}$
(b) $\int_{\sqrt{e}}^{e^2} \frac{dx}{x}$	(e) $\int_0^1 \frac{dx}{x+1}$	(h) $\int_1^2 \frac{3}{5-2x} dx$
(c) $\int_1^5 \frac{1}{x} dx$	(f) $\int_4^{18} \frac{dx}{x-2}$	(i) $\int_0^\pi \frac{1}{2x+\pi} dx$

3. Find primitives of the following by first writing them as separate fractions:

(a) $\frac{x+1}{x}$	(c) $\frac{3x^2-2x}{x^2}$
(b) $\frac{2-x}{3x}$	(d) $\frac{3x^3+4x-1}{x^2}$

4. Use the result $\int \frac{f'(x)}{f(x)} dx = \log f(x)$, or $\int \frac{1}{u} \frac{du}{dx} dx = \log u$, to integrate:
- (a) $\frac{2x}{x^2 - 9}$ (c) $\frac{2x + 1}{x^2 + x - 3}$ (e) $\frac{x + 3}{x^2 + 6x - 1}$
 (b) $\frac{6x + 1}{3x^2 + x}$ (d) $\frac{5 - 6x}{2 + 5x - 3x^2}$ (f) $\frac{3 - x}{12x - 3 - 2x^2}$
5. (a) Find y as a function of x , if $y' = \frac{1}{4x}$ and $y = 1$ when $x = e^2$. What is the x -intercept of this curve?
- (b) The gradient of a curve is given by $y' = \frac{2}{x + 1}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve?
- (c) Find $y(x)$, given that $y' = \frac{2x + 5}{x^2 + 5x + 4}$ and $y = 1$ when $x = 1$.
- (d) Given that the derivative of $f(x)$ is $\frac{x^2 + x + 1}{x}$ and $f(1) = 1\frac{1}{2}$, find $f(x)$.
- (e) Write down the equation of the family of curves with the property $y' = \frac{2 + x}{x}$. Hence find the curve that passes through $(1, 1)$ and evaluate y at $x = 2$ for this graph.

DEVELOPMENT

6. Use the result $\int \frac{f'(x)}{f(x)} dx = \log f(x)$, or $\int \frac{1}{u} \frac{du}{dx} dx = \log u$, to find:
- (a) $\int \frac{3x^2}{x^3 - 5} dx$ (c) $\int \frac{x^3 - 3x}{x^4 - 6x^2} dx$ (e) $\int \frac{\sqrt{x}}{x^{\frac{3}{2}} + 1} dx$
 (b) $\int \frac{4x^3 + 1}{x^4 + x - 5} dx$ (d) $\int \frac{10x^3 - 7x}{5x^4 - 7x^2 + 8} dx$ (f) $\int \frac{x^2 - \sqrt{x}}{x^3 - 2x^{\frac{3}{2}} + 1} dx$
7. Using the methods of the previous question, evaluate:
- (a) $\int_{-e}^{-2} \frac{1 - 3x^2}{x - x^3} dx$ (b) $\int_3^6 \frac{4x - 5}{2x^2 - 5x} dx$ (c) $\int_1^e \frac{2x + e}{x^2 + ex} dx$
8. (a) Differentiate $x \log x$ and hence find: (i) $\int \log x dx$ (ii) $\int_{\sqrt{e}}^e \log x dx$
- (b) Use the change of base formula and the integral in part (a) to evaluate $\int_1^{10} \log_{10} x dx$.
- (c) Differentiate $x^2 \log x$ and hence determine $\int_{\sqrt{e}}^e x \log x dx$.
- (d) Differentiate $\sqrt{x} \log x$ and hence determine the family of primitives of $\frac{\log x}{\sqrt{x}}$.
9. Find: (a) $\int_a^{a^n} \frac{1}{x} dx$ (b) $\int \frac{dx}{t(s + tx)}$ (c) $\int_0^1 \frac{dx}{b^2 x + b}$
10. (a) Given that we may write $\frac{x}{(x + 4)^2} = \frac{(x + 4) - 4}{(x + 4)^2}$, evaluate $\int_0^1 \frac{x}{(x + 4)^2} dx$.
- (b) Show that $\frac{1}{x^2 - 9} = \frac{1}{6} \left(\frac{1}{x - 3} - \frac{1}{x + 3} \right)$. Hence find $\int \frac{dx}{x^2 - 9}$.

11. Rewrite in the form $\int \frac{1}{u} \frac{du}{dx} dx = \log u$, or $\int \frac{f'(x)}{f(x)} dx = \log f(x)$, then evaluate:

(a) $\int_4^{16} \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$

(c) $\int_e^{e^e} \frac{1}{x \log(x^2)} dx$

(b) $\int_e^{e^2} \frac{dx}{x \log x}$

(d) $\int_1^{27} \frac{3}{(\sqrt[3]{x}+1)x^{\frac{2}{3}}} dx$

12. Given $y = \log(x + \sqrt{x^2 + 1})$, find y' and hence determine $\int \frac{dx}{\sqrt{x^2 + 1}}$ to within a constant.

————— EXTENSION —————

13. Determine a primitive of $\frac{1}{x + \sqrt{x}}$.

14. (a) Write down the derivatives of $\log(ax + b)$ and $\log(-ax - b)$. What do you notice? Use this to justify the result $\int \frac{a}{ax + b} dx = \log |ax + b|$, and then evaluate $\int_{-5}^{-1} \frac{1}{2x + 1} dx$.

(b) A certain curve has gradient $y' = \frac{1}{x}$ and its two branches pass through the two points $(-1, 2)$ and $(1, 1)$. Find the equation of the curve.

15. [A series converging to $\log(1 + x)$, and approximations to the log function]

(a) Use the formula for the partial sum of a GP to prove that for $t \neq -1$,

$$1 - t + t^2 - t^3 + \cdots + t^{2n} = \frac{1}{1 + t} + \frac{t^{2n+1}}{1 + t}.$$

(b) Integrate both sides of this result from $t = 0$ to $t = x$ to show that for $x > -1$,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t} dt.$$

(c) Explain why $\frac{t^{2n+1}}{1+t} \leq t^{2n+1}$, for $0 \leq t \leq 1$. Hence prove that for $0 \leq x \leq 1$, the integral $\int_0^x \frac{t^{2n+1}}{1+t} dt$ converges to 0 as $n \rightarrow \infty$. Hence show that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \text{ for } 0 \leq x \leq 1.$$

(d) (i) Use this series to approximate $\log \frac{3}{2}$ to two decimal places.

(ii) Write down the series converging to $\log 2$ — called the *alternating harmonic series*.

(e) With a little more effort, it can be shown that the series in part (c) converges to the given limit for $-1 < x \leq 1$ (the proof is a reasonable challenge). Use this to write down the series converging to $\log(1 - x)$ for $-1 \leq x < 1$, and hence approximate $\log \frac{1}{2}$ to two decimal places.

(f) Use both series to show that for $-1 < x < 1$,

$$\log \left(\frac{1+x}{1-x} \right) = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right).$$

Use this result and an appropriate value of x to find $\log 3$ to five significant figures.

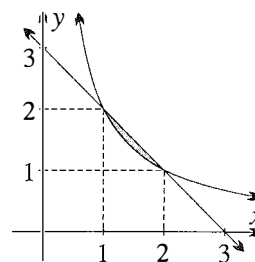
12E Applications of Integration

The usual methods of finding areas and volumes can now be applied to the reciprocal function, whose primitive was previously unavailable.

WORKED EXERCISE: Find the area contained between the hyperbola $xy = 2$ and the line $x + y = 3$.

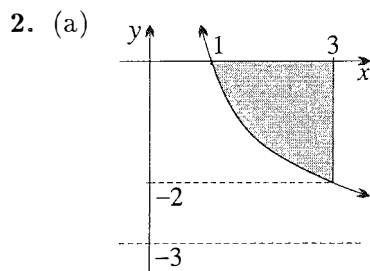
SOLUTION: Substitution shows that the curves meet at $A(1, 2)$ and $B(2, 1)$,

$$\begin{aligned} \text{so area} &= \int_1^2 \left((3 - x) - \frac{2}{x} \right) dx \\ &= \left[3x - \frac{1}{2}x^2 - 2\log x \right]_1^2 \\ &= (6 - 2 - 2\log 2) - (3 - \frac{1}{2} - 2\log 1) \\ &= 1\frac{1}{2} - 2\log 2 \text{ square units.} \end{aligned}$$

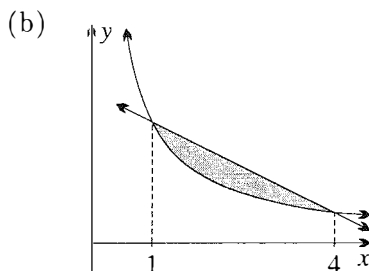


Exercise 12E

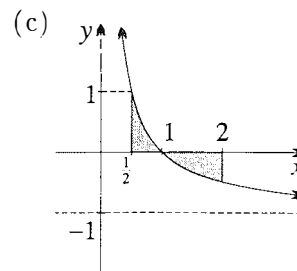
1. Find the area under the curve $y = \frac{1}{x}$ for: (a) $e \leq x \leq e^2$ (b) $2 \leq x \leq 8$



Find the area of the region bounded by $y = \frac{3}{x} - 3$, the x -axis and $x = 3$.

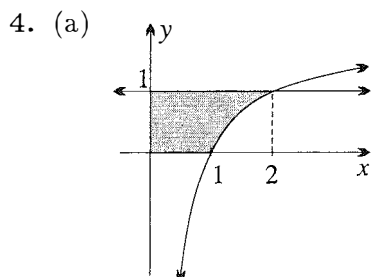


Find the area of the region between $y = \frac{2}{x}$ and the line $x + 2y - 5 = 0$.

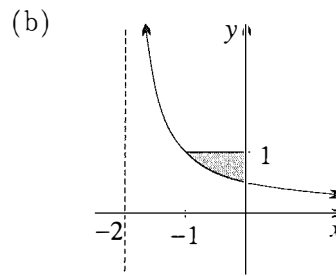


Find the area of the region bounded by $y = \frac{1}{x} - 1$, the x -axis, $x = \frac{1}{2}$ and $x = 2$.

3. (a) Find the area under the graph $y = \frac{x}{x^2 + 1}$ between $x = 0$ and $x = 2$.



Find the area of the region in the first quadrant bounded by $y = 2 - \frac{2}{x}$ and $y = 1$.



Find the area of the region bounded by the curve $y = \frac{1}{x+2}$, the y -axis and $y = 1$.

5. (a) Sketch the region bounded by the x -axis, $y = x$, $y = \frac{1}{x}$ and $x = e$.

(b) Hence find the area of this region by using two appropriate integrals.

6. (a) Sketch the area bounded by the coordinate axes, $y = 1$, $x = 8$ and the curve $y = \frac{4}{x}$.
 (b) Determine the area of this region with the aid of an appropriate integral.

DEVELOPMENT

7. (a) Find the two intersection points of the curve $y = \frac{1}{x}$ with $y = 4 - 3x$.
 (b) Determine the area between these two curves.
8. The curve $y = \frac{1}{x^2}$ (called a *truncus*) is rotated about the y -axis between $y = 1$ and $y = 6$. Evaluate the resulting volume.
9. (a) Find the volume generated when the curve $y = \frac{1}{\sqrt{x}}$ is rotated about the x -axis between $x = 2$ and $x = 4$.
 (b) A horn is created by rotating the curve $y = \frac{1}{\sqrt{4-x}}$ about the x -axis between $x = 0$ and $x = 3\frac{3}{4}$. Find the volume of the horn.
 (c) Another horn is generated by rotating the curve $y = 1 + \frac{1}{x}$ about the x -axis between $x = \frac{1}{2}$ and $x = 3$. Find its volume.
10. (a) Expand $(x+1)(x-1)(x-4)$.
 (b) Use part (a) to help find the intersection points of $y = \frac{4}{x}$ and $y = 1 + 4x - x^2$, and hence sketch the two curves.
 (c) Hence find the area in the first quadrant enclosed between these two curves.
11. Carefully graph $y = \frac{1}{x}$ and $y = 2 - x$ on the same number plane. Find the area between these two curves, the x -axis and the line $x = 4$.
12. (a) Differentiate $y = x \log x$, and hence write down a primitive of $\log x$.
 (b) Hence determine the area under $y = \log x$ between $x = e$ and $x = e^2$.
 (c) Use part (a) to help determine the area in the first quadrant above $y = \log x$ and below $y = c$, for $c > 0$.
13. (a) The area between $y = \sqrt{x}$ and $y = \frac{1}{\sqrt{x}}$, and between $x = 1$ and $x = 4$, is rotated about the x -axis. Find the volume of the resulting solid.
 (b) Compare the volume found in part (a) with the volume generated when the area below $y = \sqrt{x} - \frac{1}{\sqrt{x}}$, also between $x = 1$ and $x = 4$, is rotated about the x -axis.
14. The area in the first quadrant under $y = 1$ and above $y = 2 - \frac{2}{x}$ is rotated about the x -axis. Find the volume so formed.
15. A rod lying between $x = 1$ and $x = 3$ has density at any point x given by $\rho(x) = \frac{1}{x}$.
 (a) Calculate its mass, given the formula $M = \int_1^3 \rho(x) dx$.
 (b) Find the position $\bar{x}$ of the centre of mass, given the formula $\bar{x} = \frac{1}{M} \int_1^3 x \rho(x) dx$.

16. (a) Use upper and lower rectangles to prove that $\frac{1}{2} < \int_{2^n}^{2^{n+1}} \frac{1}{x} dx < 1$, for $n \geq 0$.
- (b) Hence prove that $\int_1^{2^n} \frac{1}{x} dx \rightarrow \infty$ as $n \rightarrow \infty$.
17. (a) Explain why $\log e = \int_1^3 \frac{1}{x} dx - \int_e^3 \frac{1}{x} dx$.
- (b) Show that Simpson's rule with five function values estimates the first integral as $\frac{11}{10}$.
- (c) Show that the trapezoidal rule with two points estimates $\int_e^3 \frac{1}{x} dx = \frac{1}{6e}(9 - e^2)$.
- (d) Combine parts (a), (b) and (c) to show that e may be approximated with the equation $5e^2 + 3e - 45 = 0$. Solve this equation to find an approximation for e (answer correct to three decimal places).
18. Sketch the region cut off in the first quadrant by $y = \frac{2x - 3}{x^2 - 3x - 4}$, and find its area.
19. (a) Show that $4x = 2(2x + 1) - 2$.
- (b) Hence evaluate the area under $y = \frac{4x}{2x + 1}$ between $x = 0$ and $x = 1$.
20. (a) Graph the region bounded by the curve $y = 1/x$, the x -axis and the lines $x = -3$ and $x = -2$. Use the fact that $y = 1/x$ is an odd function to express the area as an integral, and evaluate the area.
- (b) Find the area between the curve $y = 1/x$ and the x -axis, for:
- (i) $-1 \leq x \leq -e^{-3}$ (ii) $-9 \leq x \leq -3$
21. (a) Show that the curves $y = \frac{6}{x}$ and $y = x^2 - 6x + 11$ intersect when $x = 1, 2$ and 3 .
- (b) Graph these two curves and shade the two areas enclosed by them.
- (c) Find the total area enclosed by the two curves.

————— EXTENSION —————

22. Consider the area under $y = \frac{1}{x}$ between $x = n$ and $x = n + 1$.
- (a) Show that $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n}$. (b) Hence show that $\frac{n}{n+1} < \log(1 + \frac{1}{n})^n < 1$.
- (c) Take the limit of this last result as n tends to infinity to show that $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$.
- (d) Repeat the above steps, replacing $n + 1$ with $n + t$ and show that $\lim_{n \rightarrow \infty} (1 + \frac{t}{n})^n = e^t$.
23. What is the volume generated when the area under $y = e^{-x^2}$ between $x = 0$ and $x = 1$ is rotated about the y -axis? (A sketch will be required first.)
24. The curve $y = \frac{\sqrt{x}}{x+1}$ is rotated about the x -axis, generating a volume between $x = 0$ and $x = c$. Determine this volume. [HINT: $\frac{x}{(x+1)^2} = \frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}$]
25. (a) Find the x coordinates of the inflexion points of $y = \frac{x}{x^2 + 1}$.
- (b) Hence explain why the trapezoidal rule applied to this function between $x = 0$ and $x = \sqrt{3}$ will underestimate the area.

The Exponential Function

Exponential functions and logarithmic functions are mutual inverses, so it is now fairly straightforward to develop the calculus of exponential functions from the calculus of the logarithmic function in the previous chapter. Again the natural base to use is e , and the chapter is essentially a study of the characteristic properties of $y = e^x$.

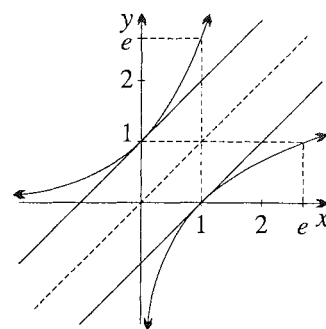
Understanding the exponential function $y = e^x$ is one of the main goals of this course. Functions involving e^x are essential for modelling some of the most common situations in the natural world, such as population growth, radioactive decay, the dying away of a note on the piano, inflation and depreciation — the phrase ‘growing exponentially’ has now entered the common vocabulary. Calculus is essential for the study of e^x , in fact the very definition of e involves calculus. This is quite unlike the study of linear and quadratic functions in earlier chapters, where algebraic and geometric techniques predominate.

STUDY NOTES: Once again, a thorough algebraic and graphical understanding that exponential and logarithmic functions are inverses is fundamental. Drawing tangents on a graph-paper sketch of $y = e^x$, or on some software package version of it, should be used to reinforce the key understanding that at each point on this curve, gradient is equal to height. More complicated versions of natural growth, such as Newton’s law of cooling, have been left until later, but they could be developed now.

13A The Exponential Function and its Derivative

As with logarithmic functions, the most natural base to use for exponential functions in calculus is e , and the function $y = e^x$ is therefore called *the exponential function* to distinguish it from exponential functions like $y = 2^x$ which have other bases. Here are the tables of values and graphs of the mutually inverse functions $y = e^x$ and $y = \log x$.

x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
$\log x$	-2	-1	0	1	2
x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2



The graphs are mutual reflections in the diagonal line $y = x$. Furthermore, the tangent to $y = \log x$ at its x -intercept is reflected into the tangent to $y = e^x$ at its y -intercept — both tangents have gradient 1 and are parallel to $y = x$.

Differentiating the Exponential Function: The most significant thing about the exponential function is that its derivative is equal to itself.

1 THE EXPONENTIAL FUNCTION IS ITS OWN DERIVATIVE: $\frac{d}{dx} e^x = e^x$

PROOF: Let $y = e^x$.

Then $x = \log y$,

so $\frac{dx}{dy} = \frac{1}{y}$, as established in the last chapter.

Hence $\frac{dy}{dx} = y$, since $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocals of each other,
 $= e^x$.

Gradient Equals Height: The geometrical interpretation of this result is a striking relationship between the gradient and the height of the curve $y = e^x$ at each point on it.

2 GRADIENT EQUALS HEIGHT: At each point on the curve $y = e^x$, the gradient of the curve is equal to the height above the x -axis.

In particular, the gradient at the y -intercept is exactly 1 (which was already clear because the curves $y = e^x$ and $y = \log x$ are mutual reflections in $y = x$).

The Standard Forms for Differentiation: These are the standard forms for differentiation. The second follows by the chain rule with $u = ax + b$, and the third restates the chain rule.

3 STANDARD FORMS FOR DIFFERENTIATION:

A. $\frac{d}{dx} e^x = e^x$

B. $\frac{d}{dx} e^{ax+b} = a e^{ax+b}$

C. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$ OR $\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$

WORKED EXERCISE: Find the derivatives of: (a) $e^{\frac{1}{2}(9-x)}$ (b) e^{x^2} (c) $x^3 e^x$

SOLUTION: (a) $\frac{d}{dx} e^{\frac{1}{2}(9-x)} = -\frac{1}{2} e^{\frac{1}{2}(9-x)}$, by the second standard form.

(b) Let $y = e^{x^2}$, then by the chain rule with $u = x^2$,
 $\frac{dy}{dx} = 2x e^{x^2}$.

(c) Let $y = x^3 e^x$, then by the product rule with $u = x^3$ and $v = e^x$,
 $\frac{dy}{dx} = 3x^2 e^x + x^3 e^x$
 $= x^2 e^x (3 + x)$.

Exponential Functions with Other Bases: Any other exponential function $y = a^x$, with base a different from e , can be expressed as a base e exponential function $y = e^{kx}$. Using the identity $a = e^{\log a}$,

$$\begin{aligned} a^x &= (e^{\log a})^x \\ &= e^{x \log a}, \text{ by the index law } (e^m)^x = e^{mx}. \end{aligned}$$

Thus a^x has been expressed in the form e^{kx} , where $k = \log a$ is a constant. In this form, the function can be differentiated:

$$\begin{aligned} \frac{d}{dx} a^x &= e^{x \log a} \log a, \text{ using the second standard form above,} \\ &= a^x \log a, \text{ since } e^{x \log a} = a^x. \end{aligned}$$

Either the process can be reproduced or the results remembered.

4 OTHER BASES: $a^x = e^{x \log a}$ and $\frac{d}{dx} a^x = a^x \log a$

A Characterisation of the Exponential Function: The derivative of the exponential function $f(x) = e^x$ is $f'(x) = e^x$ — the same function. This property characterises the exponential function amongst all other exponential functions.

5 THEOREM: The exponential function $y = e^x$ is the only exponential function whose derivative is equal to itself.

PROOF: Let $f(x) = a^x$ be any other exponential function.

Then $f'(x) = a^x \log a$,

and so $f'(x)$ and $f(x)$ are identical if and only if

$$\log a = 1,$$

that is, $a = e$.

NOTE: The fact that the derivative is equal to the function is the fundamental reason why the exponential function is so important in mathematics. The study of natural growth and decay in Section 13E will be based on this property.

Extension — An Alternative Way to Differentiate Powers of Other Bases: The function a^x can also be differentiated using the log laws and implicit differentiation.

Let $y = a^x$.

Then $\log y = \log a^x$, taking logs of both sides,

so $\log y = x \log a$, by the log laws,

$$\frac{1}{y} \frac{dy}{dx} = \log a, \text{ by implicit differentiation,}$$

$$\boxed{\times y} \quad \frac{dy}{dx} = a^x \log a.$$

WORKED EXERCISE:

- Express $y = 3 \times 2^x$ as a multiple of a power of e .
- Hence find the derivative of $y = 3 \times 2^x$.
- [Extension] Differentiate $y = 3 \times 2^x$ by the alternative method of taking logs of both sides and differentiating implicitly.

SOLUTION:

$$\begin{aligned} \text{(a)} \quad y &= 3 \times 2^x \\ &= 3 \times (e^{\log 2})^x \\ &= 3 \times e^{x \log 2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{dy}{dx} &= 3 \log 2 \times e^{x \log 2} \\ &= 3 \log 2 \times 2^x \end{aligned}$$

(c) Alternatively, taking logs of both sides,
 $\log y = \log 3 + x \log 2$,

so using implicit differentiation,

$$\frac{1}{y} \frac{dy}{dx} = \log 2,$$

$$\boxed{\times y} \quad \frac{dy}{dx} = y \log 2$$

$$= 3 \log 2 \times 2^x.$$

Exercise 13A

1. Differentiate:

(a) e^{2x}

(c) $-e^{5x}$

(e) e^{ax}

(g) $-e^{-\pi x}$

(b) e^{-3x}

(d) $e^{\frac{1}{2}x}$

(f) e^{-kx}

(h) $e^{-\frac{1}{c}x}$

2. Find the derivative of:

(a) e^{2x-1}

(d) $2e^{\frac{1}{2}x+4}$

(g) $\frac{e^x - e^{-x}}{2}$

(b) e^{1-x}

(e) e^{px+q}

(h) $\frac{e^{ax}}{a} + \frac{e^{-bx}}{b}$

(c) $3e^{-3x+4}$

(f) $e^{2x} - e^{-3x}$

3. Write as a power of e and then differentiate:

(a) $(e^x)^2$

(b) $\frac{1}{e^{3x}}$

(c) $\sqrt{e^x}$

(d) $\frac{1}{(\sqrt{e^x})^3}$

4. Use the product and chain rules as appropriate to differentiate:

(a) e^{x^2}

(d) e^{6+x-x^2}

(g) xe^{-x}

(j) xe^{x^2+1}

(b) e^{1-x^2}

(e) $\frac{1}{2}e^{3x^2-2x+1}$

(h) $(x-1)e^x$

(k) $(x^2+5x-5)e^x$

(c) e^{x^2+2x}

(f) xe^x

(i) $(x+1)e^{3x-4}$

(l) $(x^2-x^3)e^{-x}$

5. (a) Find the gradient of the curve $y = e^{5x}$ at the point $A(a, e^{5a})$, and show that the gradient is 5 times the height.
 (b) Find the gradient of the curve $y = be^{-3x}$ at the point $A(a, be^{-3a})$, and show that the gradient is -3 times the height.
 (c) Find the gradient of the curve $y = be^{kx}$ at the point $A(a, be^{ka})$, and show that the gradient is k times the height.

DEVELOPMENT

6. Use the product, quotient, chain and log rules as appropriate to differentiate:

(a) $(x^2 - x)e^{2x-1}$

(d) $e^x \log x$

(g) $\frac{x+1}{e^x}$

(i) $\frac{e^x}{(x+1)^2}$

(b) $\log(1 - e^x)$

(e) $\log(e^x + e^{-x})$

(h) $\frac{e^x + 1}{e^x - 1}$

(j) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

(c) $\log(e^x + x)$

(f) $\frac{e^x}{x}$

7. Use the identity $a = e^{\log a}$ to write each expression as a power of e . Thus differentiate it.

(a) 2^x

(c) π^x

(e) 2^{3x-1}

(g) a^{x-2}

(i) $x2^x$

(b) 10^x

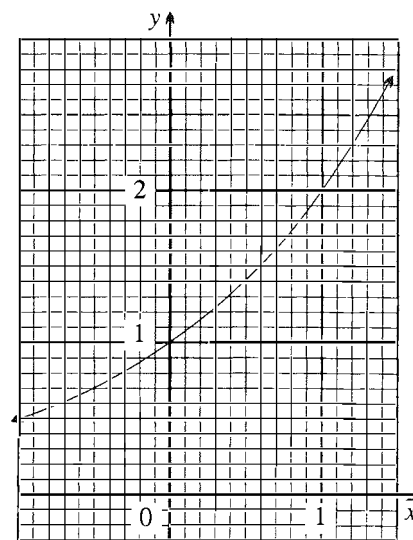
(d) a^x

(f) 5^{2-x}

(h) a^{bx+c}

(j) 3^{x^3-3x}

8. Simplify then differentiate:
 (a) $e^{\log x}$ (b) $e^{-\log x}$ (c) $\log(e^{-x})$ (d) $\log(e^{3x-7})$
9. (a) Show that $Ae^0, Ae^{-1}, Ae^{-2}, \dots$ forms a GP, and find its limiting sum.
 (b) Show that for the AP $u_1 = a, u_2 = a + d, u_3 = a + 2d, \dots$, the sequence $e^{u_1}, e^{u_2}, e^{u_3}, \dots$ is a GP.
 (c) Conversely, show that for the GP $u_1 = a, u_2 = ar, u_3 = ar^2, \dots$, the sequence $\log u_1, \log u_2, \log u_3, \dots$ is an AP.
10. (a) Show that both $y = e^{-x}$ and $y = xe^{-x}$ are solutions of $y'' + 2y' + y = 0$.
 (b) Show that $y = e^{-3x} + x - 1$ is a solution of the differential equation $y'' + 2y' - 3y = 5 - 3x$.
 (c) Find the values of λ that make $y = 5e^{\lambda x}$ a solution of:
 (i) $y'' + 3y' - 10y = 0$ (ii) $y'' + y' - y = 0$
11. (a) Show that $y = Ae^{kx}$ is a solution of: (i) $y' = ky$ (ii) $y'' - k^2y = 0$
 (b) Show that $y = Ae^{kx} + C$ is a solution of $\frac{dy}{dx} = k(y - C)$.
 (c) Show that $y = (Ax + B)e^{3x}$ is a solution of $y'' - 6y' + 9y = 0$.
12. Differentiate: (a) $e^{\frac{1}{x}}$ (b) $e^{x+\log x}$ (c) $3^{-\frac{1}{x}}$
13. (a) Find the x -coordinates of the stationary points of $y = xe^{-x^2}$.
 (b) Determine where the second derivative of $y = x^2 e^x$ changes sign.
 (c) Let $f(x) = xe^x$ and $g(x) = xe^{-x}$. Find $f'(x)g'(x)$.
14. We define two new functions $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$.
 (a) Show that each of these functions is the derivative of the other.
 (b) Show that both functions are solutions of the differential equation $y'' - y = 0$.
 (c) Show that $\cosh^2 x - \sinh^2 x = 1$.
15. (a) Make a copy of the graph of $y = 2^x$ on the right and on it draw the secant from $x = 0$ to $x = 1$. Also draw the tangent at $x = 0$.
 (b) Compare the secant and tangent and hence explain why the gradient of the tangent is less than 1. Measure both gradients to confirm this.
 (c) Differentiate $y = 2^x$ by writing $y = e^{x \log 2}$, and hence show that $y' = \log 2$ at $x = 0$. Compare this result with your answer to the previous part.
 (d) Show that the gradient of a secant from $x = 0$ to $x = h$ is given by $\frac{2^h - 1}{h}$. Use a calculator to evaluate this quantity for smaller and smaller values of h . What is the value of $\lim_{h \rightarrow 0} \frac{2^h - 1}{h}$?
 (e) Differentiate $y = 2^x$ by first principles, and use the previous part to help evaluate the limit.
16. Given $y = e^x$, it is clear that $x = \log y$. Differentiate the latter equation and hence prove that $\frac{dy}{dx} = e^x$.



17. Differentiate, either by expressing as a power of e , or by taking logs of both sides and using implicit differentiation:

(a) $y = x^x$ (b) $y = x^{2-x}$ (c) $y = x^{\log x}$ (d) $y = x^{\frac{1}{\log x}}$

————— EXTENSION —————

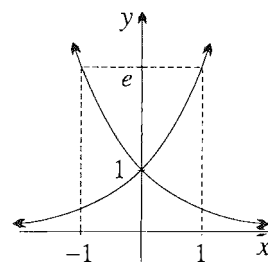
18. (a) Find the possible values of λ that make $y = e^{\lambda x}$ a solution of $ay'' + by' + cy = 0$.
 (b) When will there be no real solution for λ ?
19. (a) Prove that a function of the form $y = Ae^x$, where A is a constant, is the only function that is its own derivative. Proceed by the method of proof by contradiction as follows:
 (i) Assume that there exists another function $f(x)$ that has this property and is not a multiple of e^x . Show that $f'(x) - f(x) = 0$.
 (ii) Let $g(x) = e^{-x}f(x)$. Show that $g'(x) = 0$.
 (iii) Explain why $g(x)$ is constant, and complete the proof.
 (b) Show by direct differentiation that if $f(x) = x e^{x - \log x}$, then $f(x) = f'(x)$. Explain this result in the light of what was proven in part (a).
20. (a) In Exercise 12B of the previous chapter on logarithms, the result $\lim_{u \rightarrow 1} \frac{\log u}{u - 1} = 1$ was proven. Use the substitution $h = \log u$ to show that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.
 (b) Differentiate $y = e^x$ from first principles and use the result $y' = e^x$ to show that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.

13B Applications of Differentiation

The derivatives of algebraic functions, logarithmic functions and exponential functions are now known, and the usual applications of differentiation are possible, in particular the sketching of curves whose equations involve e^x .

The Graphs of e^x and e^{-x} : The graphs of $y = e^x$ and $y = e^{-x}$ are the essential graphs for this section. Since x is replaced by $-x$, these two graphs are reflections of each other in the y -axis:

x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
x	-2	-1	0	1	2
e^{-x}	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$. The gradient of $y = e^x$ at $(0, 1)$ is 1, and so the gradient of $y = e^{-x}$ at $(0, 1)$ must be -1 . This means that the curves are perpendicular at their point of intersection.

NOTE: The function $y = e^{-x}$ is as important as $y = e^x$ in applications, or even more important. It describes a great many physical situations where a quantity 'dies away exponentially', like the dying away of the sound of a plucked string.

Two More Special Limits — e^x Dominates x : Just as x dominates $\log x$, so we can show that e^x dominates x . Curve sketching involving exponential functions often requires this idea, which is expressed by the following limits.

6

THE FUNCTION e^x DOMINATES THE FUNCTION x :

$$\lim_{x \rightarrow -\infty} x e^x = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} x e^{-x} = 0$$

More colourfully, ‘in a battle between x and e^x , e^x always wins’.

PROOF:

A. We know already from Section 12C that $\lim_{u \rightarrow 0^+} u \log u = 0$.

Substitute $u = e^x$, so $\log u = x$,

then $u \rightarrow 0^+$ as $x \rightarrow -\infty$, so that $\lim_{x \rightarrow -\infty} e^x x = 0$.

B. Similarly, we proved in Section 12C that $\lim_{u \rightarrow \infty} \frac{\log u}{u} = 0$.

Using the same substitution, $u \rightarrow \infty$ as $x \rightarrow \infty$,

$$\text{so } \lim_{x \rightarrow \infty} \frac{x}{e^x} = 0.$$

The more general result is that e^x dominates x^k , for all $k > 0$, but the proof is left to a question in the following exercises.

7

THE FUNCTION e^x DOMINATES THE FUNCTION x^k , FOR $k > 0$:

$$\lim_{x \rightarrow -\infty} x^k e^x = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} x^k e^{-x} = 0$$

An Example of Curve Sketching: The application of the standard curve sketching menu to the sketch of $y = x^2 e^x$ will illustrate the use of these limits.

1. The domain is the whole real number line.
2. $f(-x) = x^2 e^{-x}$, which is neither $f(x)$ nor $-f(x)$, so the function is neither even nor odd.
3. The only zero is $x = 0$. Also, y is positive for all $x \neq 0$.
4. $\lim_{x \rightarrow -\infty} y = 0$, since e^x dominates x^2 . Also, $y \rightarrow \infty$ as $x \rightarrow \infty$.

5. Differentiating twice by the product rule,

$$\begin{aligned} f'(x) &= 2x e^x + x^2 e^x & f''(x) &= (2x + 2) e^x + (x^2 + 2x) e^x \\ &= x(x + 2) e^x, & &= (x^2 + 4x + 2) e^x, \end{aligned}$$

so $f'(x) = 0$ when $x = 0$ and $x = -2$ (notice that e^x can never be zero).

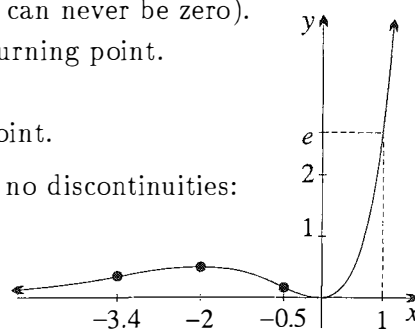
Since $f''(0) = 2 > 0$, the point $(0, 0)$ is a minimum turning point.

Also, $f''(-2) = -2e^{-2} < 0$,

so $(-2, 4e^{-2}) \doteq (-2, 0.541)$ is a maximum turning point.

6. $f''(x)$ has zeroes at $-2 - \sqrt{2}$ and $-2 + \sqrt{2}$, and has no discontinuities:

x	-4	$-2 - \sqrt{2}$	-2	$-2 + \sqrt{2}$	0
$f''(x)$	$2e^{-4}$	0	$-2e^{-2}$	0	2
	∪	.	∩	.	∪



so there are inflexions at $\left(-2 - \sqrt{2}, (6 + 4\sqrt{2})e^{-2-\sqrt{2}}\right) \doteq (-3.414, 0.384)$,
 and also at $\left(-2 + \sqrt{2}, (6 - 4\sqrt{2})e^{-2+\sqrt{2}}\right) \doteq (-0.586, 0.191)$.

Exercise 13B

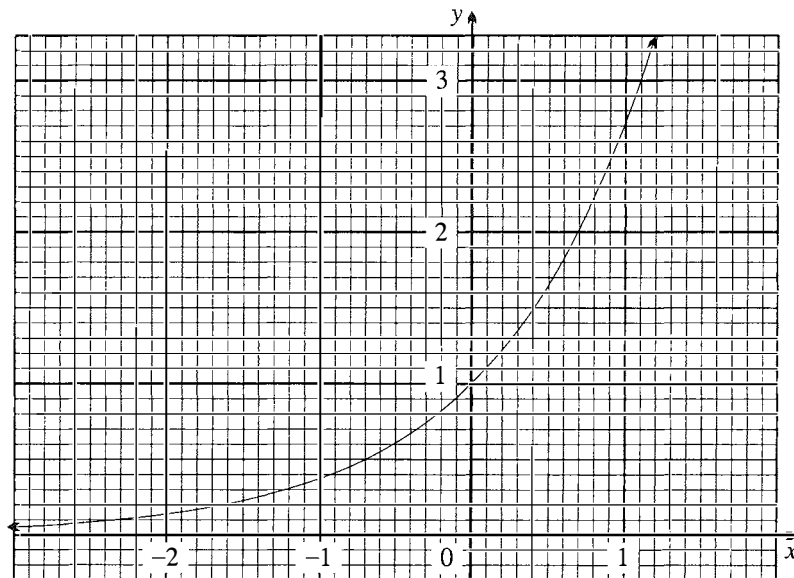
1. (a) Copy and complete these tables of values of the functions $y = e^x$ and $y = \log x$.

x	-2	-1	0	1	2
e^x					

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log x$					

- (b) Sketch both curves on the one set of axes, choosing appropriate scales on the axes.
 (c) Add the line $y = x$ to your graph. What transformation transforms the graph of $y = e^x$ into the graph of $y = \log x$?
 (d) What are the domain and range of $y = e^x$ and $y = \log x$?

2.



- (a) Copy the graph of $y = e^x$ and on it draw the tangent at $x = 0$, extending the tangent down to the x -axis.
 (b) Measure the gradient of this tangent and confirm that it is equal to the height of the exponential graph at the point of contact.
 (c) Repeat for the tangents at $x = -2, -1$ and 1 .
 (d) What do you notice about the x -intercepts of the tangents?
3. Use your knowledge of transformations to help sketch the graphs of the given functions:
- | | | | |
|-------------------|-------------------|----------------------|----------------------------|
| (a) $y = e^{x-1}$ | (c) $y = -e^x$ | (e) $y = 1 - e^x$ | (g) $y = e^{\frac{1}{2}x}$ |
| (b) $y = e^{-x}$ | (d) $y = e^{x-2}$ | (f) $y = e^{-x} - 1$ | (h) $y = e^{- x }$ |
4. (a) Find the equation of the tangent to $y = e^x$ at its y -intercept.
 (b) Show that the tangent to $y = x - e^x$ at $x = 1$ passes through the origin.
5. (a) Find the equation of the normal to $y = e^{-x}$ at the point $P(-1, e)$.
 (b) Find the x and y intercepts of the normal.
 (c) Find the area of the triangle whose vertices lie at the intercepts and the origin.

6. (a) Find the first and second derivatives for the curve $y = x - e^x$.
 (b) Deduce that the curve is concave down for all values of x .
 (c) What is its maximum value? (d) Sketch the curve and write down its range.
7. Consider the curve $y = x e^x$.
 (a) Show that $y' = (1 + x)e^x$ and $y'' = (2 + x)e^x$.
 (b) Show that there is one stationary point, and determine its nature.
 (c) Find the coordinates of the lone point of inflexion.
 (d) Examine the behaviour of y as $x \rightarrow -\infty$.
 (e) Hence sketch the curve, and then write down its range.
 (f) Hence also sketch $y = -x e^{-x}$ by recognising the simple transformation.
8. (a) Given that $y = e^{-\frac{1}{2}x^2}$, find y' and y'' .
 (b) Show that this curve has a maximum turning point at its y -intercept, and has two points of inflexion.
 (c) Examine the behaviour of y as $x \rightarrow -\infty$ and $x \rightarrow \infty$.
 (d) Sketch the graph and write down its range.

DEVELOPMENT

9. (a) Given that $y = (1 - x)e^x$, find y' and y'' .
 (b) Show that this curve has a maximum turning point at its y -intercept, and an inflexion point at $(-1, 2e^{-1})$.
 (c) Sketch the graph and write down its range.
10. Find the x -intercept of the tangent to $y = (1 - x)e^x$ at $x = -1$.
11. [Another characterisation of $y = e^x$ — it is the only exponential function whose gradient at its y -intercept is 1.]
 (a) Prove that $y = a^x$ has derivative $y = a^x \log a$.
 (b) Prove that $y = a^x$ has gradient 1 at its y -intercept if and only if $a = e$.
 (c) Prove that $y = Aa^x$ has gradient 1 at its y -intercept if and only if $a = e^{1/A}$.
12. The line $y = mx$ is tangent to the curve $y = \frac{e^x}{x}$. Show that the point of contact is $A(2, \frac{1}{2}e^2)$ by showing that the gradient of OA is equal to the gradient of the tangent to the curve at A .
13. (a) Find the equation of the tangent to $y = e^x$ at $x = t$.
 (b) Hence show that the x -intercept of this tangent is $t - 1$. Does this agree with your answers to question 2(d)?
14. (a) Show that $y = e - x^2 e^x$ has an x -intercept at $x = 1$.
 (b) Show that the curve has two turning points and classify them.
 (c) Examine the behaviour as $x \rightarrow \infty$ and deduce that it also has two inflexion points.
 (d) Sketch the curve and write down its range.
15. (a) Find the intercepts of $y = (1 + x)^2 e^{-x}$.
 (b) Show that the curve has two turning points and classify them.
 (c) Examine the behaviour of y as $x \rightarrow \infty$ and hence deduce that it also has two inflexions.
 (d) Sketch the curve and write down its range.

16. (a) Classify the stationary points of $y = xe^{-x^2}$.
 (b) Locate the three inflexion points, sketch the curve and write down its range.
17. (a) Find the equation of the normal to $y = e^{-x^2}$ at the point where $x = t$.
 (b) Determine the x -intercept of the normal.
 (c) Hence find the values of t for which the normal passes through the origin.

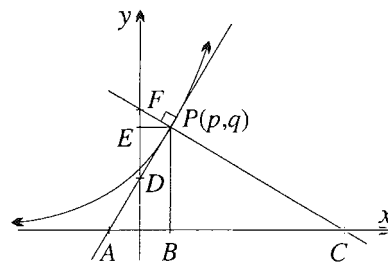
18. On the graph of $y = e^x$ are drawn the tangent and normal to the curve at the point $P(p, q)$.

(a) Find the coordinates of each of the points A , B , C , D , E and F in terms of p and q .

(b) Hence show: (i) $AB = 1$ (ii) $BC = q^2$

(iii) $DE = pq$ (iv) $EF = \frac{p}{q}$

(c) What is the area of: (i) $\triangle ACP$? (ii) $\triangle DFP$?



19. Find the x -coordinates of the stationary points of $y = xe^{-|x|}$ by considering positive and negative values of x separately.
20. Show that $y = (x^2 + 3x + 2)e^x$ has an inflexion point at one of its x -intercepts. Sketch the curve and label all important features. Do not find the y -coordinates of the stationary points.
21. (a) What is the natural domain of $y = \frac{e^x}{x}$?
 (b) Show that the curve has a local minimum at $(1, e)$ but no inflexion points.
 (c) Sketch the curve and state its range.
22. (a) What is the natural domain of $y = e^{\frac{1}{x}}$?
 (b) Carefully determine the behaviour of y and y' as $x \rightarrow -\infty$, $x \rightarrow 0$ and $x \rightarrow \infty$.
 (c) Deduce that there must be an inflexion point and find it.
 (d) Sketch the curve and give its range.
23. Follow the steps in the previous question in order to sketch the graph of $y = xe^{\frac{1}{x}}$.

EXTENSION

24. If the positive base a of $y = a^x$ and $y = \log_a x$ is small enough, then the two curves will intersect. What base must be chosen so that the two are tangent at the point of contact? Proceed as follows:
- (a) Rewrite both equations with base e , and let $k = \log a$.
 (b) Explain why the gradient of the tangent at the point of contact must be 1.
 (c) Use the last part to obtain two equations for the gradient.
 (d) Solve these simultaneously to find k , and hence write down the base a .
25. [Here are two proofs of the more general dominance result mentioned in the theory above.]
- (a) One of the results from the previous chapter is $\lim_{u \rightarrow 0} u^{\frac{1}{k}} \log u = 0$. Substitute $u = e^x$ and hence prove that $\lim_{x \rightarrow -\infty} x^k e^x = 0$.
- (b) Given that $\lim_{u \rightarrow -\infty} u e^u = 0$, substitute $u = x/k$ and hence prove that $\lim_{x \rightarrow -\infty} x^k e^x = 0$.

26. One of the fundamental functions in the study of statistics is $y = \int_0^x e^{-\frac{1}{2}t^2} dt$.
- What is the y -intercept of this function? (b) Use the fundamental theorem of calculus to find y' and show that there are no stationary points.
 - Show that y' is even. What can be said about y ?
 - Show that there is a point of inflexion at the y -intercept.
 - Investigate y' as $x \rightarrow \infty$. Is it possible to deduce from this alone whether y has any horizontal asymptote? (f) It can be shown (with more advanced techniques) that y has horizontal asymptote $y = \sqrt{\frac{\pi}{2}}$. Now sketch the function and state its range.

13C Integration of the Exponential Function

The standard forms for differentiation can be reversed to provide standard forms for integration.

8

STANDARD FORMS FOR INTEGRATION:

$$\text{A. } \int e^x dx = e^x + C$$

$$\text{B. } \int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

$$\text{C. } \int e^u \frac{du}{dx} dx = e^u + C \quad \text{OR} \quad \int f'(x) e^{f(x)} dx = e^{f(x)} + C$$

WORKED EXERCISE: Find: (a) $\int_0^2 e^x dx$ (b) $\int_2^3 e^{5-2x} dx$ (c) $\int_{-1}^1 x e^{x^2} dx$

SOLUTION:

$$\begin{aligned} \text{(a) } \int_0^2 e^x dx &= [e^x]_0^2 \\ &= e^2 - e^0 \\ &= e^2 - 1 \end{aligned}$$

$$\begin{aligned} \text{(b) } \int_2^3 e^{5-2x} dx &= -\frac{1}{2} [e^{5-2x}]_2^3 \\ &= -\frac{1}{2} (e^{-1} - e) \\ &= \frac{e^2 - 1}{2e} \end{aligned}$$

$$\begin{aligned} \text{(c) } \int_{-1}^1 x e^{x^2} dx &= \frac{1}{2} \int_{-1}^1 2x e^{x^2} dx, \\ &= \frac{1}{2} [e^{x^2}]_{-1}^1 \\ &= \frac{1}{2} (e - e) \\ &= 0, \text{ since the integrand is odd.} \end{aligned}$$

$$\left| \begin{array}{l} \text{Let } u = x^2. \\ \text{Then } \frac{du}{dx} = 2x. \\ \int e^u \frac{du}{dx} dx = e^u \end{array} \right.$$

WORKED EXERCISE: Find $f(x)$ and $f(1)$, if $f'(x) = 1 + 2e^{-x}$ and $f(0) = 1$.

SOLUTION: $f'(x) = 1 + 2e^{-x}$
 so $f(x) = x - 2e^{-x} + C$, for some constant C .
 Since $f(0) = 1$, $1 = 0 - 2e^0 + C$
 so $C = 3$ and $f(x) = x - 2e^{-x} + 3$.
 Hence $f(1) = 1 - 2e^{-1} + 3$

$$= 4 - \frac{2}{e}.$$

Integrals of Exponential Functions with Other Bases: Since $\frac{d}{dx} a^x = a^x \log a$, it follows (omitting the constant of integration) that:

9 **OTHER BASES:** $\int a^x dx = \frac{a^x}{\log a}$, for all positive bases $a \neq 1$.

Either this result or the process of obtaining it should be learnt. The primitive can also be obtained by expressing a^x as a power of e :

$$\begin{aligned} \int a^x dx &= \int (e^{\log a})^x dx \\ &= \int e^{x \log a} dx \\ &= \frac{1}{\log a} e^{x \log a}, \text{ since } \int e^{kx} = \frac{1}{k} e^{kx}, \\ &= \frac{1}{\log a} a^x, \text{ since } e^{x \log a} = a^x. \end{aligned}$$

NOTE: The formulae for differentiation and integration of a^x both involve $\log a$:

$$\frac{d}{dx} a^x = a^x \log a \quad \text{and} \quad \int a^x dx = \frac{a^x}{\log a}.$$

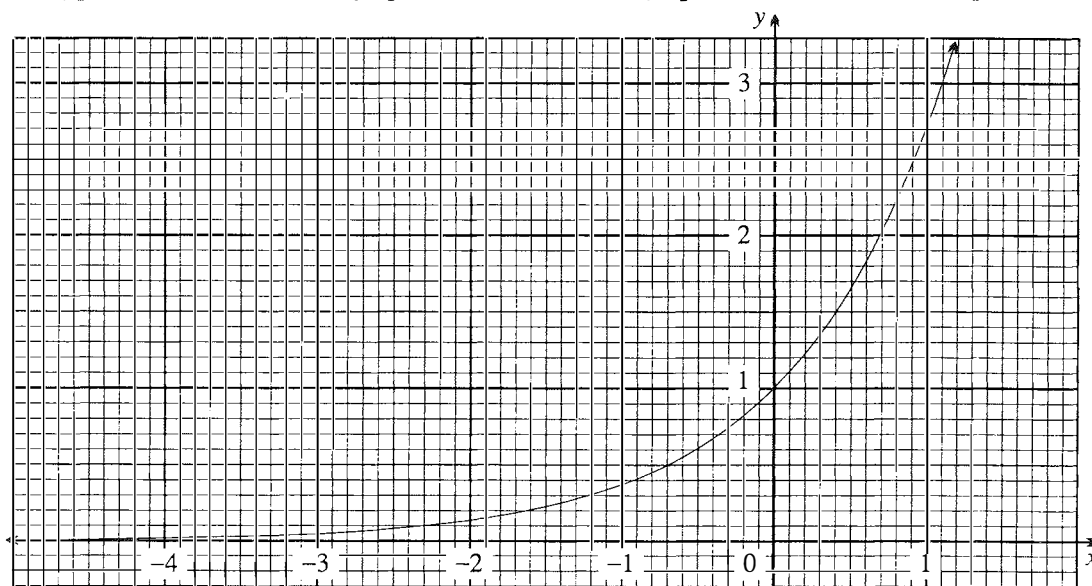
Since $\log a = 1$ when $a = e$, the formulae are simplest when the base is e , which confirms that e is the appropriate base to use for the calculus of exponential functions.

Exercise 13C

1. (a) Evaluate these definite integrals, then approximate them to two decimal places:

(i) $\int_0^1 e^x dx$ (ii) $\int_{-1}^0 e^x dx$ (iii) $\int_{-2}^0 e^x dx$ (iv) $\int_{-3}^0 e^x dx$

(b)



The graph above shows that $y = e^x$ from $x = -5$ to $x = 1$, with a scale of 10 divisions to 1 unit, so that 100 little squares equal 1 square unit. By counting squares under the curve from $x = 0$ to $x = 1$, find an estimate for $\int_0^1 e^x dx$, and compare it with the approximation obtained in part (a).

(c) Count squares to the left of the y -axis to obtain estimates of:

$$(i) \int_{-1}^0 e^x dx, \quad (ii) \int_{-2}^0 e^x dx, \quad (iii) \int_{-3}^0 e^x dx,$$

and compare the results with the approximations obtained in part (a).

(d) Continue counting squares to the left of $x = -3$, and estimate the total area under the curve to the left of the y -axis.

2. Find the following indefinite integrals:

$$\begin{array}{llll} (a) \int e^{2x} dx & (e) \int e^{4x-2} dx & (i) \int e^{3-x} dx & (m) \int \pi e^{3x-\sqrt{2}} dx \\ (b) \int e^{\frac{1}{3}x} dx & (f) \int e^{x-1} dx & (j) \int e^{7-2x} dx & (n) \int \bullet e^{b-ax} dx \\ (c) \int e^{4x+5} dx & (g) \int 6e^{3x+2} dx & (k) \int e^{\pi x-1} dx & (o) \int \bullet e^{bx+c} dx \\ (d) \int e^{1+3x} dx & (h) \int 2e^{2x-1} dx & (l) \int \frac{1}{2} e^{1-ex} dx & (p) \int -\pi e^{a-\pi x} dx \end{array}$$

3. Evaluate the following definite integrals:

$$\begin{array}{llll} (a) \int_1^2 e^x dx & (c) \int_0^2 e^{x-1} dx & (e) \int_{\log 2}^{\log 3} e^x dx & (g) \int_{-\pi}^0 e^{1-\frac{1}{x}} dx \\ (b) \int_{-1}^3 e^{-x} dx & (d) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x} dx & (f) \int_{\log 4}^{\log 6} e^{-x} dx & (h) \int_{\bullet}^{2a} e^{bx} dx \end{array}$$

4. Use the index laws to simplify each integrand and hence find the indefinite integral of:

$$(a) \int \frac{1}{(e^x)^2} dx \quad (b) \int \frac{e^x + 1}{e^x} dx \quad (c) \int \sqrt{e^x} dx \quad (d) \int \frac{e^x - e^{-x}}{\sqrt{e^x}} dx$$

5. Use the identity $\bullet = e^{\log a}$ to express each of the following as a power of e , and hence find its primitive.

$$(a) 2^x \quad (b) 3^x \quad (c) 5^{-x} \quad (d) \pi^x$$

6. (a) Find y as a function of x , if $y' = e^{x-1}$ and $y = 1$ when $x = 1$. What is the y -intercept of this curve?

(b) The gradient of a curve is given by $y' = e^{2-x}$ and the curve passes through the point $(0, 1)$. What is the equation of this curve? What is its horizontal asymptote?

(c) Find y , given that $y' = 2^{-x}$, and $y = \frac{1}{2 \log 2}$ when $x = 1$.

(d) It is known that $f'(x) = e^x + \frac{1}{e}$ and that $f(-1) = -1$. Find $f(0)$.

7. Use the standard form $\int f'(x)e^{f(x)} dx = e^{f(x)} + C$, or $\int e^u \frac{du}{dx} dx = e^u + C$, to integrate:

$$\begin{array}{ll} (a) 2xe^{x^2+3} & (c) (3x+2)e^{3x^2+4x+1} \\ (b) (10x-2)e^{5x^2-2x} & (d) (x^2-2x)e^{x^3-3x^2} \end{array}$$

DEVELOPMENT

8. Find a primitive of each function:

(a) $(6x^2 - 8x + 6)e^{x^3 - 2x^2 + 3x - 5}$	(d) $(e^x + 1)^2$	(g) $\sqrt{x}e^{x\sqrt{x}}$
(b) $\frac{1}{x} + e^{3x}$	(e) $(e^x - e^{-x})^2$	(h) $e^{2\log x}$
(c) $\frac{1}{\sqrt{x}} - xe^{-x^2}$	(f) $\frac{e^{\frac{1}{x}}}{x^2}$	(i) $\log(e^{2kx})$

9. (a) Differentiate xe^x . (b) Hence find $\int_0^2 xe^x dx$.

10. (a) Differentiate $e^x + e^{-x}$. (b) Hence find $\int_0^2 \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$.

11. Find the indefinite integral $\int \frac{e^x}{e^x + 1} dx$.

12. Use the identity $a = e^{\log a}$ to help find a primitive of:

(a) $2x + \frac{2}{x} + 2^x$	(b) $a^x + ax$	(c) $2(x+1)3^{x^2+2x}$

13. (a) Differentiate $y = x^2 e^{-x^2}$.

(b) Hence show that $\int x^3 e^{-x^2} dx = \frac{-e^{-x^2}}{2}(x^2 + 1) + C$, and calculate $\int_1^2 x^3 e^{-x^2} dx$.

14. (a) Show that $xe^x = e^{x+\log x}$.

(b) Hence differentiate xe^x without using the product rule.

15. (a) The gradient of a certain curve is $y' = -\sqrt{x}e^{-x\sqrt{x}}$. Given that its y -intercept is 1, determine the equation of this curve.

(b) Another curve has gradient $3^{-x} \log 3$ and an horizontal asymptote of $y = 2$. Find the equation of the curve and its y -intercept.

16. (a) Find $\frac{d}{dx} e^{ax}$ and hence write down $\int e^{ax} dx$.

(b) Show that $\int e^{ax+b} dx = e^b \int e^{ax} dx$.

(c) Hence confirm the standard form $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$.

17. Find $\int_0^1 2^{\log x} dx$. [HINT: Given that $e^{\log x} = x$, what is $2^{\log x}$?]

EXTENSION

18. Show that $\int \frac{e^x + 1}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}} dx = 2e^{\frac{1}{2}x} + C$.

19. The intention of this question is to outline a reasonably rigorous proof of the famous result that e^x can be written as the limit of the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

- (a) For any positive number R , we know that $1 < e^t < e^R$ for $0 < t < R$, because e^x is an increasing function. Integrate this inequality over the interval $t = 0$ to $t = x$, where $0 < x < R$, and hence show that $x < e^x - 1 < e^R x$.
- (b) Change the variable to t , giving $t < e^t - 1 < e^R t$. Then integrate this new inequality from $t = 0$ to $t = x$, and hence show that $\frac{x^2}{2!} < e^x - 1 - x < \frac{e^R x^2}{2!}$.
- (c) Do this process twice more, and prove that:
- (i) $\frac{x^3}{3!} < e^x - 1 - x - \frac{x^2}{2!} < \frac{e^R x^3}{3!}$
- (ii) $\frac{x^4}{4!} < e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} < \frac{e^R x^4}{4!}$
- (d) Now use induction to prove that $\frac{x^{n+1}}{(n+1)!} < e^x - 1 - x - \frac{x^2}{2!} - \dots - \frac{x^n}{n!} < \frac{e^R x^{n+1}}{(n+1)!}$.
- (e) Show that as $n \rightarrow \infty$, the left and right expressions converge to zero. Hence prove that the infinite power series converges to e^x for $x > 0$. [HINT: Let k be the smallest integer greater than x and show that for $n > k$, each term in the sequence is less than the corresponding term of a geometric sequence with ratio $\frac{x}{k}$.]
- (f) Prove that the power series also converges to e^x for $x < 0$.
20. (a) Use the power series in the previous question to show that
- $$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \dots$$
- (b) Find α , the value of the right-hand side, to four decimal places when $x = 0.5$.
- (c) Let $u = e^{0.5}$. Show that $u^2 - 2\alpha u + 1 = 0$.
- (d) Solve this quadratic equation and hence estimate both $e^{0.5}$ and $e^{-0.5}$ to two decimal places. Compare your answers with the values obtained directly from the calculator.

13D Applications of Integration

The normal applications of integration to areas, volumes, and primitives are now available with functions involving algebraic, logarithmic and exponential functions.

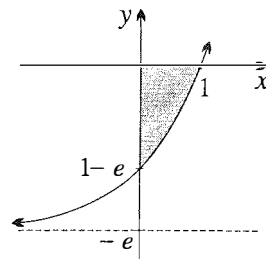
WORKED EXERCISE: Sketch the graph of $y = e^x - e$, then find the area between the curve, the x -axis and the y -axis.

SOLUTION: Move the graph of $y = e^x$ down e units.

The x -intercept is $x = 1$, because then $y = e^1 - e = 0$.

$$\begin{aligned} \int_0^1 (e^x - e) dx &= [e^x - ex]_0^1 \\ &= (e - e) - (1 - 0) \\ &= -1 \quad (\text{negative, being below the } x\text{-axis}) \end{aligned}$$

so the required area is 1 square unit.



WORKED EXERCISE: Find the volume generated when the area between the curve $y = 3e^{1-2x}$ and the ordinates $x = -1$ and $x = 1$ is rotated about the x -axis.

SOLUTION: Volume $= \int_{-1}^1 \pi y^2 dx$

$$= \pi \int_{-1}^1 9e^{2-4x} dx, \text{ since } y^2 = 9e^{2-4x},$$

$$= -\frac{9}{4}\pi \left[e^{2-4x} \right]_{-1}^1$$

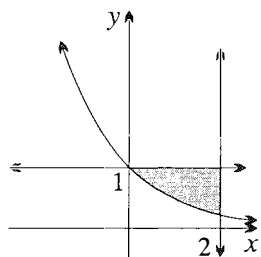
$$= -\frac{9}{4}\pi (e^{-2} - e^6)$$

$$= \frac{9\pi(e^6 - e^{-2})}{4} \text{ cubic units.}$$

Exercise 13D

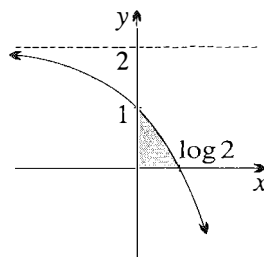
- Find the area between $y = e^x$ and the x -axis for:
 - $-1 \leq x \leq 0$
 - $1 \leq x \leq 3$
 - $\log 2 \leq x \leq \log 5$
 - $\log \frac{1}{2} \leq x \leq \log 4$
- Find the area under the graph $y = e^{-x} + 1$ between $x = 0$ and $x = 2$.
- Use the trapezoidal rule with five function values to estimate the area under the curve $y = e^{-x^2}$ between $x = 0$ and $x = 4$. Give your answer to four decimal places.
 - Use Simpson's rule with five function values to estimate the area in part (a).

4. (a)



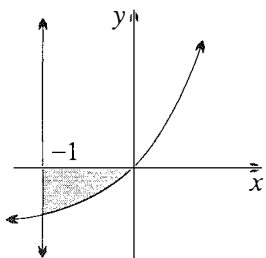
Find the area of the region bounded by the curve $y = e^{-x}$ and the lines $x = 2$ and $y = 1$.

(b)



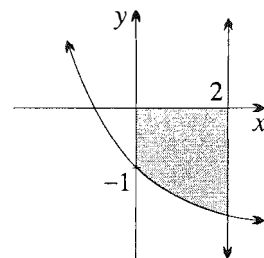
Find the area of the region in the first quadrant bounded by $y = 2 - e^x$ and the coordinate axes.

(c)



Find the area between the x -axis, the curve $y = e^x - 1$, and the line $x = -1$.

(d)



What is the area bounded by $x = 2$, $y = e^{-x} - 2$, the x -axis and the y -axis?

- Change the subject of $y = \log x$ to x , and hence find the area between $y = \log x$ and the y -axis, and between $y = 0$ and $y = 1$.
- Sketch a graph of $y = e^x$ and $y = x + 1$, and shade the area between these curves, $x = 0$ and $x = 1$. Then write down the area of this region as an integral and evaluate it.
- The region under $y = e^x$ between $x = 0$ and $x = 1$ is rotated about the x -axis. Write down the volume of the resulting solid as an integral, and evaluate it.

DEVELOPMENT

8. (a) Show that the curves $y = x^2$ and $y = e^{x+1}$ intersect at $x = -1$.
(b) Hence sketch the region in the second quadrant between these two curves and the y -axis, and find its area.
9. Sketch the region between the graphs of $y = e^x$ and $y = x$, between the y -axis and $x = 2$, then find its area.
10. Sketch the region bounded by the x -axis, the lines $y = x$ and $x = 2$ and the curve $y = e^{1-x}$. Find its area.
11. The shape of a metal stud is created by rotating the curve $y = e^x - e^{-x}$ about the x -axis between $x = 0$ and $x = \frac{1}{2}$. Find its volume.
12. A horn is generated by rotating the curve $y = 1 + e^{-x}$ about the x -axis between $x = 1$ and $x = 3$. Find its volume to three decimal places.
13. A certain bulb and capillary tube are generated when the curve $y = \sqrt{2x} e^{-\frac{1}{2}x^2}$ is rotated about the x -axis between $x = 0$ cm and $x = 2$ cm. Find the capacity of liquid the apparatus could hold. Give your answer to four significant figures.
14. Find the intercepts of the curve $y = 8 - 2^x$ and hence find the area of the region bounded by this curve and the coordinate axes.
15. Consider the two curves $y = ae^x$ and $y = be^{-x}$, where $0 < a < b$.
(a) Find the point of intersection of these two curves.
(b) Hence find the area of the region bounded by these two curves and the y -axis.
16. A rubber seal has the shape generated by rotating the region under $y = \log x$ between $x = 1$ and $x = 2$ about the y -axis. Find the volume of the washer.
17. A sheet of plywood cut out by a jigsaw occupies the region bounded by $y = x$, the x -axis and $y = x - e^x$ between $x = -1$ and $x = 1$, all units being in metres. If the cost of the plywood is \$15 for cutting plus \$8 per square metre, what is the total cost of the sheet to the nearest cent?
18. (a) (i) Find $\int_0^N e^{-x} dx$. (ii) Take the limit of part (i) as $N \rightarrow \infty$. [This is an amazing result: a region that extends to infinity has a *finite* area! There are many other such examples of unbounded regions with finite areas.]
(b) Similarly find the area of the region in the second quadrant under the curve $y = e^x$, and compare your answer with that to the previous part.
(c) (i) Likewise evaluate $\int_0^N 2xe^{-x^2} dx$. (ii) Show that $y = xe^{-x^2}$ is odd.
(iii) Hence show that the total area between the curve $y = xe^{-x^2}$ and the x -axis is 1.

EXTENSION

19. (a) Determine $\int_{\delta}^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$. (b) What happens to the integral as $\delta \rightarrow 0^+$?
20. (a) If the graph of the function $y = ce^{-bx} - a$ has intercepts $y(0) = 1$ and $y(1) = 0$, show that $y = \frac{e^{b-bx} - 1}{e^b - 1}$.
(b) Calculate the area in the first quadrant cut off by the above function.
(c) Use a calculator to evaluate this area for smaller and smaller values of b and hence predict the limit as $b \rightarrow 0$. Hence describe the shape of the area in this limit.

21. (a) Differentiate $x e^{-x}$ and hence find $\int_0^N x e^{-x} dx$.
 (b) Determine the limit of this integral as $N \rightarrow \infty$.
 (c) Differentiate $x^2 e^{-x}$, and hence find $\int_0^\infty x^2 e^{-x} dx$.

13E Natural Growth and Decay

The exponential function is its own derivative, meaning that at each point on the curve, the gradient is equal to the height. This property is the reason why it occurs so often in the modelling of natural phenomena.

Consider a growing population, of people in some country, or rabbits on an island, or bacteria in a laboratory culture. Regard the population P as a function of time t . The rate at which the population is growing at any time is proportional to the value of the population at that time. That is, the gradient of the population graph is proportional to the height of the graph:

$$\frac{dP}{dt} = kP, \text{ where } k \text{ is a constant of proportionality.}$$

Such a situation is called *natural growth*, and a population growing in this way is said to obey the *law of natural growth*. To model this situation, we need a function whose rate of change is proportional rather than equal to the quantity.

The Natural Growth Theorem: The following theorem gives the complete answer.

10	NATURAL GROWTH: Suppose the rate of change of y is proportional to y :
	$\frac{dy}{dt} = ky$, where k is a constant of proportionality.
	Then $y = y_0 e^{kt}$, where y_0 is the value of y at time $t = 0$.

PROOF:

- A. The function $y = y_0 e^{kt}$ certainly satisfies $\frac{dy}{dt} = ky$, since

$$\begin{aligned} \frac{dy}{dt} &= y_0 \frac{d}{dt} e^{kt} \\ &= y_0 k e^{kt} \\ &= ky. \end{aligned}$$

Also, substituting $t = 0$ gives $y = y_0$, as required.

- B. The proof of the converse is harder (and would not be required).

Suppose that a function y of t satisfies $\frac{dy}{dt} = ky$.

Let $u = y e^{-kt}$.

$$\begin{aligned} \text{Then } \frac{du}{dt} &= \frac{dy}{dt} e^{-kt} - ky e^{-kt}, \text{ by the product rule,} \\ &= ky e^{-kt} - ky e^{-kt} \\ &= 0. \end{aligned}$$

Hence $u = C$, for some constant C ,

$$ye^{-kt} = C$$

$$\boxed{\times e^{kt}} \quad y = C e^{kt},$$

and substituting $t = 0$ shows that $C = y_0$, as required.

NOTE: Questions often require a proof that a given function is a solution of a differential equation, by substitution of the function into the differential equation, as in Part A above. The 3 Unit course, however, would not require a proof that there are no other solutions, as in Part B.

Problems Involving Natural Growth: If only the differential equation is given, one can use the natural growth theorem to write down the solution with no further working. Usually, the constant k should be calculated from given values of P at different times, then the approximate value of k can be held in the memory of the calculator.

WORKED EXERCISE: The rabbit population P on an island was estimated to be 1000 at the start of 1995 and 3000 at the start of 2000.

- Assuming natural growth, find P as a function of the time t years after the start of 1995, and sketch the graph.
- How many rabbits are there at the start of 2003 (answer to the nearest 10 rabbits)?
- When will the population be 10 000 (answer to the nearest month)?
- Find the rate of growth (to the nearest 10 rabbits per year):
 - when there are 8000 rabbits,
 - at the start of 1997.

SOLUTION:

- With natural growth, $\frac{dP}{dt} = kP$, for some constant $k > 0$,

so by the theorem, $P = 1000 e^{kt}$, since $P = 1000$ when $t = 0$.

When $t = 5$, $P = 3000$, so $3000 = 1000 e^{5k}$

$$e^{5k} = 3$$

$$5k = \log 3$$

$$k = \frac{1}{5} \log 3$$

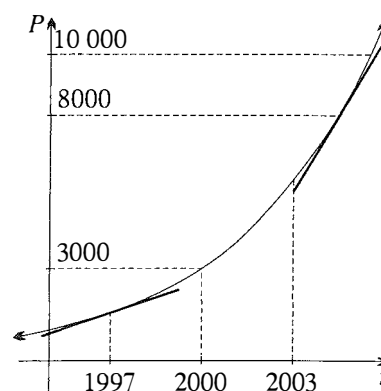
(approximate k and store it in the memory).

- When $t = 8$, $P = 1000 e^{8k}$
 $\doteq 5800$ rabbits.

- Substituting $P = 10\,000$, $10\,000 = 1000 e^{kt}$
 $e^{kt} = 10$
 $kt = \log 10$
 $t = \frac{\log 10}{k}$

$\doteq 10$ years and 6 months,

so the population will reach 10 000 about 6 months into 2005.



- 11** **NATURAL DECAY:** In situations of natural decay, let the constant of proportionality be $-k$, where k is a positive constant.

It is perfectly acceptable to omit the minus sign so that k is a negative constant, but the arithmetic of logarithms is easier if the minus sign is built in and k is a positive constant.

WORKED EXERCISE: A paddock has been contaminated with strontium-90, which has a half-life of 28 years (meaning that exactly half of any quantity of the isotope will decay in 28 years).

- Find the mass of strontium-90 as a function of time, and sketch the graph.
- Find what proportion of the radioactivity will remain after 100 years (answer correct to the nearest 0.1%).
- How long will it take for the radioactivity to drop to 0.001% of its original value (answer correct to the nearest year)?

SOLUTION:

- Let M be the quantity of the isotope at time t years.

Then $\frac{dM}{dt} = -kM$, for some positive constant k of proportionality,

so $M = M_0 e^{-kt}$, where M_0 is the quantity present at time $t = 0$.

When $t = 28$, $M = \frac{1}{2}M_0$, so $\frac{1}{2}M_0 = M_0 e^{-28k}$
 $e^{-28k} = \frac{1}{2}$.

Taking reciprocals,

$$e^{28k} = 2$$

$$28k = \log 2$$

$$k = \frac{1}{28} \log 2$$

(approximate k and store it in the memory).

- When $t = 100$, $M = M_0 e^{-100k}$
 $\doteq 0.084M_0$,

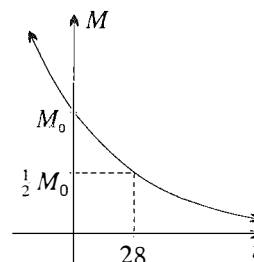
and so the radioactivity has dropped to about 8.4% of its original value.

- When $M = 10^{-5}M_0$, $10^{-5}M_0 = M_0 e^{-kt}$
 $e^{-kt} = 10^{-5}$

$$-kt = -5 \log 10$$

$$t = \frac{5 \log 10}{k}$$

$$\doteq 465 \text{ years.}$$



Exercise 13E

1. Given that $y = 5e^{\frac{1}{2}t}$, show that $\frac{dy}{dt} = \frac{1}{2}y$. Use these equations to solve the following.

- Find y to three significant figures when: (i) $t = 2$ (ii) $t = -3$ (iii) $t = 4.5$
- Find t to two decimal places when: (i) $y = 10$ (ii) $y = 1$ (iii) $y = 30$
- Find the exact value of $\frac{dy}{dt}$ when: (i) $y = 8$ (ii) $y = 11$ (iii) $y = \frac{1}{3}$
- Find $\frac{dy}{dt}$ to three decimal places when: (i) $t = 7$ (ii) $t = 3.194$ (iii) $t = \log 3$

2. In the following questions, give exact answers where appropriate or else approximate to four significant figures.
 - (a) Given that $y = 3e^{2t}$, show that $\frac{dy}{dt} = 2y$. Find y when $t = 1.5$.
 - (b) Given that $y = e^{-t}$, show that $\frac{dy}{dt} = -y$. Find t when $y = \frac{1}{2}$.
 - (c) Given that $y = 10e^{-\frac{1}{2}t}$, show that $\frac{dy}{dt} = -\frac{1}{2}y$. Find $\frac{dy}{dt}$ when $y = 6$.
 - (d) Given that $y = \sqrt{2}e^{\sqrt{2}t}$, show that $\frac{dy}{dt} = \sqrt{2}y$. Find $\frac{dy}{dt}$ when $t = \frac{1}{\sqrt{2}}$.
3. The population P of a city rose from 1 million at the beginning of 1975 to 2.5 million at the beginning of 1985. Assuming natural growth, $P = 10^6 \times e^{kt}$ where t is the time in years since the beginning of 1975.
 - (a) Find the value of the positive constant k , and sketch the curve.
 - (b) What was the population of the city at the beginning of 1998, to three significant figures?
 - (c) In what year is the population 10 million?
 - (d) Find the rate $\frac{dP}{dt}$ at which the population is increasing at the beginning of that year. Give your answer to the nearest thousand.
4. Ten kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount of undissolved sugar remaining is given by $S = 10e^{-kt}$.
 - (a) Calculate k , given that $S = 3.2$ when $t = 4$, and sketch the graph.
 - (b) At what time will there be 1 kg of sugar remaining?
 - (c) How fast is the sugar dissolving after 1 hour? Give your answer in units of kilograms per hour to the nearest half kilogram.
5. The value V of a factory machine depreciated with time t years such that $\frac{dV}{dt} = -kV$, for some constant $k > 0$.
 - (a) Show that $V = V_0 e^{-kt}$ satisfies the given differential equation.
 - (b) The initial value of a particular item of machinery is \$15 000. Explain why $V_0 = 15\,000$.
 - (c) In the first year the machine depreciates in value by 30%. Find the value of the constant k .
 - (d) The company that bought the machine writes off any machine when it has depreciated to 5% of its initial value. How many years does this take? Round your answer up to the nearest whole year.
6. In an experiment in which bacteria are grown on a petri dish, it is found that the area A in cm^2 covered by the bacteria increases from 0.5 cm^2 to 1 cm^2 in a period of 3 hours.
 - (a) Assuming that the area covered obeys the law of natural growth, show that A at time t hours after the initial observation is given by $A = \frac{1}{2}e^{kt}$.
 - (b) Find the value of the positive constant k .
 - (c) What area of the petri dish will be covered after 7 hours? Answer to three significant figures.
 - (d) If the diameter of the petri dish is 10 cm, how long will it take for the bacteria to cover the dish? Answer to the nearest 10 minutes.

7. When a liquid is placed in a refrigerator kept at 0°C , the rate at which it cools is proportional to its temperature h at time t , thus $\frac{dh}{dt} = -kh$ where k is a positive constant.
- Show that $h = h_0 e^{-kt}$ is a solution of the differential equation.
 - Find h_0 , given that the liquid is initially at 100°C .
 - After 5 minutes the temperature has dropped to 40°C . Find the value of k .
 - Find the temperature of the liquid after 15 minutes.
8. The height H of a wave decays away so that $H = H_0 e^{-\frac{1}{3}t}$, where H_0 is the initial height of the wave. Giving your answer to the nearest whole percent, what percentage of the initial height is the height of the wave when: (a) $t = 1$? (b) $t = 3$? (c) $t = 8$?

DEVELOPMENT

9. A quantity Q of radium at time t years is given by $Q = Q_0 e^{-kt}$, where k is a positive constant and Q_0 is the amount of radium at time $t = 0$.
- Given that $Q = \frac{1}{2}Q_0$ when $t = 1690$ years, calculate k .
 - After how many years does only 20% of the initial amount of radium remain? Give your answer to the nearest year.
10. The most efficient way of boiling water is to add heat at a rate proportional to the temperature H of the water, thus $\frac{dH}{dt} = kH$.
- Show $H = H_0 e^{kt}$ is a solution of the differential equation.
 - The water is initially at room temperature 20°C , and after $1\frac{1}{2}$ minutes reaches 30°C . Find the value of k .
 - Find how long, to the nearest second, it takes for the water to reach boiling point.
11. The amount A in grams of carbon-14 isotope in a dead tree trunk after t years is given by $A = A_0 e^{-kt}$, where A_0 and k are positive constants.
- Show that A satisfies the equation $\frac{dA}{dt} = -kA$.
 - The amount of isotope is halved every 5750 years. Find the value of k .
 - For a certain dead tree trunk the amount of isotope is only 15% of the original amount in the living tree. How long ago did the tree die (answer to the nearest 1000 years)?
12. Current research into Alzheimer's disease suggests that the rate of loss of percentage brain function is proportional to the percentage brain function already lost. That is, if L is the percentage brain function lost, then $\frac{dL}{dt} = kL$, for some constant $k > 0$.
- Two years ago a patient was initially diagnosed with Alzheimer's disease, with a 15% loss of brain function. This year the patient was diagnosed with 20% loss of brain function. Show that $L = 15 e^{kt}$, where $k = \frac{1}{2} \log \frac{4}{3}$.
 - The nearby care centre will admit patients to 24-hour nursing care when a patient reaches 60% loss of brain function. In how many more years will that be? Answer to the nearest year.
13. A wet porous substance loses moisture at a rate that is proportional to the moisture content M , that is $\frac{dM}{dt} = -kM$, where $k > 0$. On a particular day a wet towel on a clothes-line loses half its moisture in the first $1\frac{1}{2}$ hours.
- Show that $M = M_0 e^{-kt}$ is a solution of the differential equation.

- (b) Find the value of k .
- (c) How long, in total, will it take for the towel to become 99% dry? Answer to the nearest hour.
14. Air pressure P in millibars is a function of the altitude a in metres, with $\frac{dP}{da} = -\mu P$. The pressure at sea level is 1013.25 millibars.
- (a) Show that $P = 1013.25 e^{-\mu a}$ is a solution to this problem.
- (b) One reference book quotes the pressure at 1500 metres to be 845.6 millibars. Find the value of μ for the data in that book.
- (c) Another reference book quotes the pressure at 6000 metres to be half that at sea level. Find the value of μ in this case.
- (d) Are the data in the two books consistent?
- (e) Assuming the first book to be correct:
- (i) What is the pressure at 4000 metres?
- (ii) What is the pressure 1 km down a mine shaft?
- (iii) At what altitude is the pressure 100 millibars?
15. A certain radioactive isotope decays at such a rate that after 68 minutes only a quarter of the initial amount remains.
- (a) Find the half-life of this isotope.
- (b) What proportion of the initial amount will remain after 3 hours? Give your answer as a percentage to one decimal place.
16. The half-life of naturally occurring radioactive carbon C^{14} is 5730 years. In a living organism, the rate of C^{14} decay is typically constant at around 15.3 disintegrations per minute. When the organism dies the level of C^{14} decays away. Let C be the amount of radioactive carbon measured in disintegrations per minute at time t years.
- (a) Assuming natural decay, $C = C_0 e^{-kt}$. Find the values of k and C_0 .
- (b) In an archaeological dig, bones are found which exhibit 11 disintegrations per minute. How old are the bones, to the nearest year?
- (c) Carbon dating is only deemed reliable for ages between 1000 and 10 000 years. In another archaeological expedition, artefacts are found in a cave which exhibit 2.25 disintegrations per minute.
- (i) How old do these artefacts seem to be, to the nearest year?
- (ii) Is this figure reliable or should other tests be carried out to confirm the age of the artefacts?
17. In 1980 the population of Bedsworth was $B = 25\,000$ and the population of Yarra was $Y = 12\,500$. That year the mine was closed in Bedsworth and the population began falling, while the population of Yarra continued to grow, so that
- $$B = 25\,000 \times e^{-pt} \quad \text{and} \quad Y = 12\,500 \times e^{qt}.$$
- (a) Ten years later it was found the populations of the two towns were $B = 20\,000$ and $Y = 15\,000$. Find the values of p and q .
- (b) In what year were the populations of the two towns equal?

EXTENSION

18. Measurements of the radioactivity of isotopes of certain elements are used to determine the *radiometric age* t of a sample, which is an estimate of its actual age. The rate of decay is proportional to the amount P of the parent isotope, thus $\frac{dP}{dt} = -\lambda P$.
- If the sample originally had an amount P_0 of the parent isotope, show that $P = P_0 e^{-\lambda t}$ satisfies the given differential equation.
 - In a sample that has not been contaminated by outside sources, the sum of the amount P of parent isotope and amount D of daughter isotope must equal P_0 , thus $P_0 = P + D$. The ratio $\frac{D}{P}$ is easily measured; find a formula for the radiometric age in terms of this ratio.
 - In a sample of crystal mica, the ratio of daughter strontium Sr^{87} to parent rubidium Rb^{87} is 0.74%. The half-life of Rb^{87} is 47×10^9 years.
 - Find the value of λ to three significant figures.
 - Find the radiometric age of the sample to the nearest million years.
19. [Here we introduce the technique of *integrating factors* for solving differential equations. The quantity e^{kx} is known as the integrating factor.]
- Given that $z = e^{kx}y$, show that $\frac{dz}{dx} = e^{kx} \left(\frac{dy}{dx} + ky \right)$.
 - If $\frac{dy}{dx} + ky = 0$, use part (a) to show that $z = A$, where A is a constant.
 - Hence show that $y = A e^{-kx}$.
20. [Here we justify guessing a solution of the form $y = e^{kt}$ as a solution to the second-order differential equation $y'' + ay' + by = 0$.]
- Show that if λ is one solution of $k^2 + ak + b = 0$, then $\mu = -(\lambda + a)$ is the other solution.
 - In the next parts, assume that $\lambda \neq \mu$. Let $z = y' - \lambda y$. Find z' , and hence show that $y'' = z' + \lambda z + \lambda^2 y$.
 - Given that $y'' + ay' + by = 0$, show that $z' - \mu z = 0$.
 - Solve the above equation for z , and hence show that $y' - \lambda y = A e^{\mu t}$, where A is a constant.
 - Multiply the equation for y by $e^{-\lambda t}$, and hence show that $y = B e^{\mu t} + C e^{\lambda t}$, where B and C are constants.
 - How would the above change if $\lambda = \mu$?

The Trigonometric Functions

This chapter will extend the calculus to the sine and cosine functions and the other trigonometric functions. The sine and cosine functions are extremely important because their graphs are waves. Mathematically, they are the simplest wave forms, and every other wavy graph can be constructed from combinations of them. They are therefore essential in the modelling of all the many wave-like phenomena such as sound waves, light and radio waves, vibrating strings, tides and economic cycles.

In the last two chapters, we saw how e was the right number to use for the base of the exponential and logarithmic functions in calculus. In this chapter, a new measure of angles based on the number π will turn out to be appropriate for the calculus of the trigonometric functions. This is the reason why the irrational numbers e and π are so important in calculus.

STUDY NOTES: This chapter has two parts. The first part (Sections 14A–14F) leads towards the proof that the derivative of $\sin x$ is $\cos x$. On this basis, Sections 14G–14J then develop the differentiation and integration of the trigonometric functions, applying the derivative and the integral in the usual ways. Establishing that the derivative of $\sin x$ is $\cos x$ requires a connected body of theory to be developed. In Sections 14A and 14B, radian measure is introduced and applied to the mensuration of circles. This allows the six trigonometric functions to be sketched in their true form in Section 14C. In Section 14D the formulae for the trigonometric functions of compound angles are developed, and are applied in Section 14E to the angle between two lines. In Section 14F the fundamental limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ is proven using an appeal to geometry and the mensuration of the circle. Combining all this material finally allows the derivative of $\sin x$ to be established in Section 14G. In all study of the trigonometric functions, sketches of the graphs should be made whenever possible. Computers and other machines may help here in generating the graphs of a number of functions quickly, so that the effect of changing the formulae or the parameters can be discovered by experimentation.

14A Radian Measure of Angle Size

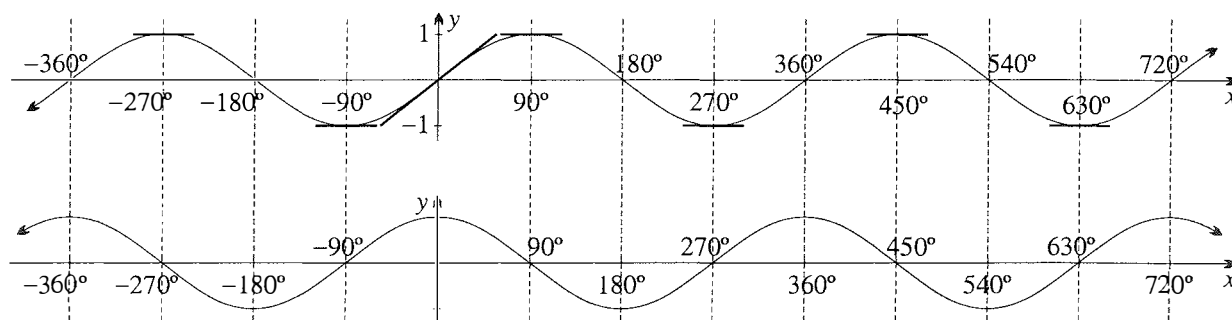
The use of degrees to measure angle size is based on astronomy, not on mathematics. There are 360 days in the year, to the nearest convenient number, so 1° is the angle through which our sun moves against the fixed stars each day, or (after Copernicus) the angle swept out by the Earth each day in its orbit around the sun. Mathematics is far too general a discipline to be tied to the particularities

of our solar system, so it is quite natural that we should pass to a new system for measuring angles once the mathematics becomes a little more sophisticated.

The Problem of How to Measure Angles: The need for a new system for measuring angles arises when one attempts to differentiate the trigonometric functions. The upper graph in the sketch below is $y = \sin x^\circ$. The lower graph is a rough sketch of the derivative of $y = \sin x^\circ$, paying attention to where the gradient of $y = \sin x^\circ$ is zero, maximum and minimum.

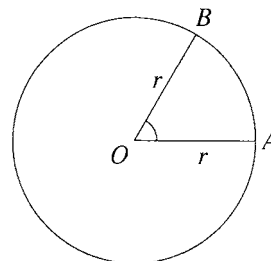
The lower graph seems unmistakably to be some sort of cosine graph, but it is not at all clear what the scale on its vertical axis should be. Most importantly, the y -intercept of the lower graph is the gradient of the upper graph at the origin. But gradients are not properly defined on the upper graph yet, — there are numbers on the vertical axis, but degrees on the horizontal axis, and there is no obvious way to set up the relationship between the scales on the two axes. If 1 unit on the x -axis were chosen to be 1 degree, then 90° would be placed about a metre off the page, making the upper graph very flat indeed. If one unit on the x -axis were chosen to be 90° , then the gradient of $\sin x$ at the origin would be somewhat steeper than 1.

We shall make the choice of units on the horizontal axis so that the gradient of $y = \sin x$ at the origin is exactly 1. The result will be the most convenient situation, because the derivative of $\sin x$ will turn out to be exactly $\cos x$, giving the simple formula $\frac{d}{dx} \sin x = \cos x$. A bit of experimentation stretching the graph horizontally shows that we need 90° to be about $1\frac{1}{2}$ units — the precise value will turn out to be $\frac{\pi}{2}$. This is the real purpose of the new units for measuring angles introduced in the next paragraph, and then developed over the next six sections in Sections 14A–14F.



Radian Measure: The new units for measuring angles are called *radians*. Their definition is purely mathematical, being the ratio of two lengths, and is thus very similar to the definitions of the trigonometric functions. Given an angle with vertex O , construct a circle with centre O meeting the two arms of the angle at A and B . Then:

1 **RADIAN MEASURE:** Size of $\angle AOB = \frac{\text{arc length } AB}{\text{radius } OA}$



Since the whole circumference is $2\pi r$, there must be 2π radians in a revolution, π radians in a straight angle, and $\frac{\pi}{2}$ radians in a right angle.

2

$$\text{CONVERSIONS: } 360^\circ = 2\pi, \quad 180^\circ = \pi, \quad 90^\circ = \frac{\pi}{2}$$

One radian is then the angle subtended at the centre of a circle by an arc of length equal to the radius. The sector OAB in the diagram must then be almost an equilateral triangle, and so 1 radian is about 60° . More precisely:

3

$$\text{MORE CONVERSIONS: } 1 \text{ radian} = \frac{180^\circ}{\pi} \div 57^\circ 18', \quad 1 \text{ degree} = \frac{\pi}{180} \div 0.017453$$

NOTE: The measure of an angle in radians is a ratio of lengths, and so is a dimensionless real number. The units are therefore normally omitted. For example, 'an angle of size 1.3' means an angle of 1.3 radians. Calculators set to the wrong mode routinely cause havoc at this point!

WORKED EXERCISE:

- (a) Express 60° , 495° and 37° in radians.
 (b) Express the angles $\frac{\pi}{6}$, $\frac{3\pi}{4}$ and 0.3 in degrees.
 (c) Evaluate $\cos \frac{\pi}{6}$, $\sec \frac{\pi}{4}$ and $\sin 1$ (to four decimal places).

SOLUTION:

$$\begin{aligned} \text{(a) } 60^\circ &= 60 \times \frac{\pi}{180} \\ &= \frac{\pi}{3} \\ 495^\circ &= 495 \times \frac{\pi}{180} \\ &= \frac{11\pi}{4} \\ 37^\circ &= 37 \times \frac{\pi}{180} \\ &= \frac{37\pi}{180} \end{aligned}$$

$$\begin{aligned} \text{(b) } \frac{\pi}{6} &= \frac{\pi}{6} \times \frac{180^\circ}{\pi} \\ &= 30^\circ \\ \frac{3\pi}{4} &= \frac{3\pi}{4} \times \frac{180^\circ}{\pi} \\ &= 135^\circ \\ 0.3 &= 0.3 \times \frac{180^\circ}{\pi} \\ &= \frac{54^\circ}{\pi} \end{aligned}$$

$$\begin{aligned} \text{(c) } \cos \frac{\pi}{6} &= \frac{1}{2}\sqrt{3} \\ &\quad (\text{notice } \frac{\pi}{6} = 30^\circ) \\ \sec \frac{\pi}{4} &= \sqrt{2} \\ &\quad (\text{notice } \frac{\pi}{4} = 45^\circ) \\ \sin 1 &\div 0.8415 \\ &\quad (\text{set calculator to radians}) \end{aligned}$$

Solving Trigonometric Equations in Radians: The steps for giving the solution in radians to a trigonometric equation are unchanged. First, establish the quadrants in which the angle can lie. Secondly, find the related angle, but use radian measure, not degrees.

WORKED EXERCISE:

- (a) Solve $\cos x = -\frac{1}{2}$, for $0 \leq x \leq 2\pi$.
 (b) Solve $\operatorname{cosec} x = -3$ correct to five significant figures, for $0 \leq x \leq 2\pi$.

SOLUTION:

$$\begin{aligned} \text{(a) Since } \cos x &= -\frac{1}{2}, \\ x &\text{ is in the second or third quadrant, and the related angle is } \frac{\pi}{3}. \\ \text{Hence } x &= \pi - \frac{\pi}{3} \text{ or } \pi + \frac{\pi}{3} \\ &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}. \end{aligned}$$

$$\begin{aligned} \text{(b) Since } \operatorname{cosec} x &= -3 \\ \sin x &= -\frac{1}{3}, \\ \text{so } x &\text{ is in the third or fourth quadrant, and the related angle is } \sin^{-1} \frac{1}{3}. \\ \text{Hence } x &= \pi + \sin^{-1} \frac{1}{3} \text{ or } 2\pi - \sin^{-1} \frac{1}{3} \\ &\div 3.4814 \text{ or } 5.9433. \end{aligned}$$

Exercise 14A

1. Express the following angles in radians as multiples of π :

- (a) 90° (c) 30° (e) 120° (g) 135° (i) 360° (k) 270°
 (b) 45° (d) 60° (f) 150° (h) 225° (j) 300° (l) 210°

2. Express the following angles in degrees:

- (a) π (d) $\frac{\pi}{2}$ (f) $\frac{\pi}{4}$ (h) $\frac{5\pi}{6}$ (j) $\frac{3\pi}{2}$ (l) $\frac{7\pi}{4}$
 (b) 2π (e) $\frac{\pi}{3}$ (g) $\frac{2\pi}{3}$ (i) $\frac{3\pi}{4}$ (k) $\frac{4\pi}{3}$ (m) $\frac{11\pi}{6}$
 (c) 4π

3. Use your calculator to express in radians correct to three decimal places:

- (a) 73° (b) 14° (c) 168° (d) $21^\circ 36'$ (e) $95^\circ 17'$ (f) $211^\circ 12'$

4. Use your calculator to express in degrees and minutes:

- (a) 2 radians (c) 1.44 radians (e) 3.1985 radians
 (b) 0.3 radians (d) 0.123 radians (f) 5.64792 radians

5. Use your calculator to evaluate correct to two decimal places:

- (a) $\sin 2$ (c) $\tan 3.21$ (e) $\sec 1.23$
 (b) $\cos 2.5$ (d) $\operatorname{cosec} 0.7$ (f) $\cot 5.482$

6. Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:

- (a) $\sin \frac{\pi}{3}$ (c) $\cos \frac{5\pi}{6}$ (e) $\tan \frac{3\pi}{4}$ (g) $\sin \frac{5\pi}{4}$
 (b) $\sin \frac{\pi}{4}$ (d) $\tan \frac{4\pi}{3}$ (f) $\cos \frac{5\pi}{3}$ (h) $\tan \frac{7\pi}{6}$

7. Solve for x over the domain $0 \leq x \leq 2\pi$:

- (a) $\sin x = \frac{1}{2}$ (d) $\sin x = 1$ (g) $\cos x + 1 = 0$
 (b) $\cos x = -\frac{1}{2}$ (e) $2 \cos x = \sqrt{3}$ (h) $\sqrt{2} \sin x + 1 = 0$
 (c) $\tan x = -1$ (f) $\sqrt{3} \tan x = 1$ (i) $\cot x = 1$

DEVELOPMENT

8. Express in radians as multiples of π :

- (a) 20° (c) 36° (e) 112.5° (g) $32^\circ 30'$
 (b) 22.5° (d) 100° (f) 252° (h) $65^\circ 45'$

9. Express in degrees:

- (a) $\frac{\pi}{12}$ (b) $\frac{2\pi}{5}$ (c) $\frac{20\pi}{9}$ (d) $\frac{11\pi}{8}$ (e) $\frac{17\pi}{10}$ (f) $\frac{23\pi}{15}$

10. Find: (a) the complement of $\frac{\pi}{6}$, (b) the supplement of $\frac{\pi}{6}$.

11. Two angles of a triangle are $\frac{\pi}{3}$ and $\frac{2\pi}{9}$. Find, in radians, the third angle.

12. Find, correct to three decimal places, the angle in radians through which:

- (a) the second hand of a clock turns in 7 seconds,
 (b) the hour hand of a clock turns between 6 am and 6.40 am.

13. If $f(x) = \sin x$, $g(x) = \cos 2x$ and $h(x) = \tan 3x$, find, correct to three significant figures:

- (a) $f(1) + g(1) + h(1)$ (b) $f(g(h(1)))$

14. (a) Copy and complete the table, giving values correct to three decimal places.

x	1	1.5	2
$\sin x$			

- (b) Hence use Simpson's rule with three function values to find $\int_1^2 \sin x \, dx$ (give your answer to one decimal place).

15. Simplify:

- (a) $\sin(\pi - x)$ (c) $\cos(\pi - x)$ (e) $\tan(\pi + x)$ (g) $\sec(\pi - x)$
 (b) $\sin(\frac{\pi}{2} - x)$ (d) $\cos(\pi + x)$ (f) $\tan(2\pi - x)$ (h) $\operatorname{cosec}(\frac{\pi}{2} - x)$

16. Find the exact value of:

- (a) $\cos(-\frac{\pi}{3})$ (c) $\tan \frac{29\pi}{6}$ (e) $\operatorname{cosec} \frac{3\pi}{2}$
 (b) $\sin \frac{11\pi}{4}$ (d) $\cot \frac{2\pi}{3}$ (f) $\sec \frac{5\pi}{4}$

17. Solve each of these equations for $0 \leq x \leq 2\pi$:

- (a) $\tan x = \sqrt{2} - 1$ (c) $\cos^2 x + \cos x = 0$ (e) $3 \sin^2 x = \cos^2 x$
 (b) $2 \sin x \cos x = \cos x$ (d) $2 \cos^2 x = \cos x + 1$ (f) $\tan^2 x + \tan x = \sqrt{3} \tan x + \sqrt{3}$

18. Solve for $-\pi \leq x \leq \pi$:

- (a) $\sin 2x = \frac{1}{2}$ (c) $\tan(x - \frac{\pi}{6}) = \sqrt{3}$ (e) $\operatorname{cosec}(x - \frac{3\pi}{4}) = 1$
 (b) $\cos 3x = -1$ (d) $\sec(x + \frac{\pi}{4}) = -\sqrt{2}$ (f) $\cot(x + \frac{5\pi}{6}) = \sqrt{3}$

19. Express $\frac{11\pi}{320}$ radians in degrees, minutes and seconds.

20. Express $25^\circ 21'$ in radians in terms of π .

21. The angles of a certain pentagon are in arithmetic progression, and the largest angle is double the smallest. Find, in radians, the size of each angle of the pentagon.

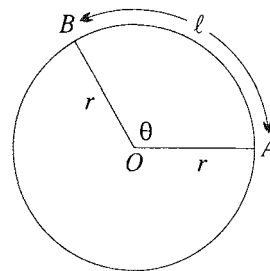
EXTENSION

22. (a) Explain why $\sin n \neq 0$, for all integers n . (b) Use your calculator to find the first positive integer n for which $|\sin n| < 0.01$, and explain your result.
 23. Given that $0 \leq \theta \leq 2\pi$, solve the equation $4 \cos^2 \theta + 2 \sin \theta = 3$ (give your solutions in terms of π).

14B Mensuration of Arcs, Sectors and Segments

The calculation of arc lengths and the areas of sectors and segments has always been possible, but radian measure allows the three formulae to be expressed in more elegant forms.

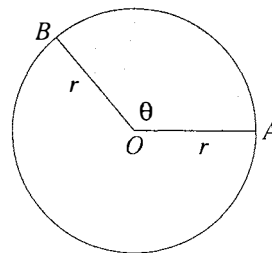
Arc Length: In the diagram on the right, AB is an arc of length ℓ subtending an angle θ at the centre O of a circle with radius r . The definition of angle size is $\theta = \frac{\ell}{r}$, which means that $\ell = r\theta$.



4 ARC LENGTH: $\ell = r\theta$

Area of a Sector: In the next diagram, the sector AOB is the shaded area bounded by the arc AB and the two radii OA and OB . Its area is a proportion of the total area:

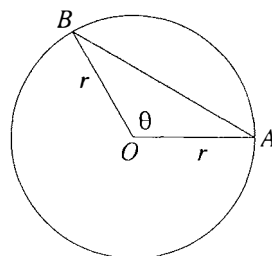
$$\begin{aligned}\text{area of sector} &= \frac{\theta}{2\pi} \times (\text{area of circle}) \\ &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{1}{2} r^2 \theta.\end{aligned}$$



5 **AREA OF SECTOR:** Area = $\frac{1}{2} r^2 \theta$

Area of a Segment: In the third diagram, the segment is the shaded area between the arc AB and the chord AB . Its area is the area of the sector OAB minus the area of the isosceles triangle $\triangle OAB$, whose area is given by the usual area formula for a triangle:

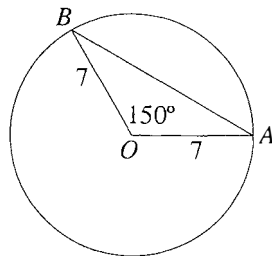
$$\begin{aligned}\text{area of segment} &= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta \\ &= \frac{1}{2} r^2 (\theta - \sin \theta).\end{aligned}$$



6 **AREA OF SEGMENT:** Area = $\frac{1}{2} r^2 (\theta - \sin \theta)$

WORKED EXERCISE:

- Find the lengths of the minor and major arcs formed by two radii of a circle of radius 7 metres meeting at 150° .
- Find the areas of the minor and major sectors.
- Find the length of the chord AB .
- Find the areas of the major and minor segments.



NOTE: An arc, sector or segment is called *minor* if the angle subtended at the centre is less than 180° , and *major* if the angle is a reflex angle. Obviously the angles subtended at the centre by a minor arc and the corresponding major arc add to 360° .

SOLUTION:

- The minor arc subtends 150° at the centre, which in radians is $\frac{5\pi}{6}$, and the major arc subtends an angle of 210° , which in radians is $\frac{7\pi}{6}$,

$$\begin{aligned}\text{so minor arc} &= r\theta & \text{and major arc} &= r\theta \\ &= 7 \times \frac{5\pi}{6} & &= 7 \times \frac{7\pi}{6} \\ &= \frac{35\pi}{6} \text{ metres.} & &= \frac{49\pi}{6} \text{ metres.}\end{aligned}$$

$$\begin{aligned}\text{(b) Minor sector} &= \frac{1}{2} r^2 \theta & \text{Major sector} &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2} \times 7^2 \times \frac{5\pi}{6} & &= \frac{1}{2} \times 7^2 \times \frac{7\pi}{6} \\ &= \frac{245\pi}{12} \text{ m}^2. & &= \frac{343\pi}{12} \text{ m}^2.\end{aligned}$$

- Using the cosine rule, $AB^2 = 7^2 + 7^2 - 2 \times 7 \times 7 \times \cos \frac{5\pi}{6}$
 $= 98(1 + \frac{1}{2}\sqrt{3})$
 $= 49(2 + \sqrt{3}),$

$$AB = 7\sqrt{2 + \sqrt{3}} \text{ metres.}$$

$$\begin{aligned}
 \text{(d) Minor segment} &= \frac{1}{2}r^2(\theta - \sin \theta) & \text{Major segment} &= \frac{1}{2}r^2(\theta - \sin \theta) \\
 &= \frac{1}{2} \times 49 \left(\frac{5\pi}{6} - \sin \frac{5\pi}{6} \right) & &= \frac{1}{2} \times 49 \left(\frac{7\pi}{6} - \sin \frac{7\pi}{6} \right) \\
 &= \frac{49}{2} \left(\frac{5\pi}{6} - \frac{1}{2} \right) & &= \frac{49}{2} \left(\frac{7\pi}{6} + \frac{1}{2} \right) \\
 &= \frac{49}{12} (5\pi - 3) \text{ m}^2. & &= \frac{49}{12} (7\pi + 3) \text{ m}^2.
 \end{aligned}$$

WORKED EXERCISE: Find, to the nearest millimetre, the radius of a circle in which:

- (a) a sector, (b) a segment,
subtending an angle of 90° at the centre has area 1 square metre.

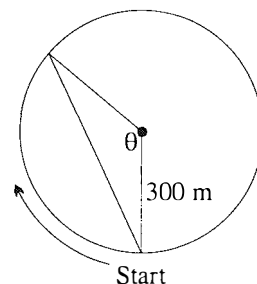
SOLUTION:

$$\begin{aligned}
 \text{(a) Area of sector} &= \frac{1}{2}r^2\theta & \text{(b) Area of segment} &= \frac{1}{2}r^2(\theta - \sin \theta) \\
 1 &= \frac{1}{2} \times r^2 \times \frac{\pi}{2} & 1 &= \frac{1}{2} \times r^2 \times \left(\frac{\pi}{2} - 1 \right) \\
 r^2 &= \frac{4}{\pi} & r^2 &= \frac{4}{\pi - 2} \\
 r &\doteq 1.128 \text{ metres.} & r &\doteq 1.872 \text{ metres.}
 \end{aligned}$$

WORKED EXERCISE: An athlete runs at a steady 4 m/s around a circular track of radius 300 metres. She runs clockwise, starting at the southernmost point on the track. (a) How far has she run after 10 minutes? (b) What total angle does her complete path subtend at the centre? (c) How far, in a direct line across the field, is she from her start? (d) What is then her bearing from the centre?

SOLUTION:

- (a) Distance run $= 4 \times 10 \times 60$
 $= 2400$ metres.
- (b) Substituting into $\ell = r\theta$,
 $2400 = 300\theta$
 $\theta = 8$ (which is about $458^\circ 22'$).
- (c) (Distance from start) $^2 = 300^2 + 300^2 - 2 \times 300^2 \times \cos 8$
 $= 300^2(2 - 2 \cos 8)$,
distance $= 300\sqrt{2 - 2 \cos 8}$
 $\doteq 454.08$ metres.
- (d) The original bearing was 180° T,
so final bearing $\doteq 180^\circ + 458^\circ 22'$
 $\doteq 278^\circ 22'$ T.



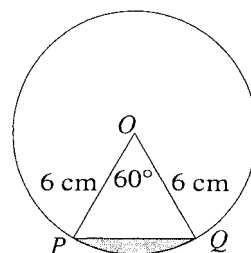
Exercise 14B

- A circle has radius 6 cm. Find the length of an arc of this circle that subtends an angle at the centre of: (a) 2 radians (b) $\frac{\pi}{3}$ radians
- A circle has radius 8 cm. Find the area of a sector of this circle that subtends an angle at the centre of: (a) 1 radian (b) $\frac{3\pi}{8}$ radians
- What is the radius of the circle in which an arc of length 10 cm subtends an angle of 2.5 radians at the centre?
- If the area of a sector of a circle of radius 4 cm is 12 cm^2 , find the angle at the centre in radians.

5. A circle has radius 3.4 cm. Find, correct to the nearest millimetre, the length of an arc of this circle that subtends an angle at the centre of: (a) 40° (b) $73^\circ 38'$
6. Find, correct to the nearest square metre, the area of a sector of a circle of radius 100 metres if the angle at the centre is 100° .
7. A circle has radius 12 cm. Find, in exact form:
 - (a) the length of an arc that subtends an angle of 120° at the centre,
 - (b) the area of a sector in which the angle at the centre is 40° .
8. An arc of a circle of radius 7.2 cm is 10.6 cm in length. Find the angle subtended at the centre by this arc, correct to the nearest degree.
9. A sector of a circle has area 52 cm^2 and contains an angle of $44^\circ 16'$. Find, in cm correct to one decimal place, the radius of the circle.

10. In the diagram on the right:

- (a) Find the exact area of sector OPQ .
- (b) Find the exact area of $\triangle OPQ$.
- (c) Hence find the exact area of the shaded minor segment.



11. A chord of a circle of radius 4 cm subtends an angle of 150° at the centre. Use the formula $A = \frac{1}{2}r^2(\theta - \sin \theta)$ to find the areas of the minor and major segments cut off by the chord.
12. A circle has centre C and radius 5 cm, and an arc AB of this circle has length 6 cm. Find the area of the sector ACB .

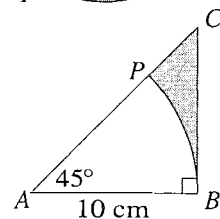
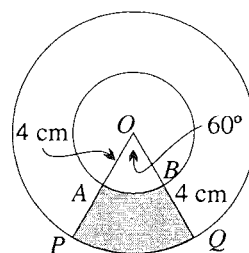
DEVELOPMENT

13. The diagram on the right shows two concentric circles with common centre O .

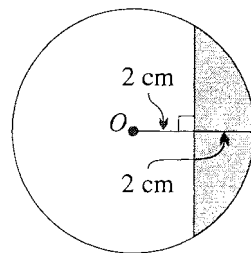
- (a) Find the exact perimeter of the region $APQB$.
- (b) Find the exact area of the region $APQB$.

14. In the diagram on the right, $\triangle ABC$ is a triangle that is right-angled at B , $AB = 10 \text{ cm}$ and $\angle A = 45^\circ$. The circular arc BP has centre A and radius AB . It meets the hypotenuse AC at P .

- (a) Find the exact area of sector ABP .
- (b) Hence find the exact area of the shaded portion BCP .



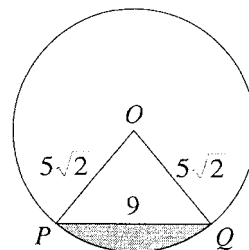
15. (a) Through how many radians does the minute hand of a watch turn between 7:10 am and 7:50 am?
- (b) If the minute hand is 1.2 cm long, find, to the nearest cm, the distance travelled by its tip in that time.
16. (a) A wheel on a fixed axle is rotating at 200 revolutions per minute. Convert this angular velocity into radians per second, giving your answer to the nearest whole number.
- (b) If the radius of the wheel is 0.3 metres, how far, to the nearest metre, will a point on the outside edge of the wheel travel in one second?



17. Find the exact area of the shaded region.

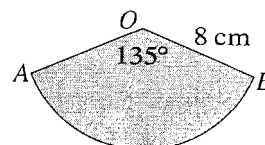
18. In a circle with centre O and radius $5\sqrt{2}$ cm, a chord of length 9 cm is drawn.

- (a) Use the cosine rule to find $\angle POQ$ in radians, correct to two decimal places.
 (b) Hence find, to the nearest cm^2 , the area of the minor segment cut off by the chord.



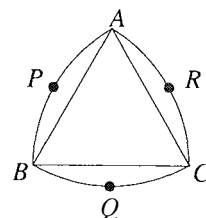
19. A piece of paper is in the shape of a sector of a circle. The radius is 8 cm and the angle at the centre is 135° . The straight edges of the sector are placed together so that a cone is formed.

- (a) Show that the base of the cone has radius 3 cm.
 (b) Show that the cone has perpendicular height $\sqrt{55}$ cm.
 (c) Hence find, in exact form, the volume of the cone.
 (d) Find the curved surface area of the cone.



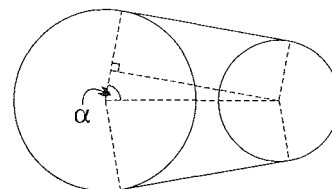
20. $\triangle ABC$ is equilateral and its side length is 2 cm. AB , BC and CA are circular arcs with centres C , A and B respectively.

- (a) Find the length of the arc AB .
 (b) Find the area of the sector $CAPBC$.
 (c) Find the length of the perimeter $APBQCR$.
 (d) Find the area of $\triangle ABC$ and hence find the area enclosed by the perimeter $APBQCR$. (Give all answers in exact form.)



21. The diameters of two circular pulleys are 6 cm and 12 cm and their centres are 10 cm apart.

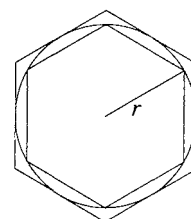
- (a) Calculate the angle α in radians, correct to four decimal places.
 (b) Hence find, in cm correct to one decimal place, the length of a taut belt that goes round the two pulleys.



22. OP and OQ are radii of length r centimetres of a circle centred at O . The arc PQ of the circle subtends an angle of θ radians at O and the perimeter of the sector OPQ is 12 cm.

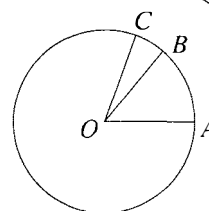
- (a) Show that the area $A \text{ cm}^2$ of the sector is given by $A = \frac{72\theta}{(2 + \theta)^2}$.
 (b) Hence find the maximum area of the sector.

23. Consider two regular hexagons drawn inside and outside a circle of radius r , as shown in the diagram opposite. By considering the perimeters of the circle and the two hexagons, show that $3 < \pi < 2\sqrt{3}$.

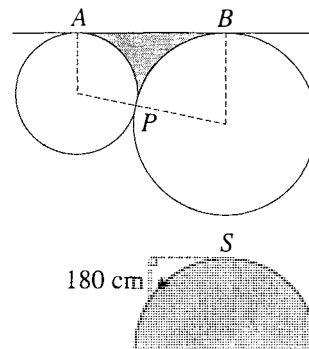


24. In the diagram to the right, O is the centre of the circle and the arc AB is equal in length to the radius of the circle. C is a point on the circle such that $\triangle OAC$ is equilateral.

- (a) By definition, what is the size of $\angle AOB$?
 (b) Explain why B must lie on the minor arc AC , and hence show geometrically that one radian is less than 60° .

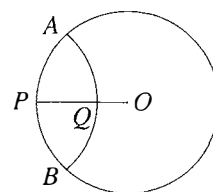


25. Two circles of radii 2 cm and 3 cm touch externally at P . AB is a common tangent. Calculate, in cm^2 correct to two decimal places, the area of the region bounded by the tangent and the arcs AP and BP .
26. A certain hill is represented by a hemisphere of radius 1 km. A man 180 cm tall walks down the hill from the summit S at 6 km/h. How long (to the nearest second) will it be before he is invisible to a person lying on the ground at S ?



EXTENSION

27. A right circular cone is to be constructed by cutting a major sector from a circular sheet of paper and joining the two straight edges. Show that the volume of the cone is maximised when the central angle of the sector is $\frac{2\sqrt{2}\pi}{\sqrt{3}}$ radians.
28. In the diagram, P is a point on a circle with centre O . A circular arc AQB , with centre P , is drawn so as to divide the area of the given circle in the ratio 1 : 2. If $\angle POA = \theta$, show that $\sin \theta + (\pi - \theta) \cos \theta = \frac{2\pi}{3}$. (You may need to use the result $\sin 2\theta = 2 \sin \theta \cos \theta$.)



14C Graphs of the Trigonometric Functions in Radians

Now that angle size has been defined as a ratio, that is, as a pure number, the trigonometric functions can be drawn in their true shape. On the next full page, the graphs of the six functions have been drawn using the same scale on the x -axis and y -axis. This means that the gradient of the tangent at each point now equals the true value of the derivative there, and the areas under the graphs faithfully represent the appropriate definite integrals.

Place a ruler on the graph of $y = \sin x$ to represent the gradient of the tangent at the origin. The ruler should lie along the line $y = x$, meaning that the gradient of the tangent is 1. As discussed earlier, this is necessary if the derivative of $y = \sin x$ is to be exactly $\cos x$. But all this needs to be proven properly, which will be done in later sections.

Amplitude of the Sine and Cosine Functions: The *amplitude* of a wave is the maximum height of the wave above the mean position. As we have seen, $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 , and a mean value of 0, so both have amplitude 1.

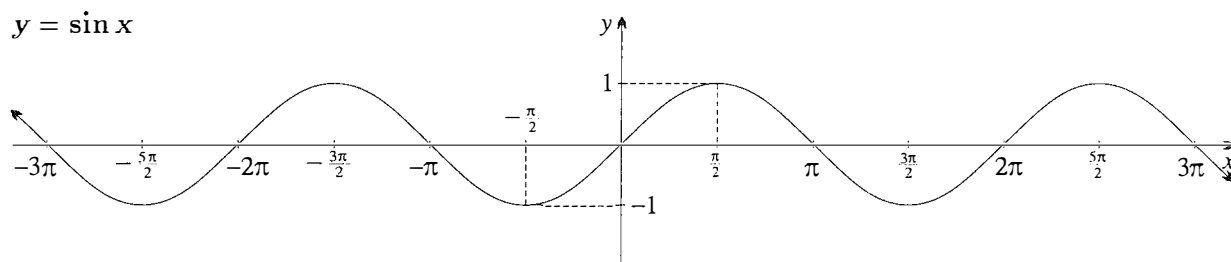
More generally, for positive constants a and b , the functions $y = a \sin bx$ and $y = a \cos bx$ have maximum value a and minimum value $-a$, so their amplitude is a .

7

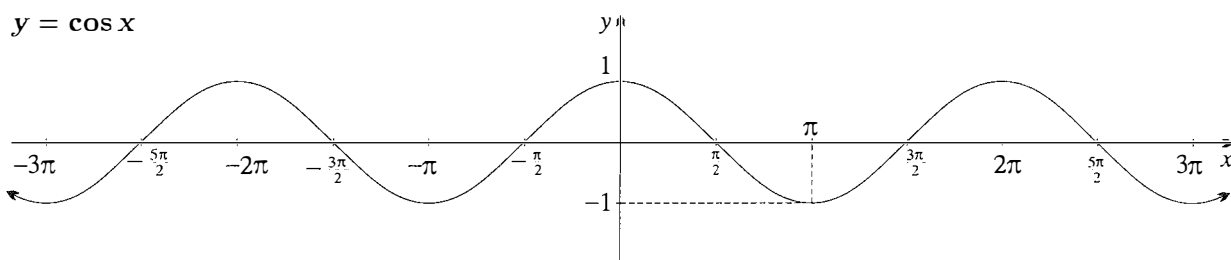
AMPLITUDE: The functions $y = \sin x$ and $y = \cos x$ have amplitude 1.
The functions $y = a \sin bx$ and $y = a \cos bx$ have amplitude a .

The other four trigonometric functions increase without bound near their asymptotes, and so the idea of amplitude makes no sense. The vertical scale of $y = a \tan bx$, however, can conveniently be tied down using the fact that $\tan \frac{\pi}{4} = 1$.

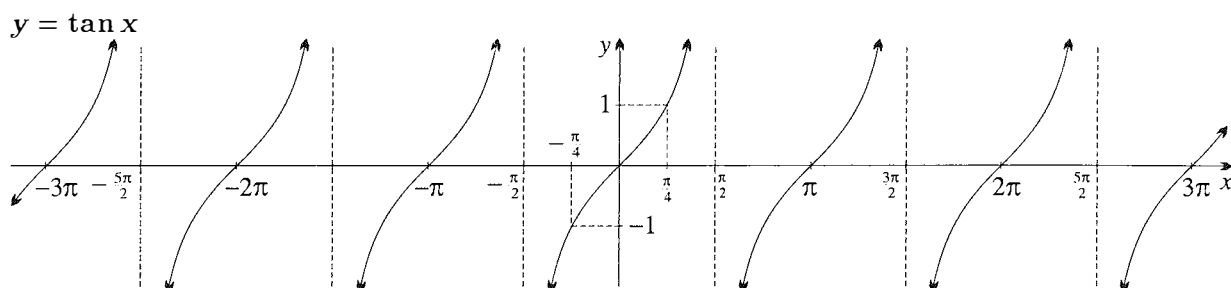
$$y = \sin x$$



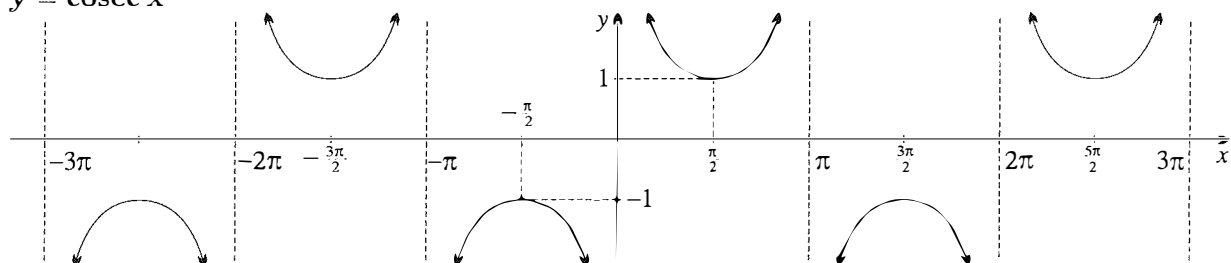
$$y = \cos x$$



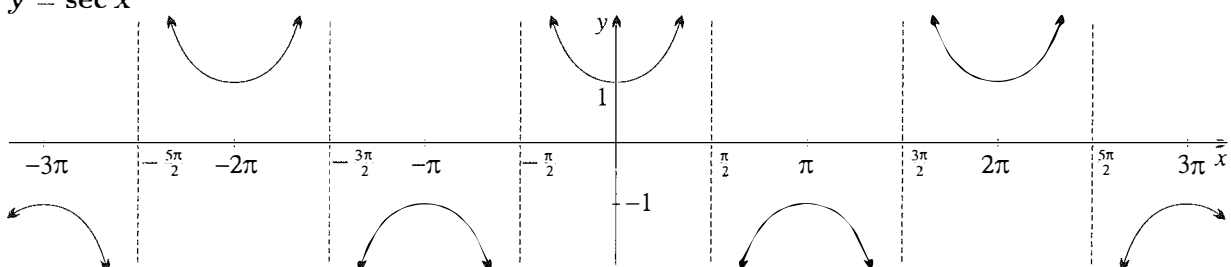
$$y = \tan x$$



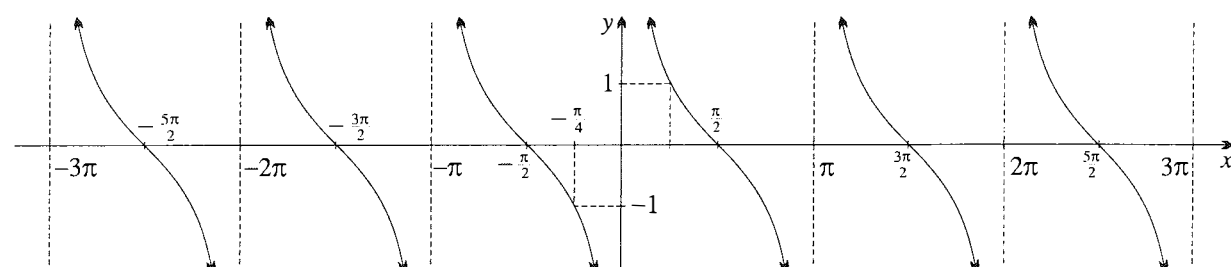
$$y = \operatorname{cosec} x$$



$$y = \sec x$$



$$y = \cot x$$



The Periods of the Trigonometric Functions: The six trigonometric functions are called *periodic functions* because each of them repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit. The sine and cosine curves on the previous page are waves that repeat every revolution, so they have period 2π . Their reciprocal functions secant and cosecant also have period 2π . The tangent and cotangent functions, on the other hand, repeat every half revolution, so they have period π .

For positive constants a and b , the function $y = a \sin bx$ is also periodic. As explained in Section 2I, replacing x by bx stretches the graph horizontally by a factor of $\frac{1}{b}$, so $y = a \sin bx$ must have period $\frac{2\pi}{b}$. The same applies to the other five trigonometric functions, and in summary:

THE PERIODS OF THE TRIGONOMETRIC FUNCTIONS:

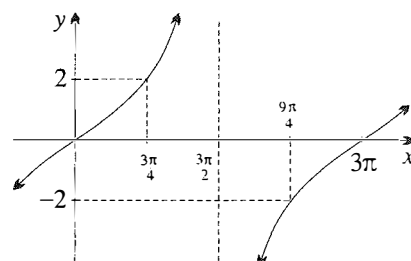
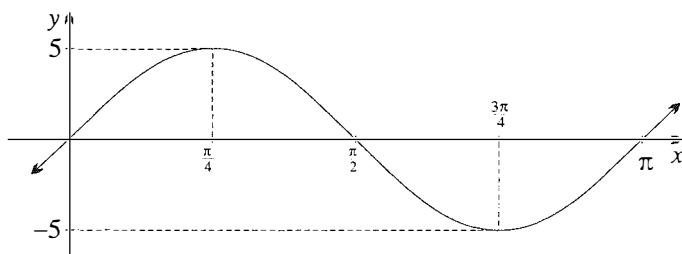
- 8 A. $y = \sin x$, $y = \cos x$, $y = \sec x$ and $y = \operatorname{cosec} x$ each have period 2π .
 $y = a \sin bx$, $y = a \cos bx$, $y = a \sec bx$ and $y = a \operatorname{cosec} bx$ have period $\frac{2\pi}{b}$.
 B. $y = \tan x$ and $y = \cot x$ each have period π .
 $y = a \tan bx$ and $y = a \cot bx$ each have period $\frac{\pi}{b}$.

WORKED EXERCISE: Sketch one period of: (a) $y = 5 \sin 2x$, (b) $y = 2 \tan \frac{1}{3}x$, showing all intercepts, turning points and asymptotes.

SOLUTION: 7 5

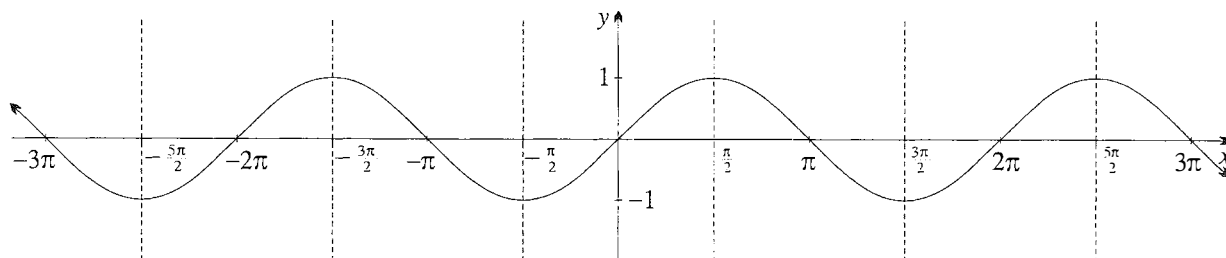
(a) $y = 5 \sin 2x$ has an amplitude of 5, and a period of $\frac{2\pi}{2} = \pi$.

(b) $y = 2 \tan \frac{1}{3}x$ has period $\frac{\pi}{1/3} = 3\pi$, and when $x = \frac{3\pi}{4}$, $y = 2$.



The Symmetries of the Sine Wave: The wavy graph of $y = \sin x$ has a great number of symmetries. Essentially one can build the whole curve by taking many copies of the rising part from $x = 0$ to $x = \frac{\pi}{2}$, and joining them all together in different orientations.

What follows is an account of these various transformations of the curve. When the symmetries are expressed algebraically, properties including oddness, the period, and the various identities known from the All Stations To Central diagram fall out as consequences. Below is another sketch of $y = \sin x$.



Rotations of the Sine Curve: The sine curve has rotational symmetry about the origin, meaning that a rotation of 180° about the origin maps the curve onto itself. That is, $y = \sin x$ is odd, which can be stated algebraically as

$$\sin(-x) = -\sin x.$$

The curve is also mapped onto itself under rotations of 180° about each of its x -intercepts.

Translations of the Sine Curve: The period of the sine curve is 2π , and the curve is stable under translations to the right or left by 2π or by integer multiples of 2π . The stability under translation to the left by 2π can be written algebraically as

$$\sin(x + 2\pi) = \sin x.$$

If the curve is translated to the left by only π , it is the same as reflecting the graph in the x -axis (taking opposites). This is represented algebraically by:

$$\sin(x + \pi) = -\sin x.$$

Translations to the right or left by odd multiples of π will all change $y = \sin x$ to $y = -\sin x$.

Reflections of the Sine Curve: The sine curve is mapped onto itself under reflections in every one of the dotted lines in the diagram above, that is in the vertical lines through any turning point of the wave. The symmetry in the vertical line $x = \frac{\pi}{2}$ is expressed algebraically by

$$\sin(\pi - x) = \sin x,$$

because any point x on the horizontal axis is reflected to $\pi - x$ by reflection in $x = \frac{\pi}{2}$. On the other hand, reflection in the vertical line through any x -intercept has the same effect as reflecting the curve in the x -axis (taking opposites). In particular, reflection in the vertical line $x = \pi$ is represented algebraically by

$$\sin(2\pi - x) = -\sin x.$$

Transforming the Sine Wave into the Cosine Wave: The sine and cosine waves are sketched below on one set of axes. Clearly they have exactly the same shape and size, or in geometric language are *congruent*. The most obvious way of moving one wave onto the other is by translation:

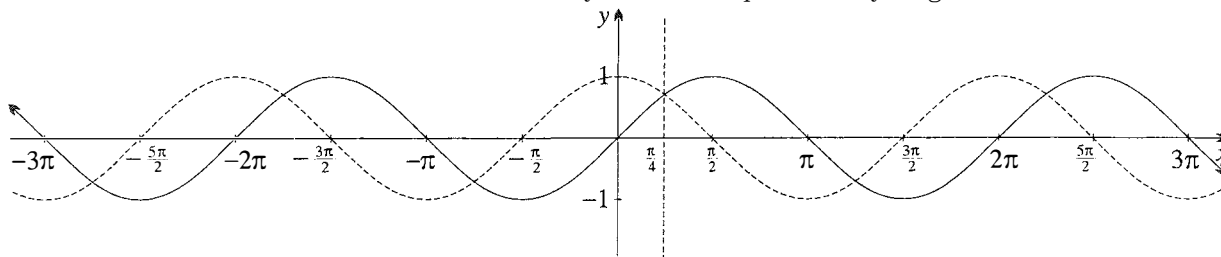
$$\cos(x - \frac{\pi}{2}) = \sin x \text{ translates the cosine graph to the right by } \frac{\pi}{2},$$

$$\sin(x + \frac{\pi}{2}) = \cos x \text{ translates the sine graph to the left by } \frac{\pi}{2}.$$

The sine and cosine curves can also be reflected into each other by reflection in the vertical line $x = \frac{\pi}{4}$, as is clear in the diagram below:

$$\sin(\frac{\pi}{2} - x) = \cos x \quad \text{or} \quad \cos(\frac{\pi}{2} - x) = \sin x.$$

This of course is the well-known identity about complementary angles.



Symmetries of the Other Trigonometric Functions: This rich variety of symmetry can be seen with each of the other five trigonometric functions, and the same treatment could well be given to each of them. One should not try to memorise these results. The intention is that they can easily be written down using knowledge of the various symmetries. Of particular interest are their evenness or oddness:

9

$$\begin{array}{lll} \sin(-x) = -\sin x & \cos(-x) = \cos x & \tan(-x) = -\tan x \\ \operatorname{cosec}(-x) = -\operatorname{cosec} x & \sec(-x) = \sec x & \cot(-x) = -\cot x \end{array}$$

their periods, associated with translations to the left by 2π or π :

10

$$\begin{array}{lll} \sin(x + 2\pi) = \sin x & \cos(x + 2\pi) = \cos x & \tan(x + \pi) = \tan x \\ \operatorname{cosec}(x + 2\pi) = \operatorname{cosec} x & \sec(x + 2\pi) = \sec x & \cot(x + \pi) = \cot x \end{array}$$

the identities associated with the All Stations To Central diagram:

11

$$\begin{array}{lll} \sin(\pi - x) = \sin x & \cos(\pi - x) = -\cos x & \tan(\pi - x) = -\tan x \\ \sin(\pi + x) = -\sin x & \cos(\pi + x) = -\cos x & \tan(\pi + x) = \tan x \\ \sin(2\pi - x) = -\sin x & \cos(2\pi - x) = \cos x & \tan(2\pi - x) = -\tan x \\ \operatorname{cosec}(\pi - x) = \operatorname{cosec} x & \sec(\pi - x) = -\sec x & \cot(\pi - x) = -\cot x \\ \operatorname{cosec}(\pi + x) = -\operatorname{cosec} x & \sec(\pi + x) = -\sec x & \cot(\pi + x) = \cot x \\ \operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x & \sec(2\pi - x) = \sec x & \cot(2\pi - x) = -\cot x \end{array}$$

and the three identities associated with reflection in the vertical line $x = \frac{\pi}{4}$, identities that involve exchanging an angle with its complement, and are the origin of the names of the three co-functions:

12

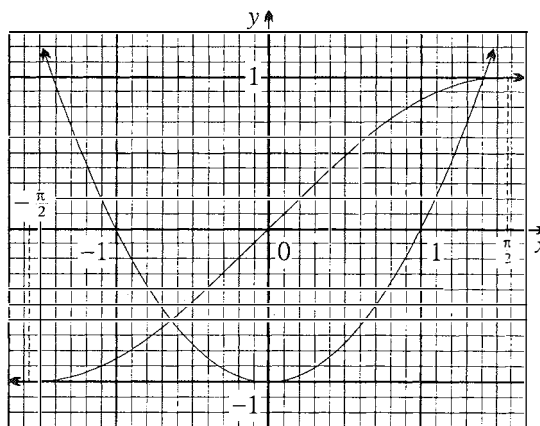
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \tan\left(\frac{\pi}{2} - x\right) = \cot x \quad \sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$

Graphical Solutions of Trigonometric Equations: Many trigonometric equations cannot be solved by algebraic methods. As usual, a graph-paper sketch will show where the solutions are, and their approximate values.

WORKED EXERCISE: Find, by drawing a graph, the number of solutions to $\sin x = x^2 - 1$. Then use the graph to find approximations correct to one decimal place.

SOLUTION: Here are $y = \sin x$ and $y = x^2 - 1$. Clearly the equation has two solutions. The positive solution is $x \doteq 1.4$, and the negative solution is $x \doteq -0.6$.

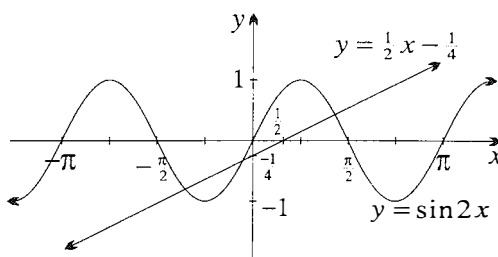
NOTE: Graphics calculators and computer packages are particularly useful here. They allow sketches to be drawn quickly, and many programs will give the approximate coordinates of the intersections.



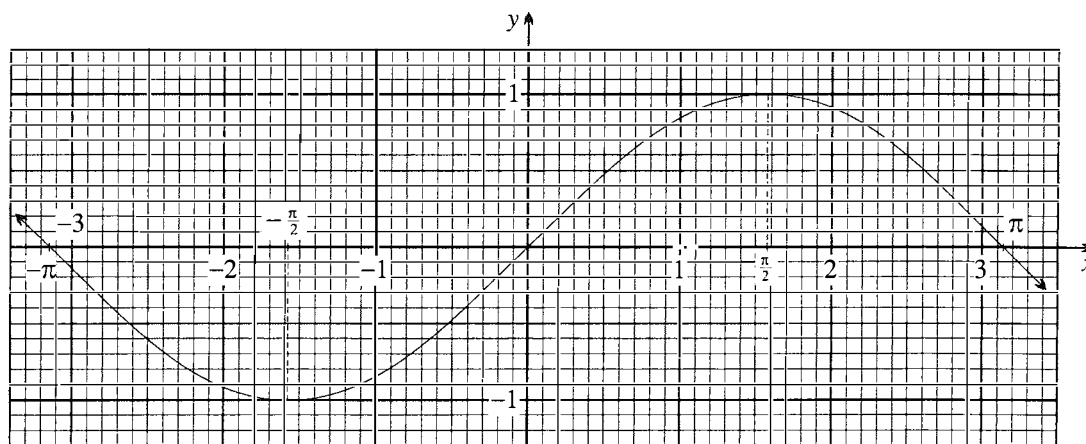
Exercise 14C

NOTE: Machine sketching would allow experience of many more similar examples, and may help elucidate the importance of period and amplitude.

- Sketch on separate diagrams the graphs of the following functions for $-2\pi \leq x \leq 2\pi$. State the period in each case.
 - $y = \sin x$
 - $y = \cos x$
 - $y = \tan x$
- On the same diagram, sketch each of the following functions for $0 \leq \theta \leq 2\pi$:
 - $y = \sin \theta$
 - $y = 2 \sin \theta$
 - $y = 4 \sin \theta$
- On the same diagram, sketch each of the following functions for $0 \leq \alpha \leq 2\pi$:
 - $y = \cos \alpha$
 - $y = \cos 2\alpha$
 - $y = \cos 4\alpha$
- On the same diagram, sketch each of the following functions for $0 \leq t \leq 2\pi$:
 - $y = \cos t$
 - $y = \cos(t - \pi)$
 - $y = \cos(t - \frac{\pi}{4})$
- State the period and amplitude, then sketch on separate diagrams for $0 \leq x \leq 2\pi$:
 - $y = \sin 2x$
 - $y = 2 \cos 2x$
 - $y = 4 \sin 3x$
 - $y = 3 \cos \frac{1}{2}x$
 - $y = \tan 2x$
- Sketch $y = \sin 2\pi x$ for $-1 \leq x \leq 2$.
- (a) Sketch $y = \sin x$ for $-\pi \leq x \leq \pi$. (b) On the same diagram sketch $y = \sin(x + \frac{\pi}{2})$ for $-\pi \leq x \leq \pi$. (c) Hence simplify $\sin(x + \frac{\pi}{2})$.
- In the given diagram, the curve $y = \sin 2x$ is graphed for $-\pi \leq x \leq \pi$ and the line $y = \frac{1}{2}x - \frac{1}{4}$ is graphed.
 - In how many points does the line $y = \frac{1}{2}x - \frac{1}{4}$ meet the curve $y = \sin 2x$?
 - State the number of solutions of the equation $\sin 2x = \frac{1}{2}x - \frac{1}{4}$. How many of these solutions are positive?
 - Briefly explain why the line $y = \frac{1}{2}x - \frac{1}{4}$ will not meet the curve $y = \sin 2x$ outside the domain $-\pi \leq x \leq \pi$.



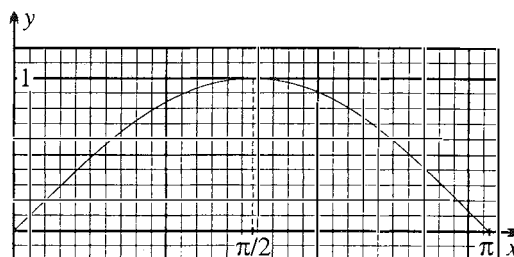
9.



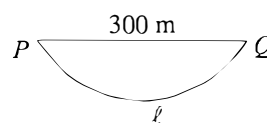
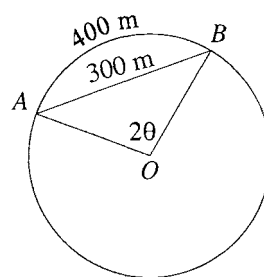
Photocopy the above graph of $y = \sin x$ for $-\pi \leq x \leq \pi$, and on it graph the line $y = \frac{1}{2}x$. Hence find the three solutions of the equation $\sin x = \frac{1}{2}x$, giving answers to one decimal place where necessary.

DEVELOPMENT

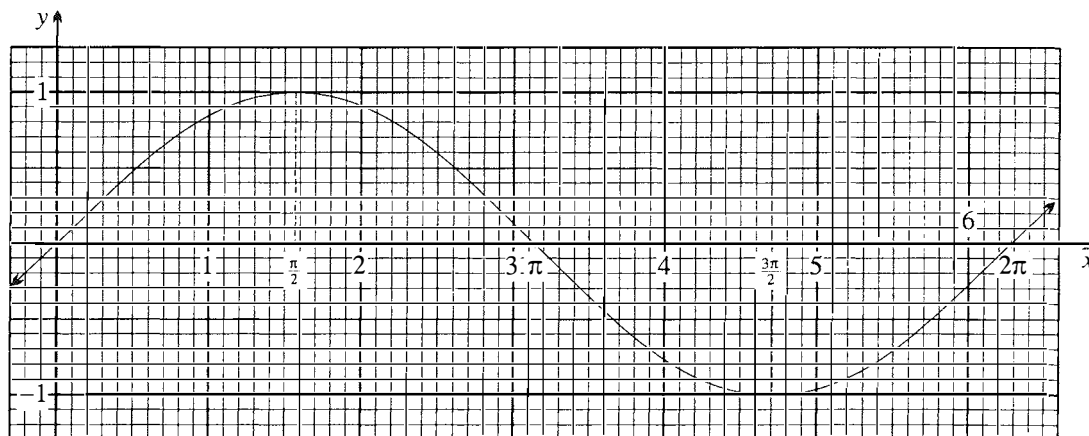
10. (a) Sketch the curve $y = \cos 3x$ for $-\pi \leq x \leq \pi$.
 (b) Hence sketch, on the same diagram, $y = 4 \cos 3(x - \frac{\pi}{6})$ for $-\pi \leq x \leq \pi$.
11. Sketch the curve $y = 3 - \cos 2x$ for $0 \leq x \leq 2\pi$.
12. Sketch on separate diagrams for $0 \leq \alpha \leq 2\pi$:
 (a) $y = \operatorname{cosec} 2\alpha$ (b) $y = 3 \sec \frac{1}{2}\alpha$ (c) $y = \cot 2\alpha$
13. (a) Carefully sketch the curve $y = \sin^2 x$ for $0 \leq x \leq 2\pi$ after completing the table.
 (b) Explain why $y = \sin^2 x$ has range $0 \leq y \leq 1$.
 (c) Write down the period and amplitude of $y = \sin^2 x$.
- | | | | | | |
|-----|---|-----------------|-----------------|------------------|-------|
| x | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | π |
| y | | | | | |
14. (a) Sketch the graph of $y = 2 \cos x$ for $-2\pi \leq x \leq 2\pi$.
 (b) On the same diagram, carefully sketch the line $y = 1 - \frac{1}{2}x$, showing its x - and y -intercepts.
 (c) How many solutions does the equation $2 \cos x = 1 - \frac{1}{2}x$ have?
 (d) Mark with the letter P the point on the diagram from which the negative solution of the equation in (c) is obtained.
 (e) Prove algebraically that if n is a solution of the equation in (c), then $-2 \leq n \leq 6$.
15. (a) What is the period of the function $y = \sin \frac{\pi}{2}x$?
 (b) Sketch the curve $y = 1 + \sin \frac{\pi}{2}x$ for $0 \leq x \leq 4$.
 (c) Through what fixed point does the line $y = mx$ always pass for varying values of m ?
 (d) By considering possible points of intersection of the graphs of $y = 1 + \sin \frac{\pi}{2}x$ and $y = mx$, find the range of values of m for which the equation $\sin \frac{\pi}{2}x = mx - 1$ has exactly one real solution in the domain $0 \leq x \leq 4$.
16. (a) Sketch the curve $y = 2 \cos 2x$ for $0 \leq x \leq 2\pi$.
 (b) Sketch the line $y = 1$ on the same diagram.
 (c) How many solutions does the equation $\cos 2x = \frac{1}{2}$ have in the domain $0 \leq x \leq 2\pi$?
 (d) What is the first positive solution to $\cos 2x = \frac{1}{2}$?
 (e) Use your diagram to help you find the values of x in the domain $0 \leq x \leq 2\pi$ for which $\cos 2x < \frac{1}{2}$.
17. (a) On the same diagram, sketch $y = \sin x$ and $y = \cos x$, for $0 \leq x \leq 2\pi$.
 (b) Hence, on the same diagram, carefully sketch $y = \sin x + \cos x$ for $0 \leq x \leq 2\pi$.
 (c) What is the period of $y = \sin x + \cos x$?
 (d) Estimate the amplitude of $y = \sin x + \cos x$ to one decimal place.
18. (a) Sketch $y = 3 \sin 2x$ and $y = 4 \cos 2x$ on the same diagram for $-\pi \leq x \leq \pi$.
 (b) Hence sketch the graph of $y = 3 \sin 2x - 4 \cos 2x$ on the same diagram, for $-\pi \leq x \leq \pi$.
 (c) Estimate the amplitude of the graph in (b).
19. (a) (i) Photocopy this graph of $y = \sin x$ for $0 \leq x \leq \pi$, and on it graph the line $y = \frac{3}{4}x$.
 (ii) Measure the gradient of $y = \sin x$ at the origin. Then use calculus to find that gradient.



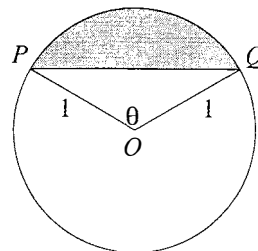
- (iii) For what values of k does $\sin x = kx$ have a solution for $0 < x < \pi$?
- (b) The diagram shows points A and B on a circle with centre O . $\angle AOB = 2\theta$, chord AB is of length 300 metres and the minor arc AB is of length 400 metres.
- Show that $\sin \theta = \frac{3}{4}\theta$.
 - Use the graph from (a)(i) to determine θ correct to one decimal place.
 - Hence find $\angle AOB$ in radians, correct to one decimal place, and show that the radius of the circle is about 154 metres.
- (c) P and Q are two points 300 metres apart. The circular arc PQ is of length ℓ metres.
- If C is the centre of the arc and $\angle PCQ = 2\alpha$, show that $\sin \alpha = \frac{300\alpha}{\ell}$.
 - Use your answer to (a)(iii) to find the possible range of values of ℓ .



20.



- Photocopy the above graph of $y = \sin x$ for $0 \leq x \leq 2\pi$, and on it carefully graph the line $y = x - 2$.
 - Use your graph to estimate, to two decimal places, the value of x for which $\sin x = x - 2$.
- The diagram shows points P and Q on a circle with centre O whose radius is one unit. $\angle POQ = \theta$. If the area of the shaded segment is one square unit, use part (a) to find θ correct to the nearest degree.
- Suppose instead that the area of the segment is 2 square units.
 - Show that $\sin \theta = \theta - 4$.
 - By drawing a suitable line on the graph in part (a), find θ to the nearest degree.



21. Show graphically that $x^2 - 2x + 4 > 3 \sin x$ for all real values of x .

22. Sketch $y = \cos x$ and then answer these questions:

- Give the equations of all axes of symmetry. ($x = -2\pi$ to 2π will do.)
- Around which points does the graph have rotational symmetry?

- (c) What translations will leave the graph unchanged?
- (d) Describe two translations that will move the graph of $y = \cos x$ to $y = -\cos x$.
- (e) Describe two translations that will move the graph of $y = \cos x$ to the graph of $y = \sin x$.
- (f) Name two vertical lines such that reflection of $y = \cos x$ in either of these lines will reflect the graph into the graph of $y = \sin x$.
- 23.** Sketch $y = \tan x$ and then answer these questions:
- (a) Explain whether $y = \tan x$ has any axes of symmetry.
- (b) Name the points about which $y = \tan x$ has rotational symmetry.
- (c) What translations will leave the graph unchanged?
- (d) Name two vertical lines such that reflection of $y = \tan x$ in either of these lines will reflect the graph into the graph of $y = \cot x$.

EXTENSION

- 24.** (a) Use a graphical approach to determine the number of positive solutions of $\sin x = \frac{x}{200}$.
- (b) Find a positive integer value of n such that $\sin x = \frac{x}{n}$ has 69 positive solutions.

14D Trigonometric Functions of Compound Angles

Before the differentiation of $\sin x$ at the origin can be extended to differentiation at all points on the curve, we will need to develop formulae for expanding objects like $\sin(x + h)$, $\tan(x - y)$, $\cos 2x$ and so forth. These trigonometric identities are most important for all sorts of other reasons as well, and must be thoroughly memorised.

As with all fundamental results in trigonometry, these formulae require an appeal to geometry, in this case Pythagoras' theorem in the form of the distance formula, the cosine rule, and the Pythagorean trigonometric identity. The approach given here begins with the expansion of $\cos(\alpha - \beta)$ and uses that result to derive the other expansions. There are many alternative approaches.

The Expansion of $\cos(\alpha - \beta)$: We shall prove that for all real numbers α and β ,

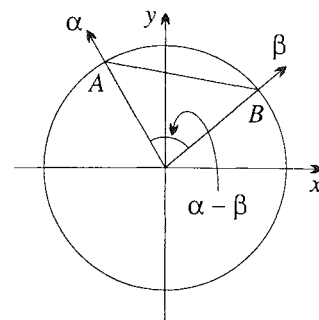
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

PROOF: Let the rays corresponding to the angles α and β intersect the circle $x^2 + y^2 = r^2$ at the points A and B respectively. Then by the definitions of the trigonometric functions for general angles,

$$A = (r \cos \alpha, r \sin \alpha) \quad \text{and} \quad B = (r \cos \beta, r \sin \beta).$$

Now we can use the distance formula to find AB^2 :

$$\begin{aligned} AB^2 &= r^2(\cos \alpha - \cos \beta)^2 + r^2(\sin \alpha - \sin \beta)^2 \\ &= r^2(\cos^2 \alpha + \cos^2 \beta + \sin^2 \alpha + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta) \\ &= 2r^2(1 - \cos \alpha \cos \beta - \sin \alpha \sin \beta), \quad \text{since } \sin^2 \theta + \cos^2 \theta = 1. \end{aligned}$$



But the angle $\angle AOB$ is $\alpha - \beta$, and so by the cosine rule,

$$\begin{aligned} AB^2 &= r^2 + r^2 - 2r^2 \cos(\alpha - \beta) \\ &= 2r^2(1 - \cos(\alpha - \beta)). \end{aligned}$$

Equating these two expressions for AB^2 ,

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

NOTE: It was claimed in the proof that $\angle AOB = \alpha - \beta$. This is not necessarily the case, because it's also possible that $\angle AOB = \beta - \alpha$, or that $\angle AOB$ differs from either of these two values by a multiple of 2π . But the cosine function is even, and it is periodic with period 2π . So it will still follow in every case that $\cos \angle AOB = \cos(\alpha - \beta)$, which is all that is required in the proof.

The Six Compound Angle Formulae: Here are the compound angle formulae for the sine, cosine and tangent functions, followed by the remaining five proofs.

THE COMPOUND ANGLE FORMULAE:

- 13
- A. $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
 - B. $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
 - C. $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
 - D. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
 - E. $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
 - F. $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

PROOF: We proceed from formula E, which has already been proven.

- B. Replacing β by $-\beta$ in E, which is the expansion of $\cos(\alpha - \beta)$,
- $$\begin{aligned} \cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) \\ &= \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta) \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta, \text{ since cosine is even and sine is odd.} \end{aligned}$$

- A. Using the identity $\sin \theta = \cos(\frac{\pi}{2} - \theta)$,
- $$\begin{aligned} \sin(\alpha + \beta) &= \cos(\frac{\pi}{2} - (\alpha + \beta)) \\ &= \cos((\frac{\pi}{2} - \alpha) - \beta) \\ &= \cos(\frac{\pi}{2} - \alpha) \cos \beta + \sin(\frac{\pi}{2} - \alpha) \sin \beta \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta. \end{aligned}$$

- D. Replacing β by $-\beta$, and noting that cosine is even and sine is odd,
- $$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

- C. Since $\tan \theta$ is the ratio of $\sin \theta$ and $\cos \theta$,

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\ &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}, \text{ dividing top and bottom by } \cos \alpha \cos \beta. \end{aligned}$$

F. Replacing β by $-\beta$, and noting that the tangent function is odd,

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$

WORKED EXERCISE: Express $\sin(x + \frac{2\pi}{3})$ in the form $a \cos x + b \sin x$.

$$\begin{aligned}\text{SOLUTION: } \sin(x + \frac{2\pi}{3}) &= \sin x \cos \frac{2\pi}{3} + \cos x \sin \frac{2\pi}{3} \\ &= -\frac{1}{2} \sin x + \frac{1}{2} \sqrt{3} \cos x\end{aligned}$$

WORKED EXERCISE: Given that $\sin \alpha = \frac{1}{3}$ and $\cos \beta = \frac{4}{5}$, where α is acute and $-\frac{\pi}{2} < \beta < 0$, find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

SOLUTION: First, using the diagrams on the right,

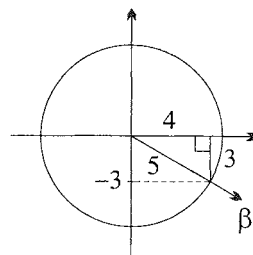
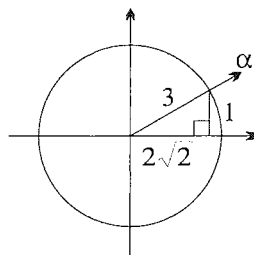
$$\cos \alpha = \frac{2}{3} \sqrt{2} \text{ and } \sin \beta = -\frac{3}{5}.$$

$$\text{So } \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\begin{aligned}&= \frac{1}{3} + \frac{2}{3} \sqrt{2} \\ &= \frac{1}{3}(2 + 3\sqrt{2}),\end{aligned}$$

$$\text{and } \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\begin{aligned}&= \frac{2}{3} \sqrt{2} + \frac{1}{5} \\ &= \frac{1}{15}(8\sqrt{2} + 3).\end{aligned}$$



The Double Angle Formulae: Replacing both α and β by θ in the compound angle formula $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ gives

$$\sin 2\theta = \sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta.$$

The same process gives expansions of $\cos 2\theta$ and $\tan 2\theta$.

14

$$\begin{aligned}\text{THE DOUBLE ANGLE FORMULAE: } \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

The expansion of $\cos 2\theta$ can then be combined with the Pythagorean identity to give two other forms of the expansion, first by using $\sin^2 \theta = 1 - \cos^2 \theta$, then by using $\cos^2 \theta = 1 - \sin^2 \theta$.

15

$$\begin{aligned}\text{THE } \cos 2\theta \text{ FORMULAE: } \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta\end{aligned}$$

WORKED EXERCISE: Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

$$\begin{aligned}\text{SOLUTION: LHS} &= (\sin x + \cos x)^2 \\ &= \sin^2 x + \cos^2 x + 2 \sin x \cos x \\ &= 1 + \sin 2x \\ &= \text{RHS.}\end{aligned}$$

WORKED EXERCISE: Express $\cot 2x$ in terms of $\cot x$.

$$\begin{aligned}
 \text{SOLUTION: } \cot 2x &= \frac{1}{\tan 2x} \\
 &= \frac{1 - \tan^2 x}{2 \tan x} \\
 &= \frac{\cot^2 x - 1}{2 \cot x}, \text{ after division of top and bottom by } \tan^2 x.
 \end{aligned}$$

Further Exact Values of Trigonometric Functions: The various compound angle formulae can be used to find expressions in surds for trigonometric functions of many angles other than ones whose related angles are the standard 30° , 45° and 60° .

WORKED EXERCISE: Find exact values of: (a) $\sin 75^\circ$ (b) $\cos 75^\circ$ (c) $\tan 22\frac{1}{2}^\circ$

SOLUTION: [There are many alternative methods.]

$$\begin{aligned}
 \text{(a) } \sin 75^\circ &= \sin(30^\circ + 45^\circ) \\
 &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \\
 &= \frac{1}{2} \times \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} \times \frac{1}{2}\sqrt{2} \\
 &= \frac{1}{4}(\sqrt{2} + \sqrt{6}) \\
 \text{(b) } \cos 75^\circ &= \cos(135^\circ - 60^\circ) \\
 &= \cos 135^\circ \cos 60^\circ + \sin 135^\circ \sin 60^\circ \\
 &= -\frac{1}{2}\sqrt{2} \times \frac{1}{2} + \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} \\
 &= \frac{1}{4}(\sqrt{6} - \sqrt{2}) \\
 \text{(c) } \tan 45^\circ &= \frac{2 \tan 22\frac{1}{2}^\circ}{1 - \tan^2 22\frac{1}{2}^\circ} \\
 1 \times (1 - \tan^2 22\frac{1}{2}^\circ) &= 2 \tan 22\frac{1}{2}^\circ \\
 \tan^2 22\frac{1}{2}^\circ + 2 \tan 22\frac{1}{2}^\circ - 1 &= 0. \\
 \Delta &= 4 + 4 \\
 &= 4 \times 2 \\
 \tan 22\frac{1}{2}^\circ &= \frac{-2 + 2\sqrt{2}}{2} \text{ or } \frac{-2 - 2\sqrt{2}}{2} \\
 &= \sqrt{2} - 1, \text{ since } \tan 22\frac{1}{2}^\circ > 0.
 \end{aligned}$$

Exercise 14D

1. Expand:

$$\begin{array}{lll}
 \text{(a) } \sin(x - y) & \text{(c) } \sin(3\alpha + 5\beta) & \text{(e) } \tan(A + 2B) \\
 \text{(b) } \cos(2A + 3B) & \text{(d) } \cos(\theta - \frac{\phi}{2}) & \text{(f) } \tan(3\alpha - 4\beta)
 \end{array}$$

2. Express as a single trigonometric function:

$$\begin{array}{ll}
 \text{(a) } \cos x \cos y - \sin x \sin y & \text{(d) } \sin 5A \cos 2A - \cos 5A \sin 2A \\
 \text{(b) } \sin 3\alpha \cos 2\beta + \cos 3\alpha \sin 2\beta & \text{(e) } \cos 70^\circ \cos 20^\circ + \sin 70^\circ \sin 20^\circ \\
 \text{(c) } \frac{\tan 40^\circ - \tan 20^\circ}{1 + \tan 40^\circ \tan 20^\circ} & \text{(f) } \frac{\tan \alpha + \tan 10^\circ}{1 - \tan \alpha \tan 10^\circ}
 \end{array}$$

3. Use the double angle formulae to simplify:

$$\begin{array}{lll}
 \text{(a) } 2 \sin x \cos x & \text{(d) } 2 \sin 20^\circ \cos 20^\circ & \text{(g) } 2 \sin 3\theta \cos 3\theta \\
 \text{(b) } \cos^2 \theta - \sin^2 \theta & \text{(e) } 2 \cos^2 50^\circ - 1 & \text{(h) } 1 - 2 \sin^2 2A \\
 \text{(c) } \frac{2 \tan \alpha}{1 - \tan^2 \alpha} & \text{(f) } \frac{2 \tan 70^\circ}{1 - \tan^2 70^\circ} & \text{(i) } \frac{2 \tan 4x}{1 - \tan^2 4x}
 \end{array}$$

4. Use the compound angle results to prove:

$$\begin{array}{lll}
 \text{(a) } \sin(90^\circ + A) = \cos A & \text{(c) } \tan(360^\circ - A) = -\tan A & \text{(e) } \cos(270^\circ - A) = -\sin A \\
 \text{(b) } \cos(90^\circ - A) = \sin A & \text{(d) } \tan(180^\circ + A) = \tan A & \text{(f) } \sin(360^\circ - A) = -\sin A
 \end{array}$$

5. Prove the identities:

$$\begin{array}{ll}
 \text{(a) } \sin(A + 45^\circ) = \frac{1}{\sqrt{2}}(\sin A + \cos A) & \text{(c) } \tan(\frac{\pi}{4} - A) = \frac{1 - \tan A}{1 + \tan A} \\
 \text{(b) } 2 \cos(\theta + \frac{\pi}{3}) = \cos \theta - \sqrt{3} \sin \theta & \text{(d) } \sin(A - 30^\circ) = \frac{1}{2}(\sqrt{3} \sin A - \cos A)
 \end{array}$$

6. Prove:

$$(a) (\cos A - \sin A)(\cos A + \sin A) = \cos 2A \quad (c) \sin 2\theta = 2 \sin \theta \sin\left(\frac{\pi}{2} - \theta\right)$$

$$(b) (\sin \alpha - \cos \alpha)^2 = 1 - \sin 2\alpha \quad (d) \frac{1}{1 - \tan \theta} - \frac{1}{1 + \tan \theta} = \tan 2\theta$$

7. (a) By expressing 15° as $(45^\circ - 30^\circ)$, show that:

$$(i) \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}} \quad (ii) \cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}} \quad (iii) \tan 15^\circ = 2 - \sqrt{3}$$

(b) Hence find surd expressions for: (i) $\sin 75^\circ$ (ii) $\cos 75^\circ$ (iii) $\tan 75^\circ$

8. (a) If $\cos \alpha = \frac{4}{5}$, find $\cos 2\alpha$.

(c) If $\sin \theta = \frac{5}{13}$ and θ is acute, find $\sin 2\theta$.

(b) If $\sin x = \frac{2}{3}$, find $\cos 2x$.

(d) If $\tan A = \frac{1}{2}$, find $\tan 2A$.

9. (a) If $\tan \alpha = \frac{1}{2}$ and $\tan \beta = \frac{1}{3}$, find $\tan(\alpha + \beta)$.

(b) If $\cos A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, where A and B are both acute, find $\sin(A + B)$.

(c) If $\sin \theta = \frac{2}{3}$ and $\cos \phi = \frac{1}{4}$, where θ and ϕ are both acute, find $\cos(\theta - \phi)$.

10. Prove the identities:

$$(a) \sin(A + B) + \sin(A - B) = 2 \sin A \cos B$$

$$(b) \cos(x - y) - \cos(x + y) = 2 \sin x \sin y$$

$$(c) \sin(x + y) + \cos(x - y) = (\sin x + \cos x)(\sin y + \cos y)$$

DEVELOPMENT

11. If $\sin x = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\sin 2x$.

12. If $\sin A = \frac{2}{3}$, $\frac{\pi}{2} < A < \pi$, and $\tan B = \frac{2}{3}$, $\pi < B < \frac{3\pi}{2}$, show that $\cos(A + B) = \frac{3\sqrt{5} + 4}{3\sqrt{13}}$.

13. Find the exact value of: (a) $\cos 105^\circ$ (b) $\sin \frac{13\pi}{12}$ (c) $\cot 285^\circ$

14. Show that: (a) $\sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta) + \cos(\frac{\pi}{4} - \theta)\sin(\frac{\pi}{4} + \theta) = 1$

$$(b) \tan 35^\circ + \tan 10^\circ + \tan 35^\circ \tan 10^\circ = 1$$

15. Show that:

$$(a) \cos^4 \alpha - \sin^4 \alpha = \cos 2\alpha$$

$$(d) \cos \theta - \sin \theta \sin 2\theta = \cos \theta \cos 2\theta$$

$$(b) \cos 2x + \cos x = (\cos x + 1)(2 \cos x - 1)$$

$$(e) \tan(\frac{\pi}{4} + \alpha) - \tan(\frac{\pi}{4} - \alpha) = 2 \tan 2\alpha$$

$$(c) \frac{\sin 2A}{1 - \cos 2A} = \cot A$$

$$(f) \frac{\sin 2\theta + \sin \theta}{\cos 2\theta + \cos \theta + 1} = \tan \theta$$

16. Prove: (a) $\frac{2 \sin(x - y)}{\cos(x + y) - \cos(x - y)} = \cot x - \cot y$

$$(b) \frac{\sin(\theta + \phi)}{\cos(\theta - \phi)} = \frac{\tan \theta + \tan \phi}{1 + \tan \theta \tan \phi} \quad (c) \frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta} + \frac{\sin(\beta - \gamma)}{\sin \beta \sin \gamma} + \frac{\sin(\gamma - \alpha)}{\sin \gamma \sin \alpha} = 0$$

17. (a) Show that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

(b) Hence simplify $\sin^2 75^\circ - \sin^2 15^\circ$.

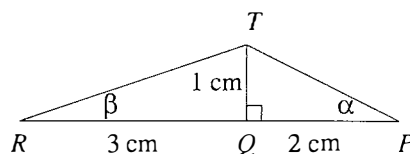
18. Show that: (a) $2 \sin \frac{4\pi}{5} \cos \frac{\pi}{5} = \sin \frac{2\pi}{5}$ (b) $\cos^2 \frac{4\pi}{7} - \sin^2 \frac{3\pi}{7} = \cos \frac{6\pi}{7}$

19. (a) Find the value of $\tan 135^\circ$.

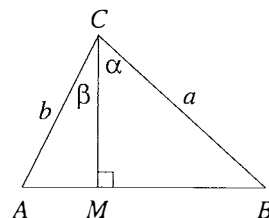
(b) Let $\tan 67\frac{1}{2}^\circ = t$. Use the double angle formula for $\tan 2\theta$ to show that $t^2 - 2t - 1 = 0$.

(c) Hence find the exact value of $\tan 67\frac{1}{2}^\circ$.

20. (a) Show that $\frac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$.
 (b) Hence find the exact value of $\tan \frac{\pi}{8}$.
21. (a) Write down the formula for $\sin(\alpha + \beta)$.
 (b) Hence show that the angles α and β in the diagram to the right have a sum of 45° .

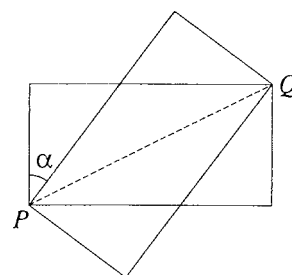


22. [The expansion of $\sin(\alpha + \beta)$ using areas] The diagram shows the altitude CM of $\triangle ABC$, with $\angle MCB = \alpha$ and $\angle MCA = \beta$.
- (a) Show that $CM = b \cos \beta = a \cos \alpha$.
 (b) Find the areas of $\triangle MCB$, $\triangle MCA$ and $\triangle CAB$ in terms of a , b , α and β .
 (c) Hence prove the compound angle formula



$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

23. The diagram shows two rectangles. Each rectangle is 6 cm long and 3 cm wide, and they share a common diagonal PQ . Show that $\tan \alpha = \frac{3}{4}$.



EXTENSION

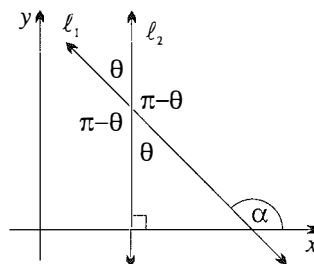
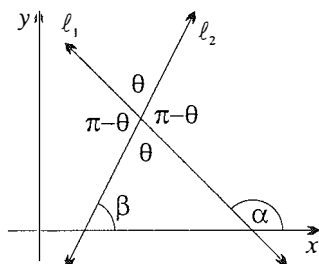
24. (a) Prove the trigonometric identity

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

- (b) Hence show that in any triangle ABC , $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

14E The Angle Between Two Lines

Suppose ℓ_1 and ℓ_2 are lines that are neither parallel nor perpendicular, as in the diagrams below. When the lines meet, they form four angles adding to 2π , with vertically opposite angles equal, and adjacent angles supplementary. Let one of the acute angles be θ , then the four angles are θ , $\pi - \theta$, θ and $\pi - \theta$. Our task is to use trigonometry to find an exact expression for θ in terms of the gradients of the lines.



A Formula for the Angle Between Two Lines: Suppose first that neither line is vertical, as in the diagram on the left. Let the angles of inclination of ℓ_1 and ℓ_2 be α and β respectively, and let the gradients be m_1 and m_2 respectively. Then $m_1 = \tan \alpha$ and $m_2 = \tan \beta$. On the diagram given here, $\theta = \alpha - \beta$, but depending on which angle is larger, and whether the difference is acute or obtuse, θ could be

$$\alpha - \beta \quad \text{or} \quad \beta - \alpha \quad \text{or} \quad \pi - (\alpha - \beta) \quad \text{or} \quad \pi - (\beta - \alpha).$$

The tangents of these four angles are the same apart from negative signs. Since θ is acute, $\tan \theta$ must be positive, so we can deal with all cases at once by using absolute value notation:

$$\tan \theta = |\tan(\alpha - \beta)| = \left| \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \right| = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

Now suppose that the second line is vertical and the first line has gradient m , as in the diagram on the right. The acute angle θ is either $\frac{\pi}{2} - \alpha$ or $\alpha - \frac{\pi}{2}$, so in both cases

$$\tan \theta = \left| \tan\left(\frac{\pi}{2} - \alpha\right) \right| = |\cot \alpha| = \left| \frac{1}{\tan \alpha} \right| = \left| \frac{1}{m} \right|.$$

ANGLES BETWEEN TWO LINES: Suppose that two lines are neither parallel nor perpendicular, and let θ be the acute angle between them.

1. If neither line is vertical, then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|, \text{ where } m_1 \text{ and } m_2 \text{ are the respective gradients.}$$

2. If one line is vertical, then

$$\tan \theta = \left| \frac{1}{m} \right|, \text{ where } m \text{ is the gradient of the other line.}$$

NOTE: If substitution into the first formula gives the answer zero, then $m_1 = m_2$ and the lines are parallel, so the lines either coincide or don't meet at all. If substitution into that formula gives a zero denominator, then $m_1 m_2 = -1$ and the lines are perpendicular.

WORKED EXERCISE: Find the tangents of the angles of the triangle formed by the three lines $\ell_1 : 2x + y = 1$, $\ell_2 : 2x - y = 5$ and $\ell_3 : x - 2y = 8$, then approximate the angles to the nearest minute.

SOLUTION: The gradients of the three lines are $m_1 = -2$, $m_2 = 2$ and $m_3 = \frac{1}{2}$.

First, $m_1 \times m_3 = -1$, so the lines ℓ_1 and ℓ_3 are perpendicular.

Let θ be the acute angle between ℓ_1 and ℓ_2 , and ϕ the acute angle between ℓ_2 and ℓ_3 .

$$\begin{aligned} \text{Then } \tan \theta &= \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right| \\ &= \left| \frac{2 + 2}{1 + 2 \times (-2)} \right| \\ &= \frac{4}{3}, \end{aligned}$$

$$\text{so } \theta \doteq 53^\circ 8'.$$

$$\begin{aligned} \text{Also, } \tan \phi &= \left| \frac{m_3 - m_2}{1 + m_3 m_2} \right| \\ &= \left| \frac{\frac{1}{2} - 2}{1 + \frac{1}{2} \times 2} \right| \\ &= \frac{3}{4}, \end{aligned}$$

$$\text{so } \phi \doteq 36^\circ 52'.$$

Notice that $\theta + \phi = 90^\circ$, as expected.

The Angle Between Two Curves: The angle between two curves can be defined clearly using tangents.

17 ANGLE BETWEEN CURVES: The angle between two curves at a point of intersection is the angle between the two tangents at this point.

WORKED EXERCISE: Find the points of intersection of $y = x^3 - x$ and $y = x - x^2$, and the exact values of the acute angles between the curves at these points.

SOLUTION: Solving simultaneously, $x^3 - x = x - x^2$

$$x^3 + x^2 - 2x = 0$$

$$x(x+2)(x-1) = 0,$$

so they intersect at $O(0,0)$, $A(1,0)$ and $B(-2,-6)$.

The two derivatives are $\frac{dy}{dx} = 3x^2 - 1$ and $\frac{dy}{dx} = 1 - 2x$,

so the gradients of the tangents at the three points are:

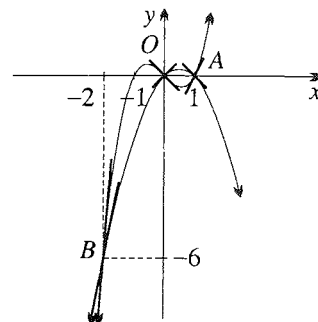
point	$O(0,0)$	$A(1,0)$	$B(-2,-6)$
gradient of cubic	-1	2	11
gradient of quadratic	1	-1	5

The curves meet at right angles at O .

Let θ and ϕ be the acute angles between the curves at A and B respectively.

$$\begin{aligned} \text{At } A, \tan \theta &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| & \text{At } B, \tan \phi &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \\ &= \left| \frac{2 - (-1)}{1 + 2 \times (-1)} \right| & &= \left| \frac{11 - 5}{1 + 11 \times 5} \right| \\ &= 3, & &= \frac{3}{28}, \end{aligned}$$

$$\text{and so } \theta = \tan^{-1} 3 [\div 71^\circ 34'] \quad \text{and so } \phi = \tan^{-1} \frac{3}{28} [\div 6^\circ 7'].$$



WORKED EXERCISE: [Questions like this are a little harder because they result in an absolute value equation.] Find the gradients of the lines making an angle of 60° with $3x - y + 4 = 0$.

SOLUTION: Let the gradient be m , then since the given line has gradient 3,

$$\tan 60^\circ = \left| \frac{m - 3}{1 + 3m} \right|$$

$$|1 + 3m|\sqrt{3} = |m - 3|.$$

Squaring both sides, $27m^2 + 18m + 3 = m^2 - 6m + 9$

$$13m^2 + 12m - 3 = 0.$$

Using the quadratic formula, $\Delta = 144 + 156 = 300$,

$$\begin{aligned} m &= \frac{1}{26}(-12 + 10\sqrt{3}) \text{ or } \frac{1}{26}(-12 - 10\sqrt{3}) \\ &= \frac{1}{13}(-6 + 5\sqrt{3}) \text{ or } \frac{1}{13}(-6 - 5\sqrt{3}). \end{aligned}$$

Exercise 14E

NOTE: In each question that asks for an angle, first find and simplify the tangent of the angle, which is known exactly, before finding an approximation to the angle.

- Find, to the nearest degree, the acute angle between two lines whose gradients are:
 - $m_1 = 2$ and $m_2 = \frac{1}{2}$
 - $m_1 = 4$ and $m_2 = -2$
 - $m_1 = 1\frac{1}{4}$ and $m_2 = \frac{1}{3}$
- Find the acute angle between each pair of lines:
 - $y = 3x - 1$ and $y = -2x + 5$
 - $x - 2y + 1 = 0$ and $x + 3y + 2 = 0$
 - $4x - 3y + 2 = 0$ and $7x + y - 1 = 0$
 - $4x + 3y + 5 = 0$ and $3x - 4y - 2 = 0$
 - $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$
 - $y + 4x - 7 = 0$ and $y = 5 - 4x$
- Find, to the nearest minute, the acute angle between the lines $3x + 4y = 8$ and $2x + 3y = 5$.

4. Use the formula $\tan \theta = \left| \frac{1}{m} \right|$ to find, correct to the nearest minute, the acute angle between the vertical line $x = -1$ and the line $4x - 3y + 5 = 0$.
5. (a) Find the point of intersection of the parabolas $y = x^2$ and $y = (x - 2)^2$.
 (b) Sketch the two curves, showing their point of intersection.
 (c) Find the acute angle, to the nearest degree, at which the curves intersect.
6. (a) Show that the curves $y = e^x$ and $y = x^2 - 3x + 1$ have a common point at $(0, 1)$.
 (b) Find, to the nearest degree, the acute angle between the tangents to the curves at $(0, 1)$.

DEVELOPMENT

7. Find the acute angle and the obtuse angle between the lines:
 (a) $2x - y + 1 = 0$ and $11x - (8 + 5\sqrt{3})y + 5\sqrt{3} = 0$
 (b) $y = mx + b$ and $(m - 1)x - (m + 1)y = (m^2 - 1)b$
8. A , B and C are the points $(-2, 1)$, $(1, 7)$ and $(5, 0)$ respectively. Calculate the three interior angles of $\triangle ABC$ correct to the nearest minute and check their sum. Explain why the sum is not exactly 180° .
9. (a) The line $y = mx$ is inclined at 45° to $y = 2x - 4$. Find the two possible values of m .
 (b) Explain geometrically why these two values of m represent perpendicular gradients.
10. (a) Show that the line $y - 1 = m(x - 1)$ always passes through the point $(1, 1)$.
 (b) Find, in general form, the equations of the two lines through $(1, 1)$ that are inclined at 45° to the line $y = 5x + 6$.
11. (a) Find, in general form, the equations of the lines that pass through the point $(1, 2)$ and are inclined at an angle of 60° to the line $x + y = 3$.
 (b) Find, in general form, the equations of the lines that pass through $(2, 3)$ and are inclined at an angle whose tangent ratio is $\frac{1}{3}$ to the line $y = 2x + 4$.
12. (a) Sketch the parabolas $y = 3x^2$ and $y = 4x - x^2$, showing their points of intersection.
 (b) Find, to the nearest minute, the acute angles between these curves at their points of intersection.
13. (a) Show that the curves $y = \log_e(x - 1)$ and $y = \frac{2 - x}{x}$ meet at $(2, 0)$.
 (b) Find $\tan \alpha$, where α is the acute angle between the tangents at $(2, 0)$.
14. Find the points of intersection of $y = x^2$ with the curve defined parametrically by $x = t^3$ and $y = t^2$. Find the angle between the curves at these points. Sketch the situation.

EXTENSION

15. Find the equations of the lines that pass through the origin and are inclined at 75° to the line $x + y + (y - x)\sqrt{3} = a$.
16. If ϕ is the acute angle between the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, prove that

$$\cos \phi = \frac{|A_1A_2 + B_1B_2|}{\sqrt{(A_1^2 + B_1^2)(A_2^2 + B_2^2)}}.$$

14F The Behaviour of $\sin x$ Near the Origin

The next step in proving that $\sin x$ has derivative $\cos x$ is to establish that the curve $y = \sin x$ has gradient exactly 1 at the origin. Geometrically, this means that the line $y = x$ is the tangent to $y = \sin x$ at the origin.

A Fundamental Inequality: First, we make an appeal to geometry and use areas to establish the following inequality concerning x , $\sin x$ and $\tan x$.

AN INEQUALITY CONCERNING $\sin x$ AND $\tan x$ NEAR THE ORIGIN:

- 18** A. $\sin x < x < \tan x$, for $0 < x < \frac{\pi}{2}$ (that is, x acute).
 B. $\sin x > x > \tan x$, for $-\frac{\pi}{2} < x < 0$.

PROOF:

- A. Suppose that x is an acute angle.

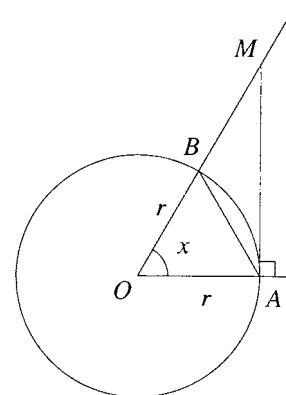
Construct a circle of centre O and radius r ,
 and a sector AOB subtending the angle x at the centre O .

Let the tangent at A meet the radius OB at M
 (OB will need to be produced), and join the chord AB .

Then $\text{area } \triangle OAB < \text{area sector } OAB < \text{area } \triangle OAM$

$$\frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x < \frac{1}{2}r^2 \tan x$$

$$\div \frac{1}{2}r^2 \quad \sin x < x < \tan x.$$



- B. Since x , $\sin x$ and $\tan x$ are all odd functions,
 $\sin x > x > \tan x$, for $-\frac{\pi}{2} < x < 0$.

The Main Theorem: This inequality now allows the fundamental limits to be proven:

19 THE FUNDAMENTAL LIMITS: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

PROOF: Suppose first that x is acute, so that $\sin x < x < \tan x$.

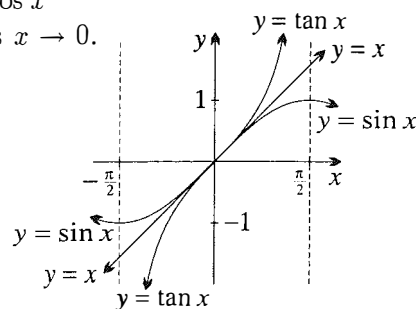
Dividing through by $\sin x$ gives $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

But $\cos x \rightarrow 1$ as $x \rightarrow 0^+$, and so $\frac{x}{\sin x} \rightarrow 1$ as $x \rightarrow 0^+$, as required.

Since $\frac{x}{\sin x}$ is even, it follows also that $\frac{x}{\sin x} \rightarrow 1$ as $x \rightarrow 0^-$.

Finally, $\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x} \rightarrow 1 \times 1$, as $x \rightarrow 0$.

The diagram on the right shows what has been proven about the graphs of $y = x$, $y = \sin x$ and $y = \tan x$ near the origin. The line $y = x$ is a common tangent at the origin to both $y = \sin x$ and $y = \tan x$. On both sides of the origin, $y = \sin x$ curves away from the tangent towards the x -axis, and $y = \tan x$ curves away from the tangent in the opposite direction.



20

THE BEHAVIOUR OF $\sin x$ AND $\tan x$ NEAR THE ORIGIN: The line $y = x$ is a tangent to both $y = \sin x$ and $y = \tan x$ at the origin.

When $x = 0$, the derivatives of both $\sin x$ and $\tan x$ are exactly 1.

WORKED EXERCISE: Find: (a) $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$ (b) $\lim_{h \rightarrow 0} \frac{2h}{\sin \frac{1}{2}h}$ (c) $\lim_{x \rightarrow 0} \frac{\tan 5x}{\sin \frac{1}{3}x}$

SOLUTION:

$$(a) \lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$$

$$= \frac{5}{3} \times \lim_{x \rightarrow 0} \frac{\sin 5x}{5x}$$

$$= \frac{5}{3} \times 1$$

$$= \frac{5}{3}$$

$$(b) \lim_{h \rightarrow 0} \frac{2h}{\sin \frac{1}{2}h}$$

$$= 4 \times \lim_{h \rightarrow 0} \frac{\frac{1}{2}h}{\sin \frac{1}{2}h}$$

$$= 4 \times 1$$

$$= 4$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan 5x}{\sin \frac{1}{3}x}$$

$$= 15 \lim_{x \rightarrow 0} \left(\frac{\tan 5x}{5x} \times \frac{\frac{1}{3}x}{\sin \frac{1}{3}x} \right)$$

$$= 15 \times 1 \times 1$$

$$= 15$$

Approximations to the Trigonometric Functions for Small Angles: For small angles, positive or negative, the limits above yield good approximations for the three trigonometric functions (the angle must of course be expressed in radians).

21

SMALL ANGLE APPROXIMATIONS: $\sin x \doteq x$
 $\cos x \doteq 1$
 $\tan x \doteq x$

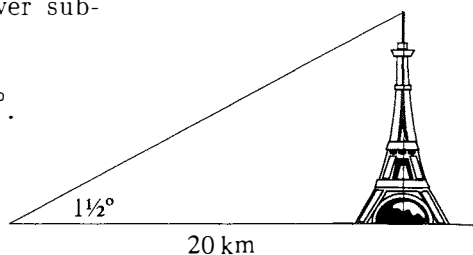
Question 1 in the following exercise asks for tables of values for $\sin x$, $\cos x$ and $\tan x$ for progressively smaller angles. This will give some idea about how good the approximations are.

WORKED EXERCISE: Approximately how high is a tower subtending an angle of $1\frac{1}{2}^\circ$ if it is 20 km away?

SOLUTION: From the diagram, height = $20\,000 \tan 1\frac{1}{2}^\circ$.

But $1\frac{1}{2}^\circ$ is $\frac{\pi}{120}$, so $\tan 1\frac{1}{2}^\circ \doteq \frac{\pi}{120}$.

Hence, approximately, height $\doteq \frac{500\pi}{3}$ metres
 $\doteq 524$ metres.



WORKED EXERCISE: The sun subtends an angle of $0^\circ 31'$ at the Earth, which is 150 000 000 km away. What is the sun's approximate diameter?

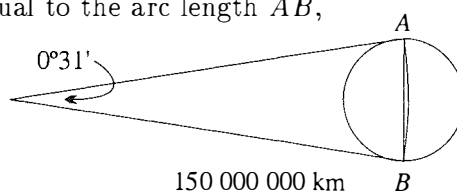
NOTE: This problem can be done similarly to the previous problem, but like most small-angle problems, it can also be done by approximating the diameter to an arc of the circle.

SOLUTION: Since the diameter AB is approximately equal to the arc length AB ,

$$\text{diameter} \doteq r\theta$$

$$\doteq 150\,000\,000 \times \frac{31}{60} \times \frac{\pi}{180}$$

$$\doteq 1\,353\,000 \text{ km.}$$



Exercise 14F

1. (a) Copy and complete the following table of values, giving entries to four significant figures. For each column, hold x in the calculator's memory until the column is complete:

angle size in degrees	60°	30°	10°	5°	2°	1°	20'	5'	1'	30''	10''
angle size x in radians											
$\sin x$											
$\frac{\sin x}{x}$											
$\tan x$											
$\frac{\tan x}{x}$											
$\cos x$											

- (b) Write x , $\sin x$ and $\tan x$ in ascending order, for acute angles x .
- (c) Although $\sin x \rightarrow 0$ and $\tan x \rightarrow 0$ as $x \rightarrow 0$, what are the limits, as $x \rightarrow 0$, of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$?
- (d) Experiment with your calculator to find how small x must be for $\frac{\sin x}{x} > 0.999$.
2. Find the following limits:
- (a) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ (c) $\lim_{x \rightarrow 0} \frac{\sin x}{2x}$ (e) $\lim_{x \rightarrow 0} \frac{5x}{\sin 3x}$
- (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{x}$ (d) $\lim_{x \rightarrow 0} \frac{\sin 3x}{2x}$ (f) $\lim_{x \rightarrow 0} \frac{\sin 3x + \sin 5x}{x}$
3. Find the following limits:
- (a) $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$ (b) $\lim_{\theta \rightarrow 0} \frac{\tan 7\theta}{5\theta}$ (c) $\lim_{\theta \rightarrow 0} \frac{7\theta}{3 \tan 5\theta}$
- DEVELOPMENT**
4. Find: (a) $\lim_{x \rightarrow 0} \frac{\sin x}{\frac{x}{2}}$ (b) $\lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x}$ (c) $\lim_{x \rightarrow 0} \frac{\sin \frac{x}{3}}{2x}$
5. Find: (a) $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x}$ (b) $\lim_{y \rightarrow 0} \frac{\tan 3y}{\tan 2y}$ (c) $\lim_{t \rightarrow 0} \frac{\sin 5t}{\tan 7t}$
6. Find: (a) $\lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta^2}$ (b) $\lim_{\theta \rightarrow 0} \frac{\tan a\theta}{\sin b\theta}$ (c) $\lim_{\theta \rightarrow 0} \frac{\theta^2}{1 - \sin^2 \theta}$
7. Show that: (a) $\lim_{n \rightarrow 0} \frac{\sin n^\circ}{n} = \frac{\pi}{180}$ (b) $\lim_{n \rightarrow \infty} n \sin \frac{\pi}{n} = \pi$
8. (a) Write down the compound angle formula for $\cos(A + B)$.
- (b) By replacing both A and B with x , show that $\cos 2x = 1 - 2 \sin^2 x$.
- (c) Hence find $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$.

9. (a) Express 2° in radians.
 (b) Explain why $\sin 2^\circ \doteq \frac{\pi}{90}$.
 (c) Taking π as 3.142, find $\sin 2^\circ$ correct to four decimal places *without* using a calculator.
10. A car travels 1 km up a road which is inclined at 5° to the horizontal. Through what vertical distance has the car climbed? (Use the fact that $\sin x \doteq x$ for small angles, and give your answer correct to the nearest metre.)
11. A tower is 30 metres high. What angle, correct to the nearest minute, does it subtend at a point 4 km away? (Use the fact that when x is small, $\tan x \doteq x$.)
12. The moon subtends an angle of $31'$ at an observation point on Earth 400 000 km away. Use the fact that the diameter of the moon is approximately equal to an arc of a circle whose centre is the point of observation to show that the diameter of the moon is approximately 3600 km.
13. A regular polygon of 300 sides is inscribed in a circle of radius 60 cm. Show that each side is approximately 1.26 cm.
14. (a) Show that $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ by multiplying top and bottom by $1 + \cos x$.
 (b) Explain why this means that $\cos x \doteq 1 - \frac{1}{2}x^2$ for small angles.
 (c) Use the calculator to approximate the difference between $\cos x$ and $1 - \frac{1}{2}x^2$ for small angles, and find the smallest angle, in degrees, for which this difference exceeds 0.001.
 (d) A post 5 metres long leans at 10° to the vertical. Find how high the top is above the ground using the approximation $\cos x \doteq 1 - \frac{1}{2}x^2$. Give your answer correct to four significant figures, and then check using the calculator's cosine function.

EXTENSION

15. (a) Write down the compound-angle formula for $\sin(A - B)$.
 (b) Hence show that, for small x , $\sin(\theta - x) \doteq \sin \theta - x \cos \theta$.
 (c) Use the result in (b) to show that $\sin 29^\circ 57' \doteq \frac{3600 - \sqrt{3}\pi}{7200}$.
 (d) To how many decimal places is the approximation in (c) accurate?
 (e) Use similar methods to obtain approximations to $\sin 29^\circ$, $\cos 31^\circ$, $\tan 61^\circ$, $\cot 59^\circ$ and $\sin 46^\circ$, checking the accuracy of your approximations using the calculator.
16. The chord AB of a circle of radius r subtends an angle x at the centre O .
 (a) Find AB^2 by the cosine rule, and find the length of the arc AB .
 (b) By equating arc and chord, show that for small angles, $\cos x \doteq 1 - \frac{x^2}{2}$. Explain whether the approximation is bigger or smaller than $\cos x$.

NOTE: The approximation $\cos x = 1 - \frac{1}{2}x^2$ is an excellent small-angle approximation for $\cos x$. It can be shown further that $\cos x$ is exactly the limit of the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

14G The Derivatives of the Trigonometric Functions

We can now establish the derivative of $\sin x$, which is the central theorem of the chapter. From this, the derivatives of $\cos x$ and $\tan x$ can quickly be calculated.

The Derivative of $\sin x$: The proof that $\frac{d}{dx} \sin x = \cos x$ uses the compound angle formulae to bring everything back to the fundamental limit proven earlier, $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$. The first part of the proof develops a further identity involving compound angles. This identity is part of a group of ‘sums to products’ identities, which are part of the 4 Unit course, but play no further role in the 3 Unit course.

PROOF:

A. We know that $\sin(A + B) = \sin A \cos B + \cos A \sin B$,
 and $\sin(A - B) = \sin A \cos B - \cos A \sin B$.
 Subtracting these, $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$.
 Now let $S = A + B$
 and $T = A - B$,
 then adding and halving, $A = \frac{1}{2}(S + T)$
 subtracting and halving, $B = \frac{1}{2}(S - T)$,
 giving the identity, $\sin S - \sin T = 2 \cos \frac{1}{2}(S + T) \sin \frac{1}{2}(S - T)$.

B. Using the first-principles definition of the derivative,

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \lim_{u \rightarrow x} \frac{\sin u - \sin x}{u - x} & \text{OR} & \quad \frac{d}{dx}(\sin x) = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} \\ &= \lim_{u \rightarrow x} \frac{2 \cos \frac{1}{2}(u + x) \sin \frac{1}{2}(u - x)}{u - x} & & = \lim_{h \rightarrow 0} \frac{2 \cos(x + \frac{1}{2}h) \sin \frac{1}{2}h}{h} \\ &= \cos x \times \lim_{u \rightarrow x} \frac{\sin \frac{1}{2}(u - x)}{\frac{1}{2}(u - x)} & & = \cos x \times \lim_{h \rightarrow 0} \frac{\sin \frac{1}{2}h}{\frac{1}{2}h} \\ &= \cos x, \text{ because the limit is 1.} & & = \cos x, \text{ because the limit is 1.} \end{aligned}$$

The Derivatives of $\cos x$ and $\tan x$: Once the derivative of $\sin x$ is proven, it is straightforward to differentiate the other trigonometric functions.

A. Let $y = \cos x$
 $= \sin(\frac{\pi}{2} - x)$.

Applying the chain rule,

let $u = \frac{\pi}{2} - x$,

then $y = \sin u$.

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= -1 \times \cos u \\ &= -1 \times \cos(\frac{\pi}{2} - x) \\ &= -\sin x. \end{aligned}$$

B. Let $y = \tan x$
 $= \frac{\sin x}{\cos x}$.

Let $u = \sin x$ and $v = \cos x$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \quad (\text{quotient rule}) \\ &= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \quad (\text{Pythagoras}) \\ &= \sec^2 x. \end{aligned}$$

22

STANDARD DERIVATIVES:

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

WORKED EXERCISE: Use the chain rule to differentiate $y = \sin(ax + b)$.

SOLUTION: Let $u = ax + b$, so that $y = \sin u$.

Then applying the chain rule, $\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$

$$= a \times \cos u$$

$$= a \cos(ax + b).$$

WORKED EXERCISE: Use the chain rule to differentiate $y = \tan^2 x$.

SOLUTION: Let $u = \tan x$, so that $y = u^2$.

Then applying the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 2u \times \sec^2 x$$

$$= 2 \tan x \sec^2 x.$$

WORKED EXERCISE: Use product and quotient rules to differentiate:

(a) $y = e^x \cos x$

(b) $y = \frac{\sin x}{x}$

SOLUTION:

(a) Applying the product rule,
let $u = e^x$ and $v = \cos x$.

Then $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$= e^x \cos x - e^x \sin x$$

$$= e^x (\cos x - \sin x).$$

(b) Applying the quotient rule,
let $u = \sin x$ and $v = x$.

Then $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$$= \frac{x \cos x - \sin x}{x^2}.$$

Replacing x by $ax + b$: The first result in the previous worked exercise can be extended to the other trigonometric functions, giving the following standard forms.

23

FUNCTIONS OF $ax + b$:

$$\frac{d}{dx} \sin(ax + b) = a \cos(ax + b)$$

$$\frac{d}{dx} \cos(ax + b) = -a \sin(ax + b)$$

$$\frac{d}{dx} \tan(ax + b) = a \sec^2(ax + b)$$

WORKED EXERCISE: Differentiate the following functions:

(a) $y = 4 \sin(3x - \frac{\pi}{3})$

(b) $y = \frac{3}{2} \tan \frac{3}{2}x$

(c) $y = 5 \cos 2x \cos \frac{1}{2}x$

SOLUTION:

(a) $\frac{dy}{dx} = 12 \cos(3x - \frac{\pi}{3})$

(b) $\frac{dy}{dx} = \frac{9}{4} \sec^2 \frac{3}{2}x$

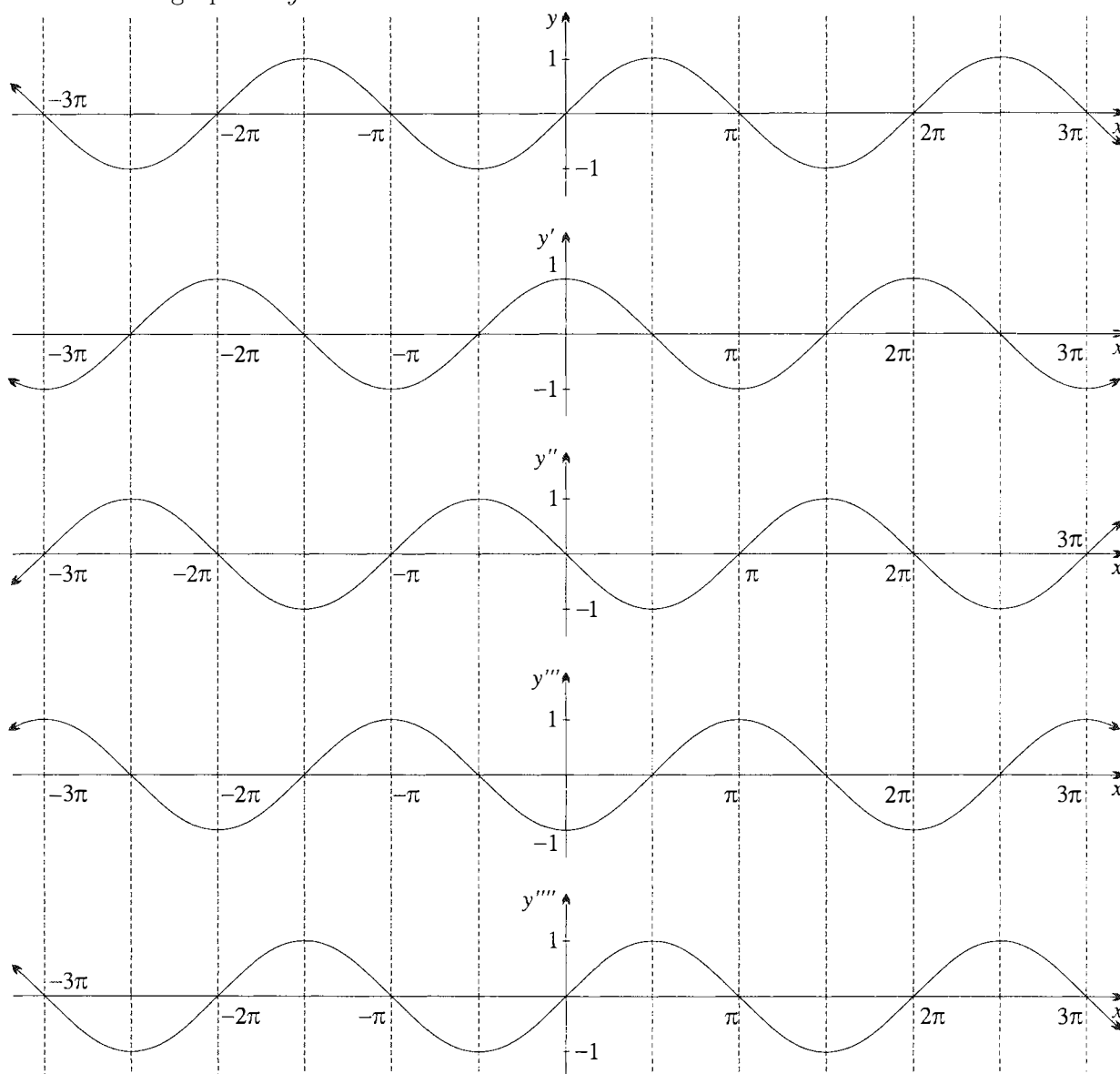
(c) Applying the product rule, let $u = 5 \cos 2x$ and $v = \cos \frac{1}{2}x$.

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= v \frac{du}{dx} + u \frac{dv}{dx} \\ &= -10 \cos \frac{1}{2}x \sin 2x - \frac{5}{2} \cos 2x \sin \frac{1}{2}x.\end{aligned}$$

Successive Differentiation of Sine and Cosine: Differentiating $y = \sin x$ repeatedly,

$$\frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x.$$

So differentiation is an order 4 operation on the sine function, which means that when differentiation is applied four times, the original function returns. Below are the graphs of $y = \sin x$ and its first four derivatives.



Each application of the differentiation operator shifts the wave backwards a quarter revolution, so four applications shift it backwards one revolution, where it coincides with itself again. Thus differentiation is behaving like a rotation of 90° anticlockwise, and of course a rotation of 90° has order 4.

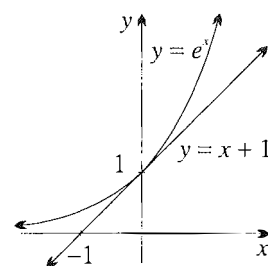
Notice that double differentiation exchanges $y = \sin x$ with its opposite function $y = -\sin x$, with each graph being the reflection of the other in the x -axis. It has a similar effect on the cosine function. So both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'' = -y$. This idea will later be central to simple harmonic motion.

The exponential function should be mentioned in this context. The first derivative of $y = e^x$ is $y' = e^x$, and the second derivative of $y = e^{-x}$ is $y'' = e^{-x}$. This means that we now have four functions whose fourth derivatives are equal to themselves:

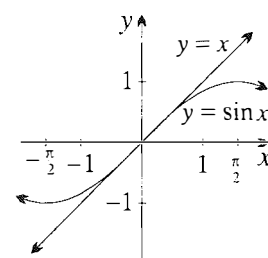
$$y = \sin x, \quad y = \cos x, \quad y = e^x, \quad y = e^{-x}.$$

This is one clue amongst many others in our course that the trigonometric functions and the exponential functions are very closely related.

Some Analogies between π and e : In the previous chapter, we found that by choosing the base of our exponential function to be the special number e , then the derivative of $y = e^x$ was exactly $y' = e^x$. In contrast, the derivative of any other exponential function $y = a^x$ is $y = a^x \log a$, which is only a multiple of itself. In particular, the tangent of $y = e^x$ at the y -intercept has gradient exactly 1.



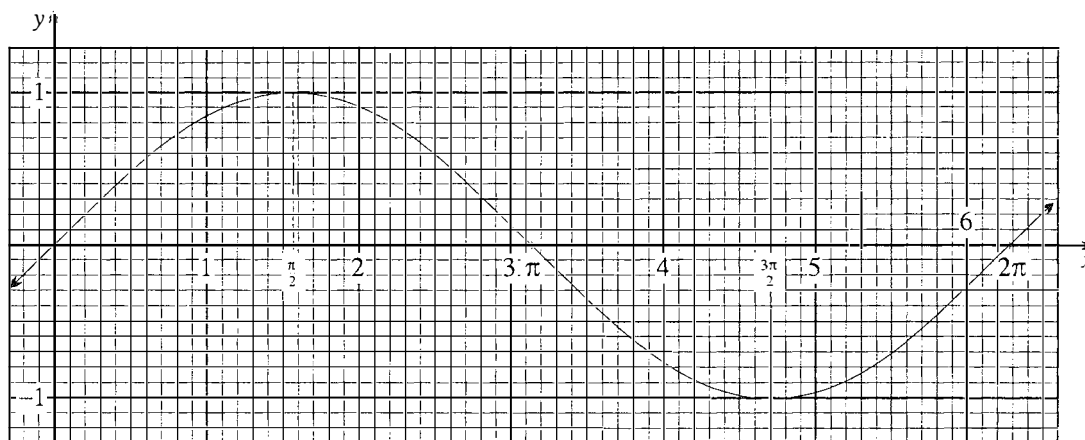
The choice of radian measure, based on the special number π , was motivated exactly the same way — as we have just proven, the derivative of $y = \sin x$ using radian measure is exactly $y' = \cos x$, but would only be a multiple of $\cos x$ were another system of measuring angles used. In particular, the gradient of $y = \sin x$ at the origin is exactly 1.



Both numbers $\pi \doteq 3.1416$ and $e \doteq 2.7183$ are irrational (although this is not easy to prove), with π defined using areas of circles, and e defined using areas associated with the rectangular hyperbola. All these things are further hints of deeper connections between trigonometric and exponential functions that lie beyond this course.

Exercise 14G

1.



Photocopy the sketch above of $f(x) = \sin x$. Carefully draw tangents at the points where $x = 0, 0.5, 1, 1.5, \dots, 3$, and also at $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. Measure the gradient of each tangent to two decimal places, and copy and complete the following table.

11. (a) Show that $\frac{1}{2}(\sin(m+n)x + \sin(m-n)x) = \sin mx \cos nx$.
 (b) Hence, without using the product rule, find the derivative of $\sin mx \cos nx$.
 (c) Simplify $\frac{1}{2}(\cos(m+n)x + \cos(m-n)x)$, and hence differentiate $\cos mx \cos nx$.
12. (a) If $y = \sin x$, prove: (i) $\frac{dy}{dx} = \sin(\frac{\pi}{2} + x)$ (ii) $\frac{d^2y}{dx^2} = \sin(\pi + x)$ (iii) $\frac{d^3y}{dx^3} = \sin(\frac{3\pi}{2} + x)$
 (b) Deduce an expression for $\frac{d^ny}{dx^n}$.
13. (a) If $y = \ln(\tan 2x)$, show that $\frac{dy}{dx} = 2 \sec 2x \operatorname{cosec} 2x$.
 (b) Show that $\frac{d}{dx}(\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x) = \sin^4 x \cos^3 x$.
 (c) Show that $\frac{d}{dx} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right) = \sec^2(\frac{\pi}{4} + x)$.
 (d) (i) Show that $\cos 2x = \cos^2 x - \sin^2 x$ by using the formula for $\cos(A+B)$.
 (ii) Hence show that $\frac{d}{dx} \left(\frac{1 + \cos 2x}{1 - \cos 2x} \right) = -2 \cot x \operatorname{cosec}^2 x$.
 (e) Show that $\frac{d}{dx} \left(\ln \frac{\sqrt{2} - \cos x}{\sqrt{2} + \cos x} \right) = \frac{2\sqrt{2} \sin x}{1 + \sin^2 x}$.
14. Given the parametric equations $x = e^t \cos t$, $y = e^t \sin t$, show that $\frac{dy}{dx} = \tan(t + \frac{\pi}{4})$.
15. [First-principles differentiation of $\sin x$, $\cos x$ and $\tan x$]
 (a) Use the definition $\frac{d}{dx} \tan x = \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan x}{h}$ and the usual expansion of $\tan(x+h)$ to prove that the derivative of $\tan x$ is $\sec^2 x$.
 (b) (i) Show that $\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0$ by multiplying top and bottom by $1 + \cos h$.
 (ii) Use the $h \rightarrow 0$ definition of the derivative to show that
- $$\frac{d}{dx} \sin x = \cos x \times \lim_{h \rightarrow 0} \frac{\sin h}{h} - \sin x \times \lim_{h \rightarrow 0} \frac{1 - \cos h}{h},$$
- and then use part (i) to show that the derivative of $\sin x$ is $\cos x$.
 (c) Use methods similar to those in part (b) to prove, using the $h \rightarrow 0$ definition of the derivative, that the derivative of $\cos x$ is $-\sin x$.

EXTENSION

16. Find $\frac{dy}{dx}$ in terms of x and y by differentiating implicitly:
 (a) $\sin x + \cos y = 1$ (b) $x \sin y + y \sin x = 1$ (c) $\sin(x+y) = \cos(x-y)$
17. (a) Use the expansions of $\cos(A+B)$ and $\cos(A-B)$ to prove that
- $$\cos S - \cos T = -2 \sin \frac{1}{2}(S+T) \sin \frac{1}{2}(S-T).$$
- (b) Hence differentiate $\cos x$ from first principles, using $f'(x) = \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x}$.

18. (a) It can be shown that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$. Making the assumption that the series can be differentiated term by term (which is true here, but not for all series), show that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$. Show also that the fourth derivative of each series is the original series.
- (b) Using the first few terms, find $\sin 1$ and $\cos 1$ to four significant figures.

14H Applications of Differentiation

We have now reached an important point in the course where the differentiation of the trigonometric functions can be applied to the analysis of a number of very significant functions. With trigonometric functions, it is usually easier to determine the nature of stationary points by using the second derivative rather than from a table of values of the derivative.

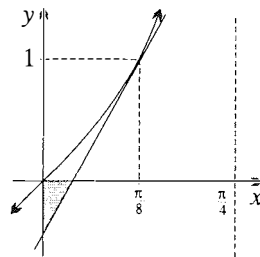
WORKED EXERCISE: Find the x -intercept and y -intercept of the tangent to $y = \tan 2x$ at the point on the curve where $x = \frac{\pi}{8}$, and find the area of the triangle formed by this tangent and the coordinate axes.

SOLUTION: Since $y = \tan 2x$,
 $y' = 2 \sec^2 2x$.

When $x = \frac{\pi}{8}$, $y = 1$
 and $y' = 4$,
 so the tangent is $y - 1 = 4(x - \frac{\pi}{8})$.

When $x = 0$, $y - 1 = -\frac{\pi}{2}$
 $y = \frac{1}{2}(2 - \pi)$,
 and when $y = 0$, $-1 = 4(x - \frac{\pi}{8})$
 $x = \frac{1}{8}(\pi - 2)$,

so area of triangle $= \frac{1}{2} \times \frac{1}{2}(\pi - 2) \times \frac{1}{8}(\pi - 2)$
 $= \frac{1}{32}(\pi - 2)^2$ square units.

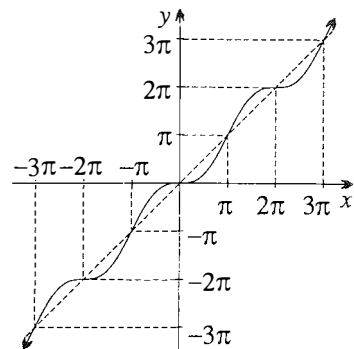


WORKED EXERCISE: Use the standard curve sketching menu to sketch $y = x - \sin x$.

NOTE: This function is essentially the function describing the area of a segment, if the radius in the formula $A = \frac{1}{2}r^2(x - \sin x)$ is held constant while the angle x at the centre varies.

SOLUTION:

1. The domain is the set of all real numbers.
2. $f(x)$ is odd, since both $\sin x$ and x are odd.
3. The function is zero at $x = 0$ and nowhere else, since $\sin x < x$ for $x > 0$, and $\sin x > x$ for $x < 0$.
4. $\sin x$ always remains between -1 and 1 , so $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.
5. $f'(x) = 1 - \cos x$, so $f'(x)$ has zeroes whenever $\cos x = 1$, that is, for $x = \dots, -2\pi, 0, 2\pi, 4\pi, \dots$. But $f'(x)$ is never negative, since $\cos x$ is always between -1 and 1 ,



so each stationary point is a stationary inflexion,
and these points are $\dots, (-2\pi, -2\pi), (0, 0), (2\pi, 2\pi), (4\pi, 4\pi), \dots$

6. $f''(x) = \sin x$,

which is zero for $x = \dots, -\pi, 0, \pi, 2\pi, 3\pi, \dots$

We know that $\sin x$ changes sign around each of these points,

so $\dots, (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi), \dots$ are also inflexions.

Since $f'(\pi) = 1 - (-1) = 2$, the gradient at these other inflexions is 2.

WORKED EXERCISE: [A harder example] Find, as multiples of π , the x -coordinates of the turning points of $y = e^{-x} \sin 4x$, for $0 \leq x \leq 2\pi$. Then sketch the curve, ignoring inflexions.

NOTE: Various forms of this important function describe *damped oscillations*, like the decaying vibrations of a plucked guitar string, or a swing gradually slowing down. The $\sin x$ factor supplies the oscillations, and the e^{-x} factor provides the natural decay.

SOLUTION: For $y = e^{-x} \sin 4x$, using the product rule,

$$\begin{aligned} y' &= -e^{-x} \sin 4x + 4e^{-x} \cos 4x \\ &= e^{-x}(-\sin 4x + 4 \cos 4x). \end{aligned}$$

This has no discontinuities, and has zeroes when

$$\sin 4x = 4 \cos 4x,$$

that is, $\tan 4x = 4$.

The related angle here is about 0.42π , so

$$\begin{aligned} 4x &\doteq 0.42\pi, 1.42\pi, 2.42\pi, 3.42\pi, 4.42\pi, 5.42\pi, 6.42\pi \text{ or } 7.42\pi, \\ x &\doteq 0.11\pi, 0.36\pi, 0.61\pi, 0.86\pi, 1.11\pi, 1.36\pi, 1.61\pi \text{ or } 1.86\pi. \end{aligned}$$

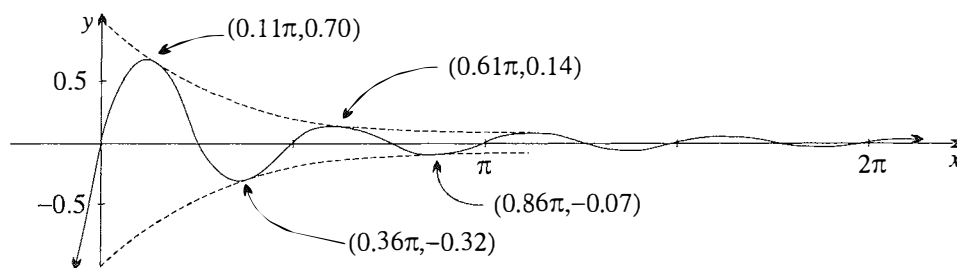
Using the product rule again,

$$\begin{aligned} y'' &= -e^{-x}(-\sin 4x + 4 \cos 4x) + e^{-x}(-4 \cos 4x - 16 \sin 4x) \\ &= -e^{-x}(15 \sin 4x + 8 \cos 4x) \\ &= -e^{-x} \cos 4x(15 \tan 4x + 8) \end{aligned}$$

so when $\tan 4x = 4$, the sign of y'' is opposite to the sign of $\cos 4x$,

and the stationary points are alternately maxima and minima.

Thus $(0.11\pi, 0.6965)$, $(0.61\pi, 0.1448)$, $(1.11\pi, 0.0301)$, $(1.61\pi, 0.0063)$ are maxima, and $(0.36\pi, -0.3175)$, $(0.86\pi, 0.0660)$, $(1.36\pi, -0.0137)$, $(1.86\pi, -0.0029)$ are minima.



WORKED EXERCISE: Two radii OP and OQ of a circle of radius 3 cm meet at an angle θ at the centre O . The angle $\theta = \angle POQ$ between them is increasing at 0.2 radians per minute. Find, when $\theta = \frac{\pi}{3}$, the rate of increase of:

(a) the area of $\triangle OPQ$,

(b) the length of the chord PQ .

SOLUTION: Let A be the area of $\triangle OPQ$, and ℓ be the length of the chord PQ .

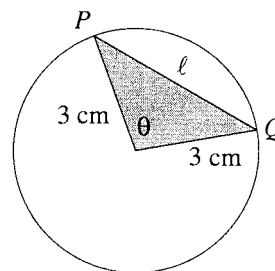
$$\begin{aligned} \text{(a) Since } A &= \frac{1}{2}r^2 \sin \theta = \frac{9}{2} \sin \theta, & \frac{dA}{dt} &= \frac{dA}{d\theta} \times \frac{d\theta}{dt} \\ & & &= \frac{9}{2} \cos \theta \times \frac{d\theta}{dt}, \end{aligned}$$

$$\begin{aligned} \text{and substituting } \frac{d\theta}{dt} &= \frac{1}{5} \text{ and } \theta = \frac{\pi}{3}, & \frac{dA}{dt} &= \frac{9}{2} \times \frac{1}{2} \times \frac{1}{5} \\ & & &= \frac{9}{20} \text{ cm}^2/\text{min}. \end{aligned}$$

$$\begin{aligned} \text{(b) By the cosine rule,} & & \ell^2 &= 9 + 9 - 18 \cos \theta \\ & & \ell &= 3\sqrt{2 - 2 \cos \theta}, \end{aligned}$$

$$\begin{aligned} \text{so by the chain rule,} & & \frac{d\ell}{dt} &= \frac{3 \sin \theta}{\sqrt{2 - 2 \cos \theta}} \times \frac{d\theta}{dt} \end{aligned}$$

$$\begin{aligned} \text{and substituting } \frac{d\theta}{dt} &= \frac{1}{5} \text{ and } \theta = \frac{\pi}{3}, & \frac{d\ell}{dt} &= \frac{\frac{3}{2}\sqrt{3}}{\sqrt{2 - 1}} \times \frac{1}{5} \\ & & &= \frac{3}{10}\sqrt{3} \text{ cm/min}. \end{aligned}$$



Exercise 14H

NOTE: The large number of sketches should allow the alternative of machine sketching of some graphs, followed by algebraic explanation of the features.

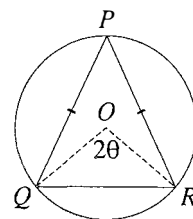
- Find the gradient of the tangent to each of the following curves at the point indicated:
 - $y = \cos x$ at $x = \frac{\pi}{6}$
 - $y = \sin x$ at $x = \frac{\pi}{4}$
 - $y = \tan x$ at $x = 0$
 - $y = \cos 2x$ at $x = \frac{\pi}{4}$
 - $y = \sin \frac{x}{2}$ at $x = \frac{2\pi}{3}$
 - $y = \tan 2x$ at $x = \frac{\pi}{6}$
- Find the equation of the tangent at the given point on each of the following curves:
 - $y = \cos 2x$ at $(\frac{\pi}{4}, 0)$
 - $y = \sin 2x$ at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$
 - $y = x \sin x$ at $(\pi, 0)$
- Find the equations of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.
 - If the tangent meets the x -axis at P and the normal meets the y -axis at Q , find the area of $\triangle OPQ$, where O is the origin.
- Find, in the domain $0 \leq x \leq 2\pi$, the x -coordinates of the points on each of the following curves where the gradient of the tangent is zero.
 - $y = 2 \sin x$
 - $y = 2 \cos x + x$
 - $y = 2 \sin x + \sqrt{3}x$
 - $y = e^{\sin x}$
- Find the equations of the tangent and normal to the curve $y = 2 \sin x - \cos 2x$ at $(\frac{\pi}{6}, \frac{1}{2})$.
 - Find the first and second derivatives of $y = \cos x + \sqrt{3} \sin x$.
 - Find the stationary points in the domain $0 \leq x \leq 2\pi$ and use the second derivative to determine their nature.
 - Find the points of inflexion.
 - Hence sketch the curve for $0 \leq x \leq 2\pi$.
- Repeat the previous question for $y = \cos x - \sin x$. Verify your results by sketching $y = \cos x$ and $y = -\sin x$ on the same diagram and then sketching $y = \cos x - \sin x$ by addition of heights. (Notice that the final graph is a sine wave shifted sideways. This situation will be discussed more fully in the Year 12 Volume.)

8. (a) Find the first and second derivatives of $y = x + \sin x$.
 (b) Find the stationary points in the domain $-2\pi < x < 2\pi$ and determine their nature.
 (c) Find the points of inflexion for $-2\pi < x < 2\pi$.
 (d) Sketch the curve for $-2\pi \leq x \leq 2\pi$.

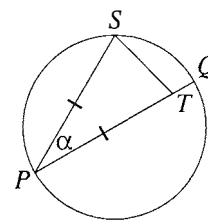
9. Repeat the previous question for $y = x - \cos x$.

DEVELOPMENT

10. (a) Find the first and second derivatives of $y = 2 \sin x + \cos 2x$.
 (b) Show that $y' = 0$ when $\cos x = 0$ or $\sin x = \frac{1}{2}$. (You will need to use the formula $\sin 2x = 2 \sin x \cos x$.)
 (c) Hence find the stationary points in the interval $-\pi \leq x \leq \pi$ and determine their nature.
 (d) Sketch the curve for $-\pi \leq x \leq \pi$ using this information.
11. Repeat the previous question for $y = 2 \cos x + \sin 2x$. (You will need to use the formula $\cos 2x = 1 - 2 \sin^2 x$ to find the stationary points.)
12. (a) Find the first and second derivatives of $y = e^{-x} \cos x$. (Note that this function models damped oscillations.)
 (b) Find the stationary points for $-\pi \leq x \leq \pi$ and determine their nature.
 (c) Find the points of inflexion for $-\pi \leq x \leq \pi$.
 (d) Hence sketch the curve for $-\pi \leq x \leq \pi$.
13. Repeat the previous question for $y = e^x \sin x$. (Note that this function models oscillations such as feedback loops which grow exponentially.)
14. The angle θ between two radii OP and OQ of a circle of radius 6 cm is increasing at the rate of 0.1 radians per minute.
 (a) Show that the area of sector OPQ is increasing at the rate of $1.8 \text{ cm}^2/\text{min}$.
 (b) Find the rate at which the area of $\triangle OPQ$ is increasing at the instant when $\theta = \frac{\pi}{4}$.
 (c) Find the value of θ for which the rate of increase of the area of the segment cut off by the chord PQ is at its maximum.
15. An isosceles triangle has equal sides of length 10 cm. The angle θ between these equal sides is increasing at the rate of 3° per minute. Show that the area of the triangle is increasing at $\frac{5\sqrt{3}\pi}{12} \text{ cm}^2$ per minute at the instant when $\theta = 30^\circ$.
16. A rotating light L is situated at sea 180 metres from the nearest point P on a straight shoreline. The light rotates through one revolution every 10 seconds. Show that the rate at which a ray of light moves along the shore at a point 300 metres from P is $136\pi \text{ m/s}$.
17. PQR is an isosceles triangle inscribed in a circle with centre O of radius one unit, as shown in the diagram. Let $\angle QOR = 2\theta$, where θ is acute.
 (a) Show that the area A of $\triangle PQR$ is given by $A = \sin \theta (\cos \theta + 1)$.
 (b) Hence show that, as θ varies, $\triangle PQR$ has its maximum possible area when it is equilateral.



18. PQ is a diameter of the given circle and S is a point on the circumference. T is the point on PQ such that $PS = PT$. Let $\angle SPT = \alpha$.



- (a) Show that the area A of $\triangle SPT$ is $A = \frac{1}{2}d^2 \cos^2 \alpha \sin \alpha$, where d is the diameter of the circle.
- (b) Hence show that the maximum area of $\triangle SPT$ as S varies on the circle is $\frac{1}{9}d^2\sqrt{3}$ units².
19. A straight line passes through the point $(2, 1)$ and has positive x - and y -intercepts at P and Q respectively. Suppose $\angle OPQ = \alpha$, where O is the origin.
- (a) Explain why the line has gradient $-\tan \alpha$.
- (b) Find the x - and y -intercepts in terms of α .
- (c) Show that the area of $\triangle OPQ$ is given by $A = \frac{(2 \tan \alpha + 1)^2}{2 \tan \alpha}$.
- (d) Hence show that this area is maximised when $\tan \alpha = \frac{1}{2}$.
20. Find the maximum and minimum values of the expression $\frac{2 - \sin \theta}{\cos \theta}$ for $0 \leq \theta \leq \frac{\pi}{4}$, and state the values of θ for which they occur.
21. Find any stationary points and inflexions, then sketch each curve for $0 \leq x \leq 2\pi$:
- (a) $y = 2 \sin x + x$ (b) $y = 2x - \tan x$
22. Find any stationary points and any other important features, then sketch for $0 \leq x \leq 2\pi$:
- (a) $y = \sin^2 x + \cos x$ (b) $y = \sin^3 x \cos x$ (c) $y = \tan^2 x - 2 \tan x$

EXTENSION

23. Find the equation of the tangent to the curve $y = \pi \tan \frac{y}{x} + x - 4$ at the point $(4, \pi)$.
24. (a) Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.
- (b) Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.
- (c) Let $f(x) = \frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$. Find $f'(x)$ and show that $f'(x) < 0$ in the given domain.
- (d) Sketch the graph of $f(x)$ over the given domain and explain why $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$.
25. (a) Find the derivative of $y = e^{\lambda x} \sin nx$, where $\lambda \neq 0$ and $n > 0$.
- (b) Show that in each interval of length $\frac{2\pi}{n}$ there are two stationary points, and explain why one is a maximum and the other is a minimum.
- (c) What happens to the x values of the stationary points as: (i) $\lambda \rightarrow \infty$ (ii) $\lambda \rightarrow 0$?
26. Let $f(x) = \frac{\sin x}{x}$. Remember that we proved in Section 14F that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.
- (a) Write down the domain of $f(x)$, show that $f(x)$ is even, find the zeroes of $f(x)$, and determine $\lim_{x \rightarrow \infty} f(x)$.
- (b) Differentiate $f(x)$, and hence show that $f(x)$ has stationary points when $\tan x = x$.
- (c) Sketch $y = \tan x$ and $y = x$ on one set of axes, and hence use your calculator to estimate the turning points for $0 \leq x \leq 4\pi$. Give the x -coordinates in the form $\lambda\pi$, with λ to no more than two decimal places.
- (d) Using this information, sketch $y = f(x)$.

14I Integration of the Trigonometric Functions

The Standard Forms for Integrating the Trigonometric Functions: Since the derivative of $\sin x$ is $\cos x$, the primitive of $\cos x$ is $\sin x$. Each of the standard forms for differentiation can be reversed in this way, giving the following three standard integrals.

24	STANDARD INTEGRALS:	$\int \cos x \, dx = \sin x$
		$\int \sin x \, dx = -\cos x$
		$\int \sec^2 x \, dx = \tan x$

WORKED EXERCISE: Show that the shaded area of the first arch of $y = \sin x$ is 2 square units.

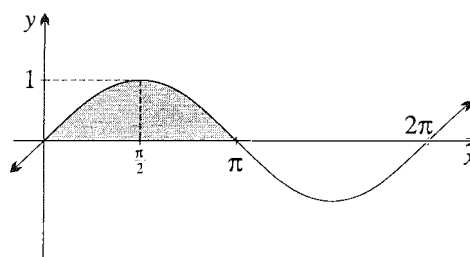
SOLUTION: Area = $\int_0^{\pi} \sin x \, dx$

$$= \left[-\cos x \right]_0^{\pi}$$

$$= -\cos \pi + \cos 0$$

$$= -(-1) + 1$$

$$= 2 \text{ square units.}$$



NOTE: The fact that this area is such a simple number is another confirmation that radians are the correct angle unit to use for the calculus of the trigonometric functions. Comparable simple results were obtained earlier when e was used as the base for logarithms and powers, for example,

$$\int_1^e \log x \, dx = 1 \quad \text{and} \quad \lim_{N \rightarrow \infty} \int_0^N e^{-x} \, dx = 1.$$

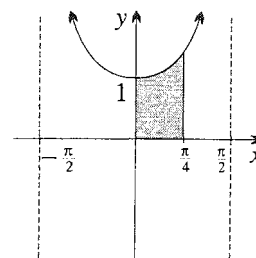
WORKED EXERCISE: Find the volume generated when the shaded area under $y = \sec x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x -axis.

SOLUTION: Volume = $\int_0^{\frac{\pi}{4}} \pi \sec^2 x \, dx$

$$= \pi \left[\tan x \right]_0^{\frac{\pi}{4}}$$

$$= \pi \left(\tan \frac{\pi}{4} - \tan 0 \right)$$

$$= \pi \text{ cubic units.}$$



Replacing x by $ax + b$: Reversing the standard forms for derivatives:

25	FUNCTIONS OF $ax + b$:	$\int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b)$
		$\int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b)$
		$\int \sec^2(ax + b) \, dx = \frac{1}{a} \tan(ax + b)$

WORKED EXERCISE: Evaluate: (a) $\int_0^{\frac{3\pi}{2}} \cos \frac{1}{3}x \, dx$ (b) $\int_0^{\frac{\pi}{8}} \sec^2(2x + \pi) \, dx$

SOLUTION:

$$\begin{aligned}
 \text{(a)} \quad \int_0^{\frac{3\pi}{2}} \cos \frac{1}{3}x \, dx &= 3 \left[\sin \frac{1}{3}x \right]_0^{\frac{3\pi}{2}} \\
 &= 3(\sin \frac{\pi}{2} - 3 \sin 0) \\
 &= 3
 \end{aligned}
 \qquad
 \begin{aligned}
 \text{(b)} \quad \int_0^{\frac{\pi}{8}} \sec^2(2x + \pi) \, dx &= \frac{1}{2} \left[\tan(2x + \pi) \right]_0^{\frac{\pi}{8}} \\
 &= \frac{1}{2}(\tan \frac{5\pi}{4} - \tan \pi) \\
 &= \frac{1}{2}
 \end{aligned}$$

Given a Derivative, Find an Integral: As always, the results of chain-rule and product-rule differentiations can be reversed to give a primitive.

WORKED EXERCISE:

(a) Use the product rule to differentiate $x \sin x$.

(b) Hence find $\int x \cos x \, dx$.

SOLUTION:

(a) Let $y = x \sin x$. By the product rule, $y' = vu' + uv'$ $\qquad\qquad\qquad = \sin x + x \cos x$.	Let $u = x$ and $v = \sin x$. Then $u' = 1$ and $v' = \cos x$.

(b) Reversing this result,

$$\begin{aligned}
 \int (\sin x + x \cos x) \, dx &= x \sin x \\
 \int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\
 &= x \sin x + \cos x + C, \text{ for some constant } C.
 \end{aligned}$$

WORKED EXERCISE:

(a) Use the chain rule to differentiate $\cos^5 x$.

(b) Hence find $\int_0^{\pi} \sin x \cos^4 x \, dx$.

SOLUTION:

(a) Let $y = \cos^5 x$. By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\qquad\qquad\qquad = -5 \sin x \cos^4 x$.	Let $u = \cos x$, then $y = u^5$. So $\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = 5u^4$.

(b) Reversing this result,

$$\begin{aligned}
 \int_0^{\pi} (-5 \sin x \cos^4 x) \, dx &= \left[\cos^5 x \right]_0^{\pi} \\
 \int_0^{\pi} \sin x \cos^4 x \, dx &= -\frac{1}{5} \left[\cos^5 x \right]_0^{\pi} \\
 &= -\frac{1}{5}(-1 - 1) \\
 &= \frac{2}{5}.
 \end{aligned}$$

Using the Reverse Chain Rule: The integral in the previous worked exercise could have been evaluated directly using the reverse chain rule.

WORKED EXERCISE: Use the reverse chain rule to evaluate:

$$(a) \int_0^{\pi} \sin x \cos^4 x \, dx$$

$$(b) \int_0^{\frac{\pi}{3}} \tan^7 x \sec^2 x \, dx$$

SOLUTION:

$$\begin{aligned} (a) \int_0^{\pi} \sin x \cos^4 x \, dx &= - \int_0^{\pi} (-\sin x) \cos^4 x \, dx \\ &= -\frac{1}{5} \left[\cos^5 x \right]_0^{\pi} \\ &= \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \cos x. \\ \text{Then } \frac{du}{dx} &= -\sin x. \\ \int u^4 \frac{du}{dx} dx &= \frac{1}{5} u^5 \end{aligned}$$

$$\begin{aligned} (b) \int_0^{\frac{\pi}{3}} \tan^7 x \sec^2 x \, dx &= \frac{1}{8} \left[\tan^8 x \right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{8} (\tan^8 \frac{\pi}{3} - \tan^8 0) \\ &= \frac{1}{8} \times (\sqrt{3})^8 \\ &= \frac{81}{8} \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \tan x. \\ \text{Then } \frac{du}{dx} &= \sec^2 x. \\ \int u^7 \frac{du}{dx} dx &= \frac{1}{8} u^8 \end{aligned}$$

The Primitives of $\tan x$ and $\cot x$: The primitives of $\tan x$ and $\cot x$ can be found by using the ratio formulae $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$, followed by the reverse chain rule.

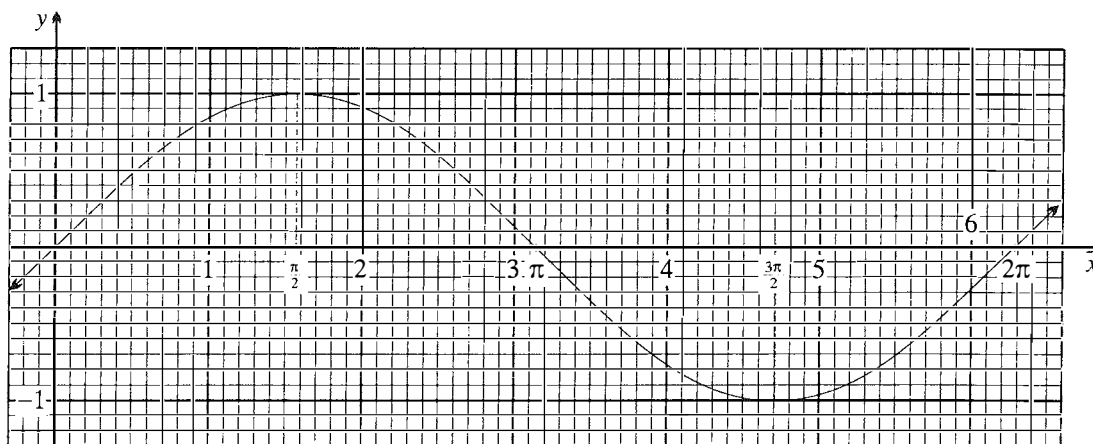
WORKED EXERCISE: Find the primitive of $\tan x$.

$$\begin{aligned} \text{SOLUTION: } \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= - \int \frac{-\sin x}{\cos x} \, dx \\ &= -\log(\cos x) + C \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \cos x. \\ \text{Then } \frac{du}{dx} &= -\sin x. \\ \int \frac{1}{u} \frac{du}{dx} dx &= \log u \end{aligned}$$

Exercise 14I

1.



- (a) The first worked exercise in the theory for this section proved that $\int_0^{\pi} \sin x \, dx = 2$. Count squares on the graph of $y = \sin x$ drawn above to confirm this result.

(b) Count squares and use symmetry to find:

$$\begin{array}{lll} \text{(i)} \int_0^{\frac{\pi}{4}} \sin x \, dx & \text{(iii)} \int_0^{\frac{3\pi}{4}} \sin x \, dx & \text{(v)} \int_0^{\frac{3\pi}{2}} \sin x \, dx \\ \text{(ii)} \int_0^{\frac{\pi}{2}} \sin x \, dx & \text{(iv)} \int_0^{\frac{5\pi}{4}} \sin x \, dx & \text{(vi)} \int_0^{\frac{7\pi}{4}} \sin x \, dx \end{array}$$

(c) Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$ and confirm the results of (b).

2. Find:

$$\begin{array}{llll} \text{(a)} \int \sec^2 x \, dx & \text{(c)} \int \sin 2x \, dx & \text{(e)} \int \cos(3x - 2) \, dx & \text{(g)} \int \sec^2(4 - x) \, dx \\ \text{(b)} \int \cos(x + 2) \, dx & \text{(d)} \int \sec^2\left(\frac{1}{3}x\right) \, dx & \text{(f)} \int \sin(7 - 5x) \, dx & \text{(h)} \int \sec^2\left(\frac{1-x}{3}\right) \, dx \end{array}$$

3. Find the value of:

$$\begin{array}{llll} \text{(a)} \int_0^{\frac{\pi}{6}} \cos x \, dx & \text{(c)} \int_0^{\frac{\pi}{3}} \sec^2 x \, dx & \text{(e)} \int_0^{\frac{\pi}{3}} \sin 2x \, dx & \text{(g)} \int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{1}{2}x\right) \, dx \\ \text{(b)} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \, dx & \text{(d)} \int_0^{\frac{\pi}{4}} 2 \cos 2x \, dx & \text{(f)} \int_0^{\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right) \, dx & \text{(h)} \int_0^{\pi} (2 \sin x - \sin 2x) \, dx \end{array}$$

4. Find: (a) $\int (6 \cos 3x - 4 \sin \frac{1}{2}x) \, dx$ (b) $\int (8 \sec^2 2x - 10 \cos \frac{1}{4}x + 12 \sin \frac{1}{3}x) \, dx$

DEVELOPMENT

5. Find the following indefinite integrals, where a , b , u and v are constants:

$$\text{(a)} \int a \sin(ax + b) \, dx \quad \text{(b)} \int \pi^2 \cos \pi x \, dx \quad \text{(c)} \int \frac{1}{u} \sec^2(v + ux) \, dx \quad \text{(d)} \int \frac{a}{\cos^2 ax} \, dx$$

6. (a) Copy and complete $1 + \tan^2 x = \dots$, and hence find $\int \tan^2 x \, dx$.

(b) Simplify $1 - \sin^2 x$, and hence find the value of $\int_0^{\frac{\pi}{3}} \frac{2}{1 - \sin^2 x} \, dx$.

7. Comment on the validity and the result of the following computation:

$$\int_0^{\pi} \sec^2 x \, dx = [\tan x]_0^{\pi} = \tan \pi - \tan 0 = 0.$$

8. (a) Copy and complete $\int \frac{f'(x)}{f(x)} \, dx = \dots$, and hence show that $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x} \, dx \doteq 0.4$.

(b) Write $\tan x$ in terms of $\sin x$ and $\cos x$, and hence show that $\int_0^{\frac{\pi}{4}} \tan x \, dx = \frac{1}{2} \ln 2$.

(c) Differentiate $e^{\sin x}$, and hence find the value of $\int_0^{\frac{\pi}{2}} \cos x e^{\sin x} \, dx$.

(d) Copy and complete $\int f'(x)e^{f(x)} \, dx = \dots$, and hence find $\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x} \, dx$.

9. (a) Find $\frac{d}{dx}(\sin x^2)$, and hence find $\int 2x \cos x^2 \, dx$.

(b) Find $\frac{d}{dx}(\cos x^3)$, and hence find $\int x^2 \sin x^3 \, dx$.

(c) Find $\frac{d}{dx}(\tan \sqrt{x})$, and hence find $\int \frac{1}{\sqrt{x}} \sec^2 \sqrt{x} \, dx$.

10. (a) Show that $\frac{d}{dx}(\sin x - x \cos x) = x \sin x$, and hence find $\int_0^{\frac{\pi}{2}} x \sin x \, dx$.

(b) Show that $\frac{d}{dx}(\frac{1}{3} \cos^3 x - \cos x) = \sin^3 x$, and hence find $\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$.

11. (a) Find $\frac{d}{dx}(\sin^5 x)$, and hence find $\int \sin^4 x \cos x \, dx$.

(b) Find $\frac{d}{dx}(\tan x)^{-3}$, and hence find $\int \frac{\sec^2 x}{\tan^4 x} \, dx$.

12. Use the rule $\int u^n \frac{du}{dx} \, dx = \frac{u^{n+1}}{n+1}$ or $\int f'(x)f(x)^n \, dx = \frac{(f(x))^{n+1}}{n+1}$, to evaluate:

(a) $\int_0^{\pi} \sin x \cos^8 x \, dx$ (c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin^7 x \, dx$ (e) $\int_0^{\frac{\pi}{3}} \sec^2 x \tan^7 x \, dx$

(b) $\int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx$ (d) $\int_0^{\frac{\pi}{6}} \cos x \sin^n x \, dx$ (f) $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^n x \, dx$

13. (a) Use the compound angle formula for $\sin(A+B)$ to show that $\sin 2x = 2 \sin x \cos x$.

(b) Hence find: (i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \cos x \, dx$ (ii) $\int_0^{\frac{\pi}{4}} \sin 2x \cos 2x \, dx$ (iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 2x \cos 2x \, dx$

14. (a) Use the compound angle formula for $\cos(A+B)$ to show that $\cos 2x = \cos^2 x - \sin^2 x$.

(b) Hence show that: (i) $\cos 2x = 1 - 2 \sin^2 x$ (ii) $\cos 2x = 2 \cos^2 x - 1$

(c) Hence show that: (i) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (ii) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

(d) Hence find: (i) $\int \sin^2 x \, dx$ (ii) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

(e) Use part (c)(i) to write $\sin^2 2x$ in terms of $\cos 4x$, and hence find $\int_0^{\frac{\pi}{2}} \sin^2 2x \, dx$.

(f) Find: (i) $\int_0^{\frac{\pi}{3}} \cos^2 \frac{x}{2} \, dx$ (ii) $\int_0^{\frac{\pi}{4}} \sin^2 \frac{x}{2} \, dx$

15. Find:

(a) $\int e^{2x} \cos e^{2x} \, dx$ (c) $\int \frac{\sec^2 x}{3 \tan x + 1} \, dx$ (e) $\int \frac{1 - \cos^3 x}{1 - \sin^2 x} \, dx$

(b) $\int \frac{\sin e^{-2x}}{e^{2x}} \, dx$ (d) $\int \frac{3 \sin x}{4 + 5 \cos x} \, dx$ (f) $\int_0^{\pi} \sin x \cos^2 x \, dx$

16. Find $\frac{d}{dx}(x \sin 2x)$, and hence find $\int_0^{\frac{\pi}{4}} x \cos 2x \, dx$.

17. (a) Show that $\frac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$. (b) Hence find $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$.

18. (a) Show that $\tan^3 x = \tan x \sec^2 x - \tan x$.

(b) Hence find: (i) $\int \tan^3 x \, dx$ (ii) $\int \tan^5 x \, dx$

19. (a) Show that $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$.

(b) Hence find: (i) $\int_0^{\frac{\pi}{2}} 2 \sin 3x \cos 2x \, dx$ (ii) $\int_0^{\pi} \sin 3x \cos 4x \, dx$

- (c) Show that $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ for positive integers m and n :
- (i) using the primitive, (ii) using symmetry arguments.
20. (a) Show that $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$.
- (b) Hence find: (i) $\int_0^{\frac{\pi}{2}} 2 \cos 3x \cos 2x \, dx$ (ii) $\int_0^{\pi} \cos 3x \cos 4x \, dx$
- (c) Find $\int \cos mx \cos nx \, dx$, where m and n are:
- (i) distinct positive integers, (ii) equal positive integers.
- (d) Show that for positive integers m and n , $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$
21. (a) Show, by finding the integral in two different ways, that for constants C and D ,
- $$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C = -\frac{1}{4} \cos 2x + D.$$
- (b) How may the two answers be reconciled?

————— EXTENSION —————

22. (a) Find the values of A and B in the identity
- $$A(2 \sin x + \cos x) + B(2 \cos x - \sin x) = 7 \sin x + 11 \cos x.$$
- (b) Hence show that $\int_0^{\frac{\pi}{2}} \frac{7 \sin x + 11 \cos x}{2 \sin x + \cos x} \, dx = \frac{5\pi}{2} + \ln 8$.
23. (a) If $y = Ae^{\lambda x} \sin x + Be^{\lambda x} \cos x$, show that $y'' - 2\lambda y' + (\lambda^2 + 1)y = 0$.
- (b) A function satisfies $y'' - 2\lambda y' + (\lambda^2 + 1)y = 0$, and $y(0) = y'(0) = 1$. Evaluate A and B , and hence find the equation of the function.
24. [The power series for $\sin x$ and $\cos x$] Here is the outline of a proof of the two power series for $\sin x$ and $\cos x$, introduced informally in the Extension group of Exercise 14G:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

- (a) We know that $\cos t \leq 1$, for t positive, since $\cos t$ was defined as the ratio of a semichord of a circle over the radius. Integrate this inequality over the interval $0 \leq t \leq x$, where x is positive, and hence show that $\sin x \leq x$.
- (b) Change the variable to t , integrate the inequality $\sin t \leq t$ over $0 \leq t \leq x$, and hence show that $\cos x \geq 1 - \frac{x^2}{2!}$.

- (c) Do it twice more, and show that: (i) $\sin x \geq x - \frac{x^3}{3!}$ (ii) $\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

- (d) Now use induction (informally) to show that for all positive integers n ,

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \leq \sin x + \frac{x^{4n+3}}{(4n+3)!},$$

and use this inequality to conclude that $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converges, with limit $\sin x$.

- (e) Proceeding similarly, prove that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ converges, with limit $\cos x$.

- (f) Use evenness and oddness to extend the results of (d) and (e) to negative values of x .

14J Applications of Integration

The integrals of the trigonometric functions can now be applied to finding areas and volumes, and to finding functions whose derivatives are known.

WORKED EXERCISE: If $f'(x) = \cos x + \sin 2x$ and $f(\pi) = 0$, find $f(\frac{\pi}{4})$.

SOLUTION: $f(x) = \sin x - \frac{1}{2} \cos 2x + C$, for some constant C .

Substituting $f(\pi) = 0$, $0 = 0 - \frac{1}{2} + C$,

so $C = \frac{1}{2}$ and $f(x) = \sin x - \frac{1}{2} \cos 2x + \frac{1}{2}$.

Substituting $x = \frac{\pi}{4}$, $f(\frac{\pi}{4}) = \frac{1}{2}\sqrt{2} - 0 + \frac{1}{2}$
 $= \frac{1}{2}(1 + \sqrt{2})$.

WORKED EXERCISE: Find where the curves $y = \sin x$ and $y = \cos x$ intersect in the interval $0 \leq x \leq 2\pi$, sketch the curves, and find the area contained between them.

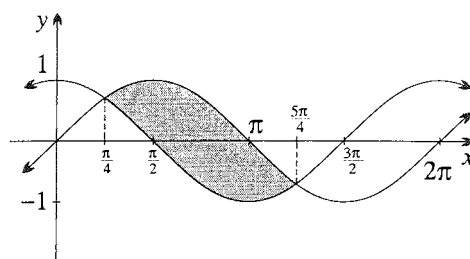
SOLUTION: Put $\sin x = \cos x$,

then $\tan x = 1$

$$x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4},$$

so they intersect at $(\frac{\pi}{4}, \frac{1}{2}\sqrt{2})$ and $(\frac{5\pi}{4}, -\frac{1}{2}\sqrt{2})$.

$$\begin{aligned} \text{Area between} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) dx \\ &= \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} + \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \\ &= 4 \times \frac{1}{2}\sqrt{2} \\ &= 2\sqrt{2} \text{ square units.} \end{aligned}$$



WORKED EXERCISE:

(a) Differentiate $y = \log(\cos x)$, and hence find $\int \tan x dx$.

(b) Sketch $y = 1 + \tan x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, then, using the integral in part (a), find the volume generated when the area under the curve from $x = 0$ to $x = \frac{\pi}{4}$ is rotated about the x -axis.

SOLUTION:

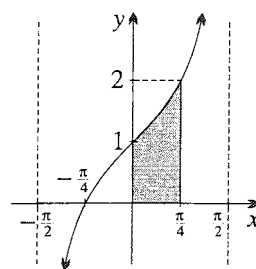
(a) By the chain rule, $\frac{d}{dx} \log(\cos x) = \frac{1}{\cos x} \times (-\sin x)$,
 $= -\tan x$.

Reversing this, $\int \tan x dx = -\log(\cos x) + C$, for some constant C .

$$\begin{aligned} \text{(b) Volume} &= \int_0^{\frac{\pi}{4}} \pi(1 + \tan x)^2 dx \\ &= \pi \int_0^{\frac{\pi}{4}} (1 + 2 \tan x + \tan^2 x) dx. \end{aligned}$$

Using the Pythagorean identity $1 + \tan^2 x = \sec^2 x$,

$$\begin{aligned} \text{volume} &= \pi \int_0^{\frac{\pi}{4}} (\sec^2 x + 2 \tan x) dx \\ &= \pi \left[\tan x - 2 \log(\cos x) \right]_0^{\frac{\pi}{4}} \end{aligned}$$



$$\begin{aligned}
 &= \pi \left(\tan \frac{\pi}{4} - 2 \log \frac{1}{\sqrt{2}} - \tan 0 + 2 \log 1 \right) \\
 &= \pi \left(1 + 2 \times \frac{1}{2} \log 2 - 0 + 0 \right) \\
 &= \pi(1 + \log 2) \text{ cubic units.}
 \end{aligned}$$

Exercise 14J

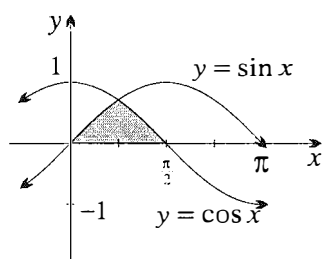
1. Find, using a diagram, the area bounded by one arch of each curve and the x -axis:

(a) $y = \sin x$

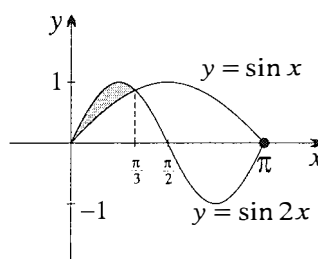
(b) $y = \cos 2x$

2. Calculate the area of the shaded region in each diagram below:

(a)



(b)



3. Sketch the area enclosed between each curve and the x -axis over the specified domain, and then find the exact value of the area. (Make use of symmetry wherever possible.)

(a) $y = \cos x$ from $x = 0$ to $x = \pi$

(e) $y = \sin 2x$ from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$

(b) $y = \sin x$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$

(f) $y = \sec^2 \frac{1}{2}x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

(c) $y = \sec^2 x$ from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$

(g) $y = \sin x$ from $x = -\frac{5\pi}{6}$ to $x = \frac{7\pi}{6}$

(d) $y = \cos 2x$ from $x = 0$ to $x = \pi$

(h) $y = \cos 3x$ from $x = \frac{\pi}{6}$ to $x = \frac{2\pi}{3}$

4. (a) The gradient function of a certain curve is given by $y' = \cos x - 2 \sin 2x$. If the curve passes through the origin, find its equation.

(b) If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(\frac{1}{6})$.

(c) If $f'(x) = \pi \cos \pi x$ and $f(0) = 0$, find $f(\frac{1}{3})$.

(d) If $f''(x) = 18 \cos 3x$ and $f'(0) = f(\frac{\pi}{2}) = 1$, find $f(x)$.

5. Find the exact volume of the solid of revolution formed when each region described below is rotated about the x -axis:

(a) the region bounded by the curve $y = \sec x$ and the x -axis from $x = 0$ to $x = \frac{\pi}{4}$,

(b) the region bounded by the curve $y = \sqrt{\cos 4x}$ and the x -axis from $x = 0$ to $x = \frac{\pi}{8}$,

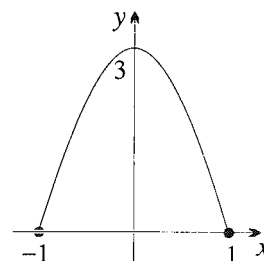
(c) the region bounded by the curve $y = \sqrt{1 + \sin 2x}$ and the x -axis from $x = 0$ to $x = \frac{\pi}{4}$.

DEVELOPMENT

6. (a) Sketch the curve $y = 2 \cos \pi x$ for $-1 \leq x \leq 1$, clearly marking the two x -intercepts.

(b) Find the exact area bounded by the curve $y = 2 \cos \pi x$ and the x -axis, between the two x -intercepts.

7. An arch window 3 metres high and 2 metres wide is made in the shape of the curve $y = 3 \cos(\frac{\pi}{2}x)$. Find the area of the window in square metres, correct to one decimal place.



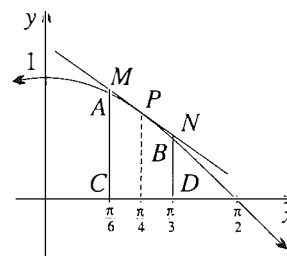
8. (a) Show that $\int_0^{2\pi} \sin nx \, dx = \int_0^{2\pi} \cos nx \, dx = 0$, for all positive integers n .

- (b) Sketch each graph, and then find the area from $x = 0$ to $x = 2\pi$ between the x -axis and: (i) $y = \sin x$ (ii) $y = \sin 2x$ (iii) $y = \sin 3x$ (iv) $y = \sin nx$ (v) $y = \cos nx$
9. (a) Show that $\int_0^1 \sin \pi x \, dx = \frac{2}{\pi}$. (b) Use Simpson's rule with five function values to approximate $\int_0^1 \sin \pi x \, dx$. (c) Hence show that $\pi(1 + 2\sqrt{2}) \div 12$.
10. The graphs of $y = x - \sin x$ and $y = x$ are sketched together in the notes to Section 14H. Find the total area enclosed between these graphs from $x = 0$ to $x = 2\pi$.
11. The region R is bounded by the curve $y = \tan x$, the x -axis and the vertical line $x = \frac{\pi}{3}$.
 (a) Sketch R and then find its area.
 (b) Find the volume generated when R is rotated about the x -axis.
12. (a) Express $\cot x$ as the ratio of $\cos x$ and $\sin x$, and hence find a primitive of $\cot x$.
 (b) Find the area between the curve $y = \cot x$ and the x -axis, from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.
 (c) Comment on the validity and the results of the following calculations:
 (i) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x \, dx = \left[\log \sin x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \log\left(\frac{1}{2}\sqrt{2}\right) - \log\left(\frac{1}{2}\sqrt{2}\right) = 0$
 (ii) $\int_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} \cot x \, dx = \left[\log \sin x \right]_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} = \log\left(\frac{1}{2}\sqrt{2}\right) - \log\left(\frac{1}{2}\sqrt{2}\right) = 0$
13. The region R is bounded by the curve $y = \sin x$, the x -axis and the vertical line $x = \frac{\pi}{2}$.
 (a) Sketch R and then find its area.
 (b) Find the exact volume of the solid generated when R is rotated through one complete revolution about the x -axis.
14. The region R is bounded by the curve $y = \cos 2x$, the x -axis and the lines $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{6}$. (a) Sketch R and then find its area.
 (b) Find the exact volume generated when the region R is rotated about the x -axis.
15. Show that for $0 < b < \frac{\pi}{2}$, the volume generated by rotating $y = \tan x$ from $x = 0$ to $x = b$ about the x -axis is πb less than the volume generated by rotating $y = \sec x$.
16. (a) Sketch the region bounded by the graphs of $y = \sin x$, $y = \cos x$, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.
 (b) Find the area of the region in part (a).
17. (a) Show that the curves $y = \sin x$ and $y = \cos 2x$ meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.
 (b) Sketch the curves $y = \sin x$ and $y = \cos 2x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{6}$.
 (c) Hence find the area of the region bounded by the two curves.
18. (a) Show that $\int_0^n (1 + \sin 2\pi x) \, dx = n$, for all positive integers n .
 (b) Sketch $y = 1 + \sin 2\pi x$, and interpret the result geometrically.
19. (a) Find $\lim_{n \rightarrow \infty} \int_0^a \sin nx \, dx$, where a is any real number. (b) Explain geometrically what is happening in part (a).
20. (a) Show that $\sqrt{2} \sin(x + \frac{\pi}{4}) = \sin x + \cos x$. (b) Hence, or otherwise, find:
 (i) the exact area under one arch of the curve $y = \sin x + \cos x$,
 (ii) the exact volume generated when the arch is rotated about the x -axis.

21. (a) Sketch $y = 1 - \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes. (b) Find:
- (i) the area of R , (ii) the volume generated when R is rotated about the x -axis.
22. A champagne glass is designed by rotating the curve $y = 4 + 4 \sin \frac{x}{4}$ from $x = 4\pi$ to $x = 6\pi$ about the x -axis through 360° . Find, in millilitres correct to two decimal places, the capacity of the glass, if 1 unit = 1 cm.
23. Sketch $y = |\sin x|$ for $0 \leq x \leq 6\pi$, and hence evaluate $\int_0^{6\pi} |\sin x| dx$.
24. (a) Explain why $x^2 \sin x < x^3 < x^2 \tan x$ for $0 < x < \frac{\pi}{2}$.
- (b) Hence show that $\int_0^{\frac{\pi}{4}} x^2 \sin x dx < \frac{\pi^4}{4^5} < \int_0^{\frac{\pi}{4}} x^2 \tan x dx$.
25. (a) Given that $y = \frac{1}{1 + \sin x}$, show that $y' = -\frac{\cos x}{(1 + \sin x)^2}$.
- (b) Hence explain why the function $y = \frac{1}{1 + \sin x}$ is decreasing for $0 < x < \frac{\pi}{2}$.
- (c) Sketch the curve for $0 \leq x \leq \frac{\pi}{2}$, and hence show that $\frac{\pi}{4} < \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx < \frac{\pi}{2}$.
26. Use symmetry arguments to help evaluate:
- (a) $\int_{-4\pi}^{4\pi} \sin 3x dx$ (c) $\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}} \cos x dx$ (e) $\int_{-\pi}^{\pi} (3 + 2x + \sin x) dx$
- (b) $\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x dx$ (d) $\int_{-\pi}^{\pi} \sec^2 \frac{1}{3} x dx$ (f) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 2x + \cos 3x + 3x^2) dx$

EXTENSION

27. In the diagram, P is the point where $x = \frac{\pi}{4}$ on the curve $y = \cos x$. The vertical lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ meet the curve at A and B respectively, and they meet the tangent at P at M and N respectively.
- (a) Show that the tangent at P is $x + \sqrt{2}y = 1 + \frac{\pi}{4}$.
- (b) Show that M has y -coordinate $\frac{1}{24}\sqrt{2}(12 + \pi)$ and find the y -coordinate of N .
- (c) Find the areas of the trapezia $ABDC$ and $MNDC$.
- (d) Hence show, without using a calculator, that $3\sqrt{2}(\sqrt{3} - 1) < \pi < 6(\sqrt{3} - 1)^2$.



28. (a) Show that $\frac{d}{dx} \left(-\frac{1}{2} e^{-x} (\sin x + \cos x) \right) = e^{-x} \sin x$.
- (b) Find $\int_0^N e^{-x} \sin x dx$, and show that $\int_0^\infty e^{-x} \sin x dx$ converges to $\frac{1}{2}$.
- (c) Find $\int_0^\pi e^{-x} \sin x dx$, $\int_{2\pi}^{3\pi} e^{-x} \sin x dx$, ..., and show that the areas of the arches above the x -axis form a GP with limiting sum $\frac{e^\pi}{2(e^\pi - 1)}$.
- (d) Show that the areas of the arches below the x -axis also form a GP, and hence show that the total area contained between the curve and the x -axis, to the right of the y -axis, is $\frac{e^\pi + 1}{2(e^\pi - 1)}$. Also confirm by subtraction the result of part (b).

Answers to Exercises

Chapter One

Exercise 1A (Page 2)

- 1(a) $8x + 4y$ (b) $-3a^2 + 4a$ (c) $-5x^2 - 12x - 3$
(d) $9a - 3b - 5c$
2(a) $6x$ (b) $20a$ (c) $5ab + bc$
(d) $2x^3 - 5x^2y + 2xy^2 + 3y^3$
3(a) $2x$ (b) $4x$ (c) $-6a$ (d) $-4b$
4(a) $2x^2 - 2x + 4$ (b) $3a - 5b - 4c$
(c) $-3a + 2b - 2c + 2d$ (d) $2ab - 2bc + 2cd$
5(a) $2x^2 - 2x$ (b) $6x^2y + 2y^3$ (c) $a^3 - c^3 - abc$
(d) $4x^4 - 5x^3 - 2x^2 - x + 2$
6(a) $10a$ (b) $-18x$ (c) $-3a^2$ (d) $6a^3b$ (e) $-8x^5$
(f) $-6p^3q^4$
7(a) $6a^5b^6$ (b) $-24a^4b^8$ (c) $9a^6$ (d) $-8a^{12}b^3$
8(a) 18 (b) 2
9(a) 59 (b) 40
10(a) 5 (b) $-7x^2$ (c) $12a$ (d) $-3x^3y^4z$
11(a) -2 (b) $3x$ (c) xy (d) $-a^4$ (e) $-7ab^3$
(f) $5ab^2c^6$
12(a) $3a^2$ (b) $5c^4$ (c) a^2bc^6
13(a) $2x^5$ (b) $9xy^5$ (c) b^4 (d) $2a^3$
14 $-x^3 + 3x^2 + 7x - 8$
15 $-b + 11c$
16 $8d - 14c - 2b$
17 $-18x^{25}y^{22}$
18(a) $0 \leq x \leq 2$ (b) $x \leq -\sqrt{3}$ or $0 \leq x \leq \sqrt{3}$

Exercise 1B (Page 4)

- 1(a) $4a + 8b$ (b) $x^2 - 7x$ (c) $-3x + 6y$ (d) $-a^2 - 4a$
(e) $5a + 15b - 10c$ (f) $-6x + 9y - 15z$
(g) $-2x^4 + 4x^3 + 6x^2 - 2x$ (h) $6x^3y^2 - 15x^4y$
(i) $-2a^4b^4 + 4a^5b^2$
2(a) $x + 4$ (b) $-8a - 3b + 5c$
(c) $-x^5 - 25x^4 + 10x^3 + 13x^2 - 6x$
(d) $-12x^5y^5 + 14x^4y^6 - 13x^3y^8$

- 3(a) $x^2 + 5x + 6$ (b) $2a^2 + 13a + 15$ (c) $x^2 - 2x - 8$
(d) $2b^2 - 13b + 21$ (e) $12x^2 + 17x - 40$
(f) $30 - 71x + 42x^2$
5(a) $x^2 - 2xy + y^2$ (b) $a^2 + 6a + 9$ (c) $n^2 - 10n + 25$
(d) $c^2 - 4$ (e) $4a^2 + 4a + 1$ (f) $9p^2 - 12p + 4$
(g) $9x^2 - 16y^2$ (h) $16y^2 - 40xy + 25x^2$
6(a) $a^2 - 4b^2$ (b) $10 - 17x - 20x^2$ (c) $16x^2 + 56x + 49$
(d) $x^4 - x^2y - 12y^2$ (e) $a^2 - ac - b^2 + bc$ (f) $27x^3 + 1$
7(a) $t^2 + 2 + \frac{1}{t^2}$ (b) $t^2 - 2 + \frac{1}{t^2}$ (c) $t^2 - \frac{1}{t^2}$
8(a) $a^2 - b^2 - c^2 + 2bc$ (b) $x^2 - 2x + 3$
9(a) $11x - 3$ (b) $-4b + 8c - 8$
10(a) 10404 (b) 998001 (c) 39991
11(a) $2ab - b^2$ (b) $2x + 3$ (c) $18 - 6a$ (d) $4pq$
(e) $x^2 + 2x - 1$ (f) $a^2 - 2a - 6$
12 $7x^2 + 16ax + 4a^2$
13(a) $x^3 - 6x^2 + 12x - 8$ (b) $x^2 + y^2 + z^2$
(c) $x^2 - y^2 - z^2 + 2yz$ (d) $a^3 + b^3 + c^3 - 3abc$

Exercise 1C (Page 6)

- 1(a) $a(x - y)$ (b) $x(x + 3)$ (c) $3a(a - 2b)$
(d) $6x(2x + 3)$ (e) $2a^3(3 + a + 2a^2)$
(f) $7xy(x^2 - 2xy + 3y)$
2(a) $(x - y)(a + b)$ (b) $(a + b)(a + c)$ (c) $(x - 3)(x - y)$
(d) $(2a - b)(x - y)$ (e) $(b + c)(a - 1)$
(f) $(x - 3)(2x^2 - a)$
3(a) $(x - 3)(x + 3)$ (b) $(1 - a)(1 + a)$
(c) $(2x - y)(2x + y)$ (d) $(5x - 4)(5x + 4)$
(e) $(1 - 7k)(1 + 7k)$ (f) $(9ab - 8)(9ab + 8)$
4(a) $(x + 3)(x + 5)$ (b) $(x - 1)(x - 3)$ (c) $(a + 4)(a - 2)$
(d) $(y - 7)(y + 4)$ (e) $(c - 3)(c - 9)$ (f) $(p + 12)(p - 3)$
(g) $(u - 20)(u + 4)$ (h) $(x - 3)(x - 17)$
(i) $(t + 25)(t - 2)$ (j) $(x - 15)(x + 6)$
(k) $(x - 2y)(x - 3y)$ (l) $(x + 2y)(x + 4y)$
(m) $(a - 3b)(a + 2b)$ (n) $(p + 8q)(p - 5q)$
(o) $(c - 11d)(c - 13d)$
5(a) $(2x + 1)(x + 2)$ (b) $(3x + 2)(x + 2)$
(c) $(3x - 1)(2x - 3)$ (d) $(3x - 1)(x + 5)$

- (e) $(3x-4)(3x+2)$ (f) $(2x-3)(3x+1)$
 (g) $(3x-1)(2x-1)$ (h) $(3x-5)(x+6)$
 (i) $(3x-4)(4x+3)$ (j) $(12x-5)(x+3)$
 (k) $(4x-5)(6x-5)$ (l) $(2x-y)(x+y)$
 (m) $(2a-b)(2a-3b)$ (n) $(3p+4q)(2p-q)$
 (o) $(9u+4v)(2u-3v)$
 6(a) $3(a-2)(a+2)$ (b) $(x-y)(x+y)(x^2+y^2)$
 (c) $x(x-1)(x+1)$ (d) $5(x+2)(x-3)$
 (e) $y(5-y)(5+y)$ (f) $(2-a)(2+a)(4+a^2)$
 (g) $2(2x-3)(x+5)$ (h) $x(x-1)(x-7)$
 (i) $(x-2)(x+2)(x^2+1)$ (j) $(x-1)(x+1)(a-2)$
 (k) $m(4m-n)(4m+n)$ (l) $a(x-5a)(x+4a)$
 7(a) $(9-x)(8+x)$ (b) $(a-b-c)(a-b+c)$
 (c) $a(a-4b)(a-6b)$ (d) $(a-b)(a+b-1)$
 (e) $(x-4)(x+4)(x^2+16)$
 (f) $(2p-q-r)(2p+q+r)$ (g) $x^2(3x-2)(2x+1)$
 (h) $(c+1)(a^2-b)$ (i) $9(x+5)(x-1)$
 (j) $(2x-1)(2x+1)(x-3)(x+3)$
 (k) $(xy-16)(xy+3)$ (l) $x(x-y-z)(x-y+z)$
 (m) $(4-5x)(5+4x)$ (n) $(2x-1)(2x+1)(x-3)$
 (o) $(2x-3y)(6x+5y)$ (p) $(x+a-b)(x+a+b)$
 (q) $9(x-7)(x+5)$ (r) $(x^2-x-1)(x^2+x+1)$
 (s) $x(5x-9y)(2x+y)$
 (t) $(x+2y-a+b)(x+2y+a-b)$ (u) $4xy$
 8(a) $(a+b)(a+b^2)$ (b) $(a+c)(b-d)$ (c) $4ab(a-b)^2$
 (d) $(2x^2+3y^2)(2x-3y)(x+y)$
 (e) $(x-y)(x+y)^3$
 (f) $(a-b-c)(a+b+c)(a-b+c)(a+b-c)$
 (g) $(x^2+y^2)(a^2+b^2+c^2)$
 (h) $(x+ay)(x-by)$ (i) $(a^2-ab+b^2)(a^2+ab+b^2)$
 (j) $(a^2-2ab+2b^2)(a^2+2ab+2b^2)$

Exercise 1D (Page 8)

- 1(a) $\frac{1}{2}$ (b) $\frac{1}{a}$ (c) $\frac{x}{3y}$ (d) $\frac{3}{a}$ (e) $\frac{4y}{5xz}$ (f) $\frac{w}{u^2v}$
 2(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{x}$ (d) $\frac{b}{2}$ (e) $\frac{3x}{2y}$ (f) 1 (g) $\frac{1}{2a}$
 (h) $\frac{2c}{3b}$ (i) $6a^2$ (j) $\frac{5c}{2a}$ (k) $\frac{x^2}{yz}$ (l) $\frac{1}{2a^2}$
 3(a) $\frac{7x}{10}$ (b) $\frac{a}{6}$ (c) $\frac{3x-2y}{24}$ (d) $\frac{13a}{6}$ (e) $\frac{b}{15}$ (f) $-\frac{xy}{45}$
 (g) $\frac{3}{2x}$ (h) $\frac{25}{12x}$ (i) $\frac{b-a}{ab}$ (j) $\frac{x^2+1}{x}$ (k) $\frac{a^2+b}{a}$ (l) $\frac{x-2}{2x^2}$
 4(a) $\frac{5x+7}{6}$ (b) $\frac{-x-17}{10}$ (c) $\frac{9x+26}{12}$ (d) $\frac{12x+3}{5}$
 (e) $\frac{2x-16}{15x}$ (f) $\frac{1}{x(x+1)}$ (g) $\frac{2x}{x^2-1}$ (h) $\frac{5x-13}{(x-2)(x-3)}$
 (i) $\frac{-10}{(x+3)(x-2)}$ (j) $\frac{x^2+y^2}{x^2-y^2}$ (k) $\frac{ax-bx}{(x+a)(x+b)}$ (l) $\frac{2x}{x^2-1}$
 5(a) $\frac{1}{x+y}$ (b) $\frac{3}{2b}$ (c) $\frac{x}{x-2}$ (d) $\frac{a+3}{a+4}$ (e) $\frac{x+y}{x-y}$
 (f) $\frac{x+5}{x+4}$ (g) $\frac{c+d}{a}$ (h) $\frac{y-5}{2y+1}$ (i) $\frac{3a+2b}{3x+2y}$
 6(a) $\frac{3x}{2(x-1)}$ (b) a (c) $\frac{c+2}{c+4}$ (d) x (e) $\frac{3x-1}{a+b}$ (f) $\frac{x-7}{3(x+3)}$
 7(a) $\frac{2}{x^2-1}$ (b) $\frac{2x}{(x-2)^2(x+2)}$ (c) $\frac{3x}{x^2-y^2}$
 (d) $\frac{x+1}{(x-2)(x+3)(x+4)}$ (e) $\frac{bx}{a(a-b)(a+b)}$

- (f) $\frac{x}{(x-1)(x-2)(x-3)}$
 8(a) -1 (b) $-u-v$ (c) $3-x$ (d) $\frac{2}{a-b}$ (e) 1
 (f) $\frac{-1}{2x+y}$
 9(a) $\frac{1}{3}$ (b) $\frac{7}{13}$ (c) $\frac{3}{11}$ (d) $\frac{1}{5}$ (e) $\frac{1}{x+2}$ (f) $\frac{t^2-1}{t^2+1}$
 (g) $\frac{ab}{a+b}$ (h) $\frac{x^2+y^2}{x^2-y^2}$ (i) $\frac{x^2}{2x+1}$ (j) $\frac{x-1}{x-3}$
 11(a) $x+y$ (b) $\frac{2x+3}{3x-1}$ (c) $\frac{a-b+c}{ab}$ (d) $\frac{x^4-y^4}{x^2y^2}$ (e) $\frac{16x}{x^2-16}$
 (f) $\frac{4}{x+2y}$ (g) $\frac{-13x}{(x+1)(x+2)(x+3)}$ (h) $\frac{2}{(x+1)^2(x-1)}$
 12(a) $x^2+2+\frac{1}{x^2}$ (b) 7
 13(a) 0 (b) 3 (c) $\frac{3m-m}{2}$ (d) $\frac{1}{x}$

Exercise 1E (Page 11)

- 3(a) $a^3+3a^2b+3ab^2+b^3$ (b) $x^3-3x^2y+3xy^2-y^3$
 (c) b^3-3b^2+3b-1 (d) $p^3+6p^2+12p+8$
 (e) $1-3c+3c^2-c^3$ (f) $t^3-9t^2+27t-27$
 (g) $8x^3+60x^2y+150xy^2+125y^3$
 (h) $27a^3-108a^2b+144ab^2-64b^3$
 4(a) $(x+y)(x^2-xy+y^2)$ (b) $(a-b)(a^2+ab+b^2)$
 (c) $(y+1)(y^2-y+1)$ (d) $(g-1)(g^2+g+1)$
 (e) $(b-2)(b^2+2b+4)$ (f) $(2c+1)(4c^2-2c+1)$
 (g) $(3-t)(9+3t+t^2)$ (h) $(5+a)(25-5a+a^2)$
 (i) $(3h-1)(9h^2+3h+1)$
 (j) $(u-4v)(u^2+4uv+16v^2)$
 (k) $(abc+10)(a^2b^2c^2-10abc+100)$
 (l) $(6x+5y)(36x^2-30xy+25y^2)$
 5(a) $2(x+2)(x^2-2x+4)$ (b) $a(a-b)(a^2+ab+b^2)$
 (c) $3(2t+3)(4t^2-6t+9)$
 (d) $y(x-5)(x^2+5x+25)$
 (e) $2(5p-6q)(25p^2+30pq+36q^2)$
 (f) $x(3x+10y)(9x^2-30xy+100y^2)$
 (g) $5(xy-1)(x^2y^2+xy+1)$
 (h) $x^3(x+y)(x^2-xy+y^2)$
 6(a) $\frac{x^2+x+1}{x+1}$ (b) $\frac{a-5}{a^2-2a+4}$ (c) $\frac{a^2-a+1}{2a^2}$ (d) $\frac{1}{x}$
 7(a) $\frac{12a+12}{a^3-8}$ (b) $\frac{x^2}{x^3-1}$ (c) $\frac{x^2-3x+8}{(x-4)(x+2)(x^2-2x+4)}$
 (d) $\frac{3a^2-ab}{a^3+b^3}$
 8(a) $(a+b)(a^2-ab+b^2+1)$
 (b) $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$
 (c) $(2a-3)(a+2)(a^2-2a+4)$ (d) $2y(3x^2+y^2)$
 (e) $(s-t)(s^2+st+t^2+s+t)$ (f) $2t(t^2+12)$
 (g) $9(a-b)(a^2-ab+b^2)$
 (h) $(x-2)(x+1)(x^2+2x+4)(x^2-x+1)$
 (i) $(u+1)(u^2+1)(u^4-u^2+1)$
 (j) $(1-x)(1+x+x^2)(2+3x^3)$
 (k) $(x-1)(x+1)(x+2)(x^2+1)(x^2-2x+4)$
 (l) $(a+1)(a^2+a+1)(a^2-a+1)$
 9(a) $\frac{6}{a-3}$ (b) 1 (c) $\frac{2}{a}$ (d) $\frac{x}{x^3-27}$ (e) $\frac{3}{x^3-1}$ (f) $\frac{1}{(1-x)^2}$

$$10 \quad (A+B)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4, \\ (A-B)^4 = A^4 - 4A^3B + 6A^2B^2 - 4AB^3 + B^4, \\ A^4 + B^4 = (A^2 - \sqrt{2}AB + B^2)(A^2 + \sqrt{2}AB + B^2), \\ A^4 - B^4 = (A-B)(A+B)(A^2 + B^2)$$

$$11(a) \quad x(x^2+1)(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1) \\ (b) \quad (x-y)(x+y)(x^2+y^2)(x^2+xy+y^2) \\ (x^2-xy+y^2)(x^2-\sqrt{3}xy+y^2)(x^2+\sqrt{3}xy+y^2)$$

12 13

13 $8x^3$

15 $a+b$

16 $1+a-a^3$

Exercise 1F (Page 13)

$$1(a) \quad x=10 \quad (b) \quad x > \frac{2}{3} \quad (c) \quad a=-5 \quad (d) \quad x \geq 4 \\ (e) \quad x=-1 \quad (f) \quad y=50 \quad (g) \quad t < 0 \quad (h) \quad x=-16$$

$$2(a) \quad x=9 \quad (b) \quad x \geq -5 \quad (c) \quad x > -4 \quad (d) \quad x=-7 \\ (e) \quad a \geq -\frac{7}{5} \quad (f) \quad t < -30 \quad (g) \quad y = -\frac{16}{7} \quad (h) \quad u \leq 48$$

$$3(a) \quad x < 4 \quad (b) \quad x = -11 \quad (c) \quad a > -\frac{1}{2} \quad (d) \quad y \geq 2 \\ (e) \quad x \leq \frac{7}{9} \quad (f) \quad x = -\frac{3}{5} \quad (g) \quad x < \frac{23}{6} \quad (h) \quad x = -\frac{2}{5} \\ (i) \quad \text{There are no solutions.}$$

$$(j) \quad \text{All real numbers are solutions.} \quad (k) \quad x \leq \frac{19}{6}$$

$$(l) \quad x = \frac{3}{14} \quad (m) \quad x > -1 \quad (n) \quad x = \frac{17}{6}$$

$$4(a) \quad x=4 \quad (b) \quad a=8 \quad (c) \quad y < 16 \quad (d) \quad x = \frac{1}{3}$$

$$(e) \quad a = \frac{2}{5} \quad (f) \quad y = \frac{3}{2} \quad (g) \quad x \geq -8 \quad (h) \quad a \geq 1\frac{7}{8}$$

$$(i) \quad x > \frac{1}{4} \quad (j) \quad a = -\frac{5}{4} \quad (k) \quad t = \frac{3}{5} \quad (l) \quad c < \frac{9}{2}$$

$$(m) \quad a = -1 \quad (n) \quad x = \frac{1}{5} \quad (o) \quad x = \frac{7}{17} \quad (p) \quad t = -\frac{26}{27}$$

$$5(a) \quad x \geq 15 \quad (b) \quad a > -15 \quad (c) \quad x = \frac{9}{2} \quad (d) \quad x = \frac{1}{6}$$

$$(e) \quad x = \frac{1}{2} \quad (f) \quad x > 6 \quad (g) \quad x > 20 \quad (h) \quad x = -\frac{23}{2}$$

$$(i) \quad x = -\frac{7}{3} \quad (j) \quad x = \frac{5}{6} \quad (k) \quad a > -11 \quad (l) \quad x \leq 2$$

$$(m) \quad x > \frac{34}{57} \quad (n) \quad x = -\frac{7}{3} \quad (o) \quad x = -\frac{5}{2} \quad (p) \quad x \leq -\frac{43}{69}$$

$$6(a) \quad a=3 \quad (b) \quad s=16 \quad (c) \quad v = \frac{2}{3} \quad (d) \quad \ell = 21$$

$$(e) \quad C=35 \quad (f) \quad c = -\frac{14}{5}$$

$$7(a) \quad x=1, 2, 3 \quad (b) \quad x=-5, -4, -3, -2, -1$$

$$(c) \quad x > -4 \quad (d) \quad x \leq 2 \quad (e) \quad x=2, 3, 4, 5, 6$$

$$(f) \quad x=-3, -2, -1, 0, 1, 2 \quad (g) \quad 0 < x \leq 5$$

$$(h) \quad 1 \leq x \leq 4$$

$$8(a) \quad -4 \quad (b) \quad 7 \quad (c) \quad 36 \quad (d) \quad 80 \text{ litres} \quad (e) \quad 15 \text{ min}$$

$$(f) \quad 16 \quad (g) \quad 30 \text{ km/h} \quad (h) \quad 5 \text{ hours}$$

$$9(a) \quad b = \frac{a+d}{c} \quad (b) \quad n = \frac{t-a+d}{d} \quad (c) \quad r = \frac{p-qt}{t}$$

$$(d) \quad v = \frac{3}{u-1} \quad (e) \quad a = -\frac{2b}{3} \quad (f) \quad g = \frac{2fh}{5f-h}$$

$$(g) \quad y = \frac{2x}{1-x} \quad (h) \quad b = \frac{4a+5}{a-1}$$

$$(i) \quad d = \frac{5c-7}{3c+2} \quad (j) \quad v = \frac{1+u-w-uw}{1-u}$$

$$10(a) \quad x = \frac{14}{5} \quad (b) \quad a=4$$

11(b) $x=6$

Exercise 1G (Page 16)

$$1(a) \quad x=3 \text{ or } -3 \quad (b) \quad a=2 \text{ or } -2 \quad (c) \quad t=1 \text{ or } -1$$

$$(d) \quad x = \frac{3}{2} \text{ or } -\frac{3}{2} \quad (e) \quad x = \frac{1}{2} \text{ or } -\frac{1}{2}$$

$$(f) \quad y = \frac{4}{5} \text{ or } -\frac{4}{5}$$

$$2(a) \quad x=0 \text{ or } 5 \quad (b) \quad c=0 \text{ or } -2 \quad (c) \quad t=0 \text{ or } 1$$

$$(d) \quad a=0 \text{ or } 3 \quad (e) \quad b=0 \text{ or } \frac{1}{2} \quad (f) \quad u=0 \text{ or } -\frac{1}{3}$$

$$(g) \quad y=0 \text{ or } \frac{2}{3} \quad (h) \quad u=0 \text{ or } -\frac{12}{5}$$

$$3(a) \quad x=1 \text{ or } 2 \quad (b) \quad x=-4 \text{ or } -2$$

$$(c) \quad a=-5 \text{ or } 3 \quad (d) \quad y=-5 \text{ or } 1 \quad (e) \quad p=-2 \text{ or } 3$$

$$(f) \quad a=-11 \text{ or } 12 \quad (g) \quad c=3 \text{ or } 6 \quad (h) \quad t=-2 \text{ or } 10$$

$$(i) \quad u=-8 \text{ or } 7 \quad (j) \quad h=-25 \text{ or } -2$$

$$(k) \quad k=-4 \text{ or } 15 \quad (l) \quad \alpha=-22 \text{ or } 2$$

$$4(a) \quad a = \frac{1}{3} \text{ or } 2 \quad (b) \quad x = -5 \text{ or } -\frac{1}{2}$$

$$(c) \quad b = -\frac{2}{3} \text{ or } 2 \quad (d) \quad y = -4 \text{ or } \frac{3}{2} \quad (e) \quad x = \frac{1}{5} \text{ or } 5$$

$$(f) \quad t = \frac{3}{4} \text{ or } 3 \quad (g) \quad t = -\frac{5}{2} \text{ or } 3 \quad (h) \quad u = -\frac{4}{5} \text{ or } \frac{1}{2}$$

$$(i) \quad x = \frac{3}{5} \quad (j) \quad x = -\frac{2}{3} \text{ or } -\frac{3}{2} \quad (k) \quad b = -\frac{3}{2} \text{ or } -\frac{1}{6}$$

$$(l) \quad k = -\frac{8}{3} \text{ or } \frac{1}{2}$$

$$5(a) \quad x = \frac{1}{2}(1 + \sqrt{5}) \div 1.618$$

$$\text{or } x = \frac{1}{2}(1 - \sqrt{5}) \div -0.6180$$

$$(b) \quad x = \frac{1}{2}(-1 + \sqrt{13}) \div 1.303$$

$$\text{or } x = \frac{1}{2}(-1 - \sqrt{13}) \div -2.303 \quad (c) \quad a=3 \text{ or } 4$$

$$(d) \quad u = -1 + \sqrt{3} \div 0.7321$$

$$\text{or } u = -1 - \sqrt{3} \div -2.732$$

$$(e) \quad c = 3 + \sqrt{7} \div 5.646 \text{ or } c = 3 - \sqrt{7} \div 0.3542$$

$$(f) \quad x = -\frac{1}{2} \quad (g) \quad a = \frac{1}{2}(2 + \sqrt{2}) \div 1.707$$

$$\text{or } a = \frac{1}{2}(2 - \sqrt{2}) \div 0.2929 \quad (h) \quad x = -3 \text{ or } \frac{2}{5}$$

$$(i) \quad b = \frac{1}{4}(-3 + \sqrt{17}) \div 0.2808$$

$$\text{or } b = \frac{1}{4}(-3 - \sqrt{17}) \div -1.781$$

$$(j) \quad c = \frac{1}{3}(2 + \sqrt{13}) \div 1.869$$

$$\text{or } c = \frac{1}{3}(2 - \sqrt{13}) \div -0.5352$$

$$(k) \quad t = \frac{1}{4}(1 + \sqrt{5}) \div 0.8090$$

$$\text{or } t = \frac{1}{4}(1 - \sqrt{5}) \div -0.3090 \quad (l) \quad \text{no solutions}$$

$$6(a) \quad x = -1 \text{ or } 2 \quad (b) \quad a = 2 \text{ or } 5 \quad (c) \quad y = \frac{1}{2} \text{ or } 4$$

$$(d) \quad b = -\frac{2}{5} \text{ or } \frac{2}{3} \quad (e) \quad k = -1 \text{ or } 3 \quad (f) \quad u = \frac{4}{3} \text{ or } 4$$

$$7(a) \quad x = 1 + \sqrt{2} \text{ or } x = 1 - \sqrt{2}$$

$$(b) \quad a = 2 + \sqrt{3} \text{ or } a = 2 - \sqrt{3}$$

$$(c) \quad a = 1 + \sqrt{5} \text{ or } a = 1 - \sqrt{5}$$

$$(d) \quad m = \frac{1}{5}(2 + \sqrt{14}) \text{ or } m = \frac{1}{5}(2 - \sqrt{14})$$

$$(e) \quad y = 1 + \sqrt{6} \text{ or } y = 1 - \sqrt{6}$$

$$(f) \quad k = \frac{1}{4}(-5 + \sqrt{73}) \text{ or } k = \frac{1}{4}(-5 - \sqrt{73})$$

$$8(a) \quad p = \frac{1}{2} \text{ or } 1 \quad (b) \quad x = -3 \text{ or } 5 \quad (c) \quad n = 5$$

$$9(a) \quad a = 2b \text{ or } a = 3b \quad (b) \quad a = -2b \text{ or } a = \frac{b}{3}$$

$$10(a) \quad y = 2x \text{ or } y = -2x \quad (b) \quad y = \frac{x}{11} \text{ or } y = -\frac{x}{2}$$

$$11(a) \quad x = 15 \quad (b) \quad 7 \quad (c) \quad 6 \text{ and } 9 \quad (d) \quad 4 \text{ cm}$$

$$(e) \quad \frac{2}{5} \text{ or } -\frac{9}{6} \quad (f) \quad 3 \text{ cm} \quad (g) \quad 2 \text{ hours, } 4 \text{ hours}$$

$$(h) \quad 55 \text{ km/h and } 60 \text{ km/h}$$

$$12(a) \quad a = -\frac{7}{3} \text{ or } 3 \quad (b) \quad k = -4 \text{ or } 15$$

- (c) $t = 2\sqrt{3}$ or $-\sqrt{3}$
 (d) $m = \frac{1}{3}(1 + \sqrt{2})$ or $m = \frac{1}{3}(1 - \sqrt{2})$
 13(a) $x = 2c$ or $x = \frac{11c}{14}$ (b) $x = a$ or $x = \frac{ab}{a-2b}$

Exercise 1H (Page 20)

- 1(a) $x = 2, y = 4$ (b) $x = -1, y = 3$
 (c) $x = 2, y = 2$ (d) $x = 9, y = 1$
 (e) $x = 3, y = 4$ (f) $x = 4, y = -1$
 (g) $x = 5, y = 3\frac{3}{5}$ (h) $x = 13, y = 7$
 2(a) $x = -1, y = 3$ (b) $x = 5, y = 2$
 (c) $x = -4, y = 3$ (d) $x = 2, y = -6$
 (e) $x = 1, y = 2$ (f) $x = 16, y = -24$
 (g) $x = 1, y = 6$ (h) $x = 5, y = -2$
 (i) $x = 5, y = 6$ (j) $x = 7, y = 5$
 (k) $x = \frac{1}{2}, y = \frac{3}{2}$ (l) $x = 5, y = 8$
 3(a) $x = 1, y = 1$ or $x = -2, y = 4$
 (b) $x = 2, y = 1$ or $x = 4, y = 5$
 (c) $x = 0, y = 0$ or $x = 1, y = 3$
 (d) $x = -2, y = -7$ or $x = 3, y = -2$
 (e) $x = -3, y = -5$ or $x = 5, y = 3$
 (f) $x = 1, y = 6$ or $x = 2, y = 3$
 (g) $x = 5, y = 3$ or $x = 5, y = -3$
 or $x = -5, y = 3$ or $x = -5, y = -3$
 (h) $x = 9, y = 6$ or $x = 9, y = -6$
 or $x = -9, y = 6$ or $x = -9, y = -6$
 4(a) Each apple cost 40 cents, each orange cost 60 cents. (b) 44 adults, 22 children
 (c) The man is 36, the son is 12.
 (d) 189 for, 168 against
 (e) $\frac{2}{15}$ (f) 9 \$20 notes, 14 \$10 notes
 (g) 5 km/h, 3 km/h (h) 72
 5(a) $x = 12, y = 20$ (b) $x = 3, y = 2$
 6(a) $x = 6, y = 3, z = 1$ (b) $x = 2, y = -1, z = 3$
 (c) $a = 3, b = -2, c = 2$ (d) $p = -1, q = 2, r = 5$
 (e) $x = 5, y = -3, z = -4$
 (f) $u = -2, v = 6, w = 1$
 7(a) $x = 5, y = 10$ or $x = 10, y = 5$
 (b) $x = -8, y = -11$ or $x = 11, y = 8$
 (c) $x = \frac{1}{2}, y = 4$ or $x = 2, y = 1$
 (d) $x = 4, y = 5$ or $x = 5, y = 4$
 (e) $x = 1, y = 2$ or $x = \frac{3}{2}, y = \frac{7}{4}$
 (f) $x = 2, y = 5$ or $x = \frac{10}{3}, y = 3$
 8(a) $x = 1, y = \frac{5}{4}$
 (b) $x = 2, y = 4$ or $x = -2, y = -4$
 or $x = \frac{4}{3}, y = 6$ or $x = -\frac{4}{3}, y = -6$
 9(b) $x = 1, y = -2$ or $x = -1, y = 2$
 or $x = \frac{7}{3}, y = \frac{2}{3}$ or $x = -\frac{7}{3}, y = -\frac{2}{3}$

Exercise 1I (Page 22)

- 1(a) 1 (b) 9 (c) 25 (d) 81 (e) $\frac{9}{4}$ (f) $\frac{1}{4}$ (g) $\frac{25}{4}$
 (h) $\frac{81}{4}$
 2(a) $(x+2)^2$ (b) $(y+1)^2$ (c) $(p+7)^2$ (d) $(m-6)^2$
 (e) $(t-8)^2$ (f) $(20-u)^2$ (g) $(x+10y)^2$ (h) $(ab-12)^2$
 3(a) $x^2+6x+9 = (x+3)^2$ (b) $y^2+8y+16 = (y+4)^2$
 (c) $a^2 - 20a + 100 = (a-10)^2$
 (d) $b^2 - 100b + 2500 = (b-50)^2$
 (e) $u^2 + u + \frac{1}{4} = (u + \frac{1}{2})^2$
 (f) $t^2 - 7t + \frac{49}{4} = (t - \frac{7}{2})^2$
 (g) $m^2 + 50m + 625 = (m+25)^2$
 (h) $c^2 - 13c + \frac{169}{4} = (c - \frac{13}{2})^2$
 4(a) $x = -1$ or 3 (b) $x = 0$ or 6 (c) $a = -4$ or -2
 (d) $y = -5$ or 2 (e) $b = -2$ or 7
 (f) $x = -2 + \sqrt{3}$ or $x = -2 - \sqrt{3}$
 (g) $x = 5 + \sqrt{5}$ or $x = 5 - \sqrt{5}$
 (h) no solution for y
 (i) $a = \frac{1}{2}(-7 + \sqrt{21})$ or $a = \frac{1}{2}(-7 - \sqrt{21})$
 5(a) $p^2 - 2pq + q^2 = (p-q)^2$
 (b) $a^2 + 4ab + 4b^2 = (a+2b)^2$
 (c) $x^2 - 6xy + 9y^2 = (x-3y)^2$
 (d) $c^2 + 40cd + 400d^2 = (c+20d)^2$
 (e) $u^2 - uv + \frac{1}{4}v^2 = (u - \frac{1}{2}v)^2$
 (f) $m^2 + 11mn + \frac{121}{4}n^2 = (m + \frac{11}{2}n)^2$
 6(a) $x = 2$ or 3 (b) $x = \frac{1}{2}(2 + \sqrt{6})$ or $x = \frac{1}{2}(2 - \sqrt{6})$
 (c) no solution for x
 (d) $x = \frac{1}{2}(-4 + \sqrt{10})$ or $x = \frac{1}{2}(-4 - \sqrt{10})$
 (e) $x = -\frac{3}{2}$ or $\frac{1}{2}$
 (f) $x = \frac{1}{4}(1 + \sqrt{5})$ or $x = \frac{1}{4}(1 - \sqrt{5})$
 (g) $x = -\frac{1}{3}$ or 3 (h) $x = -3$ or $\frac{5}{2}$
 (i) $x = \frac{1}{2}(5 + \sqrt{11})$ or $x = \frac{1}{2}(5 - \sqrt{11})$
 7(b) $a = 3, b = 4$ and $c = 25$
 (d) $A = -5, B = 6$ and $C = 8$
 8(a) $x^3 + 12x^2 + 48x + 64 = (x+4)^3$
 (b) $u = x + 4, u^3 - 18u + 12 = 0$

Exercise 1J (Page 27)

- 1(a) infinite (b) finite, 10 members
 (c) finite, 0 members (d) infinite
 (e) finite, 18 members (f) infinite
 (g) finite, 6 members (h) finite, 14 members
 2(a) false (b) true (c) true (d) false (e) true
 (f) false
 3(a) false (b) true (c) true (d) true (e) false
 4(a) true (b) true
 5(a) $\emptyset, \{a\}$ (b) $\emptyset, \{a\}, \{b\}, \{a, b\}$ (c) $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

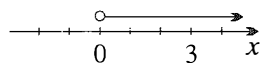
(d) $\emptyset$ 6(a) $\{m, n\}, \{m\}$ (b) $\{2, 4, 6, 8\}, \{4, 6\}$ (c) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \{4, 9\}$ (d) $\{c, o, m, p, u, t, e, r, s, f, w, a\}, \{o, t, e, r\}$ (e) $\{1, 2, 3, 5, 7, 9, 11\}, \{3, 5, 7, 11\}$

7(a) students who study both Japanese and History (b) students who study either Japanese or History or both

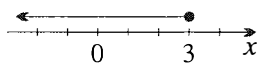
8(a) Q (b) P 9(a) $\{2, 4, 5, 7, 9, 10\}$ (b) $\{1, 2, 5, 8, 9\}$ (c) $\{1, 2, 4, 5, 7, 8, 9, 10\}$ (d) $\{2, 5, 9\}$ (e) $\{2, 5, 9\}$ (f) $\{1, 2, 4, 5, 7, 8, 9, 10\}$

10(a) III (b) I (c) I (d) II (e) IV

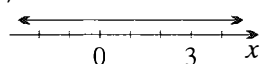
11(a)(i)



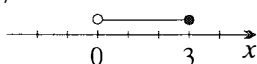
(ii)



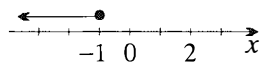
(iii)



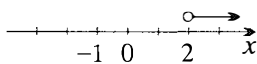
(iv)



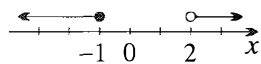
(b)(i)



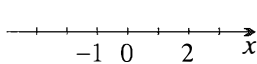
(ii)



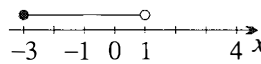
(iii)



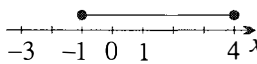
(iv)



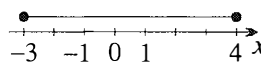
(c)(i)



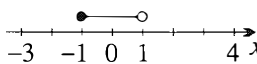
(ii)



(iii)

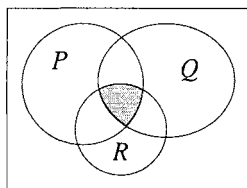


(iv)

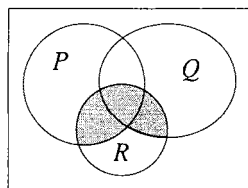
12(a) $|A \cap B|$ is subtracted so that it is not counted twice. (b) 5 (c) LHS = 7, RHS = $5 + 6 - 4 = 7$

13(a) 10 (b) 22 (c) 12

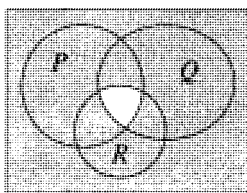
14(a)



(b)



(c)



15 4

16(a) Every subset of a four member set can become a subset of a five member set in two ways — leaving it alone, and adding the fifth member.

(b) An n -member set has 2^n subsets.17 $2^8 = 256$

18 'The set whose only member is the empty set is not equal to the empty set because the empty set is a member of the set whose only member is the empty set.' It is true.

19 It is true.

20 $A \cup B$ 21 If $A \in A$, then $A \notin A$. If $A \notin A$, then $A \in A$. Hence A is not well-defined.

Chapter Two

Exercise 2A (Page 33)

- 1(a) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 (b) 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199
- 2(a) $2^3 \times 3$ (b) $2^2 \times 3 \times 5$ (c) $2^3 \times 3^2$ (d) $2 \times 3^2 \times 7$
 (e) $2^3 \times 13$ (f) $3^3 \times 5$ (g) $3^3 \times 7$ (h) $2 \times 3 \times 7^2$
 (i) $3^2 \times 5 \times 7$ (j) 5×11^2
- 3(a) $8, \frac{9}{8}$ (b) $6, \frac{14}{15}$ (c) $26, \frac{3}{4}$ (d) $16, \frac{7}{9}$ (e) $24, \frac{7}{9}$
 (f) $21, \frac{14}{15}$
- 4(a) $24, \frac{5}{24}$ (b) $90, \frac{13}{90}$ (c) $72, \frac{1}{72}$ (d) $210, \frac{173}{105}$
 (e) $216, \frac{5}{72}$ (f) $780, \frac{401}{780}$
- 5(a) 0.625 (b) 0.6 (c) 0.4375 (d) 0.5 (e) 0.15
 (f) 0.583 (g) 4.64 (h) $5.\dot{3}\dot{6}$ (i) 2.875 (j) $2.8\dot{3}$
- 6(a) $\frac{3}{20}$ (b) $\frac{7}{9}$ (c) $\frac{27}{250}$ (d) $\frac{2}{11}$ (e) $\frac{78}{25}$ (f) $\frac{60}{11}$ (g) $\frac{5}{3}$
 (h) $\frac{40}{33}$ (i) $\frac{19}{90}$ (j) $\frac{98}{15}$
- 7(a) $1.8\dot{3}$ (b) $1.08\dot{3}$ (c) $0.4\dot{6}$ (d) $0.43\dot{2}$ (e) $0.07\dot{4}$
 (f) $0.541\dot{6}$ (g) $3.14285\dot{7}$ (h) $1.214285\dot{7}$ (i) $2.07692\dot{3}$
 (j) $1.23809\dot{5}$
- 8(a) $\frac{25}{33}$ (b) $\frac{28}{27}$ (c) $\frac{169}{37}$ (d) $\frac{44}{101}$ (e) $\frac{2}{1}$ (f) $\frac{5}{2}$ (g) $\frac{137}{90}$
 (h) $\frac{129}{55}$ (i) $\frac{257}{36}$ (j) $\frac{5}{44}$
- 9 The digits of each cycle are in the same order, but start at a different place in the cycle.
- 10(a) $2^8, 2^4 = 16$ (b) $2^6 \times 3^2, 2^3 \times 3 = 24$
 (c) $5^2 \times 7^2, 5 \times 7 = 35$ (d) $2^4 \times 11^2, 2^2 \times 11 = 44$
- 11(a) $\text{HCF} = 2^2 \times 3^2 \times 11, \text{LCM} = 2^3 \times 3^3 \times 11$
 (b) $\text{HCF} = 7 \times 13, \text{LCM} = 2^4 \times 13^2 \times 7$
 (c) $\text{HCF} = 3^2 \times 7^2, \text{LCM} = 2 \times 3^3 \times 7^3$
 (d) $\text{HCF} = 2 \times 5^2 \times 7, \text{LCM} = 2^5 \times 3^2 \times 5^2 \times 7$
- 12(a) the primes $< \sqrt{250}$, namely 2, 3, 5, 7, 11, 13
 (b) It is prime since $22 > \sqrt{457}$.
 (c) $247 = 13 \times 19, 329 = 7 \times 47, 451 = 11 \times 41, 503$ is prime, 727 is prime, $1001 = 7 \times 11 \times 13$.
- 14(a) $1 + 2 + 4 + 7 + 14 = 28$
 15(c) $3.000\ 300\ 03 \neq 3$, showing that some fractions are not stored exactly. (The number you obtain may vary depending on the calculator used.)

Exercise 2B (Page 37)

- 2(a) rationals: $4\frac{1}{2} = \frac{9}{2}, 5 = \frac{5}{1}, -5\frac{3}{4} = \frac{-23}{4}, 0 = \frac{0}{1}, \sqrt{4} = \frac{2}{1}$ (b) rationals: $\sqrt[3]{27} = \frac{3}{1}, \sqrt{\frac{4}{9}} = \frac{2}{3}, -3 = \frac{-3}{1}, 16\frac{1}{2} = \frac{4}{1}$
- 3 $\frac{ad+bc}{2bd}$, which is in the form $\frac{p}{q}$, where p and q are integers.
- 8(b) $\sqrt{11}$

- 11(c) $\pi \doteq \frac{333}{106}$, with error less than 10^{-4} .

$$12 \quad 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}, 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \dots}}}},$$

$$2 + \frac{1}{4 + \frac{1}{4 + \frac{1}{4 + \frac{1}{4 + \dots}}}}$$

- 13 $\pi \doteq 3.11$

- 14(a) Clearly $\frac{p+1}{n} > a$. $\frac{p+1}{n} = \frac{p}{n} + \frac{1}{n} < a + b - a = b$
 (b) $n = 63\ 293, p = 2000$ (c) $\frac{2001}{63\ 293}$

Exercise 2C (Page 40)

- 1 The graph is steeper there.

- 2(a) 4 (b) 9 (c) 6 (d) $2\sqrt{3}$ (e) $3\sqrt{3}$ (f) $2\sqrt{5}$
 (g) $x\sqrt{6}$ (h) $2y\sqrt{2}$ (i) 22 (j) $5x$ (k) $6\sqrt{2}$ (l) $y\sqrt{y}$
 (m) 4 (n) 7 (o) $2x$ (p) $y\sqrt[3]{4}$
- 3(a) $\sqrt{6}$ (b) $2\sqrt{3}$ (c) $3\sqrt{5}$ (d) 5 (e) $6\sqrt{15}$ (f) 84
 (g) $15\sqrt{3}$ (h) 12 (i) $6\sqrt{15}$ (j) $20\sqrt{21}$ (k) $36\sqrt{6}$
 (l) $420\sqrt{3}$ (m) $6\pi\sqrt{2}$ (n) $2a^2\pi\sqrt{\pi}$ (o) $336x^2\sqrt{33}$
- 4(a) $\sqrt{20}$ (b) $\sqrt{27}$ (c) $\sqrt{72}$ (d) $\sqrt{150}$ (e) $\sqrt{48}$
 (f) $\sqrt{32}$ (g) $\sqrt{567}$ (h) $\sqrt{68}$ (i) $\sqrt{275x^2}$ (j) $\sqrt{216\pi^2}$
 (k) $\sqrt{117y^3}$ (l) $\sqrt{864a^4}$
- 5(a) $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{5}\sqrt{7}$ (d) $\frac{5}{2}$ (e) $\frac{7}{3}$ (f) $\frac{1}{4}\sqrt[3]{7}$ (g) $\frac{3}{2}$
 (h) $\frac{4}{3}$
- 6(a) $\sqrt{2}$ (b) $\sqrt{6}$ (c) $\sqrt{5}$ (d) $2\sqrt{3}$ (e) $\frac{1}{2}$ (f) $\frac{2}{5}$
 (g) $\frac{2}{3}$ (h) $\frac{6}{7}$ (i) $2\sqrt{2}$ (j) $\frac{1}{5}\sqrt{7}$ (k) $\frac{1}{3}\sqrt{11}$ (l) $\frac{1}{2}\sqrt{7}$
- 7(a) $2\sqrt{2} \doteq 2.82$ (b) $2\sqrt{3} \doteq 3.46$ (c) $2\sqrt{5} \doteq 4.48$
 (d) $3\sqrt{2} \doteq 4.23$ (e) $3\sqrt{3} \doteq 5.19$ (f) $3\sqrt{5} \doteq 6.72$
 (g) $5\sqrt{2} \doteq 7.05$ (h) $5\sqrt{3} \doteq 8.65$
- 8 If $a = 3$ and $b = 4$, then $\text{LHS} = 5$, but $\text{RHS} = 7$.
 If one of a or b is zero, then they are equal.
- 9(a) $2\sqrt{2}$ (b) $20\sqrt{3}$ (c) $3\sqrt{7}$ (d) $\sqrt{6}$
- 10(a) $2\sqrt{3} - 1$ (b) $-\sqrt{6}$ (c) 0 (d) $2\sqrt{10}$ (e) $2\sqrt{5}$
 (f) $4\sqrt{3} - 5\sqrt{2}$ (g) $3\sqrt{6} + 6\sqrt{2}$ (h) $3\sqrt{3} - \sqrt{13}$
 (i) $6\sqrt{2} - 2\sqrt{7}$
- 11(a) $a = 192$ (b) $x = 275$ (c) $y = 15$ (d) $m = 24$
- 12(a) $\sqrt{6} + 3$ (b) $5 + 5\sqrt{3}$ (c) $6\sqrt{3} - 12$
 (d) $3\sqrt{21} - 7\sqrt{2}$ (e) $a + \sqrt{ab}$ (f) $4\sqrt{a} - 4a$
 (g) $\sqrt{x^2 + 2x} + x$ (h) $x - 1 + \sqrt{x^2 - 1}$
- 13(a) $\sqrt{15} + \sqrt{6} - \sqrt{10} - 2$ (b) $\sqrt{10} + \sqrt{15} + \sqrt{2} + \sqrt{3}$
 (c) $\sqrt{6} - \sqrt{3} - \sqrt{2} + 1$ (d) $3\sqrt{2} + 2\sqrt{3} - \sqrt{6} - 2$
 (e) $26 + 6\sqrt{6}$ (f) $19 + \sqrt{7}$ (g) $4\sqrt{5} - 2\sqrt{15} + 2\sqrt{3} - 3$
 (h) $6\sqrt{3} - 3\sqrt{10} - \sqrt{30} + 5$
- 14(a) $2\sqrt{2} + 3$ (b) 2 (c) -4 (d) $4 - 2\sqrt{3}$ (e) $5 + 2\sqrt{6}$
 (f) -2 (g) -1 (h) $14 - 8\sqrt{3}$ (i) $4a + 1 - 4\sqrt{a}$
 (j) $a + 6 + 4\sqrt{a + 2}$ (k) $x - 2$

- (l) $2x - 1 + 2\sqrt{(x+1)(x-2)}$ (m) $\frac{3}{2} + \frac{1}{2}\sqrt{5}$
 (n) $\frac{3}{2} - \frac{1}{2}\sqrt{5}$
 15(a) 2 (b) $2\sqrt{3}$ (c) $\frac{5\sqrt{3}}{2}$ (d) 3 (e) 5 (f) 5 (g) $\sqrt{2}$
 (h) $\frac{5\sqrt{7}}{9}$
 16(a) 3 (b) $\sqrt{15}$ (c) 4 (d) $2\sqrt{15}$
 17(a) $-\frac{b}{a}$ (b) $\frac{c}{a}$
 18(a) $xy\sqrt{y}$ (b) x^2y^3 (c) $x + 3$ (d) $(x+1)\sqrt{x}$
 (e) $x(x+1)y^2$ (f) $x(x+1)$

Exercise 2D (Page 43)

- 1(a) $\frac{\sqrt{3}}{3}$ (b) $\frac{2\sqrt{7}}{7}$ (c) $\frac{5\sqrt{11}}{11}$ (d) $\sqrt{5}$ (e) $\frac{\sqrt{2}}{2}$ (f) $\frac{\sqrt{6}}{2}$
 (g) $\frac{\sqrt{15}}{3}$ (h) $\frac{7\sqrt{10}}{5}$
 2(a) 4, 1 (b) 6, 4 (c) 2, -1 (d) 10, 1 (e) 2, -11
 (f) -2, -6
 3(a) $1 + \sqrt{2}$ (b) $-1 - \sqrt{2}$ (c) $\frac{3(\sqrt{5}+1)}{4}$ (d) $-\frac{3(\sqrt{5}+1)}{4}$
 (e) $\sqrt{3} - \sqrt{2}$ (f) $\frac{\sqrt{5}-\sqrt{3}}{2}$ (g) $\frac{4+\sqrt{7}}{3}$ (h) $\frac{3(\sqrt{11}+\sqrt{6})}{5}$
 4(a) $\frac{\sqrt{2}}{3}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\sqrt{6}$ (d) $\frac{2\sqrt{3}}{3}$ (e) $\frac{\sqrt{21}}{7}$ (f) $\frac{\sqrt{14}}{35}$
 (g) $\frac{\sqrt{17}}{2}$ (h) $\frac{6\sqrt{11}}{11}$
 5(a) $2\sqrt{2} + \sqrt{5}$ (b) $3\sqrt{2} + 4$ (c) $2(\sqrt{5} + \sqrt{3})$
 (d) $\frac{3\sqrt{15}-9}{2}$ (e) $\frac{28-10\sqrt{7}}{3}$ (f) $2\sqrt{2} - 3$
 (g) $1 + \sqrt{3}$ (h) $\frac{9\sqrt{2}+\sqrt{21}-3\sqrt{42}-7}{47}$ (i) $2 - \sqrt{3}$
 (j) $4 + \sqrt{15}$ (k) $\frac{3\sqrt{x}-6}{x-4}$ (l) $\frac{q+\sqrt{p}}{q^2-p}$
 (m) $\sqrt{x+1} - \sqrt{x-1}$ (n) $\frac{x+y-2\sqrt{xy}}{x-y}$ (o) $\frac{a+b+2\sqrt{ab}}{a-b}$
 (p) $\frac{\sqrt{2}(\sqrt{5}+1)}{2}$ (q) $\sqrt{21} - \sqrt{15}$ (r) $\sqrt{x^2+2x-x}$
 6(a) 3 (b) 1
 7(a) $\sqrt{3}$ (b) $\frac{\sqrt{5}}{4}$ (c) $\frac{-2}{17}$
 8 $\frac{\sqrt{x+h}-\sqrt{x}}{h}$
 9(a) $2\sqrt{2}$ (b) 4 (c) 4 (d) $\frac{2}{x-1}$
 10(b) $2\sqrt{5}$ (c) 18 (d)(i) 6 (ii) 14
 11(a) $2\sqrt{3}$ (b) $3\sqrt{11}$ (c) $3 + 2\sqrt{2}$ (d) $7 - 2\sqrt{6}$
 12(a) $\frac{1}{12}(2\sqrt{3} + 3\sqrt{2} - \sqrt{30})$ (b) $\frac{1}{2}\sqrt[3]{4}$
 (c) $\sqrt[3]{4} + \sqrt[3]{2} + 1$
 14(a) 8.33 (b) 8.12

Exercise 2E (Page 46)

- 1(a) $a = 7, b = -2$ (b) $a = 2, b = 3$
 (c) $a = -7, b = -4$ (d) $a = 3, b = -2$
 (e) $a = \frac{5}{7}, b = \frac{1}{2}$ (f) $a = \frac{2}{3}, b = 3$
 2(a) $a = 2, b = 18$ (b) $a = -1, b = 2$
 (c) $a = 3, b = -2$ (d) $a = 5, b = 20$
 (e) $a = \frac{4}{3}, b = \frac{1}{2}$ (f) $a = -\frac{1}{5}, b = -\frac{3}{4}$
 3(a) $x = 7, y = 28$ (b) $x = 4, y = 12$
 (c) $x = 39, y = 12$ (d) $x = 11, y = 5$
 (e) $x = 9, y = 6$ (f) $x = 14, y = 180$
 4(a) $x = 0, y = -3, z = 2$
 (b) $x = 20, y = 10, z = 3$

- (c) $x = -7, y = 1, z = 10$
 (d) $x = 20, y = -10, z = 3$
 5(a) $a = 2, b = 1$ (b) $a = -2, b = 1$
 (c) $a = \frac{1}{2}, b = \frac{1}{2}$ (d) $a = -\frac{1}{2}, b = \frac{1}{2}$
 (e) $a = 3, b = 2$ (f) $a = \frac{9}{5}, b = \frac{8}{15}$
 6(a) $x = 2, y = 3$ (b) $x = \frac{5}{2}, y = 8$
 (c) $x = \frac{9}{2}, y = \frac{3}{2}$ (d) $x = 12, y = -6$
 (e) $x = 1, y = \frac{1}{2}$ (f) $x = \frac{1}{3}, y = \frac{1}{2}$
 (g) $x = -2$ and $y = -5$, or $x = 5$ and $y = 2$
 (h) $x = \frac{5}{2}$ and $y = \frac{1}{2}$, or $x = \frac{1}{2}$ and $y = \frac{5}{2}$
 7(a) $a = 1, b = 1$ (b) $a = 2, b = -1$
 (c) $a = 1, b = -2$ (d) $a = \frac{3}{2}, b = 1$
 8(a) $3 - \sqrt{6}$
 (b)(i) $5 - \sqrt{3}$ (ii) $7 + \sqrt{17}$ (iii) $-\frac{1}{2} + \frac{1}{\sqrt{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{3}$

Exercise 2F (Page 50)

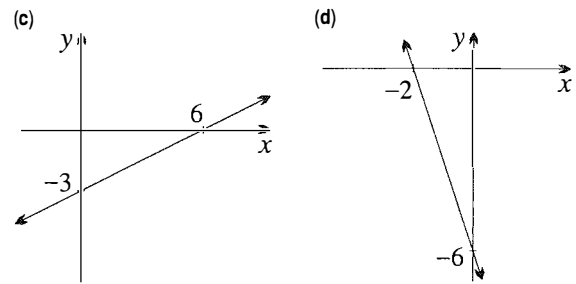
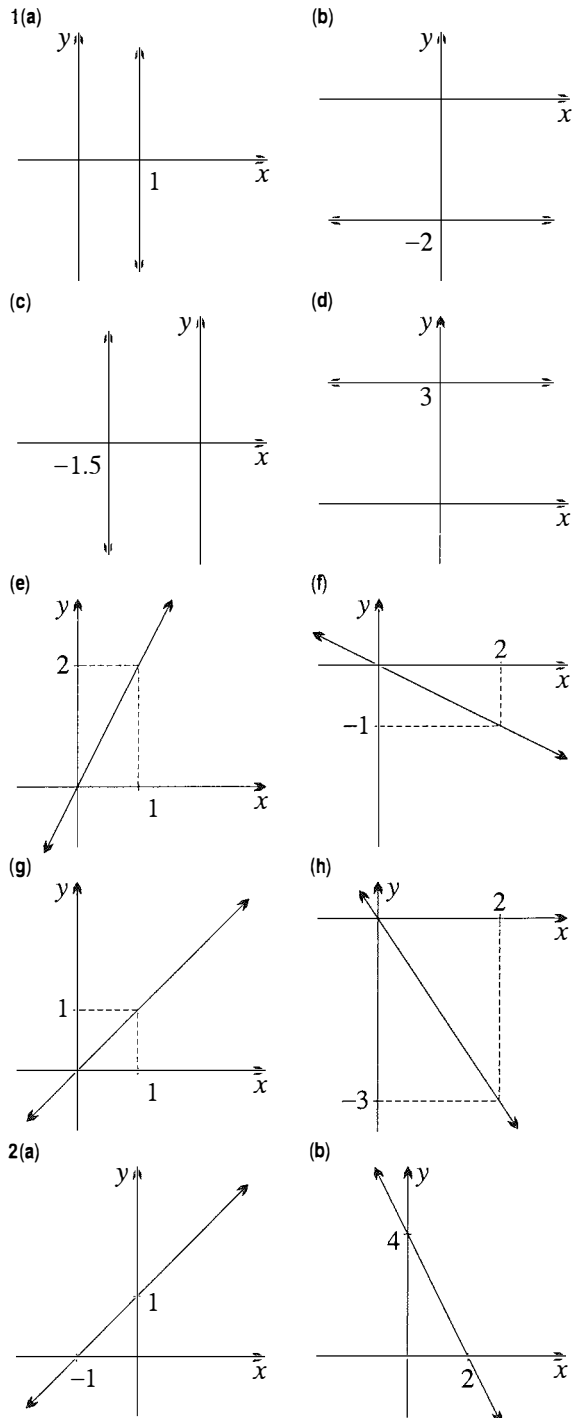
- 1 a, b, d, e, g, h, j, l
 2(a) domain: all real numbers, range: $y \geq -1$
 (b) domain: all real numbers, range: $y > -1$
 (c) domain: all real numbers, range: all reals
 (d) domain: all real numbers, range: $y = 2$
 (e) domain: all real numbers, range: $y < 2$
 (f) domain: $x \geq -1$, range: all real numbers
 (g) domain: $x \neq 0$, range: $y \neq 0$
 (h) domain: all real numbers, range: all reals
 (i) domain: $0 \leq x \leq 3$, range: $-3 \leq y \leq 3$
 (j) domain: $x < 4$, range: $y > 0$
 (k) domain: all reals, range: $y \leq -1, y > 1$
 (l) domain: all real numbers, range: $y < 1$
 3(a) 2 (b) 0 (c) $a^2 - 2$ (d) $a^2 - 2$ (e) $a^2 + 4a + 2$
 (f) $x^2 - 2x - 1$ (g) $-1\frac{3}{4}$ (h) $9t^2 + 12t + 2$ (i) $t^4 - 2$
 (j) $t^2 + \frac{1}{t^2}$
 4(a) 0 (b) -1 (c) 8 (d) 0 (e) $t^2 - 2t$ (f) $t^2 + 2t$
 (g) $w^2 - 4w + 3$ (h) $w^2 - 2w - 1$ (i) $w^2 - 1$
 (j) $x^2 - 2x = g(x)$
 5(a) -3, -2, -1, 0, 1, 0, -1 (b) 3, 0, 0, 1, 4
 6(a) all real numbers (b) $x \neq 3$ (c) $x \geq 0$ (d) $x > 0$
 (e) $x \leq 2$ (f) $x < 2$
 8(a) $h^3 - h + 1$ (b) $-h^3 + h + 1$ (c) $h^2 - 1$ (d) $h^2 - 1$
 (e) $\frac{13}{16}$ (f) $\frac{3}{4}$
 9(a) 1, 3, 6, 10, 15, 21 (b) 1, 2, 2, 3, 2, 4
 (c) 1, 3, 4, 7, 6, 12
 10(a) $-3 \leq x \leq 3$ (b) $x \leq -2$ or $x \geq 2$
 (c) $x > 0$ (d) $x \neq -1$ (e) $x \neq 3$ and $x \neq 2$
 (f) $x \neq 3$ and $x \neq -3$
 11(a) 64 (b) 28 (c) $(x+3)^2$ (d) $x^2 + 3$ (e) 64
 (f) 12 (g) 2^{3x} (h) 3×2^x

- 13(a) all real values of a and b
 (b) all real values of a and b (c) no solutions
 (d) no solutions (e) $a = 0$ and b is any real number, or $b = 0$ and a is any real number.

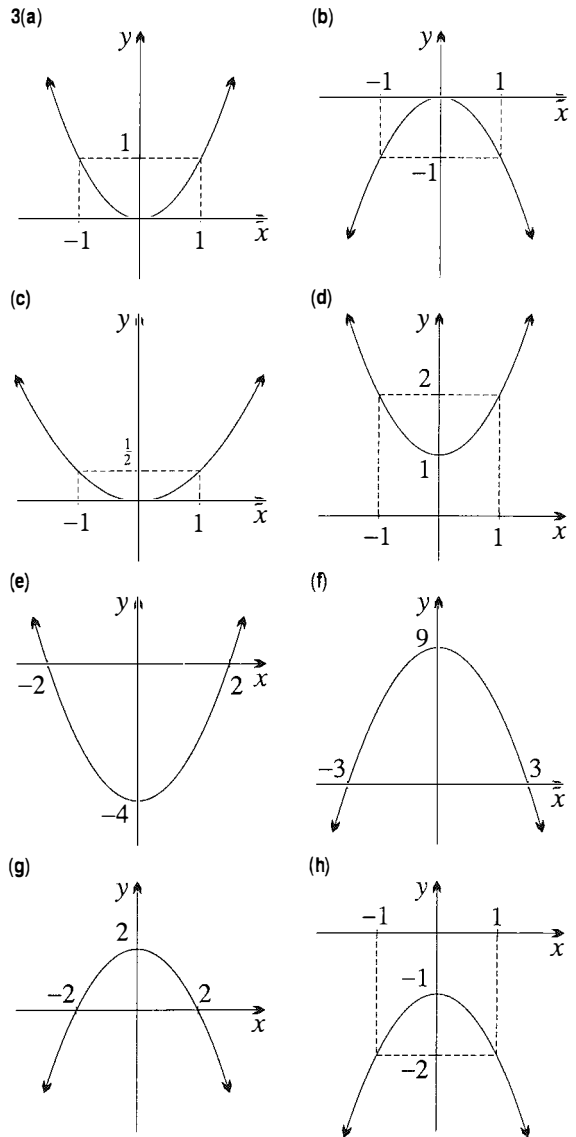
14 It approaches 2.72.

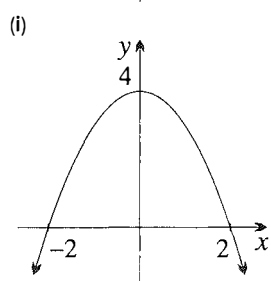
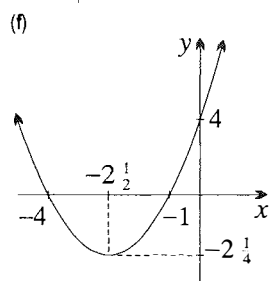
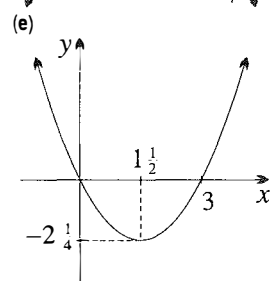
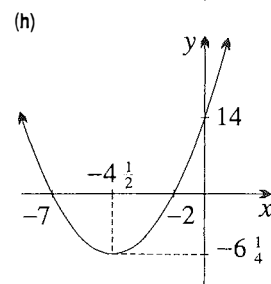
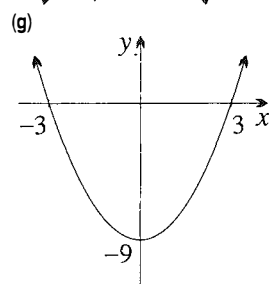
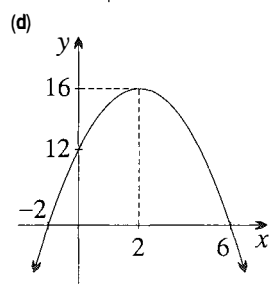
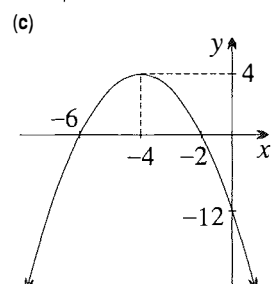
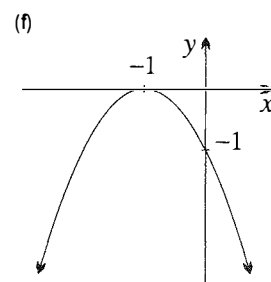
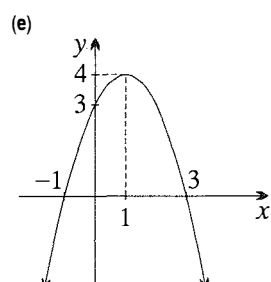
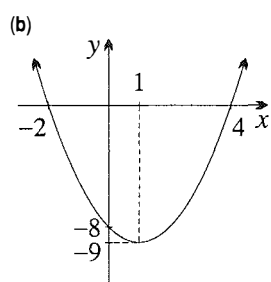
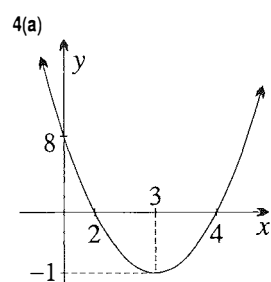
15(b) $(s(x))^2 = \frac{1}{2}(c(2x) - 1)$

Exercise 2G (Page 56)



- (e) intercepts: (1, 0) and (0, 1)
 (f) intercepts: (-1, 0) and (0, 2)
 (g) intercepts: (3, 0) and (0, -1)
 (h) intercepts: (4, 0) and (0, -2)
 (i) intercepts: (6, 0) and (0, -4)
 (j) intercepts: (-6, 0) and (0, -1½)
 (k) intercepts: (2, 0) and (0, 5)
 (l) intercepts: (3, 0) and (0, -7½)



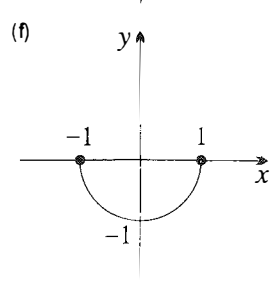
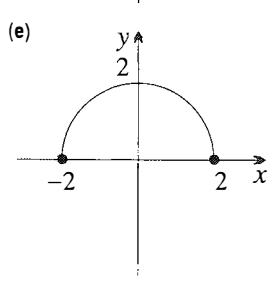
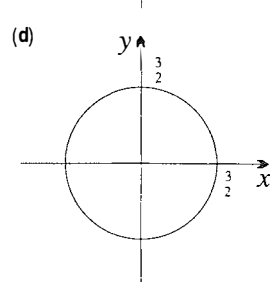
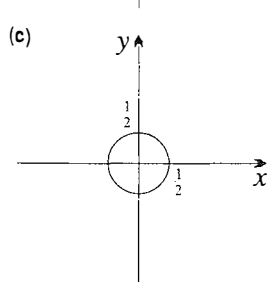
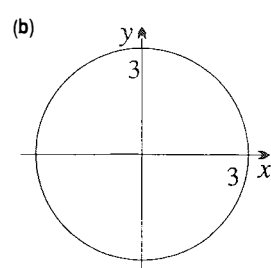
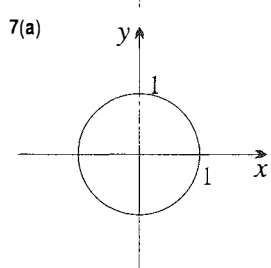
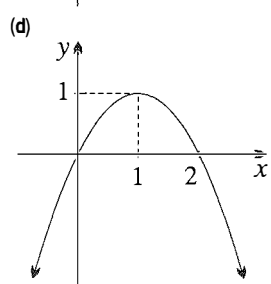
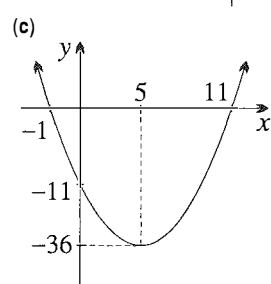
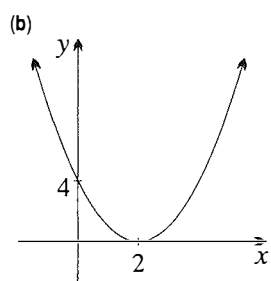
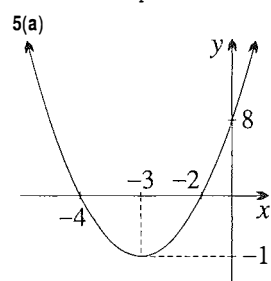


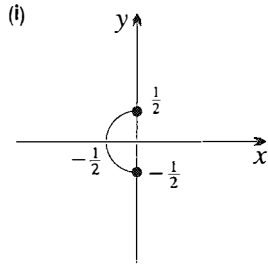
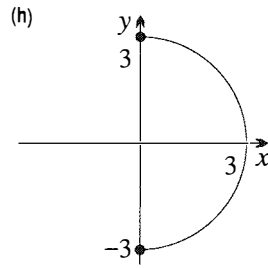
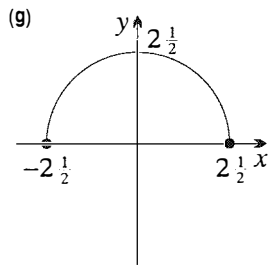
6 The curves are all sketched in the notes of Section 2G.

The ranges are:

(a) $y \geq -1$ (b) $y \geq -9$ (c) $y \leq 4$ (d) $y \leq 16$

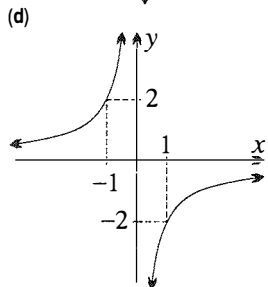
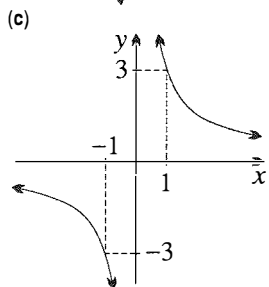
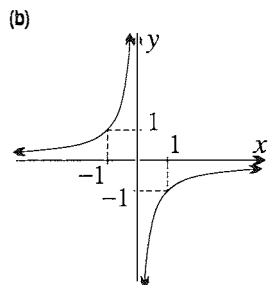
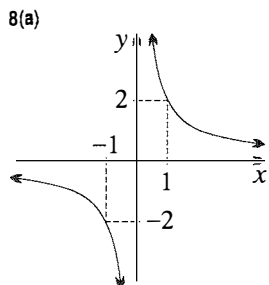
(e) $y \geq -2\frac{1}{4}$ (f) $y \geq -2\frac{1}{4}$



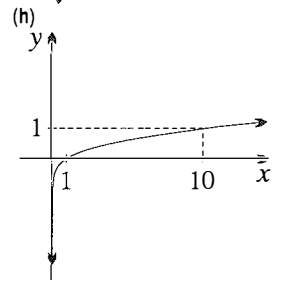
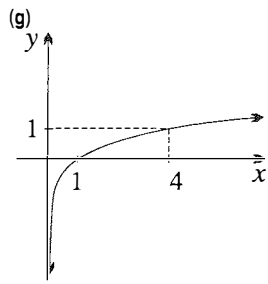
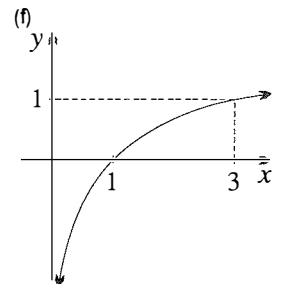
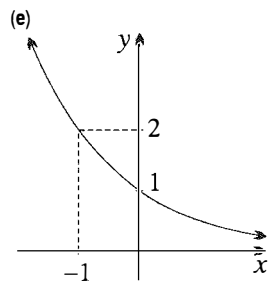
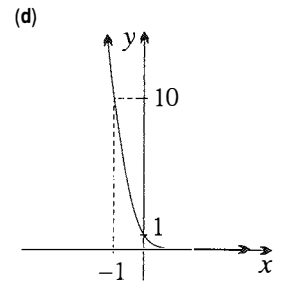
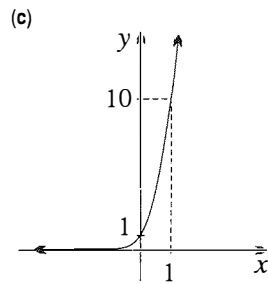
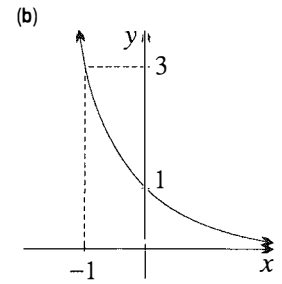
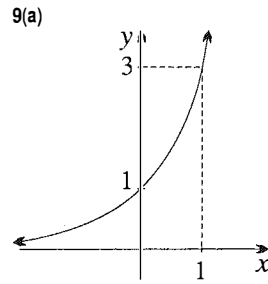


The domains and ranges are respectively:

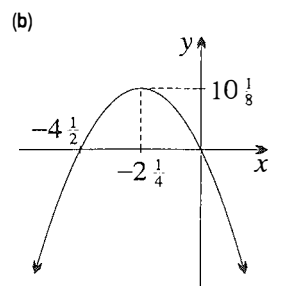
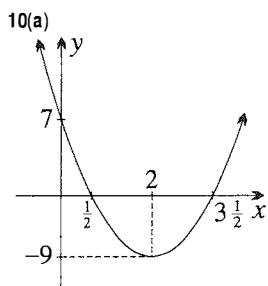
- (a) $-1 \leq x \leq 1$, $-1 \leq y \leq 1$
 (b) $-3 \leq x \leq 3$, $-3 \leq y \leq 3$
 (c) $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$
 (d) $-\frac{3}{2} \leq x \leq \frac{3}{2}$, $-\frac{3}{2} \leq y \leq \frac{3}{2}$
 (e) $-2 \leq x \leq 2$, $0 \leq y \leq 2$
 (f) $-1 \leq x \leq 1$, $-1 \leq y \leq 0$
 (g) $-2\frac{1}{2} \leq x \leq 2\frac{1}{2}$, $0 \leq y \leq 2\frac{1}{2}$
 (h) $0 \leq x \leq 3$, $-3 \leq y \leq 3$
 (i) $-\frac{1}{2} \leq x \leq 0$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$

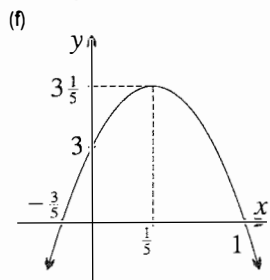
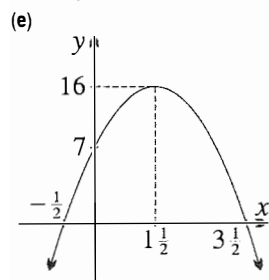
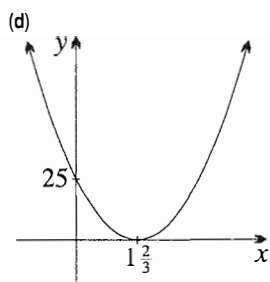
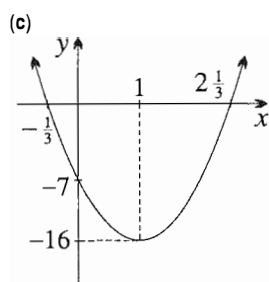


Each domain is $x \neq 0$, each range is $y \neq 0$.

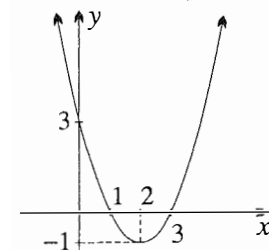


For parts (a)–(e), the domain is all real x , and the range is $y > 0$. For parts (f)–(h), the domain is $x > 0$, and the range is all real y .

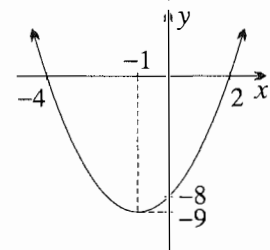




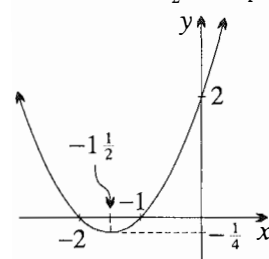
11(a) $y = (x - 2)^2 - 1$



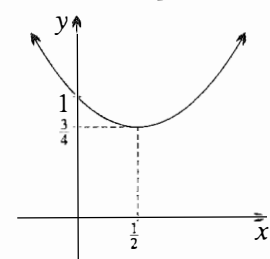
(b) $y = (x + 1)^2 - 9$



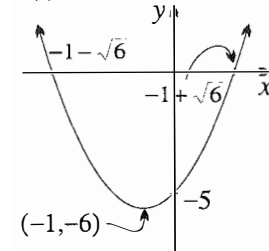
(c) $y = (x + 1 \frac{1}{2})^2 - \frac{1}{4}$



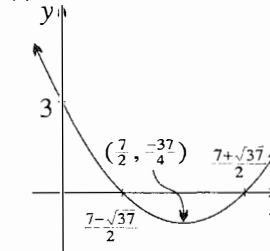
(d) $y = (x - \frac{1}{2})^2 + \frac{3}{4}$



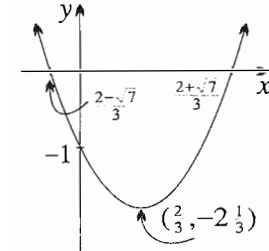
12(a)



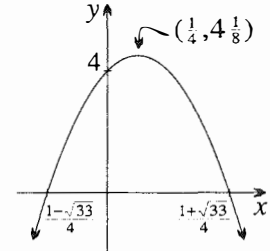
(b)



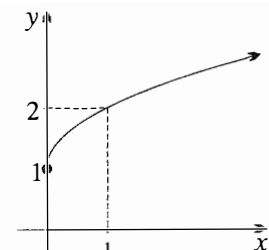
(c)



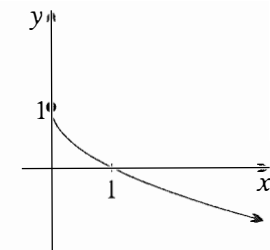
(d)



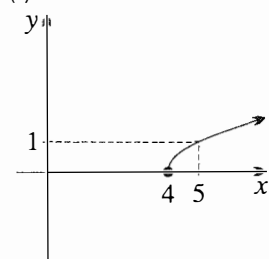
13(a)



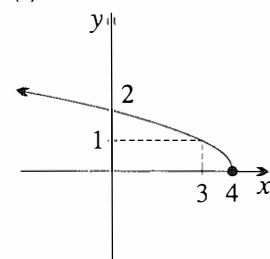
(b)



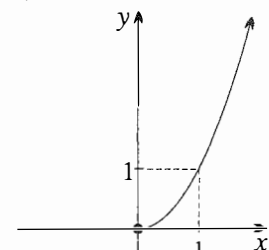
(c)



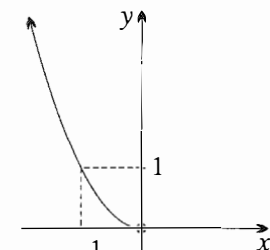
(d)



(e)



(f)



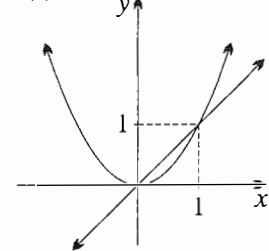
The domains and ranges are respectively:

(a) $x \geq 0, y \geq 1$ (b) $x \geq 0, y \leq 1$

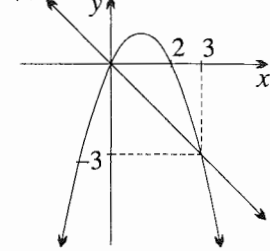
(c) $x \geq 4, y \geq 0$ (d) $x \leq 4, y \geq 0$

(e) $x \geq 0, y \geq 0$ (f) $x \leq 0, y \geq 0$

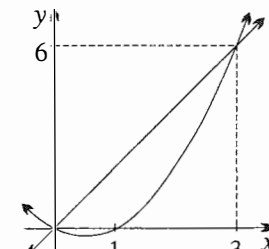
14(a)



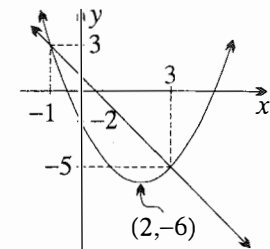
(b)

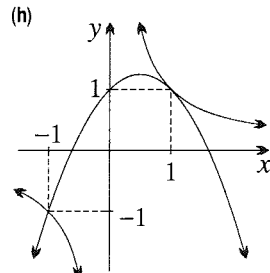
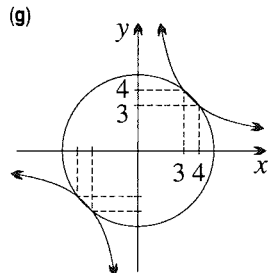
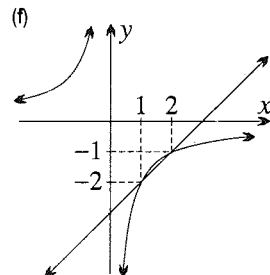
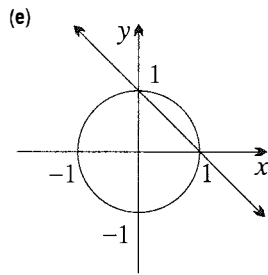


(c)



(d)





intersection points: (a) $(0,0)$, $(1,1)$ (b) $(0,0)$, $(3,-3)$ (c) $(0,0)$, $(3,6)$ (d) $(-1,3)$, $(3,-5)$
 (e) $(1,0)$, $(0,1)$ (f) $(1,-2)$, $(2,-1)$
 (g) $(-4,-3)$, $(-3,-4)$, $(3,4)$, $(4,3)$
 (h) $(-1,-1)$, $(1,1)$

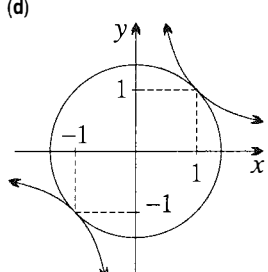
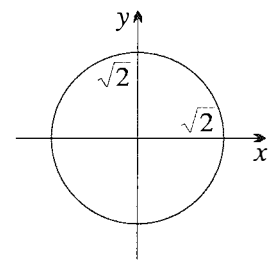
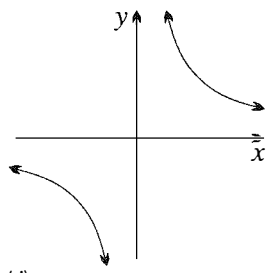
15(a) $r = \sqrt{5}$, $(2,1)$, $(1,2)$, $(-1,2)$, $(-2,1)$, $(-2,-1)$, $(-1,-2)$, $(1,-2)$, $(2,-1)$

(b) $r = \sqrt{2}$, $(1,-1)$, $(-1,-1)$

(c) $r = \sqrt{10}$, $(3,1)$, $(1,3)$, $(1,-3)$, $(3,-1)$

(d) $r = \sqrt{17}$, $(4,1)$, $(1,4)$, $(-1,4)$, $(-4,1)$, $(-4,-1)$, $(-1,-4)$, $(1,-4)$, $(4,-1)$

16(a) $y = \frac{1}{x}$ (b) $x^2 + y^2 = 2$



17(a) $(0, 2\sqrt{\lambda^2 - \alpha^2})$ (b) $r = \lambda$

18(a) $P(\frac{2}{b}, \frac{b}{2})$ (c) 2 units²

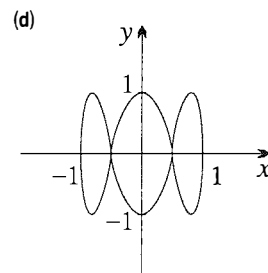
19(a) $a = \frac{1}{4}$, $b = \frac{3}{4}$, $c = 1$ (b) $\sqrt{2} \div \frac{23}{16}$, $\frac{1}{\sqrt{2}} \div \frac{11}{16}$

20(a) $y = (4x^2 - 1)\sqrt{1 - x^2}$

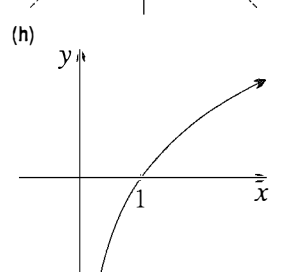
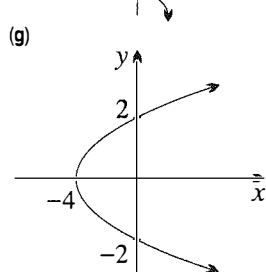
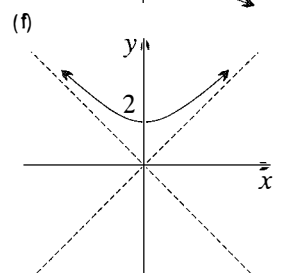
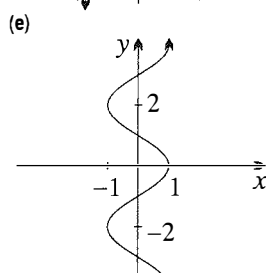
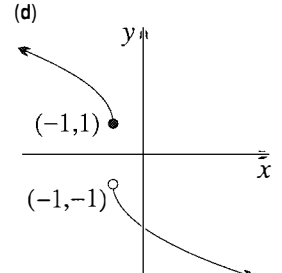
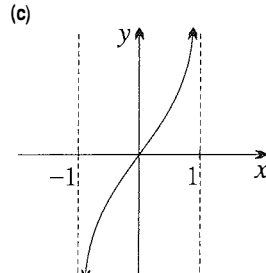
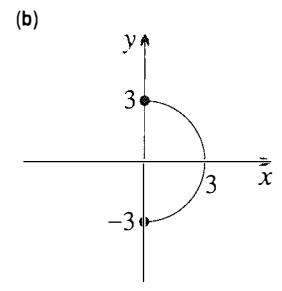
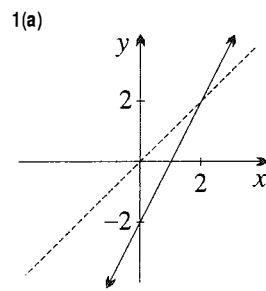
or $y = -(4x^2 - 1)\sqrt{1 - x^2}$

(b) $-1 \leq x \leq 1$

(c) $(-1,0)$, $(-\frac{1}{2},0)$, $(\frac{1}{2},0)$, $(1,0)$, $(0,-1)$, $(0,1)$



Exercise 2H (Page 62)



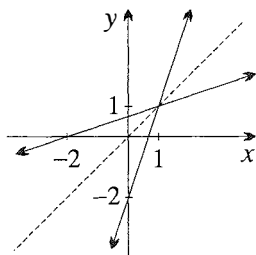
2 Original is a function: all except (f)

Inverse is a function: (a), (c), (d), (f), (h)

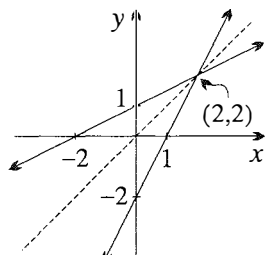
3(a) $y = \frac{x+2}{3}$ (b) $y = 2x - 2$ (c) $y = 6 - 2x$

(d) $y - x + 1 = 0$ (e) $2y + 5x - 10 = 0$ (f) $x = 2$

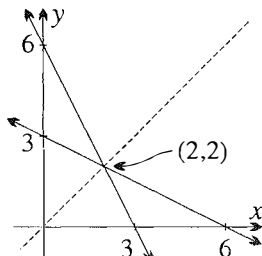
4(a)



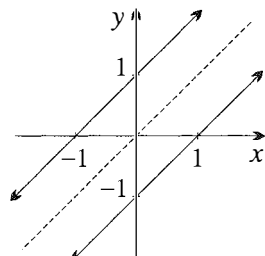
(b)



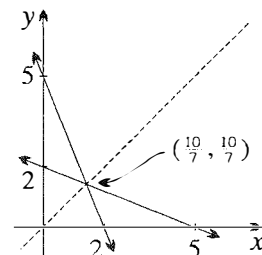
(c)



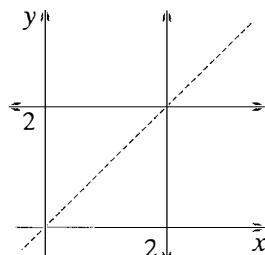
(d)



(e)



(f)

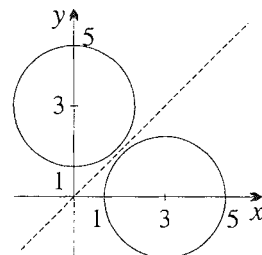


5(a) $y = \frac{1}{x-1}$ (b) $y = \frac{1}{x} - 1$ (c) $y = \frac{2x+2}{x-1}$

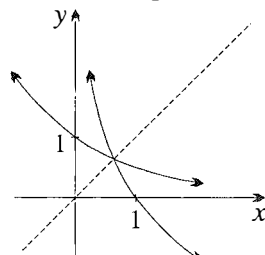
(d) $y = \frac{2x}{3-x}$

6 Each inverse is identical to the original function. Therefore the graph must be symmetric about the line $y = x$.

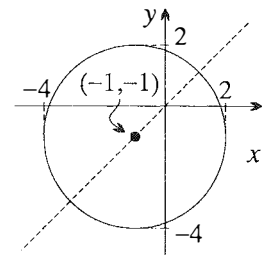
8(a) $x^2 + (y-3)^2 = 4$



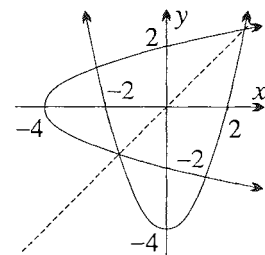
(b) $y = -\log_2 x$



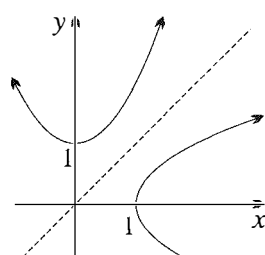
(c) $(x+1)^2 + (y+1)^2 = 9$



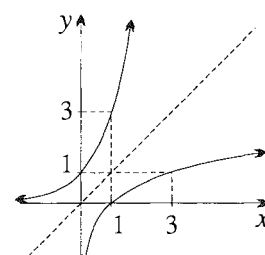
(d) $x = y^2 - 4$



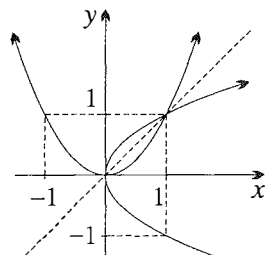
(e) $x = y^2 + 1$



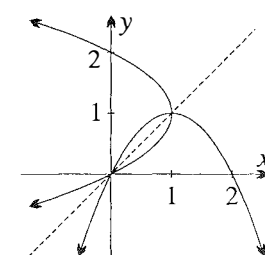
(f) $y = 3^x$



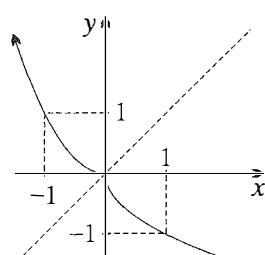
9(a) $x = y^2$



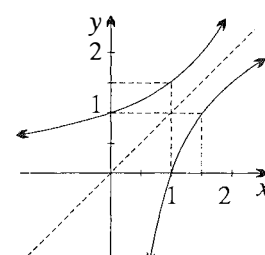
(b) $x = 2y - y^2$



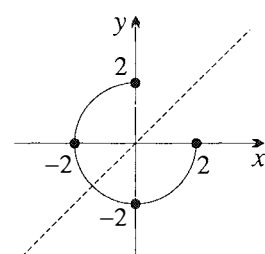
(c) $y = x^2$, where $x \leq 0$



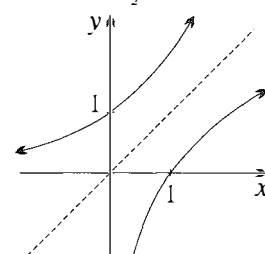
(d) $y = \log_2 x$



(e) $x = -\sqrt{4 - y^2}$



(f) $y = \log_{\frac{1}{3}} x$



10(a) It fails the horizontal line test, for example $f(1) = f(-1) = 1$, so the inverse is not a function.

(b) $f^{-1}(x) = x^2$, where $x \geq 0$.

(c) It fails the horizontal line test, for example $f(1) = f(-1) = 1$, so the inverse is not a function.

(d) $f^{-1}(x) = (x-1)^{\frac{1}{3}}$

(e) It fails the horizontal line test, for example $f(1) = f(-1) = 8$, so the inverse is not a function.

(f) $f^{-1}(x) = \sqrt{9-x}$

(g) It fails the horizontal line test, for example $f(1) = f(-1) = \frac{1}{3}$, so the inverse is not a function.

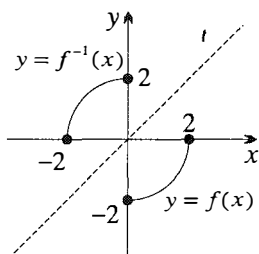
(h) $f^{-1}(x) = \frac{1-3x}{1+x}$

(i) $f^{-1}(x) = -\sqrt{x}$ (j) $f^{-1}(x) = 1 + \sqrt{1+x}$

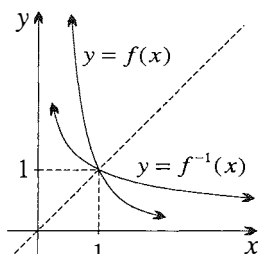
(k) $f^{-1}(x) = 1 - \sqrt{1+x}$ (l) $f^{-1}(x) = \frac{x+1}{x-1}$

12(b) The inverse of the first, $x = -y^2$, is not a function. The second is a natural restriction of the domain of the first in order that its inverse $y = \sqrt{-x}$ is a function.

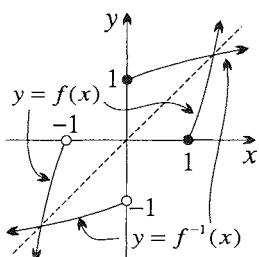
13(a) $0 \leq x \leq 2$



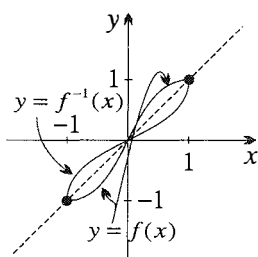
(b) $x > 0$



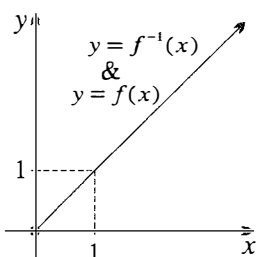
(c) $x < -1$ or $x \geq 1$



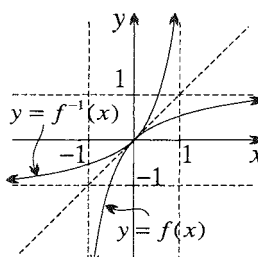
(d) $-1^{\circ} \leq x^{\circ} \leq 1^{\circ}$



(e) $x \geq 0$



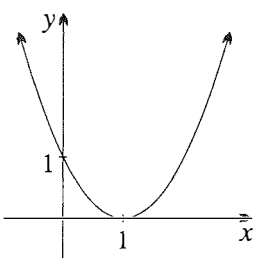
(f) $-1^{\circ} < x^{\circ} < 1^{\circ}$



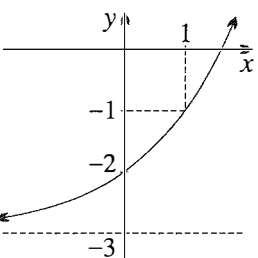
14 $\log_3(x^2) \neq 2\log_3(x)$ if $x < 0$. Instead we must write $\log_3(x^2) = 2\log_3(\sqrt{x^2})$, and neither of these functions has an inverse that is a function.

Exercise 21 (Page 66)

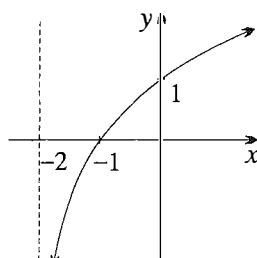
1(a) $y = (x-1)^2$



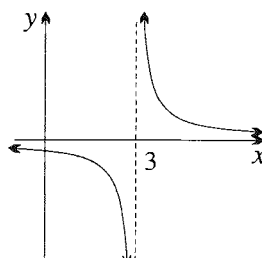
(b) $y = 2^x - 3$



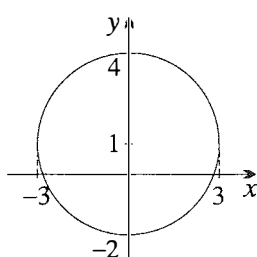
(c) $y = \log_2(x+2)$



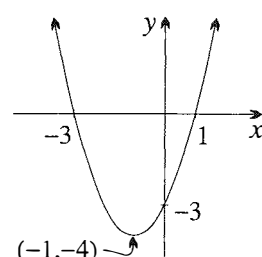
(d) $y = \frac{1}{x-3}$



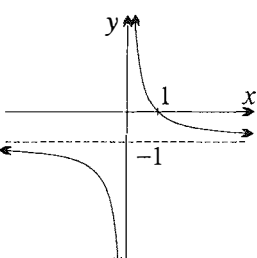
(e) $x^2 + (y-1)^2 = 9$



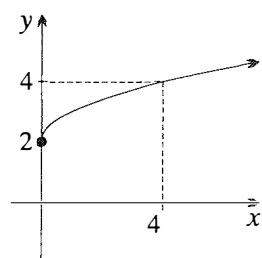
(f) $y = (x+1)^2 - 4$



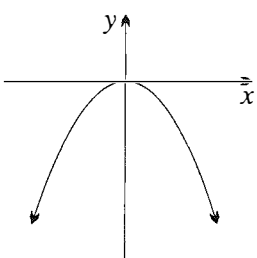
(g) $x(y+1) = 1$



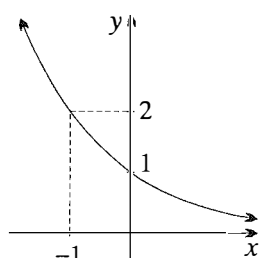
(h) $y = \sqrt{x} + 2$



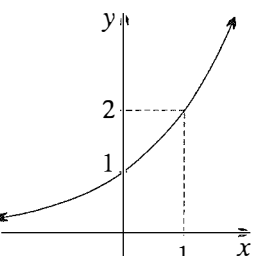
2(a) $y = -x^2$



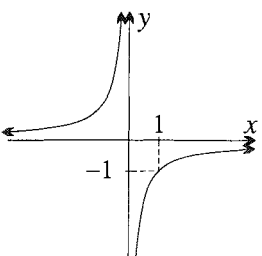
(b) $y = 2^{-x}$



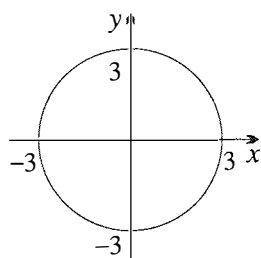
(c) $y = 2^x$



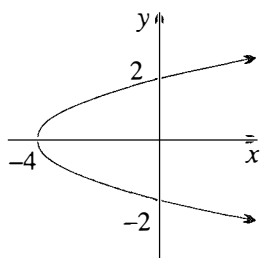
(d) $y = -\frac{1}{x}$



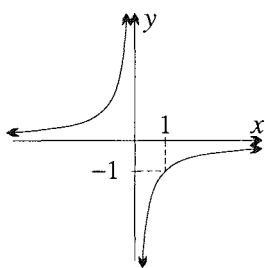
(e) $x^2 + y^2 = 9$



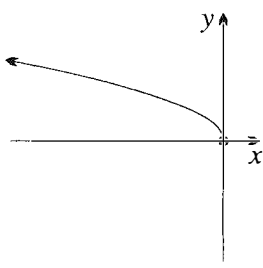
(f) $x = y^2 - 4$



(g) $-xy = 1$



(h) $y = \sqrt{-x}$

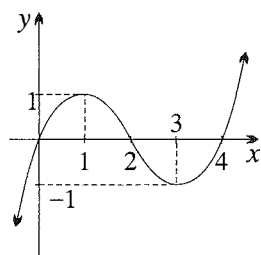


3(a) $r = 2, (-1, 0)$ (b) $r = 1, (1, 2)$

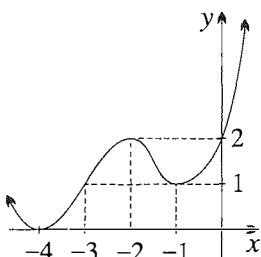
(c) $r = 3, (1, 2)$ (d) $r = 5, (-3, 4)$

(e) $r = 3, (5, -4)$ (f) $r = 6, (-7, 1)$

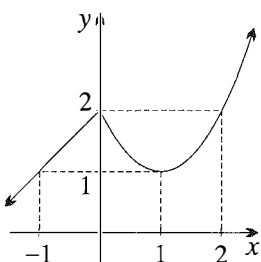
4(a)(i)



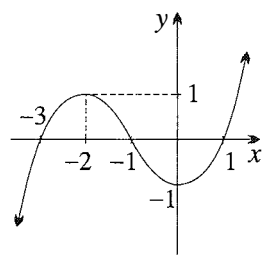
(b)(i)



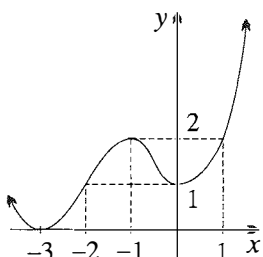
(c)(i)



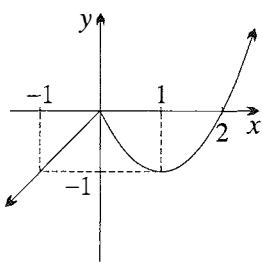
(ii)



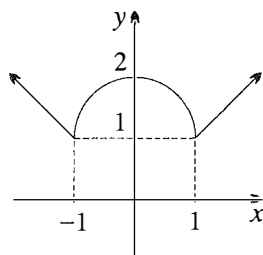
(ii)



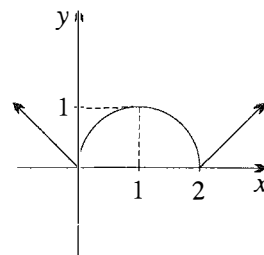
(ii)



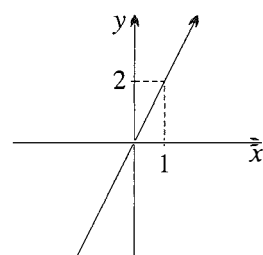
(d)(i)



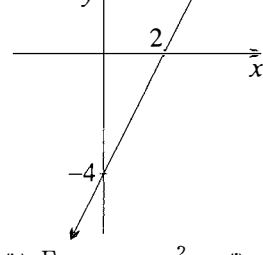
(ii)

5(a) From $y = 2x$: (i) shift up 4 (or left 2)

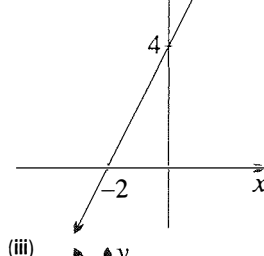
(ii) shift down 4 (or right 2)

(iii) reflect in y -axis and shift up 4

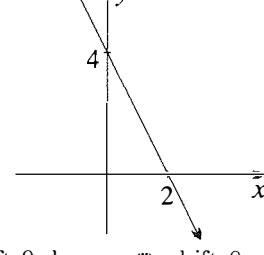
(ii)



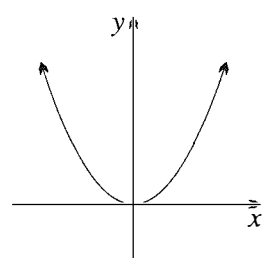
(i)



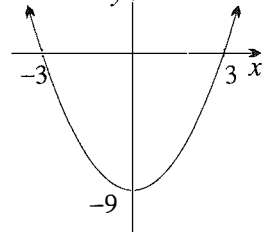
(iii)

(b) From $y = x^2$: (i) shift 9 down (ii) shift 9 up

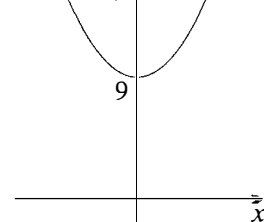
(iii) shift 3 right



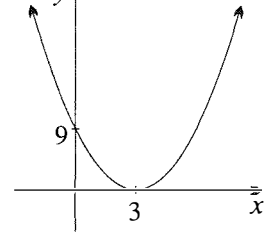
(ii)

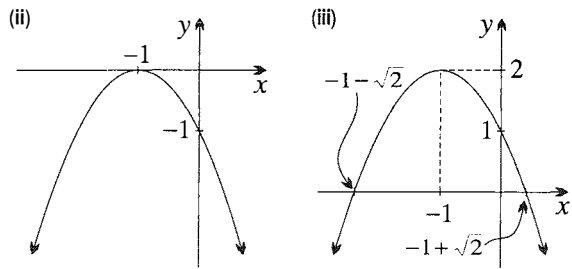
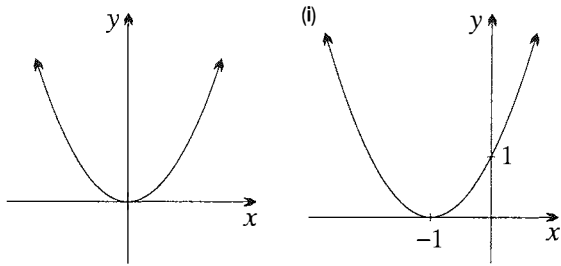


(i)

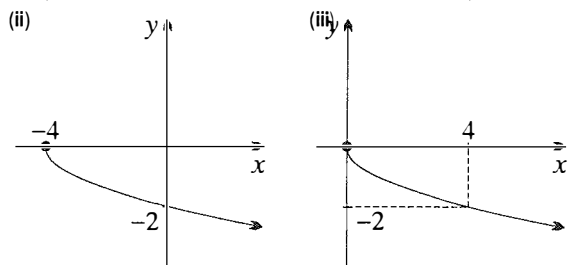
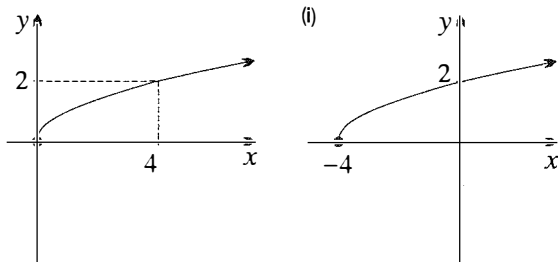


(iii)

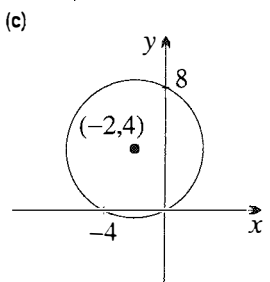
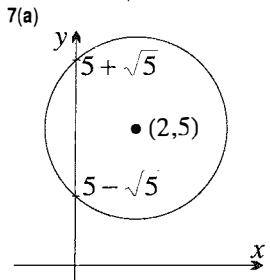
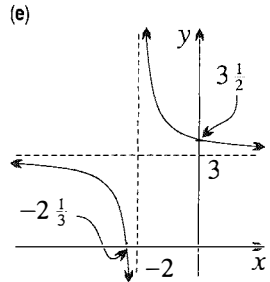
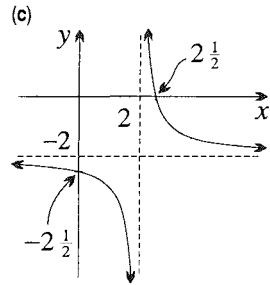
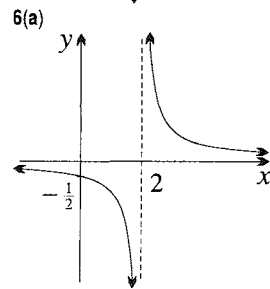
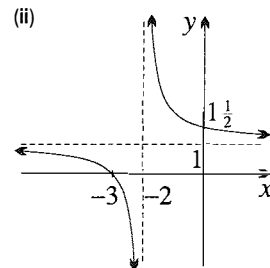
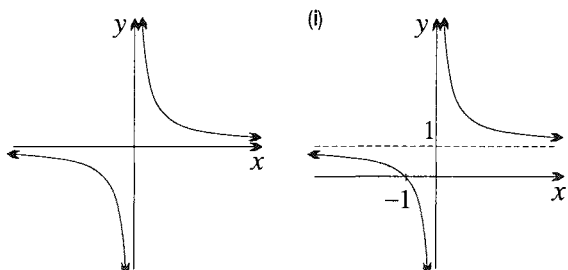
(c) From $y = x^2$: (i) shift 1 left(ii) shift 1 left and reflect in x -axis(iii) shift 1 left, reflect in x -axis and shift up 2



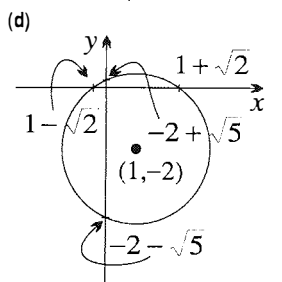
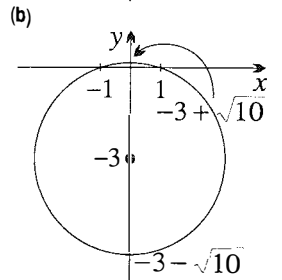
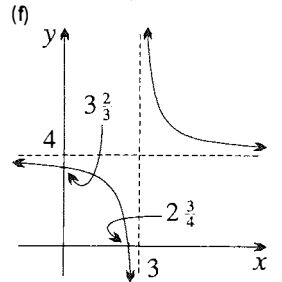
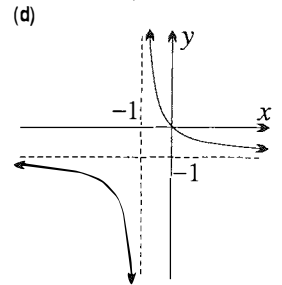
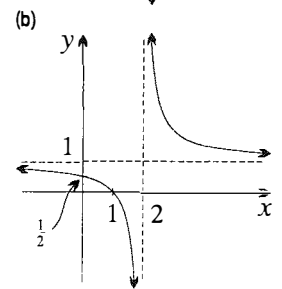
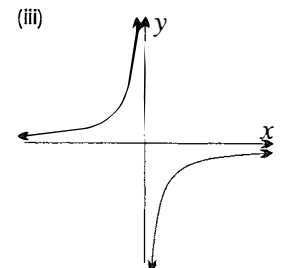
- (d) From $y = \sqrt{x}$: (i) shift 4 left
 (ii) shift 4 left and reflect in x -axis
 (iii) reflect in x -axis



- (e) From $y = \frac{1}{x}$: (i) shift up 1
 (ii) shift up 1, left 2
 (iii) reflect in the x -axis or in the y -axis



- (a) $(x-2)^2 + (y-5)^2 = 9$, $r = 3$, centre: $(2, 5)$
 $x^2 + (y+3)^2 = 10$, $r = \sqrt{10}$, centre: $(0, -3)$



$$(x+2)^2 + (y-4)^2 = 20, r = 2\sqrt{5}, \text{ centre: } (-2, 4)$$

$$(x-1)^2 + (y+2)^2 = 6, r = \sqrt{6}, \text{ centre: } (1, -2)$$

8(a) $(x+1)^2 + (y-2)^2 = 25$ (b) On the line $y = x$,
so $(\frac{1+\sqrt{41}}{2}, \frac{1+\sqrt{41}}{2})$ and $(\frac{1-\sqrt{41}}{2}, \frac{1-\sqrt{41}}{2})$

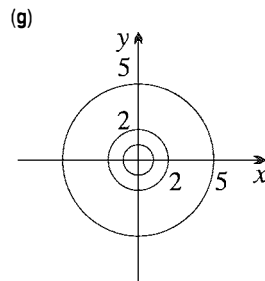
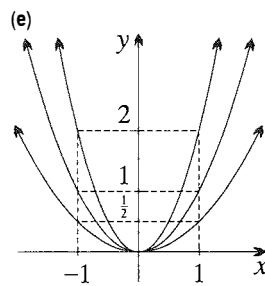
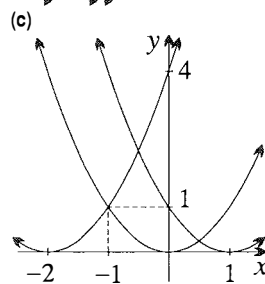
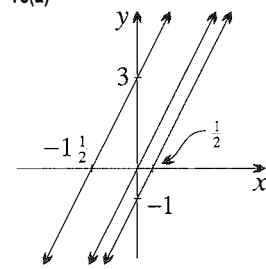
9(a) The parabola $y = x^2$ shifted left 2, down 1.
 $y + 1 = (x + 2)^2$

(b) The hyperbola $xy = 1$ shifted right 2, down 1.
 $y + 1 = \frac{1}{x-2}$

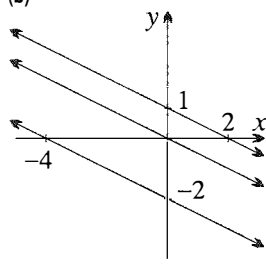
(c) The exponential $y = 2^x$ reflected in the x -axis,
shifted 1 up. $y = 1 - 2^x$

(d) The parabola $y = x^2$ reflected in $y = 0$, then
shifted 3 right and 1 up. $y = -(x-3)^2 + 1$

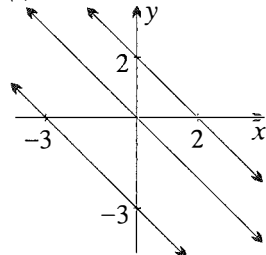
10(a)



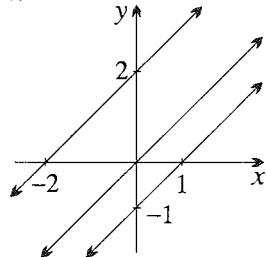
(b)



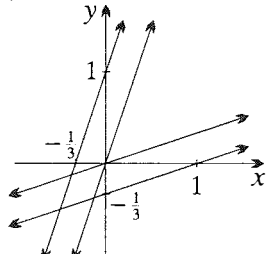
(d)



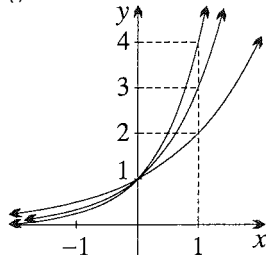
(f)



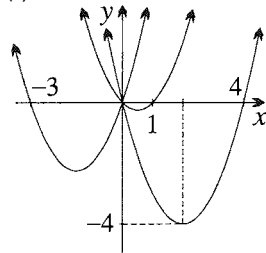
(h)



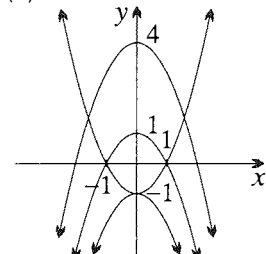
(i)



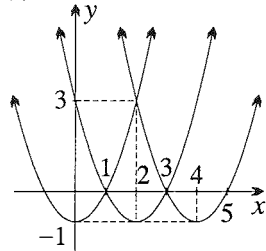
(k)



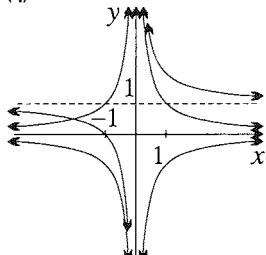
(m)



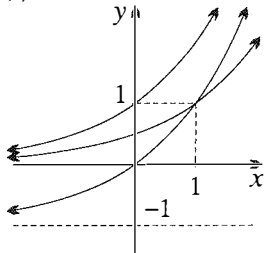
(o)



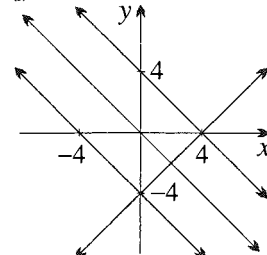
(q)



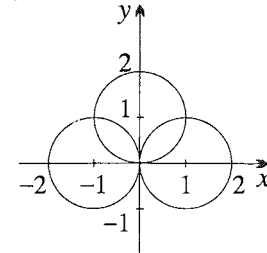
(s)



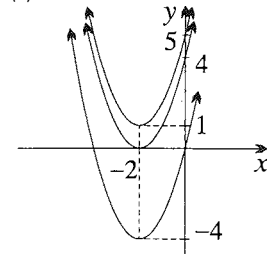
(j)



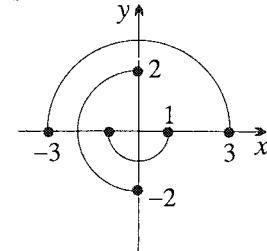
(l)



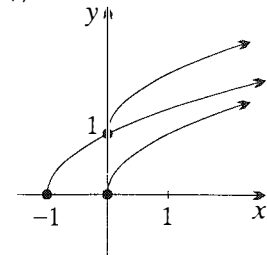
(n)



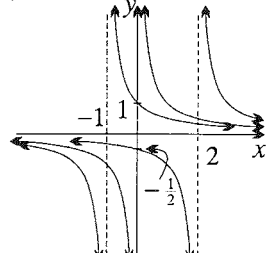
(p)

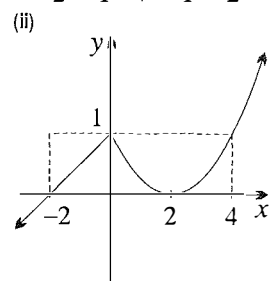
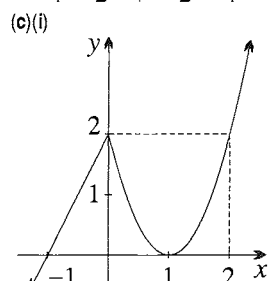
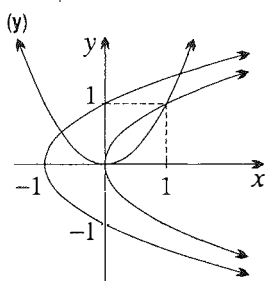
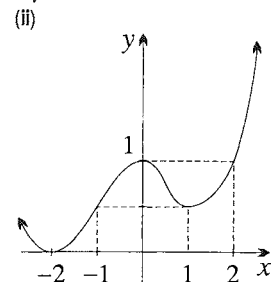
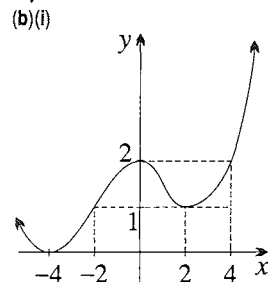
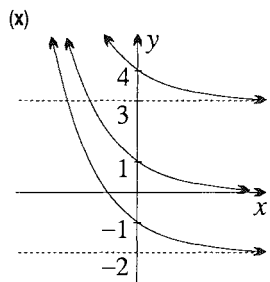
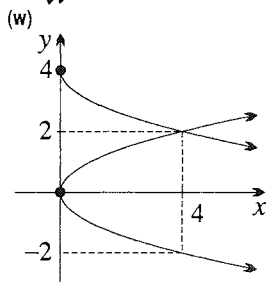
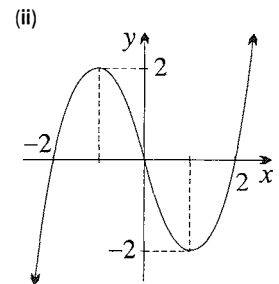
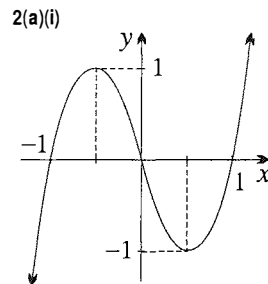
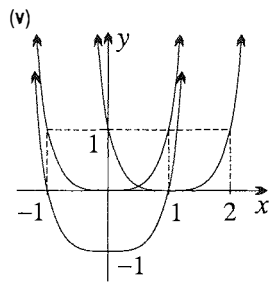
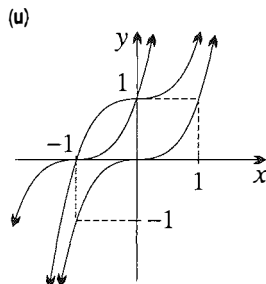


(r)



(t)





11(a) $x + 2y - 2 = 0$ (b) $x + 2y - 2 = 0$

(c) Both shifts yield the same result.

12 $y - y_1 = m(x - x_1)$ is the line $y = mx$ shifted right by x_1 and up by y_1 .

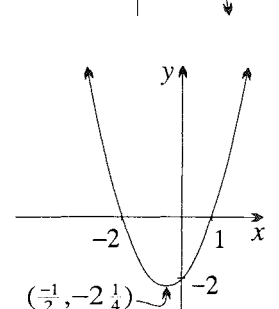
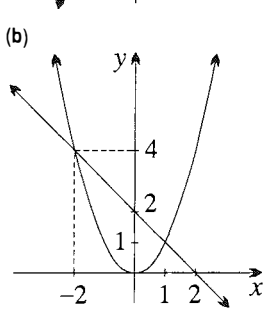
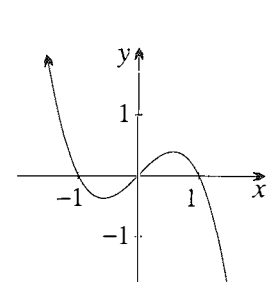
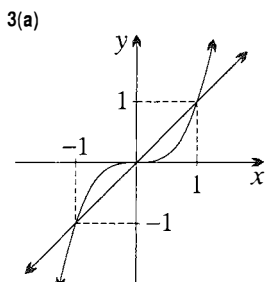
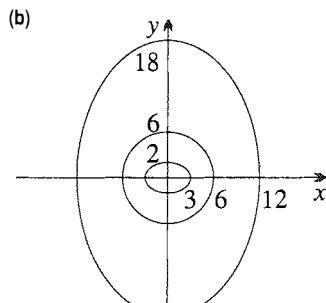
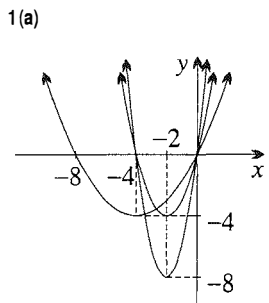
13(a) $y - a = f(x)$, $-y - a = f(x)$, $-y = f(x)$, $y = f(x)$

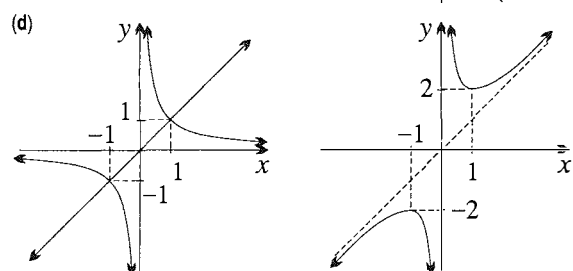
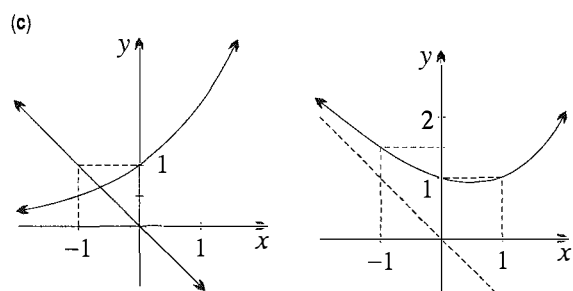
(b) $y = f(x - a)$, $-y = f(x - a)$, $-y = f(x - 2a)$, $y = f(x - 2a)$

(c) $x = f(y)$, $-x = f(y)$, $-y = f(x)$, $y = f(x)$

(d) $-x = f(y)$, $-y = f(-x)$, $x = f(-y)$, $y = f(x)$

Exercise 2J (Page 71)

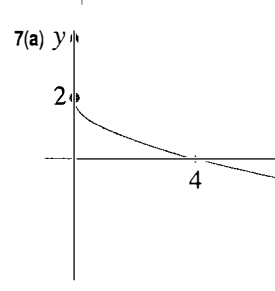
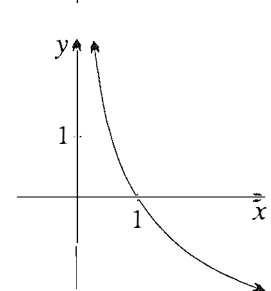
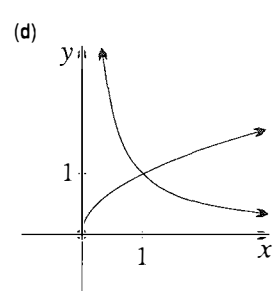
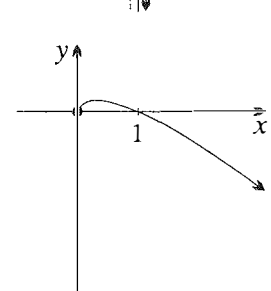
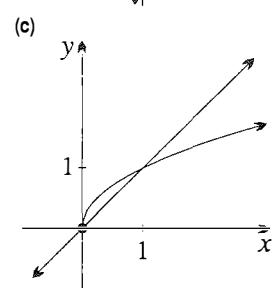
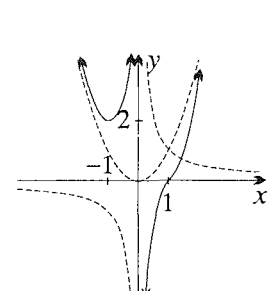
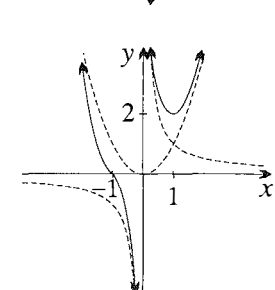
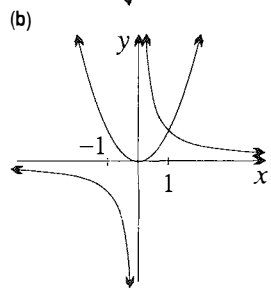
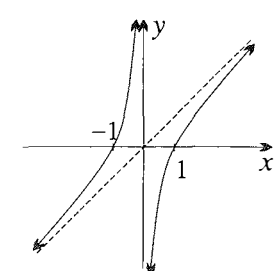
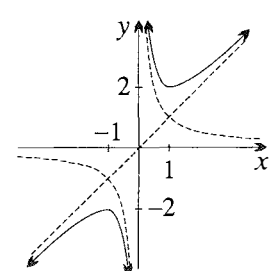
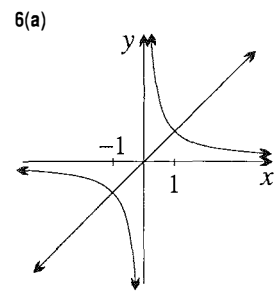
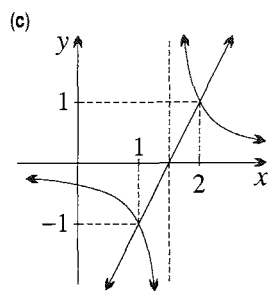
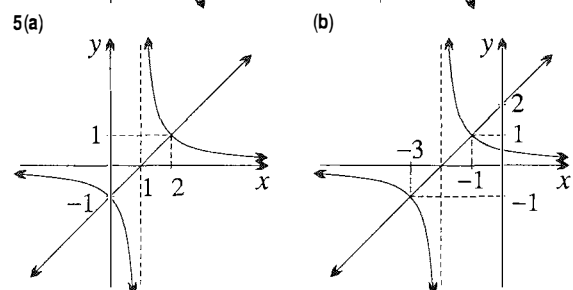
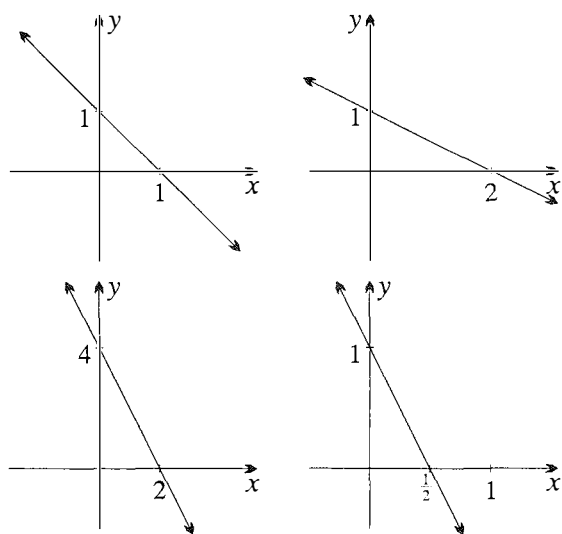


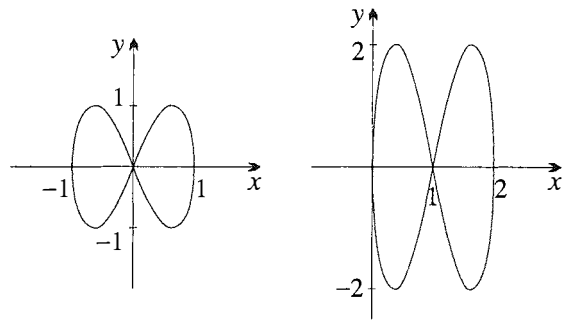
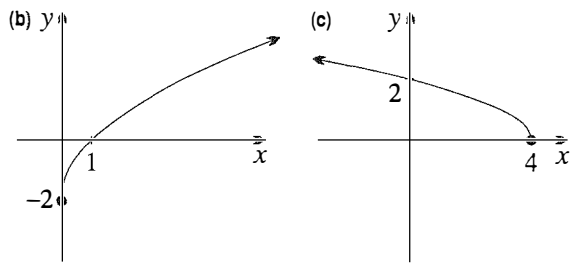


4(a) stretch horizontally by factor 2

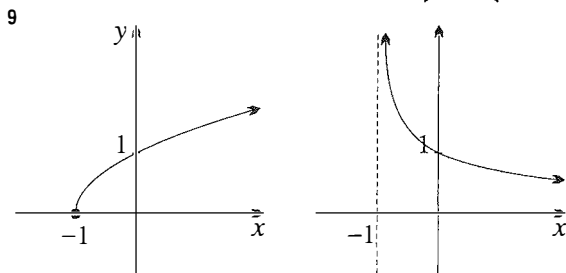
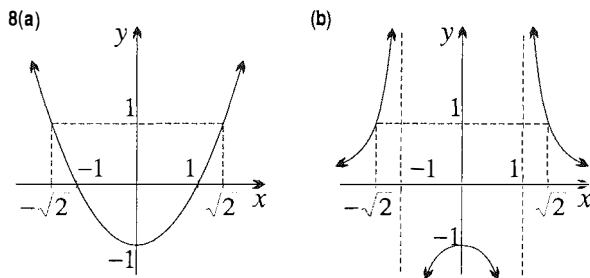
(b) stretch horizontally by factor 2,
vertically by factor 4

(c) stretch horizontally by factor $\frac{1}{2}$





- (a) reflect in the x -axis and translate up 2
 (b) stretch vertically by a factor of 2, and translate down 2
 (c) reflect in y -axis and translate right 4, or translate left 4 and reflect in y -axis



- 10(a)(i) stretch vertically by factor 2, $\frac{y}{2} = 2^x$, or translate left by 1, $y = 2^{(x+1)}$

- (ii) stretch along both axes by k , $\frac{y}{k} = \frac{1}{\frac{x}{k}}$,

or stretch horizontally by k^2 , $y = \frac{1}{\frac{x}{k^2}}$

- (iii) reciprocal, $y = \frac{1}{3^x}$,

or reflect in the y -axis, $y = 3^{-x}$

- 11 stretch horizontally by factor $\sqrt{3}$ and vertically by factor $3\sqrt{3}$

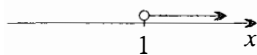
- 12(c) $(-1, 0)$, $(0, 0)$, $(1, 0)$ (d) a figure eight

- (e) $\left((x-1)^2 + \left(\frac{y}{2}\right)^2\right)^2 = (x-1)^2 - \left(\frac{y}{2}\right)^2$

Chapter Three

Exercise 3A (Page 76)

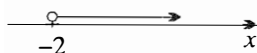
1(a) $x > 1$



(b) $x \leq 2$



(c) $x > -2$



(d) $x < 3$



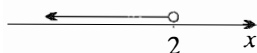
(e) $x \geq -1$



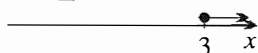
(f) $x < 2$



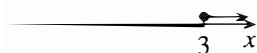
(g) $x < 2$



(h) $x \geq 3$



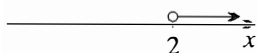
(i) $x \geq 3$



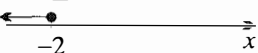
(j) $x \leq -2$



(k) $x > 2$



(l) $x \leq -2$



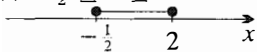
2(a) $-2 \leq x < 3$



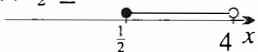
(b) $\frac{4}{3} < x \leq 5$



(c) $-\frac{1}{2} \leq x \leq 2$



(d) $\frac{1}{2} \leq x < 4$



3(a) $x > 4$

(b) $x \leq 2$

(c) $x < 2$

(d) $x \leq -1$

(e) $-2 \leq x < 1$

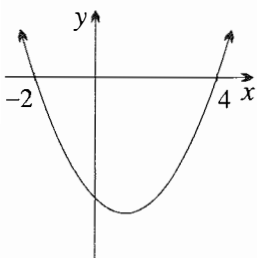
(f) $-6 \leq x \leq 15$

4(a) $0 < x < 4$

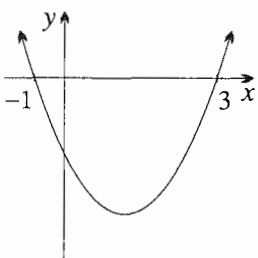
(b) $x \leq -1$ or $x \geq 3$

(c) $x \leq 0$ or $x \geq 2$

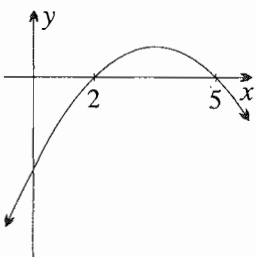
5(a) $-2 < x < 4$



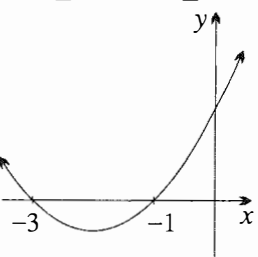
(b) $x < -1$ or $x > 3$



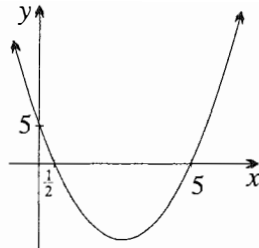
(c) $2 \leq x \leq 5$



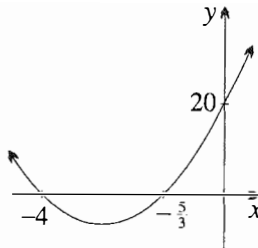
(d) $x \leq -3$ or $x \geq -1$



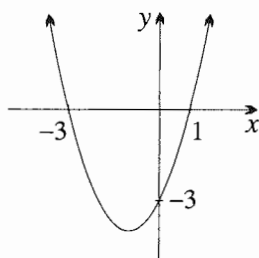
(e) $x < \frac{1}{2}$ or $x > 5$



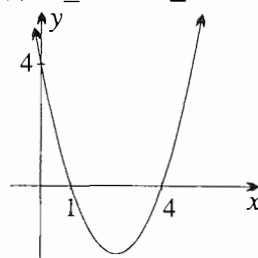
(f) $-4 \leq x \leq -\frac{5}{3}$



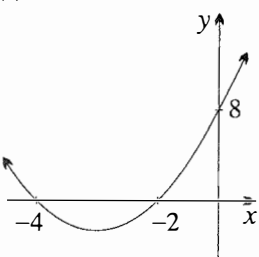
6(a) $-3 < x < 1$



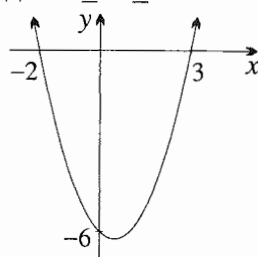
(b) $x \leq 1$ or $x \geq 4$



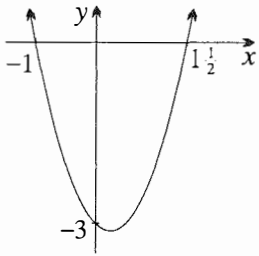
(c) $x < -4$ or $x > -2$



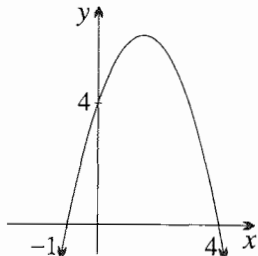
(d) $-2 \leq x \leq 3$



(e) $-1 < x < 1\frac{1}{2}$



(f) $-1 < x < 4$



7(a) $-1 \leq x \leq 1$

(b) $x < 0$ or $x > 3$

(c) $x \leq -12$ or $x \geq 12$

(d) $x < 0$ or $x > 0$ (or simply $x \neq 0$)

(e) $x = 3$

(f) $1 \leq x \leq 3$

8(a) $x < 0$ or $x \geq \frac{1}{2}$

(b) $3 < x < 5$

(c) $-4 < x \leq -2\frac{1}{2}$

(d) $x < \frac{3}{2}$ or $x > 4$

(e) $1 < x < 3$

(f) $\frac{5}{3} < x \leq 3$

9 The curve is always above the line.

10(a) false: $x = 0$

(b) false: $x = \frac{1}{2}$

(c) true

(d) false: $x = \frac{1}{2}$ or $x = -2$

(e) false: $x = -1$

(f) true

(g) false: $x = -1$

(h) true

13(a) $\frac{1}{2} < x \leq 3$

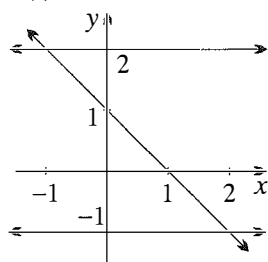
(b) $-3 < x < -2$

(c) $x < 1$ or $x \geq 3$

(d) $x < -\frac{1}{7}$ or $x > 2$

14 The two lines are parallel and thus the first is always below the second.

15(a)



(b) $-1 \leq x < 2$. The solution to the inequality is where the diagonal line lies between the horizontal lines.

16 $5x - 4 < 7 - \frac{1}{2}x$, with solution $x < 2$

17(a) $x \geq 3$ (b) $0 < x \leq 3$ (c) $-4 \leq x \leq 4$

(d) $x < -4$ (e) $0 < x < 8$ (f) $\frac{1}{25} \leq x \leq 625$

18(a) true (b) false: $a = -2$, $b = -1$ (c) true

(d) false: $a = -1$, $b = 1$ (e) true

(f) false: $a = 1$, $b = 2$

19(a) $-4 \leq 4t < 12$ (b) $-3 < -t \leq 1$

(c) $6 \leq t + 7 < 10$ (d) $-3 \leq 2t - 1 < 5$

(e) $0 \leq \frac{1}{2}(t + 1) < 2$ (f) $-2 \leq \frac{1}{2}(3t - 1) < 4$

(g) $\frac{1}{2} \leq 2^t < 8$ (h) $0 \leq \sqrt{t + 1} < 2$

20(a) $7 < x^2 + 3 < 19$ (b) $3 \leq x^2 + 3 \leq 12$

21(b)(i) Either $x^2 > xy > y^2$,
or $x^2 - y^2 = (x + y)(x - y) > 0$ so $x^2 > y^2$,
or otherwise. (ii) $n > 0$

22(a) Put $x = \sqrt{a}$ and $y = \frac{1}{\sqrt{a}}$.

(b) Put $x = \sqrt{a}$ and $y = \sqrt{b}$.

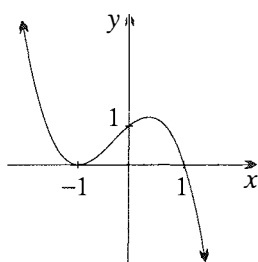
23 $x^2 + xy + y^2 = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x + y)^2$
or otherwise.

25(a) $2(a^2 + b^2 + c^2 - ab - bc - ac)$

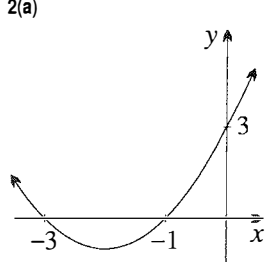
(b) $2(a^3 + b^3 + c^3 - 3abc)$

Exercise 3B (Page 81)

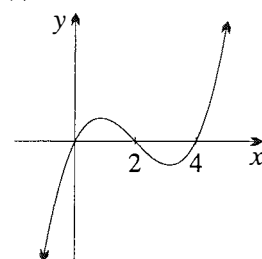
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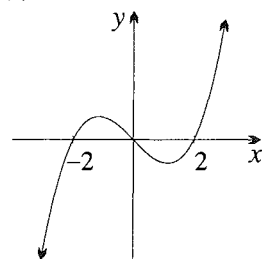
2(a)



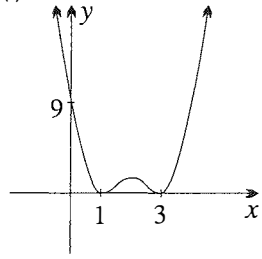
(b)



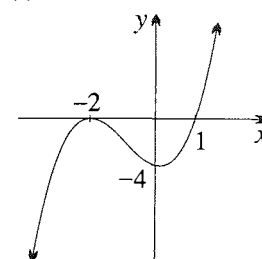
(d)



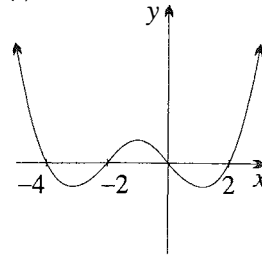
(f)



(c)



(e)



3(a) $x \leq 0$ or $1 \leq x \leq 2$

(b) $-2 < x < 0$

or $2 < x < 4$

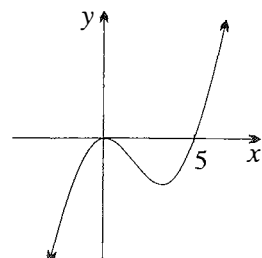
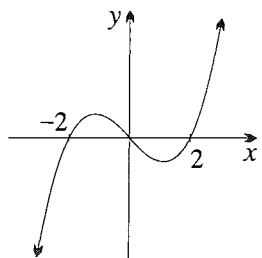
(c) $0 < x < 3$ or $x > 3$

(d) $x = 0$ or $x \geq 4$

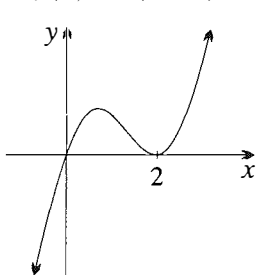
(e) $x = -3$ or $x = 3$

(f) $x = -2$ or $x \geq 0$

4(a) $f(x) = x(x - 2)(x + 2)$ (b) $f(x) = x^2(x - 5)$



(c) $f(x) = x(x - 2)^2$



5(a) $-2 < x < 0$ or $x > 2$ (b) $x < 0$ or $0 < x < 5$

(c) $x \leq 0$ or $x = 2$

6(a) $x < 1$ or $3 < x < 5$ (b) $x \neq 1$ and $x \neq 3$
(alternatively, $x < 1$ or $1 < x < 3$ or $x > 3$)

(c) $-2 < x \leq 4$ (d) $-3 < x < 0$ or $x > 3$

(e) $-3 < x < -1$ (f) $x < 0$ or $0 < x < 5$

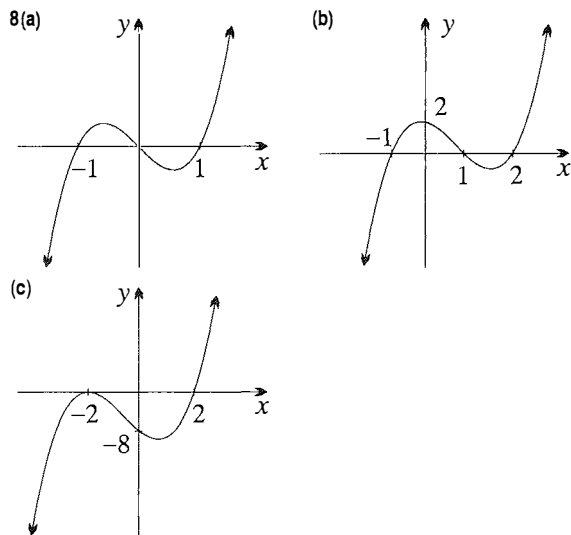
(g) $x \leq 0$ or $x \geq 5$ (h) $-2 \leq x < 0$ or $x \geq 2$

(i) $x < -3$ or $0 < x \leq 2$

7(a) $y = x(x+1)(x-1)$, $x = -1, 0$ or 1

(b) $y = (x-2)(x-1)(x+1)$, $x = -1, 1$ or 2

(c) $y = (x+2)^2(x-2)$, $x = -2$ or 2



9(a) zero for $x = 0$, undefined at $x = 3$, positive for $x < 0$ or $x > 3$, negative for $0 < x < 3$

(b) zero for $x = 4$, undefined at $x = -2$, positive for $x < -2$ or $x > 4$, negative for $-2 < x < 4$

(c) zero for $x = -3$, undefined at $x = -1$, positive for $x < -3$ or $x > -1$, negative for $-3 < x < -1$

10(a) $x \leq -4$ or $-3 < x \leq 1$

(b) $-2 < x < -1\frac{1}{2}$ or $x > \frac{1}{2}$

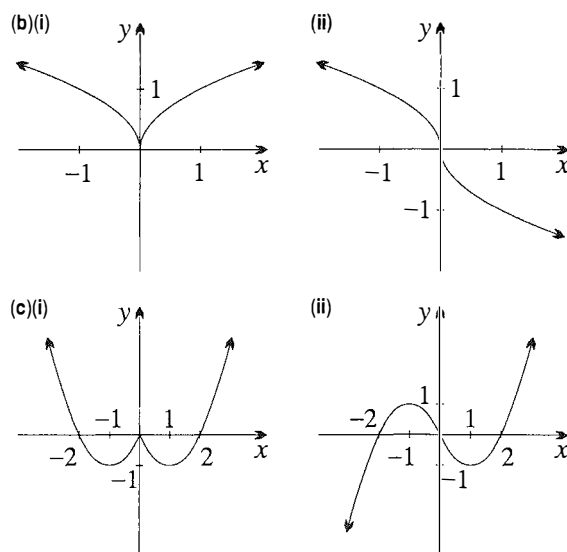
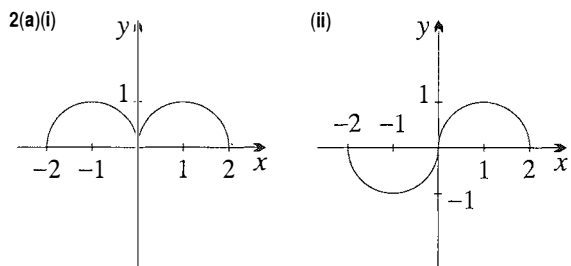
(c) $-\frac{1}{2} \leq x < 1\frac{1}{2}$ or $x \geq 2\frac{1}{2}$

Exercise 3C (Page 84)

1(a) $x \neq -1$ (b) $x \neq \frac{3}{2}$ (c) all real numbers

(d) all real numbers (e) $x \geq 0$ (f) $x \geq 1$

(g) $x \leq 7$ (h) $x \geq -4$

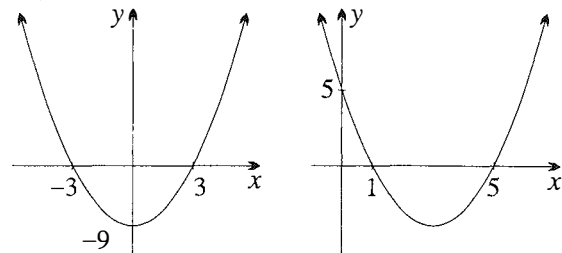


4(a) even (b) neither (c) odd (d) even

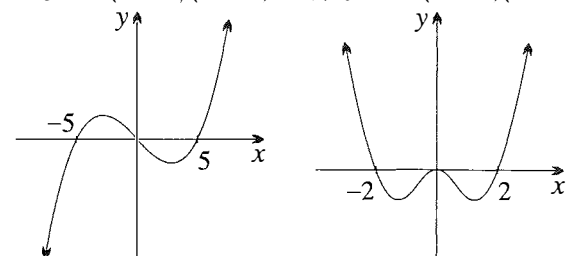
(e) neither (f) odd (g) odd (h) neither

If a function is a sum of multiples of odd powers of x , then it is odd. If it is a sum of multiples of even powers, then it is even. If the sum involves even and odd powers, then it is neither.

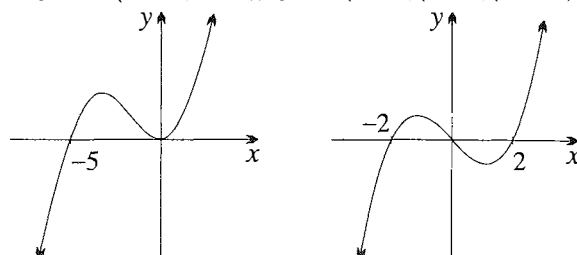
5(a) $y = (x+3)(x-3)$ (b) $y = (x-1)(x-5)$



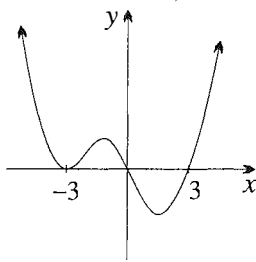
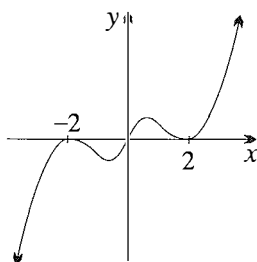
(c) $y = x(x-5)(x+5)$ (d) $y = x^2(x-2)(x+2)$



(e) $y = x^2(x+5)$ (f) $y = x(x-2)(x+2)(x^2+4)$



(g) $y = x(x-2)^2(x+2)^2$ (h) $y = x(x-3)(x+3)^2$



6(a) $x \neq 1$ (b) $x \neq 4$ (c) $x \neq -1$ (d) $x \neq 2$

(e) $x > -4$ (f) $x > 1$ (g) all real x (h) $x \neq 3$

7(a) $x \leq -2$ or $x \geq 2$ (b) $x < -2$ or $x > 2$

8(a) $-2 \leq x \leq 2$ (b) $-2 < x < 2$ (c) $-5 \leq x \leq 5$

(d) $-5 < x < 5$ (e) $x \leq -2$ or $x \geq 2$

(f) $x < -2$ or $x > 2$

9(a) even (b) even (c) odd (d) neither

10(a)(i) even (ii) even (iii) odd (b)(i) even (ii) odd (iii) in general, neither

11(a) Suppose $f(0) = c$. Then since $f(x)$ is odd, $f(0) = -f(0) = -c$. So $c = -c$, and hence $c = 0$.

(b) It is not defined at the origin (it is 1 for $x > 0$, and -1 for $x < 0$).

12(a) Let $y = f^{-1}(-x)$. Then $-x = f(y)$, from which it follows $x = f(-y)$ since f is odd. Hence $-y = f^{-1}(x)$, and thus $f^{-1}(-x) = -f^{-1}(x)$ as required. (b) The graph will fail the horizontal line test unless it is a single point on the y -axis.

13(b)(i) $g(x) = 1 + x^2$ and $h(x) = -2x$

(ii) $g(x) = \frac{2^x + 2^{-x}}{2}$ and $h(x) = \frac{2^x - 2^{-x}}{2}$

(c) In the first, $g(x)$ and $h(x)$ are not defined for all x in the natural domain of $f(x)$, specifically at $x = -1$. In the second, $x = 0$ is the only place at which $g(x)$ and $h(x)$ are defined.

Exercise 3D (Page 89)

1(a) For $|x-2|$: 3, 2, 1, 0, 1.

For $|x|-2$: -1, -2, -1, 0, 1.

(b) The first is $y = |x|$ shifted right 2 units, the second is $y = |x|$ shifted down 2 units.

2(a) 5 (b) 3 (c) 7 (d) 3 (e) 3 (f) 3 (g) 16 (h) -3

3(a) $x = 3$ or -3 (b) $x = 5$ or -5

(c) $x = 10$ or -4 (d) $x = 5$ or -7

(e) $x = 6$ or -5 (f) $x = 2$ or $-3\frac{1}{3}$

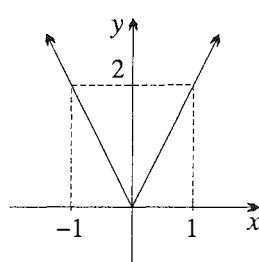
(g) $x = \frac{7}{5}$ or $-\frac{11}{5}$ (h) $x = 2$ or $-\frac{8}{7}$

4(a) false: $x = 0$ (b) true (c) true

(d) false: $x = -2$ (e) true (f) true

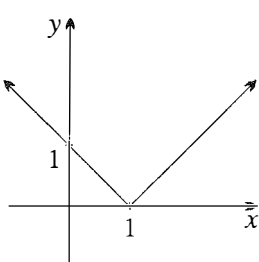
(g) false: $x = -2$ (h) true

5(a)



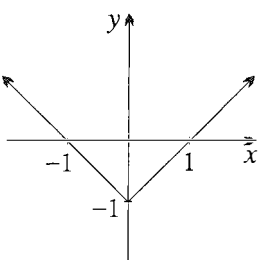
$y = \begin{cases} 2x, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$

(c)



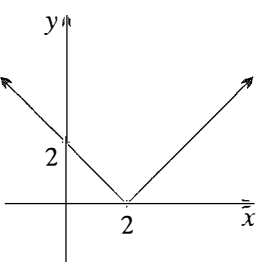
$y = \begin{cases} x-1, & \text{for } x \geq 1, \\ 1-x, & \text{for } x < 1. \end{cases}$

(e)



$y = \begin{cases} x-1, & \text{for } x \geq 0, \\ -x-1, & \text{for } x < 0. \end{cases}$

(g)

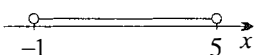


$y = \begin{cases} x-2, & \text{for } x \geq 2, \\ 2-x, & \text{for } x < 2. \end{cases}$

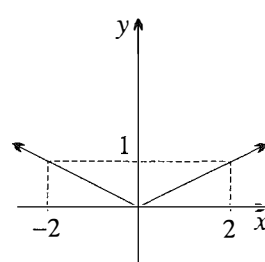
6 An absolute value can not be negative.

7(a) even (b) neither (c) odd (d) even

8(a) $-1 < x < 5$

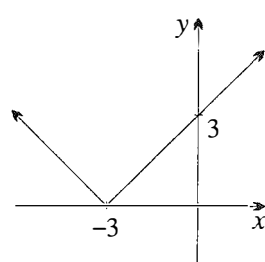


(b)



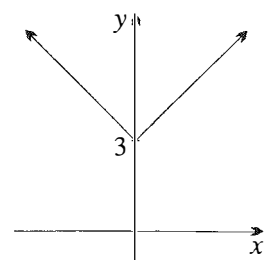
$y = \begin{cases} \frac{1}{2}x, & \text{for } x \geq 0, \\ -\frac{1}{2}x, & \text{for } x < 0. \end{cases}$

(d)



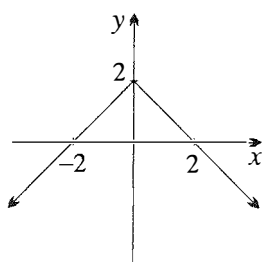
$y = \begin{cases} x+3, & \text{for } x \geq -3, \\ -x-3, & \text{for } x < -3. \end{cases}$

(f)



$y = \begin{cases} x+3, & \text{for } x \geq 0, \\ 3-x, & \text{for } x < 0. \end{cases}$

(h)

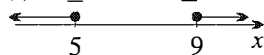


$y = \begin{cases} 2-x, & \text{for } x \geq 0, \\ 2+x, & \text{for } x < 0. \end{cases}$

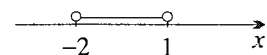
(b) $\frac{1}{3} \leq x \leq 3$



(c) $x \geq 9$ or $x \leq 5$



(d) $-2 < x < 1$



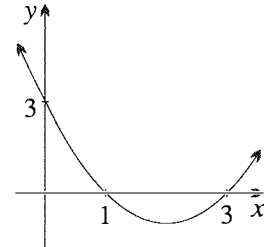
(e) $x > 2$ or $x < \frac{1}{3}$



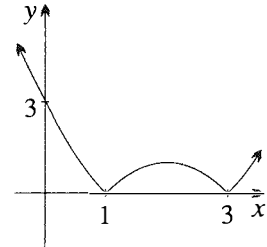
(f) $x \geq \frac{2}{5}$ or $x \leq -2$



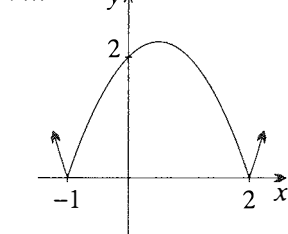
9(a)(i)



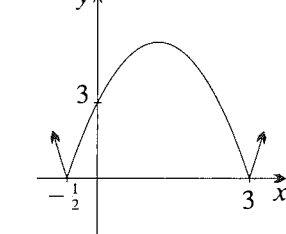
(ii)



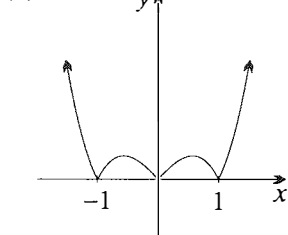
(b)(i)



(ii)



(iii)

10(a) The first holds when x is positive, the second when x is negative.

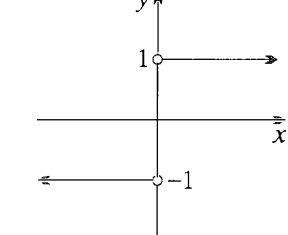
(b)(i) $-2 < x < 2$ or $-10 < x < -6$

(ii) $3 \leq x < 4\frac{1}{2}$

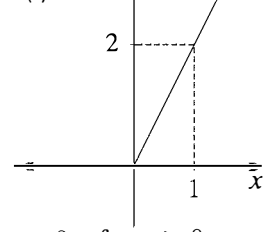
or $\frac{1}{2} < x \leq 2$

(a) y is undefined for $x = 0$.(c) $y = 1$, for $x > 0$, and $y = -1$, for $x < 0$.

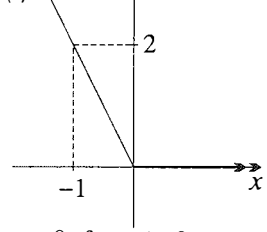
11



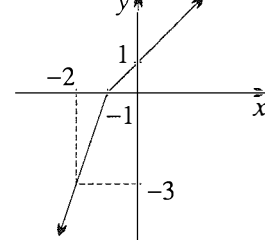
12(a)

 $y = 2x$, for $x \geq 0$,
 $y = 0$, for $x < 0$.

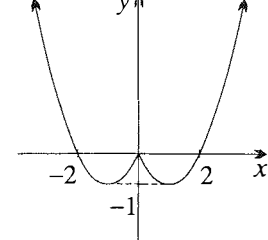
(b)

 $y = 0$, for $x \geq 0$,
 $y = -2x$, for $x < 0$.13(a) false: $x = 2$ and $y = -2$ (b) true(c) false: $x = 2$ and $y = -2$ (d) true (e) true(f) false: $x = -2$ 14(a) $x \neq 1$ (b) $y = \frac{1}{x-1}$, for $x > 1$, and $y = \frac{1}{1-x}$, for $x < 1$.

17(a)



(b)

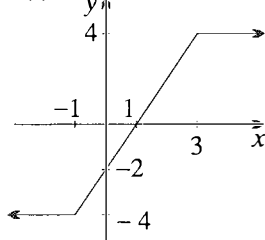


$$y = \begin{cases} x+1, & \text{for } x \geq -1, \\ 3(x+1), & \text{for } x < -1. \end{cases} \quad y = \begin{cases} x^2 - 2x, & \text{for } x \geq 0, \\ x^2 + 2x, & \text{for } x < 0. \end{cases}$$

18(a) An absolute value must be positive.

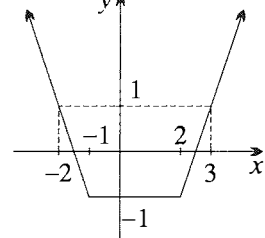
(b) $x > 1$

19(a)



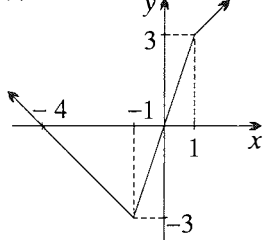
$$y = \begin{cases} -4, & \text{for } x < -1, \\ 2x - 2, & \text{for } -1 \leq x < 3, \\ 4, & \text{for } x \geq 3. \end{cases}$$

(b)



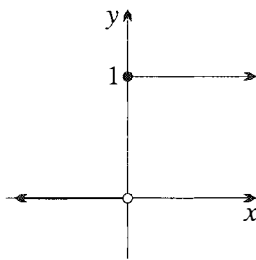
$$y = \begin{cases} -2x - 3, & \text{for } x < -1, \\ -1, & \text{for } -1 \leq x < 2, \\ 2x - 5, & \text{for } x \geq 2. \end{cases}$$

(c)

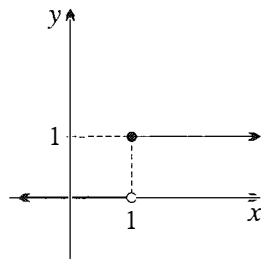


$$y = \begin{cases} -x - 4, & \text{for } x < -1, \\ 3x, & \text{for } -1 \leq x < 1, \\ x + 2, & \text{for } x \geq 1. \end{cases}$$

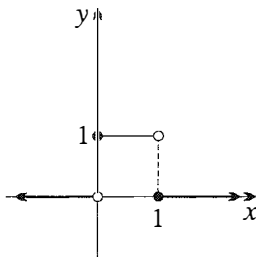
20(a)(i)



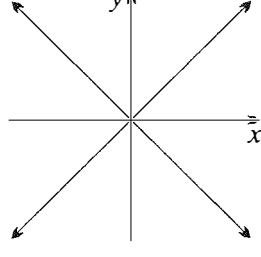
(ii)



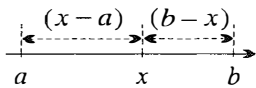
(b)



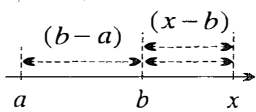
21



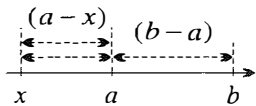
22(a) $|x-a| + |x-b| = (x-a) + (b-x) < c$



(b) $|x-a| + |x-b| = (x-a) + (x-b) = (b-a) + 2(x-b) < c$



(c) $|x-a| + |x-b| = (a-x) + (b-x) = (b-a) + 2(a-x) < c$


 (d) The result follows directly from parts (a), (b) and (c). (e) $-3 < x < 7$

Exercise 3E (Page 93)

1(a) $\sqrt{2} \doteq 1.4$, $\sqrt{3} \doteq 1.7$ (b) $y = 2$ and $y = 3$

(c) $x = -1$ or $x = 2$ (d) $x < -1$ or $x > 2$

(e) $x = -2$ or $x = 1$, $-2 \leq x \leq 1$

(f) $x \doteq 1.62$ or $x \doteq -0.62$

(g)(i) Draw $y = -x$, $x = 0$ or $x = -1$.

(ii) Draw $y = x + \frac{1}{2}$, $x \doteq 1.37$ or $x \doteq -0.37$.

(iii) Draw $y = \frac{1}{2}x + \frac{1}{2}$, $x = 1$ or $x = -\frac{1}{2}$.

2(a) $x \leq -3$ (b) $0 \leq x \leq 2$ (c) $x = 1$

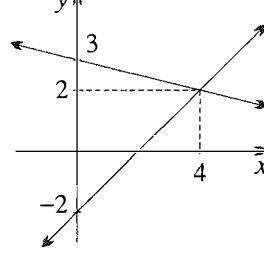
3(a) $x < -2$ or $x > 1$ (b) $0 \leq x \leq 1$

(c) $-1 < x < 0$ or $x > 1$

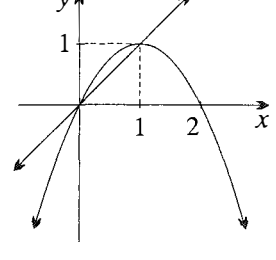
4(a) $(4, 2)$, $x - 2 = 3 - \frac{1}{4}x$ (b) $(0, 0)$ and $(1, 1)$, $x = 2x - x^2$ (c) $(-1, -2)$ and $(2, 1)$, $\frac{2}{x} = x - 1$

(d) $(-1, -1)$ and $(0, 0)$ and $(1, 1)$ $x^3 = x$

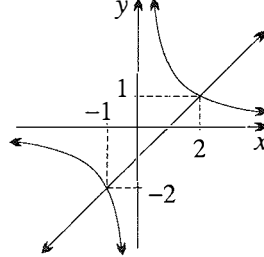
(a)



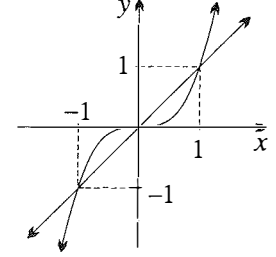
(b)



(c)

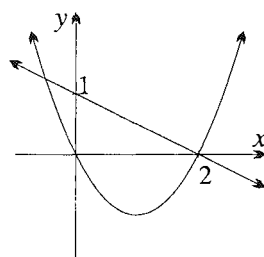


(d)

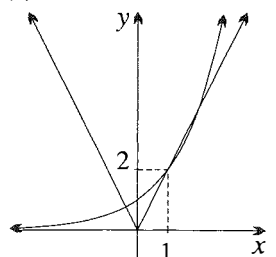


5(a) $x \geq 4$ (b) $0 < x < 1$ (c) $x < -1$ or $0 < x < 2$
 (d) $-1 < x < 0$ or $x > 1$

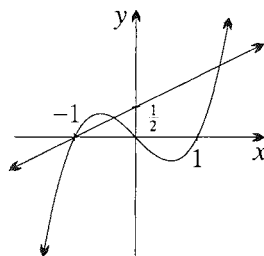
6(a) 2 solutions



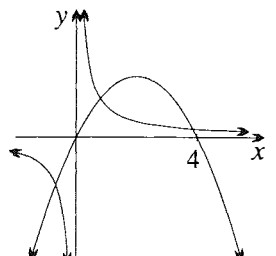
(b) 3 solutions



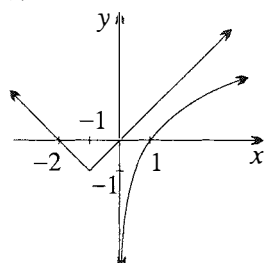
(c) 3 solutions



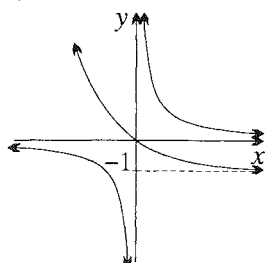
(d) 3 solutions

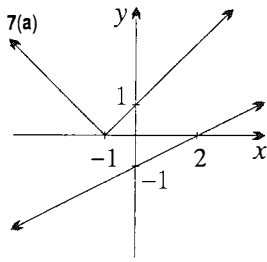


(e) no solutions



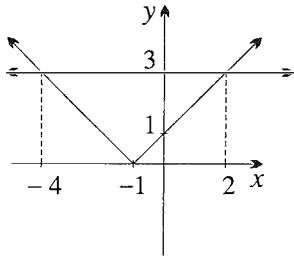
(f) no solutions



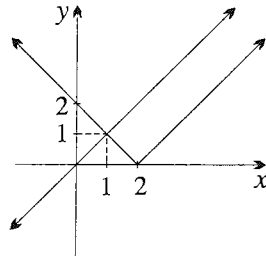


(b) The graph of $y = |x + 1|$ is always above the graph of $y = \frac{1}{2}x - 1$.

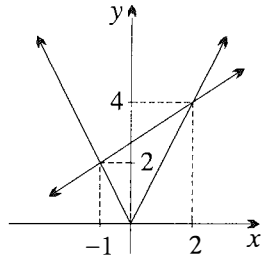
8(a) $(-4, 3), (2, 3)$



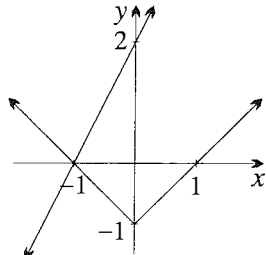
(b) $(1, 1)$



(c) $(-1, 2), (2, 4)$



(d) $(-1, 0)$

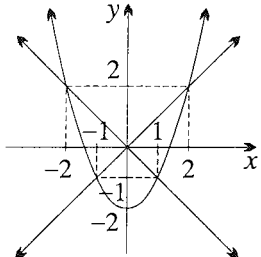


9(a) $-4 \leq x \leq 2$ (b) $x < 1$ (c) $x \leq -1$ or $x \geq 2$
(d) $x < -1$

10(b) The right-hand branch is $y = x$, which gives solution $x = 3$, and the left-hand branch is $y = -x$, which gives solution $x = -3$.

(c) $-3 \leq x \leq 3$

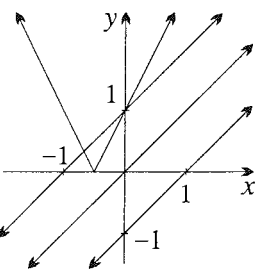
11(a)



(b) $x = 2$ or -2

(c) $x < -2$ or $x > 2$

12



(c) $c > \frac{1}{2}$

13(b) $b^2 < \frac{9}{2}$

14(a) 2

(b) The solutions are not integers.

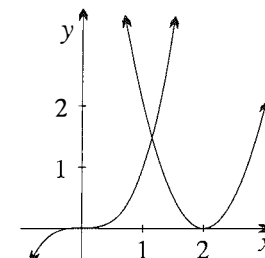
(c) $x = \frac{1}{11}$ or $\frac{7}{3}$

15(a) $x \leq 2\frac{1}{2}$ (b) $x \leq -4$ or $x \geq 0$

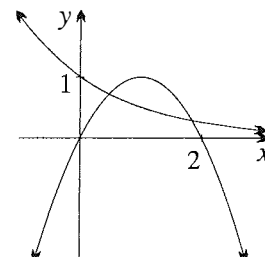
(c) $-2 < x < \frac{2}{3}$ (d) $x < -4$ or $x > \frac{2}{3}$

16(a) $x \neq 1.1$

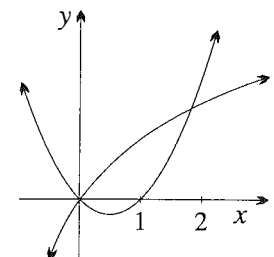
(b) $x \neq 1.2$



(c) $x \neq 0.5$ or $x \neq 1.9$



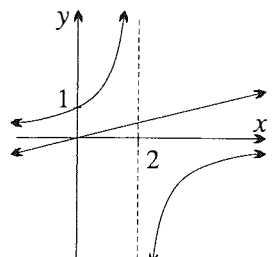
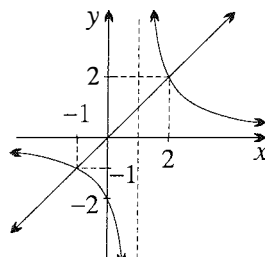
(d) $x = 0$ or $x \neq 1.8$



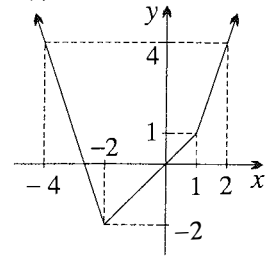
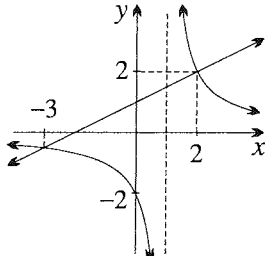
17(a) $-1 \leq x < 1$ or $x \geq 2$

(b) $x < 2$

(c) $-3 < x < 1$ or $x > 2$



18(a)

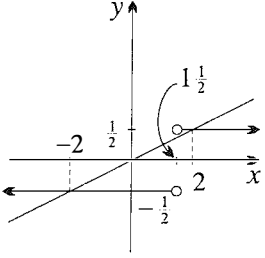


$$y = \begin{cases} -3x - 8, & \text{for } x < -2, \\ x, & \text{for } -2 \leq x < 1, \\ 3x - 2, & \text{for } x \geq 1. \end{cases}$$

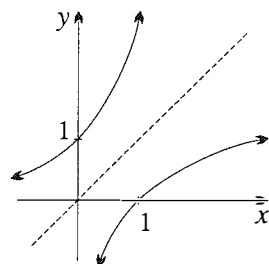
(b) $-3\frac{1}{3} \leq x \leq -2\frac{1}{3}$ or $-1 \leq x \leq 1\frac{1}{3}$

19(b) $b < m$ (c) $-p \leq m \leq p$ and $b < -\frac{qm}{p}$

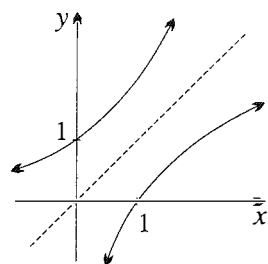
20 $x \leq -2$ or $1\frac{1}{2} < x \leq 2$



21(a)(i)

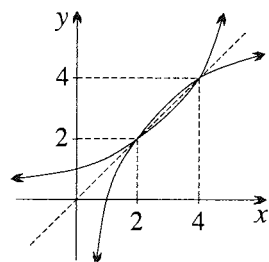


(ii)



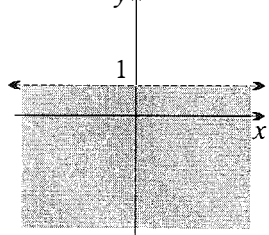
(b) 0, 1 or 2

(iii)

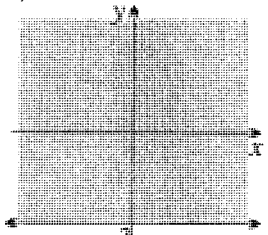


Exercise 3F (Page 98)

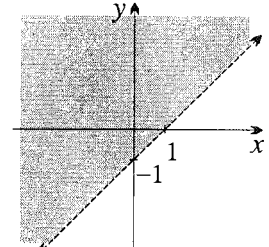
1(a)



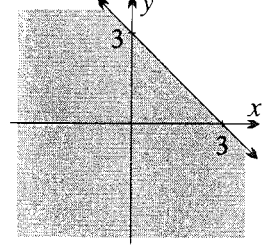
(b)



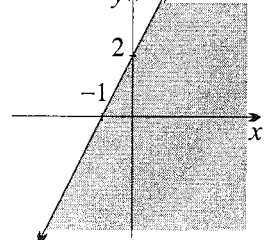
(c)



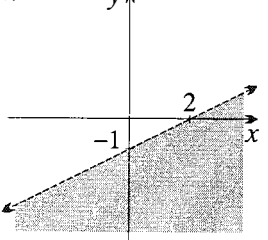
(d)



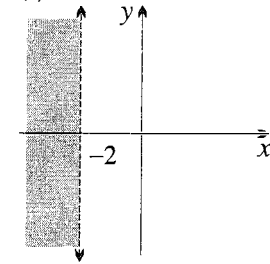
(e)



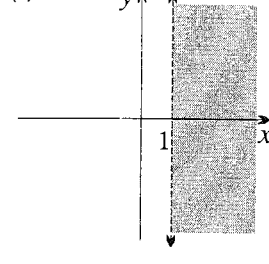
(f)



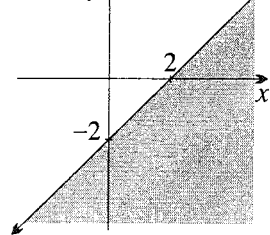
2(a)



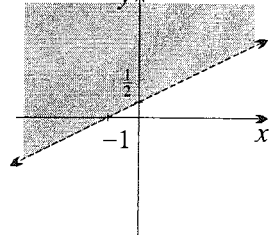
(b)



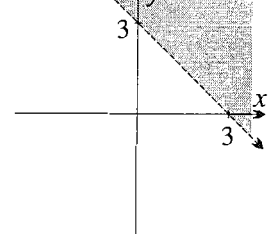
(c)



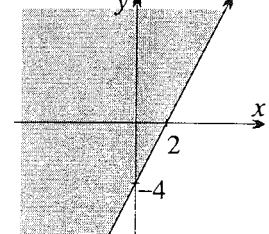
(d)



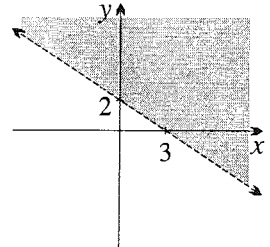
(e)



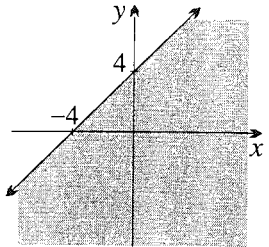
(f)



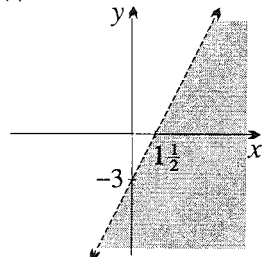
3(a)



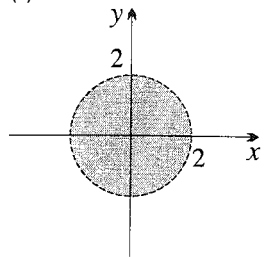
(b)



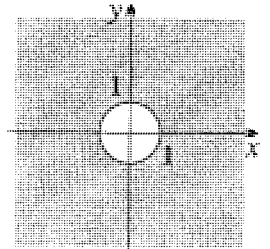
(c)



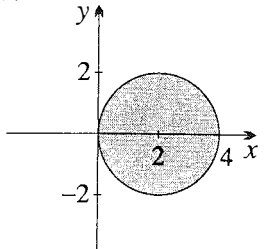
4(a)

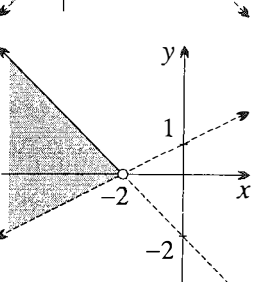
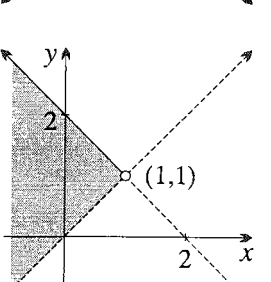
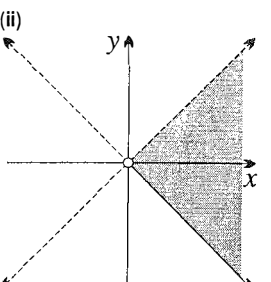
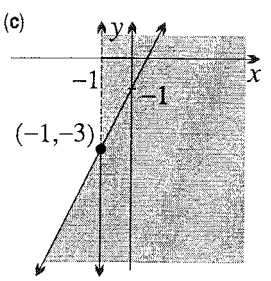
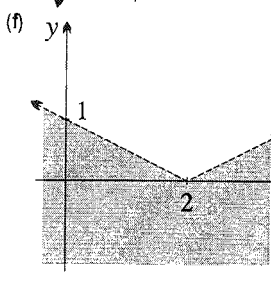
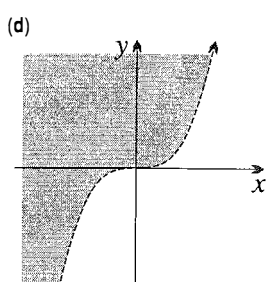
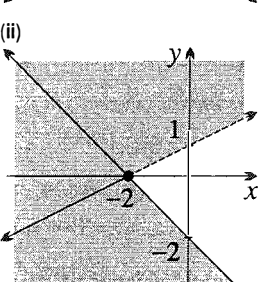
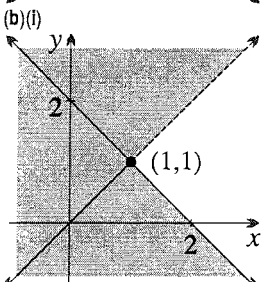
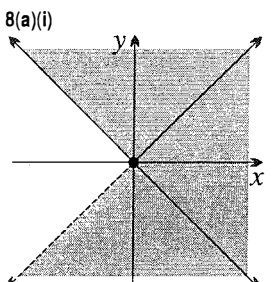
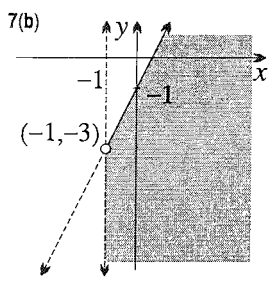
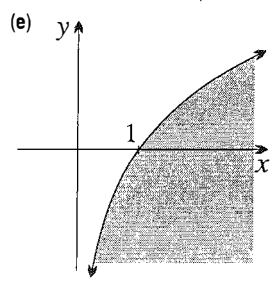
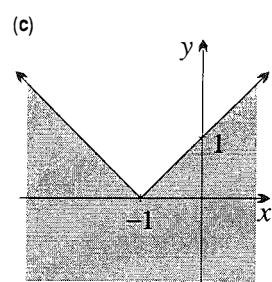
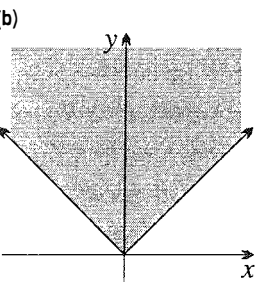
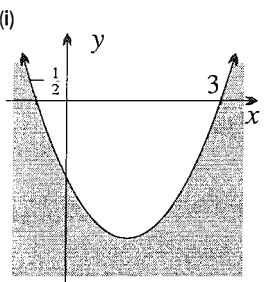
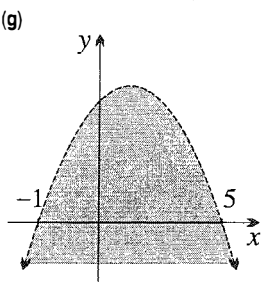
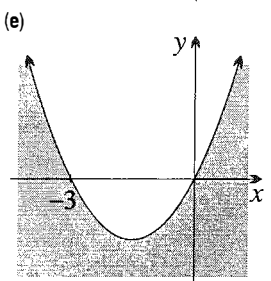
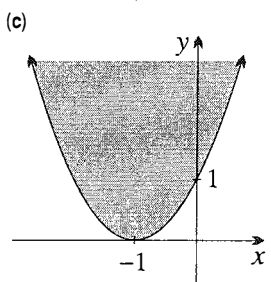
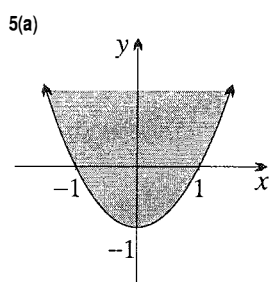
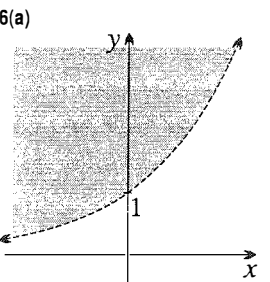
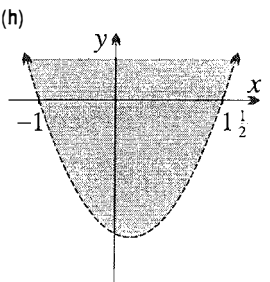
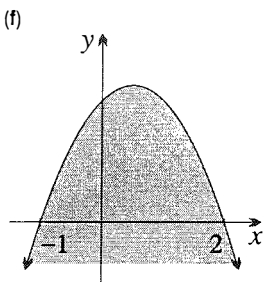
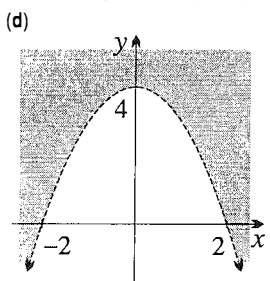
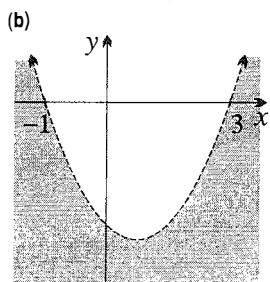
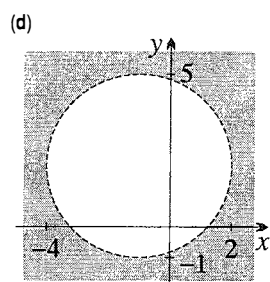


(b)



(c)





9(a) $x \geq 0$ and $y \geq 0$ (b) $x \leq 0$ and $y \geq 0$

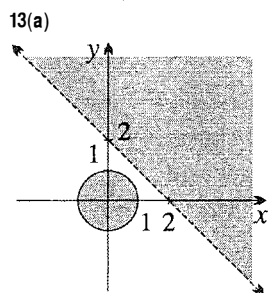
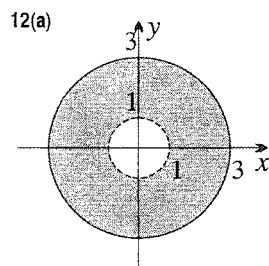
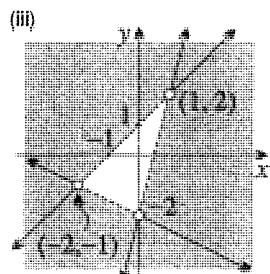
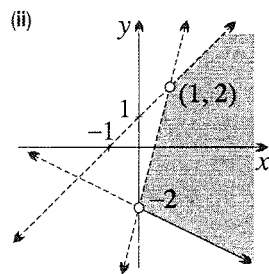
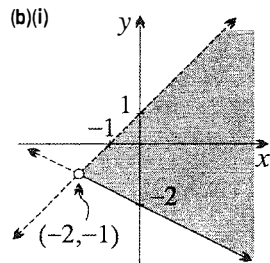
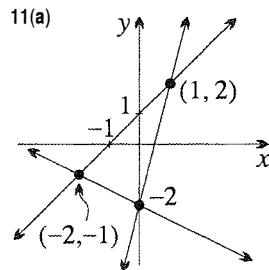
(c) $x \leq 0$ and $y \leq 0$ (d) $x \geq 0$ and $y \leq 0$

(e) $x \geq 0$ or $y \geq 0$ (f) $x \geq 0$ or $y \leq 0$

10(a) $y < x$ and $y \leq 2 - x$

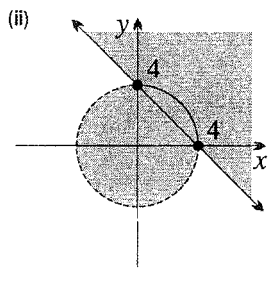
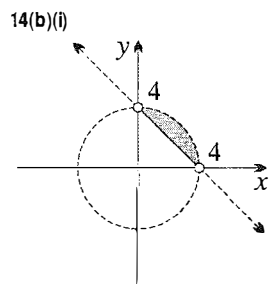
(b) $y \leq -\frac{1}{2}x - 1$ or $y \geq 2 - 2x$

(c) $y < x + 2$ or $y > 4x - 1$



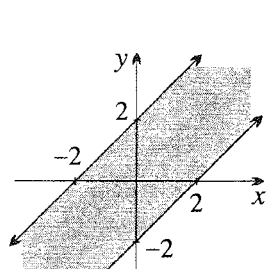
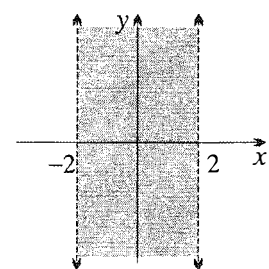
(b) whole plane

(b) no intersection

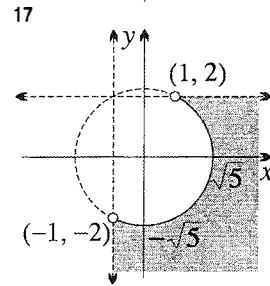
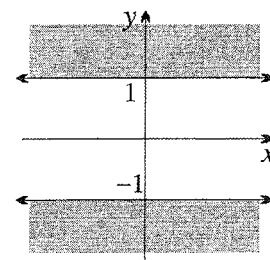


15(a) $x < 2$ and $x > -2$

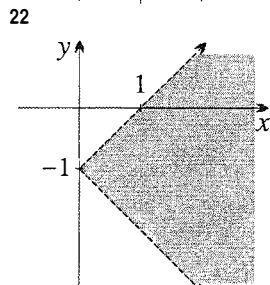
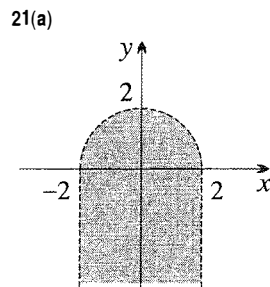
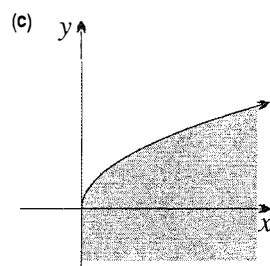
(b) $x - y \leq 2$ and $x - y \geq -2$



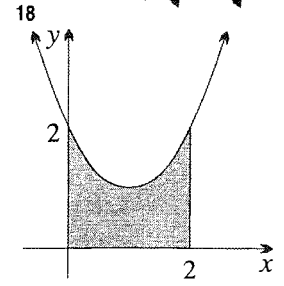
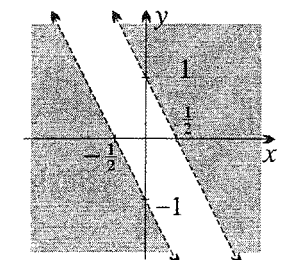
16(a) $y \geq 1$ or $y \leq -1$



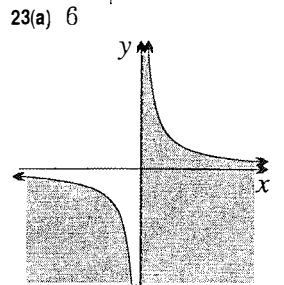
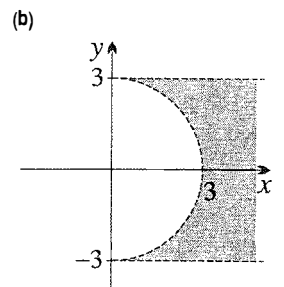
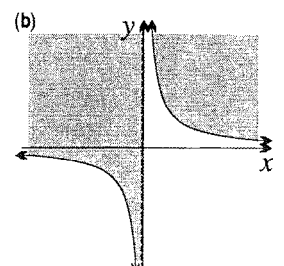
19(b) The curve is undefined for $x < 0$.



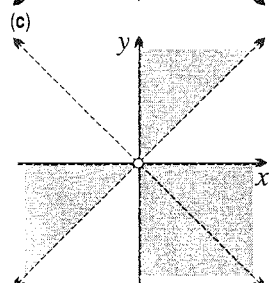
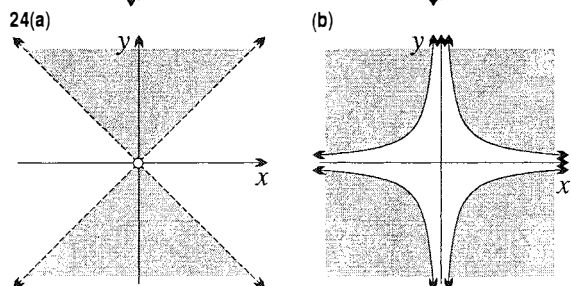
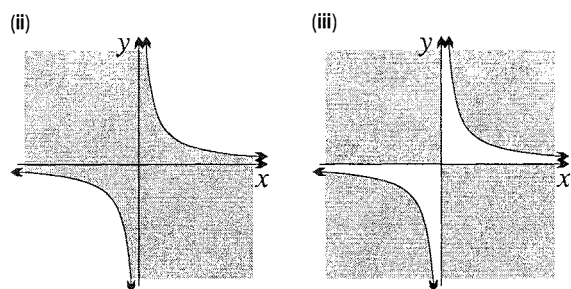
(b) $y + 2x > 1$ or $y + 2x < -1$



20(a) The curve is undefined when $x = 0$.



(b)(i)



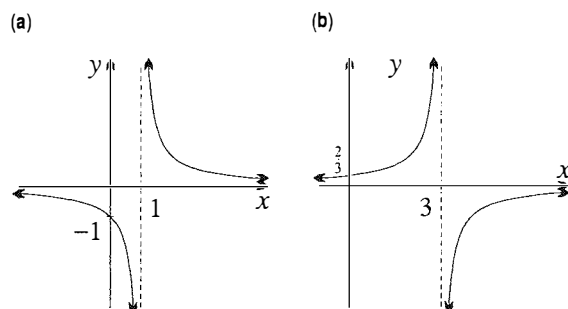
25(a) A region is connected if every pair of points within the region can be joined by a curve that lies within the region.

(b) A region is not convex if there exist two points in the region which may be joined by a straight line that goes outside of the region.

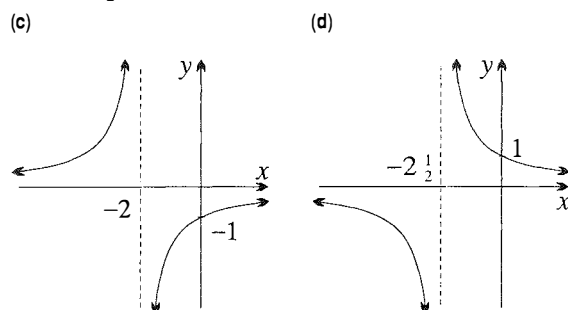
Exercise 3G (Page 105)

- 1(a) $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$
 (b) $f(x) \rightarrow 1$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$
 (c) $f(x) \rightarrow -2$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$
 (d) $f(x) \rightarrow \frac{1}{2}$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$
 (e) $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$
 (f) $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$

- 2(a) domain: $x \neq 1$, vertical asymptote: $x = 1$,
 $y \rightarrow \infty$ as $x \rightarrow 1^+$, and $y \rightarrow -\infty$ as $x \rightarrow 1^-$
 (b) domain: $x \neq 3$, vertical asymptote: $x = 3$,
 $y \rightarrow -\infty$ as $x \rightarrow 3^+$, and $y \rightarrow \infty$ as $x \rightarrow 3^-$



- (c) domain: $x \neq -2$, vertical asymptote: $x = -2$,
 $y \rightarrow -\infty$ as $x \rightarrow -2^+$, and $y \rightarrow \infty$ as $x \rightarrow -2^-$
 (d) domain: $x \neq -2\frac{1}{2}$, vertical asymptote: $x = -2\frac{1}{2}$,
 $y \rightarrow \infty$ as $x \rightarrow -2\frac{1}{2}^+$, and $y \rightarrow -\infty$ as $x \rightarrow -2\frac{1}{2}^-$



3(a) $x \neq 2$

(b) $x = 0$ and $y = 0$

(c) $y \rightarrow 1$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

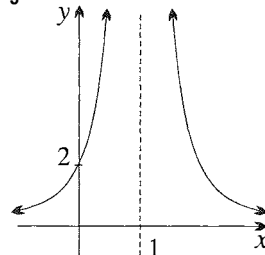
(d) $x = 2$ is a vertical asymptote, $y \rightarrow \infty$ as $x \rightarrow 2^+$, $y \rightarrow -\infty$ as $x \rightarrow 2^-$.

4(a) $x = -3$

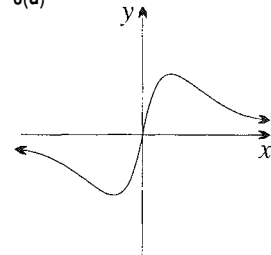
(b) $x = 1$ and $y = -\frac{1}{3}$

(c) $y \rightarrow 1$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$,
 $y \rightarrow -\infty$ as $x \rightarrow -3^+$,
 $y \rightarrow \infty$ as $x \rightarrow -3^-$.

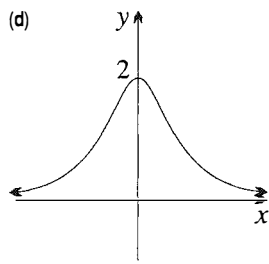
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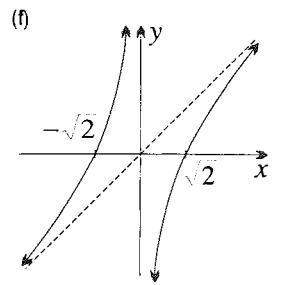
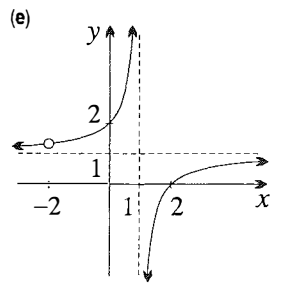
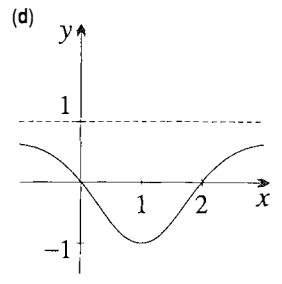
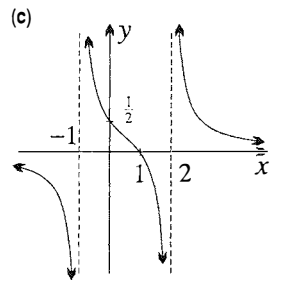
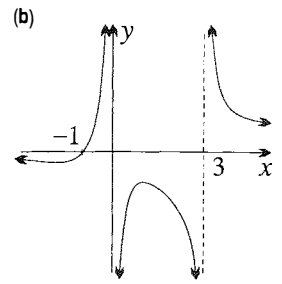
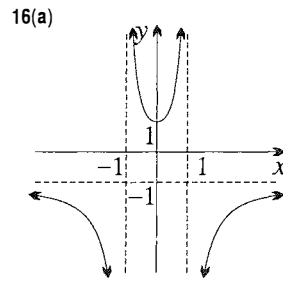
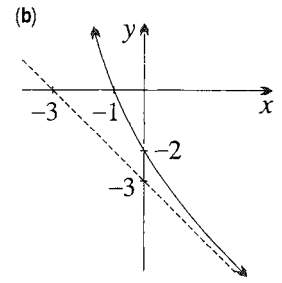
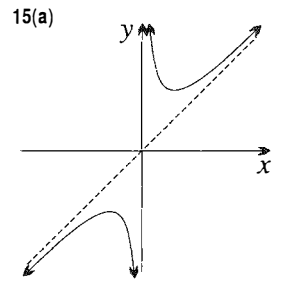
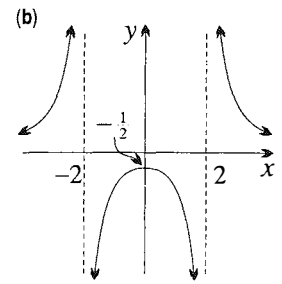
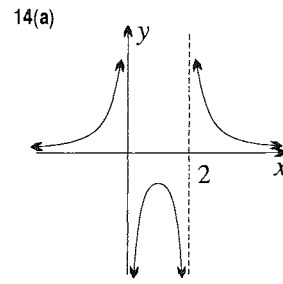
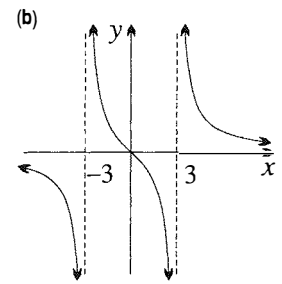
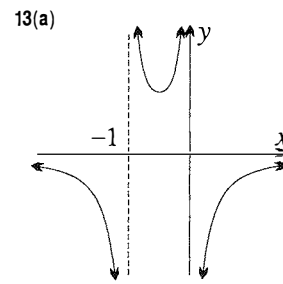
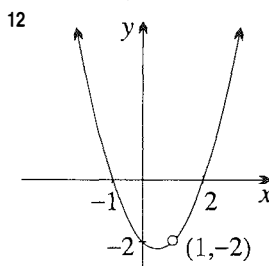
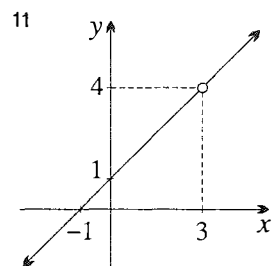
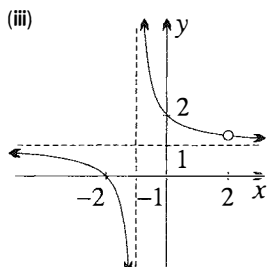
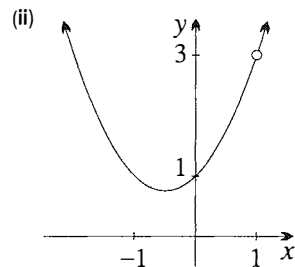
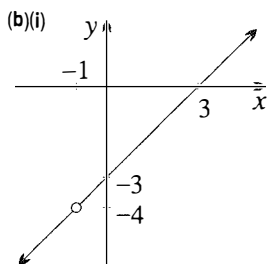
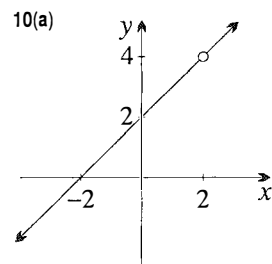
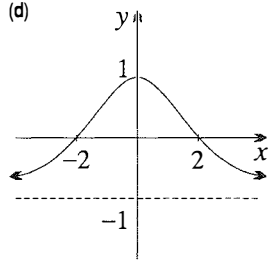
6(d)

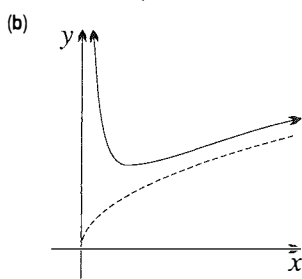
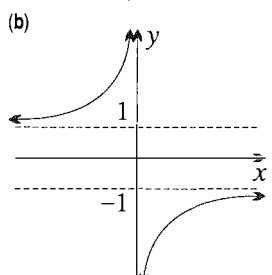
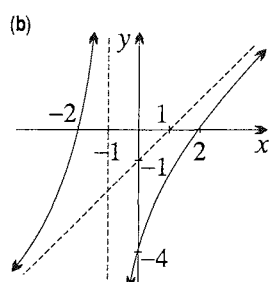
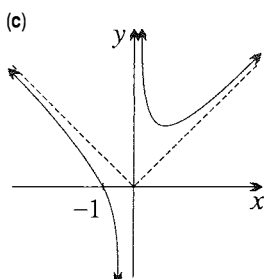
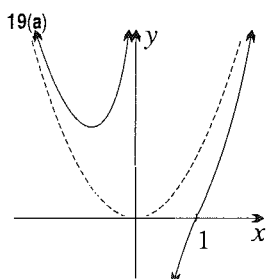
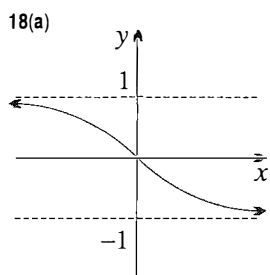
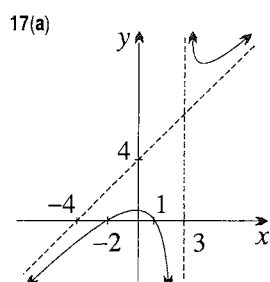


- 7(a) even
(b) $y = 2$



- 8(a) $x = 1$, $x = 3$ and $y = 2$
(b) $x = \frac{1}{3}$, $x = -\frac{1}{3}$ and $y = \frac{4}{9}$
(c) $x = -1$, $x = -4$ and $y = 1$
(d) $x = -5$, $x = 2$ and $y = 0$
9(b) $x = -2$, 2 and $y = 1$ (d)
(c) $y = -1$





Chapter Four

Exercise 4A (Page 111)

1(a) 0.4067 (b) 0.4848 (c) 0.7002 (d) 0.8443

(e) 4.9894 (f) 0.9571 (g) 2.9238 (h) 1.4945

(i) 0.6745 (j) 1.8418 (k) 2.6372 (l) 1.0119

2(a) 76° (b) 27° (c) 39° (d) 71° (e) 10° (f) 21°

3(a) $41^\circ 25'$ (b) $16^\circ 42'$ (c) $46^\circ 29'$ (d) $77^\circ 3'$

(e) $40^\circ 32'$ (f) $75^\circ 24'$

4(a) $\frac{12}{13}$ (b) $\frac{5}{12}$ (c) $\frac{13}{12}$ (d) $\frac{5}{12}$ (e) $\frac{13}{12}$ (f) $\frac{13}{5}$

5(a) 6 and 17 (b)(i) $\frac{15}{17}$ (ii) $\frac{4}{5}$ (iii) $\frac{3}{4}$ (iv) $\frac{17}{8}$ (v) $\frac{5}{3}$

(vi) $\frac{15}{8\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) 2 (e) $\sqrt{2}$ (f) $\sqrt{3}$

7(a) 1 (b) $\frac{1}{2}$ (c) 4 (d) 1

8(a) $x \doteq 4.4$ (b) $a \doteq 10.4$ (c) $h \doteq 19.0$, $j \doteq 16.2$

(d) $k \doteq 17.4$, $\ell \doteq 12.6$

9(a) $\alpha \doteq 58^\circ 24'$, $\beta \doteq 31^\circ 36'$

(b) $x \doteq 31^\circ 47'$, $y \doteq 58^\circ 13'$

(c) $\theta \doteq 57^\circ 16'$, $\phi \doteq 32^\circ 44'$

(d) $\alpha \doteq 54^\circ 19'$, $\beta \doteq 35^\circ 41'$

10(a) 0.61 (b) 2.86 (c) 0.26 (d) 0.31 (e) 1.09

(f) 3.65

11(b) 3 (c)(i) $\frac{1}{3}\sqrt{5}$, $\frac{2}{3}$

12(a)(i) $\frac{1}{2}\sqrt{22}$ (ii) $\frac{3}{2}\sqrt{2}$

14(a) $71^\circ 34'$ (b) 21.98 (c) $\alpha = 54^\circ 19'$, $\beta = 35^\circ 41'$

15(a) $b \doteq 8.452$ (b) $\ell \doteq 8.476$

(c) $s \doteq 10.534$, $h \doteq 17.001$

(d) $a \doteq 16.314$, $b \doteq 7.607$

16 73°

17 11°

18(a) $\angle PQR = 20^\circ + 70^\circ = 90^\circ$ (using alternate angles on parallel lines and the fact that due west is 270°). (b) $110^\circ + 39^\circ = 149^\circ$

19(a) 5.1 cm (b) 16 cm (c) $PQ = 18 \sin 40^\circ$, $63^\circ 25'$

20(a) $69^\circ 5'$, $69^\circ 5'$ and $41^\circ 51'$ (b) 0.4838 (c) $60^\circ 31'$

(d) 3.172 (e) $64^\circ 1'$ and $115^\circ 59'$ (f) 0.2217

21(b) $16\sqrt{3} \text{ cm}^2$

23 457 metres

24 1.58 nautical miles

25(a) $y = x \tan 39^\circ$ and $y + 7 = x \tan 64^\circ$

26(a) 108°

Exercise 4B (Page 115)

 2(b)(i) $BD = a \cos B$

 3(a) $\angle QPR = 90^\circ - \theta$, so $\angle RPS = \theta$.

 (b) $\frac{h}{a}$ and $\frac{b}{h}$

 5(a) $\angle OTP = 90^\circ$ (radius $\perp$ tangent)

 and $\angle OTA = 90^\circ - \theta$ (angle sum of $\triangle OTA$),

 so $\angle ATP = \theta$.

 6 $\frac{1}{2}q$, $\frac{1}{2}p\sqrt{3}$ and $\frac{1}{2}(q + p\sqrt{3})$

 8(a) If $\angle RBQ = \alpha$, then $\angle RQB = 90^\circ - \alpha$ (angle sum of $\triangle BQR$) and so $\angle RQP = \alpha$ (complementary angles). Therefore $\angle QPR = 90^\circ - \alpha$ (angle sum of $\triangle PQR$) and so $\angle QPC = \alpha$ (complementary angles). Thus $\angle RBQ = \angle RQP = \angle QPC$.

 12(a) If $OA = OB = x$ and $OP = y$, then

 $AP - PB = (x + y) - (x - y) = 2y = 2 \times OP$.

 13(a)(i) $\angle NSO = 90^\circ - \alpha$ (angle sum of $\triangle NOS$), so $\angle QSR = 90^\circ - \alpha$ (vertically opposite) and so $\angle RPQ = \alpha$ (angle sum of $\triangle PQS$). (ii) $NR = MQ$ (opposite sides of rectangle $MNRQ$).

 So $NP = NR + RP = MQ + RP$.

Exercise 4C (Page 119)

 3(a) -320° (b) -250° (c) -170° (d) -70°

 (e) -300° (f) -220°

 4(a) 310° (b) 230° (c) 110° (d) 10° (e) 280° (f) 170°

 5(a) $70^\circ, 430^\circ, -290^\circ, -650^\circ$

 (b) $100^\circ, 460^\circ, -260^\circ, -620^\circ$

 (c) $140^\circ, 500^\circ, -220^\circ, -580^\circ$

 (d) $200^\circ, 560^\circ, -160^\circ, -520^\circ$

 (e) $240^\circ, 600^\circ, -120^\circ, -480^\circ$

 (f) $340^\circ, 700^\circ, -20^\circ, -380^\circ$

 6(a) $\sin \theta = \frac{4}{5}$, $\cos \theta = \frac{3}{5}$, $\tan \theta = \frac{4}{3}$,

 $\operatorname{cosec} \theta = \frac{5}{4}$, $\sec \theta = \frac{5}{3}$, $\cot \theta = \frac{3}{4}$

 (b) $\sin \theta = \frac{3}{5}$, $\cos \theta = -\frac{4}{5}$, $\tan \theta = -\frac{3}{4}$,

 $\operatorname{cosec} \theta = \frac{5}{3}$, $\sec \theta = -\frac{5}{4}$, $\cot \theta = -\frac{4}{3}$

 (c) $\sin \theta = -\frac{2}{5}\sqrt{5}$, $\cos \theta = -\frac{1}{5}\sqrt{5}$, $\tan \theta = 2$,

 $\operatorname{cosec} \theta = -\frac{1}{2}\sqrt{5}$, $\sec \theta = -\sqrt{5}$, $\cot \theta = \frac{1}{2}$

 (d) $\sin \theta = -\frac{5}{13}$, $\cos \theta = \frac{12}{13}$, $\tan \theta = -\frac{5}{12}$,

 $\operatorname{cosec} \theta = -\frac{13}{5}$, $\sec \theta = \frac{13}{12}$, $\cot \theta = -\frac{12}{5}$

8(a)(i) 0.5 (ii) -0.5 (iii) 0.95 (iv) 0.95 (v) 0.59

(vi) 0.81 (vii) -0.89 (viii) 0.45 (ix) -0.81 (x) 0.59

 (b)(i) $30^\circ, 150^\circ$ (ii) $120^\circ, 240^\circ$

 (iii) $64^\circ, 116^\circ$ (iv) $53^\circ, 307^\circ$ (v) $53^\circ, 127^\circ$

 (vi) $143^\circ, 217^\circ$ (vii) $204^\circ, 336^\circ$ (viii) $107^\circ, 253^\circ$

 (c) $45^\circ, 225^\circ$

 10(b) A circle of radius r_0 .

 (c) The curve is called a *rectangular hyperbola*, and is the same curve as the curve $y = 1/x$ in the two-dimensional coordinate plane.

Exercise 4D (Page 125)

1(a) + (b) + (c) - (d) - (e) - (f) - (g) - (h) + (i) + (j) + (k) + (l) - (m) + (n) - (o) - (p) +

 2(a) 36° (b) 30° (c) 50° (d) 20° (e) 60° (f) 30° (g) 60° (h) 70° (i) 40° (j) 60°

 3(a) $-\tan 50^\circ$ (b) $\cos 50^\circ$ (c) $-\sin 40^\circ$ (d) $\cot 80^\circ$ (e) $-\sec 10^\circ$ (f) $-\operatorname{cosec} 40^\circ$ (g) $-\cos 5^\circ$ (h) $\operatorname{cosec} 55^\circ$ (i) $-\tan 40^\circ$ (j) $-\sin 85^\circ$ (k) $\sec 80^\circ$ (l) $\cot 20^\circ$

4(a) 1 (b) -1 (c) 0 (d) 0 (e) undefined (f) 1 (g) -1 (h) 0 (i) 1 (j) 0 (k) undefined (l) undefined

 5(a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\frac{\sqrt{3}}{2}$ (e) $\sqrt{3}$ (f) $-\frac{2}{\sqrt{3}}$ (g) -2 (h) $\frac{1}{\sqrt{2}}$ (i) -2 (j) $-\frac{1}{\sqrt{3}}$ (k) $-\frac{2}{\sqrt{3}}$ (l) $-\frac{1}{\sqrt{2}}$ (m) $\frac{1}{2}$ (n) $\sqrt{3}$ (o) $-\sqrt{2}$ (p) $-\frac{2}{\sqrt{3}}$

6(a) 0.42 (b) -0.91 (c) 0.91 (d) -0.42 (e) 0.49 (f) 0.49

7(a) -0.70 (b) -1.22 (c) -0.70 (d) -0.52 (e) 1.92 (f) -0.52

 8(a) 1 (b) $-\frac{7}{2}$ (c) $\frac{3}{4}$

 10(a) $(2, 2\sqrt{3})$ (b) $(-\sqrt{3}, 1)$ (c) $(1, -1)$ (d) $(-5, -5\sqrt{3})$

 11(a) $53^\circ 8'$ (b) $138^\circ 11'$ (c) 300° (d) $213^\circ 41'$

 13(a) $-\sin A$ (b) $\cos A$ (c) $-\tan A$ (d) $\sec A$ (e) $\sin A$ (f) $-\sin A$ (g) $-\cos A$ (h) $\tan A$ (i) $-\sec A$ (j) $-\operatorname{cosec} A$ (k) $-\cot A$ (l) $\sec A$

 14(a) $y = \sin \theta$ and $y = \cos \theta$ have range $-1 \leq y \leq 1$, $y = \tan \theta$ and $y = \cot \theta$ have range $\mathbf{R}$, $y = \sec \theta$ and $y = \operatorname{cosec} \theta$ have range $y \geq 1$ or $y \leq -1$.

 (b) $\sin \theta$, $\cos \theta$, $\operatorname{cosec} \theta$ and $\sec \theta$ have period 360° ; $\tan \theta$ and $\cot \theta$ have period 180° .

 (c) $\sin \theta$, $\operatorname{cosec} \theta$, $\tan \theta$ and $\cot \theta$ are odd; $\cos \theta$ and $\sec \theta$ are even.

 (d) The graphs have point symmetry about every θ -intercept, and about every point where an asymptote crosses the θ -axis.

 (e) $\sin \theta$, $\cos \theta$, $\operatorname{cosec} \theta$ and $\sec \theta$ have line symmetry in every vertical line through a maximum or minimum; $\tan \theta$ and $\cot \theta$ have no axes of symmetry.

 15(a) $\cos \theta$ (b) $\cos \theta$ (c) $-\sin \theta$ (d) $-\cos \theta$

 (e) $-\tan \theta$ (f) $-\operatorname{cosec} \theta$

16(a) -1 (b) $\tan \alpha$ (c) $-\cot \alpha$ (d) $-\cos \alpha$

Exercise 4E (Page 128)

- 1(a) $\sin \theta = \frac{4}{5}$, $\tan \theta = \frac{4}{3}$
 (b) $\sin \theta = \frac{5}{13}$, $\sec \theta = -\frac{13}{12}$
 2(a) $\cos \alpha = \frac{15}{17}$ or $-\frac{15}{17}$, $\cot \alpha = \frac{15}{8}$ or $-\frac{15}{8}$
 (b) $\tan x = -\frac{1}{3}\sqrt{7}$, $\operatorname{cosec} x = \frac{4}{7}\sqrt{7}$
 3(a) $\cos \beta = -\frac{3}{13}\sqrt{13}$ (b) $\cot \alpha = -\frac{1}{2}\sqrt{21}$
 (c) $\operatorname{cosec} \theta = \frac{1}{2}\sqrt{5}$ or $-\frac{1}{2}\sqrt{5}$ (d) $\sec A$ is undefined.
 4(a) $\operatorname{cosec} P = -\frac{3}{4}\sqrt{2}$ (b) $\tan \theta = 0$
 (c) $\sin \alpha = \frac{1}{3}\sqrt{5}$ or $-\frac{1}{3}\sqrt{5}$, $\cot \alpha = \frac{2}{5}\sqrt{5}$ or $-\frac{2}{5}\sqrt{5}$
 (d) $\operatorname{cosec} x = \frac{1}{5}\sqrt{34}$ or $-\frac{1}{5}\sqrt{34}$,
 $\sec x = \frac{1}{3}\sqrt{34}$ or $-\frac{1}{3}\sqrt{34}$
 5 $\cos \theta = -\frac{\sqrt{q^2 - p^2}}{q}$, $\tan \theta = -\frac{p}{\sqrt{q^2 - p^2}}$
 6 $\sin \alpha = \pm \frac{k}{\sqrt{1 + k^2}}$, $\sec \alpha = \pm \sqrt{1 + k^2}$
 7(b) $\sin x = \frac{2t}{1 + t^2}$, $\tan x = \frac{2t}{1 - t^2}$
 8 $\tan(\theta + 90^\circ) = \frac{\sqrt{1 - k^2}}{k}$
 9 HINT: $\tan \theta = a - \frac{1}{4a}$ or $\frac{1}{4a} - a$

Exercise 4F (Page 131)

- 2(a) $\operatorname{cosec} \theta$ (b) $\cot \alpha$ (c) $\tan \beta$ (d) $\cot \phi$
 3(a) 1 (b) 1 (c) 1
 6(a) $\cos^2 \alpha$ (b) $\sin^2 \alpha$ (c) $\sin A$ (d) $\cos A$
 7(a) $\cos^2 \theta$ (b) 1 (c) $\tan^2 \beta$ (d) $\cot^2 A$
 8(a) $\cos \theta$ (b) $\operatorname{cosec} \alpha$ (c) $\cot \beta$ (d) $\tan \phi$
 9(a) 1 (b) $\sin^2 \beta$ (c) $\sec^2 \phi$ (d) 1
 10(a) $\cos^2 \beta$ (b) $\operatorname{cosec}^2 \phi$ (c) $\cot^2 A$ (d) -1
 14(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (b) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
 (c) $(x - 2)^2 + (y - 1)^2 = 1$ (d) $x^2 + y^2 = 2$
 15(a) 2 (b) 0 (c) 1 (d) 0
 17(a) $y - x = 1$ (b) $x^2 + 2xy + 2y^2 = 5$
 (c) $x^2y = y + 2$

Exercise 4G (Page 137)

- 1(a) $\theta = 60^\circ$ or 120° (b) $\theta = 45^\circ$ or 225°
 (c) $\theta = 135^\circ$ or 225° (d) $\theta = 120^\circ$ or 300°
 (e) $\theta = 210^\circ$ or 330° (f) $\theta = 150^\circ$ or 210°
 2(a) $\theta = 90^\circ$ (b) $\theta = 180^\circ$ (c) $\theta = 90^\circ$ or 270°
 (d) $\theta = 0^\circ$ or 360° (e) $\theta = 0^\circ$ or 180° or 360°
 (f) $\theta = 90^\circ$ or 270°
 3(a) $x \doteq 65^\circ$ or 295° (b) $x \doteq 7^\circ$ or 173°
 (c) $x \doteq 98^\circ$ or 278° (d) $x \doteq 114^\circ$ or 294°
 (e) $x \doteq 222^\circ$ or 318° (f) $x \doteq 80^\circ$ or 280°

- 4(a) $\alpha \doteq 5^\circ 44'$ or $174^\circ 16'$
 (b) $\alpha \doteq 95^\circ 44'$ or $264^\circ 16'$ (c) $\alpha = -45^\circ$ or 135°
 (d) $\alpha = 270^\circ$ (e) no solutions
 (f) $\alpha = 45^\circ$ or 315° (g) $\alpha = 90^\circ$ or -90°
 (h) $\alpha \doteq 243^\circ 26'$ (i) $\alpha = 150^\circ$ (j) $\alpha = 210^\circ$ or 330°
 (k) $\alpha = 60^\circ$ or 300° (l) $\alpha \doteq 18^\circ 26'$ or $198^\circ 26'$
 (m) $\alpha = -360^\circ, -180^\circ, 0^\circ, 180^\circ$ or 360°
 (n) $\alpha \doteq -16^\circ 42'$ or $163^\circ 18'$
 (o) $\alpha \doteq 224^\circ 26'$, $315^\circ 34'$, $584^\circ 26'$ or $675^\circ 34'$
 (p) $\alpha = 157^\circ 30'$ or $337^\circ 30'$
 5(a) $\theta = 0^\circ, 180^\circ$ or 360°
 (b) $\theta = 30^\circ, 150^\circ, 210^\circ$ or 330°
 (c) $\theta \doteq 72^\circ, 108^\circ, 252^\circ$ or 288°
 (d) $\theta = 45^\circ, 135^\circ, 225^\circ$ or 315°
 6(a) $x = 15^\circ, 75^\circ, 195^\circ$ or 255°
 (b) $x = 67^\circ 30', 112^\circ 30', 247^\circ 30'$ or $292^\circ 30'$
 (c) $x = 20^\circ, 80^\circ, 140^\circ, 200^\circ, 260^\circ$ or 320°
 (d) no solutions
 7(a) $\alpha = 75^\circ$ or 255° (b) $\alpha = 210^\circ$ or 270°
 (c) $\alpha = 345^\circ$ or 165° (d) $\alpha = 285^\circ$ or 45°
 8(a) $\theta = 45^\circ$ or 225° (b) $\theta = 150^\circ$ or 330°
 (c) $\theta = 60^\circ, 120^\circ, 240^\circ$ or 300°
 (d) $\theta = 45^\circ, 135^\circ, 225^\circ$ or 315°
 9(a) $\theta = 0^\circ, 90^\circ, 270^\circ$ or 360°
 (b) $\theta = 30^\circ, 90^\circ, 210^\circ$ or 270°
 (c) $\theta = 0^\circ, 60^\circ, 180^\circ, 300^\circ$ or 360°
 (d) $\theta = 135^\circ$ or 315° , or $\theta \doteq 63^\circ 26'$ or $243^\circ 26'$
 (e) $\theta = 90^\circ, 210^\circ$ or 330°
 (f) $\theta = 60^\circ$ or 300° , or $\theta \doteq 104^\circ 29'$ or $255^\circ 31'$
 (g) $\theta \doteq 70^\circ 32'$ or $289^\circ 28'$
 (h) $\theta \doteq 23^\circ 35', 156^\circ 25', 221^\circ 49'$ or $318^\circ 11'$
 (i) $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$ or 360°
 10(a) $x = 60^\circ, 90^\circ, 270^\circ$ or 300°
 (b) $x = 135^\circ$ or 315° , or $x \doteq 71^\circ 34'$ or $251^\circ 34'$
 (c) $x = 210^\circ$ or 330° , or $x \doteq 14^\circ 29'$ or $165^\circ 31'$
 (d) $x \doteq 48^\circ 11'$ or $311^\circ 49'$
 (e) $x \doteq 56^\circ 19', 116^\circ 34', 236^\circ 19'$ or $296^\circ 34'$
 11(a) $\alpha = 90^\circ$, or $\alpha \doteq 199^\circ 28'$ or $340^\circ 32'$
 (b) $\alpha \doteq 63^\circ 26', 161^\circ 34', 243^\circ 26'$ or $341^\circ 34'$
 12(a) $A \doteq 48^\circ 11'$ or $311^\circ 49'$
 (b) $A \doteq 23^\circ 35'$ or $156^\circ 25'$
 13(a) $x = 45^\circ, 180^\circ$ or 225°
 (b) $x = 120^\circ$ or 240° , or $x \doteq 19^\circ 28'$ or $160^\circ 32'$
 14(a) $x = 71^\circ 34'$ or $251^\circ 34'$
 (b) $x \doteq 75^\circ 58', 116^\circ 34', 255^\circ 58'$ or $296^\circ 34'$
 (c) $x \doteq 21^\circ 48', 116^\circ 34', 201^\circ 48'$ or $296^\circ 34'$
 (d) $x = 0^\circ, 135^\circ, 180^\circ, 315^\circ$ or 360°

- 15(a) $\theta = 54^\circ, 126^\circ, 198^\circ$ or 342°
 (b) $\theta \doteq 13^\circ 41', 121^\circ 19', 193^\circ 41'$ or $301^\circ 19'$
 (c) $\theta \doteq 33^\circ 41', 63^\circ 26', 213^\circ 41'$ or $243^\circ 26'$
 (d) $\theta = 45^\circ$ or 225° , or $\theta \doteq 18^\circ 26'$ or $198^\circ 26'$
 (e) $\theta = 30^\circ, 45^\circ, 135^\circ, 150^\circ, 210^\circ, 225^\circ, 315^\circ$
 or 330° (f) $\theta = 120^\circ, 225^\circ, 300^\circ$ or 315°
 (g) $\theta \doteq 78^\circ 28', 228^\circ 35', 281^\circ 32'$ or $311^\circ 25'$
 (h) $\theta = 60^\circ, 180^\circ$ or 300°
 (i) $\theta = 45^\circ, 120^\circ, 225^\circ$ or 300°
 (j) $\theta = 135^\circ$ or 315° , or $\theta \doteq 161^\circ 34'$ or $341^\circ 34'$

Exercise 4H (Page 143)

- 1(a) 1.9 (b) 9.2 (c) 8.9
 2(a) 49° (b) 53° (c) 43°
 3(a) 5 cm^2 (b) 22 cm^2
 4 $42^\circ 24', 137^\circ 36'$
 5(a) $49^\circ 46'$ (b) $77^\circ 53'$ (c) 3.70 cm^2
 6(a) $69^\circ 2'$ or $110^\circ 58'$ (b) 16.0 units or 11.0 units
 7 Either $B \doteq 62^\circ 38', C \doteq 77^\circ 22', c \doteq 11.5$
 or $B \doteq 117^\circ 22', C \doteq 22^\circ 38', c \doteq 4.6$.
 8(b) 28 metres
 9(b) 6 cm
 10 32
 11(a) $3\sqrt{6}$ (b) $3\sqrt{2}$ (c) $2\sqrt{6}$ (d) $6\sqrt{2}$
 12(b) 79.3 metres
 13 11.0 cm
 14(a) $\angle PJK = \angle PBQ = 20^\circ$ (corresponding angles on parallel lines).
 But $\angle PJK = \angle PAJ + \angle APJ$ (exterior angle of triangle). So $\angle APJ = 20^\circ - 5^\circ = 15^\circ$.
 (d) 53 metres
 17(a) $\angle QSM = 36^\circ$ (angle sum of $\triangle QRS$) and $\angle PSM = 48^\circ$ (angle sum of $\triangle PSM$),
 so $\angle PSQ = 48^\circ - 36^\circ = 12^\circ$. $\angle SPQ = 24^\circ$.
 So $\angle PQS = 180^\circ - 24^\circ - 12^\circ = 144^\circ$ (angle sum of $\triangle PQS$). (c) 473 metres
 18(b) 12 km (c) 9:52 am
 19(a) $\sin \angle BMA = \sin(180^\circ - \theta) = \sin \theta$
 20(d) $\frac{\sqrt{3}-1}{2\sqrt{2}}$
 21 If the related angle for θ is α and the known angle is β and $\alpha < \beta$, then $\theta = \alpha$ is one solution and $\theta = 180^\circ - \alpha$ is the other possibility.
 But $(180^\circ - \alpha) + \beta = 180^\circ + (\beta - \alpha) > 180^\circ$, and so $\theta = 180^\circ - \alpha$ is impossible.
 23(a) $8\sqrt{3}$ (b) As the triangle varies, the circle remains unchanged, because $\angle A$ is an angle at the circumference standing on the chord BC .

- (c) 2 : 1
 24(a) Combine the formulae $D_C = c/\sin C$ and $\Delta = \frac{1}{2}ab\sin C$.
 (c) $D_C : D_I = abcs : 4\Delta^2$, $D_C D_I = \frac{abc}{s}$
 (d) $\frac{16\Delta^3}{\pi a^2 b^2 c^2}$, $\frac{s^2}{\pi \Delta}$

Exercise 4I (Page 148)

- 1(c) $\sqrt{13}$ 14.43 cm $\sqrt{10}$
 2(c) 57° $101^\circ 32'$ $-\frac{1}{5}$
 3 11.5 km
 4 167 nautical miles
 5 20°
 6(a) $94^\circ 48'$ (b) $84^\circ 33'$
 7(a) $101^\circ 38'$ (b) $78^\circ 22'$
 9 $13^\circ 10'$
 10(a) 19 cm (b) $\frac{37}{38}$
 11 $\cos A = \frac{3}{4}$, $\cos B = \frac{9}{16}$, $\cos C = \frac{1}{8}$
 12(b) 108 km (c) $\angle ACB \doteq 22^\circ$, bearing $\doteq 138^\circ$
 13(a) $\angle DAP = \angle DPA = 60^\circ$ (angle sum of isosceles triangle), so $\triangle ADP$ is equilateral.
 So $AP = 3$ cm. (b) $3\sqrt{7}$ cm
 15(a)(ii) $9\sqrt{3}\text{ units}^2$
 17(a) $D_C = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}$,
 $D_I = \frac{2\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}$

Exercise 4J (Page 152)

- 1(a)(i) $44^\circ 25'$ (ii) 9.8 cm^2 (b)(i) 11.6 cm (ii) 49°
 2(b) 10.61 metres
 3(a) 9.85 metres (b) 5.30 metres (c) 12.52 metres
 4(b) 8.7 nautical miles
 5(c) 34 metres
 6(a) $34^\circ 35'$ (b) $\angle PDA = \angle ABP$ (base angles of isosceles $\triangle ABD$) and $\angle ABP = \angle PDC$ (alternate angles on parallel lines), so $\angle PDA = \angle PDC$ and $\angle PDC = \frac{1}{2}\angle ADC$. (c) $65^\circ 35'$
 7(a) $46^\circ 59'$ or $133^\circ 1'$
 (b) 66.4 metres or 52.7 metres
 8(a) $\frac{1}{2}\sqrt{37}$ cm (b) $25^\circ 17'$
 9 P_1 by 2.5 min
 10(a) 42 km (b) 78°
 11(a) $\frac{86\sin 60^\circ 45'}{\sin 65^\circ 45'}$ (b) 66 metres
 13(a) $CQ = x \tan 48^\circ$, $PD = x \tan 52^\circ$
 (b) $QD = x \tan \alpha$ and $CQ - CP = PD - QD$

(both = PQ). (c) $42^\circ 42'$

15(a) $-\cos \theta$

16 120°

21(b) $17^\circ 52'$

22(f) As $d \rightarrow 0$, the quadrilateral becomes a triangle whose circumcircle remains the original circle.

Chapter Five

Exercise 5A (Page 159)

1(a) 5 (b) 13 (c) 10 (d) $\sqrt{8} = 2\sqrt{2}$ (e) $\sqrt{80} = 4\sqrt{5}$
(f) 13

2(a) $(3, 2\frac{1}{2})$ (b) $(\frac{1}{2}, 1)$ (c) $(-1, 1)$ (d) $(4, 5)$
(e) $(0, 1)$ (f) $(2\frac{1}{2}, -6)$

3(a)(i) $(3, 3)$ (ii) $(5, 4)$ (iii) $(9, 6)$ (iv) $(9, 6)$

(b)(i) $(0, \frac{1}{2})$ (ii) $(2, -\frac{1}{2})$ (iii) $(-3, 2)$ (iv) $(5, -2)$

(c)(i) $(-1, 1)$ (ii) $(3, -1)$ (iii) $(11, -5)$ (iv) $(37, -18)$

(d)(i) $(-6, 3)$ (ii) $(-4, -1)$ (iii) $(-10, 11)$

(iv) $(-8\frac{1}{2}, 8)$

4(a) $2:3$ (b) $3:2$ (c) $3:-1$ (or $-3:1$)

(d) $1:-3$ (or $-1:3$) (e) $4:-1$ (or $-4:1$)

(f) $1:-4$ (or $-1:4$)

5(a)(i) $2:-5$ (ii) $5:-3$ (b)(i) $3:-5$ (ii) $5:-2$

(c)(i) $3:-2$ (ii) $2:1$ (d)(i) $1:2$ (ii) $2:-3$

(e)(i) $4:-3$ (ii) $3:1$ (f)(i) $1:3$ (ii) $3:-4$

6(a) $(13, 7)$ (b) $(-7, -3)$ (c) $(-1, 0)$ (d) $(15, 8)$

7 $AB = BC = \sqrt{10}$ and $CD = DA = 5$.

Such a quadrilateral is sometimes called a *kite*.

8(a) $XY = 2\sqrt{13}$, $YZ = 2\sqrt{13}$, $ZX = 2\sqrt{26}$, so
 $XY^2 + YZ^2 = 104 = ZX^2$ (b) 26 square units

9(a) ABC is an equilateral triangle.

(b) PQR is a right triangle.

(c) DEF is none of these.

(d) XYZ is an isosceles triangle.

10(a) $(0, -4)$, $(-2, -3)$, $(-4, -2)$

(b) $(0, 6)$, $(2, 9)$, $(4, 12)$, $(6, 15)$

11(a) Both midpoints are $M(2\frac{1}{2}, 2\frac{1}{2})$. It must be a parallelogram, since its diagonals bisect each other.

(b) $AB = AD = \sqrt{5}$. $ABCD$ is a rhombus, since it is a parallelogram with a pair of adjacent sides equal.

12 $\sqrt{17}$, $2\sqrt{17}$, $2\pi\sqrt{17}$, 17π

13(a) $(x-5)^2 + (y+2)^2 = 45$

(b) $(x+2)^2 + (y-2)^2 = 74$

14(a) $S(-5, -2)$ (b)(i) $P(-1, -17)$

(ii) $P(7, -7)$ (c) $B(0, 7)$ (d) $R(12, -9)$

15(a) Check the results using the distance formula — there are eight such points.

(b) $y = 4$ or 10 (c) $a = 1 + \sqrt{2}$ or $1 - \sqrt{2}$

16(a) $P(1, -2\frac{1}{2})$ $(0, -1)$

17(a) $7:2$ (b)(i) $-3:5$ (ii) $2:3$ (iii) $-5:2$

18(a) $AB:BM = -2:1$ (b) $AB:BM = -4:3$

(c) $AB:BM = -11:4$ (d) $AB:BM = 1:1$

(e) $AB:BM = -2:3$ (f) $AB:BM = 1:3$

- 19(a)(i) $2:1$ (ii) $1:3$
 20(a) $(-11, 5)$ and $(17, -7)$ (b) $R(\frac{1}{4}, -3)$
 21(a) $P(\frac{-1+2k}{1+k}, \frac{4-2k}{1+k})$ (b) $k = 2$, $P(1, 0)$
 22(a) $(-2, 1)$ (b) $M = (4\frac{1}{2}, 1\frac{1}{2})$
 23(a) $Q(-\frac{2}{t}, \frac{1}{t^2})$
 (b)(i) $S(\frac{a}{r^2}, \frac{b}{r^2})$ (ii) $S(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2})$
 24(a) $x = \frac{3}{2}a$, a vertical straight line through the midpoint of AB . (b) $(x - 4a)^2 + y^2 = (2a)^2$, a circle with centre $(4a, 0)$ and radius $2a$.
 25(a) Q (b)(i) $k > 0$ (ii) $k < -1$ (iii) $-1 < k < 0$
 (iv) As $k \rightarrow (-1)^+$, M moves infinitely far along the ray QP . As $k \rightarrow (-1)^-$, M moves infinitely far along the ray PQ .

Exercise 5B (Page 165)

- 1(a) $2, -\frac{1}{2}$ (b) $-1, 1$ (c) $\frac{3}{4}, -\frac{4}{3}$ (d) $-\frac{p}{q}, \frac{q}{p}$
 2(a) $-1, 1$ (b) $2, -\frac{1}{2}$ (c) $\frac{1}{2}, -2$ (d) $-\frac{1}{2}, 2$
 (e) $3, -\frac{1}{3}$ (f) $-\frac{b}{2a}, \frac{2a}{b}$
 3(a) 0.27 (b) -1.00 (c) 0.41 (d) 3.08
 4(a)(i) 45° , upwards (ii) 120° , downwards
 (iii) 76° , upwards (iv) 30° , upwards
 (b)(i) 45° (ii) 30° (iii) 14° (iv) 60°
 5(a) $m = 3$, $\alpha \doteq 72^\circ$ (b) $m = -\frac{1}{2}$, $\alpha \doteq 153^\circ$
 (c) $m = -\frac{3}{4}$, $\alpha \doteq 143^\circ$ (d) $m = \frac{2}{3}$, $\alpha \doteq 34^\circ$
 (e) $m = \frac{4}{5}$, $\alpha \doteq 39^\circ$ (f) $m = -\frac{5}{2}$, $\alpha \doteq 112^\circ$
 6 Check your answers by substitution into the gradient formula.
 7(a) 3.73 (b) 1.00 (c) 2.41 (d) 0.32
 8(a) non-collinear (b) collinear with gradient $\frac{2}{3}$
 9(a) $m_{AB} = \frac{1}{2}$, $m_{BC} = -2$ and $m_{AC} = 0$, so $AB \perp BC$.
 (b)(i) $m_{PQ} = 4$, $m_{QR} = -\frac{1}{4}$ and $m_{PR} = -\frac{5}{3}$, so $PQ \perp QR$. Area = $8\frac{1}{2}$ units²
 (ii) $m_{XY} = \frac{7}{3}$, $m_{YZ} = \frac{2}{5}$ and $m_{XZ} = -\frac{5}{2}$, so $XZ \perp YZ$. Area = $14\frac{1}{2}$ units²
 10(d) square
 11 In each case, show that each pair of opposite sides is parallel. (a) Show also that two adjacent sides are equal. (b) Show also that two adjacent sides are perpendicular. (c) Show that it is both a rhombus and a rectangle.
 12(a) -5 (b) 5
 13 $\lambda = -\frac{1}{2}$
 14 $k = 2$ or -1

- 15(a) OA has gradient $\frac{1}{2}$, and OB has gradient 2 . Their product is $\frac{1}{2} \times (-2) = -1$, thus they are perpendicular. (b) $OA = AB = \sqrt{5}$ (c) $D(\frac{3}{2}, -\frac{1}{2})$
 (d) $C(1, -2)$ (e) square
 16(b) $WZ = 5$, $XY = 10$. It is a trapezium, but not a parallelogram.
 17(a) $P = (2, -1)$, $Q = (-1, 4)$, $R = (-3, 2)$, $S = (0, -3)$
 (b) $m_{PQ} = m_{RS} = -\frac{5}{3}$ and $m_{PS} = m_{QR} = 1$
 18(a) $P = (2, 3)$, $Q = (3, 5)$
 (b) $m_{PQ} = m_{BC} = 2$ and $PQ = \sqrt{5}$
 19 $\frac{1}{2}(p + q)$
 20 $x^2 + (y - 1)^2 = 5^2$, a circle with centre $(0, 1)$ and radius 5 .
 21(a) $\frac{y^2}{x(x - 4)} = -1$ (products of gradients is -1), and $(x - 4)^2 + y^2 = 1$
 (b) $Q(\frac{15}{4}, \frac{1}{4}\sqrt{15})$ or $Q(\frac{15}{4}, -\frac{1}{4}\sqrt{15})$
 23(b) They are collinear if and only if $\Delta = 0$, that is $a_1b_2 + a_2b_3 + a_3b_1 = a_2b_1 + a_3b_2 + a_1b_3$.
 24(a) $x = \frac{4p}{1-p^2}$ (b) $x = p - \frac{1}{p}$

Exercise 5C (Page 169)

- 1(a) not on the line (b) on the line (c) on the line
 2 Check your answer by substitution.
 3(a) $x = 1$, $y = 2$ (b) $x = -1$, $y = 1$
 (c) $x = 3$, $y = -4$ (d) $x = 5$, $y = 1$
 (e) $x = -2$, $y = -3$ (f) $x = -4$, $y = 1$
 4(a) $m = 4$, $b = -2$ (b) $m = \frac{1}{5}$, $b = -3$
 (c) $m = -1$, $b = 2$
 5(a) $x - y + 3 = 0$ (b) $2x + y - 5 = 0$
 (c) $x - 5y - 5 = 0$ (d) $x + 2y - 6 = 0$
 6(a) $m = 1$, $b = 3$ (b) $m = -1$, $b = 2$
 (c) $m = 2$, $b = -5$ (d) $m = \frac{1}{3}$, $b = 2$
 (e) $m = -\frac{3}{4}$, $b = \frac{5}{4}$ (f) $m = 1\frac{1}{2}$, $b = -2$
 7 The sketches are clear from the intercepts.
 (a) $A(-3, 0)$, $B(0, 3)$ (b) $A(2, 0)$, $B(0, 2)$
 (c) $A(2\frac{1}{2}, 0)$, $B(0, -5)$ (d) $A(-6, 0)$, $B(0, 2)$
 (e) $A(1\frac{2}{3}, 0)$, $B(0, 1\frac{1}{4})$ (f) $A(1\frac{1}{3}, 0)$, $B(0, -2)$
 8(a) $\alpha = 45^\circ$ (b) $\alpha = 135^\circ$ (c) $\alpha \doteq 63^\circ 26'$
 (d) $\alpha \doteq 18^\circ 26'$ (e) $\alpha \doteq 143^\circ 8'$ (f) $\alpha \doteq 56^\circ 19'$
 10(a) $y = -2x + 3$, $y = \frac{1}{2}x + 3$
 (b) $y = \frac{5}{2}x + 3$, $y = -\frac{2}{5}x + 3$
 (c) $y = -\frac{3}{4}x + 3$, $y = \frac{4}{3}x + 3$
 11(a) $x - y + 3 = 0$ (b) $-\sqrt{3}x + y + 1 = 0$
 (c) $x - \sqrt{3}y - 2\sqrt{3} = 0$ (d) $x + y - 1 = 0$

12 The angles of inclination are about 61° and 119° , so the two lines make acute angles of 61° with the x -axis.

13(a) $k = -\frac{1}{3}$ (b) $k = 3$

14(a) $2x - y = -4$ (b) $x - y = -3$ (c) $5x + y = -3$

15 $(x - a)^2 + (y - x)^2 = a^2$,
where $a = 2 - \sqrt{2}$ or $a = 2 + \sqrt{2}$,

$(x - \sqrt{2})^2 + (y + \sqrt{2})^2 = 2$,

$(x + \sqrt{2})^2 + (y - \sqrt{2})^2 = 2$

16(a) From their gradients, two pairs of lines are parallel and two lines are perpendicular.

(b) The distance between the x -intercepts of one pair of lines must equal the distance between the y -intercepts of the other pair. Thus $k = 2$ or 4 .

Exercise 5D (Page 173)

1(a) $2x - y - 1 = 0$ (b) $x + y - 4 = 0$ (c) $3x - y + 8 = 0$

(d) $5x + y = 0$ (e) $x + 3y - 8 = 0$ (f) $4x + 5y + 8 = 0$

2(a) $y = 2x - 2$ (b) $2x + y - 1 = 0$ (c) $x + 2y + 6 = 0$

(d) $3y = x + 13$

3(a) $\frac{y}{2} - x = 1$, $2x - y + 2 = 0$

(b) $\frac{x}{2} + \frac{y}{3} = 1$, $3x + 2y - 6 = 0$

(c) $-\frac{x}{4} - y = 1$, $x + 4y + 4 = 0$

(d) $\frac{x}{3} - \frac{y}{3} = 1$, $x - y - 3 = 0$

4(c)(i) No, the first two intersect at $(-4, 7)$, which does not lie on the third.

(ii) They all meet at $(5, 4)$.

5(a) $y = -2x + 5$, $y = \frac{1}{2}x + 6$

(b) $y = 2\frac{1}{2}x - 8\frac{1}{2}$, $y = -\frac{2}{5}x + 4\frac{1}{5}$

(c) $y = -1\frac{1}{3}x + 3$, $y = \frac{3}{4}x + 6\frac{1}{2}$

6(a) $3x + 2y + 1 = 0$ (b) $2x - 3y - 8 = 0$

7(a) $x - y - 1 = 0$ (b) $\sqrt{3}x + y + \sqrt{3} = 0$

(c) $x - y\sqrt{3} - 4 - 3\sqrt{3} = 0$

(d) $x + \sqrt{3}y + 2 + 5\sqrt{3} = 0$

8 $\ell_1 \parallel \ell_2$, and $\ell_3 \parallel \ell_4$, so there are two pairs of parallel sides. The vertices are $(-2, -1)$, $(-4, -7)$, $(1, -2)$, $(3, 4)$.

9 $m_{BC} \times m_{AC} = -1$ so $BC \perp AC$,

$AB: y = x - 1$, $BC: y = \frac{1}{2}x + 2$, $AC: y = 2 - 2x$

10(a)(i) $x - 3 = 0$ (ii) $y + 1 = 0$

(b) $3x + 2y - 6 = 0$ (c)(i) $x - y + 4 = 0$

(ii) $\sqrt{3}x + y - 4 = 0$ (d) $x\sqrt{3} + y + 6\sqrt{3} = 0$

11(a) $m_{AC} = \frac{2}{3}$, $\theta = 34^\circ$ (b) $3y = 2x - 2$

(c) $D(4, 2)$ (d) $m_{AC} \times m_{BD} = \frac{2}{3} \times -\frac{3}{2} = -1$, hence they are perpendicular. (e) isosceles

(f) area $= \frac{1}{2} \times AC \times BD = \frac{1}{2} \times \sqrt{52} \times \sqrt{52} = 26$

(g) $E(8, -4)$

12(b) $3x - 4y - 12 = 0$ (c) OB and AC are vertical and hence parallel, and from their gradients $(\frac{3}{4})$ OA is parallel to BC .

(d) 12 units^2 , $AB = 2\sqrt{13}$

13(b) $4y = 3x + 12$ (c) $ML = MP = 5$ (d) $N(4, 6)$

(f) $x^2 + (y - 3)^2 = 25$

14(a) $(0, 2)$

(d) gradient $= \tan(180^\circ - \theta) = -\tan \theta = -2$ so $2x + y - 6 = 0$ (e) $R(3, 0)$, hence area $= 8 \text{ units}^2$.

(f) $QR = 2\sqrt{5}$, $PS = \frac{8}{5}\sqrt{5}$

15 $k = 2\frac{1}{2}$

16(a)(i) $\mu = 4$ (ii) $\mu = -9$

(b) $\mu \neq 4$ (c)(i) $\lambda = 8$ (ii) $\lambda = 0$ or 16

17(a) $2x - 3y + k = 0$

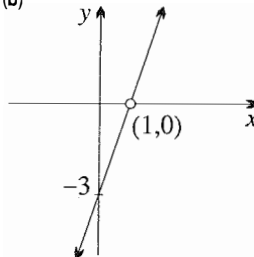
(b)(i) $2x - 3y + 2 = 0$ (ii) $2x - 3y - 9 = 0$

18(a) $4x - 3y + k = 0$

(b)(i) $4x - 3y - 8 = 0$ (ii) $4x - 3y + 11 = 0$

19(a) $x \neq 1$

(b)



20 Stretch horizontally by a factor of a and vertically by a factor of b .

21 $(\frac{1}{p+q}, \frac{1}{p+q})$ The two lines are inverse functions of each other, and so are reflections in the line $y = x$.

22 $3x + 4y - 24 = 0$

23 $y - 2 = m(x + 1)$ (a) $2x - y = -4$ (b) $x - y = -3$

(c) $5x + y = -3$

25(a) $bx + ay = 2ab$ (b) $bx + 2ay = 3ab$

(c) $\ell bx + kay = (k + \ell)ab$

28(c)(i) 1 (ii) $\frac{1}{13}\sqrt{13}$

Exercise 5E (Page 178)

1(a) $\frac{1}{2}\sqrt{10}$ (b) $\frac{4}{\sqrt{5}} = \frac{4}{5}\sqrt{5}$ (c) $\frac{5}{\sqrt{20}} = \frac{1}{2}\sqrt{5}$

2(a) 1 (b) 2 (c) $\sqrt{17}$ (d) $\frac{1}{2}\sqrt{10}$

(e) 0 (The point is on the line.) (f) $\frac{3}{2}\sqrt{5}$

3(a) D is distant $\frac{3}{10}$. (b) C is distant $4\frac{1}{10}$.

4(a) ℓ_3 is distant $\frac{3}{5}\sqrt{10}$. (b) ℓ_1 is distant $\frac{17}{13}\sqrt{13}$.

5(a) $\mu = 15$ or -5 (b) $\lambda = \frac{1}{2}$ or -1

6(a) $h > -4$ or $h < -6$ (b) $-6 \leq k \leq 4$

7(a) They do not intersect.

(b) Once, the line is tangent to the circle.

(c) Once, the line is tangent to the circle.

(d) They intersect twice.

8(a) $\frac{7}{\sqrt{10}} = \frac{7}{10}\sqrt{10}$ (b) $\frac{10}{\sqrt{17}} = \frac{10}{17}\sqrt{17}$

9(a) $x - 2y - 1 = 0$ (b) $2\sqrt{5}$ (c) $AB = 3\sqrt{5}$ so the area is 15 square units. (d) 10 square units

10(e) AC is common, $AO = AB$ and both triangles are right-angled, thus they are congruent by the RHS test. (f) 50 units² (g) $2\frac{2}{5}$

11(a) centre $(-2, -3)$ and $r = 2$, distance $\frac{4}{\sqrt{5}}$
(b) $\frac{4}{\sqrt{5}}$

12 The distances should differ. Since the distances differ, the lines are not parallel, and must intersect.

13(b) $x + 3y - 4 = 0$ and $3x - y - 2 = 0$

14(a) $y = mx$ (b) $\frac{|3m-1|}{\sqrt{m^2+1}}$

(d) $y = \frac{1}{5}(3 + 2\sqrt{6})x$ or $y = \frac{1}{5}(3 - 2\sqrt{6})x$

16(b) $\frac{|2q - q^2 - 3|}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$

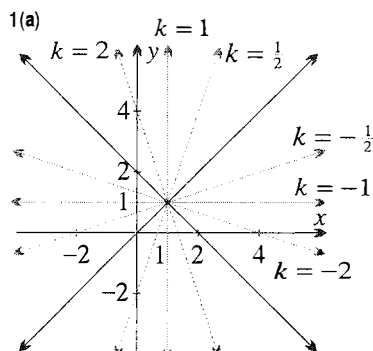
17(a) $(x-7)^2 + (y+1)^2 = 25$ (b) $\frac{7m+1}{\sqrt{m^2+1}}$

(c) $m = -\frac{4}{3}$ or $\frac{3}{4}$

(d) $4x + 3y + 6 = 0$ or $3x - 4y + 17 = 0$

19(b) Substitution gives $(p^2 + q^2)(r^2 - q^2d^2) = p^2r^2$. Rearranging this, $d^2 = \frac{r^2}{p^2 + q^2}$.

Exercise 5F (Page 182)



(b) $k = 2$: $3x + y - 4 = 0$, $k = 1$: $x = 1$,

$k = \frac{1}{2}$: $3x - y - 2 = 0$

(c) $k = -\frac{1}{2}$: $x - 3y + 2 = 0$, $k = -1$: $y = 1$,

$k = -2$: $x + 3y - 4 = 0$

2(a) $x + 2y + 9 + k(2x - y + 3) = 0$

(b) $k = -3$ gives $y = x$.

3(a) $x - 2y - 4 = 0$ (b) $2x + y - 3 = 0$ (c) $y = x - 3$

4(b) $k = -2$ gives $y = 3$. (c) $k = 1$ gives $x = 1$.

(d) $(1, 3)$

5(a) $(1, 1)$ (c) $3y + x - 4 = 0$

6(a) $2x - 3y + 6 + k(x + 3y - 15) = 0$ (b)(i) $x = 3$

(ii) $4x + 3y - 24 = 0$ (iii) $x - 6y + 21 = 0$ (iv) $3y = 4x$

7(b)(i) $3x + 4y + 5 = 0$ (ii) $3x + 2y + 7 = 0$

(iii) $2y + 5x + 13 = 0$ (iv) $x - y + 4 = 0$

8(a) $(4, 1)$ (b)(i) $(0, 4)$ (ii) $(-3, 7)$

10(a) $\sqrt{10}$ (b) $2y + x - 4 = 0$

11(a) $2x - y = 0$

(b) Using $(x + 2y + 10) + k(2x - y) = 0$ yields $k = -1$, hence the line is $3y - x + 10 = 0$.

12(a) $y = 3x - 9$ (b) $4y = x + 8$ (c) $5y = 4x - 1$

(d) $2x + 3y - 6 = 0$

13(b) $\mu = \frac{3}{2}$ and the circle is

$(x - \frac{3}{2})^2 + (y + \frac{3}{2})^2 = \frac{13}{2}$.

14(c) They are all 1 and -1 . (d) When $\lambda = -1$ the equation reduces the straight line to $x = 0$.

(e) $\lambda = -\frac{1}{9}$, giving $(x - \frac{5}{2})^2 + y^2 = \frac{29}{4}$.

15(b) $k = -\frac{3}{5}$, giving $y = 4(x^2 - 1)$. (c) Using $(h+1)y = x^2(h-1) - 2x(2h-1)$ gives $h = 1$ and the result is $y = -x$, which is not a parabola.

Exercise 5G (Page 185)

1(a)(i) $M = (4, 5)$ (ii) $OM = PM = QM = \sqrt{41}$

(iii) OM , PM and QM are three radii of the circle.

(b) $M = (p, q)$, $OM = PM = QM = \sqrt{p^2 + q^2}$

2(a) $PQ^2 = 5$, $RS^2 = 25$, $PS^2 = 17$, $QR^2 = 13$

(b) $PQ^2 = p^2 + q^2$, $RS^2 = r^2 + s^2$, $PS^2 = p^2 + s^2$, $QR^2 = q^2 + r^2$

3(a)(i) $P(2, 0)$, $Q(0, 2)$

(ii) $m_{PQ} = m_{AC} = -1$ and $AC = 4\sqrt{2}$

(b) $P(a+b, c)$, $Q(b, c)$,

$m_{PQ} = m_{AC} = 0$ and $PQ = a$

4(a) $\frac{x}{3} + \frac{y}{4} = 1$ and $4y = 3x$, thus $C(\frac{48}{25}, \frac{36}{25})$.

(b) $OA = 3$, $AB = 5$, $OC = \frac{12}{5}$, $BC = \frac{16}{5}$, $AC = \frac{9}{5}$

6(a) $AB = BC = CA = 2a$ (b) $AB = AD = 2a$

(c) $BD = 2a\sqrt{3}$

7(a) D is the origin, $AB^2 = (a-b)^2 + c^2$,

$BC^2 = (a+b)^2 + c^2$, $CD^2 = a^2$, $BD^2 = b^2 + c^2$.

(b) The sum of the squares on two sides of a triangle equals twice the square on the median to the third side plus twice the square of half the third side.

8 Both midpoints are at the origin.

9 The condition reduces to $x = q$, so that R is vertically in line with Q .

10(a) $P = (\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2))$,

$Q = (\frac{1}{2}(b_1 + c_1), \frac{1}{2}(b_2 + c_2))$,

$$R = \left(\frac{1}{2}(c_1 + d_1), \frac{1}{2}(c_2 + d_2) \right),$$

$$S = \left(\frac{1}{2}(d_1 + a_1), \frac{1}{2}(d_2 + a_2) \right)$$

(b) The midpoint of both PR and QS is

$$M \left(\frac{1}{4}(a_1 + b_1 + c_1 + d_1), \frac{1}{4}(a_2 + b_2 + c_2 + d_2) \right).$$

(c) a parallelogram

12(a) $P = (1, 4)$, $Q = (-1, 0)$ and $R = (3, 2)$,

$BQ: x - y + 1 = 0$, $CR: y - 2 = 0$, $AP: x = 1$

(b) The medians intersect at $(1, 2)$.

13(a) The median through B is

$3a(y + 6b) = (c + 3b)(x + 6a)$. The median through A is $-3a(y - 6b) = (c - 3b)(x - 6a)$.

(b) The medians intersect at $(0, 2c)$.

14(a) perpendicular bisector of $AB: x = 0$,

of $BC: c(c - y) = (b + a)(x - b + a)$,

of $AC: c(c - y) = (b - a)(x - b - a)$

(b) They all meet at $(0, \frac{c^2 + b^2 - a^2}{c})$.

(c) Any point on the perpendicular bisector of an interval is equidistant from the endpoints of that interval.

15 A suitable choice is $A(0, 0)$, $B(2b, 0)$

and $C(0, 2c)$.

16 C has coordinates $(\frac{ab^2}{a^2 + b^2}, \frac{a^2b}{a^2 + b^2})$.

Chapter Six

Exercise 6A (Page 190)

1(a) $\frac{1}{5}$ (b) $\frac{1}{5}$ (c) $3\frac{1}{2}$ (d) $\frac{4}{21}$ (e) $\frac{1}{49}$ (f) 64 (g) $\frac{81}{16}$

(h) $\frac{4}{25}$ (i) $\frac{27}{1600}$ (j) 1

2(a) 5 (b) 3 (c) 9 (d) 4 (e) 8 (f) 27 (g) 81

(h) $\frac{125}{8}$ (i) $\frac{8}{27}$ (j) $\frac{3}{2}$

3(a) 169 (b) $\frac{9}{8}$ (c) $\frac{1}{2}$ (d) 4000 (e) $\frac{1}{900}$

4(a) $\frac{7}{x}$ (b) $\frac{3}{x^3}$ (c) $\frac{x^2}{(y+1)^2}$ (d) $\frac{t+3}{s^2}$ (e) $\frac{9x^2y^2}{8}$

5(a) $\frac{y}{x^2}$ (b) $\frac{21}{x}$ (c) $\frac{s^6}{y^8}$ (d) $\frac{c^2}{5d^3}$ (e) $21m^2$ (f) $\frac{b^2}{a}$

(g) $\frac{2x}{y^2}$ (h) $\frac{p^2}{q^6}$

6(a) -1 (b) -2 (c) $\frac{1}{3}$ (d) $\frac{3}{2}$ (e) $\frac{1}{2}$ (f) $-\frac{1}{2}$

7(a) $\frac{1}{2}$ (b) $\frac{1}{128}$ (c) $\frac{25}{4}$ (d) $\frac{343}{64}$ (e) $\frac{16}{81}$ (f) $\frac{125}{8}$

(g) $\frac{25}{36}$ (h) $\frac{1}{8}$ (i) $\frac{2}{3}$ (j) $\frac{125}{27}$

8(a) $7^{\frac{1}{3}}$ (b) $5^{\frac{3}{4}}$ (c) $5^{-1}x^{\frac{4}{3}}$ (d) $11^{-\frac{1}{5}}a^{\frac{1}{2}}$ (e) $7y^{-\frac{4}{5}}$

(f) $x^{\frac{3}{2}}$ (g) $x^{\frac{5}{2}}$ (h) $2x^{-\frac{1}{2}}$ (i) $\frac{1}{8}x^{-1}$ (j) $7(3x+2)^{-1}$

9(a) 9 (b) 3 (c) $\frac{1}{20}$ (d) $\frac{3}{10}$

10(a) $x^2 + 10 + \frac{25}{x^2}$ (b) $x^4 - 14 + \frac{49}{x^4}$ (c) $9x - 12 + \frac{4}{x}$

11(a) 2^{4n} (b) 5^{n-1} (c) 3^{6x+1} (d) 11^{1-5n} (e) $7^{2n-\frac{1}{2}}$

(f) 2^{3x-5}

12(a) $x = -\frac{1}{3}$ (b) $x = \frac{1}{4}$ (c) $-\frac{2}{3}$ (d) $x = \frac{5}{12}$

(e) $x = -4$ (f) $x = -2$

13(a) $b = \frac{1}{343}$ (b) $\frac{1}{11}$ (c) $x = \frac{1}{81}$

14(a) $x = 3$ and $y = 4$ (b) $x = 0$ and $y = -1$

(c) $x = -2$ and $y = \frac{1}{2}$

15(a) $\frac{b-a}{ab}$ (b) $\frac{y}{y+1}$ (c) $\frac{x^2y^2}{y^2-x^2}$ (d) $\frac{ab}{b-a}$

(e) $\frac{x^3-y^3}{x^3y^3}$ (f) $\frac{1}{a+1}$

16(a) 2^{6n} (b) 81 (c) 2^{3x} (d) $2^{2x}3^{2x}$ (or 6^{2x})

(e) $5^{4n-4}2^{4n-5}$ (f) 2^x3^{1-x}

17(a) 50×7^n (b) 26 (c) $124 \times 5^{n-3}$ (d) 7

(e) $7 \times 2^{2n-1}$ (f) 2^n

18(a) $\frac{3^n}{2}$ (b) $\frac{1}{3^x}$ (c) -2^n3^n

19(a) $>$ (b) $>$ (c) $<$ (d) $<$ (e) $>$ (f) $>$

20(a) $1\frac{1}{2}$ (b) $4\frac{1}{2}$ (c) 5 (d) 4 (e) 6 (f) $7\sqrt{2}$

22(a) $12 < 2^{\frac{11}{3}} < 13$ (b) $13 < 2^{\frac{15}{4}} < 14$

23 $\lim_{x \rightarrow 0^+} 0^x = 0$ and $\lim_{x \rightarrow 0} x^0 = 1$

Exercise 6B (Page 195)

1(a) $3^x = 9$, $x = 2$ (b) $2^x = 16$, $x = 4$

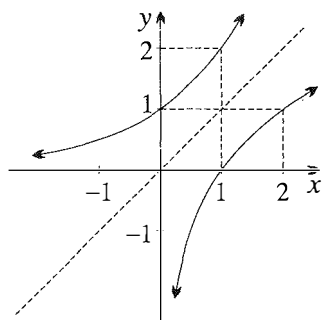
(c) $5^x = 125$, $x = 3$ (d) $10^x = \frac{1}{10}$, $x = -1$

(e) $7^x = \frac{1}{49}$, $x = -2$ (f) $(\frac{1}{3})^x = \frac{1}{81}$, $x = 4$

(g) $5^x = \sqrt{5}$, $x = \frac{1}{2}$ (h) $11^x = 1/\sqrt{11}$, $x = -\frac{1}{2}$

2 For $y = 2^x$: $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8.

For $y = \log_2 x$: -3, -2, -1, 0, 1, 2, 3.



- 3(a) $x = 4^3 = 64$ (b) $x = 13^{-1} = \frac{1}{13}$
 (c) $x = 9^{\frac{1}{2}} = 3$ (d) $x = 10^{-2} = \frac{1}{100}$
 (e) $x = (\frac{1}{16})^{-\frac{1}{4}} = 2$ (f) $x = 7^{-\frac{1}{2}} = 1/\sqrt{7}$
 (g) $x = 36^{1.5} = 216$ (h) $x = 8^{-\frac{2}{3}} = \frac{1}{4}$
 4(a) $x^3 = 27, x = 3$ (b) $x^{-1} = \frac{1}{7}, x = 7$
 (c) $x^3 = 1000, x = 10$ (d) $x^{\frac{1}{2}} = 3, x = 9$
 (e) $x^{-2} = 25, x = \frac{1}{5}$ (f) $x^2 = \frac{4}{9}, x = \frac{2}{3}$
 (g) $x^{\frac{4}{3}} = 16, x = 8$ (h) $x^{-\frac{1}{2}} = 9, x = \frac{1}{81}$
 5(a) 1 (b) -1 (c) 3 (d) -2 (e) $\frac{1}{2}$ (f) $-\frac{1}{2}$
 (g) 0 (h) $-1\frac{1}{2}$
 6(a) 3 and 4, 3.46 (b) 3 and 4, 3.01
 (c) 2 and 3, 2.21 (d) 8 and 9, 8.64
 7(a) $2\log_2 3$ (b) $1 + 2\log_2 3$ (c) $-1 - \log_2 3$
 (d) $-1 + \log_2 5$
 8(a) $x = \log_2 13 \approx 3.700$
 (b) $x = 2 + \log_3 20 \approx 4.727$
 (c) $x > \log_7 1000 \approx 3.550$
 (d) $x < -1 + \log_2 10 \approx 2.322$
 (e) $x < \log_5 0.04 = -2$
 (f) $x = -1 + \log_{\frac{1}{2}} 10 \approx -4.322$
 (g) $x < \log_{\frac{1}{3}} 100 \approx -4.192$
 (h) $x > \log_{0.06} 0.001 \approx 2.455$
 9(a) 33 powers (b) 14 powers
 10(a) $6x$ (b) $-x - y - z$ (c) $3y + 5$ (d) $2x + 2z - 1$
 (e) $y - x$ (f) $x + 2y - 2z - 1$ (g) $-2z$ (h) $3x - y - z - 2$
 11(a) $-3 + \log_2 5$ (b) $\log_2 5 + \frac{3}{2}\log_2 3$ (c) $-\frac{1}{2} - \log_2 3$
 (d) $\frac{1}{2} + \frac{3}{2}\log_2 3 - \frac{3}{2}\log_2 5$
 12(a) 5 (b) 7 (c) n (d) y
 13(a) $3 = 2^{\log_2 3}$ (b) $u = 3^{\log_3 u}$ (c) $7 = a^{\log_a 7}$
 (d) $u = v^{\log_v u}$
 14(a) $\frac{1}{2}$ (b) 49 (c) 15 (d) x^n (e) $1/x$ (f) $x \times 5^x$
 (g) x^x (h) $x^{1/x}$
 15(a) $x + y = xy$ (b) $x = 1000y$ (c) $x = y^4$
 (d) $x^2 y^3 = z^4$ (e) $2^x = y$ (f) $x = yz^n$
 (g) $64x^3 = y^2$ (h) $(2x + 1)^2 = (2x - 1)^3$
 16(b) 2, -2, -2
 18(a) $SD = \frac{1}{4}(2^{2x} - 2^{-2x}), S + D = 2^x,$
 $S - D = 2^{-x}, S^2 - D^2 = 1$

$$(b) x = \log_2 (S + \sqrt{S^2 - 1}),$$

$$x = \log_2 (D + \sqrt{D^2 + 1})$$

Exercise 6C (Page 198)

- 1(a) 21, 25, 29, 33 (b) 24, 48, 96, 192
 (c) -1, -10, -19, -28 (d) 3, 1, $\frac{1}{3}, \frac{1}{9}$
 (e) 1, -1, 1, -1 (f) 64, 81, 100, 121
 (g) $\frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$ (h) -2, 1, $-\frac{1}{2}, \frac{1}{4}$
 2(a) 3, 8, 13, 18 (b) 5, 25, 125, 625
 (c) 1, 8, 27, 64 (d) 5, -2, -9, -16
 (e) 12, 36, 108, 324 (f) 4, 12, 24, 40
 (g) -1, 2, -3, 4 (h) -3, 9, -27, 81
 3(a) 5, 17, 29, 41 (b) $\frac{3}{4}, \frac{3}{2}, 3, 6$ (c) 1, 2, 6, 24
 (d) 28, -14, 7, $-3\frac{1}{2}$ (e) 37, 13, -11, -35
 (f) $2\sqrt{2}, 4, 4\sqrt{2}, 8$ (g) 5, 15, 30, 50
 (h) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$
 4(a) $77 = T_{10}, 349$ is not a member, $1577 = T_{260}$.
 (b) 63 terms are less than 400, $T_{64} = 401$.
 5(a) 60 is not a member, $80 = T_4, 605 = T_{11}$.
 (b) 14 terms are less than 1000, $T_{15} = 1125$.
 6(a) 0, 2, 0, 2 (b) -50, 100, -200, 400
 (c) $-36x, 18x, -9x, \frac{9}{2}x$ (d) $5a, 3a, a, -a$
 (e) $4a, 8a, 16a, 32a$ (f) 1, 5, 19, 65
 (g) 3, 1, -5, 9 (h) $1, 2\sqrt{2}, 8, 16\sqrt{2}$
 (i) $\frac{3}{4}x, 3x, \frac{27}{4}x, 12x$
 7(a) $T_n = T_{n-1} + 5$ (b) $T_n = 2T_{n-1}$
 (c) $T_n = T_{n-1} - 7$ (d) $T_n = -T_{n-1}$
 8(a) -10 is not a member, $-15 = T_5$.
 (b) 106 terms
 9(a) $28 = T_7, 70 = T_{10}$ (b) 5 terms
 10(a) $1\frac{1}{2} = T_4, 96 = T_{10}$ (b) $T_7 = 12$
 11 From Q2: (a) $y = 5x - 2$ (b) $y = 5^x$ (c) $y = x^3$
 (d) $y = 12 - 7x$ (e) $y = 4 \times 3^x$ (f) $y = 2x(x + 1)$
 (g) nothing simple (h) nothing simple
 12(a) 1, 0, -1, 0, T_n where n is even.
 (b) 0, -1, 0, 1, T_n where n is odd.
 (c) -1, 1, -1, 1, no terms are zero.
 (d) 0, 0, 0, 0, all terms are zero.
 (e) -1, 0, -1, 0, T_n where n is even.
 (f) 1, 1, 0, 0, the third and fourth term in each group of 4 is zero.
 13(a) 11, 15 (b) 4, 8
 14(a) $\frac{4}{5}$ (b) $\frac{n}{n+1}$ (d) $\frac{1}{30} = T_5$
 15(a) $0.9 = T_{10}, 0.99 = T_{100}$ (b) $n^2 : (n^2 - 1)$ (d) $\frac{1}{n}$
 16(a) The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... The Lucas sequence is 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, ...

The sum of two odd integers is even, and the sum of an even and an odd is odd.

(b) The first is 2, 4, 6, 10, 16, ..., which is $2F_{n+1}$.

The second is 0, 2, 2, 4, 6, ..., which is $2F_{n-1}$.

(c) $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, $\frac{3}{2} + \frac{1}{2}\sqrt{5}$, $\frac{4}{2} + \frac{2}{2}\sqrt{5}$, $\frac{7}{2} + \frac{3}{2}\sqrt{5}$

Exercise 6D (Page 201)

1(a) $d = 3$, $T_n = 5 + 3n$, $T_{10} = 35$

(b) $d = -6$, $T_n = 27 - 6n$, $T_{10} = -33$

(c) not an AP (but $T_n = 2^{4-n}$, $T_{10} = \frac{1}{64}$)

(d) $d = 4$, $T_n = 4n - 7$, $T_{10} = 33$

(e) $d = 1\frac{1}{4}$, $T_n = \frac{1}{4}(2 + 5n)$, $T_{10} = 13$

(f) $d = -17$, $T_n = 29 - 17n$, $T_{10} = -141$

(g) $d = -\sqrt{2}$, $T_n = 5 + 2\sqrt{2} - n\sqrt{2}$, $T_{10} = 5 - 8\sqrt{2}$

(h) not an AP (but $T_n = n^2$, $T_{10} = 100$)

(i) $d = 3\frac{1}{2}$, $T_n = \frac{1}{2}(7n - 12) = \frac{7}{2}n - 6$, $T_{10} = 29$

2(a) $d = -3$, $T_n = 85 - 3n$, $T_{25} = 10$, $T_{29} = -2$

(b) $d = -8$, $T_n = 353 - 8n$, $T_{25} = 153$, $T_{45} = -7$

(c) $d = -\frac{5}{4}$, $T_n = \frac{1}{4}(103 - 5n)$, $T_{25} = -5\frac{1}{2}$, $T_{21} = -\frac{1}{2}$

3(a) $x = 23$, $d = 9$ (b) $x = -4$, $d = 18$

(c) $x = 10$, $d = 8$ (d) $x = -2$, $d = -4$

4(a) cost = $200 + 300n$ (b) cost = \$4700

(c) 32 windows

5(a) 2120, 2240, 2360, 2480

(b) $A_n = 2000 + 120n$, $A_{12} = 3440$ (c) 34 years

6(a) 667 terms (b) 44 terms (c) 81 terms

7(a) 11, 15, 19, 23, $a = 11$, $d = 4$

(b) $T_{50} + T_{25} = 314$, $T_{50} - T_{25} = 100$

(d) $815 = T_{202}$ (e) $T_{248} = 999$, $T_{249} = 1003$

(f) $T_{49} = 203$, ..., $T_{73} = 299$ lie between 200 and 300, making 25 terms.

8(a)(i) $T_{63} = 504$ (ii) $T_{106} = 848$ (iii) 44 terms

(b)(i) $T_{91} = 1001$, $T_{181} = 1991$, 91 terms

(ii) $T_{115} = 805$, $T_{285} = 1995$, 171 terms

9(a) $d = 4$, $a = -1$ (b) $d = -9$, $a = 60$

(c) $d = 3\frac{1}{2}$, $a = -4\frac{1}{2}$ (d) $d = 2 - \sqrt{5}$, $a = 7\sqrt{5} - 16$

10(a) $T_8 = 37$ (b) $T_6 = -2$

11(a) $d = 4$, $x = 1$ (b) $d = 6x$, $x = \frac{1}{3}$

(c) $d = -3 - 3x$, $x = -2$

12(a) $d = \log_3 2$, $T_n = n \log_3 2$

(b) $d = -\log_a 3$, $T_n = \log_a 2 + (4 - n) \log_a 3$

(c) $d = x + 4y$, $T_n = nx + (4n - 7)y$

(d) $d = -4 + 7\sqrt{5}$, $T_n = 9 - 4n + (7n - 13)\sqrt{5}$

(e) $d = -1.88$, $T_n = 3.24 - 1.88n$

(f) $d = -\log_a x$, $T_n = \log_a 3 + (3 - n) \log_a x$

13 The 13 terms $T_{28} = 19$, ..., $T_{40} = -17$ have squares less than 200.

14(a) $a = m + b$, $d = m$ (b) $f(x) = a + (x - 1)d$

15(a) $a = \lambda a_1 + \mu a_2$, $d = \lambda d_1 + \mu d_2$

(b) $\mathcal{A}(1, 0)$ is 1, 1, 1 ..., $\mathcal{A}(0, 1)$ is 0, 1, 2 ..., $\mathcal{A}(a, d) = a\mathcal{A}(1, 0) + d\mathcal{A}(0, 1)$.

(c) $\lambda = \frac{ad_2 - a_2d}{a_1d_2 - a_2d_1}$, $\mu = \frac{ad_1 - a_1d}{a_2d_1 - a_1d_2}$.

Note: $a_1 : a_2 \neq d_1 : d_2$ ensures $a_2d_1 - a_1d_2 \neq 0$.

Exercise 6E (Page 205)

1(a) 1, 3, 9, 27, $T_n = 3^{n-1}$

(b) 5, -10, 20, -40, $T_n = 5 \times (-2)^{n-1}$

(c) 18, 6, 2, $\frac{2}{3}$, $T_n = 18 \times (\frac{1}{3})^{n-1}$

(d) 6, -3, $1\frac{1}{2}$, $-\frac{3}{4}$, $T_n = 6 \times (-\frac{1}{2})^{n-1}$

(e) 1, $\sqrt{2}$, 2, $2\sqrt{2}$, $T_n = (\sqrt{2})^{n-1}$

(f) -7, 7, -7, 7, $T_n = -7 \times (-1)^{n-1} = 7 \times (-1)^n$

2(a) $r = 2$, $T_n = 10 \times 2^{n-1}$, $T_6 = 320$

(b) $r = \frac{1}{3}$, $T_n = 180 \times (\frac{1}{3})^{n-1}$, $T_6 = \frac{20}{27}$

(c) not a GP (but $T_n = (n + 7)^2$, $T_6 = 169$)

(d) not a GP (It's an AP with $T_n = 20 + 15n$, $T_6 = 110$.) (e) $r = 4$, $T_n = \frac{3}{4} \times 4^{n-1}$, $T_6 = 768$

(f) $r = \frac{1}{4}$, $T_n = -24 \times (\frac{1}{4})^{n-1}$, $T_6 = -\frac{3}{128}$

3(a) $r = -1$, $T_n = (-1)^{n-1}$, $T_6 = -1$

(b) $r = -2$, $T_n = -2 \times (-2)^{n-1} = (-2)^n$, $T_6 = 64$

(c) $r = -3$, $T_n = -8 \times (-3)^{n-1}$, $T_6 = 1944$

(d) $r = -\frac{1}{2}$, $T_n = 60 \times (-\frac{1}{2})^{n-1}$, $T_6 = -\frac{15}{8}$

(e) $r = -\frac{1}{2}$, $T_n = -1024 \times (-\frac{1}{2})^{n-1}$, $T_6 = 32$

(f) $r = -12$, $T_n = \frac{3}{8} \times (-12)^{n-1}$, $T_6 = -2^7 \times 3^6$

4(a) $r = 2$ (b) $r = 3$ or -3 (c) $r = \frac{1}{9}$ or $-\frac{1}{9}$

(d) $r = -\frac{3}{2}$ (e) $r = 0.1$ or -0.1

(f) $r = \sqrt{2}$ or $-\sqrt{2}$

5(a) $r = 4$, $a = \frac{1}{16}$

(b) $r = 3$ and $a = \frac{1}{9}$, or $r = -3$ and $a = -\frac{1}{9}$

(c) $r = \sqrt{2}$ and $a = \frac{3}{2}$, or $r = -\sqrt{2}$ and $a = \frac{3}{2}$

(d) $r = \frac{1}{2}$, $a = 128\sqrt{2}$

6(a) $r = \sqrt{2}$, $T_n = \sqrt{6} \times (\sqrt{2})^{n-1}$, $T_6 = 8\sqrt{3}$

(b) $r = ax^2$, $T_n = a^n x^{2n-1}$, $T_6 = a^6 x^{11}$

(c) $r = y/x$, $T_n = -x^{2-n}y^{n-2}$, $T_6 = -y^4/x^4$

7(a) 50, 100, 200, 400, 800, 1600, $a = 50$, $r = 2$

(b) $T_{50} \times T_{25} = 5^4 \times 2^{75}$, $T_{50} \div T_{25} = 2^{25}$

(d) $6400 = T_8$ (f) $T_6 = 1600$, ..., $T_{11} = 51\,200$ lie between 1000 and 100\,000, making 6 terms.

8(a) AP: $x = -48$, $d = 72$; GP: $x = 6$, $r = 4$

(b) AP: $x = 60$, $d = 36$;

GP: $x = 48$ and $r = 2$, or $x = -48$ and $r = -2$

(c) They can't form an AP. GP: $x = 9$, $r = 2$

(d) AP: $x = 2$, $d = 4$;

GP: $x = 4$ and $r = 3$, or $x = 0$ and $r = -1$

9(a) $T_n = 7 \times 2^{n-1}$, 6 terms

(b) $T_n = 2 \times 7^{n-1}$, 5 terms

(c) $T_n = 5^{n-3}$, 7 terms

10(a) 18 terms (b) 7 terms (c) 11 terms

11(a) $T_9, \dots, T_{18}$, 10 terms (b) $T_5, \dots, T_7$, 3 terms

(c) $T_8, \dots, T_{11}$, 4 terms

12(a) $P \times 1.07$, $P \times (1.07)^2$, $P \times (1.07)^3$

(b) $A_n = P \times (1.07)^n$ (c) 11 full years to double, 35 full years to increase tenfold.

13(a) $T_n = 98 \times (\frac{1}{7})^{n-1}$, 10 terms

(b) $T_n = 25 \times (\frac{1}{5})^{n-1} = (\frac{1}{5})^{n-3}$, 11 terms

(c) $T_n = (0.9)^{n-1}$, 132 terms

14(a) $W_1 = 20\,000 \times 0.8$, $W_2 = 20\,000 \times (0.8)^2$, $W_3 = 20\,000 \times (0.8)^3$, $W_n = 20\,000 \times (0.8)^n$

(b) 11 years

15 152 sheets

16(a) $T_n = 2x^n$, $x = 1$ or -1

(b) $T_n = x^{6-2n}$, $x = \frac{1}{3}$ or $-\frac{1}{3}$

(c) $T_n = 2^{-16} \times 2^{4n-4} x = 2^{4n-20} x$, $x = 6$

17(a) $a = 6\frac{1}{4}$ and $b = 2\frac{1}{2}$, or $a = 4$ and $b = -2$

(b) $a = 1$, $b = 0$ (c) $a = 66$ (d) $a = \frac{3}{16}$ (e) $a = 28$, $d = -1$ (f) $a = \frac{1}{3}$ and $r = 3$, or $a = \frac{2}{3}$ and $r = -3$

18(c) $r = 1$, $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1}{2} - \frac{1}{2}\sqrt{5}$ (d) 1, 2, 4, 8, ...

19(a) $T_n = 2^{8-3n}$

20(a) first term = 2^a , ratio = 2^d

(b) first term = $\log_2 a$, ratio = $\log_2 r$

(c) No, it can be any positive number except 1.

21(a) $a = kb$, $r = b$ (b) $f(x) = ar^{x-1}$

22(a) first term = aA , ratio = rR

23(a) first term = $a_1^\lambda a_2^\mu$, ratio = $r_1^\lambda r_2^\mu$

(b) $\mathcal{G}(2, 1)$ is 2, 2, 2, 2, ..., $\mathcal{G}(1, 2)$ is

1, 2, 4, 8, ..., $\lambda = \log_2 a$, $\mu = \log_2 r$

Exercise 6F (Page 209)

1(a) 10, 8 or -8 (b) $20\frac{1}{2}$, 20 or -20

(c) $-12\frac{1}{2}$, 10 or -10 (d) -15, no GM

(e) $3\frac{3}{4}$, 3 or -3 (f) $25a^2$, $7a^2$ or $-7a^2$

(g) 0, no GM (h) $\frac{1}{2}(a+1)$, $\sqrt{a}$ or $-\sqrt{a}$

(i) 40, 2^5 or -2^5 (j) 72 , $2^5\sqrt{2}$ or $-2^5\sqrt{2}$

(k) $\frac{1}{2}a^3(a^2+1)$, a^4 or $-a^4$ (l) $\frac{1+x^6}{2x^3}$, 1 or -1

2(a) $x = 2$ (b) $x = -4$ (c) $x = \frac{13}{3}$ (d) $x = 1$ or 6

3(a) 14, 21, 28, 35 (b) 18, 12 (c) $36\frac{1}{2}$, 33, $29\frac{1}{2}$, 26, $22\frac{1}{2}$, 19, $15\frac{1}{2}$, 12, $8\frac{1}{2}$ (d) $\sqrt{10}$, 10, $10\sqrt{10}$, 100, $100\sqrt{10}$ or $-\sqrt{10}$, 10, $-10\sqrt{10}$, 100, $-100\sqrt{10}$

4(a) $a = 14\frac{1}{4}$, $b = 25\frac{1}{2}$, $c = 36\frac{3}{4}$

(b) $a = 6$, $b = 12$, $c = 24$

or $a = -6$, $b = 12$, $c = -24$

5(a) $\sqrt{5}$, 2 or -2 (b) $\frac{3}{8}\sqrt{2}$, $\frac{1}{2}$ or $-\frac{1}{2}$

(c) x , $\sqrt{x^2 - y^2}$ or $-\sqrt{x^2 - y^2}$

(d) $x^2 + y^2$, $x^2 - y^2$ or $y^2 - x^2$

(e) $\frac{x}{x^2 - y^2}$, $\frac{1}{\sqrt{x^2 - y^2}}$ or $\frac{-1}{\sqrt{x^2 - y^2}}$

(f) $\frac{5}{2}\log_2 3$, $2\log_2 3$ or $-2\log_2 3$

(g) $2\log_2 3$, $\sqrt{3}\log_2 3$ or $-\sqrt{3}\log_2 3$

(h) $5\log_b 2$, $4\log_b 2$ or $-4\log_b 2$

(i) $\frac{1}{4}\sqrt{5}$, $\frac{1}{2}$ or $-\frac{1}{2}$

6(a) 0.100 01, 0.002 or -0.002

(b) 0.150 005, 0.100 01, 0.050 015; 0.02, 0.002, 0.0002 or -0.02, 0.002, -0.0002

7(a) $\frac{x^2 + y^2}{2xy}$, 1 (c) $x = y$

8(b) The sign of the AM is the sign of the larger in absolute value.

10(b) $(a-b)^2 \geq 0$, so $(a+b)^2 \geq 4ab$, so $a+b \geq 2\sqrt{ab}$. (c) When $a = b$.

13(b) $XP = AM$

14(a) $c : a = 5 : 3$ (b) $c : a = (1 + \sqrt{5}) : 2$

15(b) $T_8/T_1 = (\frac{1}{2})^{\frac{7}{12}} \doteq 0.6674 \doteq \frac{2}{3}$

(c) $T_5/T_1 = (\frac{1}{2})^{\frac{4}{12}} \doteq 0.7937 \doteq \frac{4}{5}$

(d) $T_6/T_1 = (\frac{1}{2})^{\frac{5}{12}} \doteq 0.7491 \doteq \frac{3}{4}$,

$T_4/T_1 = (\frac{1}{2})^{\frac{3}{12}} \doteq 0.8409 \doteq \frac{5}{6}$

(e) $T_3/T_1 = (\frac{1}{2})^{\frac{2}{12}} \doteq 0.8908 \doteq \frac{8}{9}$,

$T_2/T_1 = (\frac{1}{2})^{\frac{1}{12}} \doteq 0.9439 \doteq \frac{17}{18}$

16(a) $\lambda = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ (b) $\lambda = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ (M to the left of A), or $\lambda = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ (M to the right of B)

Exercise 6G (Page 212)

1(a) 75 (b) 55 (c) 10 (d) 40 (e) 404 (f) 0 (g) 31

(h) 10 (i) -10 (j) -1 (k) 1 (l) 80

2(a) $\sum_{n=1}^{40} n^3$ (b) $\sum_{n=1}^{40} \frac{1}{n}$ (c) $\sum_{n=1}^{20} (n+2)$ (d) $\sum_{n=1}^{12} 2^n$

(e) $\sum_{n=1}^{13} 2^{n-1}$ (f) $\sum_{n=1}^k ar^{n-1}$ (g) $\sum_{n=1}^k (a + (n-1)d)$

(h) $\sum_{n=1}^{10} (-1)^n n$ (i) $\sum_{n=1}^{10} (-1)^{n-1} n$

(j) $\sum_{n=1}^{2k+1} (-1)^{n-1} x^{n-1}$

3(c)(i) $\sum_{n=0}^6 (3n+1)$ (ii) $\sum_{n=2}^8 (3n-5)$ (iii) $\sum_{n=7}^{13} (3n-20)$

5(b) 3

6(a) 125 (b) 0 (c) 873 (d) 56 700

Exercise 6H (Page 214)

1(a) S_n : 2, 7, 15, 26, 40, 57, 77 (AP with $a = 2$ and $d = 3$) (b) T_n : 2, 4, 8, 16, 32, 64, 128 (GP with $a = 2$ and $r = 2$)

2 They are the partial sums of the AP 2, 6, 10, 14, ... For further explanation, see your chemistry teacher.

3(a) 3, 8, 15, 24, 35 (b) 3, 5, 7, 9, 11

(c) $T_n = 2n + 1$

4(a) $T_n = 5 - 2n$ (b) $T_n = 6n - 8$ (c) $T_n = 11 - 10n$

6(a) 2, 8, 26, 80, 242 (b) 2, 6, 18, 54, 162

(c) $T_n = 2 \times 3^{n-1}$

7(a) $T_n = 5 \times 2^n$ (b) $T_n = 16 \times 5^{n-1}$

(c) $T_n = 3 \times 4^{n-2}$

8(a) $T_n = 6n$, 6, 12, 18

(b) $T_n = n + 1$, 2, 3, 4 (c) $T_n = 6 - 2n$, 4, 2, 0

(d) $T_n = 4, 4, 4, 4$ (e) $T_n = 3n^2 - 3n + 1$, 1, 7, 19

(f) $T_n = 2 \times 3^{-n}$, $\frac{2}{3}$, $\frac{2}{9}$, $\frac{2}{27}$,

(g) $T_n = -6 \times 7^{-n}$, $-\frac{6}{7}$, $-\frac{6}{49}$, $-\frac{6}{343}$

(h) $T_n = a + (n - 1)d$, a , $a + d$, $a + 2d$

(i) $T_n = n^2$, 1, 4, 9 (j) $T_n = n^3$, 1, 8, 27

(k) $T_n = ar^{n-1}$, a , ar , ar^2

9(a) $T_1 = 8$, $T_n = 2n + 3$ for $n \geq 2$

(b) $T_1 = -7$, $T_n = 14 \times 3^{n-1}$ for $n \geq 2$

(c) $T_1 = 1$, $T_n = \frac{-1}{n(n-1)}$ for $n \geq 2$

(d) $T_n = 3n^2 - n + 1$ for $n \geq 1$

The formula holds for $n = 1$ when $S_0 = 0$.

Exercise 6I (Page 216)

1 185

2(a) 222 (b) -630 (c) 78 400 (d) 0 (e) 65 (f) 30

3(a) 101 terms, 10 100 (b) 13 terms, 650

(c) 11 terms, 275 (d) 100 terms, 15 250

(e) 11 terms, 319 (f) 10 terms, $61\frac{2}{3}$

4(a) 500 terms, 250 500 (b) 2001 terms, 4 002 000

(c) 3160 (d) 1440

5(a) $S_n = n(1 + 2n)$ (b) $S_n = \frac{1}{2}n(5n - 23)$

(c) $S_n = \frac{1}{4}n(21 - n)$ (d) $S_n = \frac{1}{2}n(2 + n\sqrt{2} - 3\sqrt{2})$

6(a) $\frac{1}{2}n(n + 1)$ (b) n^2 (c) $\frac{3}{2}n(n + 1)$ (d) $100n^2$

7(a) 450 legs. No creatures have the mean number of 5 legs. (b) 21 835 years (c) \$352 000

8(a) n terms, $\frac{1}{2}nx(n + 1)$ (b) $60 + 190d$

(c) 21 terms, $21(a - 50)$ (d) $40 400b$

(e) $6(13 + 24\sqrt{2})$ (f) 20 terms, $230\sqrt{3}$

9(b)(i) 16 terms (ii) more than 16 terms

(c) 5 terms or 11 terms

(d) $n = 18$ or $n = -2$, but n must be a positive integer. (e) $n = 4, 5, 6, \dots, 12$

(f) Solving $S_n > 256$ gives $(n - 8)^2 < 0$, which has no solutions.

10(b)(i) $n = 3$ (ii) $n = 10$ (iii) $n = 40$

(c) 21 or more terms (d) Solving $S_n = 50$ gives $n^2 + n - 100 = 0$, which has no integer solutions because $b^2 - 4ac = 401$ is not a square.

11(a) 20 rows, 29 logs on bottom row

(b) $S_n = 5n^2$, 7 seconds

(c) 11 trips, the deposits are 1 km apart.

12(a) 10 terms, $55 \log_a 2$ (b) 11 terms, 0

(c) 6 terms, $3(4 \log_b 3 - \log_b 2)$

(d) $15(\log_x 2 - \log_x 3)$

13(a) $d = 11$ (b) $\ell = 22$ (c) $a = -7.1$ (d) $a = -3$

(e) $d = -2$, $a = 11$, $S_{10} = 20$

(f) $a = 9$, $d = -2$, $T_2 = 7$

(g) $d = -3$, $a = 28\frac{1}{2}$, $T_4 = 19\frac{1}{2}$

14(c) $a = -27$, $d = -2$ (d) $n = 15$

15(a) $37 + 45 + \dots + 101 = 621$ (c) $n = 11$

(d) 666667 or more

16(a) $n(43 - n)$, $n = 43$ (b)(i) $\frac{3}{2}n(41 - n)$, $n = 41$

(ii) $3n(n + 14)$, $n = 3$ (iii) $\frac{1}{4}n(n + 9)$, $n = 6$

17(a)(i) 14 850 (ii) 30 000 (b) 150 000

(c) $149 700 + 150 400 = 300 100$

(d) 322 multiples, sum is 442 911

18(a) $n = 17$, $a = -32$ (b) $n = 11$, $a = 20$

19(a) 300 (c) 162

20(a) $\frac{n}{n + 1}$

(b) $\frac{3}{4} - \frac{2n + 3}{2(n + 1)(n + 2)}$ and $\frac{1}{4} - \frac{1}{2(n + 1)(n + 2)}$

Exercise 6J (Page 220)

1 2186

2 2800 kits, cats, sacks and wives

3(a) 1023, $2^n - 1$ (b) -341 , $\frac{1}{3}(1 - (-2)^n)$

(c) 242, $3^n - 1$ (d) 122, $\frac{1}{2}(1 - (-3)^n)$

(e) $\frac{1023}{64}$, $16(1 - (\frac{1}{2})^n)$ (f) $\frac{341}{64}$, $\frac{16}{3}(1 - (-\frac{1}{2})^n)$

(g) $\frac{364}{27}$, $\frac{27}{2}(1 - (\frac{1}{3})^n)$ (h) $\frac{182}{27}$, $\frac{27}{4}(1 - (-\frac{1}{3})^n)$

(i) $\frac{1820}{27}$, $\frac{135}{2}(1 - (\frac{1}{3})^n)$ (j) $-11 111$, $\frac{1}{9}(1 - 10^n)$

(k) -9091 , $\frac{1}{11}((-10)^n - 1)$ (l) $\frac{211}{24}$, $\frac{4}{3}((\frac{3}{2})^n - 1)$

4(a) $5((1.2)^n - 1)$, 25.96 (b) $20(1 - (0.95)^n)$, 8.025

(c) $100((1.01)^n - 1)$, 10.46

(d) $100(1 - (0.99)^n)$, 9.562

- 5(a)(i) 2^{63} (ii) $2^{64} - 1$ (b) 615 km^3
 6(a) $S_n = \frac{cx(3^n x^n - 1)}{3x - 1}$ (b) $S_n = \frac{x^n - 1}{(x - 1)x^{n-1}}$
 (c) $S_n = \frac{cx(1 - (-3x)^n)}{1 + 3x}$ (d) $S_n = \frac{y^n - x^n}{(y - x)y^{n-1}}$
 7(a) $S_n = ((\sqrt{2})^n - 1)(\sqrt{2} + 1)$,
 $S_{10} = 31(\sqrt{2} + 1)$
 (b) $S_n = \frac{1}{20}(1 - (-\sqrt{5})^n)(\sqrt{5} - 1)$,
 $S_{10} = -\frac{781}{5}(\sqrt{5} - 1)$
 8(a)(i) $121\frac{1}{3}$ (ii) $9 \log_a 3$ (iii) $\frac{765}{32}$ (b) $\frac{3}{4} + \frac{9}{2} + 27 = \frac{129}{4}$
 9(b) $n = 8$ (c) 14 terms (d) $S_{14} = 114681$
 10(a) 41 powers of 3 (b) 42 terms
 11(a)(i) 0.01172 tonnes (ii) 11.99 tonnes
 (b) $4.9 \times 10^{-3} \text{ g}$
 (c)(i) $S_n = 10P(1.1^{10} - 1)$ (ii) \$56.47
 12(a) 34010 and 26491 (c) 3.30
 13(a) $\frac{n+1}{n}$ (b) $\frac{3}{2^n + 1}$
 14(b) $n = 6$ (c) $T_{12} = -708588$
 (d) $S_{13} = 1594324$
 15(a)(i) 2097151 (ii) $\frac{6560}{4374}$ (b) $r = 4$ and $n = 4$
 (c) $n = 6$ and $l = -1215$
 16(a) $r = 2$ or $r = -2$ (c) $r = 3^{-\frac{1}{2}}$ or $-3^{-\frac{1}{2}}$
 17(a) $3 \times 3^n + 6 \times 2^n - 9$ (b) $2 \times 2^n + n^2 + 4n - 2$
 (c) $a = 1, d = 3, b = 3, S_n = \frac{3}{2}n^2 + \frac{5}{2}n - 6 + 6 \times 2^n$
 18 112
 19 694

Exercise 6K (Page 225)

- 1 18, 24, 26, $26\frac{2}{3}$, $26\frac{8}{9}$, $26\frac{26}{27}$, $S_\infty = 27$,
 $S_\infty - S_6 = \frac{1}{27}$
 2(a) $r = \frac{1}{2}$, $S_\infty = 2$ (b) $r = -\frac{1}{2}$, $S_\infty = \frac{2}{3}$
 (c) $r = \frac{1}{3}$, $S_\infty = 18$ (d) $r = -1$, no limiting sum
 (e) $r = \frac{9}{10}$, $S_\infty = 1000$ (f) $r = -\frac{1}{5}$, $S_\infty = -\frac{5}{3}$
 (g) $r = \frac{1}{5}$, $S_\infty = -\frac{5}{6}$
 (h) $r = 1.01$, no limiting sum
 (i) $r = -0.99$, $S_\infty = \frac{100}{199}$
 (j) $r = (1.01)^{-1}$, $S_\infty = 101$
 (k) $r = -\frac{1}{6}$, $S_\infty = \frac{108}{175}$ (l) $r = \frac{1}{4}$, $S_\infty = \frac{64}{3}\sqrt{5}$
 3(a) $x = \frac{1}{2}$ (b) $x = -\frac{2}{3}$ (c) $x = -\frac{2}{3}$
 4(a) $a = \frac{4}{3}$ (b) $a = \frac{8}{3}$ (c) $a = \frac{2}{3}$
 5(a) $0 < x < 2$, $\frac{1}{2-x}$ (b) $-2 < x < 0$, $-\frac{1}{x}$
 (c) $\frac{1}{3} < x < 1$, $\frac{1}{3-3x}$ (d) $-1 < x < -\frac{1}{3}$, $\frac{1}{3x+3}$

- 6(a) $\frac{7}{6}(7 + \sqrt{7})$ (b) $4(2 - \sqrt{2})$ (c) $5(5 - 2\sqrt{5})$
 (d) $r = \frac{1}{3}\sqrt{10} > 1$, so there is no limiting sum.
 (e) $\frac{1}{3}\sqrt{3}$ (f) $\frac{1}{2}(\sqrt{3} + 1)$ (g) $2\sqrt{5} + 4$
 (h) $r > 1$, so there is no limiting sum.
 7 $\frac{3}{7}$
 8(a) The successive down-and-up distances form a GP with $a = 15$ and $r = \frac{2}{3}$.
 (b) $S_\infty = 45$ metres
 9(a) $r = \frac{1}{2}$, $S_\infty = 14$ (b) $r = -\frac{4}{25}$, $S_\infty = -\frac{25}{9}$
 (c) The first GP has $r = \frac{1}{5}$ and $S_\infty = 5$, the second GP has $r = \frac{1}{4}$ and $S_\infty = 6\frac{2}{3}$, so the total is $11\frac{2}{3}$.
 11(a) $r = \frac{4}{5}$ (b) $18 + 6 + 2 + \dots$ or $9 + 6 + 4 + \dots$
 (c) $r = \frac{5}{8}$ (d)(i) $r = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ ($r = -\frac{1}{2} - \sqrt{5} < -1$, so it is not a possible solution.)
 (ii) $r = \frac{1}{2}$ (iii) $r = \frac{1}{2}\sqrt{2}$ or $-\frac{1}{2}\sqrt{2}$ (e) $r = 2^{-\frac{1}{3}}$
 12(b)(i) 96 (ii) 32 (iii) 64 (iv) 32
 13(a) 1 : 10 (b) 45th year
 14(a) 66667 (b) 88.2% (c) 12th month (d) 98%
 15(b) $r = -3$, which is impossible. (d)(i) $S_\infty > 3$
 (ii) $S_\infty < -4$ (iii) $S_\infty > \frac{1}{2}a$ (iv) $S_\infty < \frac{1}{2}a$
 16(a) First term is $\frac{ar}{1-r}$, ratio is r , it converges to zero because its ratio is between -1 and 1 .
 (b) $D_n = 3^{3-n}$, $D_5 = \frac{1}{9}$, 16 terms (c) 10 terms
 17(a) $-\sqrt{2} < x < \sqrt{2}$ and $x \neq 0$, $S_\infty = \frac{1}{2-x^2}$
 (b) $1 < x < \sqrt{3}$ or $-1 > x > -\sqrt{3}$, $S_\infty = \frac{1}{3-x^2}$
 (c) $x > \frac{1}{5}$ or $x < -\frac{1}{5}$, $S_\infty = \frac{5x}{5x-1}$
 (d) $x > 2$ or $x < -2$, $S_\infty = \frac{x}{x+2}$
 (e) $x \neq 0$, $S_\infty = \frac{1+x^2}{x^2}$
 (f) $x > 4$ or $x < 2$, $S_\infty = \frac{3-x}{4-x}$
 (g) $x \neq 1$ and $x \neq -1$, $S_\infty = \frac{x^2+1}{(x-1)^2}$
 19(b) $S_n = 4 - (\frac{1}{2})^{n-2} - \frac{n}{2^{n-1}}$ (c) $S'_\infty = 4$
 20(b) $S_n = \frac{ax}{x-1} + \frac{dx}{(x-1)^2} (1 - (\frac{1}{2})^{n-1})$
 $-\frac{a+(n-1)d}{(x-1)x^{n-1}}$ (c) $S_\infty = \frac{ax}{x-1} + \frac{dx}{(x-1)^2}$

Exercise 6L (Page 228)

- 1(a) $0.7 + 0.07 + 0.007 + \dots = \frac{7}{9}$ (b) $\frac{2}{3}$
 (c) $0.27 + 0.0027 + 0.000027 + \dots = \frac{3}{11}$ (d) $\frac{26}{33}$
 (e) $\frac{5}{11}$ (f) $\frac{1}{37}$ (g) $\frac{5}{37}$ (h) $\frac{5}{27}$
 2(a) $12 + (0.4 + 0.04 + \dots) = 12\frac{4}{9}$ (b) $7\frac{9}{11}$
 (c) $8.4 + (0.06 + 0.006 + \dots) = 8\frac{7}{15}$
 (d) $0.2 + (0.036 + 0.00036 + \dots) = \frac{13}{55}$
 4(a) $0.\dot{9} = 0.9 + 0.09 + 0.009 + \dots = \frac{0.9}{1-0.1} = 1$
 (b) Zero is the only number that is not negative, but is less than every positive number.
 (d) $74 = 73.\dot{9}$, $7.282 = 7.281\dot{9}$
 5(a) $\frac{29}{303}$ (b) $\frac{25}{101}$ (c) $\frac{3}{13}$ (d) $\frac{3}{7}$
 (e) $0.25 + (0.0057 + 0.000057 + \dots) = \frac{211}{825}$ (f) $1\frac{14}{135}$
 (g) $\frac{1}{3690}$ (h) $7\frac{27}{35}$
 7(a) $\frac{1}{4}, \frac{13}{16}, \frac{27}{64}$ (b) $0.1, 0.11, 0.011, 0.1011$
 (c) $\frac{2}{3}, \frac{5}{7}, \frac{1}{5}, 1$ (d) $0.\dot{0}1, 0.\dot{1}100, 0.\dot{0}01$
 8(a) Notice that $\frac{1}{9} = 0.\dot{1}$, $\frac{1}{99} = 0.0\dot{1}$, $\frac{1}{999} = 0.00\dot{1}$, and so on. If the denominator of a fraction can be made a string of nines, then the fraction will be a multiple of one of these recurring decimals.
 (b) Periods: 1, 6, 1, 2, 6, 3, 3, 5, 4, 5

Exercise 6M (Page 230)

- 1(a) $(x-1)(x+1)$ (b) $(x-1)(x^2+x+1)$
 (c) $(x-1)(x^4+x^3+x^2+x+1)$
 (d) $(t-1)(t^6+t^5+t^4+t^3+t^2+t+1)$
 (e) $(t+1)(t^2-t+1)$ (f) $(t+1)(t^4-t^3+t^2-t+1)$
 (g) $(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)$
 (h) $(x-5)(x^2+5x+25)$ (i) $(x+2)(x^2-2x+4)$
 (j) $(x-3)(x^4+3x^3+9x^2+27x+81)$
 (k) $(x+5)(x^2-5x+25)$
 (l) $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$
 (m) $(x+2)(x^4-2x^3+4x^2-8x+16)$
 (n) $(2t+1)(16t^4-8t^3+4t^2-2t+1)$
 (o) $(1-ax)(1+ax+a^2x^2+a^3x^3+a^4x^4+a^5x^5+a^6x^6)$ (p) $(3t+2a)(9t^2-6ta+4a^2)$
 2(a) x^2+xy+y^2 (b) $x^3+x^2y+xy^2+y^3$ (c) x^2-y+1
 (d) $16x^4-8x^3y+4x^2y^2-2xy^3+y^4$
 (e) $\frac{x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6}{x^4+x^3y+x^2y^2+xy^3+y^4}$
 (f) $\frac{x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6}{x^4-x^3y+x^2y^2-xy^3+y^4}$
 3(a) $(x-1)(x+1)(x^2+1)$
 (b) $(x-1)(x^2+x+1)(x+1)(x^2-x+1)$
 (c)(i) $(x^4+a^4)(x^2+a^2)(x+a)(x-a)$
 (ii) $(x-1)(x^4+x^3+x^2+x+1)(x+1)(x^4-x^3+x^2-x+1)$
 4(a) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$

- (b) $(\sqrt{x}-\sqrt{y})(x+\sqrt{xy}+y)$
 (c) $(\sqrt{x}+\sqrt{y})(x-\sqrt{xy}+y)$
 5(a) $\frac{1}{\sqrt{x}+\sqrt{y}}$ (b) $\sqrt{x}-\sqrt{y}$ (c) $x+\sqrt{xy}+y$
 (d) $\frac{1}{x-\sqrt{xy}+y}$
 6(a) $2n+1$ (b) $4n$ (c) $3n^2+3n+1$ (d) $2(3n^2+1)$
 (e) $4an$ (f) $35(n+a)(n-a)$
 7(a) $u+x$ (b) u^2+ux+x^2 (c) $u^3+u^2x+ux^2+x^3$
 (d) $-\frac{1}{ux}$ (e) $\frac{1}{\sqrt{u}+\sqrt{x}}$ (f) $-\frac{u+x}{u^2x^2}$
 8(b)(i) $(x^2+1)(x^2+x\sqrt{3}+1)(x^2-x\sqrt{3}+1)$
 (ii) $(x-1)(x+1)(x^2+1)(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)$
 (iii) $(x-1)(x+1)(x^2+1)(x^2+x+1)(x^2-x+1)(x^2+x\sqrt{3}+1)(x^2-x\sqrt{3}+1)$
 9(a) $2^{ab}-1 = (2^a)^b-1$, which factors when $a > 1$ and $b > 1$. $M_2 = 3$, $M_3 = 7$, $M_5 = 31$, $M_7 = 127$, $M_{11} = 2047 = 23 \times 89$
 (b) If b is odd, then $2^{ab}+1 = (2^a)^b+1$, which factors. $F_0 = 2^1+1 = 3$, $F_1 = 2^2+1 = 5$, $F_2 = 2^4+1 = 17$, $F_3 = 2^8+1 = 257$, $F_4 = 2^{16}+1 = 65537$, $F_5 = 2^{32}+1 = 641 \times 6700417$
 (c) The divisors of N less than N are $1, 2, 2^2, \dots, 2^{p-1}$ with sum $2^p-1 = M_p$, and $M_p, 2M_p, 2^2M_p, \dots, 2^{p-2}M_p$ with sum $(2^{p-1}-1)M_p$. The combined sum is N . Some perfect numbers: $6 = 2 \times 3$, $28 = 2^2 \times 7$, $496 = 2^4 \times 31$, $8128 = 2^6 \times 127$
 (d) $F_{n+1}-2 = (2^n+1)(2^n-1) = F_n(F_n-2)$. The result now follows, since $F_0-2 = 1$. If $n > m$, then F_m is a divisor of F_n-2 , so since F_n and F_m are both odd, they are relatively prime.

Exercise 6N (Page 234)

- 2 $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots = 1$,
 $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \dots = \frac{1}{3}$,
 $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots = \frac{1}{4}$
 5 n^3-n is divisible by 24, for odd cardinals n .
 7 $2^n > 2n^3$, for $n \geq 12$.
 17(d) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

Chapter Seven

Exercise 7A (Page 240)

1 The graph of $y = f'(x)$ should approximate a line of gradient 2 through the origin; its equation is $f'(x) = 2x$.

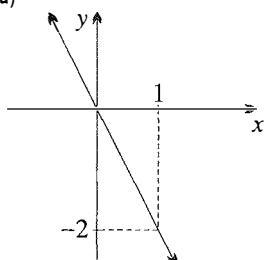
2(a) 2 (b) -3 (c) $\frac{1}{2}$ (d) 0 (e) a (f) $\frac{2}{3}$ (g) $-\frac{5}{4}$
 (h) $-\frac{10}{3}$ (i) 0

3(a) $\frac{7}{2}$ (b) 12 (c) 0

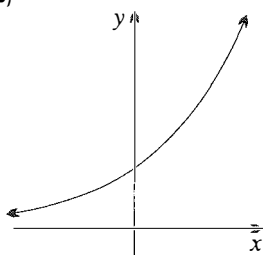
4(a) $-\frac{4}{3}$ (b) $-\frac{3}{4}$ (c) 0 (d) $\frac{4}{3}$ (e) $\frac{3}{4}$

5(a) $\frac{-x}{\sqrt{1-x^2}}$ (b) $\frac{x}{\sqrt{1-x^2}}$ (c) $\frac{-x}{\sqrt{4-x^2}}$

6(a)



(c)



7(a) $\frac{-x}{\sqrt{9-x^2}}$ (b) $\frac{x}{\sqrt{16-x^2}}$ (c) $\frac{7-x}{\sqrt{36-(x-7)^2}}$

(d) $\frac{x-1}{\sqrt{2x-x^2}}$

Exercise 7B (Page 243)

1(a) $2x + h - 4$ (c) -2 (d) 0 at C, 2 at B

2(a) 5, 5 (b) -3, -3 (c) $2x + h$, $2x$

(d) $2x + h - 4$, $2x - 4$ (e) $2x + h + 3$, $2x + 3$

(f) $4x + 2h + 3$, $4x + 3$ (g) $-8x - 4h$, $-8x$

(h) $3x^2 + 3xh + h^2$, $3x^2$

(i) $4x^3 + 6x^2h + 4xh^2 + h^3$, $4x^3$

3(a) 5, 11, $y = 5x + 1$ (b) -3, -2, $y = 4 - 3x$

(c) 4, 14, $y = 4x + 6$ (d) 0, -4, $y = -4$

(e) 7, 12, $y = 7x - 2$ (f) 11, 14, $y = 11x - 8$

(g) -16, -7, $y = -16x + 25$

(h) 12, 8, $y = 12x - 16$ (i) 32, 16, $y = 32x - 48$

4(a) none (b) none (c) $x = 0$ (d) $x = 2$

(e) $x = -1\frac{1}{2}$ (f) $x = -\frac{3}{4}$ (g) $x = 0$ (h) $x = 0$

(i) $x = 0$

5(a) 5 (b) -3 (c) $u + x$ (d) $u + x - 4$ (e) $u + x + 3$

(f) $2u + 2x + 3$ (g) $-4u - 4x$ (h) $u^2 + ux + x^2$

(i) $u^3 + u^2x + ux^2 + x^3$ (derivatives as before)

6(b)(i) $6, 80^\circ 32'$ (ii) 0, 0° (iii) 1, 45° (iv) -1, 135°

(v) -4, $104^\circ 2'$

7(b)(i) (3, -6) (ii) (2, -6) (iii) (5, 0) (iv) (0, 0)

(v) $(2\frac{1}{2}, -6\frac{1}{4})$

8(b)(i) $x = 2$ (ii) $x = -2$ (iii) $x = 2\sqrt{3}$

(iv) $x = -2\sqrt{3}$ (v) $x = \frac{2}{3}\sqrt{3}$ (vi) $x = -\frac{2}{3}\sqrt{3}$

(vii) $x = 2 \tan 37^\circ \approx 1.507$

10(b) $f'(0) = -5$, $y = -5x + 6$, whose x -intercept is $\frac{6}{5}$. (c) At (2, 0), $f'(2) = -1$ and angle of inclination is 135° . At (3, 0), $f'(3) = 1$ and angle of inclination is 45° . (d) $71^\circ 34', 108^\circ 26'$

11(b)(i) 4 (ii) -1 (iii) 0 (iv) 2.01

13(a) It is the difference-of-squares identity.

14(a) -7 (b) 6 (c) 1 (d) 5 (e) 6 or -6

15 The line is a tangent when the two points coincide, that is when $m = 2a$, so the gradient of the tangent is twice the x -coordinate.

16 They meet at $x = \frac{1}{2}(m + \sqrt{m^2 + 4b})$ and $x = \frac{1}{2}(m - \sqrt{m^2 + 4b})$. The line is a tangent when these coincide, that is, when $m^2 + 4b = 0$, in which case the tangent at $x = \frac{1}{2}m$ has gradient m , which is twice the x -coordinate.

Exercise 7C (Page 247)

1(a) $7x^6$ (b) $45x^4$ (c) $2x^5$ (d) $6x - 5$

(e) $4x^3 + 3x^2 + 2x + 1$ (f) $-3 - 15x^2$

(g) $2x^5 - 2x^3 + 2x$ (h) $x^3 + x^2 + x + 1$

(i) $4ax^3 - 2bx$ (j) $\ell x^{\ell-1}$ (k) $3b^2x^{3b-1}$

(l) $(5a + 1)x^{5a}$

2(a) 0, 7 (b) 0, 45 (c) 0, 2 (d) -5, 1 (e) 1, 10

(f) -3, -18 (g) 0, 2 (h) 1, 4 (i) 0, $4a - 2b$

(j) 0, ℓ (k) 0, $3b^2$ (l) 0, $5a + 1$

3(a) $3x^2 + 1$ (b) $6x - 6x^2 - 16x^3$ (c) $2x + 2$ (d) $8x$

(e) $4x^3 + 12x$ (f) $3x^2 - 28x + 49$ (g) $3x^2 - 10x + 3$

(h) $2a^2x - 10a$

4(a) $-3x^{-2}$ (b) $-10x^{-3}$ (c) $4x^{-4}$

(d) $-4x^{-3} - 4x^{-9}$

5(a) $-2/x^3$ (b) $-15/x^4$ (c) $-2/x^5$ (d) $3/x^6$

(e) $-c/ax^2$ (f) $-6/x^7 + 8/x^9$ (g) $-a/x^2 + 2b/x^3$

(h) $-7n/2x^{n+1}$

6(a) $-\frac{3}{x^2}$ (b) $-\frac{1}{3x^2}$ (c) $\frac{7}{3x^2}$ (d) $-\frac{a}{x^2}$

7(a) $\frac{3}{2\sqrt{x}}$ (b) $\frac{5}{\sqrt{x}}$ (c) $\frac{7}{2\sqrt{x}}$ (d) $\frac{\sqrt{7}}{2\sqrt{x}}$

- 8(a) 1, -1 (b) -1, 1 (c) $-6, \frac{1}{6}$ (d) $1/\sqrt{3}, -\sqrt{3}$
 9(a) $45^\circ, 135^\circ$ (b) $135^\circ, 45^\circ$
 (c) about $99^\circ 28', 9^\circ 28'$ (d) $30^\circ, 120^\circ$
 10(a) $y = -6x + 14, x - 6y + 47 = 0$ (b) $y = 4x - 21, x + 4y - 18 = 0$ (c) $y = -8x + 15, x - 8y + 120 = 0$
 (d) $y = -1, x = 4$
 11 $f'(x) = 3x^2$, which is positive for $x \neq 0$ and zero for $x = 0$.
 12 $2x + y = 16, A = (8, 0), B = (0, 16), AB = 8\sqrt{5}, |\triangle OAB| = 64$ square units
 13(a) (2, 8) (b) (2, 8) and (-2, 40) (c) $(2a, 4a^2)$
 (d) (0, 0) and (1, -1) and (-1, -1)
 14(a) $y = -3x + 12, x - 3y + 16 = 0$
 (b) $y = -\frac{1}{3}x + 4, y = 3x - 16$
 15 $y = -2x + 5, y = 2x + 5, (0, 5)$
 16 $y = -2x + 10, x - 2y + 15 = 0, A = (5, 0), B = (-15, 0), AB = 20, |\triangle AKB| = 80$ square units
 17 $y = 3x - 2, x + 3y = 4, P = (0, -2), Q = (0, \frac{1}{3}), |\triangle QUP| = 1\frac{2}{3}$ square units
 18 $f'(x) = 2ax + b, \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$
 19 $f'(9) = 14, f'(-5) = -14$
 20 $f'(x) = 3x^2 + a, x = \sqrt{-a/3}$ and $x = -\sqrt{-a/3}, a \leq 0$ (but no restriction on b)
 21(a) 9 (b) $-\frac{1}{4}$ (c) $a^2, -a^2, 2a^2$
 22 The tangent has gradient $2a - 6$, the normal has gradient $\frac{1}{6 - 2a}$. (a)(i) 3 (ii) 4 (iii) $3\frac{1}{4}$
 (b)(i) $2\frac{7}{8}$ (ii) 1 (iii) impossible (c) $2\frac{1}{2}$ (d) $3 - \frac{1}{2}\sqrt{3}$
 (e)(i) $3\frac{1}{3}$ (ii) $2\frac{1}{4}$
 23(a) $y = 2ax - a^2, U = (\frac{1}{2}a, 0), V = (0, -a^2)$
 (b) $T = (5, 25)$ or $(-5, 25)$
 24(a) $y = 2x_0x + 9 - x_0^2$ (b) Put the x -intercept equal to 0. (3, 18) and (-3, 18)
 25(a) $(\frac{1}{2}, 2)$ (b) $(-2, -\frac{1}{2})$ (a cannot be zero)
 (c) (1, 1) and $(3, \frac{1}{3})$ (d) impossible, as $a \neq 0$
 26(a) $2y\sqrt{t} = x + t - 2\sqrt{t}$
 (c) $t = 4, x = 4y$ ($t = 0$ is not allowed, because there is no tangent at the endpoint.)
 27 The tangent where $x = t$ is $y = 2tx + 5 - t^2$. To pass through $O, t = \sqrt{5}$ or $t = -\sqrt{5}$. The tangents are $y = 2x\sqrt{5}$ and $y = -2x\sqrt{5}$.
 31 The tangents are $y = 2ax - a^2$ and $y = 2bx - b^2$. They meet at $K = (\frac{1}{2}(a + b), ab)$.
 32(a) $y = (2ax_0 + b)x - ax_0^2 + c$. a and c must have the same sign, or $c = 0$ (b is arbitrary).

- $y = (2\sqrt{ac} + b)x$ and $y = (-2\sqrt{ac} + b)x$
 (b) Points of contact: $(\sqrt{c/a}, 2c + b\sqrt{c/a})$ and $(-\sqrt{c/a}, 2c - b\sqrt{c/a})$, whose midpoint is $(0, 2c)$.
 (c) $2\sqrt{c^3/a}$ square units

Exercise 7D (Page 252)

- 1(a) $4x^3 - 2x, -2$ (b) $2ax + b, b - 2a$
 (c) $4x - 5, -9$ (d) $3ax^2 - 2cx, 3a + 2c$
 (e) $-27/x^4, -27$
 (f) $6/\sqrt{x}, \text{undefined}$
 (g) $-a/x^2 - 2a/x^3, a$
 (h) $11/2\sqrt{x}, \text{undefined}$
 2(a) $9x^2 - 5$ (b) $5x^2 + \frac{8}{3}x$ (c) $2 + 6x^{-3}$
 (d) $a - \frac{c}{x^2} + \frac{2d}{x^3}$
 3(a) $\frac{5}{2}x^{1\frac{1}{2}}$ (b) $-\frac{1}{2}x^{-\frac{3}{2}}$ (c) $3x^{-\frac{1}{4}}$ (d) $-\frac{10}{3}x^{-1\frac{2}{3}}$
 (e) $6x^{-1\cdot6}$
 4(a) $18x^{\frac{1}{2}} = 18\sqrt{x}$ (b) $10x^{\frac{3}{2}} = 10x\sqrt{x}$
 (c) $-3x^{-\frac{3}{2}} = \frac{-3}{x\sqrt{x}}$ (d) $-\frac{15}{2}x^{-\frac{5}{2}} = \frac{-15}{2x^2\sqrt{x}}$
 (e) $3x^{-\frac{1}{5}}$
 5(a) $y = -6x, y = \frac{1}{6}x$ (b) $y = \frac{1}{4}x + 1, y = -4x + 18$
 (c) $y = 2x + 2, x + 2y + 1 = 0$ (d) $y = 0, x = 1$
 6(a) (1, 1) and (-1, -1) (b) $(1, \frac{1}{2})$
 (c) $(1, 2\frac{2}{3})$ and $(-1, 3\frac{1}{3})$ (d) none (e) $(\frac{1}{4}, -\frac{1}{2})$
 7(a) $(1, -6\frac{2}{3}), (-1, -7\frac{1}{3})$ (b) $(-1, \frac{2}{3})$
 (c) $(-\frac{1}{2}\sqrt{3}, 1\frac{3}{4})$ (d) $(3, 2\sqrt{3})$
 8(a) 1 (b) 0, 3, -3 (c) 1, -1
 9(b) At (2, -4), $71^\circ 34'$. At (-3, 6), $98^\circ 8'$.
 10(a) $x = \frac{1}{2}(\tan 22^\circ - 3) \doteq -1.298$
 (b) $x^3 = \frac{1}{4}\tan 142^\circ 17', x \doteq -0.5782$
 (c) This is impossible, because all the tangents to $y = 1/x$ have negative gradients.
 11(a) $y = (2a - 10)x - a^2 + 9, a = 3$ and $y = -4x$, or $a = -3$ and $y = -16x$ (b) $y = (2a + 15)x - a^2 + 36, a = 6$ and $y = 27x$, or $a = -6$ and $y = 3x$
 (c) $y = (4a - 7)x - 2a^2 + 6, a = \sqrt{3}$ and $y = (4\sqrt{3} - 7)x$, or $a = -\sqrt{3}$ and $y = (-4\sqrt{3} - 7)x$
 12(a) $b = 7, c = 0$ (b) $b = -2, c = -3$
 (c) $b = -10, c = 25$ (d) $b = -1, c = -2$
 (e) $b = -9, c = 17$ (f) $b = -5\frac{2}{3}, c = 7$
 13(a) $\frac{15}{2}x\sqrt{x} - 3\sqrt{x}$ (b) $48x^3$ (c) 36
 (d) $x^{\pi-1} + x^{\frac{1}{\pi}-1}$ (e) $\frac{9}{2}x^{\frac{1}{2}} - x^{-\frac{1}{2}} - 2x^{-1\frac{1}{2}}$
 (f) $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$ (g) $2x - 2x^{-3}$ (h) $\frac{3}{2}\sqrt{x}$
 (i) $4 - 4x^{-2}$ (j) $4ax^3 - 4ax^{-5}$

14(a) $a^2 - 2a$ (b) $-a - a^{-1}$ (c) $-2\frac{1}{2}$ (d) 0 (e) $4n^4$
 (f) 21 (g) -21 (h) $-3\frac{1}{2}$ (i) $2^n n$

15 $\frac{dP}{dx} = 2tx + 3u$, $\frac{dP}{du} = 6tu + 3x$,

$\frac{dP}{dt} = x^2 + 3u^2 + 1$

16(a) 12 metres (b) $x = 6$ (c) 36 metres

(d) about $85^\circ 14'$

(e) The gradients are 12 and -12 , so the acute angle with the ground is the same.

(f) about $82^\circ 52'$

(g) The gradients are $12 - 2a$ and $2a - 12$. The acute angle with the ground will be the same.

(h) $y = 12x$, $HA = 3$ metres, $HB = 36$ metres

17 At $(1, -3)$ the tangent is $x + y + 2 = 0$, at $(-1, 3)$ the tangent is $x + y - 2 = 0$.

18 The tangent is $y = x$.

19 At $(2, 1)$ the gradient is 2, which is perpendicular to $x + 2y = 4$; at $(-\frac{1}{2}, \frac{9}{4})$ the gradient is -3 .

20 $y = 2(a + 1)x - a^2 - 8$, $(1, -5)$, $(3, 7)$

21(c)(i) $y \frac{dy}{dx} = \frac{1}{2}a^2$ (ii) $y \frac{dy}{dx} = na^2 x^{2n-1}$

22 $y = 2(\alpha - 3)x + 9 - \alpha^2$, $A = (\frac{1}{2}(\alpha + 3), 0)$,
 $B = (0, 9 - \alpha^2)$, $M = (\frac{1}{4}(\alpha + 3), \frac{1}{2}(9 - \alpha^2))$,
 $\alpha = 1$

23(a) $cx + t^2y = 2ct$, $A = (2t, 0)$, $B = (0, 2c/t)$

(b) $2|c|$

(e) $AB = \frac{2}{|t|} \sqrt{t^4 + c^2}$,

perpendicular distance = $\frac{2|ct|}{\sqrt{t^4 + c^2}}$

25(b) $y = x^2 - 6x$ and $y = \frac{25}{81}x^2 + \frac{2}{9}x$

(c) $y = x^2 - x - 6$

26 The equation of the tangent at $x = t$ is a cubic in t , and every cubic has at least one solution. (Why?)

Exercise 7E (Page 258)

1(a) $12(3x + 7)^3$ (b) $-28(5 - 4x)^6$ (c) $8p(px + q)^7$

(d) $24x(x^2 + 1)^{11}$ (e) $-64x(7 - x^2)^3$

(f) $9(2x + 3)(x^2 + 3x + 1)^8$

(g) $-18(3x^2 + 1)(x^3 + x + 1)^5$ (h) $\frac{5}{2\sqrt{5x + 4}}$

(i) $\frac{-1}{\sqrt{3 - 2x}}$ (j) $\frac{7x}{\sqrt{x^2 + 1}}$ (k) $\frac{-x}{\sqrt{9 - x^2}}$

(l) $\frac{b^2x}{\sqrt{a^2 - b^2x^2}}$

2(a) $25(5x - 7)^4$ (b) $-21(4 - 3x)^6$

(c) $15(2 - 3x)^{-6}$ (d) $4p(q - x)^{-5}$ (e) $\frac{1}{(2 - x)^2}$

(f) $\frac{-5}{(3 + 5x)^2}$ (g) $\frac{15}{(x + 1)^4}$ (h) $\frac{1}{2\sqrt{x + 4}}$

(i) $\frac{-3}{2\sqrt{4 - 3x}}$ (j) $\frac{m}{2\sqrt{mx - b}}$ (k) $\frac{1}{2}(5 - x)^{-1\frac{1}{2}}$

3(a) $4t, -4$ (b) $-1/t^2, -1$ (c) $b/a, b/a$

(d) $\frac{9}{4}t, -\frac{9}{4}$

4(a) $6x(x^2 - 1)^2, (0, -1), (1, 0), (-1, 0)$

(b) $8(x - 2)(x^2 - 4x)^3, (0, 0), (2, 256), (4, 0)$

(c) $10(x + 1)(2x + x^2)^4, (0, 0), (-2, 0), (-1, -1)$

(d) $\frac{-5}{(5x + 2)^2}$, none (e) $-14(x - 5), (5, 24)$

(f) $6(x - 5)^5, (5, 4)$ (g) $2a(x - h), (h, k)$

(h) $\frac{-1}{\sqrt{3 - 2x}}$, none (i) $\frac{-2x}{(1 + x^2)^2}, (0, 1)$

(j) $\frac{x - 1}{\sqrt{x^2 - 2x + 5}}, (1, 2)$

(k) $\frac{x - 1}{\sqrt{x^2 - 2x}}$, none ($x = 1$ is outside the domain)

5(a) $y = 20x - 19, x + 20y = 21$ (b) $y = 24x - 16, x + 24y = 193$ (c) $x + 2y = 2, y = 2x - 1\frac{1}{2}$

(d) none ($x = 1$ is outside the domain)

6(a) $2\frac{1}{2}$ and 1 (b) 2 and $1\frac{1}{2}$

7(a) $y = \frac{1}{3}x + 15$ (b) $y = 3x - 4$

8(a) -5 or -7 (b) 4 or 8

9(a) $x + y(b - 4)^2 = 2b - 4$ (b)(i) $x + 4y = 0$

(ii) $x + y = 6$

10(a) $\frac{11(\sqrt{x} - 3)^{10}}{2\sqrt{x}}$ (b) $\frac{-3}{4\sqrt{4 - \frac{1}{2}x}}$

(c) $\frac{3\sqrt{2}}{(1 - x\sqrt{2})^2}$ (d) $\frac{1}{2}(5 - x)^{-1\frac{1}{2}}$

(e) $\frac{1}{2}a^2(1 + ax)^{-1\frac{1}{2}}$ (f) $\frac{1}{4}b(c - \frac{1}{2}x)^{-1\frac{1}{2}}$

(g) $-16\left(1 - \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right)^3$

(h) $6\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^5$

11(a) $a = \frac{1}{16}, b = 12$ (b) $a = \frac{1}{9}, b = -10$

13(a) $12x + 5y = 169, (\frac{169}{12}, 0), (0, \frac{169}{5})$ (c) $\frac{169^2}{120}$

(d) $\frac{13^3}{60} + \frac{169}{12} + \frac{169}{5} = \frac{169}{2}$

14(a) $4x + 3y = 25, 4x + 5y = 25$, they intersect at $(6\frac{1}{4}, 0)$. (b) $\lambda x_0x + y\sqrt{25 - x_0^2} = 25\lambda$,

$T = (25/x_0, 0), OM \times OT = 25 = OA^2$

15(a) $P = (7\frac{1}{2}, 3\frac{1}{4}), Q = (6\frac{1}{2}, 3\frac{1}{4})$

(b) area = $\frac{1}{2}PQ^2 = \frac{1}{2}$

16(a) At $P, x = h + \frac{1}{2}m$. At $Q, x = h - \frac{1}{2}m$.

(b) $\frac{1}{4}m(m^2 + 1)$

17(b) The vertical distance is $a(\alpha - h)^2$.

(c) $\alpha = \sqrt{h^2 + k/a}$ or $-\sqrt{h^2 + k/a}$

$$18(a) \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}, \frac{x}{(1 + \sqrt{1-x^2})^2 \sqrt{1-x^2}}$$

$$(b) \frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \dots \times \frac{du_{n-1}}{dx}$$

Exercise 7F (Page 261)

- 1(a) $2x^2(2x-3)$ (b) $4x-9$ (c) $4x^3$
 2(a) $3(3-2x)^4(1-4x)$, $1\frac{1}{2}$, $\frac{1}{4}$
 (b) $x^2(x+1)^3(7x+3)$, 0 , -1 , $-\frac{3}{7}$
 (c) $x^4(1-x)^6(5-12x)$, 0 , 1 , $\frac{5}{12}$
 (d) $(x-2)^2(4x-5)$, 2 , $\frac{5}{4}$
 (e) $2(x+1)^2(x+2)^3(7x+10)$, -1 , -2 , $-\frac{10}{7}$
 (f) $6(2x-3)^3(2x+3)^4(6x-1)$, $1\frac{1}{2}$, $-1\frac{1}{2}$, $\frac{1}{6}$
 3(a) $y=x$, $y=-x$ (b) $y=2x-1$, $x+2y=3$
 4(a) $(x^2+1)^4(11x^2+1)$ (b) $2\pi x^2(1-x^2)^3(3-11x^2)$
 (c) $-2(x^2+x+1)^2(7x^2+4x+1)$
 (d) $6x(3x^2-2)^3(3x^2+2)^4(27x^2-2)$
 5 $10x^3(x^2-10)^2(x^2-4)$, $(0,0)$, $(\sqrt{10},0)$,
 $(-\sqrt{10},0)$, $(2,-3456)$, $(-2,-3456)$
 6(a) $\frac{3(3x+2)}{\sqrt{x+1}}$, $-\frac{2}{3}$ (b) $\frac{4(3x-1)}{\sqrt{1-2x}}$, $\frac{1}{3}$
 (c) $\frac{10x(5x-2)}{\sqrt{2x-1}}$, 0 and $\frac{2}{5}$
 7(a) $-1 \leq x \leq 1$ (b) $\frac{1-2x^2}{\sqrt{1-x^2}}$
 (c) $\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{1}{2}}, -\frac{1}{2}\right)$
 (d) $y=x$, $y=-x$
 8(a) $y' = a(2x - \alpha - \beta)$ (b) $y'(\alpha) = a(\alpha - \beta)$
 $y'(\beta) = a(\beta - \alpha)$, $M = \left(\frac{1}{2}(\alpha + \beta), -\frac{1}{2}a(\alpha - \beta)^2\right)$
 (c) $V = \left(\frac{1}{2}(\alpha + \beta), -\frac{1}{4}a(\alpha - \beta)^2\right)$
 9 $f'(x) = (x-a)^{n-1}(nq(x) + (x-a)q'(x))$.
 The x -axis is a tangent to the curve at $x=a$.
 10 $y' = x^2(1-x)^4(3-8x)$
 12(a) $P = \left(\frac{r}{r+s}, \frac{r^r s^s}{(r+s)^{r+s}}\right)$.
 (b) When $r=s$, $P = \left(\frac{1}{2}, 2^{-2r}\right)$.
 13 $y' = u'vw + uv'w + uvw'$
 (a) $2x^4(x-1)^3(x-2)^2(3x-5)(2x-1)$,
 0 , 1 , 2 , $\frac{1}{2}$ and $\frac{5}{3}$
 (b) $\frac{(x-2)^3(11x^2-x-2)}{\sqrt{2x+1}}$,
 2 , $\frac{1}{22}(1+\sqrt{89})$, $\frac{1}{22}(1-\sqrt{89})$
 14 $y' = u_1'u_2 \dots u_n + u_1u_2' \dots u_n + \dots$
 $+u_1u_2 \dots u_n'$

Exercise 7G (Page 263)

- 1(a) $\frac{-2}{(x-1)^2}$, none (b) $\frac{4}{(x+2)^2}$, none
 (c) $\frac{-13}{(x+5)^2}$, none (d) $\frac{x(2-x)}{(1-x)^2}$, 0 , 2
 (e) $\frac{4x}{(x^2+1)^2}$, 0 (f) $\frac{m^2-b^2}{(bx+m)^2}$, none
 (g) $\frac{2x(a-b)}{(x^2-b)^2}$, 0 (provided $a \neq b$)
 (h) $\frac{6nx^{n-1}}{(x^n+3)^2}$, 0 (provided $n > 1$)
 2 $\frac{-3}{(3x-2)^2}$
 3 $\frac{20}{(5-2x)^2}$
 4(a) $y' = \frac{5}{(5-3x)^2}$, $y = 5x - 12$, $78^\circ 41'$,
 $x + 5y + 8 = 0$, $168^\circ 41'$
 (b) $y' = \frac{x^2-2x+4}{(x-1)^2}$, $4x-3y=4$, $53^\circ 8'$,
 $3x+4y=28$, $143^\circ 8'$
 5(a) $\frac{1}{2\sqrt{x}(\sqrt{x}+2)^2}$, none (b) $\frac{x+5}{2(x+1)^{\frac{3}{2}}}$, none
 ($x=-5$ is outside the domain.)
 6(a) $\frac{c^2+2c}{(c+1)^2} = -3$, $c = -\frac{1}{2}$ or $-1\frac{1}{2}$
 (b) $\frac{12k}{(9-k)^2} = 1$, $k = 3$ or 27
 7(a) $y' = \frac{\alpha - \beta}{(x - \beta)^2}$ (b) The denominator is positive, being a square, so the sign of y' is the sign of $\alpha - \beta$. (c) When $\alpha = \beta$, the curve is the horizontal line $y = 1$, and $y' = 0$ (except that y is undefined at $x = \beta$).
 8(a) $\frac{dy}{dx} = \frac{-(t+1)^2}{(t-1)^2}$, $T = (\frac{2}{3}, 2)$, $3x-27y+52=0$
 (b) $y = \frac{x}{2x-1}$, $\frac{dy}{dx} = \frac{-1}{(2x-1)^2}$, $\frac{1}{9}$
 9(a) $f'(x) = \frac{-\sqrt{2}}{\sqrt{x}(\sqrt{x}-\sqrt{2})^2}$, $f'(8) = -\frac{1}{4}$ (b) 3
 10(a) domain: $x \neq -1$, range: $y \neq 1$
 (c) $I = (-1, 0)$, $G = (1, 0)$ (d)(ii) Substitute $(c, 0)$, then $c+a^2=0$, so $a = \sqrt{-c}$ or $-\sqrt{-c}$. For $-1 < c < 0$, they are both on the right-hand branch. For $c < -1$, they are on different branches.
 12(b)(i) 54 , $\frac{3}{2}$, $9\sqrt{37}$, $\frac{3}{2}\sqrt{37}$ (ii) $\frac{1}{2}$, 8 , $\frac{1}{2}\sqrt{17}$, $2\sqrt{17}$

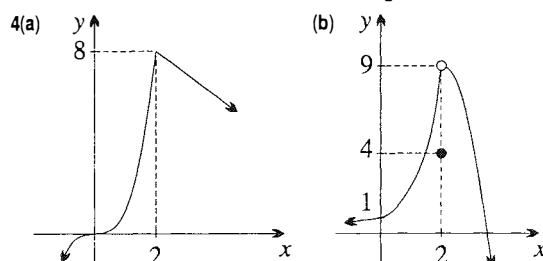
Exercise 7H (Page 266)

- 1 $\frac{dy}{dt} = (3x^2 + 1)\frac{dx}{dt}$ (a) 65 (b) $-\frac{3}{14}$
- 2(a) $240\pi \text{ cm}^2/\text{s}$ (b) $\frac{1}{12\pi} \text{ cm/s}$
- 3(a) $2/\pi \text{ cm/s}$ (b) $5/\pi \text{ cm}$ (c) $50/9\pi \text{ cm/s}$
- 4(a) $840 \text{ cm}^2/\text{s}$, $6\sqrt{2} \text{ cm/s}$
- (b) $1200 \text{ cm}^2/\text{s}$, $6\sqrt{2} \text{ cm/s}$
- 5(a)(i) $1350 \text{ cm}^3/\text{min}$, $180 \text{ cm}^2/\text{min}$, 6 cm/min
- (ii) $600 \text{ cm}^3/\text{min}$, $120 \text{ cm}^2/\text{min}$, 6 cm/min
- (b) $10\sqrt{2} \text{ cm}$
- 6(a) $\frac{dA}{dt} = \frac{1}{2}s\sqrt{3}\frac{ds}{dt}$, $\frac{dh}{dt} = \frac{1}{2}\sqrt{3}\frac{ds}{dt}$
- (b) $\frac{3}{5}\sqrt{3} \text{ cm}^2/\text{s}$, $\frac{3}{20}\sqrt{3} \text{ cm/s}$
- 7(a)(i) $\frac{1}{24\pi} \text{ cm/s}$ (ii) $1\frac{1}{2} \text{ cm}^3/\text{s}$ (b) $\frac{32000\pi}{3} \text{ cm}^3$
- 8(a) $\frac{1}{24} \text{ cm/min}$, $83\frac{1}{3} \text{ cm}^2/\text{min}$
- (b) 8 litre/min, $200 \text{ cm}^2/\text{min}$
- 9(a)(i) 40 m/s (ii) -160 m/s
- (iii) travelling horizontally
- (iv) 100 m/s or -100 m/s
- (b)(i) 1955 metres (ii) 1755 metres
- (c) $\frac{dy}{dx} = 20$, angle is about $87^\circ 8'$
- (d) 2 km high, 20 metres away
- 10 $\frac{dV}{dt} = -3\pi h^2 \frac{dh}{dt}$ (a)(i) $\frac{1}{160\pi} \text{ m/min}$
- (ii) $\frac{3}{160\pi} \text{ m/min}$ (b) $\frac{dA}{dt} = 6\pi h \frac{dh}{dt}$, $\frac{3}{20} \text{ m}^2/\text{min}$, $\frac{3}{20}\sqrt{3} \text{ m}^2/\text{min}$ (c) $\frac{96\pi}{25} \text{ m}^3/\text{min}$
- 11 $\frac{1}{125\pi} \text{ cm/s}$, $\frac{4}{5} \text{ cm}^2/\text{s}$, $\frac{4}{5}\sqrt{2} \text{ cm}^2/\text{s}$
- 12(b) $0.096 \text{ m}^3/\text{s}$, $\frac{3}{125}(\sqrt{17} + 1) \text{ m}^2/\text{s}$
- 13 $\frac{dy}{dt} = \frac{-x}{\sqrt{169 - x^2}} \frac{dx}{dt}$ (a) $\frac{5}{12} \text{ cm/s}$ (b) $1\frac{1}{5} \text{ cm/s}$
- 14(a) $\frac{dV}{dt} = \pi h(20 - h) \frac{dh}{dt}$, $\frac{1}{6\pi} \text{ cm/s}$
- (b) radius = $\sqrt{2hr - h^2}$, $\frac{8}{3} \text{ cm}^2/\text{s}$
- 15 $4\sqrt{6} \text{ cm}^2/\text{s}$
- 16 0.246 cm/s. Some exact forms are $\frac{3}{(27 + 9\sqrt{3})^{\frac{2}{3}}}$ and $\frac{1}{(3 + 3\sqrt{3})^{\frac{2}{3}}}$.

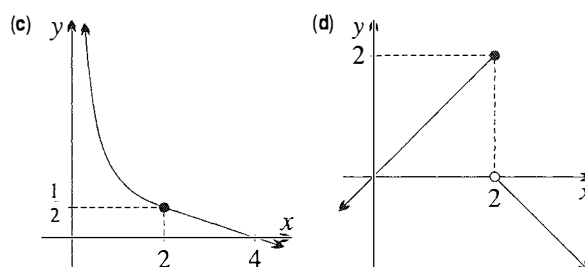
Exercise 7I (Page 271)

- 1(a)(i) $y = \frac{1 - 4/x + 3/x^2}{2 - 7/x + 6/x^2}$, $y \rightarrow \frac{1}{2}$
- (ii) $y = \frac{2/x - 5}{15 + 11/x}$, $y \rightarrow -\frac{1}{3}$
- (iii) $y = \frac{1/x + 1/x^2 + 1/x^3}{1 + 1/x + 1/x^2 + 1/x^3}$, $y \rightarrow 0$
- (iv) $y = \frac{x\sqrt{x} - 5/\sqrt{x}}{1/\sqrt{x} + 1}$, $y \rightarrow \infty$

- (b)(i) $y \rightarrow \frac{1}{2}$ (ii) $y \rightarrow -\frac{1}{3}$ (iii) $y \rightarrow 0$
- (iv) y is undefined for $x < 0$.
- 2(a) 4 (b) 27 (c) 1 (d) -6 (e) -2 (f) 0
- 3(a) $y \rightarrow \frac{1}{2}$ (b) $y \rightarrow \frac{7}{5}$ (c) $y \rightarrow -4$
- (d) $y \rightarrow \infty$ as $x \rightarrow -\frac{1}{2}^-$, $y \rightarrow -\infty$ as $x \rightarrow -\frac{1}{2}^+$
- (e) $y \rightarrow 0$ (f) $y \rightarrow -1$ (g) $y \rightarrow \frac{1}{2}$ (h) $y \rightarrow 5$

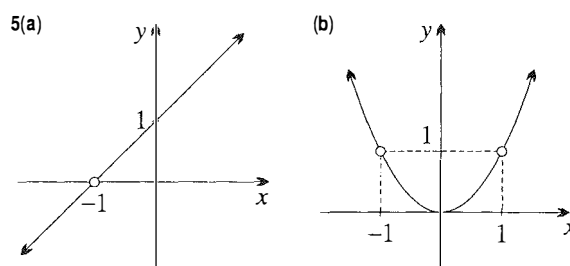


- (a) $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2) = 8$, continuous at $x=2$, domain = $\mathbf{R}$, range: $y \leq 8$
- (b) $\lim_{x \rightarrow 2^-} f(x) = 9$, $\lim_{x \rightarrow 2^+} f(x) = 9$, $f(2) = 4$, not continuous at $x=2$, domain = $\mathbf{R}$, range: $y < 9$

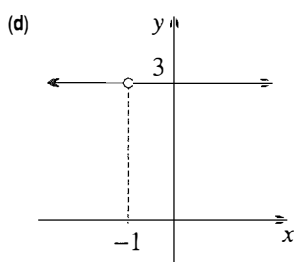
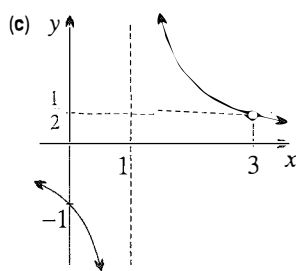


- $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = \frac{1}{2}$, $f(2) = \frac{1}{2}$, continuous at $x=2$, domain: $x > 0$, range = $\mathbf{R}$

- $\lim_{x \rightarrow 2^-} f(x) = f(2) = 2$, $\lim_{x \rightarrow 2^+} f(x) = 0$, not continuous at $x=2$, domain = $\mathbf{R}$, range: $y \leq 2$



- (a) $y = x + 1$ where $x \neq -1$, domain: $x \neq -1$, range: $y \neq 0$
- (b) $y = x^2$ where $x \neq -1$ or 1, domain: $x \neq -1$ or 1, range: $y \geq 0$, $y \neq 1$



(c) $y = \frac{1}{x-1}$ where $x \neq 3$,

domain: $x \neq 1$ or 3 , range: $y \neq 0$ or $\frac{1}{2}$

(d) $y = 3$ where $x \neq -1$,

domain: $x \neq -1$, range: $y = 3$

6(a) (gradient of PQ) $= 2x + h - 1 \rightarrow 2x - 1$ as $h \rightarrow 0$ (b) (gradient of PQ) $= u^3 + u^2x + ux^2 + x^3 - 3 \rightarrow 4x^3 - 3$ as $u \rightarrow x$

7(a)(i) $2c^2$ (ii) $\frac{4}{3}c$ (iii) $\frac{5}{3}c^2$

(b) $\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \dots + a^{n-1}$,

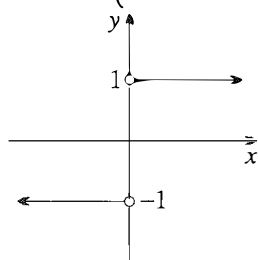
$\lim_{x \rightarrow a} = na^{n-1}$

(c) $\frac{u^{2n+1} + 2^{2n+1}}{u + 2} = u^{2n} - u^{2n-1}2 + u^{2n-2}2^2 - \dots + 2^{2n}$, $\lim_{u \rightarrow -2} = (2n+1)2^{2n}$

8(a) $a = 5$ (b) $a = -2$

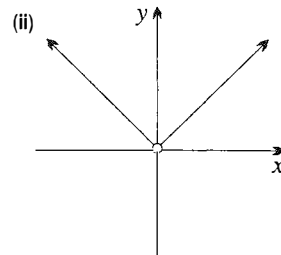
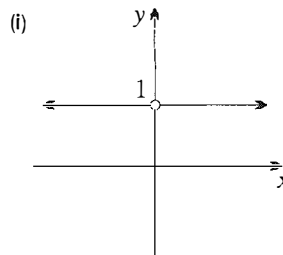
9(a) zeroes: 0, discontinuities: 3 (b) zeroes: 0, discontinuities: 7 and -1 (c) zeroes: none, discontinuities: $180n^\circ$, where $n \in \mathbf{Z}$ (d) zeroes: none, discontinuities: $360n^\circ$, where $n \in \mathbf{Z}$ (e) zeroes: $180n^\circ$, where $n \in \mathbf{Z}$, discontinuities: $90^\circ + 180n^\circ$, where $n \in \mathbf{Z}$ (f) zeroes: 1 and -1 , discontinuities: 0, 3 and -3

10(a) $y = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0, \\ \text{undefined,} & \text{for } x = 0. \end{cases}$



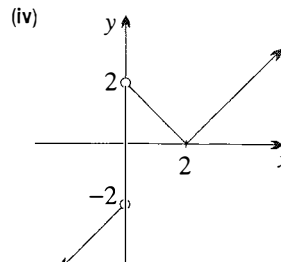
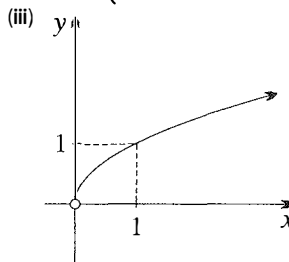
(b)(i) $y = \begin{cases} 1, & \text{for } x \neq 0, \\ \text{undefined,} & \text{for } x = 0. \end{cases}$

(ii) $y = \begin{cases} |x|, & \text{for } x \neq 0, \\ \text{undefined,} & \text{for } x = 0. \end{cases}$



(iii) $y = \begin{cases} \sqrt{x}, & \text{for } x > 0, \\ \text{undefined,} & \text{for } x \leq 0. \end{cases}$

(iv) $y = \begin{cases} x-2, & \text{for } x \geq 2, \\ 2-x, & \text{for } 0 < x < 2, \\ \text{undefined,} & \text{for } x = 0, \\ x-2, & \text{for } x < 0. \end{cases}$



12(a) They are all $\frac{1}{2\sqrt{x}}$. (b) All are examples of differentiation by first principles (using the definition of the derivative as a limit).

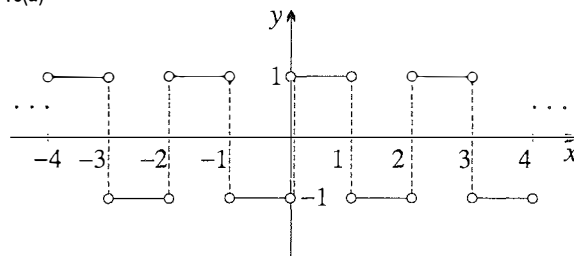
13 All except (b) are continuous in the closed interval $-1 \leq x \leq 1$.

14(a) zeroes: $135^\circ + 180n^\circ$, where $n \in \mathbf{Z}$, discontinuities: $45^\circ + 180n^\circ$, where $n \in \mathbf{Z}$

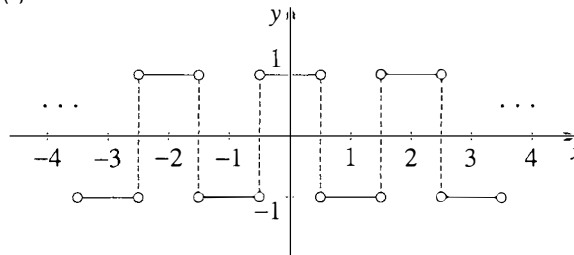
(b) zeroes: $45^\circ + 180n^\circ$, where $n \in \mathbf{Z}$, discontinuities: $135^\circ + 180n^\circ$, where $n \in \mathbf{Z}$

15(a) $\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $-\frac{1}{9}$ (d) $-\frac{1}{250}$

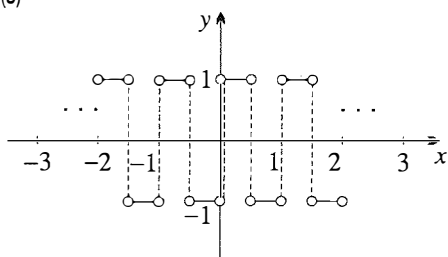
16(a)



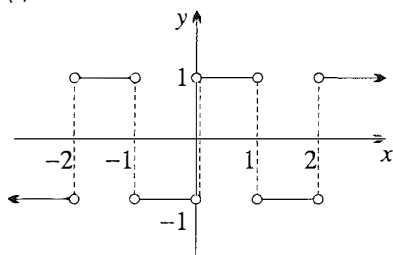
(b)



(c)

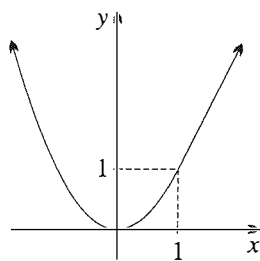
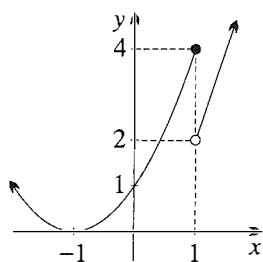
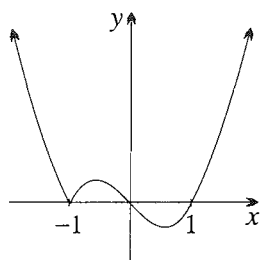
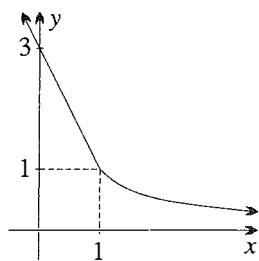
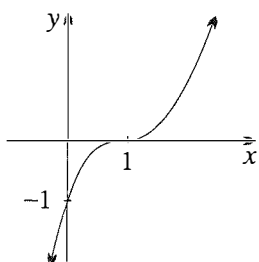
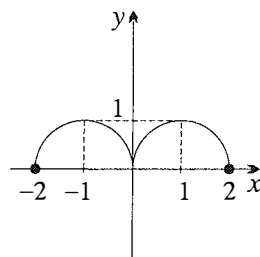
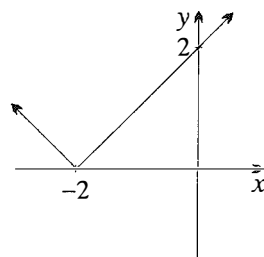
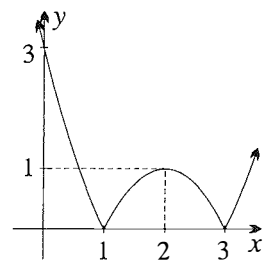
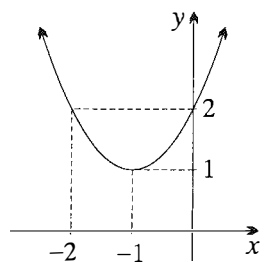
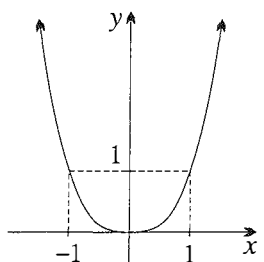
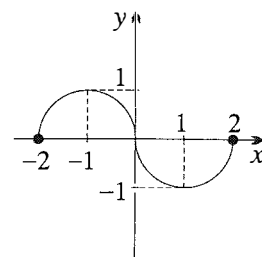
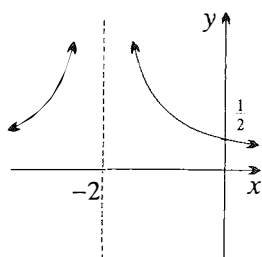
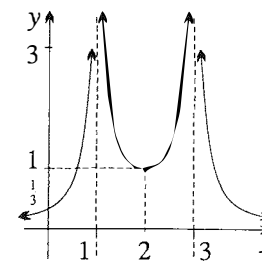
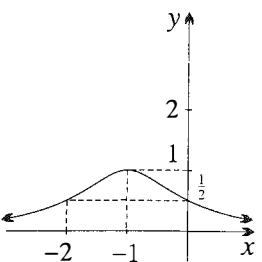
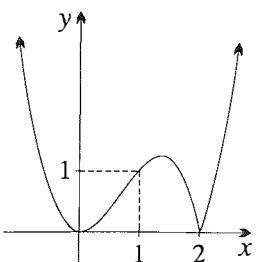


(d)

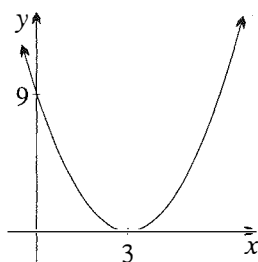


(e) same as (a)

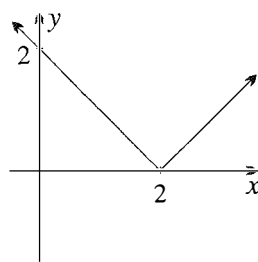
Exercise 7J (Page 275)

 1(a) differentiable
at $x = 1$

 (c) not continuous
at $x = 1$

 2 continuous but not differentiable at $x = -1$

 (b) continuous but not
differentiable at $x = 1$

 (d) differentiable
at $x = 1$

 3(a) cusp at $x = 0$

 4(a) not differentiable
at $x = -2$

 (c) not differentiable
at $x = 1$ or 3

 (e) differentiable
everywhere

 (g) differentiable
everywhere

 (b) vertical tangent
at $x = 0$

 (b) not continuous
at $x = -2$

 (d) not continuous
at $x = 1$ or 3

 (f) differentiable
everywhere

 (h) continuous but not
differentiable at $x = 2$


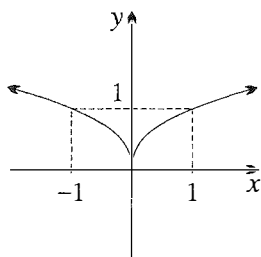
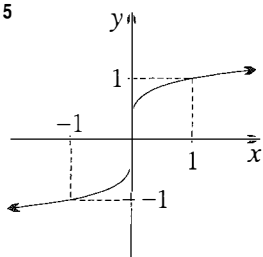
(i) differentiable everywhere



(j) continuous but not differentiable at $x = 2$



5



(a) $f'(x) = \frac{1}{5}x^{-\frac{4}{5}}$, $f'(x) \rightarrow \infty$ as $x \rightarrow 0^-$ and as $x \rightarrow 0^+$, vertical tangent at $(0, 0)$.

(b) $f'(x) = \frac{2}{5}x^{-\frac{3}{5}}$, $f'(x) \rightarrow -\infty$ as $x \rightarrow 0^-$ and $f'(x) \rightarrow \infty$ as $x \rightarrow 0^+$, cusp at $(0, 0)$.

6(a) 3 , $(4\frac{1}{2}, -6\frac{3}{4})$ (b) 3 , $(1, -8)$

(c) 1 , $(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3})$, $(-\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3})$

(d) $\frac{1}{3}$, $(\frac{9}{4}, \frac{3}{2})$ (e) $-\frac{1}{2}$, $(\sqrt{2}, 1/\sqrt{2})$ (f) 1. There are none, because all the tangents have negative gradients. (g) 0. There are none, because the tangents have gradient 1 for $x > 0$ and gradient -1 for $x < 0$.

(h) 0, $(0, 0)$ (i) $\alpha + \beta$, $(\frac{1}{2}(\alpha + \beta), \frac{1}{4}(\alpha + \beta)^2)$

(j) $-1/\alpha\beta$, $(\sqrt{\alpha\beta}, 1/\sqrt{\alpha\beta})$ when α and β are positive, $(-\sqrt{\alpha\beta}, -1/\sqrt{\alpha\beta})$ when α and β are negative, impossible when α and β have opposite signs.

7 The tangent is $y = \frac{4}{3}ka^{\frac{1}{3}}x - \frac{1}{3}ka^{\frac{4}{3}}$, $A = (\frac{1}{4}a, 0)$, $B = (0, -\frac{1}{3}ka^{\frac{4}{3}})$, $G = (a, 0)$, $H = (0, ka^{\frac{4}{3}})$, $|OGPH| : |\triangle OAB| = 24 : 1$.

8 The tangent is $y = nka^{n-1}x - (n-1)ka^n$, $A = (\frac{(n-1)a}{n}, 0)$, $B = (0, -(n-1)ka^n)$, $G = (a, 0)$ and $H = (0, ka^n)$.

$|OGPH| : |\triangle OAB| = (n-1)^2 : |2n|$, and the rectangle is bigger when $2 - \sqrt{3} < n < 2 + \sqrt{3}$. B is on the other side from H for $n > 1$ and on the same side for $n < 1$, so for $a > 0$, B is above the origin if and only if $n < 1$.

The number a can be negative provided n is a

rational number p/q with odd denominator q , in which case (taking q positive) B is above the origin for $n < 1$ and p even, or for $n > 1$ and p odd. 9(a) q must be odd. (b) $p \geq 0$ (When $p = 0$ it is reasonable to take $f(0) = 1$ and ignore the problem of 0^0 , because $\lim_{x \rightarrow 0} x^0 = 1$; thus when $p = 0$ the function is $y = 1$.) (c) no conditions on p and q (d) $p \geq 0$ and q is odd. (e) $p \geq 0$ ($p = 0$ requires the qualification above.) (f) $p \geq q$ and q is odd. (g) $0 < p < q$ and q is odd and p is odd. (h) $0 < p < q$ and q is odd and p is even.

Exercise 7K (Page 277)

1(a) $4y^3y'$ (b) $y + xy'$ (c) $-1 + y' - y - xy'$

(d) $6x + 8yy'$ (e) $y(3x^2 + y^2) + xy'(x^2 + 3y^2)$

(f) $\frac{y - xy'}{y^2}$ (g) $3(x + y)^2(1 + y')$ (h) $\frac{2(xy' - y)}{(x - y)^2}$

(i) $\frac{1 + y'}{2\sqrt{x + y}}$ (j) $\frac{x + yy'}{\sqrt{x^2 + y^2}}$

2(a) $y' = -\frac{x}{y}$ (b) $y' = -\frac{3x}{2y}$ (c) $y' = \frac{x}{y}$

(d) $y' = -\frac{2x + 3y}{3x + 4y}$ (e) $y' = \frac{(x - y)^2 + 2x^2}{(x - y)^2 + 2y^2}$

(f) $y' = -\frac{2y}{3x}$ (g) $y' = -\frac{rax^{r-1}}{sby^{s-1}}$ (h) $y' = -\frac{\sqrt{y}}{\sqrt{x}}$

(i) $y' = \frac{y}{x}$

3(a) $y' = -\frac{x}{y}$, tangent: $5x - 12y + 169 = 0$,

normal: $12x + 5y = 0$. The normal to a circle at any point is a radius, and so must pass through the centre.

(b) $A = (-\frac{169}{5}, 0)$, $B = (0, \frac{169}{12})$

(c) $|\triangle AOB| = \frac{169^2}{120}$ (ii) $AB = \frac{13^3}{60}$

4(a) $y' = -\frac{y}{x}$, tangent: $3x + 2y = 12$

normal: $2x - 3y + 5 = 0$ (b) $P(2, 3)$ is the midpoint of the interval joining $(4, 0)$ and $(0, 6)$.

5(a) $y' = \frac{1}{2y}$, tangent: $x - 6y + 9 = 0$,

normal: $6x + y = 57$ (b) $(0, 1\frac{1}{2})$ is the midpoint of the interval joining $P(9, 3)$ and $(-9, 0)$.

6(a) $\frac{dy}{dx} = \frac{t^2 + 1}{t^2 - 1}$, tangent: $5x - 3y = 8$,

normal: $3x + 5y = 15$ (b) $x^2 - y^2 = 4$

7 $x \frac{dx}{dt} + y \frac{dy}{dt} = 0$

(a) The top is slipping down at $\frac{2}{15}\sqrt{15}$ cm/s.

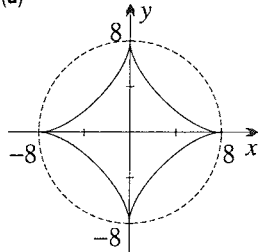
(b) The bottom is slipping out at $\frac{14}{15}\sqrt{15}$ mm/s.

8(a) $S^2 \frac{dS}{dt} = 24\pi V \frac{dV}{dt}$ or $r \frac{dS}{dt} = 2 \frac{dV}{dt}$

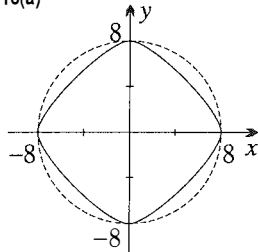
(b) $10 \text{ cm}^3/\text{s}$

9(a) The symmetries arise because the equation is unchanged when x is replaced by $-x$, or y by $-y$, or x and y are exchanged. (b) Neither $x^{\frac{2}{3}}$ nor $y^{\frac{2}{3}}$ can be negative. (c) As $x \rightarrow 8^-$, either $y \rightarrow 0^+$ and $y' \rightarrow 0^-$, or $y \rightarrow 0^-$ and $y' \rightarrow 0^+$.

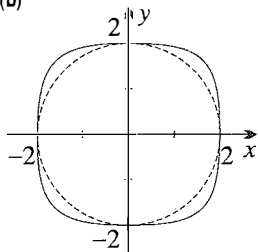
(d)



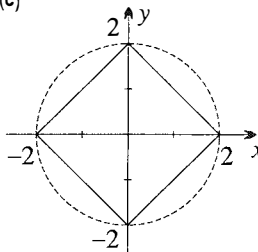
10(a)



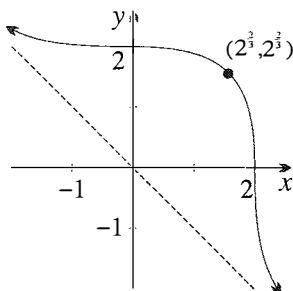
(b)



(c)

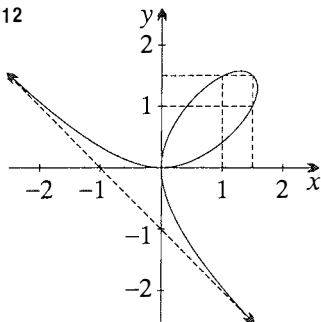


11



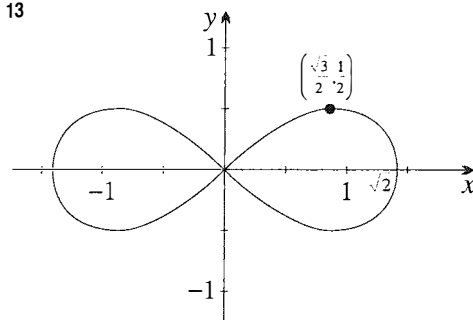
(a) The tangent at $(2, 0)$ is $x = 2$, the tangent at $(0, 2)$ is $y = 2$, and the tangent at $(2^{\frac{2}{3}}, 2^{\frac{2}{3}})$ is $x + y = 2^{\frac{5}{3}}$.

12



$(y^2 - x)y' = y - x^2$, the tangent at $(1^{\frac{1}{2}}, 1^{\frac{1}{2}})$ is $x + y = 3$, the tangent at $(2^{\frac{1}{3}}, 2^{\frac{2}{3}})$ is horizontal, the tangent at $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ is vertical.

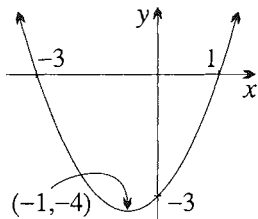
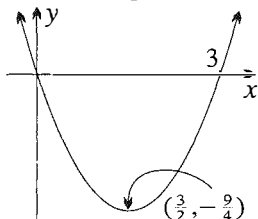
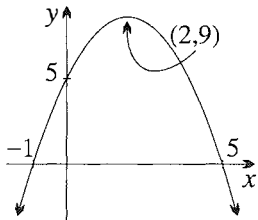
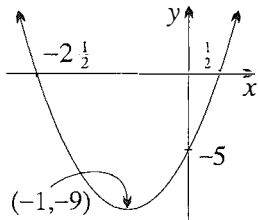
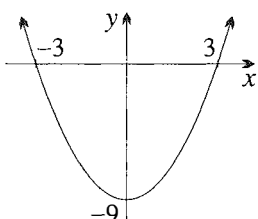
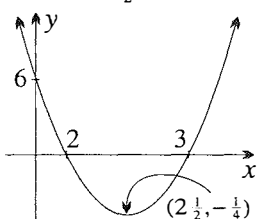
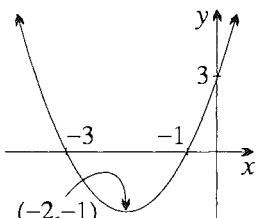
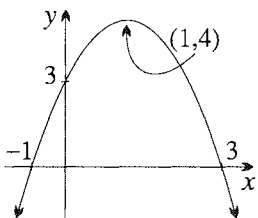
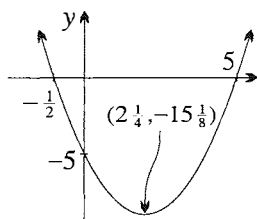
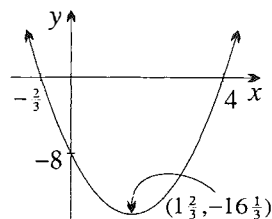
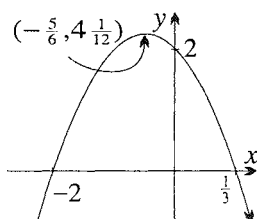
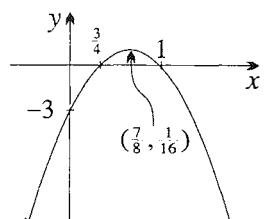
13



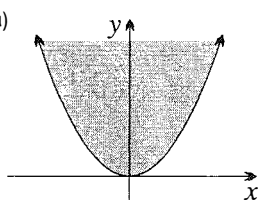
$(x^2 + y^2)(x + yy') = x - yy'$, the tangents at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are vertical. the tangents at $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$, $(\frac{1}{2}\sqrt{3}, -\frac{1}{2})$, $(-\frac{1}{2}\sqrt{3}, \frac{1}{2})$ and $(-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ are horizontal.

Chapter Eight

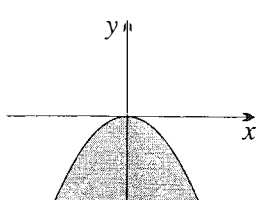
Exercise 8A (Page 283)

1(a) axis: $x = -1$ (b) axis: $x = 1\frac{1}{2}$ (c) axis: $x = 2$ (d) axis: $x = -1$ (e) axis: $x = 0$ (f) axis: $x = 2\frac{1}{2}$ (g) axis: $x = -2$ (h) axis: $x = 1$ 2(a) $x < -3$ or $x > 1$ (b) $0 \leq x \leq 3$ (c) $-1 \leq x \leq 5$ (d) $-\frac{5}{2} < x < \frac{1}{2}$ (e) $x \leq -3$ or $x \geq 3$ (f) $2 < x < 3$ (g) $-3 \leq x \leq -1$ (h) $-1 < x < 3$ 3(a) any positive multiple of $y = (x-3)(x-5)$ (b) any positive multiple of $y = x(x+4)$ (c) any positive multiple of $y = -(x+1)(x-3)$ (d) any positive multiple of $y = -x(x-2)$ 4(a) $x = 5$, $y = (x-4)(x-6)$ (b) $x = 5\frac{1}{2}$, $y = (x-3)(x-8)$ (c) $x = 1$, $y = (x+3)(x-5)$ (d) $x = -3\frac{1}{2}$, $y = (x+6)(x+1)$ 5(a) $x = 2\frac{1}{4}$ (b) $x = \frac{5}{3}$ (c) $x = -\frac{5}{6}$ (d) $x = \frac{7}{8}$ 6(a) $y = -(x-2)(x-8)$ (b) $y = -2(x-2)(x-8)$ (c) $y = -\frac{3}{16}(x-2)(x-8)$ (d) $y = 3(x-2)(x-8)$ (e) $y = \frac{4}{3}(x-2)(x-8)$ (f) $y = -\frac{20}{7}(x-2)(x-8)$ 7 $y = (x-\alpha)(x-1)$ (a) $y = x(x-1)$ (b) $y = (x-1)^2$ (c) $y = (x+15)(x-1)$ (d) $y = \frac{1}{2}(2x+3)(x-1)$ 8(a)(i) $y \geq -1$ (ii) $y \geq 3$ (iii) $-1 \leq y \leq 8$ (b)(i) $y \geq -9$ (ii) $y \geq -9$ (iii) $-8 \leq y \leq 27$ (c)(i) $y \leq 1$ (ii) $y \leq -8$ (iii) $-8 \leq y \leq 1$

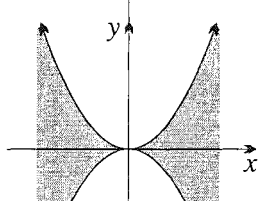
9(a)



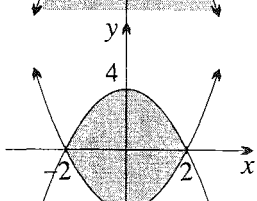
(b)



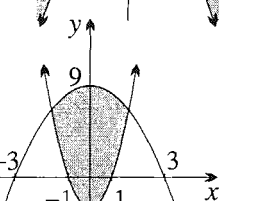
(c)



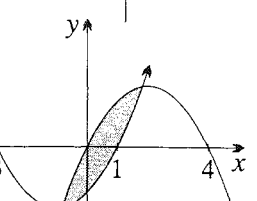
(d)



(e)



(f)

10(a) $y = x(x+3)$ (b) $y = -4x(x-2)$ (c) $y = \frac{3}{2}x^2$ (d) $y = -\frac{1}{4}x^2$ (e) $y = -\frac{2}{5}x(x-5)$ (f) $y = -2x(x+6)$

11(a) $a = \frac{c}{\alpha\beta}$ (b) $a = -\frac{b}{\alpha + \beta}$

(c) $a = \frac{2}{(1-\alpha)(1-\beta)}$

12(a) $y = (x+1)(x-2)$ (b) $y = -(x+3)(x-2)$

(c) $y = 3(x+2)(x-4)$ (d) $y = -\frac{1}{2}(x-2)(x+2)$

13(a) $-b, -c, x = -\frac{1}{2}(b+c)$

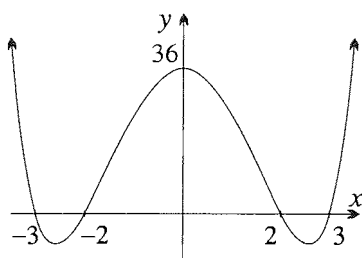
(b) $-1, a^2, x = \frac{1}{2}(a^2 - 1)$ (c) $-1, \frac{a+b}{a}, x = \frac{b}{a^2}$

(d) $c+1, c-1, x = c$

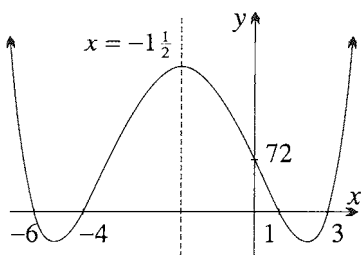
 14(a)(ii) $f'(x) = 2(x-3)$. The graph is tangent to the x -axis at $x = 3$.

 (b) The graph is tangent to the x -axis at $x = q$.

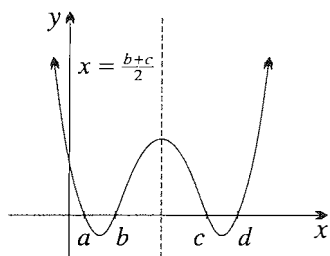
16(a)



17(d)

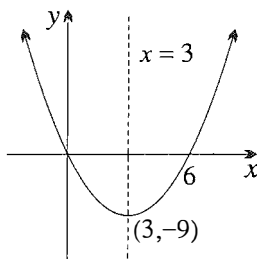


(e)

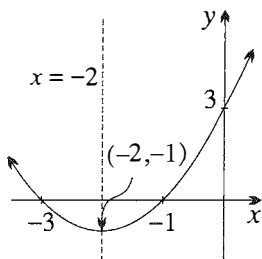


Exercise 8B (Page 287)

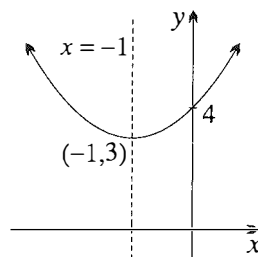
1(a)



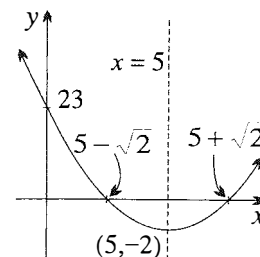
(b)



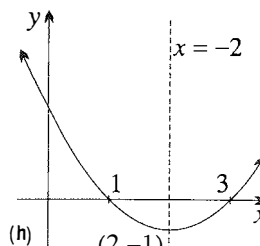
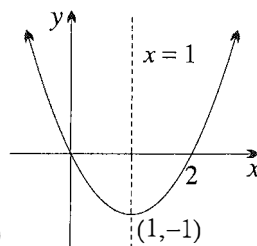
(c)



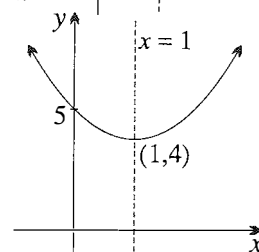
(d)



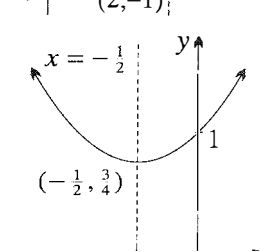
(e)



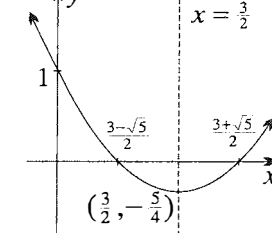
(g)



(h)



(i)


 2(a) $y = a(x-1)^2 + 2$ for any $a > 0$. ($a = 1$ is okay.)

 (b) $y = a(x+2)^2 - 3$. ($a = 1$ is okay — a must be larger than $\frac{3}{4}$.) (c) $y = -a(x-2)^2 - 1$ for any $a > 0$. ($a = 1$ is okay.)

 (d) $y = -a(x-3)^2 + 5$. ($a = 1$ is okay — a must be larger than $\frac{5}{9}$.)

 3(a) $y = (x-2)^2 + 5, x = 2, (0, 9)$ (b) $y = x^2 - 3, x = 0, (0, -3)$ (c) $y = (x+1)^2 + 7, x = -1, (0, 8)$

 (d) $y = (x-3)^2 - 11, x = 3, (0, -2)$

 4 The graph of $y = ax^2 + 1$ is the graph of $y = ax^2$ shifted up one unit. Put $h = 0$ and $k = 1$ in the formula $y = a(x-h)^2 + k$. (a) $y = 2x^2 + 1$

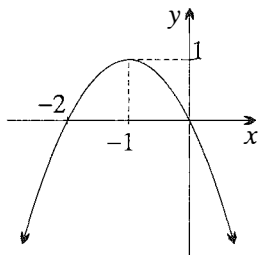
 (b) $y = -3x^2 + 1$ (c) $y = \frac{1}{3}x^2 + 1$ (d) $y = -\frac{1}{4}x^2 + 1$

 5 Put $h = -4$ and $k = 2$ in the formula $y = a(x-h)^2 + k$. (a) $y = (x+4)^2 + 2$ (b) $y = 3(x+4)^2 + 2$

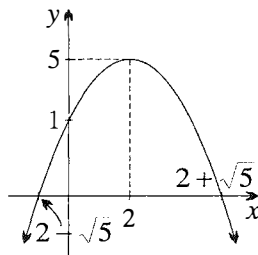
 (c) $y = -\frac{2}{49}(x+4)^2 + 2$ (d) $y = \frac{7}{8}(x+4)^2 + 2$

 (e) $y = -\frac{1}{8}(x+4)^2 + 2$ (f) $y = \frac{18}{25}(x+4)^2 + 2$

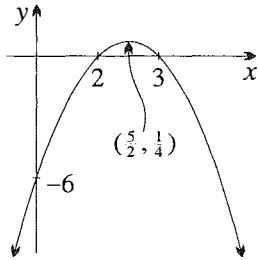
6(a) $y = -(x+1)^2 + 1$



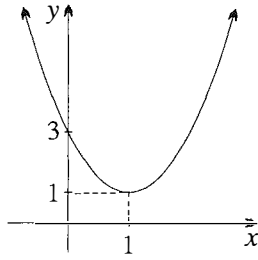
6(b) $y = -(x-2)^2 + 5$



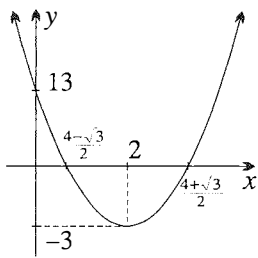
6(c) $y = -(x - 2\frac{1}{2})^2 + \frac{1}{4}$



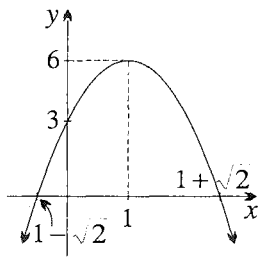
6(d) $y = 2(x-1)^2 + 1$



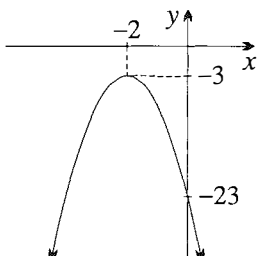
6(e) $y = 4(x-2)^2 - 3$



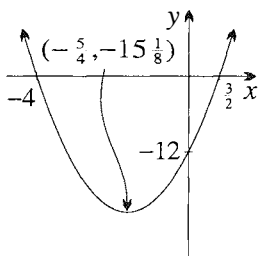
6(f) $y = -3(x-1)^2 + 6$



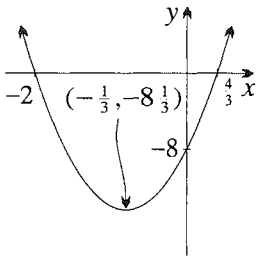
6(g) $y = -5(x+2)^2 - 3$



6(h) $y = 2(x + \frac{5}{4})^2 - 15\frac{1}{8}$



6(i) $y = 3(x + \frac{1}{3})^2 - 8\frac{1}{3}$



7(a) $y \geq 2, y \geq 6, 2 \leq y \leq 6$ (b) $y \geq 1, y \geq 33, 3 \leq y \leq 33$ (c) $y \leq 5, y \leq 5, -11 \leq y \leq 4$

8 $y = (x-3)^2 + c - 9$ (a) $c = 9$ (b) $c < 9$ (c) $c > 9$

9 $y = (x+2)^2 + k$ (a) $y = (x+2)^2 - 4$

(b) $y = (x+2)^2 - 48$ (c) $y = (x+2)^2 - 9$

(d) $y = (x+2)^2 - 10$ (e) $y = (x+2)^2 - 2$

(f) $y = (x+2)^2 + 7$ (g) $y = (x+2)^2 - 3$

(h) $y = (x+2)^2 - 2$

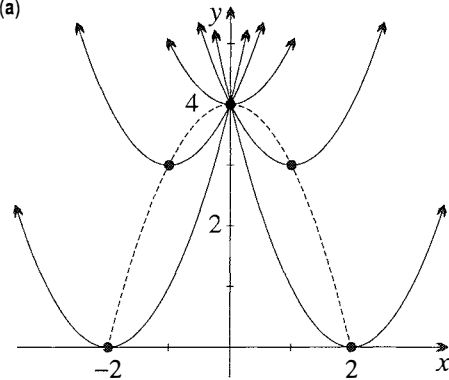
10(a) $y = 2(x-1)^2 + 1$ (b) $y = -(x-3)^2 + 2$

(c) $y = \frac{1}{2}(x+2)^2 - 4$ (d) $y = -3(x+1)^2 + 4$

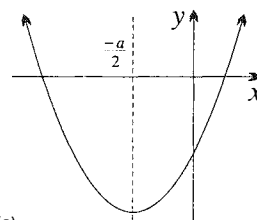
11(a) $-2, 3$ (b) $2 + \sqrt{3}, 2 - \sqrt{3}$ (c) $-\frac{5}{2}, 1$

(d) $\frac{3}{2} + \frac{1}{10}\sqrt{5}, \frac{3}{2} - \frac{1}{10}\sqrt{5}$

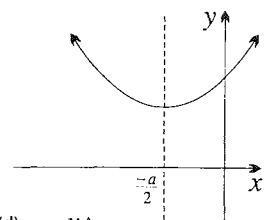
12(a)

(b) The vertex moves on the parabola $y = 4 - x^2$.

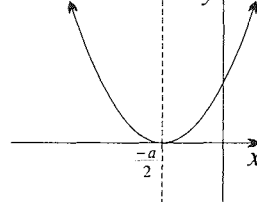
13(a)



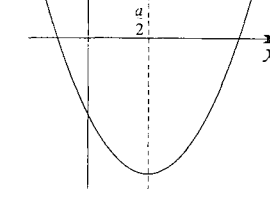
(b)



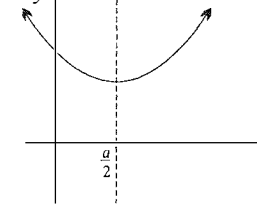
(c)



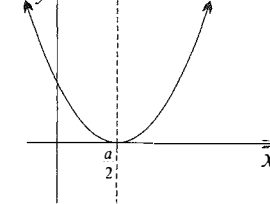
(d)



(e)



(f)



14 vertex $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$,

zeroes $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, y-intercept c

15 $y = a\left(x - \frac{1}{2}(\alpha + \beta)\right)^2 - \frac{1}{4}a(\alpha - \beta)^2$,

vertex $\left(\frac{1}{2}(\alpha + \beta), -\frac{1}{4}a(\alpha - \beta)^2\right)$

16 Tangents drawn at points equidistant from the axis of symmetry have opposite gradients.

17 $y = a(x - h)^2 + k$ (a) $a = \frac{c - k}{h^2}$

(b) $a = \frac{2 - k}{(1 - h)^2}$ (c) $a = -\frac{b}{2h}$

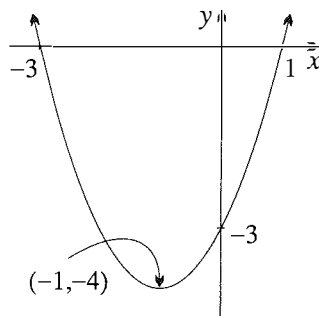
(d) $a = -\frac{k}{(\alpha - h)^2}$

18(a) $-d + \sqrt{e}$, $-d - \sqrt{e}$ (b) $2\sqrt{e}$

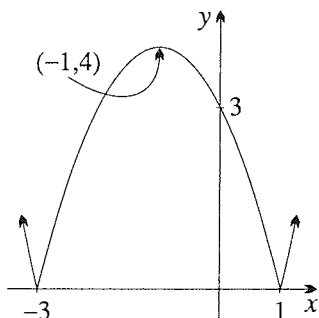
(c) $e = 1$. They have vertex on the line $y = -1$.

19 $h_1 = h_2$, but $k_1 \neq k_2$. The two curves have the same axis of symmetry, but different vertices.

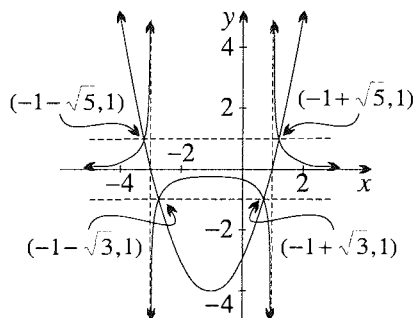
20(a)



(b)



(d)



Exercise 8C (Page 291)

1(a) -1 or -5 (b) -2 (c) -4 or 6

(d) $1 + \sqrt{2}$ or $1 - \sqrt{2}$, 2.414 or -0.4142

(e) $-2 + \sqrt{5}$ or $-2 - \sqrt{5}$, 0.2361 or -4.236

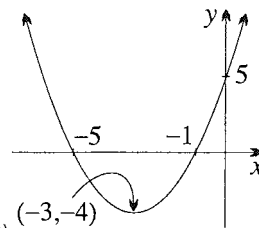
(f) $\frac{1}{2}(-1 + \sqrt{3})$ or $\frac{1}{2}(-1 - \sqrt{3})$, -1.366 or 0.3660

(g) $\frac{1}{10}(7 + \sqrt{109})$ or $\frac{1}{10}(7 - \sqrt{109})$, 0.3440 or 1.744

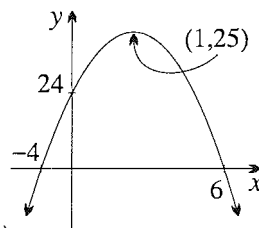
(h) $\frac{1}{8}(3 + \sqrt{57})$ or $\frac{1}{8}(3 - \sqrt{57})$, 1.319 or -0.5687

(i) $\frac{3}{2}$ or $-\frac{3}{2}$

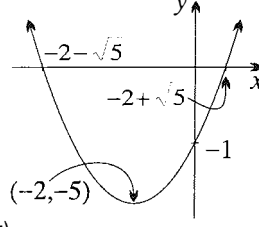
2(a)



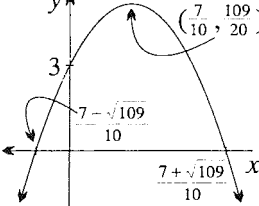
(c)



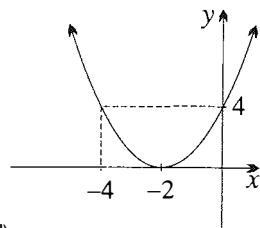
(e)



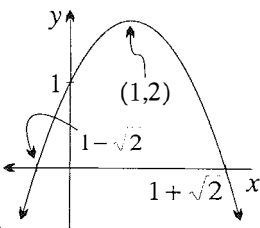
(g)



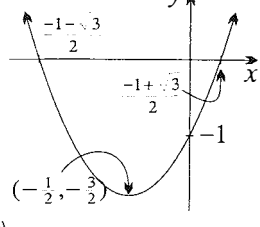
(b)



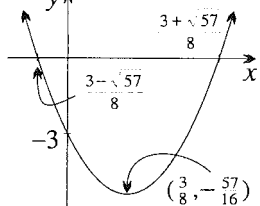
(d)



(f)



(h)



3(a) $-5 < x < -1$ (b) $x \neq -2$

(c) $x \leq -4$ or $x \geq 6$ (d) $1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}$

5(a) 16, twice (b) 5, twice (c) 0, once

(d) -31 , no times

6(a) 4 , $-\frac{2}{3}$ (b) $1 + \sqrt{5}$, $1 - \sqrt{5}$ (c) $3 + 2\sqrt{2}$, $3 - 2\sqrt{2}$

(d) 2 , $-\frac{1}{3}$

7(a) $y = (x - 3 + \sqrt{5})(x - 3 - \sqrt{5})$

(b) $y = 3(x + 1 + \frac{1}{3}\sqrt{3})(x + 1 - \frac{1}{3}\sqrt{3})$

(c) $y = -(x - \frac{3}{2} - \frac{1}{2}\sqrt{13})(x - \frac{3}{2} + \frac{1}{2}\sqrt{13})$

(d) $y = -2(x + 1)(x - \frac{1}{2})$

8(a) $(0, 3)$ and $(5, 8)$. The line and parabola intersect twice. (b) The line intersects the parabola once at $(-1, -9)$, and so it is a tangent to the parabola. (c) The line and the parabola do not intersect. (d) $(1, 2)$ and $(2, 1)$. The line and the parabola intersect twice.

9 $\frac{1}{2}p(-1 + \sqrt{5})$

10 $f'(x) = 2ax + b$. The axis of symmetry of a quadratic can be found by solving $f'(x) = 0$.

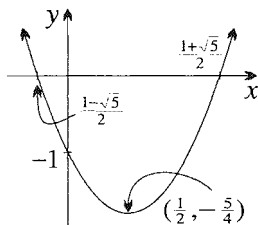
11(a) $x = h + \sqrt{-k}$ or $h - \sqrt{-k}$

12(a) $x = -\frac{1}{2}b$, vertex $(-\frac{1}{2}b, \frac{1}{4}(4c - b^2))$

(b) Difference between zeroes is $\sqrt{b^2 - 4c}$.

(c) $b^2 - 4c = 1$

13



Exercise 8D (Page 293)

1(a) 3, -3, 1 or -1 (b) 2, -2, 5 or -5

(c) $\sqrt{2}$, $-\sqrt{2}$, $\frac{2}{3}\sqrt{3}$ or $-\frac{2}{3}\sqrt{3}$ (d) 1 or 2

(e) 1 or 3 (f) $\frac{1}{4}$, $-\frac{1}{4}$, 4 or -4 (g) 3, -3, 4 or -2

(h) $2 + 2\sqrt{2}$ or $2 - 2\sqrt{2}$ (i) 1 or 2 (j) 2 or 3

2(a) 30° , 90° or 150° (b) 120° , 180° or 240°

(c) 135° or 315° (d) 30° , 150° or 270°

3(a) (1, 3) and $(\frac{9}{5}, \frac{13}{5})$ (b) (2, -1)

(c) (-2, 3) and $(\frac{100}{13}, -\frac{45}{13})$

4(a) $\frac{1}{2}$, 2, $\frac{1}{2}(-3 + \sqrt{5})$ or $\frac{1}{2}(-3 - \sqrt{5})$

(b) $\frac{1}{2}(-3 + \sqrt{29})$ or $\frac{1}{2}(-3 - \sqrt{29})$

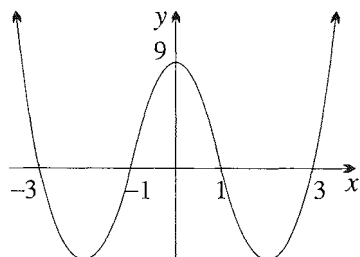
(c) $\frac{1}{2}(-5 + \sqrt{13})$, $\frac{1}{2}(-5 - \sqrt{13})$, $\frac{1}{2}(-5 + \sqrt{17})$ or $\frac{1}{2}(-5 - \sqrt{17})$

5(a) 1 (b) 3 (c) 4 (d) 11

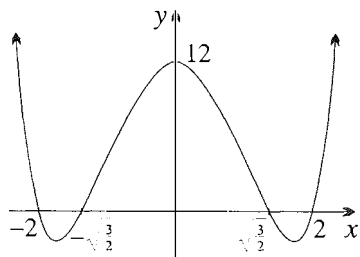
6(a) -1 or 0 (b) $5^{-\frac{3}{2}}$ or 125

7(a) $x = \frac{pr}{p+q}$, $y = \frac{qr}{p+q}$ (b) $x = \frac{6}{7}$, $y = \frac{8}{7}$

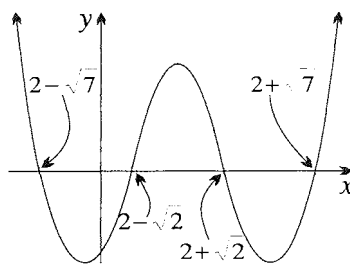
8(a)(i)



(ii)



(iii)



(b)(i) $x \leq -3$ or $-1 \leq x \leq 1$ or $x \geq 3$

(ii) $-2 \leq x \leq -\frac{1}{2}\sqrt{6}$ or $\frac{1}{2}\sqrt{6} \leq x \leq 2$

(iii) $x < 2 - \sqrt{7}$, $2 - \sqrt{2} < x < 2 + \sqrt{2}$

or $x > 2 + \sqrt{7}$

9(a) $1, \frac{1}{2}(3 + \sqrt{5})$ or $\frac{1}{2}(3 - \sqrt{5})$ (b)(i) $1, \frac{1}{2}(-5 + \sqrt{21})$

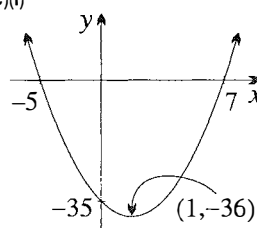
or $\frac{1}{2}(-5 - \sqrt{21})$ (ii) $\frac{1}{6}(7 + \sqrt{13})$ or $\frac{1}{6}(7 - \sqrt{13})$

Exercise 8E (Page 296)

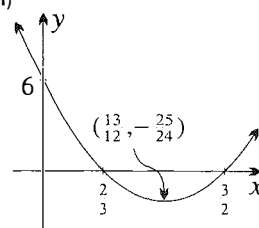
1(a)(i) -4 (ii) -9 (iii) $-\frac{1}{4}$ (iv) $-\frac{49}{8}$ (b)(i) 4 (ii) 3

(iii) -2 (iv) $\frac{11}{4}$

(c)(i)



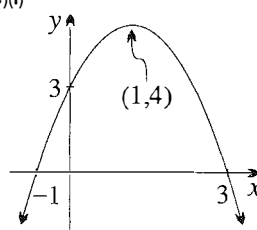
(ii)



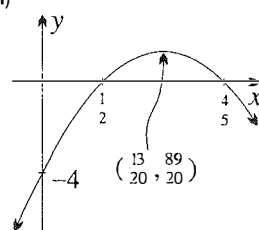
2(a)(i) 9 (ii) $\frac{25}{4}$ (iii) 9 (iv) $\frac{1}{8}$ (b)(i) -1 (ii) $-\frac{5}{4}$

(iii) $\frac{17}{4}$ (iv) $\frac{17}{8}$

(c)(i)



(ii)



3(b) $\frac{9}{4}$ when $x = \frac{3}{2}$

4 225 when the numbers are 15 and 15

5(b) 18 when $x = 3$

6 $16m^2$

7 105 metres

8 2 machines, \$7000

9(a) $2x^2 - 64x + 1024$ (b) $x = y = 16$

10(b) $15\frac{5}{8} \text{ cm}^2$

11(a) $-x^2 + 2015x$ (b) 1 015 056.25 m^2

12(a) $2x + 5y = 40$ (b) $\frac{128}{41} \text{ cm}$ and $\frac{200}{41} \text{ cm}$

13(b) $x = \frac{800}{3}$ and $y = 200$

14(b) 23

- 15 $\frac{288}{4+\pi}$ cm
 16 $\frac{m}{4r} \times \frac{m}{2(n+r)}$
 17 profit = $-\frac{5}{6}x^2 + 15x - 27$, \$40.50
 18(a) $x(16-x)$ (c) 4 (d) \$22
 19(a) $(\frac{9}{2}, \frac{9}{4})$ and $(0, 0)$ (b) $9x - 2x^2$, $\frac{81}{8}$
 20 250 metres $\times \frac{500}{\pi}$ metres
 21(a) $11300 - 1430h + 4525h^2$
 (b) approximately 94.8 minutes
 22 $(\frac{1}{3}, \frac{1}{3})$ 26(a) 1200 cm^2

Exercise 8F (Page 302)

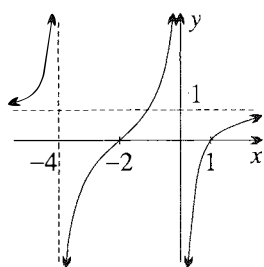
- 1(a) irrational, unequal (b) unreal (that is, no roots) (c) rational, equal (that is, a single rational root) (d) rational, unequal (e) rational, unequal (f) unreal
 2(a) $\Delta = 4$, two rational roots
 (b) $\Delta = -31$, no roots
 (c) $\Delta = 0$, one rational root
 (d) $\Delta = 32$, two irrational roots
 (e) $\Delta = 361 = 19^2$, two rational roots
 (f) $\Delta = 36$, two rational roots
 3(a) $\Delta = 100 - 4g$, $g = 25$ (b) $\Delta = 16 - 4g$, $g = 4$
 (c) $\Delta = 1 - 8g$, $g = \frac{1}{8}$ (d) $\Delta = 44 - 4g$, $g = 11$
 (e) $\Delta = g^2 - 4g$, $g = 4$ (If $g = 0$, then $y = 1$ for all x , and so y is never zero.)
 (f) $\Delta = 49 - 4g^2$, $g = \frac{7}{2}$ or $-\frac{7}{2}$
 (g) $\Delta = 16(g^2 - 6g - 7)$, $g = -1$ or 7
 (h) $\Delta = 4(g^2 + 2g - 8)$, $g = -4$ or 2
 4(a) $\Delta = 4 - 4k$, $k \leq 1$ (b) $\Delta = 64 - 8k$, $k \leq 8$
 (c) $\Delta = 4 - 12k$, $k \leq \frac{1}{3}$ (d) $\Delta = 33 - 16k$, $k \leq \frac{33}{16}$
 (e) $\Delta = k^2 - 16$, $k \leq -4$ or $k \geq 4$
 (f) $\Delta = 9k^2 - 36$, $k \leq -2$ or $k \geq 2$
 (g) $\Delta = k^2 + 12k + 20$, $k \leq -10$ or $k \geq -2$
 (h) $\Delta = k^2 - 12k$, $k \leq 0$ or $k \geq 12$
 5(a) $-4 < \ell < 4$ (b) $\ell < -3$ or $\ell > 3$
 (c) $-5 < \ell < 3$ (d) no values (e) $0 < \ell < 1$
 (f) $\ell < -6$
 6(b) $\Delta = 28$. Since $\Delta > 0$, the quadratic equation has two roots.
 7(a) They intersect twice.
 (b) They do not intersect.
 (c) They intersect once.
 (d) They intersect twice.
 8(a) $\Delta = (m+4)^2$ (b) $\Delta = (m-2)^2$
 (c) $\Delta = (2m-n)^2$ (d) $\Delta = (4m-1)^2$
 (e) $\Delta = (m-6)^2$ (f) $\Delta = 36m^2$

- 9(a) $\Delta = \lambda^2 + 4$ (b) $\Delta = 4\lambda^2 + 48$ (c) $\Delta = \lambda^2 + 16$
 (d) $\Delta = (\lambda - 1)^2 + 8$
 10(a) $m > -\frac{9}{4}$ (b) $m = \frac{39}{8}$ (c) $-1 < m < 2$
 (d) $m \leq -1$ or $m \geq -\frac{1}{2}$
 12(a) -4 (b) $\frac{17}{8}$ (c) -1 or 11
 13 $3x - y - 19 = 0$
 14 -7 and 5
 15 $y = 4x^2 - 4x - 3$
 16(a) 2 (b) 1 (c) 0
 17 If $ac < 0$, then $\Delta > 0$.
 18(a) $(x-4)^2 + y^2 = 4$ (b) $y = mx$
 (d) $m = \frac{1}{\sqrt{3}}$ or $m = -\frac{1}{\sqrt{3}}$
 (e) $(3, \sqrt{3})$ or $(3, -\sqrt{3})$
 19 $m = \frac{1}{\sqrt{3}}$ or $m = -\frac{1}{\sqrt{3}}$,
 $P(\sqrt{3}, -1)$ or $P(-\sqrt{3}, -1)$
 20 $\frac{4}{3}$ and $-\frac{3}{4}$
 21(a) $-\frac{4}{3} < a < 1$ (b) $b \neq -5$ (c) $y = 3$ or $-\frac{13}{29}$
 (d) $-1 < k < 14$
 22(b) $b^2 = ac$

Exercise 8G (Page 305)

- 1(a) $x \leq 0$ or $x \geq 1$ (b) $-\sqrt{7} < x < \sqrt{7}$ (c) $x \neq 3$
 (d) $-1 - \sqrt{34} \leq x \leq -1 + \sqrt{34}$ (e) $-2 \leq x \leq \frac{1}{3}$
 (f) $\frac{3}{2} \leq x \leq 5$ (g) $x < -\frac{1}{2}$ or $x > \frac{5}{2}$
 (h) There are no solutions.
 (i) All real numbers are solutions.
 2(a) positive definite (b) indefinite
 (c) negative definite (d) indefinite
 (e) indefinite (f) positive definite
 3(a)(i) $k > \frac{25}{32}$ (ii) $k \leq \frac{25}{32}$
 (b)(i) $-8 < k < 8$ (ii) $k \leq -8$ or $k \geq 8$
 (c)(i) $0 < k < 24$ (ii) $k \leq 0$ or $k \geq 24$
 (d)(i) $2 < k < 5$ (ii) $k \leq 2$ or $k \geq 5$
 4(a)(i) $-4 < m < 4$ (ii) $m \leq -4$ or $m \geq 4$
 (b)(i) $m > \frac{9}{8}$ (ii) $m \leq \frac{9}{8}$
 (c)(i) $-8 < m < 12$ (ii) $m \leq -8$ or $m \geq 12$
 (d)(i) $0 < m < 2$ (ii) $m \leq 0$ or $m \geq 2$
 5(a) -4 and 4
 (b) 2. (When $\ell = 0$, it is not a quadratic.)
 (c) 1. (When $\ell = -\frac{3}{10}$, the expression becomes $-\frac{1}{10}(5x-4)^2$, which is a multiple of a perfect square, but is not itself a perfect square.)
 (d) $-\frac{2}{9}$ and 2
 6(a) $2 < k < 18$ (b) no values
 (c) $k \leq 2$ or $k \geq 18$, but $k \neq \frac{5}{3}$

7(a)



(c) Whatever the value of k , every horizontal line $y = k$ intersects the graph.

8(a) $x \leq \frac{1}{2}(5 - \sqrt{21})$ or $x \geq \frac{1}{2}(5 + \sqrt{21})$

(b) no values (c) $x \leq \frac{1}{2}$ or $x \geq 4$

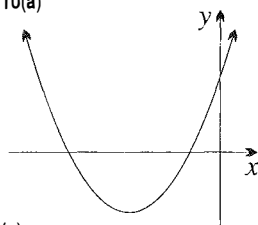
(d) $0 < x < 2$ (e) $-\frac{3}{4} \leq x \leq 2$

(f) $x < -3$ or $x > 3$

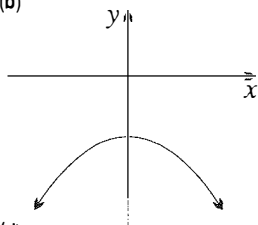
9(a) $a > 0$ and $b^2 < 3ac$

(b) $a < 0$ and $b^2 < 3ac$ (c) $b^2 \geq 3ac$

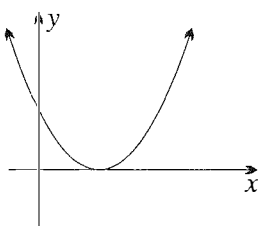
10(a)



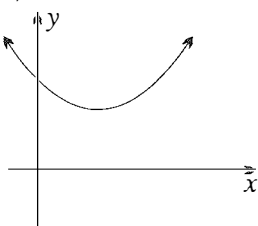
(b)



(c)



(d)



11(a) $b^2 = 3c$ (b) $b^2 > 3c$ (c) $b^2 < 3c$ (d) $c < 0$

(e) $c > 0$ and $b < 0$ (and $b^2 > 3c$)

(f) $c > 0$ and $b > 0$ (and $b^2 > 3c$)

13 $x \leq -22$ or $x \geq 2$, $y \leq -15 - 12\sqrt{2}$
or $y \geq -15 + 12\sqrt{2}$

14 $(3x - 4y + 1)(x + 2y - 3)$

15 $-\frac{5}{4}$ and $\frac{3}{4}$

16 9. Note that a^2 cannot be negative.

17(b) $(x - 1)^2 + (x + 3)^2$

Exercise 8H (Page 309)

1 $\alpha + \beta = -7$, $\alpha\beta = 10$, the roots are -2 and -5 .

2(a) $\alpha + \beta = \frac{10}{3}$, $\alpha\beta = 1$, the roots are $\frac{1}{3}$ and 3 .

(b) $\alpha + \beta = -4$, $\alpha\beta = 1$, the roots are $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$. (c) $\alpha + \beta = 1$, $\alpha\beta = -1$, the roots are $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ and $\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

3(a) 2, 5 (b) $-1, -6$ (c) $-1, 0$ (d) $-\frac{3}{2}, -\frac{1}{2}$

(e) $-1, \frac{4}{5}$ (f) $\frac{2}{3}, -\frac{4}{3}$ (g) m, n (h) $-q/p, -3r/p$

(i) $(a - 4)/a, -3/a$

4(a) $x^2 - 4x + 3 = 0$ (b) $x^2 - 4x - 12 = 0$

(c) $x^2 + 5x + 4 = 0$ (d) $4x^2 - 8x + 3 = 0$

(e) $x^2 - 4x + 1 = 0$ (f) $x^2 + 2x - 4 = 0$

5(a) 3 (b) 2 (c) 21 (d) 6 (e) 20 (f) $\frac{3}{2}$ (g) 5

(h) $\frac{5}{2}$

6(a) $\frac{5}{2}$ (b) $\frac{1}{2}$ (c) -1 (d) 5 (e) $\frac{5}{8}$ (f) 10 (g) $\frac{21}{4}$

(h) 21

7(b)(i) 3, 1, 5 (ii) $-5, -7, 53$ (iii) $\frac{7}{3}, \frac{2}{3}, 2\frac{7}{9}$

(c) $\sqrt{5}, \sqrt{53}, \frac{5}{3}$

$$9 \quad \alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

10 $\frac{5}{6}, \frac{2}{3}\sqrt{3}$

11(a) $3x^2 + 4x + 28 = 0$ (b) $7x^2 + 2x + 3 = 0$

(c) $9x^2 + 18x + 29 = 0$ (d) $9x^2 + 38x + 49 = 0$

12(a) $\frac{5}{2}$ (b) -4 (c) $\frac{7}{2}$ (d) $\frac{1}{3}$

13(a) -1 (b) 1 (c) $\frac{1}{10}$ (d) -3

14(a) $ac < 0$ (b) If $ca > 0$ the roots have the same sign. If $ab < 0$ they are both positive, if $ab > 0$ they are both negative. (c) If $ac < 0$ the roots have opposite signs. If $ab < 0$ the positive root is numerically greater, if $ab > 0$ the negative root is numerically greater.

15 $\frac{305}{27}$

16 8

17 $\frac{1}{3}(2 + 2\sqrt{10})$ or $\frac{1}{3}(2 - 2\sqrt{10})$

20(b) 6 (c) $(3, 27)$

21(a) $(\frac{1}{2}, \frac{1}{2})$ (b) $(\frac{5}{2}, \frac{11}{2})$ (c) $(-2, -2)$

22 $y = -\frac{1}{3}x$. It is the line through the centre of the circle perpendicular to the given line.

23(b) $\sqrt{14}$

24 $5\sqrt{2}$ units

25 $2 \leq m < 3$

26 When $m = 1$, $x = -1$, and when $m = -3$, $x = 1$.

27 $-1, \frac{1}{2}(1 + \sqrt{21})$ or $\frac{1}{2}(1 - \sqrt{21})$

Exercise 8I (Page 313)

2 It is true for all values of x , and hence is an identity.

4(a) $a = 1, b = 3, c = 3$ (b) $a = 1, b = 7, c = 12$

5(a) $2(x + 1)^2 - (x + 1) - 7$

(b) $a = 2, b = 16$ and $c = 35$

(c) $2(x - 2)^2 + 3(x - 2) + 1$

- 6(a) $(x+1)^2 - 2(x+1) + 1$ (b) $(n-4)^2 + 8(n-4) + 16$
 (c) $(x+2)^2 - 4(x+2) + 4$
 8(a) $a = 2, b = 1, c = -\frac{1}{2}$ or $a = 2, b = -\frac{1}{2}, c = 1$
 (b) $3(m-1)^2 - 3(m-2)^2 + (m-3)^2$
 9(a) $7(2x+1) - 12(x+1)$ (b) $\frac{1}{x+1} + \frac{1}{x+2}$
 (c) $\frac{1}{2x} + \frac{1}{2(x+2)}$
 10(a) $y = x^2 - 4x$, no
 (b) All four points lie on $y = x^2 - 5x + 6$.
 11(b) 961 (d) n^2
 12(a) 1 (b) 1

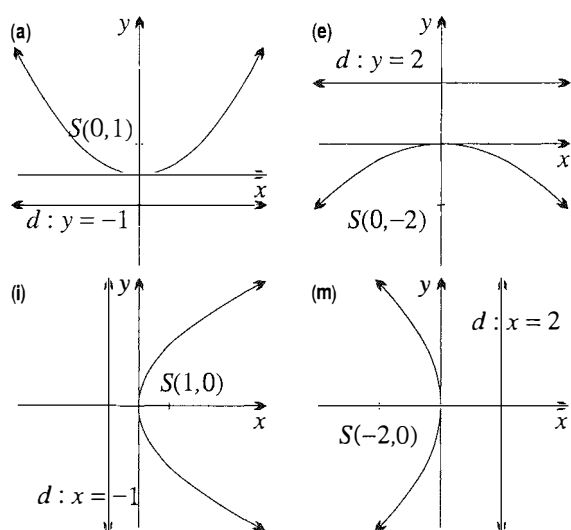
Chapter Nine

Exercise 9A (Page 318)

- 1(a) $y = -2$ (b) $x = -1$ (c) $y = 2$
 (d) $y = 3x$ or $y = -3x$ (e) $x^2 + y^2 = 9$
 (f) $y = x + 5$ (g) $(x+3)^2 + (y-1)^2 = 9$
 2 $(x-3)^2 + (y-1)^2 = 16$
 3(a) $6x - 4y + 15 = 0$ (b) $6x - 4y + 15 = 0$.
 The locus of the point P which moves so that it is equidistant from R and S is the perpendicular bisector of RS .
 4(a) $\frac{y}{x-4}, \frac{y}{x+2}$ (c) $(x-1)^2 + y^2 = 9$, which is a circle with centre $(1, 0)$ and radius 3.
 5(a) $2x + y - 1 = 0$, the perpendicular bisector of AB (b) $x^2 + y^2 + 2x - 6y + 5 = 0$, circle with centre $(-1, 3)$ and radius $\sqrt{5}$
 (c) $x^2 - 2x - 8y + 17 = 0$, parabola
 6 $(x+1)^2 = 6(y - \frac{3}{2})$, vertex: $(-1, \frac{3}{2})$
 7(b) $C(2, 1)$, radius $\sqrt{5}$
 8 $x^2 + y^2 = 1$, the circle with centre $(0, 0)$ and radius 1
 9(a) $x^2 + y^2 = 4$ (b) $3x^2 + 3y^2 - 28x + 18y + 39 = 0$
 10 $3x^2 - y^2 + 12x + 10y - 25 = 0$
 11(a) P may satisfy $x - y + 12 = 0$
 or $7x + 7y - 60 = 0$. (b) The gradients are 1 and -1 , so the lines are perpendicular.
 12 $x^2 = -4(y-1)$ and $y > 0$
 13 $8x - 2y + 3 = 0$
 14 $10x - 15y + 18 = 0$
 15(b) $y = \sqrt{3}x + 2, y = -1, y = -\sqrt{3}x + 2$
 (c) $x^2 + y^2 = 4$ (d) circle with centre O and radius 2 — the circumcircle of the triangle
 16(b) $x^2 + y^2 + z^2 = a^2$, which is the equation of a sphere with centre $(0, 0, 0)$ and radius a .
 (c) $C(0, 0, 0)$ and $r = \frac{1}{2}\sqrt{3}$

Exercise 9B (Page 323)

- 3(b) $x^2 = 12y$
 4(a) $x^2 = 20y$ (b) $x^2 = -4y$ (c) $y^2 = 8x$
 (d) $y^2 = -6x$
 5(a) $x^2 = -4ay$ (b) $y^2 = 4ax$
 6(vi) Only parts (a), (e), (i) and (m) are sketched below. The details of all the parabolas follow these sketches.



- (a) $V(0,0)$, $S(0,1)$, axis: $x = 0$,
 directrix: $y = -1$, $4a = 4$
 (b) $V(0,0)$, $S(0,2)$, axis: $x = 0$,
 directrix: $y = -2$, $4a = 8$
 (c) $V(0,0)$, $S(0, \frac{1}{4})$, axis: $x = 0$,
 directrix: $y = -\frac{1}{4}$, $4a = 1$
 (d) $V(0,0)$, $S(0, \frac{1}{3})$, axis: $x = 0$,
 directrix: $y = -\frac{1}{3}$, $4a = \frac{4}{3}$
 (e) $V(0,0)$, $S(0,-2)$, axis: $x = 0$,
 directrix: $y = 2$, $4a = 8$
 (f) $V(0,0)$, $S(0,-3)$, axis: $x = 0$,
 directrix: $y = 3$, $4a = 12$
 (g) $V(0,0)$, $S(0, -\frac{1}{2})$, axis: $x = 0$,
 directrix: $y = \frac{1}{2}$, $4a = 2$
 (h) $V(0,0)$, $S(0, -0.1)$, axis: $x = 0$,
 directrix: $y = 0.1$, $4a = 0.4$
 (i) $V(0,0)$, $S(1,0)$, axis: $y = 0$,
 directrix: $x = -1$, $4a = 4$
 (j) $V(0,0)$, $S(\frac{1}{4}, 0)$, axis: $y = 0$,
 directrix: $x = -\frac{1}{4}$, $4a = 1$
 (k) $V(0,0)$, $S(\frac{3}{2}, 0)$, axis: $y = 0$,
 directrix: $x = -\frac{3}{2}$, $4a = 6$
 (l) $V(0,0)$, $S(\frac{1}{8}, 0)$, axis: $y = 0$,
 directrix: $x = -\frac{1}{8}$, $4a = \frac{1}{2}$
 (m) $V(0,0)$, $S(-2,0)$, axis: $y = 0$,
 directrix: $x = 2$, $4a = 8$
 (n) $V(0,0)$, $S(-3,0)$, axis: $y = 0$,
 directrix: $x = 3$, $4a = 12$
 (o) $V(0,0)$, $S(-\frac{1}{4}, 0)$, axis: $y = 0$,
 directrix: $x = \frac{1}{4}$, $4a = 1$
 (p) $V(0,0)$, $S(-0.3, 0)$, axis: $y = 0$,
 directrix: $x = 0.3$, $4a = 1.2$

7 Details rather than sketches are given:

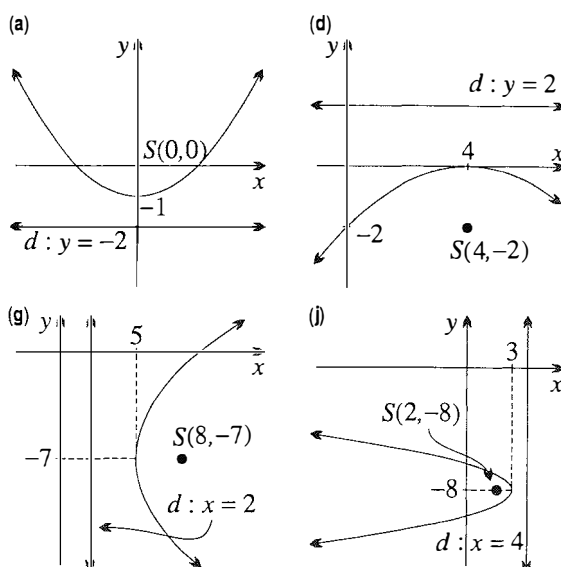
- (a) $V(0,0)$, $S(0, \frac{1}{8})$, directrix: $y = -\frac{1}{8}$
 (b) $V(0,0)$, $S(0,-1)$, directrix: $y = 1$
 (c) $V(0,0)$, $S(\frac{1}{9}, 0)$, directrix: $x = -\frac{1}{9}$
 (d) $V(0,0)$, $S(-\frac{5}{2}, 0)$, directrix: $x = \frac{5}{2}$
 8(a) $x^2 = 20y$ (b) $x^2 = -12y$ (c) $x^2 = 8y$
 (d) $x^2 = -2y$ (e) $x^2 = 4y$ (f) $x^2 = -\frac{1}{2}y$
 9(a) $y^2 = 2x$ (b) $y^2 = -4x$ (c) $y^2 = 16x$
 (d) $y^2 = -8x$ (e) $y^2 = 12x$ (f) $y^2 = -6x$
 10(a) $x^2 = 16y$ (b) $x^2 = \frac{1}{2}y$ (c) $y^2 = 2x$
 (d) $y^2 = -x$
 11(a) $x^2 = 8y$ or $x^2 = -8y$
 (b) $x^2 = 12y$, $x^2 = -12y$, $y^2 = 12x$ or $y^2 = -12x$
 (c) $x^2 = y$ or $y^2 = x$ (d) $y^2 = 2x$ or $y^2 = -2x$
 12(a) $k = 4$ (b) $y = -3x^2$
 13(a) $x^2 + y^2 + 8x - 8y + 2xy = 0$
 (b) $x^2 + y^2 - 24x + 24y + 2xy = 0$

In both cases, the distance from the focus to the origin equals the distance from the directrix to the origin, being $\sqrt{2}$ and $3\sqrt{2}$ respectively.

- 15(a) $x^2 + z^2 = 12y$ (b) $y^2 + z^2 = 4x$
 (c) $x^2 + y^2 = -8z$ (d) $x^2 + z^2 = -6y$
 16(a) focus: $(0,0,-2)$, directrix: $z = 2$
 (b) focus: $(\frac{1}{2}, 0, 0)$, directrix: $x = -\frac{1}{2}$
 (c) focus: $(0, -\frac{1}{4}, 0)$, directrix: $4y - 1 = 0$

Exercise 9C (Page 326)

- 1(b) $(x-3)^2 = 8(y-1)$
 2(a) $(x+7)^2 = 12(y+5)$ (b) $(y-2)^2 = 4(x+1)$
 3 Only the graphs of (a), (d), (h) and (j) have been sketched. The details of all graphs are given afterwards.



- (a) vertex: $(0, -1)$, focus: $(0, 0)$,

axis: $x = 0$, directrix: $y = -2$

(b) vertex: $(-2, 0)$, focus: $(-2, 1)$,

axis: $x = -2$, directrix: $y = -1$

(c) vertex: $(3, -5)$, focus: $(3, -3)$,

axis: $x = 3$, directrix: $y = -7$

(d) vertex: $(4, 0)$, focus: $(4, -2)$,

axis: $x = 4$, directrix: $y = 2$

(e) vertex: $(0, -3)$, focus: $(0, -3\frac{1}{2})$,

axis: $x = 0$, directrix: $y = -2\frac{1}{2}$

(f) vertex: $(-5, 3)$, focus: $(-5, 2)$,

axis: $x = -5$, directrix: $y = 4$

(g) vertex: $(-2, 0)$, focus: $(-\frac{1}{2}, 0)$,

axis: $y = 0$, directrix: $x = -1\frac{1}{2}$

(h) vertex: $(0, 1)$, focus: $(4, 1)$,

axis: $y = 1$, directrix: $x = -4$

(i) vertex: $(5, -7)$, focus: $(8, -7)$,

axis: $y = -7$, directrix: $x = 2$

(j) vertex: $(3, -8)$, focus: $(2, -8)$,

axis: $y = -8$, directrix: $x = 4$

(k) vertex: $(-6, 0)$, focus: $(-8\frac{1}{2}, 0)$,

axis: $y = 0$, directrix: $x = -3\frac{1}{2}$

(l) vertex: $(0, 3)$, focus: $(-\frac{1}{2}, 3)$,

axis: $y = 3$, directrix: $x = \frac{1}{2}$

4(a) $(x+2)^2 = 8(y-4)$ (b) $(y-1)^2 = 16(x-1)$

(c) $(x-2)^2 = -12(y-2)$ (d) $y^2 = -4(x-1)$

(e) $(x+5)^2 = 8(y-2)$ (f) $(y+2)^2 = 16(x+7)$

(g) $(x-8)^2 = -12(y+7)$ (h) $(y+3)^2 = -8(x+1)$

(i) $(x-6)^2 = 12(y+3)$

5(a) $(x-2)^2 = 8(y+1)$ (b) $y^2 = 4(x-1)$

(c) $(x+3)^2 = -8(y-4)$ (d) $(y-5)^2 = -12(x-2)$

(e) $(x-3)^2 = 8(y-1)$ (f) $(y-2)^2 = 12(x+4)$

(g) $x^2 = -8(y+\frac{3}{2})$ (h) $(y+4)^2 = -12(x+1)$

(i) $(x+7)^2 = 2(y+5)$

6(a) $x^2 = 8(y-2)$ (b) $y^2 = 12(x-3)$

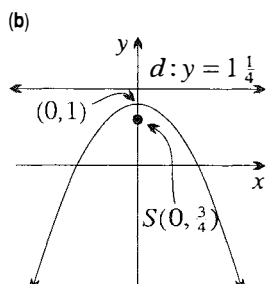
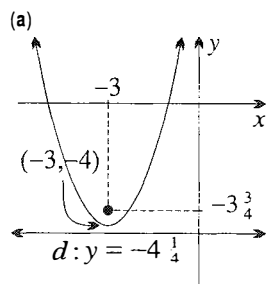
(c) $x^2 = -4(y+1)$ (d) $y^2 = -8(x+2)$

(e) $(x-1)^2 = 8(y-5)$ (f) $(y+2)^2 = 4(x-2)$

(g) $(x+1)^2 = -2(y-\frac{9}{2})$ (h) $(y-\frac{1}{2})^2 = -4(x-4)$

(i) $(x-5)^2 = 10(y+\frac{13}{2})$

7 Only graphs (a) and (b) have been sketched.



(a) $y+4 = (x+3)^2$, vertex: $(-3, -4)$,

focus: $(-3, -3\frac{3}{4})$, directrix: $y = -4\frac{1}{4}$

(b) $x^2 = -(y-1)$, vertex: $(0, 1)$,

focus: $(0, \frac{3}{4})$, directrix: $y = \frac{5}{4}$

(c) $(x-6)^2 = 6(y+6)$, vertex: $(6, -6)$,

focus: $(6, -4\frac{1}{2})$, directrix: $y = -7\frac{1}{2}$

(d) $x^2 = 4(y+\frac{1}{2})$, vertex: $(0, -\frac{1}{2})$,

focus: $(0, \frac{1}{2})$, directrix: $y = -\frac{3}{2}$

(e) $(x+3)^2 = y+25$, vertex: $(-3, -25)$,

focus: $(-3, -24\frac{3}{4})$, directrix: $y = -25\frac{1}{4}$

(f) $(x+4)^2 = 8(y-3)$, vertex: $(-4, 3)$,

focus: $(-4, 5)$, directrix: $y = 1$

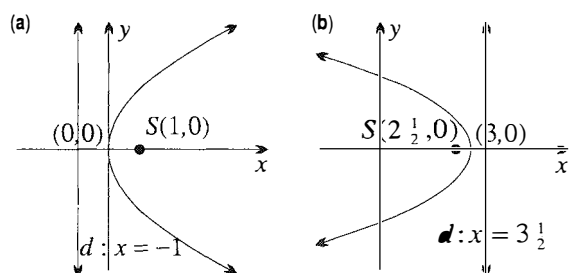
(g) $(x-3)^2 = -2(y+1\frac{1}{2})$, vertex: $(3, -1\frac{1}{2})$,

focus: $(3, -2)$, directrix: $y = -1$

(h) $(x-4)^2 = -12(y-1)$, vertex: $(4, 1)$,

focus: $(4, -2)$, directrix: $y = 4$

8 Only graphs (a) and (b) have been sketched.



(a) $y^2 = 4x$ vertex: $(0, 0)$,

focus: $(1, 0)$, directrix: $x = -1$

(b) $y^2 = -2(x-3)$ vertex: $(3, 0)$,

focus: $(2\frac{1}{2}, 0)$, directrix: $x = 3\frac{1}{2}$

(c) $y^2 = 6(x-3)$, vertex: $(3, 0)$,

focus: $(4\frac{1}{2}, 0)$, directrix: $x = 1\frac{1}{2}$

(d) $(y-1)^2 = 4(x-1)$, vertex: $(1, 1)$,

focus: $(2, 1)$, directrix: $x = 0$

(e) $(y-2)^2 = 8(x+\frac{1}{2})$, vertex: $(-\frac{1}{2}, 2)$,

focus: $(1\frac{1}{2}, 2)$, directrix: $x = -2\frac{1}{2}$

(f) $(y-3)^2 = 2(x+1)$, vertex: $(-1, 3)$,

focus: $(-\frac{1}{2}, 3)$, directrix: $x = -1\frac{1}{2}$

(g) $(y+2)^2 = -6(x-5)$, vertex: $(5, -2)$,

focus: $(3\frac{1}{2}, -2)$, directrix: $x = 6\frac{1}{2}$

(h) $(y-5)^2 = 12(x+1)$, vertex: $(-1, 5)$,

focus: $(2, 5)$, directrix: $x = -4$

9(a) $y = 2x^2 + 3x - 5$ (b) $y = -x^2 - 5x + 1$

(c) $x = y^2 - 4y + 3$ (d) $x = -2y^2 + y - 3$

10(a) $(x-1)^2 = 4(y-4)$ (b) $(x+2)^2 = -(y-3)$

(c) $(y+2)^2 = 2(x+3)$ (d) $(y-5)^2 = -\frac{1}{2}(x-2)$

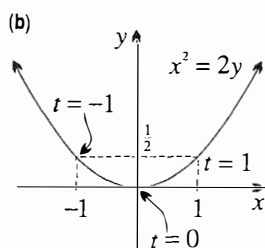
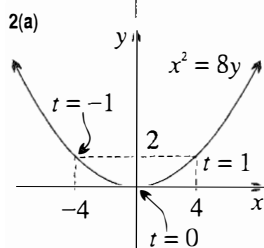
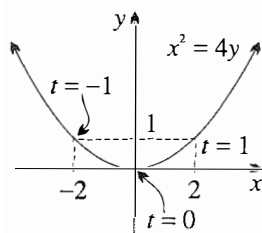
11(a) $(x-3)^2 = 8(y+1)$ or $(x-3)^2 = -8(y+1)$

or $(y+1)^2 = 8(x-3)$ or $(y+1)^2 = -8(x-3)$

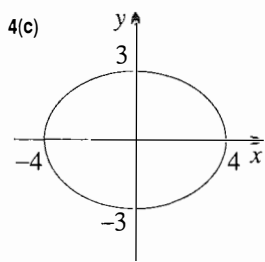
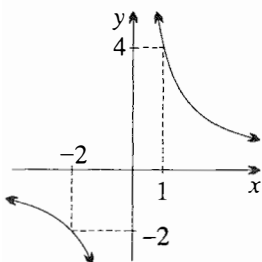
- (b) $(y+1)^2 = 8(x+1)$ or $(y+1)^2 = -8(x-3)$
 (c) $(x+2)^2 = 4(y-3)$ or $(x+2)^2 = -4(y-5)$
 (d) $(y-2)^2 = 6(x-3)$ or $(y-2)^2 = -6(x-3)$
 (e) $(x-6)^2 = -20(y-2)$ or $(y+3)^2 = 20(x-1)$
 12(a) $(x-3)^2 = y+1$ (b) $(y-2)^2 = -\frac{1}{2}(x+4)$
 13(a) $x^2 + y^2 + 6x - 18y + 2xy + 33 = 0$
 (b) $9x^2 + 16y^2 - 34x - 88y - 24xy + 121 = 0$
 14(a) $(x-1)^2 + (y-2)^2 = 8(z-1)$
 (b) $(y-2)^2 + (z+1)^2 = -4x$
 (c) $x^2 + (z-3)^2 = -14(y + \frac{3}{2})$
 (d) $(x-4)^2 + (y+3)^2 = 6(y - \frac{11}{2})$
 15(a) $V(1, 2, -1)$, $S(1, 2, 0)$, directrix: $z+2=0$
 (b) $V(1, -3, 0)$, $S(-1, -3, 0)$, directrix: $x=3$

Exercise 9D (Page 329)

t	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
x	-6	-4	-2	-1	0	1	2	4	6
y	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9
- 1(a) $x^2 = 4y$ (c) $(0, 0), (0, 1)$ (d) $t = 0$
 (e) $(2, 1), (-2, 1), t = 1$ or -1

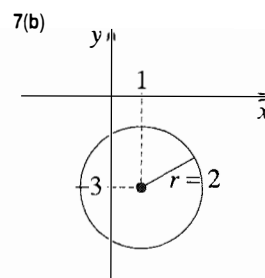
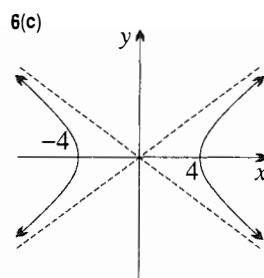


- 3(c) As $t \rightarrow \infty$, $x \rightarrow \infty$ and $y \rightarrow 0$.
 As $t \rightarrow -\infty$, $x \rightarrow -\infty$ and $y \rightarrow 0$.
 As $t \rightarrow 0^+$, $x \rightarrow 0$ and $y \rightarrow \infty$.
 As $t \rightarrow 0^-$, $x \rightarrow 0$ and $y \rightarrow -\infty$.

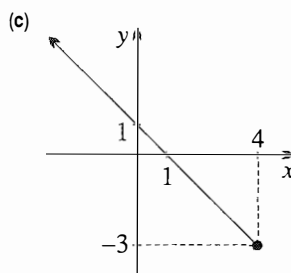
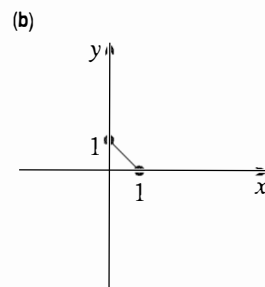
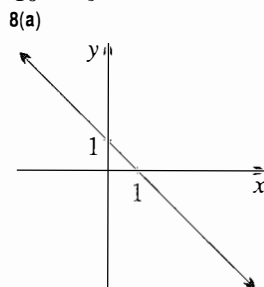


$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

- 5(a) $2x + y - 7 = 0$ (b) $4(y+4)^2 - 9(x-1)^2 = 36$
 (c) $y = x^2 - 2$ (d) $x^2 + y^2 = 2$



$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



- 9(a) $(x-3)^2 + (y+2)^2 = r^2$, circle with centre $(3, -2)$ and radius r
 (b) $y = x \tan \theta - (3 \tan \theta + 2)$, straight line with gradient $\tan \theta$

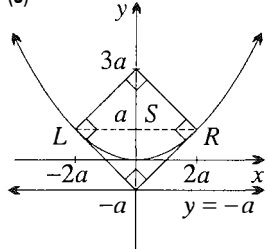
- 10(b) P_1 has parameter $\lambda = 0$. As λ moves from 0 to ∞ , P moves from P_1 to P_2 . As λ moves from $-\infty$ to -1 , P moves from P_2 infinitely far along the line. As λ moves from 0 to -1 , P moves from P_1 infinitely far along the line in the other direction. (c) $\frac{3}{10}$
 (d)(i) $(x_2^2 - 4ay_2)\lambda^2 + 2(x_1x_2 - 2ay_1 - 2ay_2)\lambda + (x_1^2 - 4ay_1) = 0$
 (ii) $(x_1 - x_2)(x_1y_2 - x_2y_1) + a(y_1 - y_2)^2 = 0$

Exercise 9E (Page 332)

- 1(a) $x + y - 3 = 0$ (b) $4y - 5x + 8 = 0$
 (c) $3x + 2y + 2 = 0$ (d) $y - x - 2 = 0$
 2(a) $4y - 3x - 12 = 0$ (b) $x^2 = 12y, (0, 3)$
 3(d) $(0, a)$ (f) $(-\frac{4a}{p+q}, -a)$

- 4(a) $P(2ap, ap^2)$, $Q(2aq, aq^2)$, $S(0, a)$
 (b) $a(p^2 + 1)$ (c) $a(q^2 + 1)$
 5(a) The x -intercept is $(4a, 0)$.
 6(a) $t = -2$ or $t = 1$ (b) $t = -3$ or $t = 1$
 8(a) Since $a = 2$, the line is $y = \frac{1}{2} \times 4x - (-5)a$, so $p + q = 4$ and $pq = -5$. $t = 5$ gives $(20, 50)$, and $t = -1$ gives $(-4, 2)$.
 (b) The point of contact is $(8, 8)$.
 (c) $p + q = 4$, $pq = 5$ has no solutions.
 9 The midpoint lies of $x = k$ if and only if k is the average of $2ap$ and $2aq$.

Exercise 9F (Page 336)

- 1(a) $y = x - 1$ (b) $x - 2y - 1 = 0$
 (c) $6x + 2y + 9 = 0$ (d) $y = qx - 3q^2$
 2(a) $x + 2y - 12 = 0$ (b) $4x - 2y + 9 = 0$
 (c) $x + my - 3m^3 - 6m = 0$
 (d) $x + qy - aq^3 - 2aq = 0$
 3(a) $y = px - ap^2$ (b) $(0, -ap^2)$, $(ap, 0)$ (c) $\frac{1}{2}a^2|p|^3$
 4(a) $x + py = 2ap + ap^3$
 (b) $(2ap + ap^3, 0)$, $(0, 2a + ap^2)$ (c) $\frac{1}{2}a^2|p|(p^2 + 2)^2$
 5(a) tangents: $y = x - a$, (c) $y = -x - a$; normals: $y = -x + 3a$, $y = x + 3a$
 (b) vertices: $(-2a, a)$, $(2a, a)$, $(0, 3a)$, $(0, -a)$; area = $8a^2$ (half product of the diagonals)

 6(a) $y = tx - t^2$ (b) 3 or -1
 (c) $y = 3x - 9$ and $x + y + 1 = 0$
 7(a) parameters: 5 and $-\frac{1}{5}$ (b) gradients: 5 and $-\frac{1}{5}$; points of contact: $(50, 125)$ and $(-2, \frac{1}{5})$
 8(a) $A = (-1, 1)$, $B = (2, 4)$ (b) at A , $y = -2x - 1$; at B , $y = 4x - 4$ (c) $M = (1\frac{1}{4}, 1)$, $D = (\frac{1}{2}, \frac{1}{4})$, AB and the tangent at D both have gradient 1.
 9(a) $t = 1$, point of contact: $(6, 3)$
 (b) $\sqrt{5}$ (d) $k = 2$ or -2
 10(b) $y + x = 3a$, $y - x = 3a$
 11(b) $(0, 0)$, $(24, 36)$, $(-24, 36)$
 14(a) $(0, -ap^2)$ (b) $(0, 2a + ap^2)$
 15(a) $(2ap, -a)$
 16(a) $M = (a(p + q), \frac{1}{2}a(p^2 + q^2))$, T has parameter $\frac{1}{2}(p + q)$ and $T = (a(p + q), \frac{1}{4}a(p + q)^2)$, $M = (a(p + q), apq)$. (d) 2 : 1
 17 When the two points are on the same side of the axis, it is the positive geometric mean, and

when they are on opposite sides of the axis, it is the negative geometric mean.

19(b) $\frac{1}{2}a^2|p - q|^3$

Exercise 9G (Page 339)

- 1(a) $y = x - 1$ (b) $y = 2x - 3$ (c) $y = -4x + 4$
 (d) $y = 4x - 7$
 2(a) $x + 2y - 3 = 0$ (b) $x - 3y + 33 = 0$
 (c) $x - 3y - 18 = 0$ (d) $x + 7y - 21 = 0$
 3(a) $(6, 9)$ (b) $(\frac{1}{2}, \frac{23}{4})$
 4(a) $x + y = 3$, $x + 2y = 12$ (b) $(-6, 9)$
 5(b) tangents: $y + x + a = 0$, $y - x + a = 0$; normals: $x + y = 3a$, $x = y - 3a$ (c) $8a^2$ square units
 6(a) $y = \frac{1}{3}x^2 + \frac{2}{3}$ (b) $2x - 3y + 1 = 0$, $3x + 2y - 5 = 0$
 7(a) $(2, 10)$ and $(-\frac{4}{5}, \frac{8}{5})$
 (b) $y = 10x - 10$ and $y = -4x - \frac{8}{5}$ (c) $(\frac{3}{5}, -4)$
 9(b) $m = 2$ gives $y = 2x - 2$, $m = -2$ gives $y = -2x - 2$.
 (c) $y = x - 2$ and $y = -x - 2$, which are perpendicular because their gradients are 1 and -1.
 10(a) $b = -6$ (b) $y = -2x - 6$ (c) (i) $y + 2x + 4 = 0$ (ii) $16y - 8x + 9 = 0$
 11 $a = -\frac{1}{4}$, $(-1, 1)$
 12(b) $y = 7x - 147$
 13(b) $y = x + 1$ and $y = -2x + 4$
 14(b) Since $\Delta = 3^2 + 4$ is positive, there are two possible gradients. Since the product of the roots is -1, these gradients are perpendicular.
 15(b) $3x - y - 27 = 0$ and $3x + y + 27 = 0$
 (c) $y = x - 3$ and $x - 3y = 1$

Exercise 9H (Page 342)

- 1(a) $y = 2$ (b) $3x - 2y = 0$ (c) $x + y - 1 = 0$
 (d) $2x - y + 6 = 0$
 2(a) $y = 3$ (b) $x + 3y = 0$ (c) $y = x - 1$ (d) $5x + 6y - 24 = 0$
 3(a) $x = 2y$ (b) $(0, 0)$, $(4, 2)$ (c) $y = 0$, $y = x - 2$
 4(a) $(0, 2)$
 (b) $x - 4y + 8 = 0$, $3x - 4y + 8 = 0$, $x + y - 2 = 0$
 5(a) $y_0y = 2a(x + x_0)$ (b) The point $(-5, 2)$ lies on the directrix $x = -5$ of the parabola.
 6(a) $y = x + 1$ (c) 4 (d) $(2, 3)$
 8(a)(i) $x = 2y$ (ii) $x - 4y + 4 = 0$
 9(a) $x_0x = 2a(y + y_0)$ (b) $x^2 - 2x_0x + 4ay_0 = 0$
 (c) $(x_0, \frac{x_0^2}{2a} - y_0)$
 10(a) $x_0x = 2a(y + y_0)$
 11(a) $y = \frac{x_0}{2a}x - y_0$

- 12(a) $x^2 - 2x - 1 = 0$, sum is 2, product is -1
 (b) $y^2 - 6y + 1 = 0$, sum is 6, product is 1
 (d) $2\sqrt{10}$ units
 13 $10\sqrt{2}$ units
 14(c) $x_0^2 > 4ay_0$, which is the condition for the point (x_0, y_0) to lie outside the parabola.
 15(a) $x_1x = 2a(y + y_1)$
 16 $x_0 = 2am$
 18(d) $4x + 5y = 25$, $(0, 5)$ and $(\frac{200}{41}, \frac{45}{41})$
 19(a) $xy_0 + x_0y = 2c^2$
 (c) $8x + 2y = 50$, $(5, 5)$ and $(\frac{5}{4}, 20)$

Exercise 9I (Page 346)

- 1(a) $y = qx - aq^2$ (e) $N(a(p+q), \frac{1}{4}a(p+q)^2)$
 2(c) q
 3(b) $B(0, at^2)$, $N(0, 2a + at^2)$ (c) $BN = 2a$ units
 5(a) $T(2at, -a)$ (e) A rhombus, because the diagonals bisect each other at right angles.
 6(a) $A(2at + at^3, 0)$, $B(0, 2a + at^2)$
 7(a) It is a focal chord. (b) $y = qx - aq^2$
 8(b) $y = tx - at^2$ (c) $R(0, -at^2)$
 9(a) $A(a(p+q), apq)$
 12(a) (x_1, y_1) satisfies $x^2 = 4ay$.
 (c) $Q(\frac{2ay_1}{x_1}, 0)$, $R(0, -y_1)$
 15(b) $Q(-2a(p + \frac{2}{p}), a(p + \frac{2}{p})^2)$
 16(a) $M = (p, 1 + p^2)$
 (b) $P = (2\sqrt{3}, 3)$ or $P = (-2\sqrt{3}, 3)$
 18(b) $p^2 + q^2 + 2$
 19 $P = (4, 4)$, $Q = (-6, 9)$, $T = (-1, -6)$
 22(a) $q = (2\sqrt{2} - 3)p$ or $(-2\sqrt{2} - 3)p$
 23(a) $K = (\frac{kx_1 + x_0}{k+1}, \frac{ky_1 + y_0}{k+1})$
 (b) $k^2(x_1^2 - 4ay_1) + (x_0^2 - 4ay_0) = 0$

Exercise 9J (Page 352)

- 1(b) $x = at$, $y = \frac{at^2}{2}$ (d) A parabola, axis the y -axis, vertex the origin, focal length $\frac{a}{2}$.
 2(b) $T(at, 0)$ (c) $M(\frac{3at}{2}, \frac{at^2}{2})$
 3(c) The locus of R is $x^2 = 2(y - 2)$.
 4(b) $M(\frac{1}{2}at, a + \frac{1}{2}at^2)$
 6(a) $Q(2at, -a)$ (c) $x^2 = 2a(y - a)$. A parabola, axis the y -axis, vertex $(0, a)$, focal length $\frac{1}{2}a$.
 7(a) $y = px + a$ (c) $y = \frac{1}{2}a$ (e) $x^2 = a(y - a)$
 8(a) $N(\frac{1}{2}at^2, \frac{3}{2}at)$
 9(d) A parabola, axis the y -axis, vertex $(0, 4a)$, focal length $\frac{1}{2}a$.
 10(b) $T(at, 0)$ (c) $y = 0$ (the x -axis)

- 12(a) $y = px - ap^2$ (b) $y = px$ (d) $y = 2px - 4ap^2$
 (f) $2x^2 = 9ay$
 13(a) $M(a(p+q), \frac{1}{2}a(p^2 + q^2))$, $x^2 = 4a(y - \frac{a}{4})$
 (b) A parabola, axis the y -axis, vertex $(0, \frac{a}{4})$, focal length a .
 14(c) $y = -4a$ (e) $x^2 = 16a(y - 6a)$
 15(a) $y = px - ap^2$, $px - p^2y - a = 0$
 (b) $y = a$, for $x > 2a$ or $x < -2a$
 16(c) $(4(p+q), 4pq)$
 17(c) $x^2 = a(y - 3a)$
 18(a) $x + py = 2p + p^3$ (d) $N(2m, 4m^2 + 3)$
 (e) $y = x^2 + 3$
 20(c) $x = 3$ and $y < \frac{3}{4}$
 (d) The points P and Q coincide, in which case T is not uniquely defined, but the limit of its position coincides with P and Q .
 21(c) $x^2 = 8ay$
 22(b) $x^2 = 4a(y - 4a)$
 (d) 3, since the cubic equation in part (c) has at most three real solutions for t .
 23(c) $y = \frac{1}{4}x^2 - x + 2$
 24(d) $N(2mk, 2a + k + 4am^2)$
 (e) $y = \frac{x}{2m} + 2a(1 + 2m^2)$
 26(c) $y = -a$, which is the directrix.

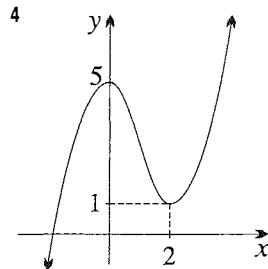
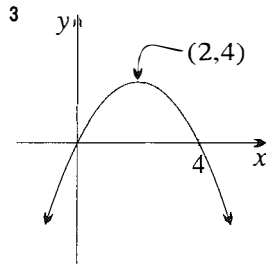
Chapter Ten

Exercise 10A (Page 359)

1(a) A, G and I (b) C and E (c) B, D, F and H

3(a) $4 - 2x$ (b)(i) $x < 2$ (ii) $x > 2$ (iii) $x = 2$

4(a) $3x^2 - 6x$ (b)(i) $x < 0$ or $x > 2$ (ii) $0 < x < 2$
(iii) $x = 0$ or $x = 2$



5(a) $\frac{3}{x^2}$

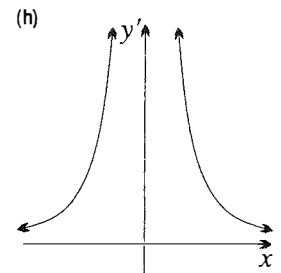
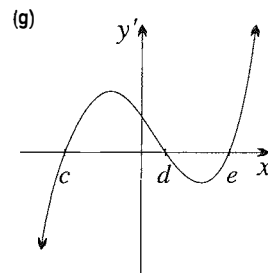
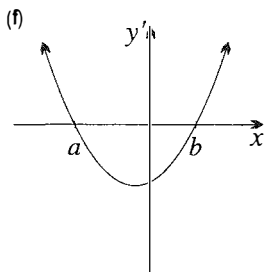
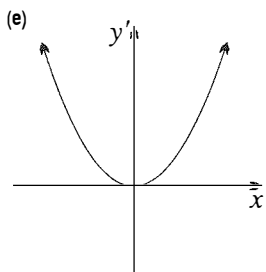
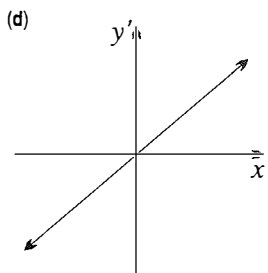
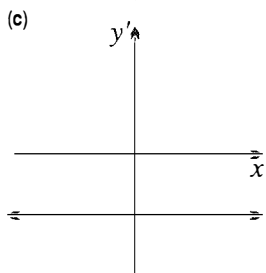
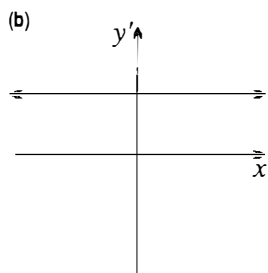
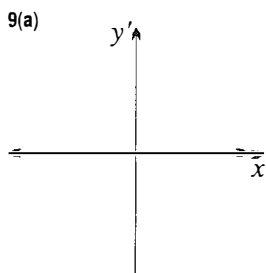
(b) The function is not continuous at $x = 0$.

6(a) $x > 2$ (b) $x < -3$ (c) $x > 1$ or $x < -1$

(d) $x < 0$ or $x > 2$

7(a) $x < -1$ or $x > -\frac{1}{3}$ (b) $x < -2$ or $0 < x < 2$

8(a) III (b) I (c) IV (d) II



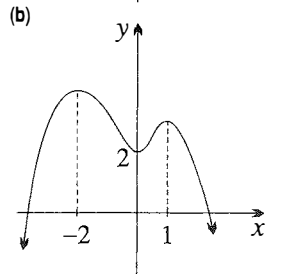
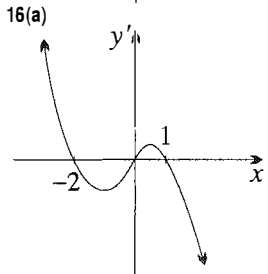
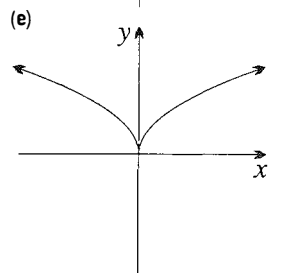
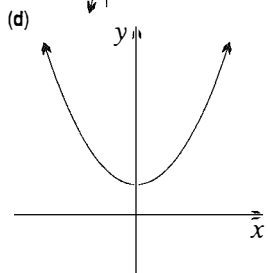
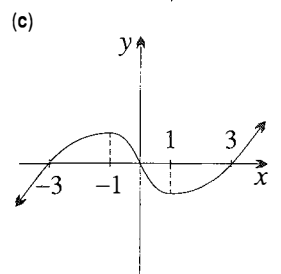
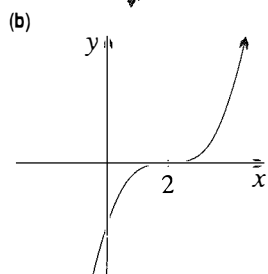
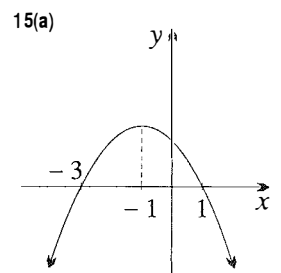
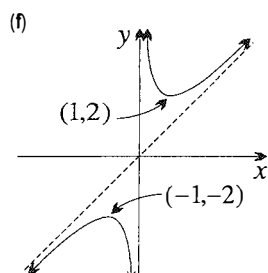
10(a) $-\frac{6}{(x-3)^2}$ (b) $\frac{x^2(x^2+3)}{(x^2+1)^2}$

11(a) $x^2 + 2x + 5$

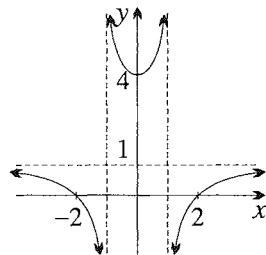
13(b) 1

14(c)(i) $x < -1$ or $x > 1$ (ii) $-1 < x < 1, x \neq 0$

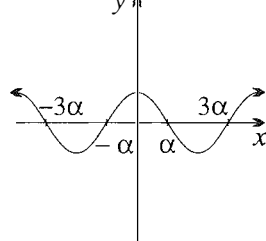
(iii) $x = -1$ or $x = 1$ (d) 2, -2 (e) $x = 0$



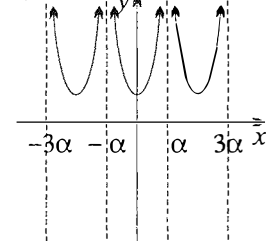
17



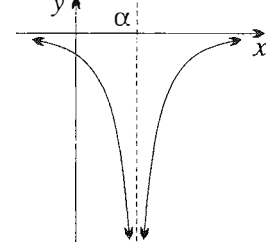
20(a)



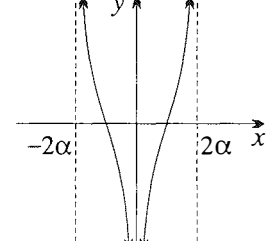
(c)



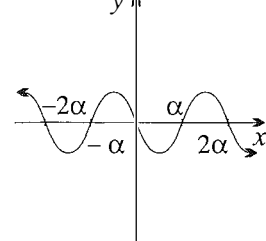
21(a)



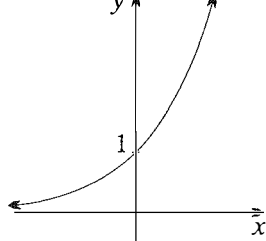
(c)

18 $-2 < x < 0$ 19(a)(i) $\frac{-4x}{(x^2 + 1)^2}$ (b) $f(x)$ is increasing for $x < 0$ and decreasing for $x > 0$.

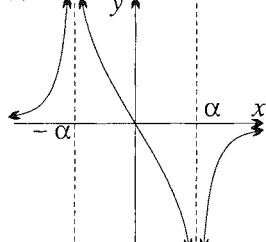
(b)



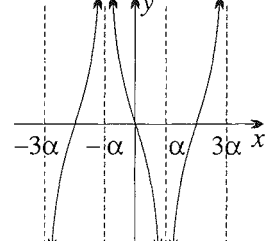
(d)



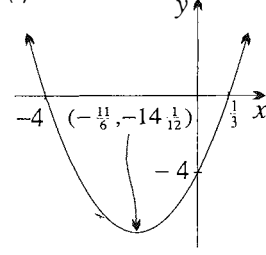
(b)



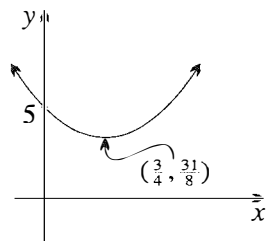
(d)



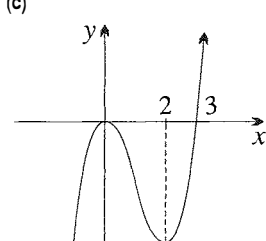
(c)



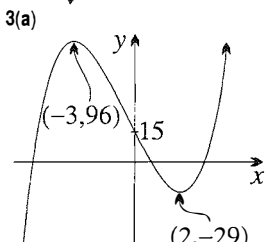
2(a)



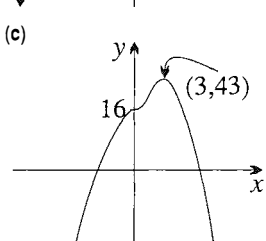
(c)



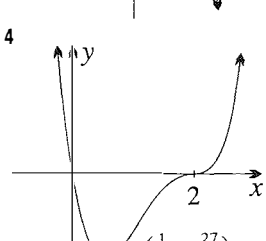
3(a)



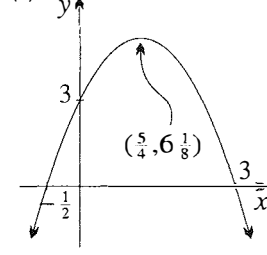
(c)



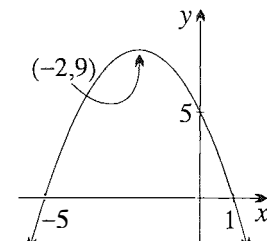
4



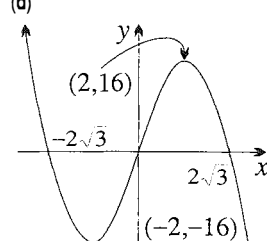
(d)



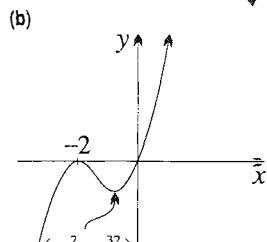
(b)



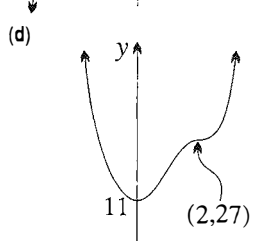
(d)



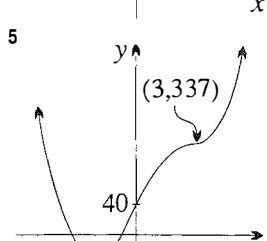
(b)



(d)

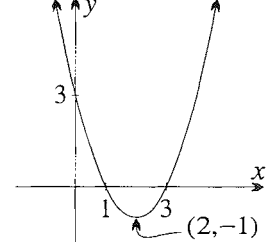


5

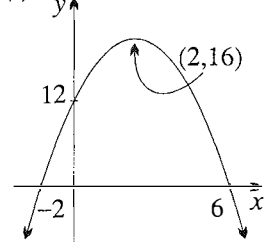


Exercise 10B (Page 365)

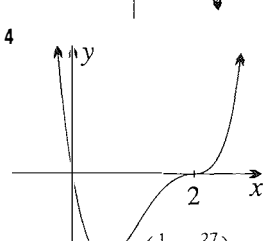
1(a)



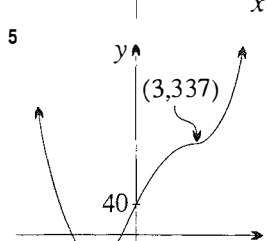
(b)



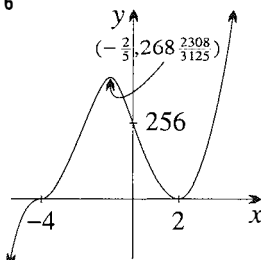
4



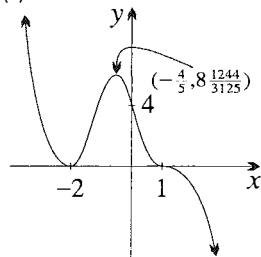
5



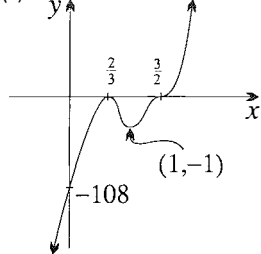
6



(b)



(d)



(d)(i) 0 (ii) 1 distinct root (iii) 2 (iv) 1

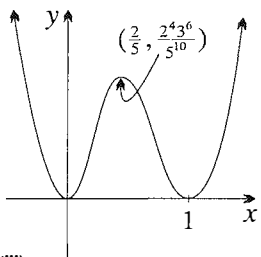
9 -8

 10 $a = b = -1, c = 6$

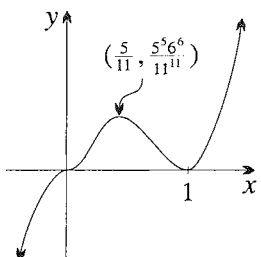
 11 $a = -1, b = 2, c = 0$

 12 $a = 2, b = 3, c = -12, d = 7$

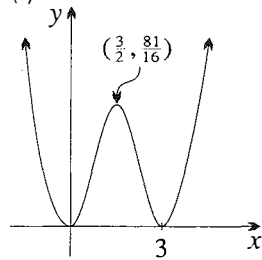
13(a)(i)



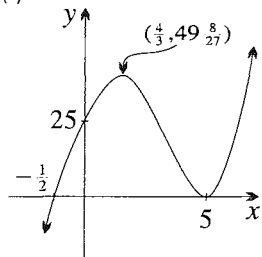
(iii)



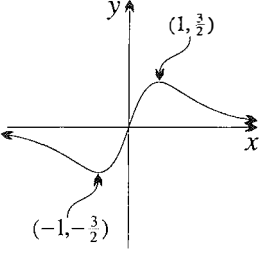
7(a)



(c)



8



Exercise 10C (Page 369)

 1(a) A relative maximum, B relative minimum
 (b) C relative minimum (c) D horizontal point of inflexion, E relative maximum

 (d) F relative minimum, G relative maximum, H relative minimum (e) I relative minimum
 (f) J horizontal point of inflexion, K relative minimum, L relative maximum

 2(a) $x = 1$ turning point

 (b) $x = 3$ turning point, $x = -\frac{1}{2}$ turning point

 (c) $x = 0$ turning point,
 $x = 3$ horizontal point of inflexion

 (d) $x = -2$ turning point, $x = 4$ turning point

 (e) $x = 0$ turning point, $x = 1$ critical value

 (f) $x = 0$ horizontal point of inflexion,
 $x = 1$ critical value

 (g) $x = 0$ turning point, $x = 1$ critical value

 (h) $x = 0$ horizontal point of inflexion,
 $x = 1$ critical value (i) $x = 0$ critical value

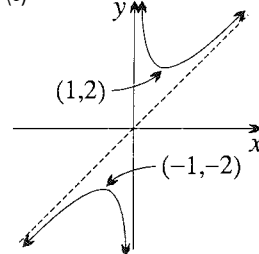
 (j) $x = -1$ turning point, $x = 1$ turning point,
 $x = 0$ critical value

 (k) $x = 1$ turning point, $x = 0$ critical value

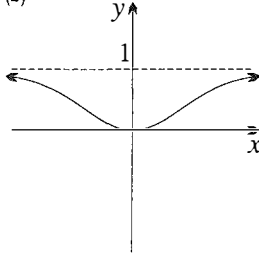
 (l) $x = 2$ turning point, $x = -2$ critical value,
 $x = 1$ critical value

 3(a) $x \neq 0$

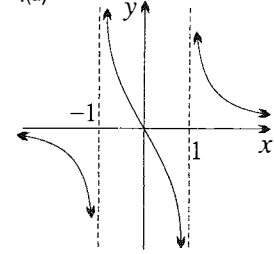
(e)



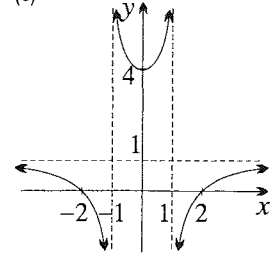
(b)

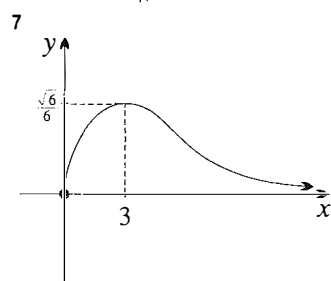
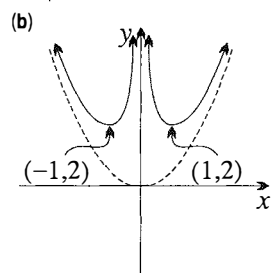
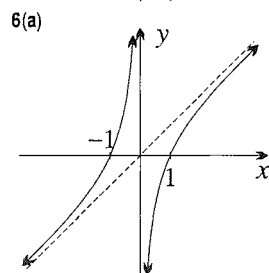
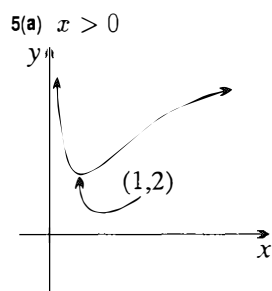
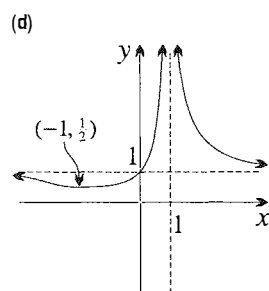


4(a)



(c)





(a) domain: $x \geq 0$.

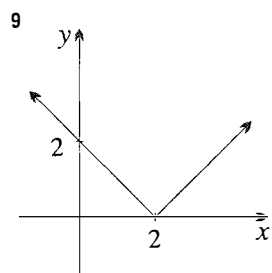
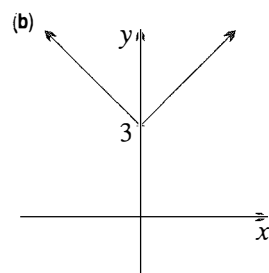
horizontal asymptotes: $x = 0$

(c) $(3, \frac{1}{6}\sqrt{6})$ is a maximum turning point.

(d) As $x \rightarrow 0^+$, $y \rightarrow 0$ and $y' \rightarrow \infty$, so the curve emerges vertically from the origin. (Notice that $y(0) = 0$, so the origin lies on the curve.)

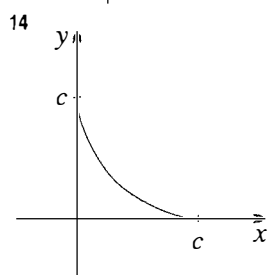
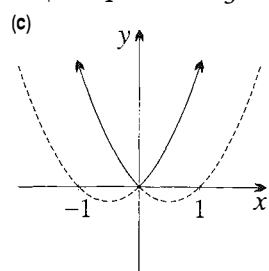
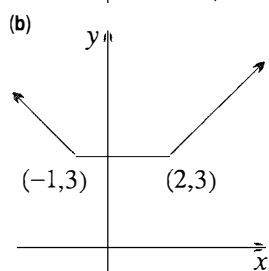
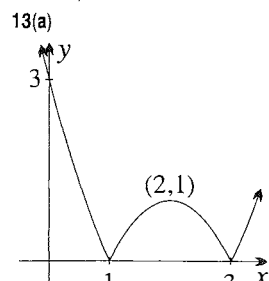
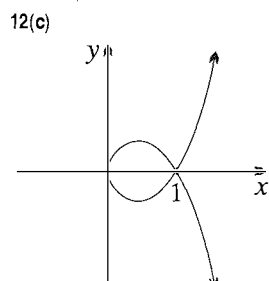
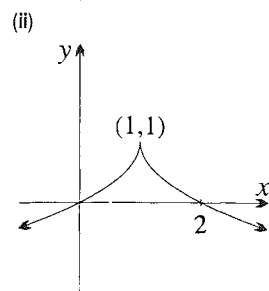
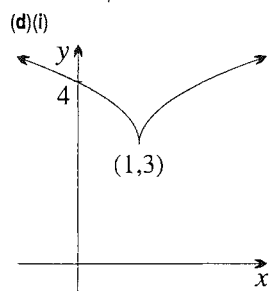
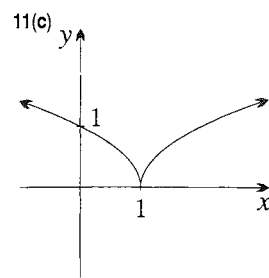
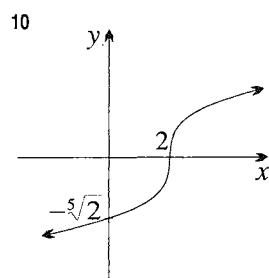
8(a) When $x < 0$, $dy/dx = -1$.

When $x > 0$, $dy/dx = 1$.



(a) When $x > 2$, $dy/dx = 1$.

When $x < 2$, $dy/dx = -1$ (b) $x = 2$ is a critical value, because y' is not defined there.



Exercise 10D (Page 371)

1(a) $10x^9$, $90x^8$, $720x^7$ (b) $15x^4$, $60x^3$, $180x^2$

(c) -3 , 0 , 0 (d) $2x - 3$, 2 , 0

(e) $12x^2 - 2x$, $24x - 2$, 24

(f) $0.3x^{-0.7}$, $-0.21x^{-1.7}$, $0.357x^{-2.7}$

(g) $-\frac{1}{x^2}$, $\frac{2}{x^3}$, $-\frac{6}{x^4}$ (h) $-\frac{2}{x^3}$, $\frac{6}{x^4}$, $-\frac{24}{x^5}$

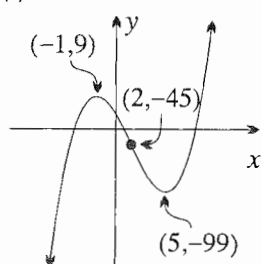
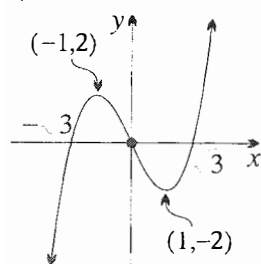
(i) $-\frac{15}{x^4}$, $\frac{60}{x^5}$, $-\frac{300}{x^6}$ (j) $2x - \frac{1}{x^2}$, $2 + \frac{2}{x^3}$, $-\frac{6}{x^4}$

- 2(a) $2(x+1)$, 2 (b) $9(3x-5)^2$, $54(3x-5)$
 (c) $8(4x-1)$, 32 (d) $-11(8-x)^1$, $110(8-x)^9$
 3(a) $\frac{-1}{(x+2)^2}$, $\frac{2}{(x+2)^3}$ (b) $\frac{2}{(3-x)^3}$, $\frac{6}{(3-x)^4}$
 (c) $\frac{12}{(5x+4)^4}$, $\frac{300}{(5x+4)^5}$
 (d) $\frac{12}{(4-3x)^3}$, $\frac{108}{(4-3x)^4}$
 4(a) $\frac{1}{2\sqrt{x}}$, $\frac{-1}{4x\sqrt{x}}$ (b) $\frac{1}{3}x^{-\frac{2}{3}}$, $-\frac{2}{9}x^{-\frac{5}{3}}$
 (c) $\frac{3}{2}\sqrt{x}$, $\frac{3}{4\sqrt{x}}$ (d) $-\frac{1}{2}x^{-\frac{3}{2}}$, $\frac{3}{4}x^{-\frac{5}{2}}$
 (e) $\frac{1}{2\sqrt{x+2}}$, $\frac{-1}{4(x+2)^{\frac{3}{2}}}$ (f) $\frac{-2}{\sqrt{1-4x}}$, $\frac{-4}{(1-4x)^{\frac{3}{2}}}$
 5(a)(i) $2\frac{13}{16}$ (ii) $\frac{3}{8}$
 (b)(i) -8 (ii) 48 (iii) -192 (iv) 384
 6 $a = -\frac{1}{2}$, $b = 3$, $c = \frac{5}{2}$
 7(a) $\frac{1}{(x+1)^2}$, $\frac{-2}{(x+1)^3}$ (b) $\frac{7}{(2x+5)^2}$, $\frac{-28}{(2x+5)^3}$
 (c) $\frac{1-x^2}{(1+x^2)^2}$, $\frac{2x(x^2-3)}{(1+x^2)^3}$
 8 $(x-1)^3(5x-1)$, $4(x-1)^2(5x-2)$
 9(a) 1, -1 (b) $-\frac{1}{3}$ (c) $-\frac{4}{3}$
 11(a) nx^{n-1} , $n(n-1)x^{n-2}$, $n(n-1)(n-2)x^{n-3}$
 (b) $n(n-1)(n-2)\dots 1, 0$
 12(a) $\frac{2}{3}$, 0 (b) $\frac{-1}{2t^2}$, $\frac{1}{2t^3}$ (c) $-\frac{3}{5}t^2$, $\frac{6}{25}t$
 (d) $\frac{t^2-1}{1+t^2}$, $\frac{4t^3}{(1+t^2)^3}$ (e) $\frac{3}{2(t-2)}$, $\frac{-3}{4(t-2)^3}$
 (f) $-\frac{(1-t)^2}{(1+t)^2}$, $\frac{2(1-t)^3}{(1+t)^3}$
 13 $a = 3$, $b = 4$
 14 $\frac{1}{r}$

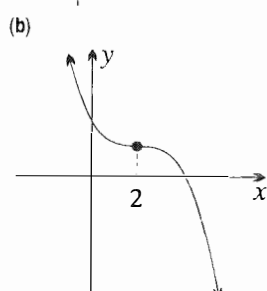
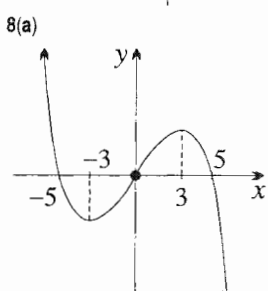
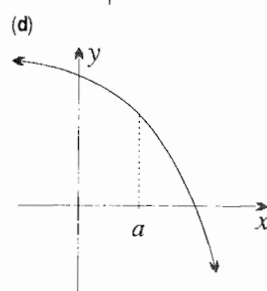
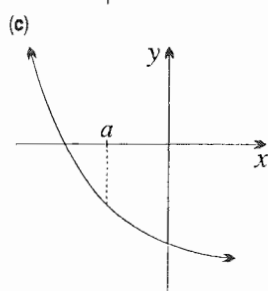
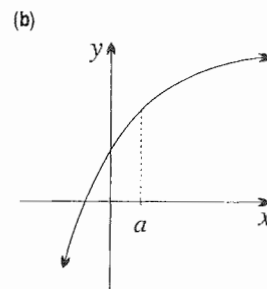
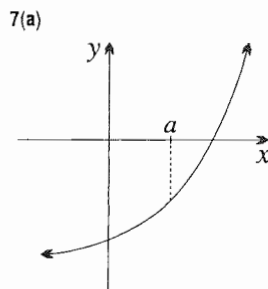
Exercise 10E (Page 376)

Point	A	B	C	D	E	F	G	H	I
1 y'	0	+	0	-	0	-	0	+	0
y''	+	0	-	0	0	0	+	0	0

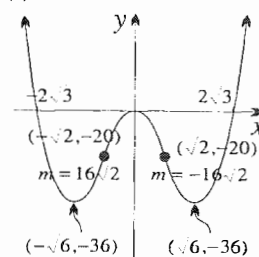
- 3(a) $f'(x) = 3x^2 - 3$, $f''(x) = 6x$ (a) $f'(x) = 3x^2 - 12x - 15$, $f''(x) = 6x - 12$
 (d)



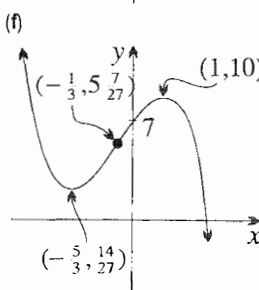
- 5(a) $x > 2$ or $x < -1$ (b) $-1 < x < 2$ (c) $x > \frac{1}{2}$
 (d) $x < \frac{1}{2}$
 6(a) $x = -5$ (b) none (c) $x = 3$, $x = -2$
 (d) $x = -2$



- 10(a) $f'(x) = 4x^3 - 24x$, $f''(x) = 12x^2 - 24$ (e)



- 11(a) $f'(x) = 5 - 2x - 3x^2$, $f''(x) = -2 - 6x$
 (e) $\frac{16}{3}$



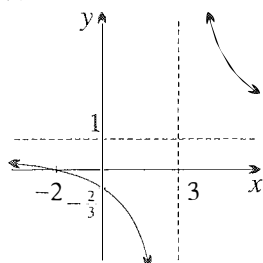
- 12(a) $y' = 3x^2 + 6x - 72$, $y'' = 6x + 6$
 (d) $75x + y - 13 = 0$
 13(b) $f''(x) = g''(x) = 0$, no
 (c) $f(x)$ has a horizontal point of inflexion.

$g(x)$ has a minimum turning point.

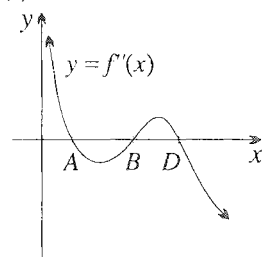
14 $a = 2$, $b = -3$, $c = 0$ and $d = 5$.

15(b) concave up when $x > 3$,
concave down when $x < 3$

(c)

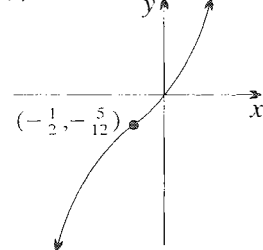


(b)

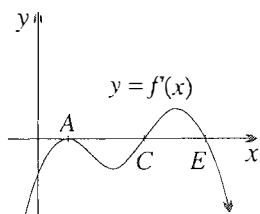


18(a) $f'(x) = x^2 + x + 1$,
 $f''(x) = 2x + 1$

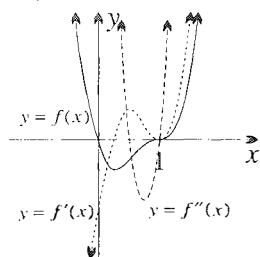
(d)



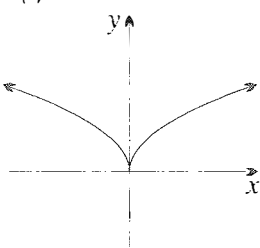
16(a)



17(b)

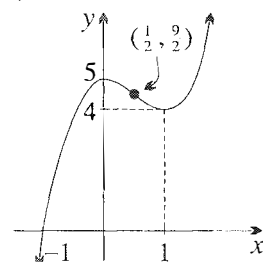


19(b)

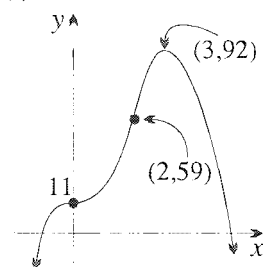


Exercise 10F (Page 379)

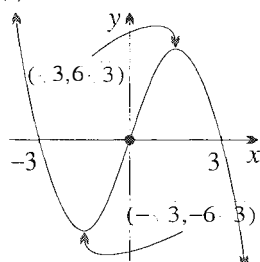
1(a)



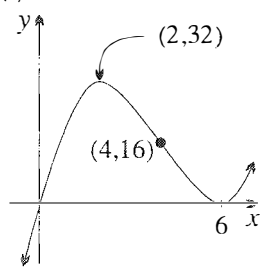
(c)



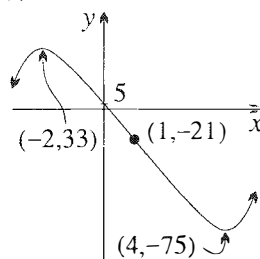
(b)



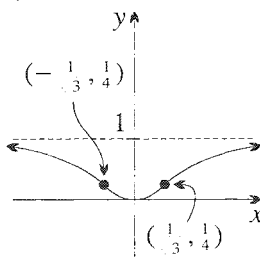
(d)



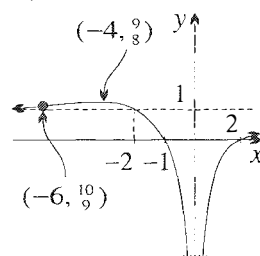
(e)



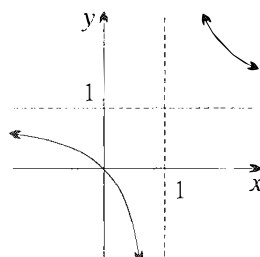
2(e)



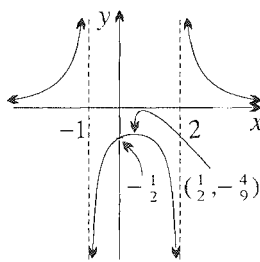
4(a)



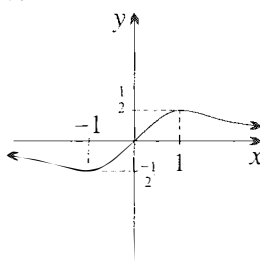
5(a)



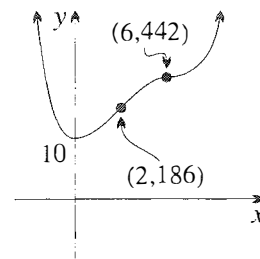
(c)



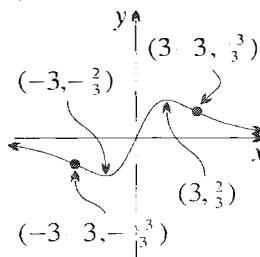
(e)



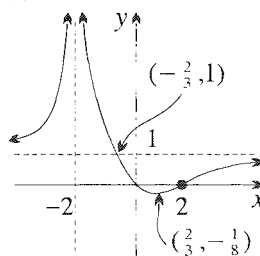
(f)



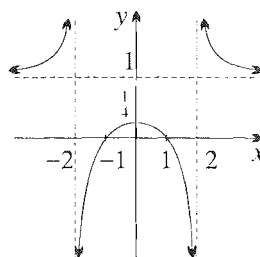
3(e)



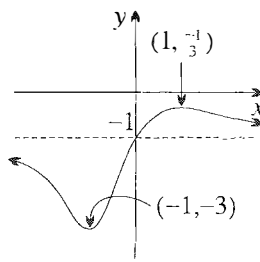
(b)



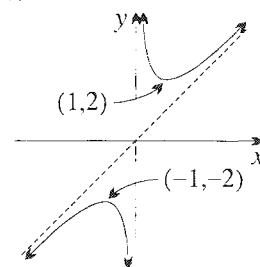
(b)

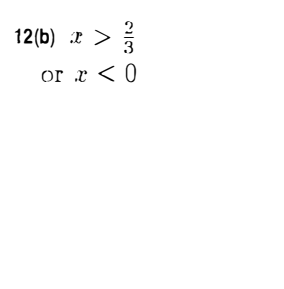
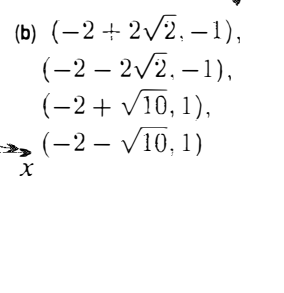
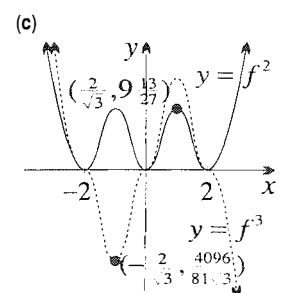
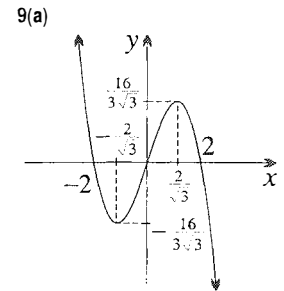
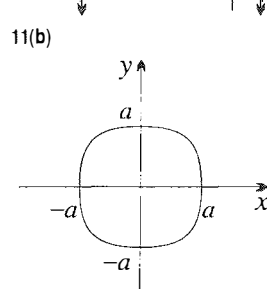
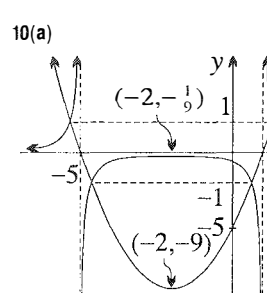
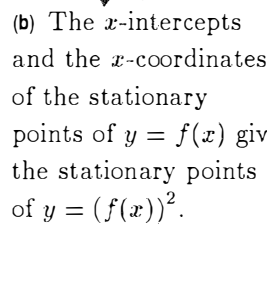
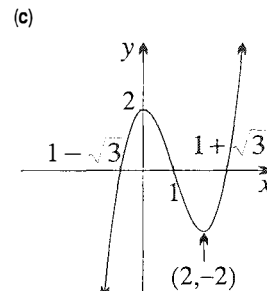
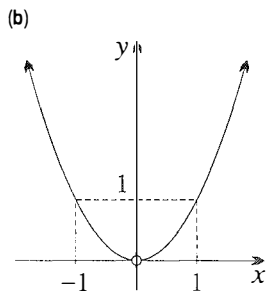
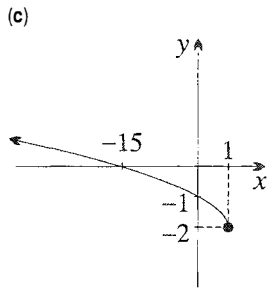
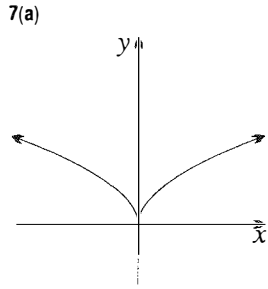
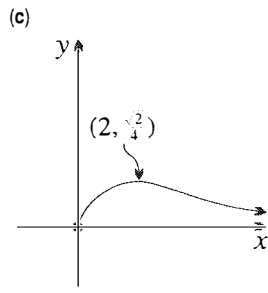
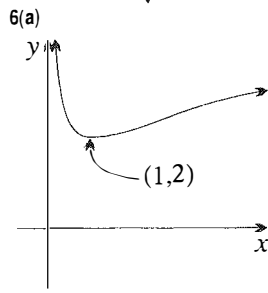
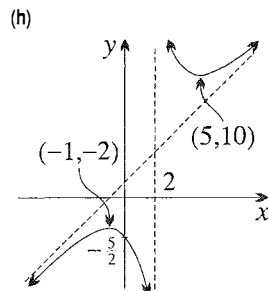
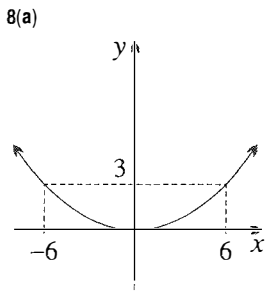
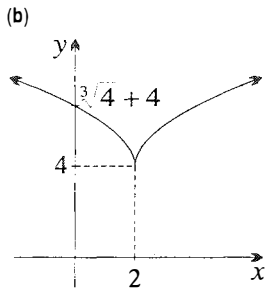
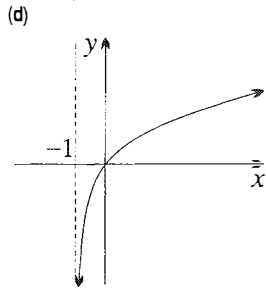
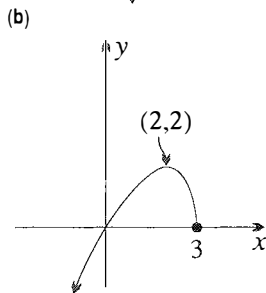
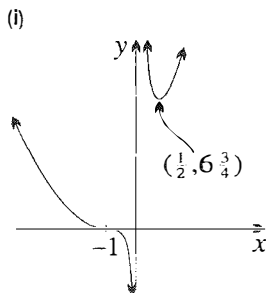
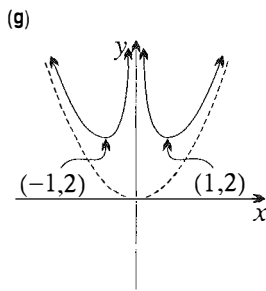


(d)



(f)





Exercise 10G (Page 382)

- 1(a) A relative maximum, B relative minimum
 (b) C absolute maximum, D relative minimum, E relative maximum, F absolute minimum
 (c) G absolute maximum, H horizontal point of inflexion (d) I horizontal point of inflexion, J absolute minimum
 2(a) 0, 4 (b) 2, 5 (c) 0, 4 (d) 0, 5 (e) 0, $2\sqrt{2}$
 (f) $-1, -\frac{1}{4}$ (g) 1, 2 (h) $-1, 2$
 3(a) 7, 7 (b) $\frac{1}{16}, \frac{1}{9}$ (c) $-1, 8$ (d) $-49, 5$ (e) 0, 4
 (f) 0, 9
 4(a) global minimum -5 , global maximum 20
 (b) global minimum -5 , local maximum 11, global maximum 139

- (c) global minimum 4, global maximum 11
 5(a) 1, 5 (b) 1, $\sqrt{5}$ (c) $\frac{1}{5}\sqrt{5}$, 1 (d) $-\frac{2}{5}\sqrt{5}$, $\frac{1}{2}\sqrt{2}$
 7(a) 2, 19, 181 (b) infinitely many
 (c) As $x \rightarrow \infty$, $y \rightarrow 0$. There is no limit as $x \rightarrow 0^+$ — however close one goes towards 0, the function takes every value in the interval $-1 \leq y \leq 1$.
 (d) The global maximum and minimum are 1 and -1 respectively, and they are attained infinitely often.

Exercise 10H (Page 384)

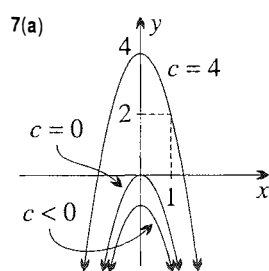
- 1(a) $2x^2 - 16x + 64$ (b) 4 (c) 32
 2(a) $11x - 2x^2$ (b) $\frac{11}{4}$ (c) $\frac{121}{8}$
 3(b) $3\frac{1}{8}$ metres
 4(b) 5 (c) 25 cm^2
 5(c) 20 (d) 200 m^2
 6(a) $R = x(47 - \frac{1}{3}x)$ (b) $-\frac{8}{15}x^2 + 32x - 10$ (c) 30
 7(a) 20 and 20 (b) 20 and 20 (c) 24 and 16
 8(b) 24 cm
 9(a) $\frac{x}{4}$, $\frac{10-x}{4}$ (c) 5 (d) $\frac{25}{8}\text{ m}^2$
 10(c) $\frac{40\,000}{\pi}\text{ m}^2$
 11(c) $4 \times 4 \times 2$
 12(c) Each of the 6 rectangles has dimensions $\frac{3}{4} \times \frac{2}{3}$.
 13 $27 \times 9 \times 18$
 14(b) 80 km/hr (c) \$400
 15 $12 \times 8 \times 8$
 16 $\frac{3}{2}$ sq units
 17 $\frac{7}{2}$ units
 18 8 units
 19(b) $y = -2ax + a^2 + 4$ (c) $\frac{2}{3}\sqrt{3}$
 20(b) $(\frac{1}{2}, \frac{1}{4})$
 21 64 cm^2
 22 width $16\sqrt{3}\text{ cm}$, depth $16\sqrt{6}\text{ cm}$
 23(b) $\frac{2}{3}a \times \frac{4}{3}a\sqrt{3}$
 24 $A(10, 0)$, $B(0, 6)$
 25 $5\sqrt{2}$ metres
 26 8 km
 27 $2(\sqrt{10} + 1) \times 4(\sqrt{10} + 1)$
 28(a) $\frac{10}{3}$ (b) $\frac{1}{6}(a + b - \sqrt{a^2 + b^2 - ab})$
 30 $2\frac{34}{39}$

Exercise 10I (Page 389)

- 1(b) $\frac{1000}{27\pi}\text{ m}^3$
 2(c) $20\sqrt{10}\pi\text{ cm}^3$
 4(d) $\frac{2}{27}\pi s^3\sqrt{3}$
 5(a) $\theta = \frac{180(\ell - 2r)}{\pi r}$
 8(b) $\frac{\pi R^2 h(r - R)}{r}$ (c) $\frac{4}{27}\pi r^2 h$
 9 $\frac{\pi rh}{2}$
 11(b) $V = \frac{5}{2}r - \frac{1}{2}\pi r^3$
 12(b) $30\text{ cm} \times 40\text{ cm}$
 13(c) $2\pi R^2$
 14(a) $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (c) $A = 2ab$, $x = \frac{1}{2}a\sqrt{2}$
 17 4:3
 18(a) $4\sqrt{3}\text{ cm}^2$
 19 $\frac{1}{2}a^2$. This problem can be done very easily without calculus — the triangle has area $\frac{1}{2}a^2 \sin \theta$, where θ is the apex angle, and $\sin \theta$ is maximum when $\theta = 90^\circ$.
 21 $3\sqrt{3}r^2$
 22 There is only one such cone. Its height is 48 cm and its radius is $12\sqrt{2}\text{ cm}$.
 23 $\frac{4}{41}x\sqrt{2}(7 - 2\sqrt{2})$

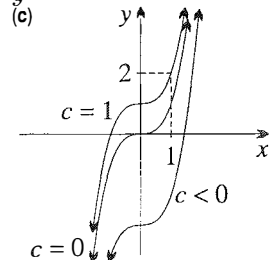
Exercise 10J (Page 394)

- 1(a) $\frac{1}{7}x^7 + C$ (b) $\frac{3}{2}x^2 + C$ (c) $5x + C$ (d) $\frac{1}{2}x^{10} + C$
 (e) $3x^7 + C$ (f) $\frac{1}{39}x^{13} + C$ (g) C
 (h) $\frac{2}{3}x^3 + \frac{5}{8}x^8 + C$ (i) $x^3 - x^4 - x^5 + C$
 (j) $\frac{1}{4}ax^4 + \frac{1}{3}bx^3 + C$ (k) $\frac{x^{a+1}}{a+1} + C$
 (l) $\frac{ax^{a+1}}{a+1} + \frac{bx^{b+1}}{b+1} + C$
 2(a) $\frac{1}{3}x^3 - \frac{3}{2}x^2 + C$ (b) $\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$
 (c) $x^3 + \frac{11}{2}x^2 - 4x + C$ (d) $\frac{5}{6}x^6 - x^4 + C$
 (e) $\frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C$ (f) $\frac{1}{4}a^2x^4 - 2ax^3 + \frac{9}{2}x^2 + C$
 3(a) $-\frac{1}{x} + C$ (b) $-\frac{1}{x^2} + C$ (c) $-\frac{1}{3x} + C$
 (d) $\frac{2}{15x^3} + C$ (e) $-\frac{1}{x} + \frac{1}{2x^2} + C$ (f) $-\frac{a}{bx} + C$
 (g) $\frac{1}{(1-a)x^{a-1}} + C$ (h) $x + \frac{x^{b-a+1}}{b-a+1} + C$
 4(a) $\frac{2}{3}x^{\frac{3}{2}} + C$ (b) $2\sqrt{x} + C$ (c) $\frac{3}{4}x^{\frac{1}{3}} + C$ (d) $\frac{4}{3}\sqrt{x} + C$
 (e) $\frac{5}{8}x^{\frac{5}{8}} + C$
 5(a) $y = x^2 + 3x + 4$ (b) $y = 3x^3 + 4x + 1$
 (c) $y = \frac{2}{3}x^{\frac{3}{2}} - 16$ 6 The rule would give the primitive of x^{-1} as $x^0/0$, which is meaningless. This problem is resolved in Chapter 12.



$$y = -2x^2 + C,$$

$$y = 4 - 2x^2$$



$$y = x^3 + C,$$

$$y = x^3 + 1$$

8(a) $\frac{1}{4}(x+1)^4 + C$ (b) $\frac{-1}{3(x-2)^3} + C$

(c) $\frac{1}{21}(3x-4)^7 + C$ (d) $-\frac{1}{28}(1-7x)^4 + C$

(e) $\frac{1}{6}(ax-b)^6 + C$ (f) $\frac{2}{81(1-9x)^9} + C$

9(a) $\frac{2}{3}(x+1)^{\frac{3}{2}} + C$ (b) $-\frac{2}{3}(1-x)^{\frac{3}{2}} + C$

(c) $\frac{1}{3}(2x-7)^{\frac{3}{2}} + C$ (d) $-\frac{2}{3}\sqrt{2-3x} + C$

(e) $\frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$ (f) $\frac{3}{4}\sqrt{4x-1} + C$

10(a) $\frac{1}{8}(2x+1)^4 - \frac{9}{8}$ (b) $y = x^3 + 2x^2 - 5x + 6$

(c) $y = \frac{4}{15}(3-x)^{\frac{5}{2}} + \frac{32}{3}x + 16\frac{2}{3}$

11(a) $\frac{x^{a+b+1}}{a+b+1} + C$ (b) $\frac{x^{a-b+1}}{a-b+1} + C$

(c) $\frac{x^{a+b+1}}{ab+1} + C$ (d) $\frac{ax^{b+1}}{b+1} + \frac{bx^{a+1}}{a+1} + C$

(e) $\frac{2}{5}x^{\frac{5}{2}} + C$ (f) $2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$

(g) $\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$ (h) $\frac{1}{2}x - \sqrt{x} + C$

12(a) $y = \frac{3}{5}x^5 - \frac{1}{4}x^4 + x$ (b) $y = -\frac{1}{4}x^4 + x^3 + 2x - 2$

(c) $y = -\frac{1}{20}(2-5x)^4 + \frac{21}{20}$ (d) $y = x^2 - 4x + 4$

13 30 14(a) -121 (b) $y = -x^3 + 4x^2 + 3$

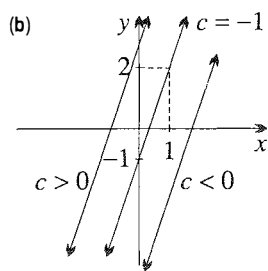
15 25 seconds

16 \$4.08. Since $p = \frac{10}{t^3} + 4$, and $t > 0$, the price will always exceed \$4 (but by a decreasing amount).

17(a) $x = -\frac{7}{2}$ (b) $x = -18\frac{2}{3}$ 18 3 seconds

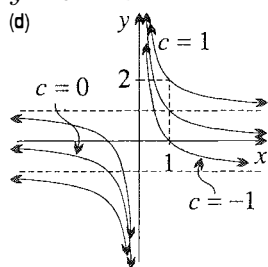
19 $f(x) = \frac{1}{x} + 1$ for $x > 0$,

and $f(x) = \frac{1}{x} + 3$ for $x < 0$



$$y = 3x + C,$$

$$y = 3x - 1$$



$$y = \frac{1}{x} + C,$$

$$y = \frac{1}{x} + 1$$

Chapter Eleven

Exercise 11A (Page 401)

1(a) 8 (b) 24 (c) 8π (d) $\frac{25}{4}\pi$ (e) 30 (f) 15

2(b) When $a = \frac{1}{4}$, the integral is $\frac{1}{192}$.

When $a = \frac{1}{2}$, it is $\frac{1}{24}$. When $a = \frac{3}{5}$, it is $\frac{9}{125}$.

When $a = \frac{4}{5}$, it is $\frac{64}{375}$.

5(a) $\frac{1}{3} + \frac{1}{6n^2}$ (b) The lines $P_0P_1, P_1P_2, \dots$ lie above the curve. Therefore the combined area of the trapezia is greater than the area under the curve.

Exercise 11B (Page 406)

1(a) 15 (b) $41\frac{2}{3}$ (c) 19 (d) 62 (e) 30 (f) 3. The

notation $\int_4^7 dx$ means $\int_4^7 1 dx$, which is $[x]_4^7$.

(g) 4 (h) $33\frac{3}{4}$ (i) $66\frac{2}{3}$

2(a)(i) $\frac{1}{10}$ (ii) $\frac{5}{36}$ (iii) $3\frac{3}{4}$ (b)(i) $\frac{1}{2}$ (ii) $\frac{15}{32}$ (iii) 7

3(a) 4 (b) $32\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $13\frac{1}{3}$ (e) $\frac{8}{15}$ (f) $27\frac{1}{2}$

4(a) $1 + \frac{\pi}{2}$ (b) $2\frac{1}{2}$

5(a) $k = 5$ (b) $k = \frac{4}{9}$ (c) $k = 5$

6 The function is discontinuous at $x = 0$, which lies in the given interval.

7(a) $4x^3 - 3x^2 + x - 1$ (b) $-(7-6x)^4$ (c)(i) $(a-x)u(x)$

8 The function defined by $f(x) = 0$ for $x \neq \frac{1}{2}$, and $f(\frac{1}{2}) = 1$ satisfies the conditions. This function, however, is not continuous, and so a slight extension of our definition of the definite integral is required.

Exercise 11C (Page 410)

1(a) $\frac{1}{4}\pi$ (b) $-\frac{9}{2}\pi$ (c) -6 (d) -8 (e) $\frac{3}{2}$ (f) -8

2(a) $3\frac{3}{4}$ (b) 0 (c) -36 (d) 0 (e) 12 (f) 0

3(a)(i) $\frac{2}{3}$ (ii) 2 (iii) $\frac{45}{4}$ (b)(i) $\frac{14}{3}$ (ii) $96\frac{4}{5}$ (iii) 4

4(a) $-\frac{2}{3}$ (b) $2\frac{2}{3}$ (c) $1\frac{1}{32}$ (d) $143\frac{3}{4}$ (e) $3\frac{1}{3}$ (f) $42\frac{3}{4}$

5(a) $\frac{\pi}{2} - \frac{1}{2}$ (b) $1 - \frac{\pi}{2}$

6(a) $k = 3$ or $k = -5$ (b) $k = 2$ or $k = -\frac{8}{5}$

(c) $k = 1$

7(a) 0 (the interval has zero width).

(b) 0 (the interval has zero width).

(c) 0 (the integrand is odd).

(d) 0 (the integrand is odd).

(e) 0 (the integrand is odd).

(f) 0 (the integrand is odd).

8(a) The function is odd, so the integral is zero.

(b) The integral can be split into the sum of the six integrals. Each odd power is an odd function, and so has integral zero. Each even power is an even

function, and so its integral is twice the integral over $0 \leq x \leq \alpha$.

9(a) The curves meet at $(0, 0)$ and at $(1, 1)$.

(b) In the interval $0 \leq x \leq 1$, the curve $y = x^2$ is always below the curve $y = \sqrt{x}$.

10(a)(i) 5 (ii) $\frac{25}{2}$ (b)(i) 25 (ii) $17\frac{1}{2}$ (iii) $27\frac{1}{2}$

11(a)(i) $\frac{1}{2}$ (ii) 18 (iii) 8 (b)(i) $-\frac{1}{2}$ (ii) -18 (iii) -8

12(b)(i) 3 (ii) 4 (iii) $-\frac{7}{3}$ (iv) $\frac{10}{3}$ (v) $60\frac{2}{3}$ (vi) 33

13(a) True, as the function is odd.

(b) True, as $\sin 4x^\circ$ is odd and $\cos 2x^\circ$ is even.

(c) False, as $2^{-x^2} > 0$ for all x .

(d) True, as $2^x < 3^x$ for $0 < x < 1$.

(e) False, as $2^x > 3^x$ for $-1 < x < 0$.

(f) True, as $t^n > t^{n+1}$ for $0 \leq t \leq 1$ and hence $\frac{1}{1+t^n} \leq \frac{1}{1+t^{n+1}}$.

14(a) The integral is $-\frac{1}{N} + 1$, which converges to 1 as $N \rightarrow \infty$.

(b) The integral is $-1 + \frac{1}{\varepsilon}$, which diverges to ∞ as $\varepsilon \rightarrow 0^+$.

(c) The integral is $2\sqrt{N} - 2$, which diverges to ∞ as $N \rightarrow \infty$.

(d) The integral is $2 - 2\sqrt{\varepsilon}$, which converges to 2 as $\varepsilon \rightarrow 0^+$.

Exercise 11D (Page 413)

1(a) $4x + C$ (b) $x^2 + C$ (c) $x^3 + C$ (d) C (e) $\frac{2}{3}x^6 + C$

(f) $\frac{5}{7}x^{1.4} + C$ (g) $\frac{1}{2}x^{14} + \frac{1}{3}x^9 + C$

(h) $4x - \frac{3}{2}x^2 + C$ (i) $x^3 - 2x^4 + \frac{7}{5}x^5 + C$

(j) $\frac{a}{3}x^3 + \frac{b}{2}x^2 + C$ (k) $\frac{1}{a+1}x^{a+1} + C$

(l) $\frac{a}{a+1}x^{a+1} + \frac{b}{b+1}x^{b+1} + C$

2(a) $-\frac{1}{x} + C$ (b) $-\frac{1}{x^3} + C$ (c) $\frac{1}{10x^2} + C$

(d) $\frac{1}{4x^4} - \frac{1}{x} + C$ (e) $\frac{x^{1-a}}{1-a} + C$ (f) $\frac{x^{a-b+1}}{a-b+1} + C$

(g) $x^a + C$ (h) $x + \frac{x^{b-a+1}}{b-a+1} + C$

3(a) $\frac{2}{3}x^{\frac{3}{2}} + C$ (b) $\frac{3}{4}x^{\frac{4}{3}} + C$ (c) $2\sqrt{x} + C$ (d) $\frac{3}{5}x^{\frac{5}{3}} + C$

4(a) $\frac{5}{3}x^3 - \frac{3}{4}x^4 + C$ (b) $\frac{4}{3}x^3 + 2x^2 + x + C$

(c) $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C$ (d) $\frac{3}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + C$

(e) $\frac{1}{2}x^2 - 4x + C$ (f) $2x^2 - \frac{8}{3}x^{\frac{3}{2}} + x + C$

(g) $\frac{1}{2}x^2 - \frac{1}{x} + C$ (h) $\frac{1}{6}x^3 - \frac{1}{16}x^4 + C$

(i) $\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + C$

5(a) $\frac{1}{6}(x+1)^6 + C$ (b) $\frac{1}{4}(x+2)^4 + C$

(c) $-\frac{1}{5}(4-x)^5 + C$ (d) $\frac{1}{15}(3x+1)^5 + C$

(e) $-\frac{5}{4}(1-\frac{x}{5})^4 + C$ (f) $-\frac{1}{2(x+1)^2} + C$

(g) $\frac{1}{11}(2x-1)^{11} + C$ (h) $-\frac{1}{2(4x+1)^4} + C$

(i) $\frac{1}{5-20x} + C$

6(a) $\frac{2}{3}(x+1)^{\frac{3}{2}} + C$ (b) $\frac{1}{3}(2x-1)^{\frac{3}{2}} + C$

(c) $-\frac{1}{6}(7-4x)^{\frac{3}{2}} + C$ (d) $\frac{3}{16}(4x-1)^{\frac{4}{3}} + C$

(e) $\frac{2}{3}\sqrt{3x+5} + C$

(f) $-\frac{4}{3}(1-\frac{x}{2})^{\frac{3}{2}} + C$ (g) $2\sqrt{x+1} + 2\sqrt{x+2} + C$

(h) $-\frac{2}{3}(4-x)^{\frac{3}{2}} - 2\sqrt{4-x} + C$ (i) $\frac{2}{3a}(ax)^{\frac{3}{2}} +$

$\frac{2}{a}\sqrt{ax} + C$

7(a) $\frac{242}{5}$ (b) 0 (c) $121\frac{1}{3}$ (d) 1 (e) $\frac{1}{3}a^2 + ab + b^2$

(f) $\frac{13}{6}$ (g) 2 (h) $\frac{45}{4}$ (i) 46 (j) $6\frac{2}{3}$ (k) $\frac{112}{9}$ (l) $8\frac{2}{5}$

8(b) $\frac{2}{3}x(1+x)^{\frac{3}{2}} - \frac{4}{15}(1+x)^{\frac{5}{2}} + C$

9 Here is one clue: $\int_{-\infty}^{-1} \frac{dx}{x^2} = \int_1^{\infty} \frac{dx}{x^2} = 1$ (an extension question in the previous exercise explains the meaning of ∞ in the limits of integration).

Exercise 11E (Page 416)

1(a) $2u^2$ (b) $\frac{21}{2}u^2$ (c) $\frac{9}{2}u^2$

2(a) $9u^2$ (b) $6\frac{2}{3}u^2$ (c) $\frac{128}{3}u^2$ (d) $6u^2$ (e) $\frac{1}{4}u^2$

(f) $57\frac{1}{6}u^2$ (g) $36u^2$ (h) $60u^2$

3(a) $\frac{9}{2}u^2$ (b) $34\frac{2}{3}u^2$ (c) $18u^2$ (d) $2u^2$

4(a) $\frac{4}{3}u^2$ (b) $\frac{27}{2}u^2$ (c) $\frac{81}{4}u^2$ (d) $46\frac{2}{5}u^2$

5(a) $\frac{9}{2}u^2$ (b) $\frac{4}{3}u^2$ (c) $\frac{45}{4}u^2$ (d) $9u^2$

6 $8u^2$

7(a) $11\frac{2}{3}u^2$ (b) $128\frac{1}{2}u^2$ (c) $4u^2$ (d) $8\frac{1}{2}u^2$

(e) $16u^2$ (f) $6\frac{3}{4}u^2$ (g) $11\frac{5}{6}u^2$ (h) $32\frac{3}{4}u^2$ (i) $\frac{17}{3}u^2$

(j) $21\frac{2}{15}u^2$

8(a) $13u^2$ (b) $2\frac{1}{2}u^2$ (c) $9\frac{1}{3}u^2$ (d) $2u^2$

9(a)(i) $64u^2$ (ii) $128u^2$ (iii) $\frac{12}{5}\sqrt{3}$ (b)(i) $50u^2$

(ii) $18u^2$ (iii) $\frac{32}{3}u^2$

10(a) $4u^2$ (b) $\frac{1024}{15}u^2$ (c) $2\sqrt{3}u^2$ (d) $\frac{5}{3}\sqrt{5}u^2$

11(a) $(2, 0)$, $(0, 4\sqrt{2})$, $(0, -4\sqrt{2})$ (b) $\frac{16}{3}\sqrt{2}u^2$

13(a) $f(x) = \frac{1}{3}x^3 - x^2 - 3x$, relative maximum at $(-1, \frac{5}{3})$, relative minimum at $(3, -9)$ (b) $16\frac{5}{6}u^2$

15(a) $2 : n + 1$ (b) $1 : n + 1$

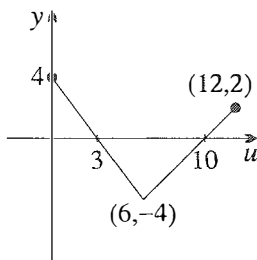
16(b) $a^2 = \frac{1}{2}(3 + \sqrt{5})$, $a^4 = \frac{1}{2}(7 + 3\sqrt{5})$,

$a^5 = \frac{1}{2}(11 + 5\sqrt{5})$

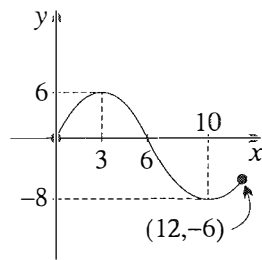
(c) Areas are $\frac{1}{5}u^2$, $\frac{1}{10}u^2$ and $\frac{1}{10}u^2$.

17(a) $\frac{2}{3}ah^3 + 2ch$

18(a)



(d)


 (b) maximum at $(3, 6)$, minimum at $(10, -8)$

 (c) 0.6 (e) $24u^2$

 19(a) $-\frac{1}{n+1}$ (b) $\frac{1}{n+1}$

Exercise 11F (Page 421)

 1(a) $20\frac{5}{8}u^2$ (b) $36u^2$ (c) $16\frac{2}{3}u^2$ (d) $\frac{9}{4}u^2$ (e) $9\frac{1}{3}u^2$

 (f) $\frac{1}{12}u^2$ (g) $\frac{1}{6}u^2$ (h) $\frac{4}{3}u^2$

 2(b) $4\frac{1}{2}u^2$

 3(a) $(-1, 16)$, $(5, 4)$ (b) $36u^2$

 4(a) $20\frac{5}{6}u^2$ (b) $57\frac{1}{6}u^2$ (c) $\frac{4}{15}u^2$ (d) $\frac{1}{32}u^2$

 5(b) $36u^2$

 6 $5\frac{5}{8}u^2$

 7(a) $4\frac{1}{2}u^2$ (b) $\frac{1}{2}u^2$ (c) $20\frac{5}{6}u^2$ (d) $21\frac{1}{3}u^2$

 8(b) $\frac{1}{3}u^2$

 9 $5\frac{1}{3}u^2$

 10(c) $108u^2$

 11(a) $4\frac{1}{2}u^2$ (b) $\frac{9}{16}u^2$

 12(a) $-1 < x < 1$ or $x > 4$ (b) $21\frac{1}{12}u^2$

 13 $\frac{1}{16}u^2$

 14(b) $y = 2x - 7$ (c) $\frac{7}{12}u^2$

 15 $1 - \frac{1}{\sqrt[3]{2}}$

Exercise 11G (Page 426)

 1(b) $81\pi u^3$

 2(b) $36\pi u^3$

 3(a) $16\pi u^3$ (b) $9\pi u^3$ (c) $\frac{32}{5}\pi u^3$ (d) $6\pi u^3$

 (e) $\frac{16}{3}\pi u^3$ (f) $\frac{1}{7}\pi u^3$ (g) $9\pi u^3$ (h) $16\pi u^3$

 4(a) $3\pi u^3$ (b) $\frac{256}{3}\pi u^3$ (c) $618\frac{2}{5}\pi u^3$ (d) $\frac{1}{2}\pi u^3$

 (e) $85\frac{1}{3}\pi u^3$ (f) $\frac{243}{5}\pi u^3$ (g) $\frac{16}{15}\pi u^3$ (h) $\frac{16}{3}\pi u^3$

 5(a) $\frac{296}{3}\pi u^3$ (b) $19\frac{5}{6}\pi u^3$ (c) $104\frac{1}{6}\pi u^3$ (d) $\frac{16}{105}\pi u^3$

 6(a) $\frac{1}{3}\pi u^3$ (b) $\frac{28}{15}\pi u^3$ (c) $\frac{81}{10}\pi u^3$ (d) $\frac{1}{2}\pi u^3$

 7(b) $\frac{1}{5}24\pi u^3$ (c) $256\pi u^3$ (d) $128\pi u^3$ (e) $128\pi u^3$

 8(a) $\frac{32}{5}\pi u^3$, $8\pi u^3$ (b) $\frac{50}{3}\pi u^3$, $\frac{5}{3}\pi u^3$

 (c) $8\pi u^3$, $\frac{128}{5}\pi u^3$ (d) $\frac{24}{5}\pi u^3$, $\frac{1}{2}\pi u^3$

 9 $682\frac{2}{3}\pi u^3$

 10 $5270\frac{2}{5}\pi u^3$

 11 $\frac{9}{2}\pi u^3$

 12(b) $\frac{4}{3}u^2$ (c)(i) $\frac{6}{5}\pi u^3$ (ii) $\frac{24}{5}\pi u^3$

 13(b)(i) $\frac{2}{35}\pi u^3$ (ii) $\frac{1}{10}\pi u^3$

 14(b) $21\frac{1}{3}\pi u^3$

 15(a)(ii) $\frac{\pi r^2(b^3 - a^3)}{3h^2}$

 16(a) $\frac{4}{3}\pi b^2 u^3$ (b) $2\pi h^2 \pi u^3$

 17(a) $2\pi a^3 u^3$ (b) $\frac{16}{5}\pi a^3 u^3$

 18(a) $x \leq 9$, $y \geq 0$ (c) $18u^2$

 (d)(i) $\frac{81}{2}\pi u^3$ (ii) $129\frac{3}{5}\pi u^3$

 19(a) $y = 3x$ (c)(i) $\frac{15}{7}\pi u^3$ (ii) $\frac{2}{5}\pi u^3$

 20(b) $72 - \frac{9}{2}\pi u^2$ (c) $1000\frac{4}{5}\pi u^3$

 21(a) relative minimum at $(1, 2)$,

 relative maximum at $(-1, -2)$ (c) $\frac{9}{4}\pi u^3$

 22(a) $(0, 0)$ and $(1, 1)$ (d) $\frac{1}{3}\pi u^3$

 23(a) maximum turning point at $(\frac{1}{3}, \frac{2}{9}\sqrt{3})$

 (c) $\frac{8}{15}u^2$ (d) $\frac{1}{12}\pi u^3$

 24(c) $8\pi u^3$

 25(b) $6\pi^2 u^3$

Exercise 11H (Page 430)

 1(a) $8x(x^2 + 3)^3$

 (b)(i) $(x^2 + 3)^4 + C$ (ii) $\frac{1}{8}(x^2 + 3)^4 + C$

 2(a) $12(x^2 + 2x)(x^3 + 3x^2 + 5)^3$

 (b)(i) $(x^3 + 3x^2 + 5)^4 + C$ (ii) $\frac{1}{12}(x^3 + 3x^2 + 5)^4 + C$

 3(a) $-7(2x + 1)(5 - x^2 - x)^6$

 (b)(i) $(5 - x^2 - x)^7 + C$ (ii) $-\frac{1}{7}(5 - x^2 - x)^7 + C$

 4(a) $15x^2(x^3 - 1)^4$

 (b)(i) $(x^3 - 1)^5 + C$ (ii) $\frac{1}{5}(x^3 - 1)^5 + C$

 5(a) $\frac{2x}{\sqrt{2x^2 + 3}}$

 (b)(i) $\sqrt{2x^2 + 3} + C$ (ii) $\frac{1}{2}\sqrt{2x^2 + 3} + C$

 6(a) $\frac{3(\sqrt{x} + 1)^2}{2\sqrt{x}}$

 (b)(i) $(\sqrt{x} + 1)^3 + C$ (ii) $\frac{2}{3}(\sqrt{x} + 1)^3 + C$

 7(a) $\frac{1}{3}(5x^2 + 3)^3 + C$ (b) $\frac{1}{4}(x^2 + 1)^4 + C$

 (c) $\frac{1}{6}(1 + 4x^3)^6 + C$ (d) $\frac{1}{30}(1 + 3x^2)^5 + C$

 (e) $\frac{1}{8}(x^2 - 4x - 5)^4 + C$ (f) $-\frac{1}{32}(1 - x^4)^8 + C$

 (g) $\frac{2}{3}(x^3 - 1)^{\frac{3}{2}} + C$ (h) $\frac{1}{15}(5x^2 + 1)^{\frac{3}{2}} + C$

 (i) $\sqrt{x^2 + 3} + C$ (j) $\frac{1}{4}\sqrt{4x^2 + 8x + 1} + C$

 (k) $-\frac{1}{4(x^2 + 5)^2} + C$ (l) $\frac{2}{5}(\sqrt{x} - 3)^5 + C$

 (m) $\frac{p}{8q}(qx^2 - 3)^4 + C$ (n) $\frac{r}{15p}(px^3 + q)^5 + C$

 8(a) $\frac{32}{15}$ (b) $\frac{3}{64}$ (c) $\frac{1}{12}$ (d) 936 (e) $\frac{a^2}{2(a^3 + 1)}$

 (f) $\frac{38}{3}b^3$

 9(a) $x \geq 1$ or $x \leq -1$ (b) $\frac{2x^2 - 1}{\sqrt{x^2 - 1}}$ (d) $\frac{16}{3}\sqrt{2}u^2$

10(a) horizontal points of inflexion at $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$, relative maximum at $(1, 216)$, relative minimum at $(-1, -216)$ (b) $600\frac{1}{4}u^2$

11(b)(i) $136\frac{8}{15}$ (ii) $\frac{16}{105}$

Exercise 11I (Page 432)

2(b) 10 (c) $10\frac{2}{3}$, the curve is concave down.

(d) $6\frac{1}{4}\%$

3(b) $10\frac{1}{10}$ (c) $y'' = 12x^{-3}$, which is positive in the interval $1 \leq x \leq 5$, so the curve is concave up.

4(b) 24.7

(c) $24\frac{2}{3}$. $y'' = -\frac{1}{4}x^{-1\frac{1}{2}}$, which is negative in the interval $9 \leq x \leq 16$, so the curve is concave down.

5(c) $1\frac{27}{260}$

6(a) 0.729 (b) 3.388

7(a) 0.7489 (b) $\pi \div 2.996$, the estimate is less than the integral, because the curve is concave down.

8 9.2 metres

9 550 m^2

10(a) $38\pi u^3$ (b) $36\pi u^3$, $5\frac{5}{9}\%$

11 $180\pi u^3$

12(e) 876 400

Exercise 11J (Page 435)

1(b) $\frac{25}{18}$ (c) $\frac{73}{90}$ (d) $\frac{11}{5}$

2(b) 2.80

3(b) 14.137 (d)(i) 12.294 (ii) 13.392

4(b) $\frac{32}{3}$ (c) $\frac{32}{3}$

5(a) $\frac{7}{15}$ (b) $\frac{22}{9}$

6(a) 7.740 (b) 0.9376 (c) 660

7(a) 0.7709 (b) 3.084

8 $6\frac{19}{30}$ metres

9 $613\frac{1}{3}\text{ m}^2$

10 $115.19u^3$

11(a) maximum turning point at $(\frac{2}{3}, \frac{4}{9}\sqrt{6})$

(c) $1.45u^2$ (d) $4.19u^3$ (e) $\frac{16}{15}\sqrt{2}u^2$, $\frac{4}{3}\pi u^3$

12 4.65 units

Chapter Twelve

Exercise 12A (Page 440)

1(c) reflection in the line $y = x$

(d) For $y = 3^x$, domain: all real numbers, range: $y > 0$. For $y = \log_3 x$, domain: $x > 0$, range: all real numbers.

2(c) reflection in the line $y = x$

(d) For $y = 10^x$, domain: all real numbers, range: $y > 0$. For $y = \log_{10} x$, domain: $x > 0$, range: all real numbers.

3(a) 4 (b) -1 (c) 2 (d) -2 (e) -2 (f) -3

(g) 3 (h) -2

4(a) 14 (b) 4 (c) 6 (d) 10 (e) $\frac{1}{5}$ (f) 2 (g) 9 (h) 49

5(a) 3 (b) 5 (c) $-2\frac{1}{2}$ (d) 10 (e) 3.5 (f) -2 (g) 2π (h) y

6(a) -2 (b) -2 (c) $\frac{5}{6}$ (d) 4.5

7(a) 1 (b) 2 (c) 2 (d) 3

8(a) -3 (b) 16 (c) -2 (d) -3 (e) 4 (f) 3 (g) 9 (h) $\frac{13}{7}$ (i) 0

9(a) 1.58 (b) 3.17 (c) 1.72 (d) 1.89 (e) 1.00 (f) -3.97

10(a) $x = 2$, $y = -1$ (b) $x = 2$, $y = -3$

(c) $x = \frac{17}{8}$, $y = \frac{15}{8}$ (d) $x = \frac{5}{4}$, $y = \frac{3}{4}$

11(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{5}{3}$ (d) $\frac{4}{3}$ (e) $-\frac{5}{2}$ (f) $-\frac{3}{2}$

(g) 4 (h) $\frac{6}{5}$

14(b)(i) $\frac{1}{3}$ (ii) $\frac{2}{3}$

15(b)(i) $\frac{7}{3}$ (ii) $\frac{8}{3}$

(c) $x = 2$, regardless of the value of a .

16(b)(i) $\log_2 3$ (ii) $\log_3 2$ (iii) $\log_3 5$

17(c) 2.3222

18(a) $x = 3$ (b) $x = 3$

Exercise 12B (Page 448)

1(a) $\frac{1}{x}$ (b) $\frac{1}{x}$ (c) $\frac{2}{x}$ (d) $1 + \frac{4}{x}$ (e) $\frac{1}{x}$ (f) $\frac{1}{x}$

(g) $\frac{3}{x}$ (h) $12x^2 - \frac{1}{x}$

2(a) $\frac{2}{2x+5}$ (b) $\frac{3}{3x-7}$ (c) $\frac{2}{3+2x}$ (d) $\frac{-1}{4-x}$ (e) $\frac{7}{4+7x}$

(f) $-\frac{5}{2-5x}$ (g) $1 + \frac{1}{1-x}$ (h) $\frac{-1}{3-x}$ (i) $\frac{a}{ax-b}$ (j) $\frac{\pi}{\pi x+1}$

(k) $\frac{\pi}{\pi x-2}$ (l) $\frac{2}{2x-ab}$

3 $e \div 2.7$

4 Check the answers using the calculator.

5(a) e (b) $-\frac{1}{e}$ (c) 6 (d) $\frac{1}{2}$ (e) $2e$ (f) 0 (g) e

(h) 1 (i) 0

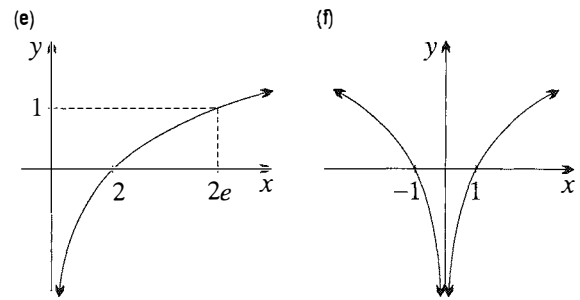
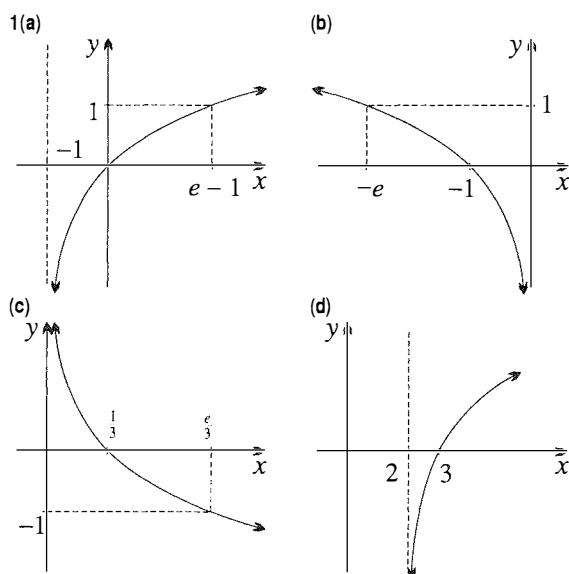
6(a) $x = \frac{1}{3}$ (b) $x = 3$ or 4

7(a) $\frac{2x}{x^2+1}$ (b) $\frac{2x+3}{x^2+3x+2}$ (c) $\frac{-2x}{2-x^2}$ (d) $\frac{1}{2x+\sqrt{x}}$

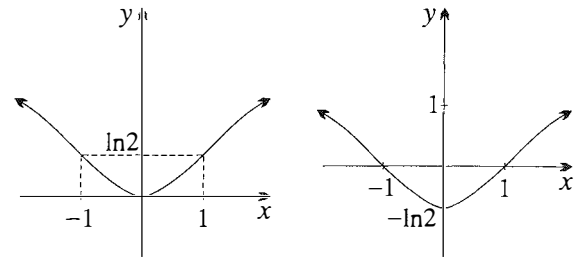
8(a) $\frac{2}{x}$ (b) $\frac{3}{x}$ (c) $\frac{1}{2x}$ (d) $\frac{-1}{x}$ (e) $\frac{1}{4+2x}$

- (f) $\frac{1}{1+x} + \frac{1}{1-x}$ (g) $\frac{1}{3x+3}$ (h) $\frac{1}{x} + \frac{1}{2(x+1)}$
 9(a) $\frac{1}{x \log 2}$ (b) $\frac{1}{x \log 10}$ (c) $\frac{1}{x \log 2}$ (d) $\frac{5}{x \log 3}$
 10(a) $1 + \log x$ (b) $\frac{2x}{2x+1} + \log(2x+1)$
 (c) $\frac{2x+1}{x} + 2 \log x$ (d) $\frac{2+\log x}{2\sqrt{x}}$
 11 $x - e^2 y + e^2 = 0$
 12 $ex + e^2 y + e^2 - 1 = 0, (\frac{1}{e} - e, 0)$
 13(a) $\frac{1}{2x-2} - \frac{2x}{x^2+1}$ (b) $\frac{2x-2}{x^2-2x} + \frac{1}{2x}$ (c) $\log \pi$
 14(a) $x(1 + 2 \log x)$ (b) $\frac{1-\log x}{x^2}$ (c) $\frac{2 \log x}{x}$
 (d) $\frac{4(\log x)^3}{x}$ (e) $\frac{-1}{x(1+\log x)^2}$ (f) $\frac{1}{2x\sqrt{\log x}}$
 (g) $\frac{8}{x}(2 \log x - 3)^3$ (h) $\frac{-1}{x(\log x)^2}$ (i) $\frac{1}{x \log x}$
 (j) $\frac{-\log 3}{x(\log x)^2}$ (k) $\frac{\log x - 1}{(\log x)^2}$
 (l) $\log_x 3 - \frac{\log 3}{(\log x)^2} = \frac{\log 3(\log x - 1)}{(\log x)^2}$
 15(a) $(\frac{1}{e}, \frac{-1}{e})$ (b) $(1, 1)$ (c) $(\frac{1}{\sqrt{e}}, \frac{1}{2e})$
 17 $x = \frac{1}{\log 10}$
 18(a) $\frac{4x-3}{2x^2-3x}$ (b) $\frac{1-4x}{x-1-2x^2}$ (c) $\frac{1}{x(1+\log x)}$
 (d) $\frac{2+\sqrt{x}}{2x(\sqrt{x}+\log x)}$ (e) $\frac{2x+x^2}{2(x-2)} + 2(x+1) \log \sqrt{x-2}$
 (f) $\frac{3-12x-7x^2}{2x(3+2x-x^2)}$
 20(a) $\frac{x(x+5)}{2\sqrt{x-1}(x+2)^2}$ (b) $\frac{(x-1)^2(x+2)(x^2-15x-4)}{(x-3)^5}$
 (c) $\frac{x(2x^2-x-2)}{(x-1)^{3/2}\sqrt{x+1}}$ (d) $\frac{(x-1)(3x^2+6x-1)}{2\sqrt{x}(x+1)^2}$
 (e) $\frac{(x+\frac{1}{x})^{\frac{1}{2}}(x^2-1)}{\pi x(x^2+1)}$ (f) $\frac{3x^2+6x+2}{2\sqrt{x}\sqrt{x+1}\sqrt{x+2}}$
 21(a) $x^x(1 + \log x)$ (b) $2x^{\log x - 1} \log x$
 (c) $x^{\frac{1}{x}-2}(1 - \log x)$
 22 $2\frac{28}{39}$
 23(d)(i) 2 (ii) 2.5937 (iii) 2.7048 (iv) 2.7169
 (v) 2.7181

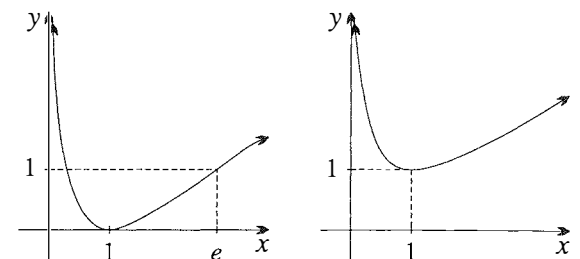
Exercise 12C (Page 452)



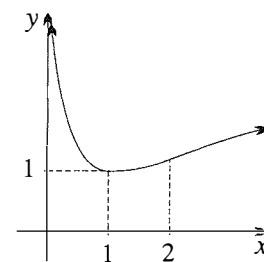
2(a) all real x (e) $y \geq 0$ (f) all real x ,
 $y \geq -\ln 2$



- 3(a) $x > 0$
 (b) $y' = \frac{2}{x} \log x$,
 $y'' = \frac{2}{x^2}(1 - \log x)$
 (c) $y \geq 0$
 4(a) $y' = 1 - \frac{1}{x}$,
 $y'' = \frac{1}{x^2}$
 (b) $x > 0$ (d) $y \geq 1$

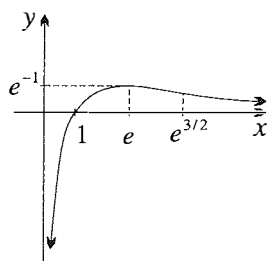


5(a) $x > 0$ (b) $y' = \frac{x-1}{x^2}$ and $y'' = \frac{2-x}{x^3}$ (d) $y \geq 1$



6(a) $x > 0$. $y \rightarrow 0^+$ as $x \rightarrow \infty$ so the x -axis is a horizontal asymptote. $y \rightarrow -\infty$ as $x \rightarrow 0^+$ so the y -axis is a vertical asymptote.

- (b) $y' = \frac{1}{x^2}(1 - \log x)$ and $y'' = \frac{1}{x^3}(2 \log x - 3)$
 (d) $(e^{\frac{3}{2}}, \frac{3}{2}e^{-\frac{3}{2}})$ (e) $y \leq e^{-1}$



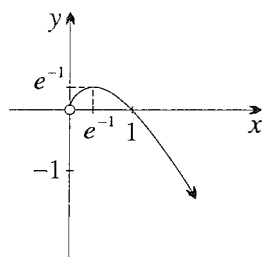
- 7(a) $y - 2 \log c = \frac{2}{c}(x - c)$, $c = e$ (b)(i) $c = 1$ or e^2
 (ii) e^{-1}

8(b) $\frac{e}{\sqrt{5}}$

9(a) $y = \frac{2 \log t}{t} x - 2 \log t + (\log t)^2$ (b) $\frac{e}{4}$

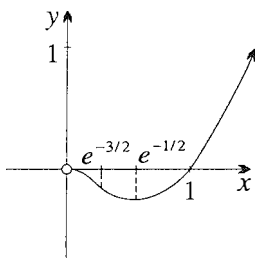
10(a) $x > 0$

(c) $y \rightarrow 0^+$ as $x \rightarrow 0^+$, $y' \rightarrow \infty$ as $x \rightarrow 0^+$, hence the graph becomes vertical approaching the origin. $y \leq e^{-1}$.

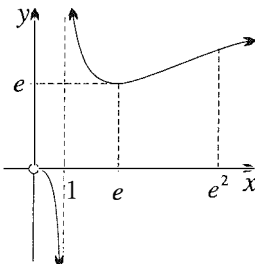


- 11(a) $x > 0$. Minimum at $(\frac{1}{\sqrt{e}}, -\frac{1}{2e})$ (c) $y \rightarrow 0$ as $x \rightarrow 0^+$, $y' \rightarrow 0$ as $x \rightarrow 0^+$, hence the graph becomes horizontal approaching the origin.

(d) $y \geq -\frac{1}{2e}$



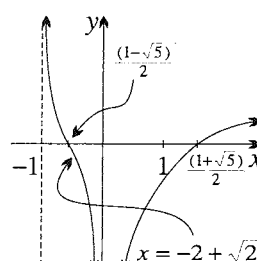
- 12 $x > 0$, $x \neq 1$, $y < 0$ or $y \geq e$. $x = 1$ is a vertical asymptote and the curve becomes horizontal approaching the origin.



13(a) $x > -1$, $x \neq 0$

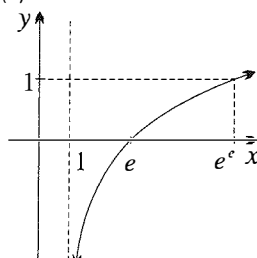
(c) $x = -2$ is outside the domain.

(d) one at $x = -2 + \sqrt{2}$



- 14(a) $x > 1$ (c) $y' = \frac{1}{x \log x}$, which can never be zero, $y'' = -\frac{1 + \log x}{(x \log x)^2}$

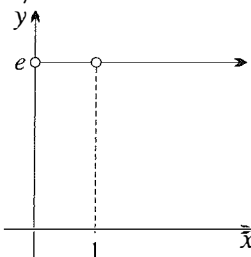
(d) The value is outside the domain.



- 15(a) $\log a$ (b) For the graph of $y = \log x$ a horizontal enlargement of factor $\frac{1}{a}$ is equivalent to a translation of $\log a$ upwards. (c) The change to base b stretches the graph vertically by a factor $\frac{1}{\log b}$, otherwise the result is as above.

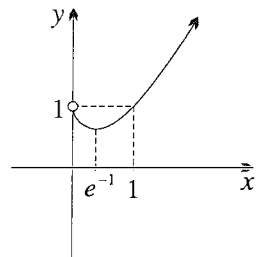
16(a) $\frac{2c}{1+c^2}$

- 18 $y = e$ for all x in the domain, which is $x > 0$, $x \neq 1$.

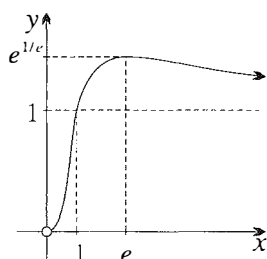


19(a) $y' = x^x(1 + \log x)$

- (b) $x > 0$, $y \geq e^{-1/e}$ and $y' = 1$ when $x = 1$.



20(b) $y' = x^{-2} x^{\frac{1}{x}}(1 - \log x)$



Exercise 12D (Page 456)

- 1(a) $\log x + C$ (b) $2 \log x + C$
 (c) $\frac{1}{3} \log x + C$ (d) $\frac{1}{5} \log(5x + 4) + C$
 (e) $\frac{5}{2} \log(3 + 2x) + C$ (f) $\frac{1}{4} \log(4x - 1) + C$
 (g) $\frac{1}{2} \log(2x - 1) + C$ (h) $\frac{3}{2} \log(2x + 1) + C$
 (i) $\log(2x - 1) + C$ (j) $-\frac{1}{5} \log(3 - 5x) + C$
 (k) $-\frac{2}{7} \log(5 - 7x) + C$ (l) $\frac{e}{\pi} \log(\pi x + 1) + C$
 (m) $-\frac{1}{e} \log(2 - ex) + C$ (n) $\frac{\sqrt{2}}{3} \log(3x - \pi) + C$
 (o) $-\frac{1}{a} \log(b - ax) + C$ (p) $-\frac{a}{c} \log(b - cx) + C$
 2(a) 1 (b) $1\frac{1}{2}$ (c) $\log 5$ (d) $\log 3$ (e) $\log 2$
 (f) $3 \log 2$ (g) $\frac{2}{3} \log 2$ (h) $\frac{3}{2} \log 3$ (i) $\frac{1}{2} \log 3$
 3(a) $x + \log x + C$ (b) $\frac{2}{3} \log x - \frac{1}{3}x + C$
 (c) $3x - 2 \log x$ (d) $\frac{3x^2}{2} + 4 \log x + \frac{1}{x} + C$
 4(a) $\log(x^2 - 9) + C$ (b) $\log(3x^2 + x) + C$
 (c) $\log(x^2 + x - 3) + C$ (d) $\log(2 + 5x - 3x^2) + C$
 (e) $\frac{1}{2} \log(x^2 + 6x - 1) + C$
 (f) $\frac{1}{4} \log(12x - 3 - 2x^2) + C$
 5(a) $y = \frac{1}{4}(\log x + 2)$, $x = e^{-2}$
 (b) $y = 2 \log(x + 1) + 1$
 (c) $y = \log\left(\frac{x^2 + 5x + 4}{10}\right) + 1$
 (d) $y = x + \log x + \frac{1}{2}x^2$
 (e) $y = 2 \log x + x + C$, $y = 2 \log x + x$,
 $y(2) = \log 4 + 2$
 6(a) $\log(x^3 - 5) + C$ (b) $\log(x^4 + x - 5) + C$
 (c) $\frac{1}{4} \log(x^4 - 6x^2) + C$ (d) $\frac{1}{2} \log(5x^4 - 7x^2 + 8) + C$
 (e) $\frac{2}{3} \log(\sqrt{x^3} + 1) + C$ (f) $\frac{1}{3} \log(x^3 - 2x^{\frac{3}{2}} + 1) + C$
 7(a) $\log \frac{6}{e^2 - 1} - 1$ (b) $\log 14$ (c) $\log\left(\frac{2e^2}{e+1}\right)$
 8(a)(i) $x \log x - x + C$ (ii) $\frac{\sqrt{e}}{2}$ (b) $10 - \frac{9}{\log 10}$ (c) $\frac{e^2}{4}$
 (d) $2\sqrt{x}(\log x - 2) + C$
 9(a) $(n - 1) \log a$ (b) $\frac{1}{t^2} \log(s + tx) + C$
 (c) $\frac{1}{b^2} \log(b + 1)$
 10(a) $\log \frac{5}{4} - \frac{1}{5}$ (b) $\frac{1}{6} \log\left(\frac{x-3}{x+3}\right) + C$
 11(a) $\log 9$ (b) $\log 2$ (c) $\frac{1}{2}$ (d) $9 \log 2$
 12 $\log(x + \sqrt{x^2 + 1}) + C$
 13 $2 \log(\sqrt{x} + 1) + C$
 14(a) They are both $\frac{a}{ax + b}$. $-\log 3$

$$(b) y = \begin{cases} \log x + 1, & \text{for } x > 0, \\ \log(-x) + 2, & \text{for } x < 0. \end{cases}$$

$$15(d)(i) \log \frac{3}{2} \doteq 0.41 \quad (ii) \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$(e) \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots, \log \frac{1}{2} \doteq -0.69 \quad (f) \text{ Using } x = \frac{1}{2}, \log 3 \doteq 1.0986.$$

Exercise 12E (Page 459)

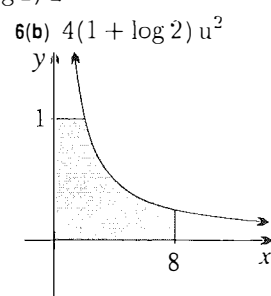
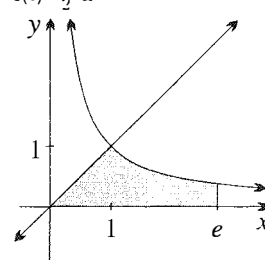
$$1(a) 1 u^2 \quad (b) 2 \log 2 u^2$$

$$2(a) (6 - 3 \log 3) u^2 \quad (b) (3\frac{3}{4} - 2 \log 4) u^2 \quad (c) \frac{1}{2} u^2$$

$$3(a) \frac{1}{2} \log 5 \doteq 0.805 u^2$$

$$4(a) 2 \log 2 u^2 \quad (b) (1 - \log 2) u^2$$

$$5(b) \frac{3}{2} u^2$$



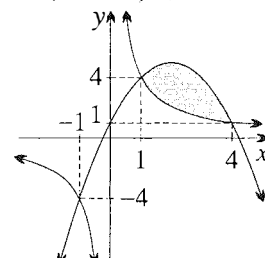
$$7(a) (\frac{1}{3}, 3) \text{ and } (1, 1) \quad (b) (\frac{4}{3} - \log 3) u^2$$

$$8 \pi \log 6 u^3$$

$$9(a) \pi \log 2 u^3 \quad (b) \pi \log 16 u^3 \quad (c) \pi (\frac{25}{6} + \log 36) u^3$$

$$10(a) x^3 - 4x^2 - x + 4$$

$$(b) (-1, -4), (1, 4) \text{ and } (4, 1)$$



$$(c) (12 - 4 \log 4) u^2$$

$$11 (\log 4 - \frac{1}{2}) u^2$$

$$12(a) x \log x - x \quad (b) e^2 u^2 \quad (c) (e^e - 1) u^2$$

$$13(a) \pi (\frac{15}{2} - \log 4) u^3 \quad (b) \pi (\frac{3}{2} + \log 4) u^3. \text{ There is a difference because } (a - b)^2 \neq a^2 - b^2.$$

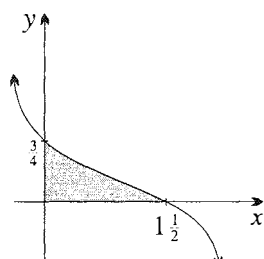
$$14 4\pi(2 \log 2 - 1) u^3$$

$$15(a) M = \log 3 \quad (b) \bar{x} = \frac{2}{\log 3}$$

$$16(a) \text{ The upper rectangle has height } 2^{-n}, \text{ the lower rectangle has height } 2^{-n-1}, \text{ both rectangles have width } 2^{n+1} - 2^n = 2^n.$$

$$17(d) 2.715$$

$$18 \log\left(\frac{25}{16}\right) u^2$$

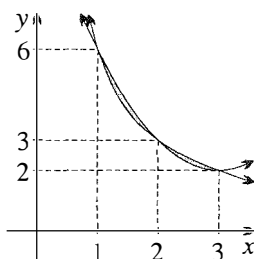


19(b) $(2 - \log 3)u^2$

20(a) Using symmetry, $\int_2^3 \frac{dx}{x} = \log\left(\frac{3}{2}\right)u^2$.

(b)(i) $3u^2$ (ii) $\log 3 u^2$

21(b) (c) $(2 - 6 \log \frac{4}{3})u^2$



23 $\frac{\pi(\epsilon-1)}{\epsilon} u^3$

24 $\pi\left(\frac{\epsilon(1+c)\log(1+c)-c}{1+c}\right)u^3$

25(a) $-\sqrt{3}, 0, \sqrt{3}$ (b) The curve is concave down.

Chapter Thirteen

Exercise 13A (Page 465)

1(a) $2e^{2x}$ (b) $-3e^{-3x}$ (c) $-5e^{5x}$ (d) $\frac{1}{2}e^{\frac{1}{2}x}$ (e) e^{ax}

(f) $-ke^{-kx}$ (g) $\pi e^{-\pi x}$ (h) $\frac{-1}{c}e^{-\frac{1}{c}x}$

2(a) $2e^{2x-1}$ (b) $-e^{1-x}$ (c) $-9e^{-3x+4}$ (d) $e^{\frac{1}{2}x+4}$

(e) pe^{px+q} (f) $2e^{2x} + 3e^{-3x}$ (g) $\frac{e^x + e^{-x}}{2}$

(h) $e^{ax} - e^{-bx}$

3(a) $2e^{2x}$ (b) $\frac{-3}{e^{3x}}$ (c) $\frac{1}{2}e^{\frac{x}{2}}$ (d) $\frac{-3}{2}e^{-\frac{3x}{2}}$

4(a) $2xe^{x^2}$ (b) $-2xe^{1-x^2}$ (c) $2(x+1)e^{x^2+2x}$

(d) $(1-2x)e^{6+x-x^2}$ (e) $(3x-1)e^{3x^2-2x+1}$

(f) $(x+1)e^x$ (g) $(1-x)e^{-x}$ (h) xe^x

(i) $(3x+4)e^{3x-4}$ (j) $(2x^2+1)e^{x^2+1}$ (k) $(x^2+7x)e^x$

(l) $(x^3-4x^2+2x)e^{-x}$

6(a) $(2x^2-1)e^{2x-1}$ (b) $\frac{-e^x}{1-e^x}$ (c) $\frac{e^x+1}{e^x+x}$

(d) $e^x(\log x + \frac{1}{x})$ (e) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ (f) $\frac{(x-1)e^x}{x^2}$

(g) $-xe^{-x}$ (h) $-\frac{2e^x}{(e^x-1)^2}$ (i) $\frac{(x-1)e^x}{(x+1)^3}$ (j) $\frac{4}{(e^x+e^{-x})^2}$

7(a) $2^x \log 2$ (b) $10^x \log 10$ (c) $\pi^x \log \pi$

(d) $e^x \log e$ (e) $2^{3x-1} 3 \log 2$ (f) $-5^{2-x} \log 5$

(g) $e^{x-2} \log e$ (h) $a^{bx+c} b \log a$ (i) $(x \log 2 + 1)2^x$

(j) $(3x^2-3)3^{x^3-3x} \log 3 = (x^2-1)3^{x^3-3x+1} \log 3$

8(a) 1 (b) $\frac{-1}{x^2}$ (c) -1 (d) 3

9(a) $\frac{Ae}{e-1}$ (b) The common ratio is e^d .

10(c)(i) -5 or 2 (ii) $-\frac{1+\sqrt{5}}{2}$ or $-\frac{1-\sqrt{5}}{2}$

12(a) $\frac{-1}{x^2}e^{\frac{1}{x}}$ (b) $(x+1)e^x$ (c) $\frac{\log 3}{x^2}3^{-\frac{1}{x}}$

13(a) $\frac{-1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$ (b) $-2-\sqrt{2}$ and $-2+\sqrt{2}$ (c) $1-x^2$

15(b) The secant has gradient 1 and the tangent is less steep. The gradient of the tangent is 0.69 to 2 decimal places.

(d) $\log 2$ (e) $y' = 2^x \lim_{h \rightarrow 0} \frac{2^{h-1}}{h} = 2^x \log 2$

17(a) $y = e^{x \log x}$, so $y' = (\log x + 1)x^x$.

(b) $y = e^{(2-x)\log x}$, so $y' = (\frac{2}{x} - 1 - \log x)x^{2-x}$.

(c) $y = e^{(\log x)^2}$, so $y' = \frac{2x^{1 \cdot 5} \log x}{x}$.

(d) $y = e^{\log x \times \frac{1}{\log x}}$, so $y' = 0$.

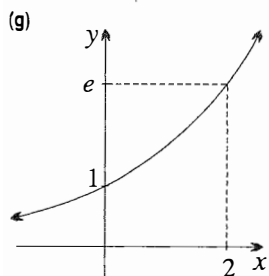
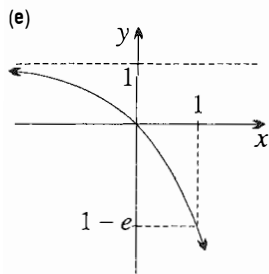
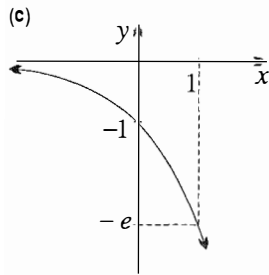
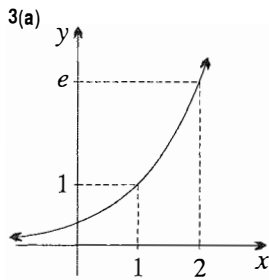
18(a) $\frac{-b+\sqrt{b^2-4ac}}{2a}$ or $\frac{-b-\sqrt{b^2-4ac}}{2a}$ (b) when $b^2 - 4ac < 0$

19(b) $f(x)$ simplifies to e^x .

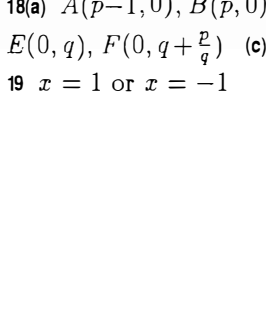
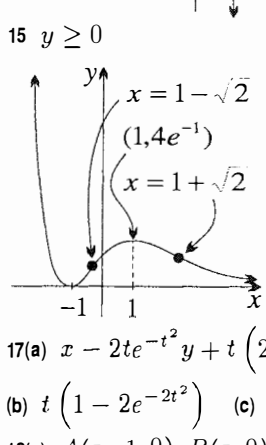
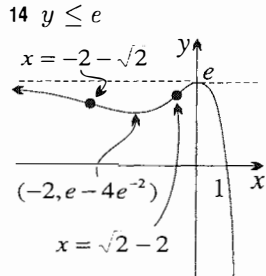
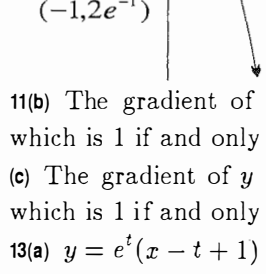
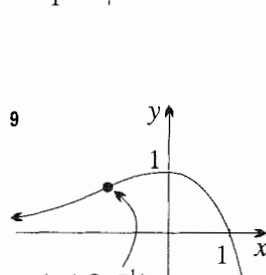
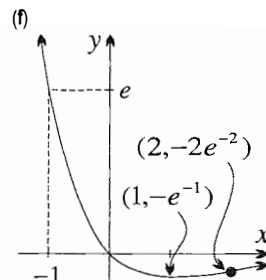
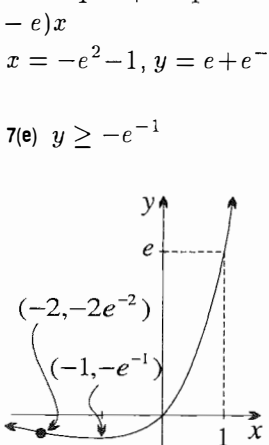
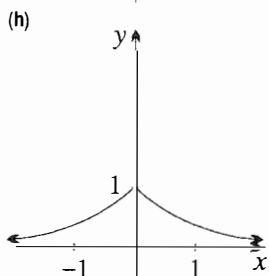
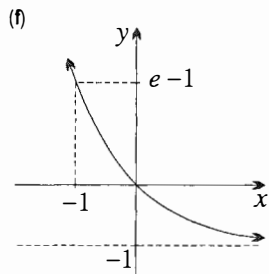
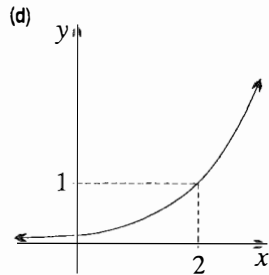
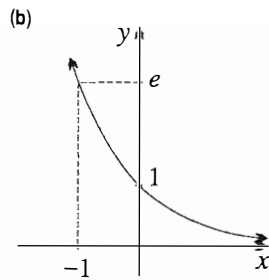
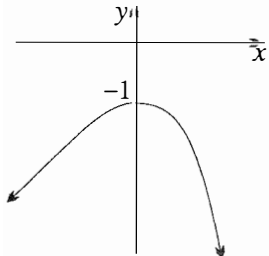
Exercise 13B (Page 469)

1(c) reflection in the line $y = x$ (d) For $y = e^x$, domain: all real numbers, range: $y > 0$. For $y = \log x$, domain: $x > 0$, range: all real numbers.

2(d) The x -intercept is 1 unit left of the point of contact.

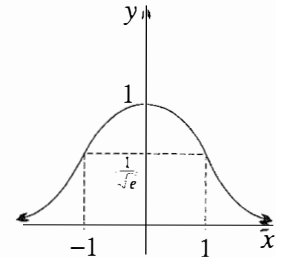


- 4(a) $y = x + 1$ (b) $y = (1 - e)x$
 5(a) $x - ey + e^2 + 1 = 0$ (b) $x = -e^2 - 1, y = e + e^{-1}$
 (c) $\frac{1}{2}(e^3 + 2e + e^{-1})$
 6(a) $y' = 1 - e^x, y'' = -e^x$ (d) $y \leq -1$



8(a) $y' = -xe^{-\frac{1}{2}x^2}, y'' = (x^2 - 1)e^{-\frac{1}{2}x^2}$

(d) $0 < y \leq 1$



(a) $y' = -xe^x, y'' = -(x + 1)e^x$
 (c) $y \leq 1$

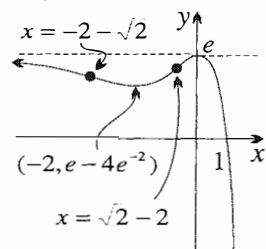
10 -3

11(b) The gradient of $y = a^x$ at $x = 0$ is $\log a$, which is 1 if and only if $a = e$.

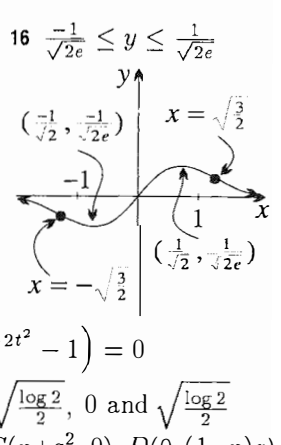
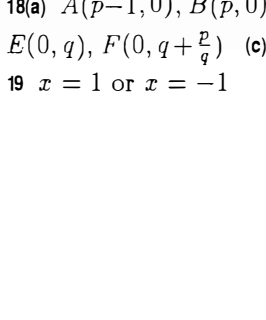
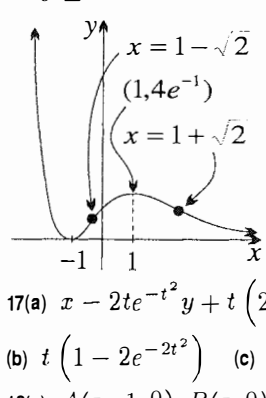
(c) The gradient of $y = Aa^x$ at $x = 0$ is $A \log a$, which is 1 if and only if $a = e^{1/A}$.

13(a) $y = e^t(x - t + 1)$

14 $y \leq e$



15 $y \geq 0$

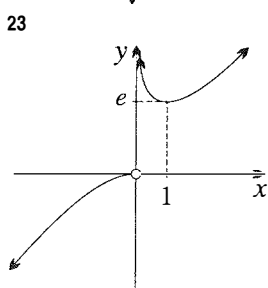
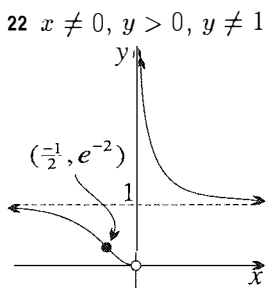
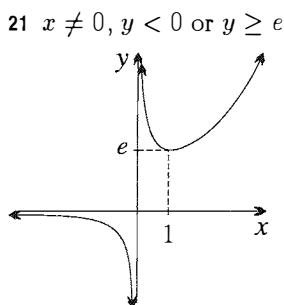
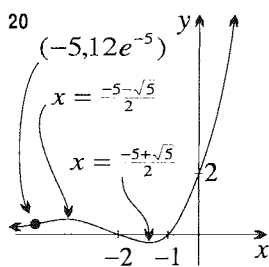


17(a) $x - 2te^{-t^2}y + t(2e^{-2t^2} - 1) = 0$

(b) $t(1 - 2e^{-2t^2})$ (c) $-\sqrt{\frac{\log 2}{2}}, 0$ and $\sqrt{\frac{\log 2}{2}}$

18(a) $A(p-1, 0), B(p, 0), C(p+q^2, 0), D(0, (1-p)q), E(0, q), F(0, q+\frac{p}{q})$ (c)(i) $\frac{q}{2}(q^2+1)$ (ii) $\frac{p^2}{2}(q+\frac{1}{q})$

19 $x = 1$ or $x = -1$



24(a) $y = e^{kx}$ and $y = \frac{1}{k} \log x$

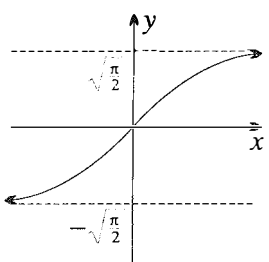
(b) Since $y = a^x$ and $y = \log_a x$ are inverse functions, they are symmetric in the line $y = x$. The common tangent must therefore be the line $y = x$, which has gradient 1. NOTE: This assertion is untrue if there is more than 1 intersection point.

(c) $ke^{kx} = 1$ and $\frac{1}{kx} = 1$ (d) $k = \frac{1}{e}, a = e^{\frac{1}{e}}$

26(a) 0 (b) $y' = e^{-\frac{1}{2}x^2}$ so $y' > 0$ for all x .

(c) y is an odd function.

(e) $y' \rightarrow 0$ as $x \rightarrow \infty$. This does not imply an asymptote. For example, $y = \log x$ has gradient $y' = \frac{1}{x}$, so $y' \rightarrow 0$ as $x \rightarrow \infty$, but $y = \log x$ clearly does not have a horizontal asymptote. (f) $-\sqrt{\frac{\pi}{2}} < y < \sqrt{\frac{\pi}{2}}$



Exercise 13C (Page 473)

1(a)(i) $e - 1 \doteq 1.72$ (ii) $1 - e^{-1} \doteq 0.63$

(iii) $1 - e^{-2} \doteq 0.86$ (iv) $1 - e^{-3} \doteq 0.95$

(d) The total area is exactly 1.

2(a) $\frac{1}{2}e^{2x} + C$ (b) $3e^{\frac{1}{3}x} + C$ (c) $\frac{1}{4}e^{4x+5} + C$

(d) $\frac{1}{3}e^{1+3x} + C$ (e) $\frac{1}{4}e^{4x-2} + C$ (f) $e^{x-1} + C$

(g) $2e^{3x+2} + C$ (h) $e^{2x-1} + C$ (i) $-e^{3-x} + C$

(j) $\frac{-1}{2}e^{7-2x} + C$ (k) $\frac{1}{\pi}e^{\pi x-1} + C$

(l) $-\frac{1}{2e}e^{1-ex} + C = -\frac{1}{2}e^{-ex} + C$ (m) $\frac{\pi}{3}e^{3x-\sqrt{2}} + C$

(n) $-e^{b-ax} + C$ (o) $\frac{a}{b}e^{bx+c} + C$ (p) $e^{a-\pi x} + C$

3(a) $e^2 - e$ (b) $e - e^{-3}$ (c) $e - e^{-1}$ (d) $\frac{e^2}{2}(e^2 - 1)$

(e) 1 (f) $\frac{1}{12}$ (g) $\pi e(e - 1)$ (h) $\frac{e^{ab}}{b}(e^{ab} - 1)$

4(a) $\frac{-1}{2}e^{-2x} + C$ (b) $x - e^{-x} + C$ (c) $2e^{\frac{x}{2}} + C$

(d) $2e^{\frac{x}{2}} + \frac{2}{3}e^{\frac{-3x}{2}} + C$

5(a) $\frac{2^x}{\log 2} + C$ (b) $\frac{3^x}{\log 3} + C$ (c) $\frac{-5^{-x}}{\log 5} + C$ (d) $\frac{\pi^x}{\log \pi} + C$

6(a) $y = e^{x-1}, y = e^{-1}$

(b) $y = e^2 + 1 - e^{2-x}, y = e^2 + 1$

(c) $y = \frac{1-2^{-x}}{\log 2}$ (d) $f(x) = e^x + \frac{x}{e} - 1, f(0) = 0$

7(a) $e^{x^2+3} + C$ (b) $e^{5x^2-2x} + C$ (c) $\frac{1}{2}e^{3x^2+4x+1} + C$

(d) $\frac{1}{3}e^{x^3-3x^2} + C$

8(a) $2e^{x^3-2x^2+3x-5} + C$ (b) $\log x + \frac{e^{3x}}{3} + C$

(c) $2\sqrt{x} + \frac{1}{2}e^{-x^2} + C$ (d) $\frac{1}{2}e^{2x} + 2e^x + x + C$

(e) $\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2} + C$ (f) $-e^{\frac{1}{x}} + C$

(g) $\frac{2}{3}e^{x\sqrt{x}} + C$ (h) $\frac{x^3}{3} + C$ (i) $kx^2 + C$

9(b) $e^2 + 1$

10(a) $e^x - e^{-x}$ (b) $\log\left(\frac{e^2+e^{-2}}{2}\right)$

11 $\log(e^x + 1) + C$

12(a) $x^2 + 2\log x + \frac{2^x}{\log 2} + C$ (b) $\frac{a^x}{\log a} + \frac{ax^2}{2} + C$

(c) $\frac{3^{x^2+2x}}{\log 3} + C$

13(a) $y' = 2(x - x^3)e^{-x^2}$ (b) $\frac{2e^{3-5}}{2e^4}$

14(b) $(x+1)e^x$

15(a) $y = \frac{2e^{-x\sqrt{x}+1}}{3}$ (b) $y = 2 - 3^{-x}, y(0) = 1$

17 $\frac{1}{\log 2+1}$

20(b) 1.1276

(d) $e^{0.5} = \bullet + \sqrt{\alpha^2 - 1}, e^{-0.5} = \bullet - \sqrt{\alpha^2 - 1}$

Exercise 13D (Page 477)

1(a) $1 - e^{-1}$ square units (b) $e(e^2 - 1)$ square units

(c) 3 square units (d) $3\frac{1}{2}$ square units

2 $3 - e^{-2}$ square units

3(a) 0.8863 square units (b) 0.8362 square units

4(a) $1 + e^{-2}$ square units

(b) $2\log 2 - 1$ square units (c) e^{-1} square units

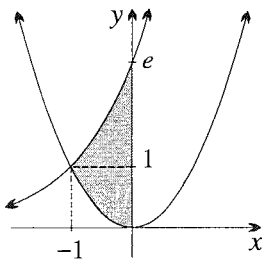
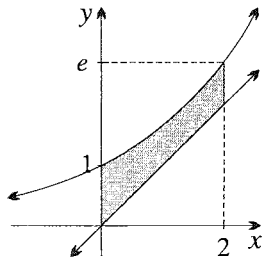
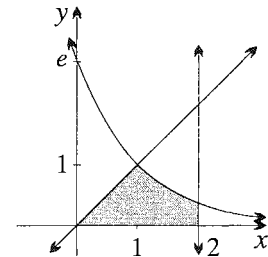
(d) $3 + e^{-2}$ square units

5 $e - 1$ square units

6 $\int_0^1 e^x - 1 - x dx = e - 2\frac{1}{2}$ square units

7 $\pi \int_0^1 (e^x)^2 dx, \frac{\pi}{2}(e^2 - 1)$ cubic units

8(b) $e - 1\frac{1}{3}$ square units


 9 $e^2 - 3$ square units

 10 $1\frac{1}{2} - e^{-1}$ square units

 11 $\pi[\frac{e-e^{-1}}{2} - 1]$ cubic units

 12 $\pi[2 + 2e^{-1}(1 - e^{-2}) + \frac{1}{2}e^{-2}(1 - e^{-4})] \doteq 8.491$ cubic units

 13 $\pi(1 - e^{-4}) \doteq 3.084$ mL

 14 intercepts (0, 7) and (3, 0) and area $24 - \frac{7}{\log 2}$ square units

 15(a) $x = \frac{1}{2} \log(\frac{b}{a})$, $y = \sqrt{ab}$ (b) $a + b - 2\sqrt{ab}$ square units

 16 $\frac{\pi}{2}(8 \log 2 - 3)$ cubic units

 17 area = $e - \frac{1}{2} - \frac{1}{e} \text{ m}^2$, so cost = \$29.80.

 18(a)(i) $1 - e^{-N}$ (ii) 1 (b) 1

 (c)(i) $\int_0^N 2xe^{-x^2} dx = 1 - e^{-N^2}$, thus in the limit as $N \rightarrow \infty$ this is just 1.

 19(a) $2(e - e^{\sqrt{5}})$ (b) It approaches $2(e - 1)$.

 20(b) $\frac{e^b - 1 - b}{b(e^b - 1)}$ (c) $\frac{1}{2}$ square units; the triangle with vertices at the origin, (1, 0) and (0, 1).

 21(a) $1 - (1 + N)e^{-N}$ (b) 1 (c) 2

Exercise 13E (Page 482)

1(a)(i) 13.6 (ii) 1.12 (iii) 47.4 (b)(i) 1.39 (ii) -3.22

 (iii) 3.58 (c)(i) 4 (ii) $5\frac{1}{2}$ (iii) $\frac{1}{6}$ (d)(i) 82.789

(ii) 12.345 (iii) 4.330

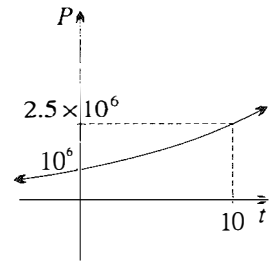
 2(a) $y \doteq 60.26$ (b) $t \doteq 0.6931$ (c) $\frac{dy}{dt} = -3$

 (d) $\frac{dy}{dt} = 2e$

 3(a) $k = \frac{1}{10} \log \frac{5}{2}$
 $\doteq 0.092$

(b) 8.23 million

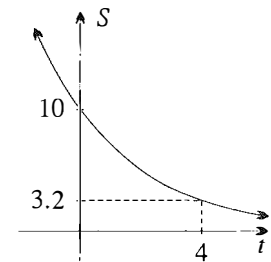
(c) during 2000

 (d) $\frac{dP}{dt} = kP$
 $\doteq 916\,000$

 4(a) $k = -\frac{1}{4} \log \frac{3.2}{10}$
 $\doteq 0.28$

 (b) $t \doteq 8.08$ hours

 $\doteq 8$ hours 5 minutes

(c) 2 kg/h


 5(b) V_0 is the value of V at $t = 0$.

 (c) $-\log \frac{7}{10} \doteq 0.36$ (d) $\frac{1}{k} \log 20 \doteq 9$ years

 6(b) $k = \frac{1}{3} \log 2 \doteq 0.23$ (c) 2.52 cm^2

(d) 21 hours 50 minutes

 7(b) $h_0 = 100$ (c) $k = -\frac{1}{5} \log \frac{2}{5} \doteq 0.18$ (d) 6.4°C

8(a) 72% (b) 37% (c) 7%

 9(a) $k = \frac{\log 2}{16.90} \doteq 4.10 \times 10^{-4}$

 (b) $t = \frac{\log 5}{k} \doteq 3924$ years

 10(b) $k = \frac{2}{3} \log \frac{3}{2} \doteq 0.27$

(c) 5 min 57 sec, to the nearest second

 11(b) $k = \frac{\log 2}{5750} \doteq 1.21 \times 10^{-4}$

 (c) $t = \frac{1}{k} \log \frac{100}{15} \doteq 16\,000$ years, to the nearest 1000 years

12(b) 8 more years

 13(b) $k = \frac{2 \log 2}{3} \doteq 0.46$ (c) 10 hours

 14(b) $\mu_1 = 1.21 \times 10^{-4}$ (c) $\mu_2 = 1.16 \times 10^{-4}$

 (d) The values of μ differ so the data are inconsistent. (e)(i) 625.5 millibars

(ii) 1143.1 millibars (iii) 19 205 metres

15(a) 34 minutes (b) 2.5%

 16(a) $k = \frac{\log 2}{5730} \doteq 1.21 \times 10^{-4}$, $C_0 = 15.3$

(b) 2728 years old (c)(i) 15 847 years old

(ii) Further tests should be carried out.

 17(a) $p = \frac{1}{10} \log \frac{5}{4} \doteq 0.022$, $q = \frac{1}{10} \log \frac{6}{5} \doteq 0.018$

 (b) $t = \frac{\log 2}{p+q} \doteq 17.10$ years, that is during 1997.

 18(b) $t = \frac{1}{\lambda} \log(1 + \frac{D}{P})$

 (c)(i) 1.47×10^{-11} (ii) 500 million

 20(b) $z' = y'' - \lambda y'$ (d) $z = A e^{\mu t}$

 (f) Only the last step would change, with $e^{-\lambda t}(y' - \lambda y) = A$ and hence $y = (At + B)e^{\lambda t}$.

Chapter Fourteen

Exercise 14A (Page 490)

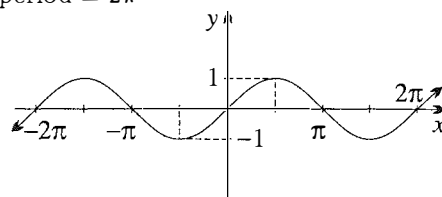
- 1(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$ (e) $\frac{2\pi}{3}$ (f) $\frac{5\pi}{6}$ (g) $\frac{3\pi}{4}$
 (h) $\frac{5\pi}{4}$ (i) 2π (j) $\frac{5\pi}{3}$ (k) $\frac{3\pi}{2}$ (l) $\frac{7\pi}{6}$
 2(a) 180° (b) 360° (c) 720° (d) 90° (e) 60°
 (f) 45° (g) 120° (h) 150° (i) 135° (j) 270°
 (k) 240° (l) 315° (m) 330°
 3(a) 1.274 (b) 0.244 (c) 2.932 (d) 0.377 (e) 1.663
 (f) 3.686
 4(a) $114^\circ 35'$ (b) $17^\circ 11'$ (c) $82^\circ 30'$ (d) $7^\circ 3'$
 (e) $183^\circ 16'$ (f) $323^\circ 36'$
 5(a) 0.91 (b) -0.80 (c) 0.07 (d) 1.55 (e) 2.99
 (f) -0.97
 6(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$ (e) -1 (f) $\frac{1}{2}$
 (g) $-\frac{1}{\sqrt{2}}$ (h) $\frac{1}{\sqrt{3}}$
 7(a) $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ (b) $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$
 (c) $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ (d) $x = \frac{\pi}{2}$ (e) $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$
 (f) $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ (g) $x = \pi$ (h) $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$
 (i) $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$
 8(a) $\frac{\pi}{9}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{5}$ (d) $\frac{5\pi}{9}$ (e) $\frac{5\pi}{8}$ (f) $\frac{7\pi}{5}$ (g) $\frac{13\pi}{72}$
 (h) $\frac{263\pi}{720}$
 9(a) 15° (b) 72° (c) 400° (d) 247.5° (e) 306°
 (f) 276°
 10(a) $\frac{\pi}{3}$ (b) $\frac{5\pi}{6}$
 11 $\frac{4\pi}{9}$
 12(a) 0.733 (b) 0.349
 13(a) 0.283 (b) 0.819
 14(a) 0.841, 0.997, 0.909 (b) 1.0
 15(a) $\sin x$ (b) $\cos x$ (c) $-\cos x$ (d) $-\cos x$
 (e) $\tan x$ (f) $-\tan x$ (g) $-\sec x$ (h) $\sec x$
 16(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{3}}$ (d) $-\frac{1}{\sqrt{3}}$ (e) -1 (f) $-\sqrt{2}$
 17(a) $x = \frac{\pi}{8}$ or $\frac{9\pi}{8}$ (b) $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$
 (c) $x = \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$ (d) $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ or 2π
 (e) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ (f) $x = \frac{\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}$ or $\frac{7\pi}{4}$
 18(a) $x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$ (b) $x = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}$ or π
 (c) $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$ (d) $x = -\pi$ or $\frac{\pi}{2}$ (e) $x = -\frac{3\pi}{4}$
 (f) $x = -\frac{2\pi}{3}$ or $\frac{\pi}{3}$
 19 $6^\circ 11' 15''$
 20 $\frac{169\pi}{1200}$
 21 $\frac{2\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \frac{7\pi}{10}, \frac{4\pi}{5}$
 22(a) The solutions of $\sin x = 0$ are $x = k\pi$ where k is an integer. Since π is irrational, $k\pi$ is never an integer when k is an integer. (b) $n = 22$ is the first positive integer solution of $|\sin n| < 0.01$. Because $\pi \div \frac{22}{7} \approx 0.01$, $\sin 22 \approx \sin 7\pi = 0$.
 23 $\theta = \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}$ or $\frac{19\pi}{10}$

Exercise 14B (Page 493)

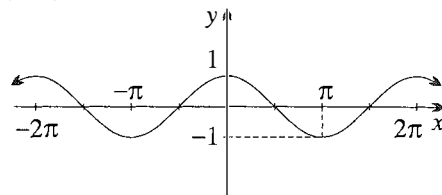
- 1(a) 12 cm (b) 2π cm
 2(a) 32 cm^2 (b) $12\pi \text{ cm}^2$
 3 4 cm
 4 1.5 radians
 5(a) 2.4 cm (b) 4.4 cm
 6 8727 m^2
 7(a) $8\pi \text{ cm}$ (b) $16\pi \text{ cm}^2$
 8 84°
 9 11.6 cm
 10(a) $6\pi \text{ cm}^2$ (b) $9\sqrt{3} \text{ cm}^2$ (c) $3(2\pi - 3\sqrt{3}) \text{ cm}^2$
 11 $\frac{4}{3}(5\pi - 3), \frac{4}{3}(7\pi + 3)$
 12 15 cm^2
 13(a) $4(\pi + 2) \text{ cm}$ (b) $8\pi \text{ cm}^2$
 14(a) $\frac{25\pi}{2} \text{ cm}^2$ (b) $\frac{25(4-\pi)}{2} \text{ cm}^2$
 15(a) $\frac{4\pi}{3}$ (b) 5 cm
 16(a) 21 rad/s (b) 6 m
 17 $\frac{4}{3}(4\pi - 3\sqrt{3}) \text{ cm}^2$
 18(a) 1.38 radians (b) 10 cm^2
 19(c) $3\sqrt{55}\pi \text{ cm}^3$ (d) $24\pi \text{ cm}^2$
 20(a) $\frac{2\pi}{3} \text{ cm}$ (b) $\frac{2\pi}{3} \text{ cm}^2$ (c) $2\pi \text{ cm}$
 (d) $\sqrt{3} \text{ cm}^2, 2(\pi - \sqrt{3}) \text{ cm}^2$
 21(a) 1.2661 radians (b) 49.2 cm
 22(b) 9 cm^2
 24(a) 1 radian (b) Since $\triangle OAC$ is equilateral, chord $AC =$ the radius $=$ arc AB , so B must lie on the minor arc AC . Since $\angle AOC = 60^\circ$ and $\angle AOB = 1$ radian, it follows that $1 \text{ radian} < 60^\circ$.
 25 2.54 cm^2
 26 36 seconds

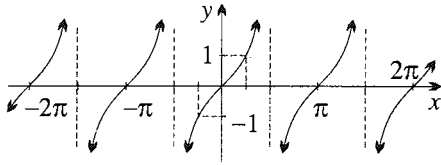
Exercise 14C (Page 501)

- 1(a) period $= 2\pi$

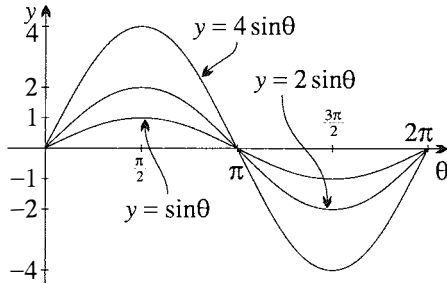


- (b) period $= 2\pi$

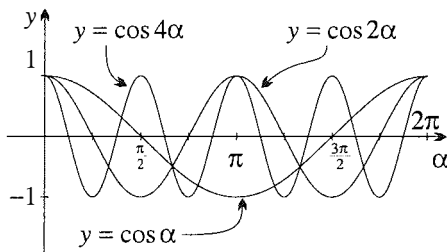


(c) period = π


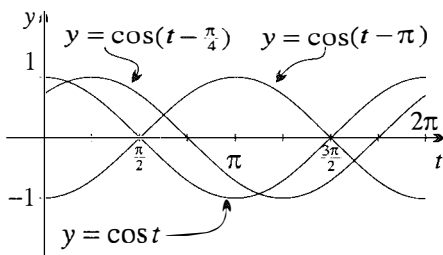
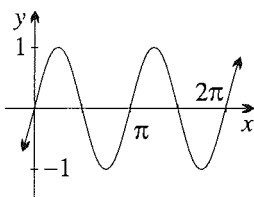
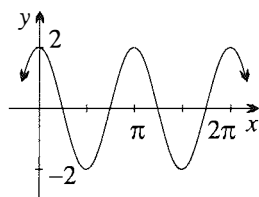
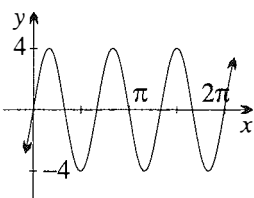
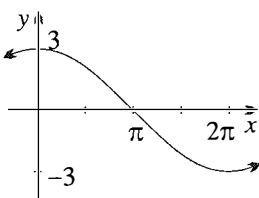
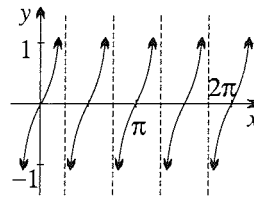
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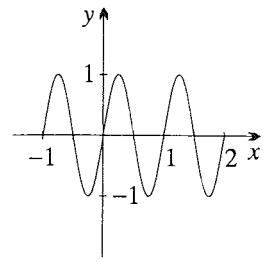
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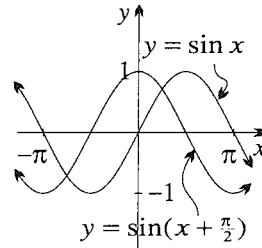
4


 5(a) period = π ,
amplitude = 1

 (b) period = π ,
amplitude = 2

 (c) period = $\frac{2\pi}{3}$,
amplitude = 4

 (d) period = 4π ,
amplitude = 3

 (e) period = $\frac{\pi}{2}$,
no amplitude


6



7

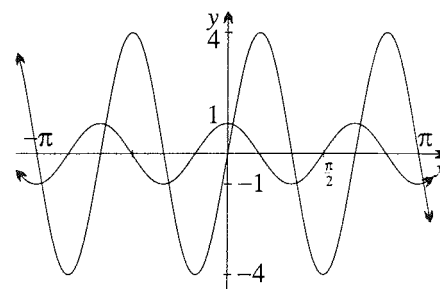

 (c) $\sin(x + \frac{\pi}{2}) = \cos x$

8(a) 3 (b) 3 solutions, 1 positive solution

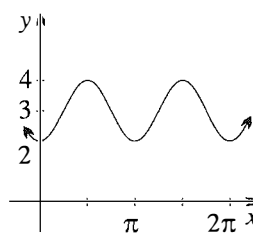
(c) Outside this domain the line is beyond the range of the sine curve.

 9 $x \doteq 1.9$, $x \doteq -1.9$ or $x = 0$

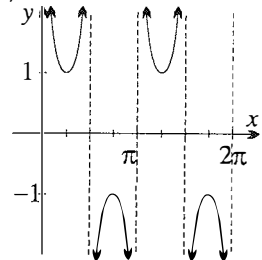
10



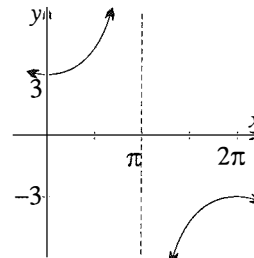
11



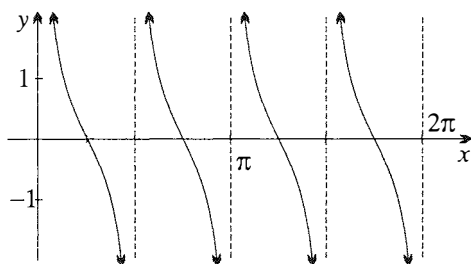
12(a)



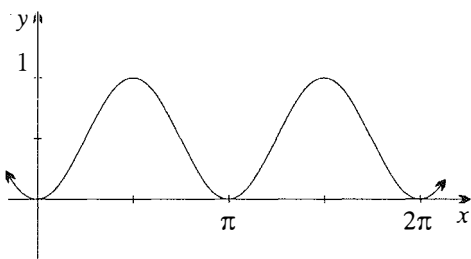
(b)



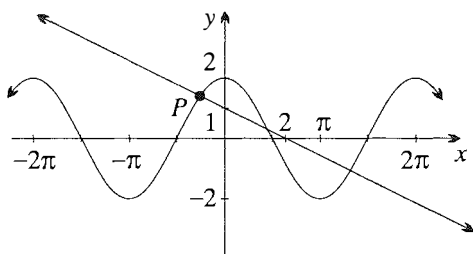
(c)



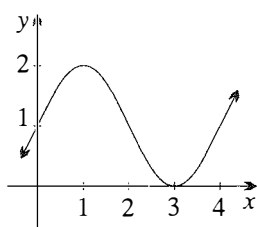
13

(a) $0, \frac{1}{2}, 1, \frac{1}{2}, 0$ (c) period = π , amplitude = $\frac{1}{2}$

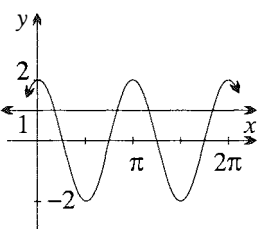
14

(c) 3 (d) P is in the second quadrant.

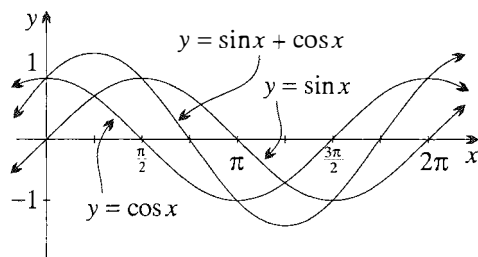
15

(a) 4 (c) the origin
(d) $m > \frac{1}{4}$

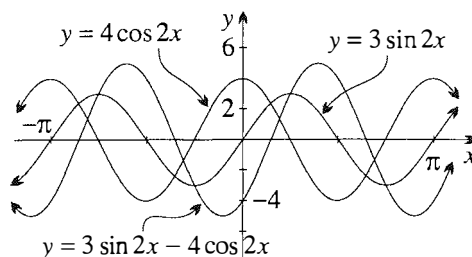
16

(c) 4 (d) $x = \frac{\pi}{6}$
(e) $\frac{\pi}{6} < x < \frac{5\pi}{6}$
OR $\frac{7\pi}{6} < x < \frac{11\pi}{6}$

17

(c) 2π (d) 1.4

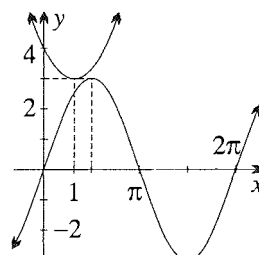
18



(c) amplitude = 5

19(a)(ii) 1 (iii) $0 < k < 1$ (b)(ii) 1.3(iii) $\angle AOB = 2\theta \doteq 2.6$ radians (c)(ii) $\ell > 300$ 20(a)(ii) 2.55 (b) 146° (c)(ii) 205°

21

22(a) $x = -2\pi, x = -\pi, x = 0, x = \pi, x = 2\pi$ (b) each of its x -intercepts (c) translations to the right or left by 2π or by integer multiples of 2π (d) translation right or left by π (e) translation to the right by $\frac{\pi}{2}$ or to the left by $\frac{3\pi}{2}$ (f) $x = \frac{\pi}{4}, x = -\frac{3\pi}{4}$ 23(a) There are none. (b) each of its x -intercepts(c) translations to the right or left by π or by integer multiples of π (d) $x = \frac{\pi}{4}, x = -\frac{\pi}{4}$

24(a) 63 (b) 218 (There is more than one answer.)

Exercise 14D (Page 507)

1(a) $\sin x \cos y - \cos x \sin y$ (b) $\cos 2A \cos 3B - \sin 2A \sin 3B$ (c) $\sin 3\alpha \cos 5\beta + \cos 3\alpha \sin 5\beta$ (d) $\cos \theta \cos \frac{\phi}{2} + \sin \theta \sin \frac{\phi}{2}$ (e) $\frac{\tan A + \tan 2B}{1 - \tan A \tan 2B}$ (f) $\frac{\tan 3\alpha - \tan 4\beta}{1 + \tan 3\alpha \tan 4\beta}$ 2(a) $\cos(x + y)$ (b) $\sin(3\alpha + 2\beta)$ (c) $\tan 20^\circ$ (d) $\sin 3A$ (e) $\cos 50^\circ$ (f) $\tan(\alpha + 10^\circ)$ 3(a) $\sin 2x$ (b) $\cos 2\theta$ (c) $\tan 2\alpha$ (d) $\sin 40^\circ$ (e) $\cos 100^\circ$ (f) $\tan 140^\circ$ (g) $\sin 6\theta$ (h) $\cos 4A$ (i) $\tan 8x$ 7(b)(i) $\frac{\sqrt{3} + 1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ (iii) $2 + \sqrt{3}$ 8(a) $\frac{7}{25}$ (b) $\frac{1}{9}$ (c) $\frac{120}{169}$ (d) $\frac{4}{3}$ 9(a) 1 (b) $\frac{63}{65}$ (c) $\frac{\sqrt{5}(1+2\sqrt{3})}{12}$ 11 $-\frac{3\sqrt{7}}{8}$ 13(a) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (b) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (c) $\sqrt{3} - 2$

- 17(b) $\frac{\sqrt{3}}{2}$
 19(a) -1 (c) $\sqrt{2} + 1$
 20(b) $\sqrt{2} - 1$
 21(a) $\sin \alpha \cos \beta + \cos \alpha \sin \beta$

Exercise 14E (Page 511)

- 1(a) 37° (b) 41° (c) 33°
 2(a) 45° (b) 45° (c) 45° (d) 90° (e) 30°
 (f) The lines are parallel and distinct, and so do not intersect at all.
 3 $3^\circ 11'$
 4 $36^\circ 52'$
 5(a) $(1, 1)$ (c) 53°
 6(b) 63°
 7(a) $30^\circ, 150^\circ$ (b) $45^\circ, 135^\circ$
 8 $\angle A = 71^\circ 34'$, $\angle B = 56^\circ 19'$ and $\angle C = 52^\circ 8'$.
 The sum is $180^\circ 1'$. The error is due to rounding.
 9(a) $m = -3$ or $\frac{1}{3}$
 (b) Let A and B be the points where $y = -3x$ and $y = \frac{1}{3}x$ meet $y = 2x - 4$. $\triangle AOB$ has two 45° angles, so the third angle, $\angle AOB$, is 90° .
 10(b) $3x + 2y - 5 = 0, 2x - 3y + 1 = 0$
 11(a) $(2 - \sqrt{3})x - y + \sqrt{3} = 0, (2 + \sqrt{3})x - y - \sqrt{3} = 0$
 (b) $x - y + 1 = 0, 7x - y - 11 = 0$
 12(b) $75^\circ 58'$ at $(0, 0)$ and $17^\circ 6'$ at $(1, 3)$
 13(b) 3
 14 At $(0, 0)$ the curves are perpendicular. At $(1, 1)$ and at $(-1, 1)$, $\tan \theta = \frac{4}{7}$, so the angle is $29^\circ 45'$.
 15 $x = 0, y = -\sqrt{3}x$

Exercise 14F (Page 515)

- 1(a) The entries under 5° are 0.087 27, 0.087 16, 0.998 7, 0.087 49, 1.003, 0.996 2
 (b) $\sin x < x < \tan x$ (c) 1 and 1
 (d) $x \leq 0.0774$ (to 4 decimal places), that is, $x \leq 4^\circ 26'$
 2(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$ (e) $\frac{5}{3}$ (f) 8
 3(a) 1 (b) $\frac{7}{5}$ (c) $\frac{7}{15}$
 4(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{6}$
 5(a) 2 (b) $\frac{3}{2}$ (c) $\frac{5}{7}$
 6(a) 1 (b) $\frac{a}{b}$ (c) 0
 8(a) $\cos A \cos B - \sin A \sin B$ (c) 2
 9(a) $\frac{\pi}{90}$ (b) $\sin 2^\circ = \sin \frac{\pi}{90} \doteq \frac{\pi}{90}$ (c) 0.0349
 10 87 metres
 11 $26'$
 14(c) 23° (d) 4.924 metres
 15(a) $\sin(A - B) = \sin A \cos B - \cos A \sin B$ (d) 6

- 16(a) $AB^2 = 2r^2(1 - \cos x)$, $\text{arc } AB = rx$
 (b) The arc is longer than the chord, so $\cos x$ is larger than the approximation.

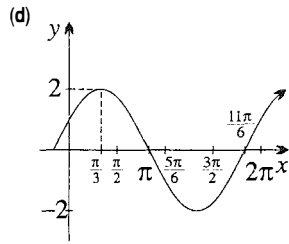
Exercise 14G (Page 520)

- 1 $y = \cos x$
 2(a) $\cos x$ (b) $-\sin x$ (c) $\sec^2 x$ (d) $2 \cos 2x$
 (e) $-2 \sin x$ (f) $4 \sec^2 2x$ (g) $2\pi \cos 2\pi x$
 (h) $\frac{\pi}{2} \sec^2 \frac{\pi}{2} x$ (i) $3 \cos x - 5 \sin 5x$
 (j) $4\pi \cos \pi x - 3\pi \sin \pi x$ (k) $-5 \sin(5x + 4)$
 (l) $-21 \cos(2 - 3x)$ (m) $-10 \sec^2(10 - x)$
 (n) $-2 \sec^2(\frac{\pi}{2} - 2\pi x)$ (o) $3 \cos(\frac{x+1}{2})$
 (p) $6 \sin(\frac{3-2x}{5})$
 3(a) $2x \cos(x^2)$ (b) $-3x^2 \sin(x^3 + 1)$ (c) $-\frac{1}{x^2} \cos(\frac{1}{x})$
 (d) $\frac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$ (e) $x \cos x + \sin x$ (f) $2x(\cos 2x - x \sin 2x)$ (g) $-2 \cos x \sin x$ (h) $3 \sin^2 x \cos x$
 (i) $-\frac{\cos x}{(1+\sin x)^2}$ (j) $\frac{1}{1+\cos x}$ (k) $\frac{-1}{1+\sin x}$
 (l) $\frac{-1}{(\cos x + \sin x)^2}$ (m) $-\tan x$ (n) $\sec^2 x \cdot e^{\tan x}$
 4(a) The graphs are reflections of each other in the x -axis. (b) The graphs are identical.
 5(a) $2 \cos 2x \cdot e^{\sin 2x}$ (b) $2e^{2x} \cos(e^{2x})$ (c) $\frac{2 \cos 4x}{\sqrt{\sin 4x}}$
 (d) $4 \cot 4x$ (e) $9 \sin 3x(1 - \cos 3x)^2$
 (f) $2(\cos 2x \sin 4x + 2 \sin 2x \cos 4x)$
 (g) $-15 \cos^4 3x \sin 3x$ (h) $\frac{20 \sin 5x}{(3+4 \cos 5x)^2}$
 (i) $15 \tan^2(5x - 4) \sec^2(5x - 4)$ (j) $\frac{x \cos \sqrt{x^2+1}}{\sqrt{x^2+1}}$
 7(a) $\frac{\pi x}{180}$ (b)(i) $\frac{\pi}{180} \cos x^\circ$ (ii) $\frac{\pi}{180} \sec^2(x^\circ + 45^\circ)$
 (iii) $-\frac{\pi}{90} \sin 2x^\circ$
 10(a) $\log_b P - \log_b Q$
 11(b) $\frac{1}{2}((m+n) \cos(m+n)x + (m-n) \cos(m-n)x)$
 (c) $\frac{1}{2}(\cos(m+n)x + \cos(m-n)x)$
 $= \cos mx \cos nx,$
 $-\frac{1}{2}((m+n) \sin(m+n)x + (m-n) \sin(m-n)x)$
 12(b) $\sin(\frac{n\pi}{2} + x)$
 16(a) $\frac{\cos x}{\sin y}$ (b) $\frac{-y \cos x - \sin y}{x \cos y + \sin x}$ (c) $\frac{\sin(x-y) + \cos(x+y)}{\sin(x-y) - \cos(x+y)}$
 18(b) $\sin 1 \doteq 0.8415, \cos 1 \doteq 0.5403$

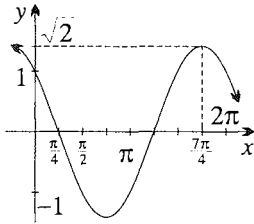
Exercise 14H (Page 525)

- 1(a) $-\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) -2 (e) $\frac{1}{4}$ (f) 8
 2(a) $y = -2x + \frac{\pi}{2}$ (b) $x + y = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$
 (c) $y = -\pi x + \pi^2$
 3(a) $x - y = \frac{\pi}{4} - \frac{1}{2}, x + y = \frac{\pi}{4} + \frac{1}{2}$ (b) $\frac{\pi^2 - 4}{32}$ units²
 4(a) $\frac{\pi}{2}, \frac{3\pi}{2}$ (b) $\frac{\pi}{6}, \frac{5\pi}{6}$ (c) $\frac{5\pi}{6}, \frac{7\pi}{6}$ (d) $\frac{\pi}{2}, \frac{3\pi}{2}$
 5 $12\sqrt{3}x - 6y = 2\sqrt{3}\pi - 3, 6x + 12\sqrt{3}y = \pi + 6\sqrt{3}$
 6(a) $y' = -\sin x + \sqrt{3} \cos x, y'' = -\cos x - \sqrt{3} \sin x$
 (b) maximum turning point $(\frac{\pi}{3}, 2)$,
 minimum turning point $(\frac{4\pi}{3}, -2)$

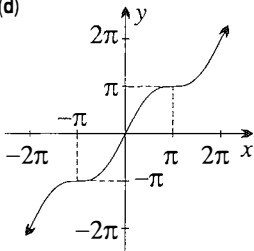
(c) $(\frac{5\pi}{6}, 0), (\frac{11\pi}{6}, 0)$



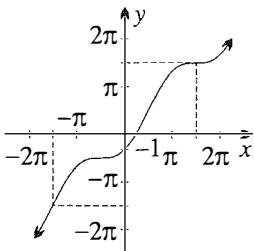
7 $y' = -\sin x - \cos x$, $y'' = -\cos x + \sin x$,
 minimum turning point $(\frac{3\pi}{4}, -\sqrt{2})$,
 maximum turning point $(\frac{7\pi}{4}, \sqrt{2})$,
 points of inflexion $(\frac{\pi}{4}, 0), (\frac{5\pi}{4}, 0)$



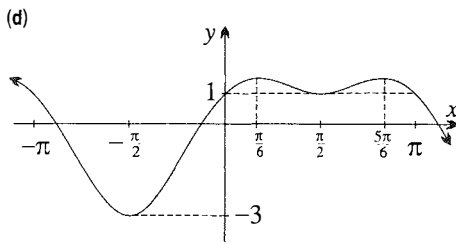
8(a) $y' = 1 + \cos x$, $y'' = -\sin x$ (b) $(-\pi, -\pi)$ and (π, π) are horizontal points of inflexion. (c) $(0, 0)$
 (d)



9 $y' = 1 + \sin x$, $y'' = \cos x$,
 horizontal points of inflexion $(-\frac{\pi}{2}, -\frac{\pi}{2}), (\frac{3\pi}{2}, \frac{3\pi}{2})$,
 points of inflexion $(-\frac{3\pi}{2}, -\frac{3\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2})$

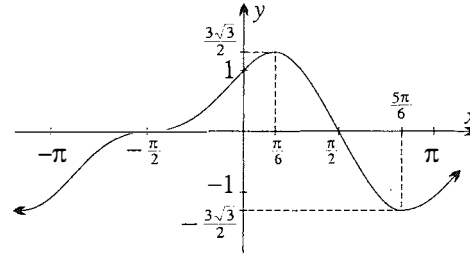


10(a) $y' = 2 \cos x - 2 \sin 2x$, $y'' = -2 \sin x - 4 \cos 2x$
 (c) maximum turning points $(\frac{\pi}{6}, \frac{3}{2})$ and $(\frac{5\pi}{6}, \frac{3}{2})$,
 minimum turning points $(-\frac{\pi}{2}, -3)$ and $(\frac{\pi}{2}, 1)$



11 horizontal point of inflexion $(-\frac{\pi}{2}, 0)$,

maximum turning point $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$,
 minimum turning point $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$

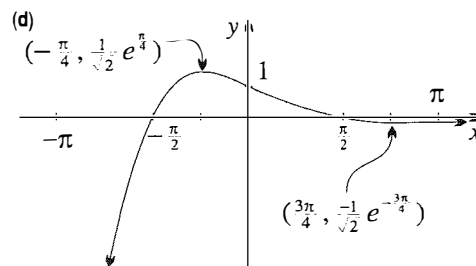


12(a) $y' = -e^{-x}(\cos x + \sin x)$, $y'' = 2e^{-x} \sin x$

(b) minimum turning point $(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}})$,

maximum turning point $(-\frac{\pi}{4}, \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}})$

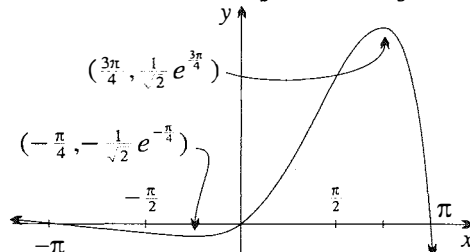
(c) $(-\pi, -e^{-\pi}), (0, 1), (\pi, -e^{-\pi})$



13 minimum turning point $(-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}})$,

maximum turning point $(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}})$,

points of inflexion $(-\frac{\pi}{2}, -e^{-\frac{\pi}{2}}), (\frac{\pi}{2}, e^{-\frac{\pi}{2}})$



14(b) $\frac{9\sqrt{2}}{10} \text{ cm}^2/\text{min}$ (c) $\theta = \pi$

19(a) The angle of inclination is $\pi - \alpha$ and so

$m = \tan(\pi - \alpha) = -\tan \alpha$.

(b) $P = (\frac{1}{\tan \alpha} + 2, 0), Q = (0, 2 \tan \alpha + 1)$

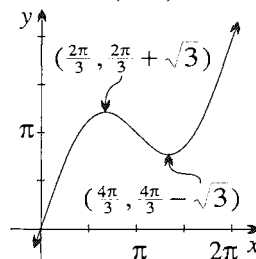
20 minimum $\sqrt{3}$ when $\theta = \frac{\pi}{6}$,

maximum 2 when $\theta = 0$

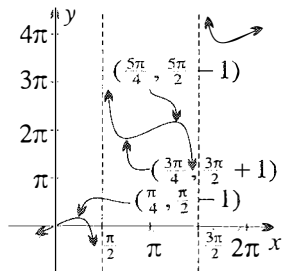
21(a) maximum turning point $(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3})$,

minimum turning point $(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3})$,

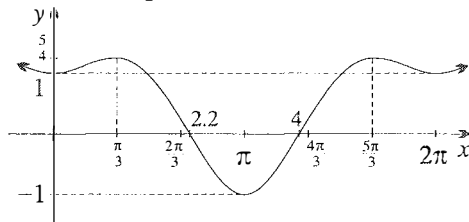
inflexion (π, π)



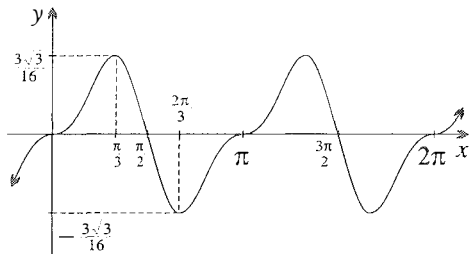
- (b) point of inflexion $(\pi, 2\pi)$,
 minimum turning points $(\frac{3\pi}{4}, \frac{3\pi}{2} + 1)$, $(\frac{7\pi}{4}, \frac{7\pi}{2} + 1)$,
 maximum turning points $(\frac{\pi}{4}, \frac{\pi}{2} - 1)$, $(\frac{5\pi}{4}, \frac{5\pi}{2} - 1)$



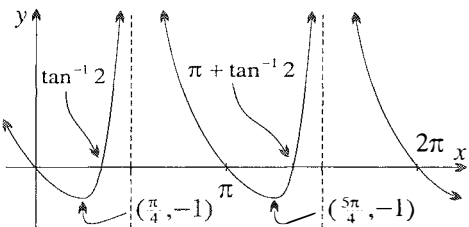
- 22(a) maximum turning points $(\frac{\pi}{3}, \frac{5}{4})$, $(\frac{5\pi}{3}, \frac{5}{4})$,
 minimum turning points $(0, 1)$, $(\pi, -1)$, $(2\pi, 1)$,
 x-intercepts $\pi - \cos^{-1} \frac{\sqrt{5}-1}{2} \approx 2.2$,
 $\pi + \cos^{-1} \frac{\sqrt{5}-1}{2} \approx 4.0$.



- (b) maximum turning points $(\frac{\pi}{3}, \frac{3\sqrt{3}}{16})$, $(\frac{4\pi}{3}, \frac{3\sqrt{3}}{16})$,
 minimum turning points $(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{16})$, $(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{16})$,
 horizontal points of inflexion $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$



- (c) minimum turning points $(\frac{\pi}{4}, -1)$, $(\frac{5\pi}{4}, -1)$,
 vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$,
 x-intercepts 0 , π , 2π , $\tan^{-1} 2 \approx 1.1$,
 $\pi + \tan^{-1} 2 \approx 4.25$



23 $(\pi^2 - 8)x - (4\pi - 8)y + (32 - 8\pi) = 0$

24(c) $f'(x) = \frac{x \cos x - \sin x}{x^2}$

25(a) $y' = e^{\lambda x}(\lambda \sin nx + n \cos nx)$

(c)(i) They approach $\frac{k\pi}{n}$ where k is an integer.

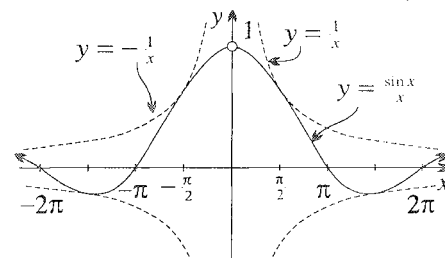
(ii) They approach $\frac{(k+\frac{1}{2})\pi}{n}$, where k is an integer.

- 26(a) Domain: $x \neq 0$, $f(x)$ is even because it is the ratio of two odd functions, the zeroes are $x = n\pi$ where n is an integer, $\lim_{x \rightarrow \infty} f(x) = 0$.

(b) $f'(x) = \frac{x \cos x - \sin x}{x^2}$, which is zero when

$\tan x = x$. (c) The graph of $y = x$ crosses the graph of $y = \tan x$ just to the left of $x = \frac{3\pi}{2}$, of $x = \frac{5\pi}{2}$ and of $x = \frac{7\pi}{2}$. Using the calculator, the three turning points of $y = f(x)$ are approximately $(1.43\pi, -0.217)$, $(2.46\pi, 0.128)$ and $(3.47\pi, -0.091)$.

(d) There is an open circle at $(0, 1)$.



Exercise 14I (Page 530)

2(a) $\tan x + C$ (b) $\sin(x+2) + C$ (c) $-\frac{1}{2} \cos 2x + C$

(d) $3 \tan \frac{1}{3}x + C$ (e) $\frac{1}{3} \sin(3x-2) + C$

(f) $\frac{1}{5} \cos(7-5x) + C$ (g) $-\tan(4-x) + C$

(h) $-3 \tan(\frac{1}{3}x) + C$

3(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{3}$ (d) 1 (e) $\frac{3}{4}$ (f) 2 (g) 1

(h) 4

4(a) $2 \sin 3x + 8 \cos \frac{1}{2}x + C$

(b) $4 \tan 2x - 40 \sin \frac{1}{4}x - 36 \cos \frac{1}{3}x + C$

5(a) $-\cos(ax+b) + C$ (b) $\pi \sin \pi x + C$

(c) $\frac{1}{u^2} \tan(v+ux) + C$ (d) $\tan ax + C$

6(a) $1 + \tan^2 x = \sec^2 x$, $\tan x - x + C$

(b) $1 - \sin^2 x = \cos^2 x$ $2\sqrt{3}$

7 The integrand $y = \sec^2 x$ is undefined at $x = \frac{\pi}{2}$, so it is not possible to form the definite integral over the interval $0 \leq x \leq \pi$. In any case, the result could hardly be 0, since the integrand is a square and can never be negative.

8(a) $\log_e f(x) + C$

(b) $\tan x = \frac{\sin x}{\cos x}$, $\int \tan x = -\ln \cos x + C$

(c) $\cos x e^{\sin x}$, $e - 1$ (d) $e^{f(x)} + C$, $e - 1$

9(a) $2x \cos x^2$, $\sin x^2 + C$

(b) $-3x^2 \sin x^3$, $-\frac{1}{3} \cos x^3 + C$

(c) $\frac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$, $2 \tan \sqrt{x} + C$

10(a) 1 (b) $\frac{5}{24}$

11(a) $5 \sin^4 x \cos x$, $\frac{1}{5} \sin^5 x + C$

(b) $-3 \sec^2 x (\tan x)^{-4}$, $-\frac{1}{3} (\tan x)^{-3} + C$

12(a) $\frac{2}{9}$ (b) $\frac{1}{n+1}$ (c) 0 (d) $\frac{1}{2^{n+1}(n+1)}$ (e) $10\frac{1}{8}$
 (f) $\frac{1}{n+1}$

13(b)(i) $\frac{1}{4}$ (ii) $\frac{1}{4}$ (iii) 0 (the integrand is odd)

14(d)(i) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ (ii) $\frac{\pi}{4}$

(e) $\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$, $\frac{\pi}{4}$

(f)(i) $\frac{1}{12}(2\pi + 3\sqrt{3})$ (ii) $\frac{1}{8}(\pi - 2\sqrt{2})$

15(a) $\frac{1}{2}\sin e^{2x} + C$ (b) $\frac{1}{2}\cos e^{-2x} + C$

(c) $\frac{1}{3}\log_e(3\tan x + 1) + C$

(d) $-\frac{3}{5}\log_e(4 + 5\cos x) + C$

(e) $\tan x - \sin x + C$ (f) $\frac{2}{3}$

16 $\sin 2x + 2x \cos 2x$, $\frac{\pi-2}{8}$

17(b) $\frac{4}{3}$

18(b)(i) $\frac{1}{2}\tan^2 x + \log_e(\cos x) + C$

(ii) $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \log_e(\cos x) + C$

19(b)(i) $\frac{6}{5}$ (ii) $-\frac{6}{7}$

20(b)(i) $1\frac{1}{5}$ (ii) 0 (c)(i) $\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C$

(ii) $\frac{\sin(m+n)x}{2(m+n)} + C$

21(b) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, so $\frac{1}{2}\sin^2 x + C$
 $= \frac{1}{4} - \frac{1}{4}\cos 2x + C = -\frac{1}{4}\cos 2x + (C + \frac{1}{4})$
 $= -\frac{1}{4}\cos 2x + D$, where $D = C + \frac{1}{4}$.

22(a) $A = 5, B = 3$

23(b) $y = (1 - \lambda)e^{\lambda x} \sin x + e^{\lambda x} \cos x$

Exercise 14J (Page 535)

1(a) $2u^2$ (b) $1u^2$

2(a) $(2 - \sqrt{2})u^2$ (b) $\frac{1}{4}u^2$

3(a) $2u^2$ (b) $\sqrt{2}u^2$ (c) $\frac{2}{3}\sqrt{3}u^2$ (d) $2u^2$ (e) $\frac{1}{2}u^2$

(f) $4u^2$ (g) $4u^2$ (h) $1u^2$

4(a) $y = \sin x + \cos 2x - 1$ (b) $\frac{1}{\pi}$ (c) $\frac{1}{2}\sqrt{3}$

(d) $f(x) = -2\cos 3x + x + (1 - \frac{\pi}{2})$

5(a) πu^3 (b) $\frac{\pi}{4}u^3$ (c) $\frac{\pi}{4}(\pi + 2)u^3$

6(b) $\frac{4}{\pi}u^2$

7 3.8 m^2

8(b) They are all $4u^2$.

9(b) $\frac{1}{6}(1 + 2\sqrt{2})$

10 $4u^2$

11(a) $\ln 2u^2$ (b) $\frac{\pi}{3}(3\sqrt{3} - \pi)u^3$

12(a) $\log \sin x$ (b) $\log 2u^2$

(c)(i) The calculation is valid. The regions above and below the x -axis have equal area, so the integral is zero.

(ii) There are asymptotes within the interval at $x = \pi$ and $x = 2\pi$, so the definite integral is not defined, and the calculation is therefore invalid.

13(a) $1u^2$ (b) $\frac{\pi^2}{4}u^3$

14(a) $\frac{\sqrt{3}}{2}u^2$ (b) $\frac{\pi}{24}(4\pi + 3\sqrt{3})u^3$

16(b) $\frac{1}{2}(3 + \sqrt{3})u^2$

17(c) $\frac{3}{4}\sqrt{3}u^2$

18(b) The curve is below $y = 1$ just as much as it is above $y = 1$, so the area is equal to the area of a rectangle n units long and one unit high.

19(a) 0 (b) As $n \rightarrow \infty$ the period of the sine curve approaches zero, and so the area approaches zero.

20(b)(i) $2\sqrt{2}u^2$ (ii) $\pi^2 u^3$

21(b)(i) $(\frac{\pi}{4} - \frac{1}{2}\ln 2)u^2$ (ii) $\pi(1 - \ln 2)u^3$

22 71.62 mL

23 12

24(a) We know that $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$. Since $x^2 > 0$, the result follows.

25(b) $\cos x$ and $(1 + \sin x)^2$ are both positive in the given domain, so y' is negative there.

26(a) 0, since the integrand is odd. (b) 0, since the integrand is odd. (c) 2, since the integrand is even. (d) $6\sqrt{3}$, since the integrand is even.

(e) 6π . The first term is even, the other two are odd. (f) $-\frac{2}{3} + \frac{\pi^3}{4}$. The first term is odd, the other two are even.

27(b) $\frac{1}{24}\sqrt{2}(12 - \pi)$ (c) $\frac{\pi}{24}(\sqrt{3} + 1)$ and $\frac{\pi}{12}\sqrt{2}$

Index

- $\equiv$ (identically equal), 312
- $\sum$ notation, 211, 224
- absolute value, 85
 - as square root of square, 88
 - defined geometrically, 85
 - defined using cases, 87
 - equations and inequations, 86
 - graphical solutions, 92
 - graphs, 87
 - identities and inequalities, 88
- adding and subtracting functions, 70
- alternating harmonic series, 458
- altitude of a triangle, 184
 - and area formula, 140
 - and cosine rule, 147
 - and sine rule, 139
 - concurrence of, 184
- Alzheimer's disease, 484
- AM and GM, 208, 236, 311
- amplitude, 496
- 'and', 25
- angle between two curves, 510
- angle between two lines, 509
- Apollonius' theorem, 186
- Apollonius, circle of, 184
- arc length, 491
- area formula, 140
- area
 - and limits, 397
 - between curves, 420
 - by integration, 415
 - of a circle, 405
- arithmetic mean, 207
 - and geometry, 208, 210
 - and the midpoint, 309
 - and the sum of the roots, 309
- arithmetic sequence, 200
 - and geometric sequences, 207
 - and linear functions, 202
 - and simple interest, 202
 - n th term of, 200
 - partial sums of, 215
- asymptotes, 55
 - horizontal, 101
 - oblique, 104
 - vertical, 100
- axis of parabola, 320
- bearings, 109
- boundary angles, 118
- boundary condition, 392
- Brahmagupta's formula, 155
- calculus, 237
- carbon dating, 484
- cardinal numbers, 29
- Cartesian equation, 328
- centroid, 185, 187
- chain rule, 254
 - generalised, 259
- chord of contact, 341
- chords of a parabola, 330
- circle
 - and trigonometric functions, 117
 - and parabola, 321
 - defined geometrically, 317
 - equation of, 54
- circumcentre, 187
- circumcircle, 146, 150, 187
 - and sine rule, 142
- collinear points, 163
- complement of a set, 25
- completing the square, 21, 285
 - and circles, 64
- composite numbers, 30
- compound angles, 504
- concavity, 373
- concurrent lines, 172
- cone
 - and Pythagoras' theorem, 121
 - area and volume of, 266, 425
- congruence
 - and trigonometry, 150
 - area formula and SAS, 141
 - cosine rule and ASS, 151

- cosine rule and SAS, 147
- sine rule and AAS, 140
- sine rule and ASS, 141
- continued fractions, 38, 42, 45
- continuity
 - and limits, 270
 - at a point, 269, 79
 - in a closed interval, 270
- coordinate plane, 156
 - and lines, 167
 - and points, 156
- Copernicus, 487
- cosine rule, 146
 - and Pythagoras' theorem, 146
- critical values, 367
- cubic identities, 11
- curvature, 372
- curve sketching, 48, 52, 64, 69
 - and calculus, 378
 - exponential functions, 487
 - logarithmic functions, 451
 - menu, 102, 378
 - quadratic functions, 280
 - trigonometric functions, 523
- cusps, 275, 369
- cyclic quadrilateral, 155
- cylinder, surface area and volume of, 266
- damped oscillations, 524
- decimals
 - periods of, 228
 - recurring, 32, 227
 - terminating, 32
- 'decimals' base 2, 228
- decreasing at a point, 357
- definite, 304
- degree, algebraic and geometric, 313
- depression, angle of, 110
- derivative, 237
 - and discriminant, 244
 - defined as a limit, 242
 - defined geometrically, 239
 - of $1/x$, 245
 - of powers of x , 245
 - of semicircle, 239
 - of $\sqrt{x}$, 245
- Descartes, René, 184
- difference of cubes, 11
- difference of powers, 229
 - sequence of, 215
- difference of squares, 3, 5
- differentiability at a point, 273
- directrix of parabola, 320
- discontinuity, 79, 269
- discriminant, 290, 299
 - and tangents, 301
- disjoint sets, 25
- domain, 48, 82
 - natural, 49, 82
 - restriction of, 62
- dominance, 451, 468
- double angle formulae, 506
- e , 441, 443
 - and exponential function, 464
 - and logarithmic functions, 447
 - and π , 520
- electron shells, 214
- elevation, angle of, 110
- ellipse, 70
- empty set, 24
- Euclid, and cosine rule, 147
- Euler, Leonhard, 49
- Euler function, 34
- even functions, 82
 - and integration, 408
 - and symmetry, 82
 - derivatives of, 377
- exponential functions, 56, 194, 438, 462
 - curve sketching, 468
 - derivative of, 462, 464
 - gradient equals height, 463
 - integral of, 472, 473
 - the exponential function, 463
- external division, 158, 349
- extremum, 362
- factorisation, 5
- feedback loop, 526
- Fermat primes, 230
- Fermat's little theorem, 228
- Fibonacci sequence, 199, 215, 236
- focal chords, 320, 331, 345
- focal length, 320
- focus of parabola, 320
- Folium of Descartes, 279
- fraction
 - algebraic, 7
 - arithmetic, 31
- functions, 48
- fundamental theorem of calculus, 402
 - differential form, 403
 - integral form, 402
- general angles, 117
- general form, 168
- geometric mean, 207
 - and geometry, 208, 210
 - and organ pipes, 210
 - and similarity, 210

- and the product of the roots, 309
- geometric sequence, 203
 - and arithmetic sequences, 207
 - and compound interest, 206
 - and depreciation, 206
 - and exponential functions, 207
 - limiting sum of, 223
 - n th term of, 203
 - partial sums of, 219
- golden mean, 210, 292
- gradient, 162
- gradient–intercept form, 168
- graphical solution of equations, 91
- grouping, factoring by, 6
- harmonic mean, 222
- HCF
 - algebraic, 5
 - arithmetic, 30
- Heron's formula, 150
- higher powers of x , 54
- homogeneous equations, 137
- horizontal line test, 60
- 'if and only if', 24
- 'if', 24
- implicit differentiation, 276
- incircle, 146, 150
- inclination, angle of, 164
- increasing at a point, 357
- indefinite, 304
- indices, 1, 188, 438
 - laws of, 189, 439
- inequalities, 73, 75
 - and definite integral, 409
 - and surds, 43
- inequations, 73
 - absolute value, 86
 - exponential, 75
 - graphical solution of, 92
 - linear, 13, 74
 - logarithmic, 75
 - quadratic, 74
 - variable in denominator, 75
- inflexion, 363, 373
 - stationary point of, 363
- initial condition, 392
- inscribed, 388
- integers, 31
- integral
 - definite, 400
 - indefinite, 412
- integration, 397, 400
- intercepts, 78
- intersection of sets, 25
- inverse function, 59
 - b^x and $\log_b x$, 439
 - differentiation of, 256
 - e^x and $\log x$, 462
- inverse relation, 59, 65
- irrational numbers, 35
- irreducible, 5
- latus rectum of parabola, 320
- LCM
 - algebraic, 7
 - arithmetic, 30
- least squares, 388
- Leibniz, Gottfried, 237, 400
- Lemniscate of Bernoulli, 72, 279
- less than, 73
- limits, 55
 - and continuity, 270
 - e^x , x and $\log x$, 451, 468
 - of definite integrals, 412
 - of functions, 268
 - of geometric series, 223
 - of $\frac{\sin x}{x}$, 513
 - of $\frac{\tan x}{x}$, 513
- line through two points, 171
- linear equations and inequations, 13, 74
- linear functions, 52
- lines
 - horizontal and vertical, 167
 - intersection of, 172
 - parallel, 163
 - perpendicular, 164
 - through the intersection of two lines, 180
- locus, 316
- locus problems, parametric, 350
- $\log x$, $\log_e x$ and $\ln x$, 445
- $\log x$ and $\ln x$ on calculators, 445
- logarithmic functions, 56, 194, 438
 - derivatives of, 441, 446
 - the logarithmic function, 445
- logarithmic inequations, 75
- logarithms, 192, 438
 - change of base, 193, 439
 - laws of, 193, 439
 - natural, 445
- Lucas sequence, 199, 215, 236
- maximisation and minimisation
 - by calculus, 383
 - in geometry, 388
 - of quadratics, 294
- maximum and minimum
 - global or absolute, 380

- local or relative, 362, 380
- maximum turning point, 364, 375
- mean value theorem, 276
- median of a triangle, 184, 185, 187
 - concurrency of, 185
- Mersenne primes, 230
- midpoint formula, 157
- minimum turning point, 364, 375
- monic quadratics, 5, 283
- music and the geometric mean, 210
- Napier, John, 445
- natural growth and decay, 479
- negative definite, 304
- Newton, Sir Isaac, 237
- Newton's method, 45
- normal
 - to a curve, 246
 - to parabola, 335
- 'not', 25
- odd functions, 83
 - and integration, 408
 - and symmetry, 83
 - derivative of, 377
- 'or', 25
- orthocentre, 184, 187
- paraboloid, 324, 327
- parabola, 316
 - defined geometrically, 320
 - reflection property, 345
 - standard forms, 321
 - translations, 325
- parallelogram, 159
- parameters, 327
- parametric differentiation, 256
- parametrisation 328
 - of circle, 328
 - of parabola, 328
 - of rectangular hyperbola, 329
- pentagon, 114
- perfect numbers, 34, 231
- period, 498
- perpendicular distance, 176
- perpendicular form, 175
- π
 - and circles, 405
 - and e , 520
 - and trigonometric functions, 487, 496, 513, 517
- piecewise defined functions, 269
- point-gradient form, 170
- polynomial, 78
- population, 479
- positive definite, 304
- power series
 - for e^x , 475
 - for $\log(1+x)$, 458
 - for $\sin x$ and $\cos x$, 516, 522, 533
- prime numbers, 30
- primitive function, 391
 - of discontinuous function, 394
 - rule for finding, 392, 393
- product rule, 260
 - generalised, 262
- proof
 - by contradiction, 31, 35
 - by mathematical induction, 231
 - by strong induction, 236
 - using coordinate geometry, 184
- pyramid, area and volume, 266
- Pythagorean identities, 129
- quadrants, 121
- quadratic equations, 16, 280
- quadratic formula, 16
- quadratic functions, 53, 280
- quadratic identities, 311
- quadratic inequations, 74
- quadratics, 280
 - AM and GM of roots of, 309
 - and lines, 313
 - axes and vertices of, 281, 286, 290
 - determined by three points, 313
 - double zeroes of, 299
 - factoring of, 5, 281
 - identically equal, 312
 - maxima and minima of, 294
 - monic, 283, 5
 - non-monic, 5, 285
 - perfect squares, 300
 - rational zeroes of, 300
 - real and unreal zeroes of, 299
 - with common vertex, 287
 - with given roots or zeroes, 282, 307
- quotient rule, 262
- radians, 487, 488
- radioactive decay, 481
- range, 48
- rates of change, 265
- ratio division formula, 157
- rational function, 101
- rational numbers, 31
- rationalising the denominator, 42
- real numbers, 35
- reciprocal function, 55
 - general primitive of, 455
 - integral of, 441, 454
- rectangle, 159

- rectangular hyperbola, 55
- recursive definition, 188, 197
- reflecting the graph, 65
- reflection in the origin, 65
- regions, graphing of, 96
- regular pentagon, 114
- related angle, 121
- relations, 49
- reverse chain rule, 429, 454, 472, 530
- rhombus, 159
- roots and zeroes, 280
- $\sec \theta$ and secant, 114
- second derivative, 371
- sector, area of, 492
- segment, area of, 492, 523
- semicircle, equation of, 54
- sequences, 196
 - partial sums of, 213
 - recursive formula for, 197
- series, 214
- sets, 23
- sign of a function, 79
- similarity
 - and the geometric mean, 210
 - and trigonometry, 108
- Simpson's rule, 434
- simultaneous equations, 18
- $\sin \theta$ and semichords, 114
- sine rule, 139
 - and the circumcircle, 142
- Snell's law, 388
- sphere
 - surface area and volume of, 266, 425
 - equation of, 319
- square, 159
- $\sqrt{x}$, 39
 - as inverse function, 62
 - graph of, 54
- $\sqrt{2}$ is not rational, 35
- St Ives, 220
- stationary point, 357
 - analysing, 364
 - and second derivative, 375
- stretching the graph, 69
- subsets, 24
 - number of, 235
- sum and product of roots, 307
- sum of cubes, 11
- sum of first n squares, 234
- sum of first n cubes, 231
- sum of powers, 229
- surds, 39
 - equality of, 45
 - inequalities, 43
- symmetric
 - in p and q , 331
 - in α and β , 308
- $\tan \theta$ and tangent, 114
- tangent to a curve, 242
 - and derivative, 242
 - and discriminant, 301, 339
 - as limits of secants, 334
 - inflectional, 374
 - to a circle, 177, 239
 - to general conic, 341
 - to parabola, 333, 338
 - vertical, 368
- terms, like and unlike, 1
- translating the graph, 64
- trapezium, 159
- trapezoidal rule, 432
- triangular numbers, 231
- trigonometry, 107
 - and right triangles, 107
 - and similarity, 108
 - and special angles, 108, 507
- trigonometric elimination, 131
- trigonometric equations, 133, 489
- trigonometric functions, 117, 487
 - a fundamental inequality, 513
 - approximations of, 514
 - as functions of real numbers, 487, 496
 - defined for acute angles, 107
 - defined for general angles, 117
 - derivatives of, 517
 - graphs of, 124, 496
 - primitives of, 528
 - symmetry of, 498
 - two fundamental limits, 513
- trigonometric identities, 129
- turning point, 363
- two-intercept form, 171
- union of sets, 25
- universal set, 25
- Venn diagram, 25
- vertex of parabola, 320
- vertical line test, 50
- volume of revolution, 423
- zeroes, 78
- zeroes and roots, 280

