YEAR 12

CAMBRIDGE Mathematics 2 Unit

Second Edition

COLOUR VERSION WITH STUDENT CD-ROM

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510

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Contents

Preface

This textbook has been written for students in Years 11 and 12 taking the 2 Unit calculus course 'Mathematics' for the NSW HSC. The book covers all the content of the course at the level required for the HSC examination. There are two volumes — the present volume is roughly intended for Year 12, and the previous volume for Year 11. Schools will, however, differ in their choices of order of topics and in their rates of progress.

Although the Syllabus has not been rewritten for the new HSC, there has been a gradual shift of emphasis in recent examination papers.

- The interdependence of the course content has been emphasised.
- Graphs have been used much more freely in argument.
- Structured problem solving has been expanded.
- There has been more stress on explanation and proof.

This text addresses these new emphases, and the exercises contain a wide variety of different types of questions.

There is an abundance of questions and problems in each exercise — too many for any one student — carefully grouped in three graded sets, so that with proper selection the book can be used at all levels of ability in the 2 Unit course.

This new second edition has been thoroughly rewritten to make it more accessible to all students. The exercises now have more early drill questions to reinforce each new skill, there are more worked exercises on each new algorithm, and some chapters and sections have been split into two so that ideas can be introduced more gradually. We have also added a review exercise to each chapter.

We would like to thank our colleagues at Sydney Grammar School and Newington College for their invaluable help in advising us and commenting on the successive drafts. We would also like to thank the Headmasters of our two schools for their encouragement of this project, and Peter Cribb, Sarah Buerckner and the team at Cambridge University Press, Melbourne, for their support and help in discussions. Finally, our thanks go to our families for encouraging us, despite the distractions that the project has caused to family life.

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How to Use This Book

This book has been written so that it is suitable for the full range of 2 Unit students, whatever their abilities and ambitions.

The Exercises: **No-one should try to do all the questions!** We have written long exercises so that everyone will find enough questions of a suitable standard — each student will need to select from them, and there should be plenty left for revision. The book provides a great variety of questions, and representatives of all types should be attempted.

Each chapter is divided into a number of sections. Each of these sections has its own substantial exercise, subdivided into three groups of questions:

FOUNDATION: These questions are intended to drill the new content of the section at a reasonably straightforward level. There is little point in proceeding without mastery of this group.

DEVELOPMENT: This group is usually the longest. It contains more substantial questions, questions requiring proof or explanation, problems where the new content can be applied, and problems involving content from other sections and chapters to put the new ideas in a wider context.

CHALLENGE: Many questions in recent 2 Unit HSC examinations have been very demanding, and this section is intended to match the standard of those recent examinations. Some questions are algebraically challenging, some require more sophistication in logic, some establish more difficult connections between topics, and some complete proofs or give an alternative approach.

The Theory and the Worked Exercises: All the theory in the course has been properly developed, but students and their teachers should feel free to choose how thoroughly the theory is presented in any particular class. It can often be helpful to learn a method first and then return to the details of the proof and explanation when the point of it all has become clear.

The main formulae, methods, definitions and results have been boxed and numbered consecutively through each chapter. They provide a bare summary only, and students are advised to make their own short summary of each chapter using the numbered boxes as a basis.

The worked examples have been chosen to illustrate the new methods introduced in the section. They should provide sufficient preparation for the questions in the following exercise, but they cannot possibly cover the variety of questions that can be asked.

The Chapter Review Exercises: A Chapter Review Exercise has been added to each chapter of the second edition. These exercises are intended only as a basic review of the chapter — for harder questions, students are advised to work through more of the later questions in the exercises.

The Order of the Topics: We have presented the topics in the order that we have found most satisfactory in our own teaching. There are, however, many effective orderings of the topics, and apart from questions that provide links between topics, the book allows all the flexibility needed in the many different situations that apply in different schools.

The time needed for the Euclidean geometry in Chapter Seven and probability in Chapter Eight will depend on students' experiences in Years 9 and 10.

We have left Euclidean geometry and probability until Year 12 for two reasons. First, we believe that functions and calculus should be developed as early as possible because these are the fundamental ideas in the course. Secondly, the courses in Years 9 and 10 already develop most of the work in Euclidean geometry and probability, at least in an intuitive fashion, so that revisiting them in Year 12, with a greater emphasis now on proof in geometry, seems an ideal arrangement.

The Structure of the Course: Recent examination papers have made the interconnections amongst the various topics much clearer. Calculus is the backbone of the course, and the two processes of differentiation and integration, inverses of each other, are the basis of most of the topics. Both processes are introduced as geometrical ideas — differentiation is defined using tangents and integration using areas — but the subsequent discussions, applications and exercises give many other ways of understanding them.

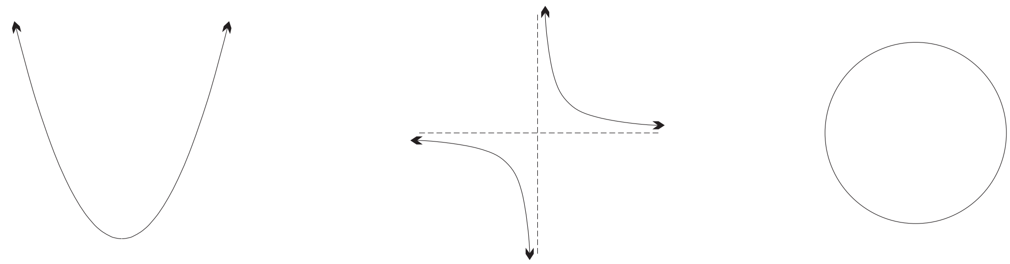
Besides linear functions, three groups of functions dominate the course:

THE QUADRATIC FUNCTIONS: These functions are known from earlier years. They are algebraic representations of the parabola, and arise naturally when areas are being considered or a constant acceleration is being applied. They can be studied without calculus, but calculus provides an alternative and sometimes quicker approach.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS: Calculus is essential for the study of these functions. We have begun the topic with the exponential function. This has the great advantage of emphasising the fundamental property that the exponential function with base e is its own derivative — this is the reason why it is essential for the study of natural growth and decay, and therefore occurs in almost every application of mathematics. The logarithmic function, and its relationship with the rectangular hyperbola $y = 1/x$, has been covered in a separate chapter.

THE TRIGONOMETRIC FUNCTIONS: Calculus is also essential for the study of the trigonometric functions. Their definitions, like the associated definition of π, are based on the circle. The graphs of the sine and cosine functions are waves, and they are essential for the study of all periodic phenomena.

Thus the three basic functions in the course, x^2, e^x and $\sin x$, and the related numbers e and π, can all be developed from the three most basic degree-2 curves — the parabola, the rectangular hyperbola and the circle. In this way, everything in the course, whether in calculus, geometry, trigonometry, coordinate geometry or algebra, can easily be related to everything else.



Algebra and Graphs: One of the chief purposes of the course, stressed heavily in recent examinations, is to encourage arguments that relate a curve to its equation. Algebraic arguments are constantly used to investigate graphs of functions. Conversely, graphs are constantly used to solve algebraic problems. We have drawn as many sketches in the book as space allowed, but as a matter of routine, students should draw diagrams for most of the problems they attempt. It is because sketches can so easily be drawn that this type of mathematics is so satisfactory for study at school.

Theory and Applications: Although this course develops calculus in a purely mathematical way using geometry and algebra, its content is fundamental to all the sciences. In particular, the applications of calculus to maximisation, motion, rates of change and finance are all parts of the syllabus. The course thus allows students to experience a double view of mathematics, as a system of pure logic on the one hand, and an essential part of modern technology on the other.

Limits, Continuity and the Real Numbers: This is a first course in calculus, and rigorous arguments about limits, continuity or the real numbers would be quite inappropriate. Any such ideas required in this course are not difficult to understand intuitively. Most arguments about limits need only the limit $\lim\limits_{x\to\infty} 1/x = 0$ and occasionally the sandwich principle. Introducing the tangent as the limit of the secant is a dramatic new idea, clearly marking the beginning of calculus, and is quite accessible. The functions in the course are too well-behaved for continuity to be a real issue. The real numbers are defined geometrically as points on the number line, and any properties that are needed can be justified by appealing to intuitive ideas about lines and curves. Everything in the course apart from these subtle issues of 'foundations' can be proven completely.

Technology: There is much discussion about what role technology should play in the mathematics classroom and what calculators or software may be effective. This is a time for experimentation and diversity. We have therefore given only a few specific recommendations about technology, but we encourage such investigation, and to this new colour version we have added some optional technology resources which can be accessed via the student CD in the back of the book. The graphs of functions are at the centre of the course, and the more experience and intuitive understanding students have, the better able they are to interpret the mathematics correctly. A warning here is appropriate — any machine drawing of a curve should be accompanied by a clear understanding of why such a curve arises from the particular equation or situation.

About the Authors

Dr Bill Pender is Subject Master in Mathematics at Sydney Grammar School, where he has taught since 1975. He has an MSc and PhD in Pure Mathematics from Sydney University and a BA (Hons) in Early English from Macquarie University. In 1973–74, he studied at Bonn University in Germany, and he has lectured and tutored at Sydney University and at the University of NSW, where he was a Visiting Fellow in 1989. He has been involved in syllabus development since the early 1990s — he was a member of the NSW Syllabus Committee in Mathematics for two years and of the subsequent Review Committee for the 1996 Years 9–10 Advanced Syllabus. More recently he was involved in the writing of the new K–10 Mathematics Syllabus. He is a regular presenter of inservice courses for AIS and MANSW, and plays piano and harpsichord.

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Derek Ward has taught Mathematics at Sydney Grammar School since 1991 and is Master in Charge of Statistics. He has an MSc in Applied Mathematics and a BScDipEd, both from the University of NSW, where he was subsequently Senior Tutor for three years. He has an AMusA in Flute, and is a lay clerk at St James', King Street, where he sings counter-tenor. He also does occasional solo work at various venues.

The Book of Nature is written in the language of Mathematics.

— The seventeenth-century Italian scientist Galileo

It is more important to have beauty in one's equations than to have them fit experiment.

— The twentieth-century English physicist Paul Dirac

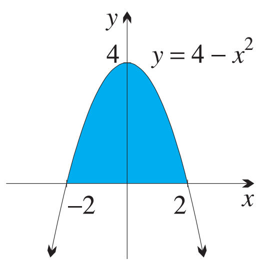
Even if there is only one possible unified theory, it is just a set of rules and equations. What is it that breathes fire into the equations and makes a universe for them to describe? The usual approach of science of constructing a mathematical model cannot answer the questions of why there should be a universe for the model to describe.

— Steven Hawking, *A Brief History of Time*

CHAPTER **ONE**

Integration

The calculation of areas has so far been restricted to regions bounded by straight lines or parts of circles. This chapter will extend the study of areas to regions bounded by more general curves. For example, it will be possible to calculate the area of the shaded region in the diagram to the right, bounded by the parabola $y = 4 - x^2$ and the x-axis.



The method developed in this chapter is called *integration*. The basis of this method is the fact that finding tangents and finding areas are inverse processes, so that integration is the inverse process of differentiation. This result is called the *fundamental theorem of calculus* and it will greatly simplify calculation of the required areas.

1 A Areas and the Definite Integral

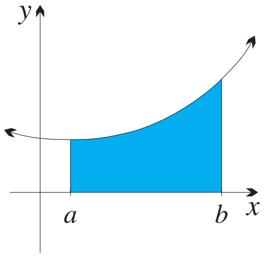
All area formulae and calculations of area are based on two principles:

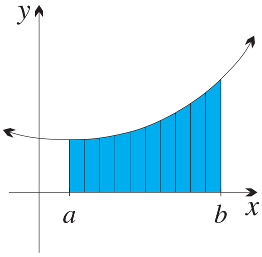
1. Area of a rectangle = length × breadth.
2. When a region is dissected, the area is unchanged.

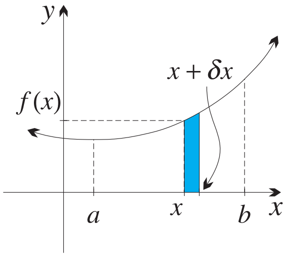
A region bounded by straight lines, like a triangle or a trapezium, can be cut up and rearranged into a rectangle with a few well-chosen cuts. Dissecting a curved region into rectangles, however, requires an infinite number of rectangles and so must be a limiting process, like differentiation.

A New Symbol — The Definite Integral: Some new notation is needed to reflect this process of infinite dissection as it applies to functions and their graphs.

The diagram on the left below shows the region contained between a given curve $y = f(x)$ and the x-axis, from $x = a$ to $x = b$. The curve must be continuous and, for the moment, entirely above the x-axis.







In the middle diagram, the region has been dissected into a number of strips. Each strip is approximately a rectangle, but only roughly so, because the upper boundary is curved. The area of the region is the sum of the areas of all the strips.

The third diagram shows just one of the strips, above the value x on the x-axis. Its height at the left-hand end is $f(x)$, and provided the strip is very thin, the height is still about $f(x)$ at the right-hand end. Let the width of the strip be δx, where δx is, as usual in calculus, thought of as being very small. Then, roughly,

$$\begin{aligned}\text{area of strip} &= \text{width} \times \text{height} \\ &= f(x)\,\delta x.\end{aligned}$$

Adding up the areas of all the strips gives the following rough formula. We need *sigma notation*, based on the Greek upper-case letter $\sum$, meaning S for sum.

$$\begin{aligned}\text{Area of shaded region} &= \sum_{x=a}^{b} \text{area of each strip} \\ &= \sum_{x=a}^{b} f(x)\,\delta x.\end{aligned}$$

If, however, there were infinitely many of these strips, each infinitesimally thin, one can imagine that the inaccuracy would disappear. This involves taking the limit so that the equality is exact:

$$\text{area of shaded region} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x.$$

At this point, the width δx is replaced by the symbol dx, which suggests an infinitesimal width, and an old form $\int$ of the letter S is used to suggest an infinite sum. The result is the strange-looking symbol $\displaystyle\int_a^b f(x)\,dx$, invented by Leibnitz. This symbol is now *defined* to denote the area of the shaded region:

$$\int_a^b f(x)\,dx = \text{area of shaded region.}$$

The Definite Integral: This new object $\displaystyle\int_a^b f(x)\,dx$ is called a *definite integral*. The rest of the chapter is concerned with evaluating definite integrals and applying them.

THE DEFINITE INTEGRAL:
Let $f(x)$ be a function that is continuous in the interval $a \le x \le b$.
For the moment, suppose that $f(x)$ is never negative in the interval.

1 The *definite integral* $\displaystyle\int_a^b f(x)\,dx$ is defined to be the area of the region between the curve and the x-axis, from $x = a$ to $x = b$.

The function $f(x)$ is called the *integrand* and the values $x = a$ and $x = b$ are called the *lower and upper bounds* of the integral.

The name 'integration' suggests putting many parts together to make a whole. The notation arises from building up the region from an infinitely large number of infinitesimally thin strips. Integration is 'making a whole' from these thin slices.

Evaluating Definite Integrals Using Area Formulae: When the function is linear or circular, the definite integral can be calculated from the graph using well-known area formulae, although a quicker method will be developed later for linear functions.

Here are the relevant area formulae:

2	FOR A TRIANGLE:	Area $= \frac{1}{2} \times$ base $\times$ height
	FOR A TRAPEZIUM:	Area $=$ width $\times$ average of parallel sides
	FOR A CIRCLE:	Area $= \pi r^2$

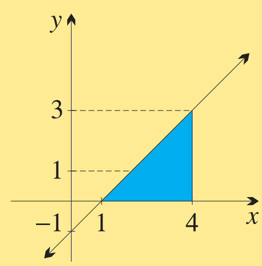
WORKED EXERCISE:

Evaluate using a graph and area formulae:

(a) $\displaystyle\int_1^4 (x-1)\,dx$ (b) $\displaystyle\int_2^4 (x-1)\,dx$

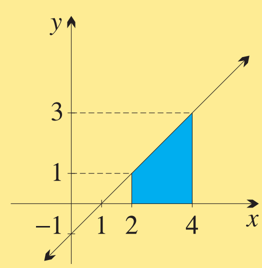
SOLUTION:

(a) The graph of $y = x - 1$ has gradient 1 and y-intercept -1.
The area represented by the integral is the shaded triangle, with base $4 - 1 = 3$ and height 3.



$$\text{Hence } \int_1^4 (x-1)\,dx = \tfrac{1}{2} \times \text{base} \times \text{height}$$
$$= \tfrac{1}{2} \times 3 \times 3$$
$$= 4\tfrac{1}{2}.$$

(b) The function $y = x - 1$ is the same as before.
The area represented by the integral is the shaded trapezium, with width $4 - 2 = 2$ and parallel sides of length 1 and 3.



$$\text{Hence } \int_2^4 (x-1)\,dx = \text{width} \times \text{average of parallel sides}$$
$$= 2 \times \frac{1+3}{2}$$
$$= 4.$$

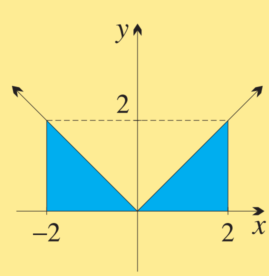
WORKED EXERCISE:

Evaluate using a graph and area formulae:

(a) $\displaystyle\int_{-2}^{2} |x|\,dx$ (b) $\displaystyle\int_{-5}^{5} \sqrt{25 - x^2}\,dx$

SOLUTION:

(a) The function $y = |x|$ is a V-shape with vertex at the origin.
Each shaded triangle has base 2 and height 2.



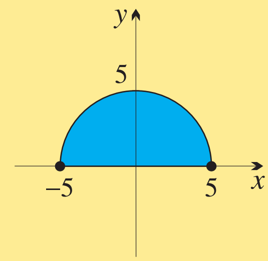
$$\text{Hence } \int_{-2}^{2} |x|\,dx = 2 \times \left(\tfrac{1}{2} \times \text{base} \times \text{height}\right)$$
$$= 2 \times \left(\tfrac{1}{2} \times 2 \times 2\right)$$
$$= 4.$$

(b) The function $y = \sqrt{25 - x^2}$ is a semicircle with centre at the origin and radius 5.

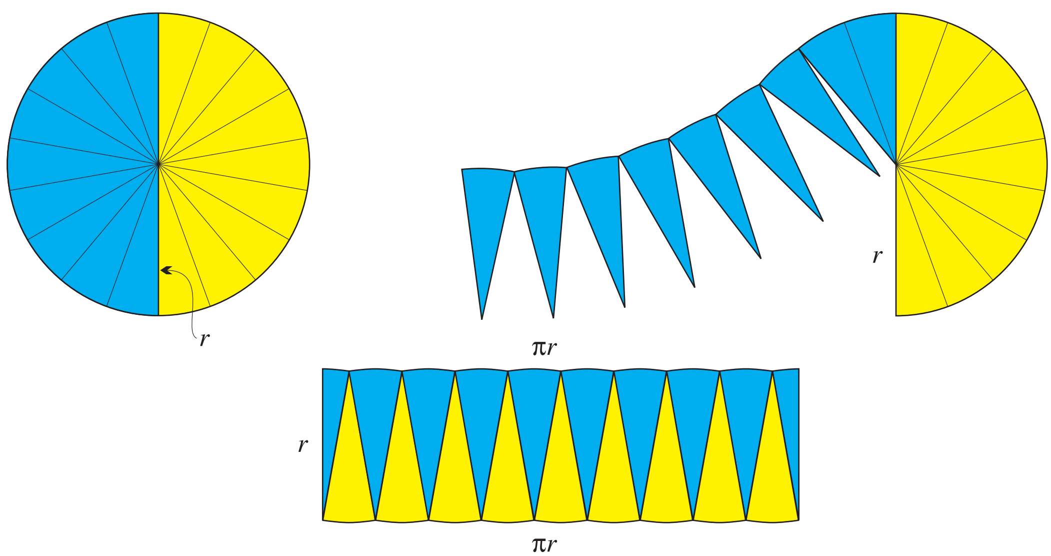
Hence $\displaystyle\int_{-5}^{5} \sqrt{25 - x^2}\, dx = \tfrac{1}{2} \times \pi r^2$

$= \tfrac{1}{2} \times 5^2 \times \pi$

$= \tfrac{25\pi}{2}$.



The Area of a Circle: In earlier years, the formula $A = \pi r^2$ for the area of a circle was proven. Because the boundary is a curve, some limiting process had to be used in that proof. For comparison with the notation for the definite integral explained at the start of this section, here is the most common version of that argument — a little rough in its logic, but very quick. It involves dissecting the circle into infinitesimally thin sectors and then rearranging them into a rectangle.

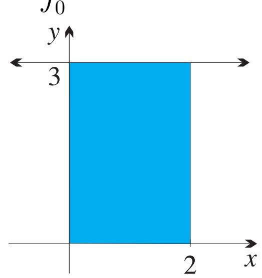


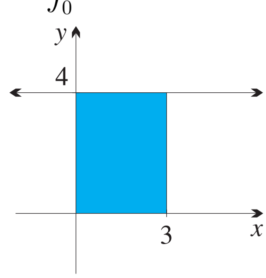
The height of the rectangle in the lower diagram is r. Since the circumference $2\pi r$ is divided equally between the top and bottom sides, the length of the rectangle is πr. Hence the rectangle has area πr^2, which is therefore the area of the circle.

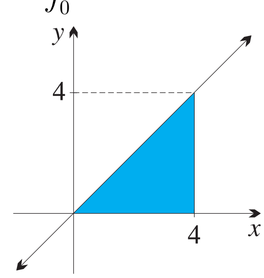
Exercise 1A

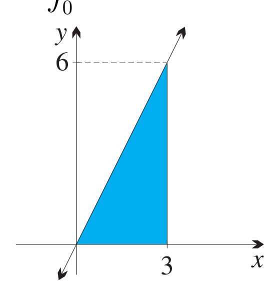
1. Use area formulae to calculate the following integrals (sketches are given):

(a) $\displaystyle\int_0^2 3\, dx$ (b) $\displaystyle\int_0^3 4\, dx$ (c) $\displaystyle\int_0^4 x\, dx$ (d) $\displaystyle\int_0^3 2x\, dx$

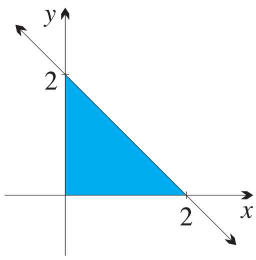


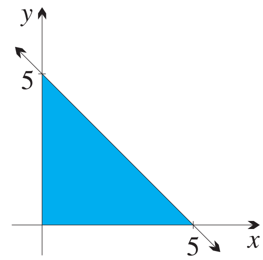


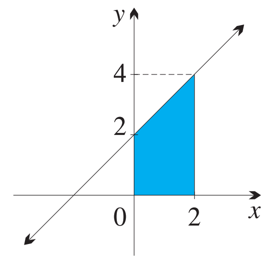


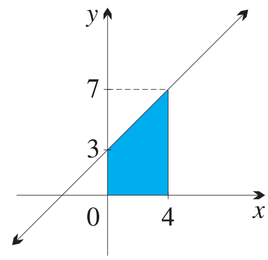


(e) $\int_0^2 (2-x)\,dx$ (f) $\int_0^5 (5-x)\,dx$ (g) $\int_0^2 (x+2)\,dx$ (h) $\int_0^4 (x+3)\,dx$



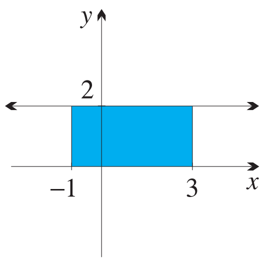


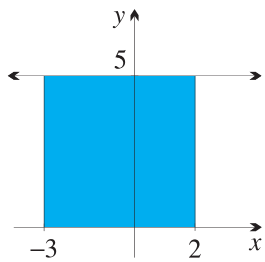


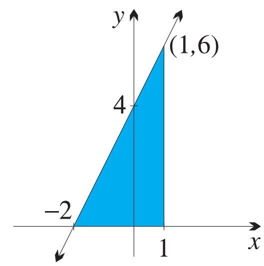


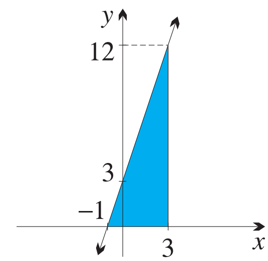
2. Use area formulae to calculate the following integrals (sketches are given):

(a) $\int_{-1}^3 2\,dx$ (b) $\int_{-3}^2 5\,dx$ (c) $\int_{-2}^1 (2x+4)\,dx$ (d) $\int_{-1}^3 (3x+3)\,dx$

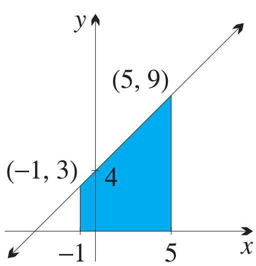


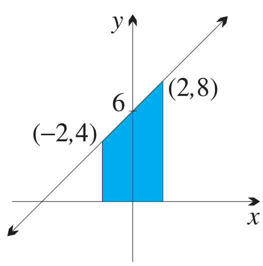


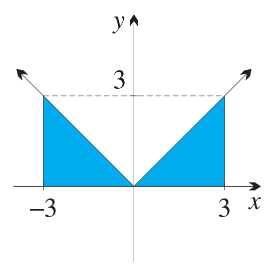


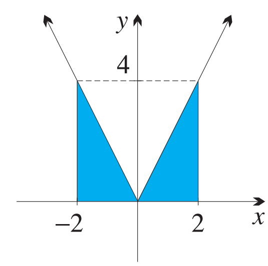


(e) $\int_{-1}^5 (x+4)\,dx$ (f) $\int_{-2}^2 (x+6)\,dx$ (g) $\int_{-3}^3 |x|\,dx$ (h) $\int_{-2}^2 |2x|\,dx$



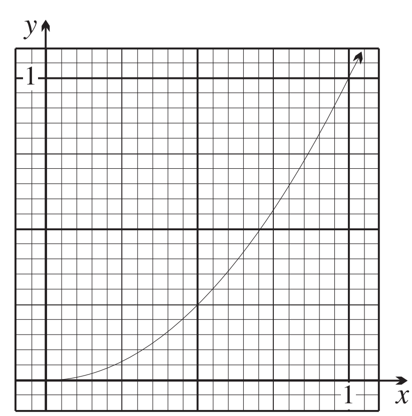






3. The diagram to the right shows the graph of $y = x^2$ from $x = 0$ to $x = 1$, drawn on graph paper.

The scale is 20 little divisions to 1 unit. This means that 400 little squares make up 1 square unit.



(a) Count how many little squares there are under the graph from $x = 0$ to $x = 1$ (keeping reasonable track of fragments of squares), then divide by 400 to approximate $\int_0^1 x^2\,dx$.

(b) By counting the appropriate squares, approximate:

(i) $\int_0^{\frac{1}{2}} x^2\,dx$ (ii) $\int_{\frac{1}{2}}^1 x^2\,dx$

Confirm that the sum of the answers to parts (i) and (ii) is the answer to part (a).

DEVELOPMENT

4. Sketch a graph of each definite integral, then use an area formula to calculate it:

(a) $\displaystyle\int_0^3 5\,dx$

(b) $\displaystyle\int_{-3}^0 5\,dx$

(c) $\displaystyle\int_{-1}^4 5\,dx$

(d) $\displaystyle\int_{-2}^6 5\,dx$

(e) $\displaystyle\int_{-5}^0 (x+5)\,dx$

(f) $\displaystyle\int_0^2 (x+5)\,dx$

(g) $\displaystyle\int_2^4 (x+5)\,dx$

(h) $\displaystyle\int_{-1}^3 (x+5)\,dx$

(i) $\displaystyle\int_4^8 (x-4)\,dx$

(j) $\displaystyle\int_4^{10} (x-4)\,dx$

(k) $\displaystyle\int_5^7 (x-4)\,dx$

(l) $\displaystyle\int_6^{10} (x-4)\,dx$

(m) $\displaystyle\int_{-2}^2 |x|\,dx$

(n) $\displaystyle\int_{-4}^4 |x|\,dx$

(o) $\displaystyle\int_0^5 |x-5|\,dx$

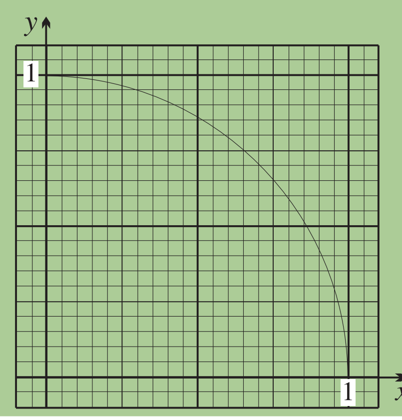
(p) $\displaystyle\int_5^{10} |x-5|\,dx$

CHALLENGE

5. [Technology] Questions 3 and 7 of this exercise involve counting squares under a curve. Many programs can do such things automatically, usually dividing the region under the curve into thin strips rather than the squares used in questions 3 and 7. Steadily increasing the number of strips should show the value converging to a limit, which can be checked either using mensuration formulae or using the exact value of the integral as calculated in the next section.

6. Sketch a graph of each definite integral, then use an area formula to calculate it:

(a) $\displaystyle\int_{-4}^4 \sqrt{16-x^2}\,dx$ (b) $\displaystyle\int_{-5}^0 \sqrt{25-x^2}\,dx$



7. The diagram to the right shows the quadrant

$$y=\sqrt{1-x^2},\quad \text{from } x=0 \text{ to } x=1.$$

As in question 3, the scale is 20 little divisions to 1 unit.

(a) Count how many little squares there are under the graph from $x=0$ to $x=1$.

(b) Divide by 400 to approximate $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.

(c) Hence find an approximation for π.

1 B The Fundamental Theorem of Calculus

The fundamental theorem is a formula for evaluating definite integrals. Its proof is rather demanding, so only the algorithm is presented in this section, by means of some worked examples. The proof is given in the appendix to this chapter.

Primitives: The formula of the fundamental theorem relies on primitives. Recall that $F(x)$ is called a *primitive* of a function $f(x)$ if the derivative of $F(x)$ is $f(x)$:

$$F(x) \text{ is a primitive of } f(x) \text{ if } F'(x)=f(x).$$

We will need the result established in the last section of the Year 11 volume:

3 FINDING PRIMITIVES: Suppose that $n \neq -1$.

$$\text{If } \frac{dy}{dx}=x^n,\ \text{then } y=\frac{x^{n+1}}{n+1}+C,\ \text{for some constant } C.$$

'Increase the index by 1 and divide by the new index.'

Statement of the Fundamental Theorem: The fundamental theorem says that a definite integral can be evaluated by writing down any primitive $F(x)$ of $f(x)$, then substituting the upper and lower bounds into it and subtracting.

4

THE FUNDAMENTAL THEOREM:
Let $f(x)$ be a function that is continuous in a closed interval $a \le x \le b$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any primitive of $f(x)$.

Using the Fundamental Theorem to Evaluate an Integral: The conventional way to set out these calculations is to enclose the primitive in square brackets, writing the upper and lower bounds as superscript and subscript respectively.

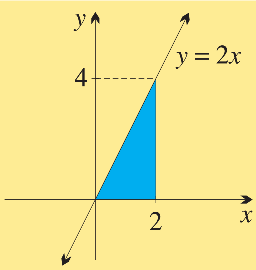
WORKED EXERCISE:

Evaluate the following definite integrals:

(a) $\displaystyle\int_0^2 2x\,dx$ (b) $\displaystyle\int_2^4 (2x-3)\,dx$

Then draw diagrams to show the regions that they represent.

SOLUTION:



(a) $\displaystyle\int_0^2 2x\,dx = \Big[x^2\Big]_0^2$ (x^2 is a primitive of $2x$.)

$= 2^2 - 0^2$ (Substitute 2, then substitute 0.)

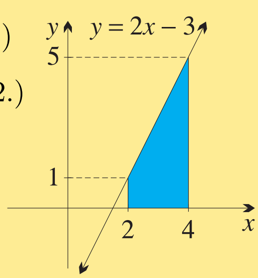
$= 4$

This value agrees with the area of the triangle shaded in the diagram to the right.

(Note that area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 2 \times 4$

$= 4$.)



(b) $\displaystyle\int_2^4 (2x-3)\,dx = \Big[x^2 - 3x\Big]_2^4$ (Take the primitive of each term.)

$= (16-12) - (4-6)$ (Substitute 4, then substitute 2.)

$= 4 - (-2)$

$= 6$

Again, this value agrees with the area of the trapezium shaded in the diagram to the right.

(Note that area of trapezium $=$ width $\times$ average of parallel sides

$= 2 \times \dfrac{1+5}{2}$

$= 2 \times 3$

$= 6$.)

NOTE: Whenever there are two or more terms in the primitive, brackets are needed when substituting the upper and lower bounds of integration. Misuse of these brackets is a common source of error.

WORKED EXERCISE:

Evaluate the following definite integrals:

(a) $\displaystyle\int_0^1 x^2\,dx$ (b) $\displaystyle\int_{-2}^2 (x^3+8)\,dx$

SOLUTION:

(a) $\displaystyle\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$ (Increase the index 2 to 3, then divide by 3.)

$= \frac{1}{3} - 0$ (Substitute 1, then substitute 0.)

$= \frac{1}{3}$

This integral was approximated by counting squares in question 3 of Exercise 1A.

(b) $\displaystyle\int_{-2}^2 (x^3+8)\,dx = \left[\frac{x^4}{4}+8x\right]_{-2}^2$ (Take the primitive of each term.)

$= (4+16)-(4-16)$ (Substitute 2, then substitute -2.)

$= 20-(-12)$

$= 32$

Expanding Brackets in the Integrand: As with differentiation, it is often necessary to expand the brackets in the integrand before finding a primitive.

WORKED EXERCISE:

Expand the brackets, then evaluate these definite integrals:

(a) $\displaystyle\int_1^6 x(x+1)\,dx$ (b) $\displaystyle\int_0^3 (x-4)(x-6)\,dx$

NOTE: Fractions arise very often in definite integrals because the standard forms for primitives involve fractions. Care is needed with the resulting common denominators, mixed numerals and cancelling.

SOLUTION:

(a) $\displaystyle\int_1^6 x(x+1)\,dx = \int_1^6 (x^2+x)\,dx$

$= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^6$

$= (72+18)-(\frac{1}{3}+\frac{1}{2})$

$= 90-\frac{5}{6}$

$= 89\frac{1}{6}$

(b) $\displaystyle\int_0^3 (x-4)(x-6)\,dx = \int_0^3 (x^2-10x+24)\,dx$

$= \left[\frac{x^3}{3}-5x^2+24x\right]_0^3$

$= (9-45+72)-(0-0+0)$

$= 36$

Writing the Integrand as Two Separate Fractions: If the integrand is a fraction with two terms in the numerator, it should normally be written as two separate fractions, as with differentiation.

WORKED EXERCISE:

Write each integrand as two separate fractions, then evaluate:

(a) $\displaystyle\int_1^2 \frac{3x^4 - 2x^2}{x^2}\,dx$ (b) $\displaystyle\int_{-3}^{-2} \frac{x^3 - 2x^4}{x^3}\,dx$

SOLUTION:

(a) $$\begin{aligned}\int_1^2 \frac{3x^4 - 2x^2}{x^2}\,dx &= \int_1^2 \big(3x^2 - 2\big)\,dx \quad \text{(Divide both terms on the top by } x^2.\text{)}\\ &= \Big[x^3 - 2x\Big]_1^2\\ &= (8-4) - (1-2)\\ &= 4 - (-1)\\ &= 5\end{aligned}$$

(b) $$\begin{aligned}\int_{-3}^{-2} \frac{x^3 - 2x^4}{x^3}\,dx &= \int_{-3}^{-2} (1-2x)\,dx \quad \text{(Divide both terms by } x^3.\text{)}\\ &= \Big[x - x^2\Big]_{-3}^{-2}\\ &= (-2-4) - (-3-9)\\ &= -6 - (-12)\\ &= -6 + 12\\ &= 6\end{aligned}$$

Negative Indices: The fundamental theorem works just as well when the indices are negative. The working, however, requires care when converting between negative powers of x and fractions.

WORKED EXERCISE:

Use negative indices to evaluate these definite integrals:

(a) $\displaystyle\int_1^5 x^{-2}\,dx$ (b) $\displaystyle\int_1^2 \frac{1}{x^4}\,dx$

SOLUTION:

(a) $$\begin{aligned}\int_1^5 x^{-2}\,dx &= \left[\frac{x^{-1}}{-1}\right]_1^5 \quad \text{(Increase the index to } -1 \text{ and divide by } -1.\text{)}\\ &= \left[-\frac{1}{x}\right]_1^5 \quad \text{(Rewrite } x^{-1} \text{ as } \frac{1}{x} \text{ before substitution.)}\\ &= -\tfrac{1}{5} - (-1)\\ &= -\tfrac{1}{5} + 1\\ &= \tfrac{4}{5}\end{aligned}$$

(b) $\displaystyle\int_1^2 \frac{1}{x^4}\,dx = \int_1^2 x^{-4}\,dx$ (Rewrite $\frac{1}{x^4}$ as x^{-4} before finding the primitive.)

$\displaystyle = \left[\frac{x^{-3}}{-3}\right]_1^2$ (Increase the index to -3 and divide by -3.)

$\displaystyle = \left[-\frac{1}{3x^3}\right]_1^2$ (Rewrite x^{-3} as $\frac{1}{x^3}$ before substitution.)

$= -\frac{1}{24} - (-\frac{1}{3})$

$= -\frac{1}{24} + \frac{8}{24}$

$= \frac{7}{24}$

Exercise 1B

TECHNOLOGY: Many programs allow definite integrals to be calculated automatically. This allows not just quick checking of the answers, but experimentation with further definite integrals. It would be helpful to generate screen sketches of the graphs and the regions involved in the integrals.

1. Evaluate the following definite integrals, using the fundamental theorem:

(a) $\displaystyle\int_0^1 2x\,dx$ (d) $\displaystyle\int_2^5 8x\,dx$ (g) $\displaystyle\int_1^2 10x^4\,dx$

(b) $\displaystyle\int_1^4 2x\,dx$ (e) $\displaystyle\int_2^3 3x^2\,dx$ (h) $\displaystyle\int_0^1 12x^5\,dx$

(c) $\displaystyle\int_1^3 4x\,dx$ (f) $\displaystyle\int_0^3 5x^4\,dx$ (i) $\displaystyle\int_0^1 11x^{10}\,dx$

2. (a) Evaluate the following definite integrals, using the fundamental theorem:

(i) $\displaystyle\int_0^1 4\,dx$ (ii) $\displaystyle\int_2^7 5\,dx$ (iii) $\displaystyle\int_4^5 dx$

(b) Check your answers by sketching the graph of the region involved.

3. Evaluate the following definite integrals, using the fundamental theorem:

(a) $\displaystyle\int_3^6 (2x+1)\,dx$ (d) $\displaystyle\int_2^3 (3x^2-1)\,dx$ (g) $\displaystyle\int_1^2 (4x^3+3x^2+1)\,dx$

(b) $\displaystyle\int_2^4 (2x-3)\,dx$ (e) $\displaystyle\int_1^4 (6x^2+2)\,dx$ (h) $\displaystyle\int_0^2 (2x+3x^2+8x^3)\,dx$

(c) $\displaystyle\int_0^3 (4x+5)\,dx$ (f) $\displaystyle\int_0^1 (3x^2+2x)\,dx$ (i) $\displaystyle\int_3^5 (3x^2-6x+5)\,dx$

4. Evaluate the following definite integrals, using the fundamental theorem. You will need to take care when finding powers of negative numbers.

(a) $\displaystyle\int_{-1}^0 (1-2x)\,dx$ (d) $\displaystyle\int_{-5}^2 dx$ (g) $\displaystyle\int_{-6}^{-2} 3x^2\,dx$

(b) $\displaystyle\int_{-1}^0 (2x+3)\,dx$ (e) $\displaystyle\int_{-1}^2 (4x^3+5)\,dx$ (h) $\displaystyle\int_{-3}^4 (12-2x)\,dx$

(c) $\displaystyle\int_{-2}^1 3x^2\,dx$ (f) $\displaystyle\int_{-2}^2 (5x^4+6x^2)\,dx$ (i) $\displaystyle\int_{-2}^{-1} (4x^3+12x^2-3)\,dx$

5. Evaluate the following definite integrals, using the fundamental theorem. You will need to take care when adding and subtracting fractions.

(a) $\int_0^3 x\,dx$ (d) $\int_0^2 (x^2+x)\,dx$ (g) $\int_{-1}^1 (x^3-x+1)\,dx$

(b) $\int_1^4 (x+2)\,dx$ (e) $\int_0^3 (x+x^2+x^3)\,dx$ (h) $\int_{-2}^3 (2x^2-3x+1)\,dx$

(c) $\int_1^3 x^2\,dx$ (f) $\int_{-1}^2 (3x+5)\,dx$ (i) $\int_{-4}^{-2} (16-x^3-x)\,dx$

DEVELOPMENT

6. By expanding the brackets where necessary, evaluate the following definite integrals:

(a) $\int_2^3 x(2+3x)\,dx$ (e) $\int_{-1}^1 x^2(5x^2+1)\,dx$ (i) $\int_{-1}^0 x(x-1)(x+1)\,dx$

(b) $\int_0^2 (x+1)(3x+1)\,dx$ (f) $\int_1^3 (x+2)^2\,dx$ (j) $\int_{-2}^{-1} x(x-2)(x+3)\,dx$

(c) $\int_0^1 3x(2+x)\,dx$ (g) $\int_{-1}^2 (x-3)^2\,dx$ (k) $\int_{-1}^0 (1-x^2)^2\,dx$

(d) $\int_2^3 2x(x-1)\,dx$ (h) $\int_{-2}^3 (4-3x)^2\,dx$ (l) $\int_4^9 \left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)\,dx$

7. By dividing each fraction through by the denominator, evaluate each integral:

(a) $\int_1^3 \frac{3x^3+4x^2}{x}\,dx$ (c) $\int_2^3 \frac{5x^2+9x^4}{x^2}\,dx$ (e) $\int_1^3 \frac{x^3-x^2+x}{x}\,dx$

(b) $\int_1^2 \frac{4x^4-x}{x}\,dx$ (d) $\int_1^2 \frac{x^3+4x^2}{x}\,dx$ (f) $\int_{-2}^{-1} \frac{x^3-2x^5}{x^2}\,dx$

8. Evaluate the following definite integrals, using the fundamental theorem. You will need to take care when finding the powers of fractions.

(a) $\int_0^{\frac{1}{2}} x^2\,dx$ (b) $\int_{\frac{2}{3}}^1 (2x+3x^2)\,dx$ (c) $\int_{\frac{3}{4}}^{\frac{4}{3}} (6-4x)\,dx$

9. (a) Evaluate the following definite integrals:

(i) $\int_5^{10} x^{-2}\,dx$ (ii) $\int_2^3 2x^{-3}\,dx$ (iii) $\int_{\frac{1}{2}}^1 4x^{-5}\,dx$

(b) By writing them with negative indices, evaluate the following definite integrals:

(i) $\int_1^2 \frac{dx}{x^2}$ (ii) $\int_1^4 \frac{dx}{x^3}$ (iii) $\int_{\frac{1}{2}}^1 \frac{3}{x^4}\,dx$

10. (a) (i) Show that $\int_2^k 3\,dx = 3k-6$.

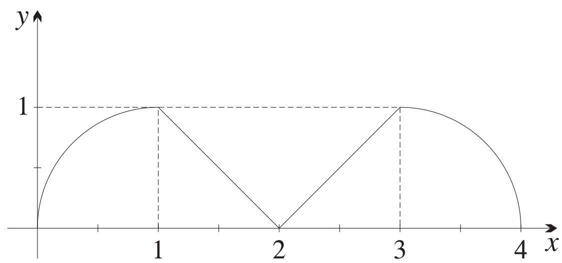
(ii) Hence find the value of k if $\int_2^k 3\,dx = 18$.

(b) (i) Show that $\int_0^k x\,dx = \frac{1}{2}k^2$.

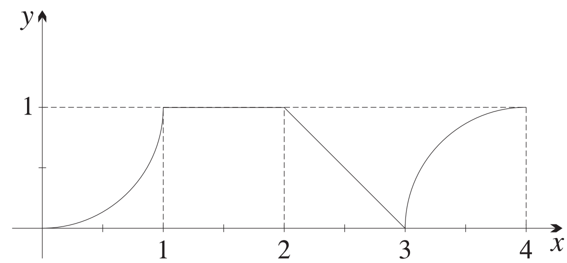
(ii) Hence find k if $k>0$ and $\int_0^k x\,dx = 18$.

11. Use area formulae to find $\int_0^4 f(x)\,dx$ in each sketch of $f(x)$:

(a)



(b)



CHALLENGE

12. By dividing each fraction through by the denominator, evaluate each integral:

(a) $\displaystyle\int_1^2 \frac{1+x^2}{x^2}\,dx$ (b) $\displaystyle\int_{-2}^{-1} \frac{1+2x}{x^3}\,dx$ (c) $\displaystyle\int_{-3}^{-1} \frac{1-x^3-4x^5}{2x^2}\,dx$

13. Evaluate the following definite integrals:

(a) $\displaystyle\int_1^3 \left(x+\frac{1}{x}\right)^2 dx$ (b) $\displaystyle\int_{-3}^{-1} \left(x^2+\frac{1}{x^2}\right)^2 dx$ (c) $\displaystyle\int_{-2}^{3} (x^2-x)^2\,dx$

14. (a) Explain why the function $y = \dfrac{1}{x^2}$ is never negative.

(b) Sketch the integrand and explain why the argument below is invalid:

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

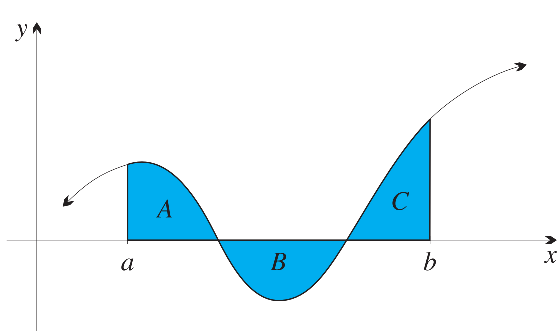
1 C The Definite Integral and its Properties

This section will first extend the theory to functions with negative values. Then some simple properties of the definite integral will be established using arguments about the dissection of the area associated with the integral.

Integrating Functions with Negative Values: When a function has negative values, its graph is below the x-axis, so the 'heights' of the little rectangles in the dissection are negative numbers. This means that any areas below the x-axis should contribute negative values to the value of the final integral.

For example, in the diagram to the right, the region B is below the x-axis and so will contribute a negative number to the definite integral:

$$\int_a^b f(x)\,dx = \text{area } A - \text{area } B + \text{area } C.$$



Because areas under the x-axis are counted as negative, the definite integral is sometimes referred to as the *signed area* under the curve, to distinguish it from area, which is always positive.

5 THE DEFINITE INTEGRAL:
Let $f(x)$ be a function that is continuous in the interval $a \le x \le b$.
Suppose now that $f(x)$ may take positive and negative values in the interval.
The *definite integral* $\int_a^b f(x)\,dx$ is the sum of the areas above the x-axis, from $x = a$ to $x = b$, minus the sum of the areas below the x-axis.

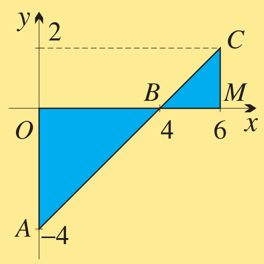
WORKED EXERCISE:

Evaluate these definite integrals:

(a) $\int_0^4 (x-4)\,dx$ (b) $\int_4^6 (x-4)\,dx$ (c) $\int_0^6 (x-4)\,dx$

Sketch the graph of $y = x - 4$ and then shade the regions associated with these integrals. Then explain how each result is related to the shaded regions.

SOLUTION:



(a) $\int_0^4 (x-4)\,dx = \Big[\tfrac{1}{2}x^2 - 4x\Big]_0^4$

$= (8 - 16) - (0 - 0)$

$= -8$

Triangle OAB has area 8 and is below the x-axis; this is why the value of the integral is -8.

(b) $\int_4^6 (x-4)\,dx = \Big[\tfrac{1}{2}x^2 - 4x\Big]_4^6$

$= (18 - 24) - (8 - 16)$

$= -6 - (-8)$

$= 2$

Triangle BMC has area 2 and is above the x-axis; this is why the value of the integral is 2.

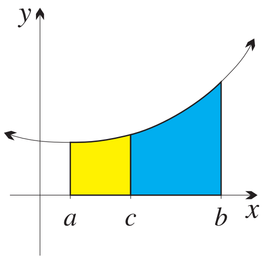
(c) $\int_0^6 (x-4)\,dx = \Big[\tfrac{1}{2}x^2 - 4x\Big]_0^6$

$= (18 - 24) - (0 - 0)$

$= -6$

This integral represents the area of $\triangle BMC$ minus the area of $\triangle OAB$; this is why the value of the integral is $2 - 8 = -6$.

Dissection of the Interval: When a region is dissected, its area remains the same. We can always dissect the region by dissecting the interval $a \le x \le b$ of integration.



Thus if $f(x)$ is continuous in the interval $a \le x \le b$, and the number c lies in this interval, then:

6 DISSECTION: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Odd and Even Functions: In the first example below, the function $y = x^3 - 4x$ is an odd function, with point symmetry in the origin. Thus the area of each shaded hump is the same. Hence the whole integral from $x = -2$ to $x = 2$ is zero, because the equal humps above and below the x-axis cancel out.

In the second diagram, the function $y = x^2 + 1$ is even, with line symmetry in the y-axis. Thus the areas to the left and right of the y-axis are equal, so there is a doubling instead of a cancelling.

7

ODD FUNCTIONS: If $f(x)$ is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

EVEN FUNCTIONS: If $f(x)$ is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

WORKED EXERCISE:

Sketch these integrals, then evaluate them using symmetry:

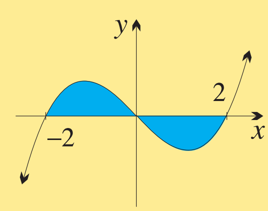
(a) $\displaystyle\int_{-2}^{2} (x^3 - 4x)\,dx$ (b) $\displaystyle\int_{-2}^{2} (x^2 + 1)\,dx$

SOLUTION:

(a) $\displaystyle\int_{-2}^{2} (x^3 - 4x)\,dx = 0$, since the integrand is odd.

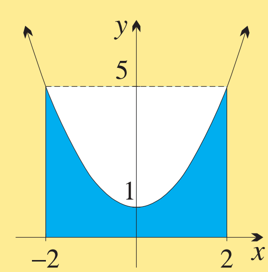
(Without this simplification, the calculation is:

$$\int_{-2}^{2} (x^3 - 4x)\,dx = \Big[\tfrac{1}{4}x^4 - 2x^2\Big]_{-2}^{2}$$
$$= (4 - 8) - (4 - 8)$$
$$= 0, \text{ as before.)}$$

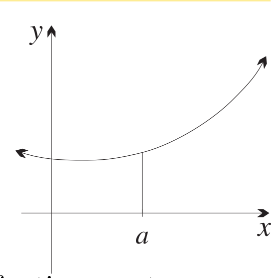


(b) Since the integrand is even,

$$\int_{-2}^{2} (x^2 + 1)\,dx = 2\int_{0}^{2} (x^2 + 1)\,dx$$
$$= 2\Big[\tfrac{1}{3}x^3 + x\Big]_{0}^{2}$$
$$= 2\Big((2\tfrac{2}{3} + 2) - (0 + 0)\Big)$$
$$= 9\tfrac{1}{3}.$$



Intervals of Zero Width: Suppose that a function is integrated over an interval $a \le x \le a$ of width zero. In this situation, the region also has width zero and so the integral is zero.



8 INTERVALS OF ZERO WIDTH: $\displaystyle\int_{a}^{a} f(x)\,dx = 0$

Running an Integral Backwards from Right to Left: A further small qualification must be made to the definition of the definite integral. Suppose that the bounds of the integral are reversed, so that the integral 'runs backwards' from right to left over the interval. Then its value reverses in sign:

9 REVERSING THE INTERVAL: Let $f(x)$ be continuous in $a \le x \le b$. Then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

This agrees perfectly with the fundamental theorem, because

$$F(a) - F(b) = -\Big(F(b) - F(a)\Big).$$

WORKED EXERCISE:

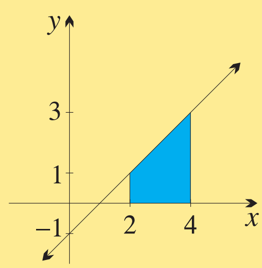
Evaluate and compare the two definite integrals:

(a) $\displaystyle\int_2^4 (x-1)\,dx$ (b) $\displaystyle\int_4^2 (x-1)\,dx$

SOLUTION:

(a) $$\int_2^4 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_2^4$$
$$= (8-4) - (2-2)$$
$$= 4,$$

which is positive, since the region is above the x-axis.



(b) $$\int_4^2 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_4^2$$
$$= (2-2) - (8-4)$$
$$= -4,$$

which is the opposite of part (a), because the integral runs backwards from right to left, from $x = 4$ to $x = 2$.

Sums of Functions: When two functions are added, the two regions are piled on top of each other, so that:

10 INTEGRAL OF A SUM: $$\int_a^b \Big(f(x) + g(x)\Big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

WORKED EXERCISE:

Evaluate these two expressions and show that they are equal:

(a) $\displaystyle\int_0^1 (x^2 + x + 1)\,dx$ (b) $\displaystyle\int_0^1 x^2\,dx + \int_0^1 x\,dx + \int_0^1 1\,dx$

SOLUTION:

(a) $$\int_0^1 (x^2 + x + 1)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_0^1$$
$$= (\tfrac{1}{3} + \tfrac{1}{2} + 1) - (0 + 0 + 0)$$
$$= 1\tfrac{5}{6}.$$

(b) $\displaystyle\int_0^1 x^2\,dx + \int_0^1 x\,dx + \int_0^1 1\,dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 + \Big[x\Big]_0^1$

$= (\tfrac{1}{3} - 0) + (\tfrac{1}{2} - 0) + (1 - 0)$

$= 1\tfrac{5}{6}$, the same as in part (a).

Multiples of Functions: Similarly, when a function is multiplied by a constant, the region is expanded vertically by that constant, so that:

11 INTEGRAL OF A MULTIPLE: $\displaystyle\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

WORKED EXERCISE:

Evaluate these two expressions and show that they are equal:

(a) $\displaystyle\int_1^3 10x^3\,dx$ (b) $\displaystyle 10\int_1^3 x^3\,dx$

SOLUTION:

(a) $\displaystyle\int_1^3 10x^3\,dx = \left[\frac{10x^4}{4}\right]_1^3$

$= \dfrac{810}{4} - \dfrac{10}{4}$

$= \dfrac{800}{4}$

$= 200.$

(b) $\displaystyle 10\int_1^3 x^3\,dx = 10 \times \left[\frac{x^4}{4}\right]_1^3$

$= 10 \times \left(\dfrac{81}{4} - \dfrac{1}{4}\right)$

$= 10 \times \dfrac{80}{4}$

$= 200.$

Inequalities with Definite Integrals: Suppose that a curve $y = f(x)$ is always underneath another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ must be less than the area under the curve $y = g(x)$.

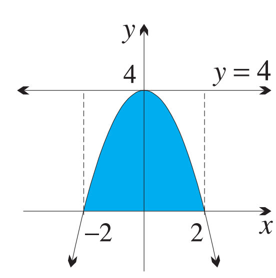
In the language of definite integrals:

12 INEQUALITY: If $f(x) \le g(x)$ in the interval $a \le x \le b$, then
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

WORKED EXERCISE:

(a) Sketch the graph of $f(x) = 4 - x^2$, for $-2 \le x \le 2$.

(b) Explain why $\displaystyle 0 \le \int_{-2}^{2} (4 - x^2)\,dx \le 16$.



SOLUTION:

(a) The parabola and line are sketched opposite.

(b) Clearly $0 \le 4 - x^2 \le 4$ over the interval $-2 \le x \le 2$.
Hence the region associated with the integral is inside the square of side length 4 in the diagram opposite.

Exercise 1C

TECHNOLOGY: All the properties of the definite integral discussed in this section have been justified visually from sketches of the graphs. Screen sketches of the graphs in this exercises would be helpful in reinforcing these explanations. Questions 6, 7, 8, 11, 12 and 13 deal with these properties. The simplification of integrals of odd and even functions is particularly important and is easily demonstrated visually by curve-sketching programs.

1. Evaluate the following definite integrals, using the fundamental theorem:

(a) $\int_{-2}^{0} 2x\,dx$ (d) $\int_{-1}^{1} 6x^5\,dx$ (g) $\int_{-1}^{4} 10x^4\,dx$

(b) $\int_{-2}^{1} 6x\,dx$ (e) $\int_{-3}^{0} 3x^2\,dx$ (h) $\int_{-3}^{-2} x^3\,dx$

(c) $\int_{-1}^{2} 4x^3\,dx$ (f) $\int_{-3}^{0} x^2\,dx$ (i) $\int_{-2}^{2} x^7\,dx$

2. Evaluate the following definite integrals, using the fundamental theorem:

(a) $\int_{-3}^{2} (1+4x)\,dx$ (e) $\int_{-1}^{1} (6x^2-8x)\,dx$ (i) $\int_{-2}^{10} (12-3x)\,dx$

(b) $\int_{-2}^{0} (3x^2-5)\,dx$ (f) $\int_{0}^{6} (x^2-6x)\,dx$ (j) $\int_{1}^{3} (3x^2-5x^4-10x)\,dx$

(c) $\int_{-1}^{1} (7-4x^3)\,dx$ (g) $\int_{-1}^{1} (x^3-x)\,dx$ (k) $\int_{-3}^{-1} (1-x-x^2)\,dx$

(d) $\int_{0}^{2} (2x-4x^3)\,dx$ (h) $\int_{0}^{3} (4x^3-2x^2)\,dx$ (l) $\int_{-2}^{2} (7-2x+x^4)\,dx$

DEVELOPMENT

3. By expanding the brackets where necessary, evaluate the following definite integrals:

(a) $\int_{1}^{3} 3x(x-4)\,dx$ (d) $\int_{0}^{2} x(1-x)\,dx$

(b) $\int_{-1}^{1} (3x-1)(3x+1)\,dx$ (e) $\int_{-2}^{2} (2-x)(1+x)\,dx$

(c) $\int_{-2}^{0} x^2(6x^3+5x^2+4x+3)\,dx$ (f) $\int_{0}^{5} x(x+1)(x-1)\,dx$

4. By dividing through by the denominator, evaluate the following definite integrals:

(a) $\int_{-2}^{-1} \frac{2x^2-5x}{x}\,dx$ (b) $\int_{-3}^{-1} \frac{3x^3+7x}{x}\,dx$ (c) $\int_{2}^{3} \frac{x^2-6x^3}{x^2}\,dx$

5. Find the value of k if:

(a) $\int_{k}^{3} 2\,dx = 4$ (d) $\int_{3}^{k} (x-3)\,dx = 0$

(b) $\int_{k}^{8} 3\,dx = 12$ (e) $\int_{1}^{k} (x+1)\,dx = 6$

(c) $\int_{2}^{3} (k-3)\,dx = 5$ (f) $\int_{1}^{k} (k+3x)\,dx = \frac{13}{2}$

6. Evaluate each group of definite integrals and use the properties of the definite integral to explain the relationships within each group:

(a) (i) $\displaystyle\int_0^2 (3x^2 - 1)\,dx$ (ii) $\displaystyle\int_2^0 (3x^2 - 1)\,dx$

(b) (i) $\displaystyle\int_0^1 20x^3\,dx$ (ii) $\displaystyle 20\int_0^1 x^3\,dx$

(c) (i) $\displaystyle\int_1^4 (4x + 5)\,dx$ (ii) $\displaystyle\int_1^4 4x\,dx$ (iii) $\displaystyle\int_1^4 5\,dx$

(d) (i) $\displaystyle\int_0^2 12x^3\,dx$ (ii) $\displaystyle\int_0^1 12x^3\,dx$ (iii) $\displaystyle\int_1^2 12x^3\,dx$

(e) (i) $\displaystyle\int_3^3 (4 - 3x^2)\,dx$ (ii) $\displaystyle\int_{-2}^{-2} (4 - 3x^2)\,dx$

7. Without finding a primitive, use the properties of the definite integral to evaluate the following, stating reasons:

(a) $\displaystyle\int_3^3 \sqrt{9 - x^2}\,dx$

(b) $\displaystyle\int_4^4 (x^3 - 3x^2 + 5x - 7)\,dx$

(c) $\displaystyle\int_{-1}^1 x^3\,dx$

(d) $\displaystyle\int_{-5}^5 (x^3 - 25x)\,dx$

(e) $\displaystyle\int_{-90^\circ}^{90^\circ} \sin x\,dx$

(f) $\displaystyle\int_{-2}^2 \frac{x}{1 + x^2}\,dx$

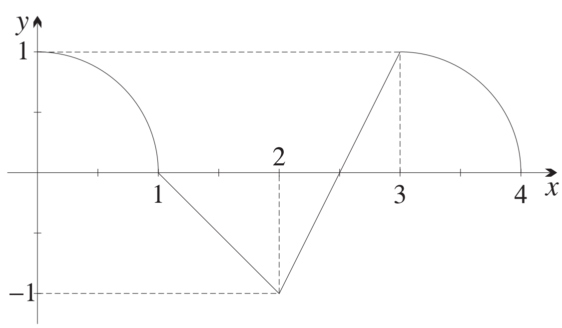
8. (a) On one set of axes sketch $y = x^2$ and $y = x^3$, clearly showing the point of intersection.

(b) Hence explain why $\displaystyle 0 < \int_0^1 x^3\,dx < \int_0^1 x^2\,dx < 1$.

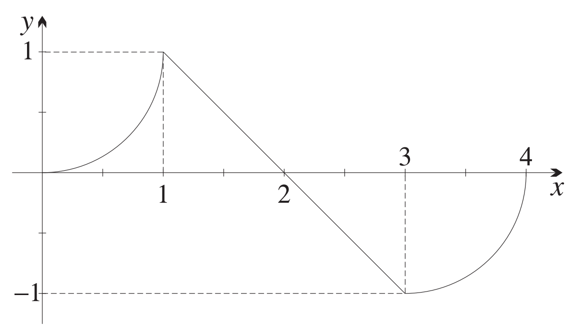
(c) Check the inequality in part (b) by evaluating each integral.

9. Use area formulae to find $\displaystyle\int_0^4 f(x)\,dx$, given the following sketches of $f(x)$:

(a)



(b)



CHALLENGE

10. By dividing through by the denominator, evaluate the following definite integrals:

(a) $\displaystyle\int_1^4 \frac{x^3 - 3}{x^3}\,dx$ (b) $\displaystyle\int_{-2}^{-1} \frac{x^5 - 2}{x^3}\,dx$ (c) $\displaystyle\int_1^2 \frac{2x^2 - 3x + 1}{x^4}\,dx$

11. Use the results of the previous question to write down the values of these definite integrals:

(a) $\displaystyle\int_4^1 \frac{x^3 - 3}{x^3}\,dx$ (b) $\displaystyle\int_{-1}^{-2} \frac{x^5 - 2}{x^3}\,dx$ (c) $\displaystyle\int_2^1 \frac{2x^2 - 3x + 1}{x^4}\,dx$

12. Sketch a graph of each integral and hence determine whether each statement is true or false:

(a) $\displaystyle\int_{-1}^1 2^x\,dx = 0$ (b) $\displaystyle\int_0^2 3^x > 0$ (c) $\displaystyle\int_{-2}^{-1} \frac{1}{x}\,dx > 0$ (d) $\displaystyle\int_2^1 \frac{1}{x}\,dx > 0$

1 D The Indefinite Integral

Now that primitives have been established as the key to calculating definite integrals, this section turns again to the task of finding primitives. First, a new and convenient notation for the primitive is introduced.

The Indefinite Integral: Because of the close connection established by the fundamental theorem between primitives and definite integrals, the term *indefinite integral* is often used for the primitive. The usual notation for the primitive of a function $f(x)$ is an integral sign without any upper or lower bounds. For example, the primitive or indefinite integral of $x^2 + 1$ is

$$\int (x^2 + 1)\,dx = \frac{x^3}{3} + x + C, \text{ for some constant } C.$$

The word 'indefinite' implies that the integral cannot be evaluated further because no bounds for the integral have yet been specified.

A definite integral ends up as a pure number. An indefinite integral, on the other hand, is a function of x — the pronumeral x is carried across to the answer. The constant is called a 'constant of integration' and is an important part of the answer. Despite being a nuisance to write down every time, it must always be included. In most problems other than definite integrals, it will not be zero.

Standard Forms for Integration: The rules for finding primitives given in the last section of the Year 11 volume can now be restated in this new notation.

13 STANDARD FORMS FOR INTEGRATION: Suppose that $n \neq -1$. Then

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ for some constant } C.$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \text{ for some constant } C.$$

The word 'integration' is commonly used to refer to both the finding of a primitive and the evaluating of a definite integral.

NOTE: Strictly speaking, the words 'for some constant C' or 'where C is a constant' should follow the first mention of the new pronumeral C, because no pronumeral should be used without having been formally introduced. There is a limit to one's patience, however, and usually in this situation it is quite clear that C is the constant of integration. If another pronumeral such as D is used, it would be wise to introduce it formally.

WORKED EXERCISE:

Use the standard form $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ to find:

(a) $\int 9\,dx$

(b) $\int 12x^3\,dx$

SOLUTION:

(a) $\int 9\,dx = 9x + C$, for some constant C

NOTE: We know that $9x$ is the primitive of 9, because $\dfrac{d}{dx}(9x) = 9$.
But the formula still gives the correct answer, because $9 = 9x^0$, and so increasing the index to 1 and dividing by this new index 1,

$$\begin{aligned}\int 9x^0\,dx &= \frac{9x^1}{1} + C, \text{ for some constant } C\\ &= 9x + C.\end{aligned}$$

(b) $$\begin{aligned}\int 12x^3\,dx &= 12 \times \frac{x^4}{4} + C, \text{ for some constant } C\\ &= 3x^4 + C\end{aligned}$$

WORKED EXERCISE:

Use the standard form $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ to find:

(a) $\displaystyle\int (3x+1)^5\,dx$ (b) $\displaystyle\int (5-2x)^2\,dx$

SOLUTION:

(a) $$\begin{aligned}\int (3x+1)^5\,dx &= \frac{(3x+1)^6}{3\times 6} + C \qquad \text{(Here } a = 3 \text{ and } b = 1.)\\ &= \tfrac{1}{18}(3x+1)^6 + C\end{aligned}$$

(b) $$\begin{aligned}\int (5-2x)^2\,dx &= \frac{(5-2x)^3}{(-2)\times 3} + C \qquad \text{(Here } a = -2 \text{ and } b = 5.)\\ &= -\tfrac{1}{6}(5-2x)^3 + C\end{aligned}$$

Negative Indices: Both standard forms apply with negative indices as well as positive indices, as in the following worked exercise.

WORKED EXERCISE:

Use negative indices to find the indefinite integrals:

(a) $\displaystyle\int \frac{12}{x^3}\,dx$ (b) $\displaystyle\int \frac{dx}{(3x+4)^2}$

SOLUTION:

(a) $$\begin{aligned}\int \frac{12}{x^3}\,dx &= \int 12x^{-3}\,dx \qquad \text{(Rewrite } \tfrac{1}{x^3} \text{ as } x^{-3} \text{ before integrating.)}\\ &= 12 \times \frac{x^{-2}}{-2} + C \qquad \text{(Increase the index to } -2 \text{ and then divide by } -2.)\\ &= -\frac{6}{x^2} + C \qquad \text{(Rewrite } x^{-2} \text{ as } \tfrac{1}{x^2}.)\end{aligned}$$

(b) $$\begin{aligned}\int \frac{dx}{(3x+4)^2} &= \int (3x+4)^{-2}\,dx \qquad \text{(Rewrite } \tfrac{1}{(3x+4)^2} \text{ as } (3x+4)^{-2}.)\\ &= \frac{(3x+4)^{-1}}{3\times(-1)} + C \qquad \text{(Here } a = 3 \text{ and } b = 4.)\\ &= -\frac{1}{3(3x+4)} + C \qquad \text{(Rewrite } (3x+4)^{-1} \text{ as } \tfrac{1}{3x+4}.)\end{aligned}$$

Special Expansions: In many integrals, brackets must be expanded before the indefinite integral can be found. The following worked exercises use the special expansions; the second also requires negative indices.

WORKED EXERCISE: Find these indefinite integrals:

(a) $\displaystyle\int (x^3 - 1)^2\, dx$ (b) $\displaystyle\int \left(3 - \frac{1}{x^2}\right)\left(3 + \frac{1}{x^2}\right) dx$

SOLUTION:

(a) $\displaystyle\int (x^3 - 1)^2\, dx = \int (x^6 - 2x^3 + 1)\, dx$ (Use $(A + B)^2 = A^2 + 2AB + B^2$.)

$$= \frac{x^7}{7} - \frac{x^4}{2} + x + C$$

(b) $\displaystyle\int \left(3 - \frac{1}{x^2}\right)\left(3 + \frac{1}{x^2}\right) dx = \int \left(9 - \frac{1}{x^4}\right) dx$ (Use $(A - B)(A + B) = A^2 - B^2$.)

$\displaystyle= \int \left(9 - x^{-4}\right) dx$ (Use $\dfrac{1}{x^4} = x^{-4}$.)

$$= 9x - \frac{x^{-3}}{-3} + C$$

$$= 9x + \frac{1}{3x^3} + C$$

Fractional Indices: The standard forms for finding primitives of powers also apply to fractional indices. These calculations require quick conversions between fractional indices and surds.

WORKED EXERCISE: Use fractional and negative indices to evaluate:

(a) $\displaystyle\int_1^4 \sqrt{x}\, dx$ (b) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\, dx$

SOLUTION:

(a) $\displaystyle\int_1^4 \sqrt{x}\, dx = \int_1^4 x^{\frac{1}{2}}\, dx$ (Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ before finding the primitive.)

$\displaystyle= \tfrac{2}{3}\Big[\, x^{\frac{3}{2}} \,\Big]_1^4$ (Increase the index to $\frac{3}{2}$ and divide by $\frac{3}{2}$.)

$= \tfrac{2}{3} \times (8 - 1)$ (Note that $4^{\frac{3}{2}} = 2^3 = 8$ and $1^{\frac{3}{2}} = 1$.)

$= 4\tfrac{2}{3}$

(b) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\, dx = \int_1^4 x^{-\frac{1}{2}}\, dx$ (Rewrite $\dfrac{1}{\sqrt{x}}$ as $x^{-\frac{1}{2}}$ before finding the primitive.)

$\displaystyle= \tfrac{2}{1}\Big[x^{\frac{1}{2}}\Big]_1^4$ (Increase the index to $\frac{1}{2}$ and divide by $\frac{1}{2}$.)

$= 2 \times (2 - 1)$ (Note that $4^{\frac{1}{2}} = \sqrt{4} = 2$ and $1^{\frac{1}{2}} = 1$.)

$= 2$

WORKED EXERCISE:

(a) Use index notation to express $\dfrac{1}{\sqrt{9-2x}}$ as a power of $9-2x$.

(b) Hence find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{9-2x}}$.

SOLUTION:

(a) $\dfrac{1}{\sqrt{9-2x}} = (9-2x)^{-\frac{1}{2}}$. (Use $\dfrac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$.)

(b) Hence $\displaystyle\int \frac{1}{\sqrt{9-2x}} = \int (9-2x)^{-\frac{1}{2}}\,dx$

$= \dfrac{(9-2x)^{\frac{1}{2}}}{-2 \times \frac{1}{2}} + C$ (Use $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}$.)

$= -\sqrt{9-2x} + C$ (Use $u^{\frac{1}{2}} = \sqrt{u}$.)

Running a Chain-Rule Differentiation Backwards: Finding primitives is the reverse process of differentiation. Thus once any differentiation has been performed, the process can then be reversed to give a primitive.

WORKED EXERCISE: [These questions are always difficult.]

(a) Differentiate $(x^2+1)^4$.

(b) Hence find a primitive of $8x(x^2+1)^3$.

SOLUTION:

(a) Let $y = (x^2+1)^4$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 4(x^2+1)^3 \times 2x$

$= 8x(x^2+1)^3$.

Let $u = x^2+1$.

Then $y = u^4$.

Hence $\dfrac{du}{dx} = 2x$

and $\dfrac{dy}{du} = 4u^3$.

(b) Hence $\dfrac{d}{dx}(x^2+1)^4 = 8x(x^2+1)^3$.

Reversing this, $\displaystyle\int 8x(x^2+1)^3\,dx = (x^2+1)^4 + C$, for some constant C.

NOTE: Questions in the 2 Unit course would never ask for such an integral without first asking for the appropriate derivative.

Exercise 1D

TECHNOLOGY: Many programs that can perform algebraic manipulation are also able to deal with indefinite integrals. They can be used to check the questions in this exercise and to investigate the patterns arising in such calculations.

1. Find the following indefinite integrals:

(a) $\int 4\,dx$ (c) $\int 0\,dx$ (e) $\int x\,dx$ (g) $\int x^3\,dx$

(b) $\int 1\,dx$ (d) $\int (-2)\,dx$ (f) $\int x^2\,dx$ (h) $\int x^7\,dx$

2. Find the indefinite integral of each function. Use the notation of the previous question.

(a) $2x$ (b) $4x$ (c) $3x^2$ (d) $4x^3$ (e) $10x^9$ (f) $2x^3$ (g) $4x^5$ (h) $3x^8$

3. Find the following indefinite integrals:

(a) $\int (x + x^2)\, dx$ (b) $\int (x^4 - x^3)\, dx$ (c) $\int (x^7 + x^{10})\, dx$

(d) $\int (2x + 5x^4)\, dx$ (e) $\int (9x^8 - 11)\, dx$ (f) $\int (7x^{13} + 3x^8)\, dx$

(g) $\int (4 - 3x)\, dx$ (h) $\int (1 - x^2 + x^4)\, dx$ (i) $\int (3x^2 - 8x^3 + 7x^4)\, dx$

4. Find the indefinite integral of each function. (Leave negative indices in your answers.)

(a) x^{-2} (b) x^{-3} (c) x^{-8} (d) $3x^{-4}$ (e) $9x^{-10}$ (f) $10x^{-6}$

5. Find the following indefinite integrals. (Leave fractional indices in your answers.)

(a) $\int x^{\frac{1}{2}}\, dx$ (b) $\int x^{\frac{1}{3}}\, dx$ (c) $\int x^{\frac{1}{4}}\, dx$ (d) $\int x^{\frac{2}{3}}\, dx$ (e) $\int x^{-\frac{1}{2}}\, dx$ (f) $\int 4x^{\frac{1}{2}}\, dx$

DEVELOPMENT

6. By expanding brackets where necessary, find the following indefinite integrals:

(a) $\int x(x+2)\, dx$ (b) $\int x(4 - x^2)\, dx$ (c) $\int x^2(5 - 3x)\, dx$

(d) $\int x^3(x - 5)\, dx$ (e) $\int (x - 3)^2\, dx$ (f) $\int (2x + 1)^2\, dx$

(g) $\int (1 - x^2)^2\, dx$ (h) $\int (2 - 3x)(2 + 3x)\, dx$ (i) $\int (x^2 - 3)(1 - 2x)\, dx$

7. By dividing through by the denominator, perform the following integrations:

(a) $\int \dfrac{x^2 + 2x}{x}\, dx$ (b) $\int \dfrac{x^7 + x^8}{x^6}\, dx$ (c) $\int \dfrac{2x^3 - x^4}{4x}\, dx$

8. Write each of these functions with negative indices and find its indefinite integral:

(a) $\dfrac{1}{x^2}$ (b) $\dfrac{1}{x^3}$ (c) $\dfrac{1}{x^5}$ (d) $\dfrac{1}{x^{10}}$ (e) $\dfrac{3}{x^4}$ (f) $\dfrac{5}{x^6}$

(g) $\dfrac{7}{x^8}$ (h) $\dfrac{1}{3x^2}$ (i) $\dfrac{1}{7x^5}$ (j) $-\dfrac{1}{5x^3}$ (k) $\dfrac{1}{x^2} - \dfrac{1}{x^5}$ (l) $\dfrac{1}{x^3} + \dfrac{1}{x^4}$

9. Write these functions with fractional indices and hence find their indefinite integrals:

(a) $\sqrt{x}$ (b) $\sqrt[3]{x}$ (c) $\dfrac{1}{\sqrt{x}}$ (d) $\sqrt[3]{x^2}$

10. Use the indefinite integrals of the previous question to evaluate:

(a) $\int_0^9 \sqrt{x}\, dx$ (b) $\int_0^8 \sqrt[3]{x}\, dx$ (c) $\int_{25}^{49} \dfrac{1}{\sqrt{x}}\, dx$ (d) $\int_0^1 \sqrt[3]{x^2}\, dx$

11. By using the formula $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, find:

(a) $\int (x+1)^5\,dx$
(b) $\int (x+2)^3\,dx$
(c) $\int (4-x)^4\,dx$
(d) $\int (3-x)^2\,dx$
(e) $\int (3x+1)^4\,dx$
(f) $\int (4x-3)^7\,dx$
(g) $\int (5-2x)^6\,dx$
(h) $\int (1-5x)^7\,dx$
(i) $\int (2x+9)^{11}\,dx$
(j) $\int 3(2x-1)^{10}\,dx$
(k) $\int 4(5x-4)^6\,dx$
(l) $\int 7(3-2x)^3\,dx$

12. By using the formula $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, find:

(a) $\int (\frac{1}{3}x-7)^4\,dx$
(b) $\int (\frac{1}{4}x-7)^6\,dx$
(c) $\int (1-\frac{1}{5}x)^3\,dx$

13. By using the formula $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, find:

(a) $\int \dfrac{1}{(x+1)^3}\,dx$
(b) $\int \dfrac{1}{(x-5)^4}\,dx$
(c) $\int \dfrac{1}{(3x-4)^2}\,dx$
(d) $\int \dfrac{1}{(2-x)^5}\,dx$
(e) $\int \dfrac{3}{(x-7)^6}\,dx$
(f) $\int \dfrac{8}{(4x+1)^5}\,dx$
(g) $\int \dfrac{2}{(3-5x)^4}\,dx$
(h) $\int \dfrac{4}{5(1-4x)^2}\,dx$
(i) $\int \dfrac{7}{8(3x+2)^5}\,dx$

14. By expanding the brackets, find:

(a) $\int \sqrt{x}\,\left(3\sqrt{x}-x\right)\,dx$
(b) $\int \left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)\,dx$
(c) $\int \left(2\sqrt{x}-1\right)^2\,dx$

15. (a) Evaluate the following definite integrals:

(i) $\int_0^1 x^{\frac{1}{2}}\,dx$
(ii) $\int_1^4 x^{-\frac{1}{2}}\,dx$
(iii) $\int_0^8 x^{\frac{1}{3}}\,dx$

(b) By writing them with fractional indices, evaluate the following definite integrals:

(i) $\int_0^4 \sqrt{x}\,dx$
(ii) $\int_1^9 x\sqrt{x}\,dx$
(iii) $\int_1^9 \dfrac{dx}{\sqrt{x}}$

16. By expanding the brackets where necessary, find:

(a) $\int_2^4 \left(2-\sqrt{x}\right)\left(2+\sqrt{x}\right)\,dx$
(b) $\int_0^1 \sqrt{x}\,\left(\sqrt{x}-4\right)\,dx$
(c) $\int_4^9 \left(\sqrt{x}-1\right)^2\,dx$

CHALLENGE

17. Explain why the indefinite integral $\int \dfrac{1}{x}\,dx$ can't be found in the usual way using the standard form $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$.

18. Find each of the following indefinite integrals:

(a) $\int \sqrt{2x-1}\,dx$
(b) $\int \sqrt{7-4x}\,dx$
(c) $\int \sqrt[3]{4x-1}\,dx$
(d) $\int \dfrac{1}{\sqrt{3x+5}}\,dx$

19. Evaluate the following:

(a) $\displaystyle\int_0^2 (x+1)^4\,dx$ (d) $\displaystyle\int_0^5 \left(1-\frac{x}{5}\right)^4 dx$ (g) $\displaystyle\int_{-2}^0 \sqrt[3]{x+1}\,dx$

(b) $\displaystyle\int_2^3 (2x-5)^3\,dx$ (e) $\displaystyle\int_0^1 \sqrt{9-8x}\,dx$ (h) $\displaystyle\int_1^5 \sqrt{3x+1}\,dx$

(c) $\displaystyle\int_{-2}^2 (1-x)^5\,dx$ (f) $\displaystyle\int_2^7 \frac{dx}{\sqrt{x+2}}$ (i) $\displaystyle\int_{-3}^0 \sqrt{1-5x}\,dx$

20. (a) (i) Find $\dfrac{d}{dx}(x^2+1)^5$. (ii) Hence find $\displaystyle\int 10x(x^2+1)^4\,dx$.

(b) (i) Find $\dfrac{d}{dx}(x^3+1)^4$. (ii) Hence find $\displaystyle\int 12x^2(x^3+1)^3\,dx$.

(c) (i) Find $\dfrac{d}{dx}(x^5-7)^8$. (ii) Hence find $\displaystyle\int 40x^4(x^5-7)^7\,dx$.

(d) (i) Find $\dfrac{d}{dx}(x^2+x)^6$. (ii) Hence find $\displaystyle\int 6(2x+1)(x^2+x)^5\,dx$.

21. (a) (i) Find $\dfrac{d}{dx}(x^2-3)^5$. (ii) Hence find $\displaystyle\int x(x^2-3)^4\,dx$.

(b) (i) Find $\dfrac{d}{dx}(x^3+1)^{11}$. (ii) Hence find $\displaystyle\int x^2(x^3+1)^{10}\,dx$.

(c) (i) Find $\dfrac{d}{dx}(x^4+8)^7$. (ii) Hence find $\displaystyle\int x^3(x^4+8)^6\,dx$.

(d) (i) Find $\dfrac{d}{dx}(x^2+2x)^3$. (ii) Hence find $\displaystyle\int (x+1)(x^2+2x)^2\,dx$.

1 E Finding Areas by Integration

The aim of this section and the next is to use definite integrals to find the areas of regions bounded by curves, lines and the coordinate axes.

Area and the Definite Integral: A definite integral is a pure number, which can be positive or negative — remember that a definite integral representing a region below the x-axis is negative in value. An area has units (called 'square units' or u^2 in the absence of any physical interpretation) and cannot be negative.

Any problem on areas requires some care when finding the correct integral or combination of integrals required. Some particular techniques are listed below, but the general rule is to draw a diagram first to see which bits need to be added or subtracted.

14 FINDING AN AREA: When using integrals to find the area of a region:

1. Draw a sketch of the curves, showing relevant intercepts and intersections.
2. Evaluate the necessary definite integral or definite integrals.
3. Write a conclusion, giving the required area in square units.

Areas Above the x-axis: When a region lies entirely above the x-axis, the relevant integral will be positive and the area will be equal to the integral, apart from needing units.

WORKED EXERCISE:

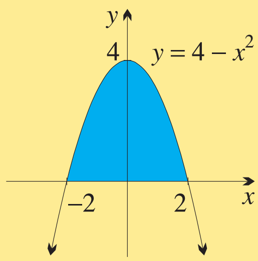
Find the area of the region bounded by the curve $y = 4 - x^2$ and the x-axis. (This was the example given on page 1 in the introduction to the chapter.)

SOLUTION:

The curve meets the x-axis at $(2, 0)$ and $(-2, 0)$.

The region lies entirely above the x-axis and the relevant integral is

$$\begin{aligned}\int_{-2}^{2} (4 - x^2)\,dx &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= (8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3}) \\ &= 5\tfrac{1}{3} - (-5\tfrac{1}{3}) \\ &= 10\tfrac{2}{3},\end{aligned}$$



which is positive because the region lies above the x-axis.

Hence the required area is $10\frac{2}{3}$ square units.

Areas Below the x-axis: When a region lies entirely below the x-axis, the relevant integral will be negative and the area will then be the opposite of this.

WORKED EXERCISE:

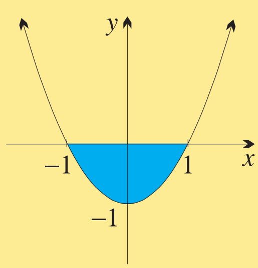
Find the area of the region bounded by the curve $y = x^2 - 1$ and the x-axis.

SOLUTION:

The curve meets the x-axis at $(1, 0)$ and $(-1, 0)$.

The region lies entirely below the x-axis and the relevant integral is

$$\begin{aligned}\int_{-1}^{1} (x^2 - 1)\,dx &= \left[\frac{x^3}{3} - x\right]_{-1}^{1} \\ &= (\tfrac{1}{3} - 1) - (-\tfrac{1}{3} + 1) \\ &= -\tfrac{2}{3} - \tfrac{2}{3} \\ &= -1\tfrac{1}{3},\end{aligned}$$



which is negative, because the region lies below the x-axis.

Hence the required area is $1\frac{1}{3}$ square units.

Areas Above and Below the x-axis: When a curve crosses the x-axis, the area of the region between the curve and the x-axis cannot usually be found by means of a single integral. This is because integrals representing regions below the x-axis have negative values.

WORKED EXERCISE:

(a) Sketch the cubic curve $y = x(x + 1)(x - 2)$, showing the x-intercepts.

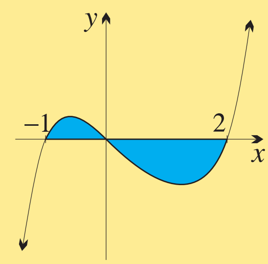
(b) Shade the region enclosed between the x-axis and the curve, and find its area.

[HINT: The expansion of the function is $\begin{aligned}y &= x(x+1)(x-2) \\ &= x(x^2 - x - 2) \\ &= x^3 - x^2 - 2x.\end{aligned}$]

(c) Find $\displaystyle\int_{-1}^{2} x(x + 1)(x - 2)\,dx$ and explain why this integral does not represent the area of the region described in part (b).

SOLUTION:

(a) The curve has x-intercepts $x = -1$, $x = 0$ and $x = 2$ and is graphed below.



(b) For the region above the x-axis,

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$
$$= (0 - 0 - 0) - (\tfrac{1}{4} + \tfrac{1}{3} - 1)$$
$$= \tfrac{5}{12},$$

and so area above $= \frac{5}{12}$ square units.

For the region below the x-axis,

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$
$$= (4 - 2\tfrac{2}{3} - 4) - (0 - 0 - 0)$$
$$= -2\tfrac{2}{3},$$

and so area below $= 2\frac{2}{3}$ square units.

Adding these, total area $= \frac{5}{12} + 2\frac{2}{3}$
$= 3\frac{1}{12}$ square units.

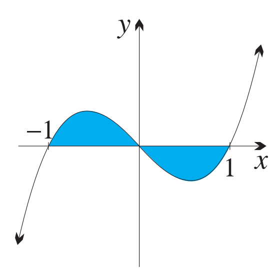
(c) $$\int_{-1}^{2} x(x+1)(x-2)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{2}$$
$$= (4 - 2\tfrac{2}{3} - 4) - (\tfrac{1}{4} + \tfrac{1}{3} - 1)$$
$$= -2\tfrac{2}{3} + \tfrac{5}{12}$$
$$= -2\tfrac{1}{4}.$$

This integral represents the area from $x = -1$ to $x = 0$ above the x-axis, *minus*, rather than plus, the area from $x = 0$ to $x = 2$ below the x-axis.

Areas Associated with Odd and Even Functions: As always, these calculations are often much easier if symmetries can be recognised.

WORKED EXERCISE:

Find the area between the curve $y = x^3 - x$ and the x-axis.



SOLUTION:

Factoring, $y = x(x^2 - 1)$
$= x(x-1)(x+1),$

and so the x-intercepts are $x = -1$, $x = 0$ and $x = 1$.

The two shaded regions have equal areas, since the function is odd.

First, $$\int_{0}^{1} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1}$$
$$= (\tfrac{1}{4} - \tfrac{1}{2}) - (0 - 0)$$
$$= -\tfrac{1}{4},$$

so area below the x-axis $= \frac{1}{4}$ square units.

Doubling, total area $= \frac{1}{2}$ square units.

Area Between a Graph and the y-axis: Integration with respect to y rather than x can often give a result more quickly without the need for subtraction.

When x is a function of y, the definite integral with respect to y represents the area of the region between the curve and the y-axis, except that areas of regions to the left of the y-axis are subtracted rather than added. The limits of integration are values of y rather than of x.

15 THE DEFINITE INTEGRAL AND INTEGRATION WITH RESPECT TO y:

Let x be a continuous function of y in some closed interval $a \le y \le b$.

Then the definite integral $\int_a^b x\,dy$ is the sum of the areas to the right of the y-axis, from $y = a$ to $y = b$, minus the sum of the areas to the left of the y-axis.

WORKED EXERCISE:

(a) Sketch the lines $y = x + 1$ and $y = 5$ and shade the region between these lines to the right of the y-axis.

(b) Use integration with respect to y to find the area of this region.

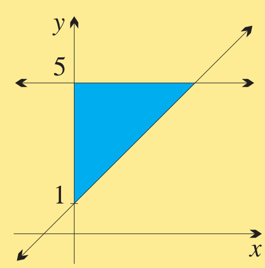
(c) Confirm the result by mensuration.

SOLUTION:

(a) The lines are sketched below. They meet at $(4, 5)$.

(b) The given equation is $y = x + 1$.
Solving for x, $x = y - 1$,
and the required integral is

$$\int_1^5 (y-1)\,dy = \left[\frac{y^2}{2} - y\right]_1^5$$
$$= \left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)$$
$$= 7\tfrac{1}{2} - (-\tfrac{1}{2})$$
$$= 8,$$



which is positive, since the region is to the right of the y-axis.

Hence the required area is 8 square units.

(c) By mensuration, area $= \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times 4 \times 4$
$= 8$ square units.

WORKED EXERCISE:

The curve in the diagram below is the cubic $y = x^3$. Use integration with respect to y to find:

(a) the areas of the shaded regions to the right and left of the y-axis,

(b) the total area of the two shaded regions.

SOLUTION:

(a) The given equation is $y = x^3$.

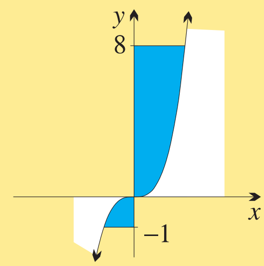
Solving for x, $x^3 = y$

$$x = y^{\frac{1}{3}}.$$

For the region to the right of the y-axis,

$$\begin{aligned}\int_0^8 y^{\frac{1}{3}}\,dy &= \tfrac{3}{4}\left[y^{\frac{4}{3}}\right]_0^8\\ &= \tfrac{3}{4}\times(16-0) \quad \text{(Note that } 8^{\frac{4}{3}} = 2^4 = 16.\text{)}\\ &= 12,\end{aligned}$$

so area $= 12$ square units.



For the region to the left of the y-axis,

$$\begin{aligned}\int_{-1}^0 y^{\frac{1}{3}}\,dy &= \tfrac{3}{4}\left[y^{\frac{4}{3}}\right]_{-1}^0\\ &= \tfrac{3}{4}\times(0-1) \quad \text{(Note that } (-1)^{\frac{4}{3}} = (-1)^4 = 1.\text{)}\\ &= -\tfrac{3}{4},\end{aligned}$$

so area $= \frac{3}{4}$ square units.

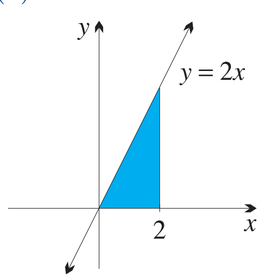
(b) Adding these, total area $= 12\frac{3}{4}$ square units.

Exercise 1E

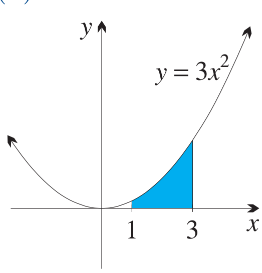
TECHNOLOGY: Any curve-sketching program will help in identifying the definite integrals that need to be evaluated to find the area of a given region. Programs that can approximate areas of regions on the screen graph can demonstrate how the final area is built up from the separate pieces.

1. Find the area of each shaded region below by evaluating the appropriate integral:

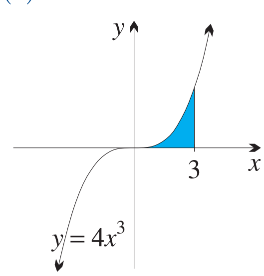
(a)



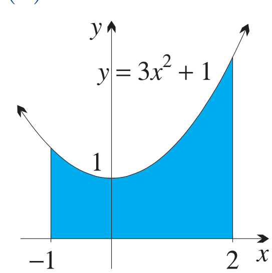
(b)



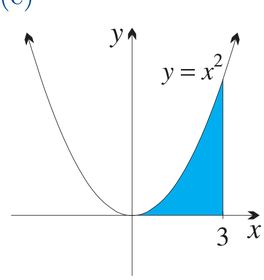
(c)



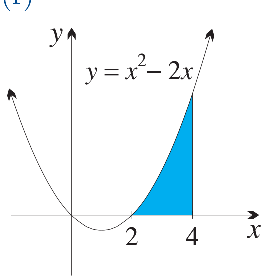
(d)



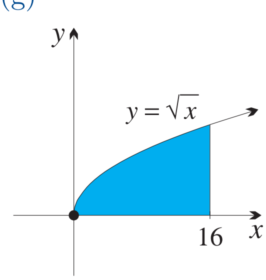
(e)



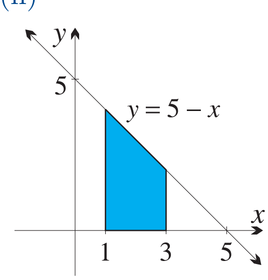
(f)

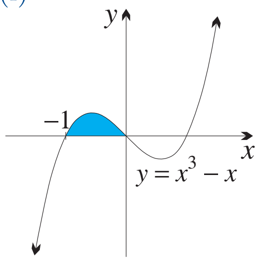


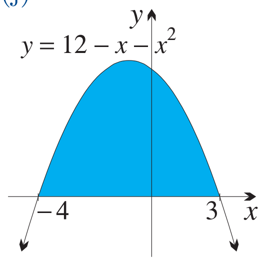
(g)

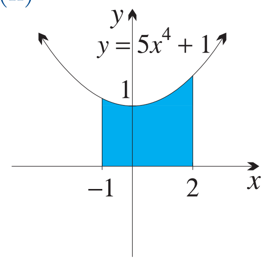


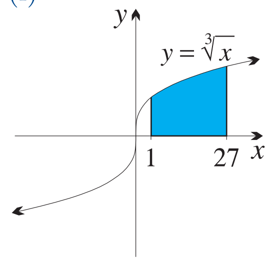
(h)



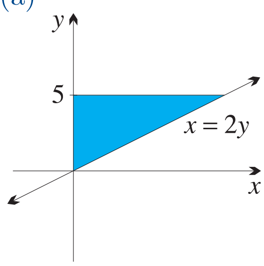
(i)


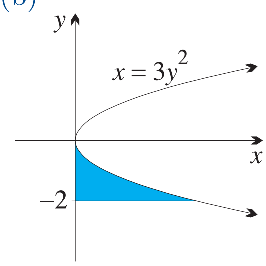
(j)


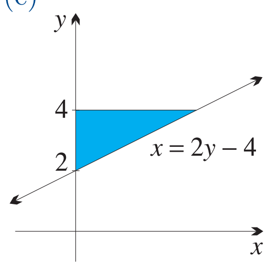
(k)


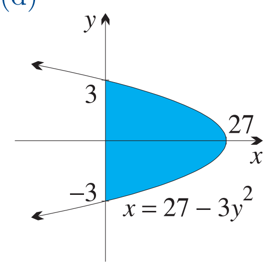
(l)


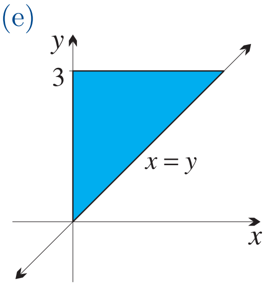
2. Find the area of each shaded region below by evaluating the appropriate integral:

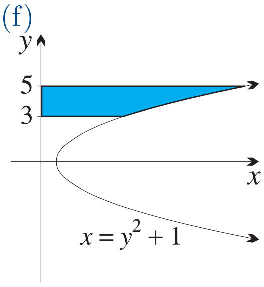
(a)


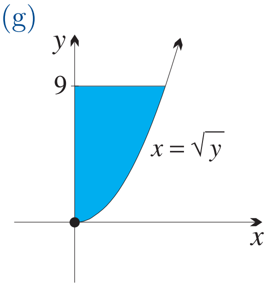
(b)


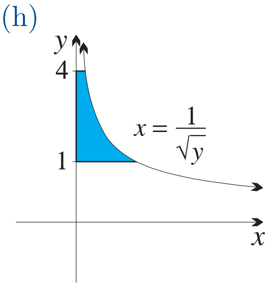
(c)


(d)


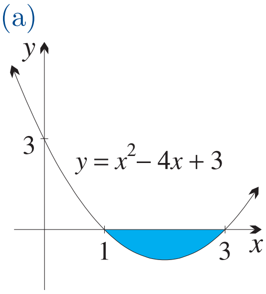
(e)


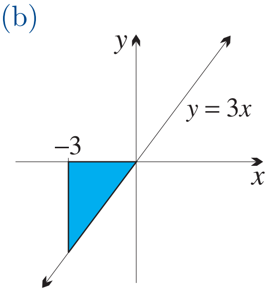
(f)


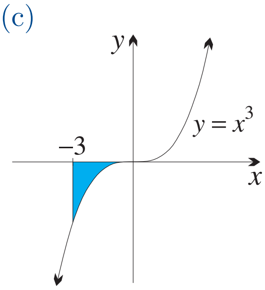
(g)


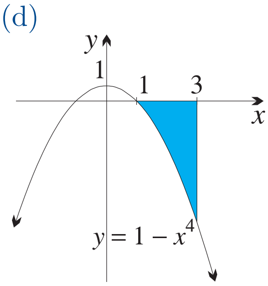
(h)


3. Find the area of each shaded region below by evaluating the appropriate integral:

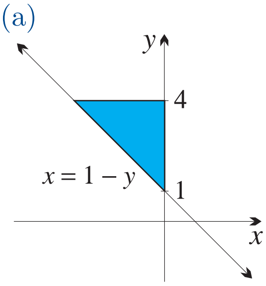
(a)


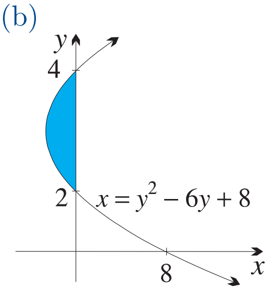
(b)


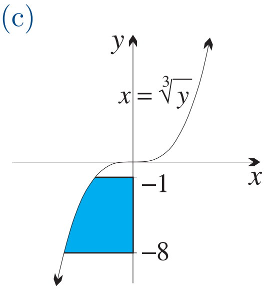
(c)


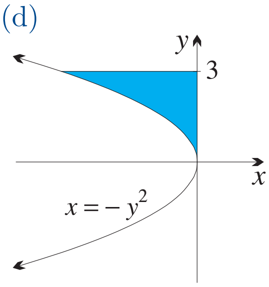
(d)


4. Find the area of each shaded region below by evaluating the appropriate integral:

(a)


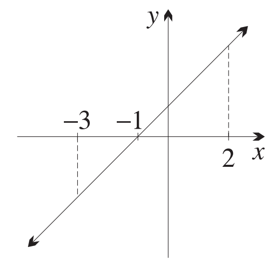
(b)


(c)


(d)


DEVELOPMENT

5. The sketch shows the line $y = x + 1$.



(a) Copy the diagram and then shade the region bounded by $y = x + 1$, the x-axis and the lines $x = -3$ and $x = 2$.

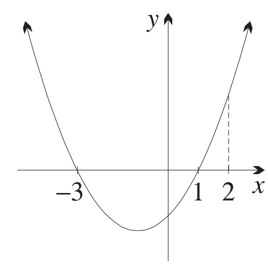
(b) By evaluating $\displaystyle\int_{-1}^{2} (x + 1)\, dx$, find the area of the shaded region above the x-axis.

(c) By evaluating $\displaystyle\int_{-3}^{-1} (x + 1)\, dx$, find the area of the shaded region below the x-axis.

(d) Hence find the area of the entire shaded region.

(e) Find $\displaystyle\int_{-3}^{2} (x + 1)\, dx$, and explain why this integral does not give the area of the shaded region.

6. The sketch shows the curve $y = (x - 1)(x + 3) = x^2 + 2x - 3$.



(a) Copy the diagram and shade the region bounded by the curve $y = (x - 1)(x + 3)$, the x-axis and the line $x = 2$.

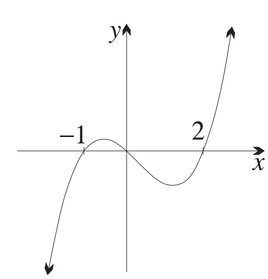
(b) By evaluating $\displaystyle\int_{-3}^{1} (x^2 + 2x - 3)\, dx$, find the area of the shaded region below the x-axis.

(c) By evaluating $\displaystyle\int_{1}^{2} (x^2 + 2x - 3)\, dx$, find the area of the shaded region above the x-axis.

(d) Hence find the area of the entire shaded region.

(e) Find $\displaystyle\int_{-3}^{2} (x^2 + 2x - 3)\, dx$, and explain why this integral does not give the area of the shaded region.

7. The sketch shows the curve $y = x(x + 1)(x - 2) = x^3 - x^2 - 2x$.



(a) Copy the diagram and shade the region bounded by the curve and the x-axis.

(b) By evaluating $\displaystyle\int_{0}^{2} (x^3 - x^2 - 2x)\, dx$, find the area of the shaded region below the x-axis.

(c) By evaluating $\displaystyle\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx$, find the area of the shaded region above the x-axis.

(d) Hence find the area of the entire region you have shaded.

(e) Find $\displaystyle\int_{-1}^{2} (x^3 - x^2 - 2x)\, dx$, and explain why this integral does not give the area of the shaded region.

8. In each part below, find the area of the region bounded by the graph of the given function and the x-axis between the specified values. You should draw a diagram for each part and check to see whether the region is above or below the x-axis.

(a) $y = x^2$, between $x = -3$ and $x = 2$

(b) $y = 2x^3$, between $x = -4$ and $x = 1$

(c) $y = 3x(x-2)$, between $x = 0$ and $x = 2$
(d) $y = x - 3$, between $x = -1$ and $x = 4$
(e) $y = (x-1)(x+3)(x-2)$, between $x = -3$ and $x = 2$
(f) $y = -2x(x+1)$, between $x = -2$ and $x = 2$
(g) $y = x(3-x)^2$, between $x = 0$ and $x = 3$
(h) $y = x^4 - 4x^2$, between $x = -5$ and $x = 0$

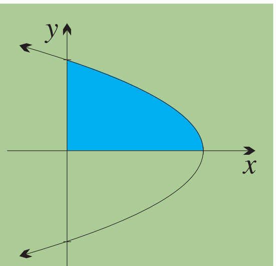
9. In each part below, find the area of the region bounded by the graph of the given function and the y-axis between the specified values. You should draw a diagram for each part and check whether the region is to the right or left of the y-axis.
(a) $x = y - 5$, between $y = 0$ and $y = 6$
(b) $x = 3 - y$, between $y = 2$ and $y = 5$
(c) $x = y^2$, between $y = -1$ and $y = 3$
(d) $x = (y-1)(y+1)$, between $y = 3$ and $y = 0$

10. In each part below you should draw a graph and look carefully for any symmetries that will simplify the calculation.
(a) Find the area of the region bounded by the curve and the x-axis:
(i) $y = x^7$, for $-2 \le x \le 2$
(ii) $y = x^3 - 16x = x(x-4)(x+4)$, for $-4 \le x \le 4$
(iii) $y = x^4 - 9x^2 = x^2(x-3)(x+3)$, for $-3 \le x \le 3$
(b) Find the area of the region bounded by the curve and the y-axis:
(i) $x = 2y$, for $-5 \le y \le 5$
(ii) $x = y^2$, for $-3 \le y \le 3$
(iii) $x = 4 - y^2 = (2-y)(2+y)$, for $-2 \le y \le 2$

11. Find the area of the region bounded by $y = |x+2|$ and the x-axis, for $-2 \le x \le 2$.

CHALLENGE

12. The diagram shows a graph of $y^2 = 16(2-x)$.
(a) Find the x-intercept and the y-intercepts.
(b) Find the area of the shaded region:
(i) by considering the region between the curve $y = 4\sqrt{2-x}$ and the x-axis,
(ii) by considering the region between the curve $x = 2 - \frac{1}{16}y^2$ and the y-axis.



13. The gradient of a curve is $y' = x^2 - 4x + 3$ and the curve passes through the origin.
(a) Find the equation of the curve.
(b) Show that the curve's turning points are $(1, 1\frac{1}{3})$ and $(3, 0)$, and sketch its graph.
(c) Find the area of the region enclosed between the curve and the x-axis between the two turning points.

14. Sketch $y = x^2$ and mark the points $A(a, a^2)$, $B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.
(a) Show that $\int_0^a x^2\,dx = \frac{2}{3}(\text{area}\,\Delta OAP)$.
(b) Show that $\int_{-a}^a x^2\,dx = \frac{1}{3}(\text{area of rectangle } ABQP)$.

1 F Areas of Compound Regions

When a region is bounded by two or more different curves, some dissection process is usually needed before integrals can be used to calculate its area.

Thus a preliminary sketch of the region becomes all the more important.

Areas of Regions Under a Combination of Curves: Some regions are bounded by different curves in different parts of the x-axis.

WORKED EXERCISE:

(a) Sketch the curves $y = x^2$ and $y = (x-2)^2$ on one set of axes.

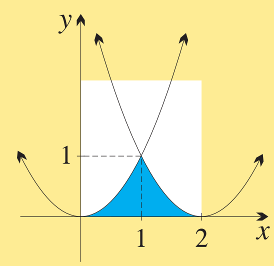
(b) Shade the region bounded by $y = x^2$, $y = (x-2)^2$ and the x-axis.

(c) Find the area of this shaded region.

SOLUTION:

(a) The two curves intersect at $(1, 1)$, because it can be checked by substitution that this point lies on both curves.

(b) The whole region is above the x-axis, but it will be necessary to find separately the areas of the regions to the left and right of $x = 1$.



(c) First,
$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \tfrac{1}{3}.$$

Secondly,
$$\int_1^2 (x-2)^2\,dx = \left[\frac{(x-2)^3}{3}\right]_1^2 = 0 - (-\tfrac{1}{3}) = \tfrac{1}{3}.$$

Combining these,
$$\text{area} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3} \text{ square units.}$$

Areas of Regions Between Curves: Suppose that one curve $y = f(x)$ is always below another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area of the region between the curves from $x = a$ to $x = b$ can be found by subtraction.

16 AREA BETWEEN CURVES: If $f(x) \le g(x)$ in the interval $a \le x \le b$, then

$$\text{area between the curves} = \int_a^b \Big(g(x) - f(x)\Big)\,dx.$$

That is, take the integral of the top curve minus the bottom curve.

The assumption that $f(x) \le g(x)$ is important. If the curves cross each other, then separate integrals will need to be taken or else the areas of regions where different curves are on top will begin to cancel each other out.

WORKED EXERCISE:

(a) Find the two points where the curve $y = (x-2)^2$ meets the line $y = x$.

(b) Draw a sketch and shade the area of the region between these two graphs.

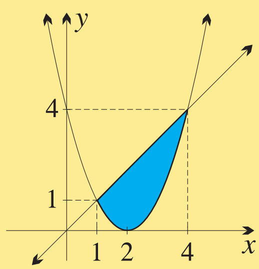
(c) Find the shaded area.

SOLUTION:

(a) Substituting $y = x$ into $y = (x-2)^2$ gives

$$\begin{aligned}(x-2)^2 &= x\\ x^2 - 4x + 4 &= x\\ x^2 - 5x + 4 &= 0\\ (x-1)(x-4) &= 0,\\ x &= 1 \text{ or } 4,\end{aligned}$$

so the two graphs intersect at $(1,1)$ and $(4,4)$.



(b) The sketch is drawn to the right.

(c) In the shaded region, the line is above the parabola.

$$\begin{aligned}\text{Hence area} &= \int_1^4 \Big(x - (x-2)^2\Big)\,dx\\ &= \int_1^4 \Big(x - (x^2 - 4x + 4)\Big)\,dx\\ &= \int_1^4 (-x^2 + 5x - 4)\,dx\\ &= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4\\ &= (-21\tfrac{1}{3} + 40 - 16) - (-\tfrac{1}{3} + 2\tfrac{1}{2} - 4)\\ &= 2\tfrac{2}{3} + 1\tfrac{5}{6}\\ &= 4\tfrac{1}{2} \text{ square units.}\end{aligned}$$

NOTE: The formula (given in Box 16 on the previous page) for the area of the region between two curves holds even if the region crosses the x-axis.

To illustrate this point, the next example is the previous example shifted down 2 units so that the region between the line and the parabola crosses the x-axis. The area of course remains the same — and notice how the formula still gives the correct answer.

WORKED EXERCISE:

(a) Find the two points where the curves $y = x^2 - 4x + 2$ and $y = x - 2$ meet.

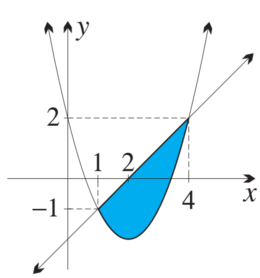
(b) Draw a sketch and find the area of the region between these two curves.

SOLUTION:

(a) Substituting $y = x - 2$ into $y = x^2 - 4x + 2$ gives

$$\begin{aligned}x^2 - 4x + 2 &= x - 2\\ x^2 - 5x + 4 &= 0\\ (x-1)(x-4) &= 0\\ x &= 1 \text{ or } 4.\end{aligned}$$

so the two graphs intersect at $(1,-1)$ and $(4,2)$.



(b) Again, the line is above the parabola.

$$\begin{aligned}\text{Hence area} &= \int_1^4 \Big((x-2)-(x^2-4x+2)\Big)\,dx \\ &= \int_1^4 (-x^2+5x-4)\,dx \\ &= \Big[-\frac{x^3}{3}+\frac{5x^2}{2}-4x\Big]_1^4 \\ &= (-21\tfrac{1}{3}+40-16)-(-\tfrac{1}{3}+2\tfrac{1}{2}-4) \\ &= 4\tfrac{1}{2} \text{ square units.}\end{aligned}$$

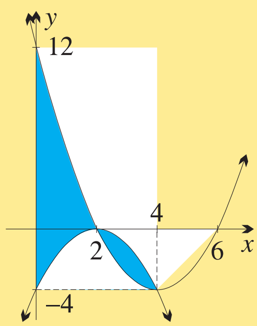
Areas of Regions Between Curves that Cross: Now suppose that one curve $y = f(x)$ is sometimes above and sometimes below another curve $y = g(x)$ in the relevant interval. In this case, separate integrals will need to be calculated.

WORKED EXERCISE:
The diagram below shows the curves

$$y = -x^2 + 4x - 4 \qquad \text{and} \qquad y = x^2 - 8x + 12$$

meeting at the points $(2, 0)$ and $(4, -4)$. Find the area of the shaded region.

SOLUTION:
In the left-hand region, the second curve is above the first.



$$\begin{aligned}\text{Hence}\qquad \text{area} &= \int_0^2 \Big((x^2-8x+12)-(-x^2+4x-4)\Big)\,dx \\ &= \int_0^2 (2x^2-12x+16)\,dx \\ &= \Big[\frac{2x^3}{3}-6x^2+16x\Big]_0^2 \\ &= 5\tfrac{1}{3}-24+32 \\ &= 13\tfrac{1}{3} \text{ square units.}\end{aligned}$$

In the right-hand region, the first curve is above the second.

$$\begin{aligned}\text{Hence}\qquad \text{area} &= \int_2^4 \Big((-x^2+4x-4)-(x^2-8x+12)\Big)\,dx \\ &= \int_2^4 (-2x^2+12x-16)\,dx \\ &= \Big[-\frac{2x^3}{3}+6x^2-16x\Big]_2^4 \\ &= (-42\tfrac{2}{3}+96-64)-(-5\tfrac{1}{3}+24-32) \\ &= -10\tfrac{2}{3}+13\tfrac{1}{3} \\ &= 2\tfrac{2}{3} \text{ square units.}\end{aligned}$$

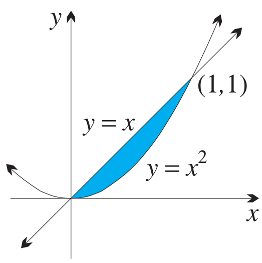
$$\begin{aligned}\text{Hence total area} &= 13\tfrac{1}{3}+2\tfrac{2}{3} \\ &= 16 \text{ square units.}\end{aligned}$$

Exercise 1F

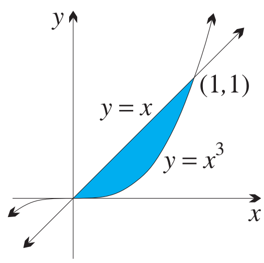
TECHNOLOGY: Screen graphing programs are particularly useful with compound regions because they allow the separate parts of the region to be identified clearly.

1. Find the area of the shaded region in each diagram below.

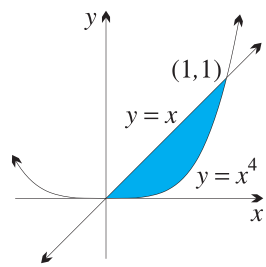
(a)



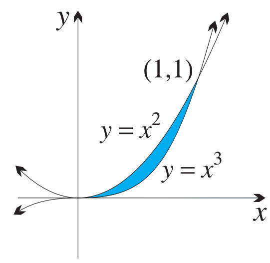
(b)



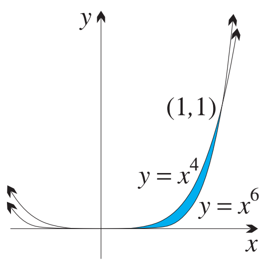
(c)



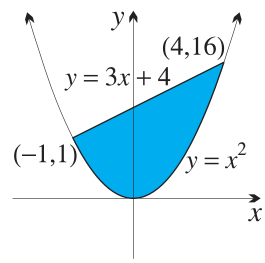
(d)



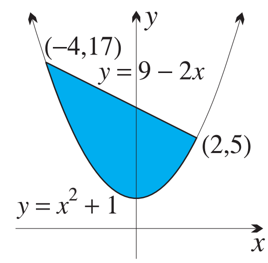
(e)



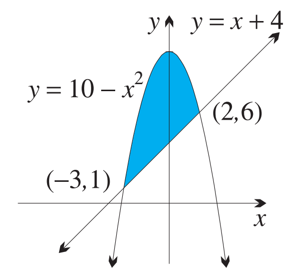
(f)



(g)

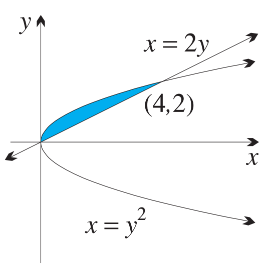


(h)

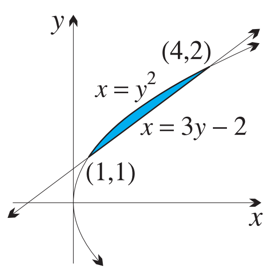


2. By considering regions between the curves and the y-axis, find the area of the shaded region in each diagram below.

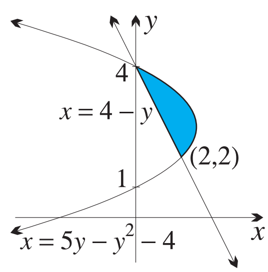
(a)



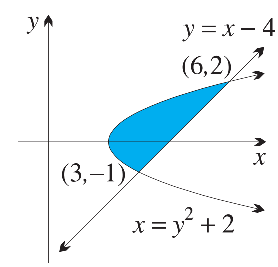
(b)



(c)

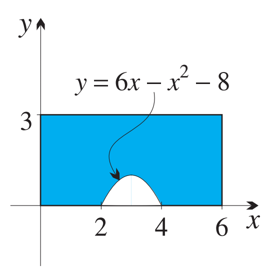


(d)

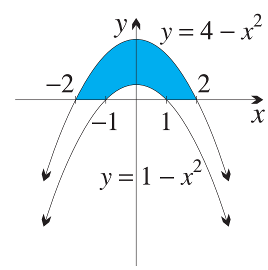


3. Find the areas of the shaded region in the diagrams below. In each case you will need to find two areas and subtract one from the other.

(a)

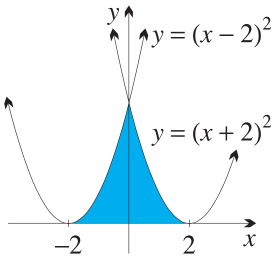


(b)

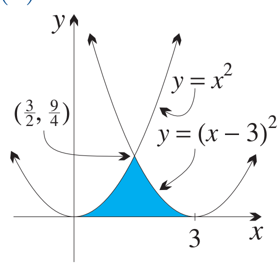


4. Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and add them.

(a)



(b)



5. (a) By solving the equations simultaneously, show that the curves $y = x^2 + 4$ and $y = x + 6$ intersect at the points $(-1, 5)$ and $(2, 8)$.

(b) Sketch the curves on the same diagram and shade the region enclosed between them.

(c) Show that this region has area

$$\int_{-1}^{2} \big((x+6) - (x^2+4)\big)\, dx = \int_{-1}^{2} (x - x^2 + 2)\, dx$$

and evaluate the integral.

6. (a) By solving the equations simultaneously, show that the curves $y = 3x - x^2 = x(3-x)$ and $y = x$ intersect at the points $(0, 0)$ and $(2, 2)$.

(b) Sketch the curves on the same diagram and shade the region enclosed between them.

(c) Show that this region has area

$$\int_{0}^{2} (3x - x^2 - x)\, dx = \int_{0}^{2} (2x - x^2)\, dx$$

and evaluate the integral.

7. (a) By solving the equations simultaneously, show that the curves $y = (x-3)^2$ and $y = 14 - 2x$ intersect at the points $(-1, 16)$ and $(5, 4)$.

(b) Sketch the curves on the same diagram and shade the region enclosed between them.

(c) Show that this region has area

$$\int_{-1}^{5} \big((14 - 2x) - (x-3)^2\big)\, dx = \int_{-1}^{5} (4x + 5 - x^2)\, dx$$

and evaluate the integral.

DEVELOPMENT

8. Solve simultaneously the equations of each pair of curves below to find their points of intersection. Sketch each pair of curves on the same diagram and shade the region enclosed between them. By evaluating the appropriate integral, find the area of the shaded region in each case.

(a) $y = x^4$ and $y = x^2$

(b) $y = 3x^2$ and $y = 6x^3$

(c) $y = 9 - x^2$ and $y = 3 - x$

(d) $y = x + 10$ and $y = (x-3)^2 + 1$

9. (a) By solving the equations simultaneously, show that the curves $y = x^2 + 2x - 8$ and $y = 2x + 1$ intersect at the points $(3, 7)$ and $(-3, -5)$.

(b) Sketch both curves on the same diagram and shade the region enclosed between them.

(c) Despite the fact that it crosses the x-axis, the region has area given by

$$\int_{-3}^{3} \Big((2x+1) - (x^2+2x-8)\Big)\,dx = \int_{-3}^{3} (9-x^2)\,dx.$$

Evaluate the integral and hence find the area of the region enclosed between the curves.

10. (a) By solving the equations simultaneously, show that the curves $y = x^2 - x - 2$ and $y = x - 2$ intersect at the points $(0, -2)$ and $(2, 0)$.

(b) Sketch both curves on the same diagram and shade the region enclosed between them.

(c) Despite the fact that it is below the x-axis, the region has area given by

$$\int_{0}^{2} \big((x-2) - (x^2-x-2)\big)\,dx = \int_{0}^{2} (2x-x^2)\,dx.$$

Evaluate this integral and hence find the area of the region between the curves.

11. Solve simultaneously the equations of each pair of curves below to find their points of intersection. Sketch each pair of curves on the same diagram and shade the region enclosed between them. By evaluating the appropriate integral, find the area of the shaded region in each case.

(a) $y = x^2 - 6x + 5$ and $y = x - 5$

(b) $y = -3x$ and $y = 4 - x^2$

(c) $y = x^2 - 1$ and $y = 7 - x^2$

(d) $y = x$ and $y = x^3$

12. Find the area bounded by the lines $y = \frac{1}{4}x$ and $y = -\frac{1}{2}x$ between $x = 1$ and $x = 4$.

13. (a) On the same number plane, sketch the graphs of the functions $y = x^2$ and $x = y^2$, clearly indicating their points of intersection. Shade the region enclosed between them.

(b) Explain why the area of this region is given by $\displaystyle\int_0^1 \left(\sqrt{x} - x^2\right)\,dx$.

(c) Find the area of the region bounded by the two curves.

CHALLENGE

14. Consider the function $x^2 = 8y$. Tangents are drawn at the points $A(4, 2)$ and $B(-4, 2)$ and intersect on the y-axis.

(a) Draw a diagram of the situation and note the symmetry about the y-axis.

(b) Find the equation of the tangent at the point A.

(c) Find the area of the region bounded by the curve and the tangents.

15. (a) Show that the tangent to $y = x^3$ at the point where $x = 2$ is $y - 12x + 16 = 0$.

(b) Show, by substituting the point into each equation, that the tangent and the curve meet again at the point $(-4, -64)$.

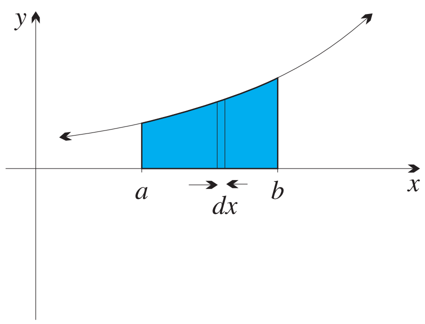
(c) Find the area of the region enclosed between the curve and the tangent.

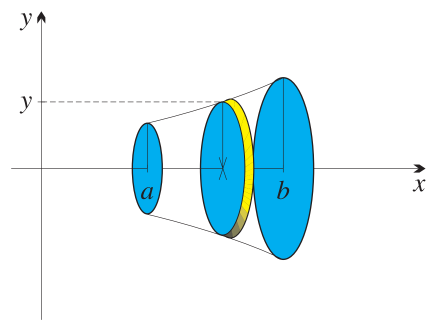
1 G Volumes of Solids of Revolution

If a region is rotated about either the x-axis or the y-axis, a solid region in three dimensions is generated, called the *solid of revolution*. The process is similar to shaping a piece of wood on a lathe, or making pottery on a wheel, because such shapes have rotational symmetry and circular cross sections.

The volumes of such solids can be found using a simple integration formula. The well-known formulae for the volumes of cones and spheres can finally be proven by this method.

Rotating a Region about the x-axis: The first diagram below shows the region under the curve $y = f(x)$ in the interval $a \le x \le b$, and the second shows the solid generated when this region is rotated about the x-axis.





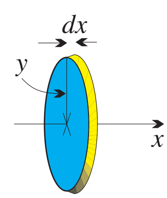
Imagine the solid sliced like salami perpendicular to the x-axis into infinitely many circular slices, each of width dx. One of the slices is shown below and the vertical strip on the first diagram above is what generates this slice when it is rotated about the x-axis.

The radius of the circular slice is the height of the strip,

so radius of circular slice $= y$.

Hence, using the formula for the area of a circle,

area of circular slice $= \pi y^2$.



But the slice is essentially a very thin cylinder of thickness dx,

so volume of circular slice $=$ area $\times$ thickness

$= \pi y^2\,dx$.

To get the total volume, simply sum all the slices from $x = a$ to $x = b$,

so $$\text{volume of solid} = \int_a^b \pi y^2\,dx.$$

17 VOLUMES OF REVOLUTION ABOUT THE x- AXIS: When the region between a curve and the x-axis, from $x = a$ to $x = b$, is rotated about the x-axis,

$$\text{volume of revolution} = \int_a^b \pi y^2\,dx \ \text{ cubic units.}$$

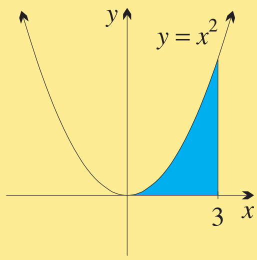
If the curve is below the x-axis, y is negative. But y^2 is positive, since it is a square, so the volume as given by the integral is still positive.

Unless other units are specified, 'cubic units', often abbreviated to u^3, are used.

WORKED EXERCISE:
Find the volume of the solid generated when the region between the curve $y = x^2$ and the x-axis, from $x = 0$ to $x = 3$, is rotated about the x-axis.

SOLUTION:
Squaring the function $y = x^2$ gives $y^2 = x^4$.

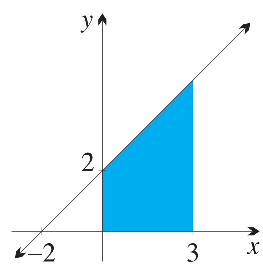
$$\begin{aligned}\text{Hence volume} &= \int_0^3 \pi y^2\,dx\\ &= \pi\int_0^3 x^4\,dx\\ &= \pi\left[\frac{x^5}{5}\right]_0^3\\ &= \pi\times\left(\frac{243}{5}-0\right)\\ &= \tfrac{243\pi}{5}\text{ cubic units.}\end{aligned}$$



WORKED EXERCISE:
Find the volume of the solid generated when the region between $y = 2 + x$ and the x-axis, from $x = 0$ to $x = 3$, is rotated about the x-axis.

SOLUTION:
Squaring, $y^2 = (2 + x)^2 = 4 + 4x + x^2$.

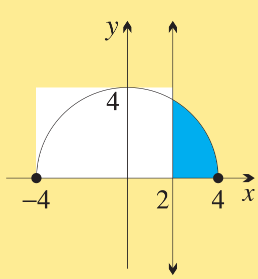
$$\begin{aligned}\text{Hence volume} &= \int_0^3 \pi y^2\,dx\\ &= \pi\int_0^3 (4+4x+x^2)\,dx\\ &= \pi\left[4x+2x^2+\frac{x^3}{3}\right]_0^3\\ &= \pi(12+18+9)-\pi(0+0+0)\\ &= 39\pi\text{ cubic units.}\end{aligned}$$



WORKED EXERCISE:
The shaded region cut off the semicircle $y = \sqrt{16 - x^2}$ by the line $x = 2$ is rotated about the x-axis. Find the volume of the solid generated generated.

SOLUTION: Squaring, $y^2 = 16 - x^2$.

$$\begin{aligned}\text{Hence}\qquad \text{volume} &= \int_2^4 \pi y^2\,dx\\ &= \pi\int_2^4 (16-x^2)\,dx\\ &= \pi\left[16x-\frac{x^3}{3}\right]_2^4\\ &= \pi(64-\tfrac{64}{3}-32+\tfrac{8}{3})\\ &= \pi(32-\tfrac{56}{3})\\ &= \tfrac{40\pi}{3}\text{ cubic units.}\end{aligned}$$



Volumes of Revolution about the y-axis: Calculating the volume of the solid generated when a region is rotated about the y-axis is simply a matter of exchanging x and y:

18 **VOLUMES OF REVOLUTION ABOUT THE y- AXIS:** When the region between a curve and the y-axis, from $y = a$ to $y = b$, is rotated about the y-axis,

$$\text{volume of revolution} = \int_a^b \pi x^2 \, dy \ \text{ cubic units.}$$

When y is given as a function of x, the equation will need to be written with x^2 as the subject.

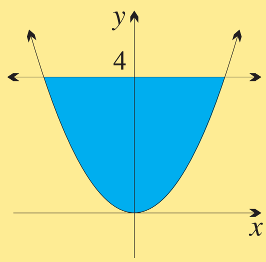
WORKED EXERCISE:

Find the volume of the solid formed by rotating the region between $y = x^2$ and the line $y = 4$ about the y-axis.

SOLUTION:

Rewriting $y = x^2$ with x^2 as the subject gives $x^2 = y$.

$$\begin{aligned}\text{Hence volume} &= \int_0^4 \pi x^2 \, dy \\ &= \pi \int_0^4 y \, dy \\ &= \pi \left[\frac{y^2}{2} \right]_0^4 \\ &= \pi \left(\frac{16}{2} - 0 \right) \\ &= 8\pi \text{ cubic units.}\end{aligned}$$



Finding Volumes by Subtraction: When rotating the region between two curves lying above the x-axis, the two integrals need to be subtracted, as if the outer volume has first been formed and the inner volume cut away from it.

The two volumes can always be calculated separately and subtracted. If the two integrals have the same limits of integration, however, it is usually more convenient to combine them:

19 **ROTATING THE REGION BETWEEN CURVES:** When the region between two curves, from $x = a$ to $x = b$, is rotated about the x-axis,

$$\text{volume of revolution} = \int_a^b \pi({y_2}^2 - {y_1}^2) \, dx, \quad \text{provided that } y_2 > y_1 > 0.$$

Similarly, when the region between two curves, from $y = a$ to $y = b$, is rotated about the y-axis,

$$\text{volume of revolution} = \int_a^b \pi({x_2}^2 - {x_1}^2) \, dy, \quad \text{provided that } x_2 > x_1 > 0.$$

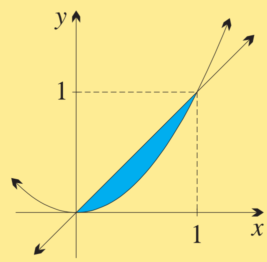
WORKED EXERCISE:

The parabola $y = x^2$ meets the line $y = x$ at $O(0, 0)$ and $A(1, 1)$. Sketch the diagram and then find the volume of the solid generated when the region between the curve and the line is rotated:

(a) about the x-axis, (b) about the y-axis.

SOLUTION:

(a) From $x = 0$ to $x = 1$, the line $y = x$ is above the parabola $y = x^2$, so take $y_2 = x$ and $y_1 = x^2$.



$$\begin{aligned}\text{Thus volume} &= \int_0^1 \pi({y_2}^2 - {y_1}^2)\,dx \\ &= \pi \int_0^1 (x^2 - x^4)\,dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 \\ &= \pi\left(\frac{1}{3} - \frac{1}{5}\right) - \pi(0 - 0) \\ &= \tfrac{2\pi}{15} \text{ cubic units.}\end{aligned}$$

(b) From $y = 0$ to $y = 1$, the parabola $x^2 = y$ is to the right of the line $x = y$, so take ${x_2}^2 = y$ and $x_1 = y$.

$$\begin{aligned}\text{Hence volume} &= \int_0^1 \pi({x_2}^2 - {x_1}^2)\,dy \\ &= \pi \int_0^1 (y - y^2) \\ &= \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 \\ &= \pi\left(\frac{1}{2} - \frac{1}{3}\right) - \pi(0 - 0) \\ &= \tfrac{\pi}{6} \text{ cubic units.}\end{aligned}$$

NOTE: The volumes of revolution about the x-axis and the y-axis are usually quite different. This is because an element of area will generate a larger element of volume if it is moved further away from the axis of rotation. In the example above, the average distance of the region from the y-axis is greater than the average distance from the x-axis, so the second answer is slightly larger.

Cones and Spheres: The formulae for the volumes of cones and spheres were learnt earlier, but proving them requires integration. The proofs of both results are developed in the following exercise and these questions should be carefully worked.

20

VOLUME OF A CONE: $V = \frac{1}{3}\pi r^2 h$,
where r is the radius of the cone and h is its perpendicular height.

VOLUME OF A SPHERE: $V = \frac{4}{3}\pi r^3$,
where r is the radius of the sphere.

Exercise 1G

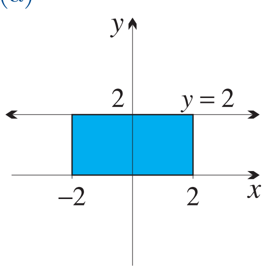
TECHNOLOGY: Relevant technologies here are lathes, which can shape a piece of wood so that all its cross-sections are circular, and programs that rotate regions on the screen. These can be better than hand-drawn diagrams at conveying the three-dimensional understanding of solids of revolution. Programs that will show the solid being sliced into thin strips would be helpful in explaining the formula for the volume of the solid of revolution.

1. Find y^2 in terms of x:

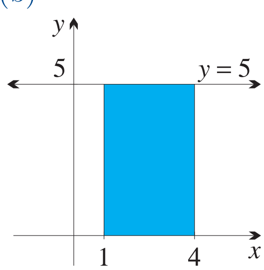
(a) $y = x$
(b) $y = x^2$
(c) $y = 3x^2$
(d) $y = 2x^5$
(e) $y = x - 1$
(f) $y = x + 5$
(g) $y = 2x - 3$
(h) $y = \sqrt{x}$
(i) $y = \sqrt{x - 4}$

2. Calculate the volume of the solid generated when each shaded region is rotated about the x-axis.

(a)



(b)



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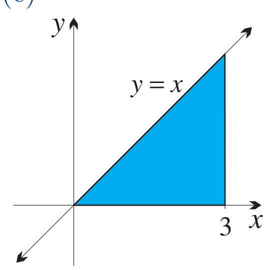

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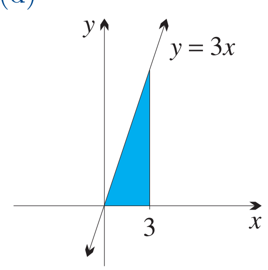

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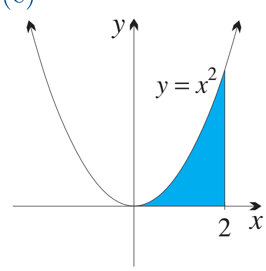
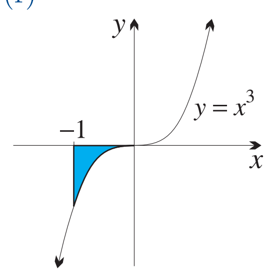
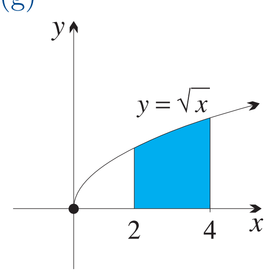


(f)



(g)



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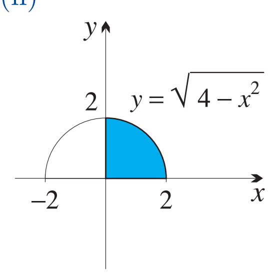
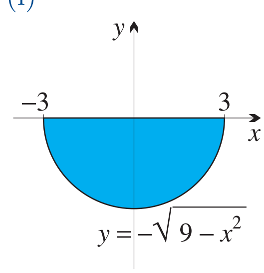


(i)



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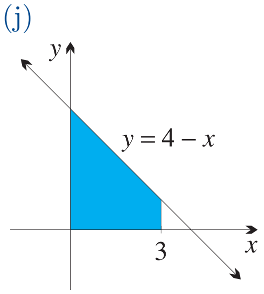
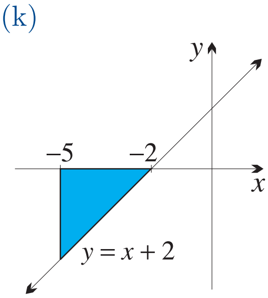
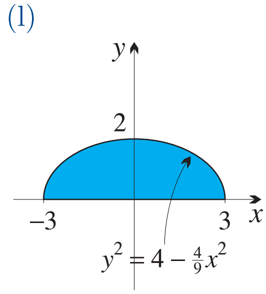


(k)

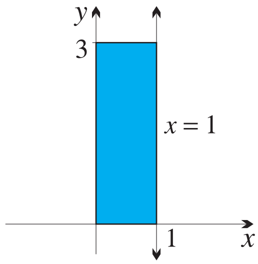


(l)

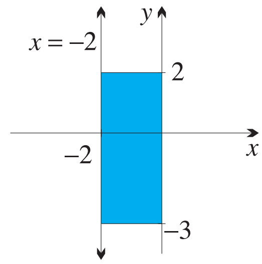


3. Calculate the volume of the solid generated when each shaded region is rotated about the y-axis.

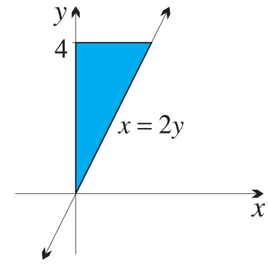
(a)



(b)



(c)



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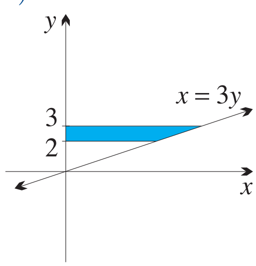

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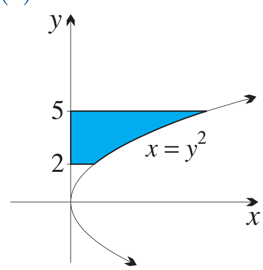
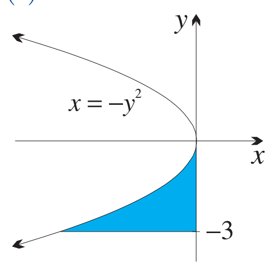
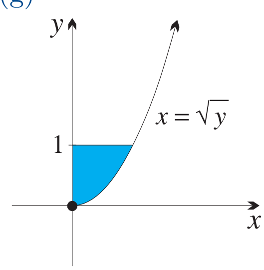


(f)



(g)



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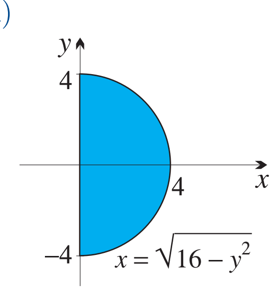
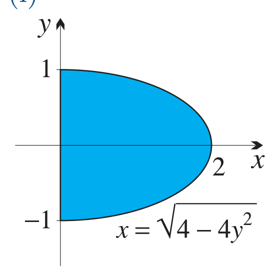
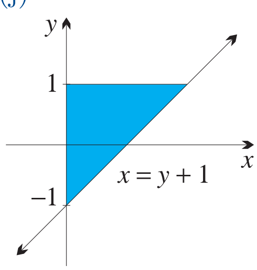


(i)



(j)



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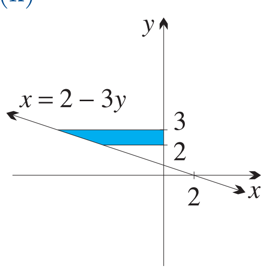

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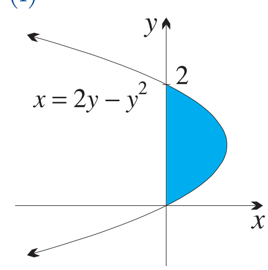


4. (a) Sketch the region bounded by the line $y = 3x$ and the x-axis, between $x = 0$ and $x = 3$.

(b) When this region is rotated about the x-axis, a right circular cone will be formed. Find the radius and height of the cone and hence find its volume.

(c) Evaluate $\displaystyle\int_0^3 \pi y^2\,dx = \pi\int_0^3 9x^2\,dx$ in order to check your answer.

5. (a) Sketch a graph of the region bounded by the curve $y = \sqrt{9 - x^2}$ and the x-axis, between $x = -3$ and $x = 3$.

(b) When this region is rotated about the x-axis, a sphere will be formed. Find the radius of the sphere and hence find its volume.

(c) Evaluate $\displaystyle\int_{-3}^3 \pi y^2\,dx = \pi\int_{-3}^3 (9 - x^2)\,dx$ in order to check your answer.

DEVELOPMENT

6. In each part, sketch the region bounded by the given curves. Find the volume of the solid generated by rotating this region about the x-axis.

(a) $y = 3x$, $x = 0$, $x = 4$ and $y = 0$
(b) $y = x^2$, $x = 1$, $x = 3$ and $y = 0$
(c) $y = \sqrt{x}$, $x = 10$, $x = 15$ and $y = 0$
(d) $y = 7x^3$, $x = 0$, $x = 2$ and $y = 0$

7. In each part, sketch the region bounded by the given curves. Find the volume of the solid generated by rotating this region about the y-axis.

(a) $x = y$, $y = 1$, $y = 3$ and $x = 0$
(b) $x = y^2$, $y = 2$, $y = 3$ and $x = 0$
(c) $x = \sqrt{y}$, $y = 0$, $y = 5$ and $x = 0$
(d) $x = -3y$, $y = 2$, $y = 6$ and $x = 0$

8. In each part, sketch the region bounded by the given curves. Find the volume of the solid generated by rotating this region about the x-axis. When finding y^2, you will need to recall that $(a + b)^2 = a^2 + 2ab + b^2$.

(a) $y = x + 3$, $x = 3$, $x = 5$ and $y = 0$
(b) $y = 2x + 1$, $x = 0$, $x = 1$ and $y = 0$
(c) $y = 5x - x^2 = x(5 - x)$ and $y = 0$
(d) $y = x^3 - x = x(x - 1)(x + 1)$ and $y = 0$

9. In each part, sketch the region bounded by the given curves. Find the volume of the solid generated by rotating this region about the y-axis. Take care when finding x^2.

(a) $x = y - 2$, $y = 1$ and $x = 0$
(b) $x = y^2 + 1$, $y = 0$, $y = 1$ and $x = 0$
(c) $x = y^2 - 3y = y(y - 3)$ and $x = 0$
(d) $y = 1 - x^2 = (1 - x)(1 + x)$ and $y = 0$

10. By evaluating the appropriate integral, find the volume of the sphere generated when the region inside the circle $x^2 + y^2 = 64$ is rotated about the x-axis.

11. A vase is formed by rotating about the y-axis the portion of the curve $x = y^2 + 6$ between $y = 6$ and $y = -6$. Find the volume of the vase.

12. (a) Sketch a graph of the region bounded by the curves $y = x^2$ and $y = x^3$, clearly showing their points of intersection at $(0, 0)$ and $(1, 1)$.

(b) Show that the volume of the solid generated when this region is rotated about the x-axis is given by $\pi \displaystyle\int_0^1 (x^4 - x^6)\, dx$.

(c) Evaluate the integral and hence find the volume of the solid.

13. (a) By solving their equations simultaneously, show that the curves $x + y = 6$ and $xy = 5$ intersect at $(1, 5)$ and $(5, 1)$.

(b) Sketch the curves on the same number plane and shade the region between them.

(c) Show that the volume of the solid generated when this region is rotated about the x-axis is given by $\pi \displaystyle\int_1^5 \left(36 - 12x + x^2 - \frac{25}{x^2}\right) dx$.

(d) Evaluate the integral and hence find the volume of the solid.

14. Sketch the curve $x^2 = 4ay$, and shade the region bounded by the curve and the latus rectum $y = a$. Find the volume of the solid generated when this region is rotated about the axis of symmetry of the parabola.

CHALLENGE

15. Three standard volume formulae will be proven in this question.

(a) A cone of height h and radius r is generated by rotating the line

$$y = \frac{r}{h}\,x$$

between $x = 0$ and $x = h$ about the x-axis. Show that the cone has volume $\frac{1}{3}\pi r^2 h$.

(b) A cylinder of height h and radius r is generated by rotating the line

$$y = r$$

between $x = 0$ and $x = h$ about the x-axis. Show that the cylinder has volume $\pi r^2 h$.

(c) A sphere of radius r is generated by rotating the semicircle

$$y = \sqrt{r^2 - x^2}$$

about the x-axis. Show that the volume of the sphere is given by $\frac{4}{3}\pi r^3$.

16. (a) State the domain and range of the function $y = \sqrt{9 - x}$.

(b) Sketch a graph of the function.

(c) Calculate the area of the region bounded by the curve and the coordinate axes in the first quadrant.

(d) Calculate the volume of the solid generated when this region is rotated:

(i) about the x-axis, (ii) about the y-axis.

17. (a) Sketch the region bounded by $y = x^2$ and the x-axis between $x = 0$ and $x = 4$.

(b) Find the volume of the solid generated when this region is rotated about the x-axis.

(c) Find the volume V_1 of the cylinder formed when the line $x = 4$ between $y = 0$ and $y = 16$ is rotated about the y-axis.

(d) Evaluate $V_2 = \displaystyle\int_0^{16} \pi x^2\,dy = \pi \int_0^{16} y\,dy$.

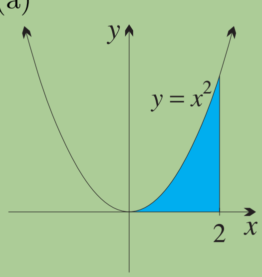
(e) Hence evaluate the volume $V = V_1 - V_2$ of the solid generated when the shaded region in part (a) is rotated about the y-axis.

18. Find the volume of the solid generated by rotating each region below:

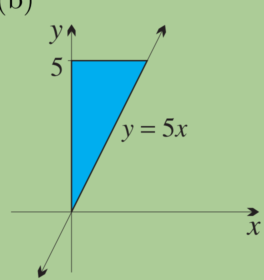
(i) about the x-axis, (ii) about the y-axis.

[HINT: In some cases a subtraction of volumes will be necessary.]

(a)



(b)



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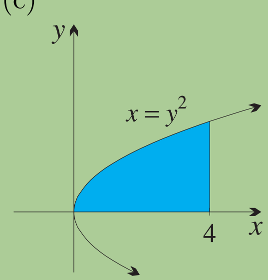

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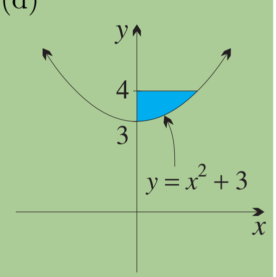


19. (a) On the same diagram, sketch the curves $y = x^2$ and $x = y^2$, clearly labelling the points of intersection.

(b) Find the volume of the solid formed when the region between the curves is rotated:

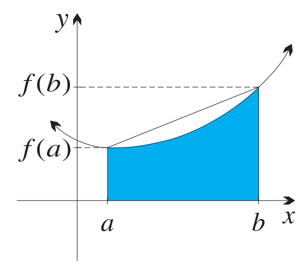
(i) about the x-axis, (ii) about the y-axis.

(c) Why are the answers identical?

1 H The Trapezoidal Rule

Methods for approximating definite integrals become necessary when exact calculation through the primitive is not possible. This can happen for two reasons. First, the primitives of many important functions cannot be written down in a form suitable for calculation. Secondly, some values of the function may be known from experiments, but the function itself may still be unknown.

The Trapezoidal Rule: The most obvious way to approximate an integral is to replace the curve by a straight line. The resulting region is then a trapezium and so the approximation method is called the *trapezoidal rule*.



Consider the trapezium in the diagram to the right.

Here $\quad\quad \text{width} = b - a,$

and $\quad \text{average of parallel sides} = \dfrac{f(a) + f(b)}{2}.$

Hence area of trapezium = width × average of parallel sides

$$= \frac{b-a}{2}\Big(f(a) + f(b)\Big).$$

Thus if the area of this trapezium is taken as an approximation to the integral:

21 TRAPEZOIDAL RULE: Given a function that is continuous in the interval $a \le x \le b$,

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}\Big(f(a) + f(b)\Big),$$

with equality when the function is linear and the region is truly a trapezium.

Always start a question by constructing a table of values.

WORKED EXERCISE:

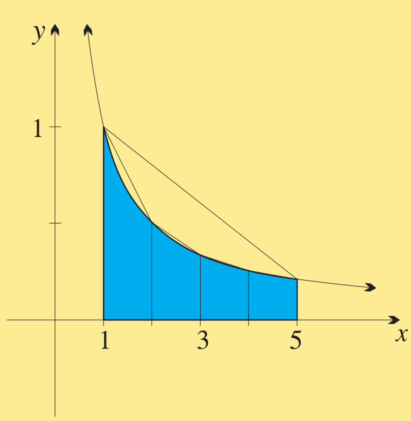
Find approximations to $\displaystyle\int_1^5 \frac{1}{x}\,dx$ using the trapezoidal rule with:

(a) one application, (b) four applications.

SOLUTION:

Always begin with a table of values of the function.

x	1	2	3	4	5
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



(a) One application of the trapezoidal rule requires just two values of the function.

$$\int_1^5 \frac{1}{x}\,dx \doteqdot \frac{5-1}{2} \times \Big(f(1) + f(5)\Big)$$

$$\doteqdot 2 \times \big(1 + \tfrac{1}{5}\big)$$

$$\doteqdot 2\tfrac{2}{5}$$

(b) Four applications of the trapezoidal rule require five values of the function. Dividing the interval $1 \le x \le 5$ into four subintervals,

$$\int_1^5 \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \int_3^4 \frac{1}{x}\,dx + \int_4^5 \frac{1}{x}\,dx$$

Each subinterval has width 1, so $\dfrac{b-a}{2} = \dfrac{1}{2}$ each time,

so applying the trapezoidal rule to each integral,

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \tfrac{1}{2}\Big(f(1)+f(2)\Big) + \tfrac{1}{2}\Big(f(2)+f(3)\Big) + \tfrac{1}{2}\Big(f(3)+f(4)\Big) + \tfrac{1}{2}\Big(f(4)+f(5)\Big)\\ &\doteqdot \tfrac{1}{2}\left(\tfrac{1}{1}+\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{3}\right) + \tfrac{1}{2}\left(\tfrac{1}{3}+\tfrac{1}{4}\right) + \tfrac{1}{2}\left(\tfrac{1}{4}+\tfrac{1}{5}\right)\\ &\doteqdot 1\tfrac{41}{60}.\end{aligned}$$

Concavity and the Trapezoidal Rule: The curve in the example above is concave up, so every approximation found using the trapezoidal rule is greater than the integral. Similarly, if a curve is concave down, every trapezoidal-rule approximation is less than the integral. The second derivative can be used to test concavity.

22

CONCAVITY AND THE TRAPEZOIDAL RULE:

If the curve is concave up, the trapezoidal rule overestimates the integral.
If the curve is concave down, the trapezoidal rule underestimates the integral.
If the curve is linear, the trapezoidal rule gives the exact value of the integral.

The second derivative $\dfrac{d^2y}{dx^2}$ can be used to test the concavity.

WORKED EXERCISE:

(a) Use one application of the trapezoidal rule to approximate $\displaystyle\int_1^5 (200x - x^4)\,dx$.

(b) Use the second derivative to explain why the approximation underestimates the integral.

SOLUTION:

(a) Construct a table of values for $y = 200x - x^4$:

x	1	5
y	199	375

$$\begin{aligned}\int_1^5 (200x - x^4)\,dx &\doteqdot \frac{5-1}{2} \times \Big(f(1)+f(5)\Big)\\ &\doteqdot 2 \times (199 + 375)\\ &\doteqdot 1148\end{aligned}$$

(b) The function is $y = 200x - x^4$.
Differentiating, $y' = 200 - 4x^3$
and $y'' = -12x^2$.
Since $y'' = -12x^2$ is negative throughout the interval $1 \le x \le 5$, the curve is concave down throughout this interval.
Hence the trapezoidal rule underestimates the integral.

Exercise 1H

TECHNOLOGY: It is not difficult to write (or download) a program that will allow the calculations of the trapezoidal rule to be automated. It can then be applied to many examples from this exercise. The number of subintervals used can be steadily increased, and the approximations should then converge to the exact value of the integral. An acompanying screen sketch showing the curve and the chords would be helpful in giving a visual impression of the size and the sign of the error.

1. Use the trapezoidal rule to approximate $\int_2^6 f(x)\,dx$ if the values of $f(x)$ are:

(a)

x	2	6
$f(x)$	8	12

(b)

x	2	6
$f(x)$	6·2	4·8

(c)

x	2	6
$f(x)$	-4	-9

2. Three values of a function $f(x)$ are known:

x	2	6	10
$f(x)$	12	20	30

(a) Use the trapezoidal rule to find approximations to $\int_2^6 f(x)\,dx$ and $\int_6^{10} f(x)\,dx$.

(b) Add these results to find an approximation to $\int_2^{10} f(x)\,dx$.

3. (a) Use the trapezoidal rule to find approximations to $\int_{-5}^0 f(x)\,dx$ and $\int_0^5 f(x)\,dx$ for a function $f(x)$ for which the following values are known:

x	-5	0	5
$f(x)$	2·4	2·6	4·4

(b) Add these results to find an approximation to $\int_{-5}^5 f(x)\,dx$.

4. Show, by means of a diagram, that the trapezoidal rule will:

(a) overestimate $\int_a^b f(x)\,dx$ if $f''(x) > 0$ for $a \le x \le b$,

(b) underestimate $\int_a^b f(x)\,dx$ if $f''(x) < 0$ for $a \le x \le b$.

5. (a) Complete this table for the function $y = x(4-x)$:

x	0	1	2	3	4
y					

(b) Using the table and the trapezoidal rule, find approximations to:

(i) $\int_0^1 x(4-x)\,dx$

(ii) $\int_1^2 x(4-x)\,dx$

(iii) $\int_2^3 x(4-x)\,dx$

(iv) $\int_3^4 x(4-x)\,dx$

(c) Add these approximations to find an approximation to $\int_0^4 x(4-x)\,dx$.

(d) What is the exact value of $\int_0^4 x(4-x)\,dx$, and why does it exceed the approximation? Sketch the curve and the four chords involved.

(e) Calculate the percentage error in the approximation (that is, divide the error by the correct answer and convert to a percentage).

6. (a) Complete this table for the function $y = \dfrac{6}{x}$:

x	1	2	3	4	5
y					

(b) Use the trapezoidal rule with the five function values above to approximate $\int_1^5 \dfrac{6}{x}\,dx$.

(c) Show that the second derivative of $y = \dfrac{6}{x}$ is $y'' = 12x^{-3}$, and use this result to explain why the approximation will exceed the exact value of the integral.

7. (a) Complete this table for the function $y = \sqrt{x}$:

x	9	10	11	12	13	14	15	16
y								

(b) Approximate $\int_9^{16} \sqrt{x}\,dx$, using the trapezoidal rule with the eight function values above. Give your answer correct to three significant figures.

(c) What is the exact value of $\int_9^{16} \sqrt{x}\,dx$? Show that the second derivative of $y = x^{\frac{1}{2}}$ is $y'' = -\frac{1}{4}x^{-1\frac{1}{2}}$, and use this result to explain why the approximation is less than the integral.

DEVELOPMENT

8. Use the trapezoidal rule with three function values to approximate each of these integrals. Answer correct to three decimal places.

(a) $\int_0^1 2^{-x}\,dx$ (b) $\int_{-2}^0 2^{-x}\,dx$ (c) $\int_1^3 \sqrt[3]{9-2x}\,dx$ (d) $\int_{-13}^{-1} \sqrt{3-x}\,dx$

9. Use the trapezoidal rule with five function values to approximate each of these integrals. Answer correct to three decimal places.

(a) $\int_2^6 \dfrac{1}{x}\,dx$

(b) $\int_0^2 \dfrac{1}{2+\sqrt{x}}\,dx$

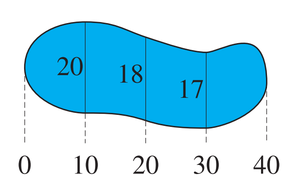
(c) $\int_4^8 \sqrt{x^2-3}\,dx$

(d) $\int_1^2 \log_{10} x\,dx$

10. An object is moving along the x-axis with values of the velocity v in m/s at various times t given in the table on the right. Given that the distance travelled may be found by calculating the area under a velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the particle in the first 5 seconds.

t	0	1	2	3	4	5
v	1·5	1·3	1·4	2·0	2·4	2·7

11. The diagram on the right shows the width of a lake at 10-metre intervals. Use the trapezoidal rule to estimate the surface area of the water.



CHALLENGE

12. A cone is generated by rotating the line $y = 2x$ about the x-axis from $x = 0$ and $x = 3$.
(a) State the integral required to evaluate the volume of the cone.
(b) Using the trapezoidal rule with four function values, approximate the volume of the cone.
(c) Calculate the exact volume of the cone and hence find the percentage error in the approximation.

13. The region under the graph $y = 2^{x+1}$ between $x = 1$ and $x = 3$ is rotated about the x-axis.
(a) State the integral required to evaluate the volume of the solid that is formed.
(b) Using the trapezoidal rule with five function values, approximate the volume of the solid.

1 I Simpson's Rule

The trapezoidal rule approximates the function that is to be integrated by a linear function, which is a polynomial of degree 1. The next most obvious method is to approximate the function by a polynomial of degree 2, that is, by a quadratic function. This approach is called *Simpson's rule*. Geometrically speaking, it approximates the curve by a parabola.

Simpson's Rule: To approximate a definite integral using Simpson's rule, the value at the midpoint of the interval as well as the values at the endpoints must be known.

23 SIMPSON'S RULE: Given a function that is continuous in the interval $a \le x \le b$,

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{6}\Big(f(a) + 4f\big(\tfrac{a+b}{2}\big) + f(b)\Big),$$

with equality when the function is truly quadratic.

Applying Simpson's Rule: The proof of Simpson's rule is not easy, so examples of its application are given first.

WORKED EXERCISE:
Experiment has found that a function $f(x)$ has the following table of values:

x	1	2	3	4	5
$f(x)$	2·31	4·56	5·34	3·02	0·22

Use Simpson's rule to approximate $\displaystyle\int_1^5 f(x)\,dx$:

(a) using one application of the rule,

(b) making the best use of the given data.

SOLUTION:

(a) To use one application of Simpson's rule to approximate $\int_1^5 f(x)\,dx$,

take $a = 1$ and $b = 5$, because these are the upper and lower bounds of the integral.

Then $\dfrac{a+b}{2} = 3$, which is the average of 1 and 5.

Applying the formula,
$$\begin{aligned}\int_1^5 f(x)\,dx &\doteqdot \frac{5-1}{6} \times \Big(f(1) + 4f(3) + f(5)\Big)\\ &\doteqdot \tfrac{2}{3} \times (2{\cdot}31 + 4 \times 5{\cdot}34 + 0{\cdot}22)\\ &\doteqdot 15{\cdot}93.\end{aligned}$$

Notice that the given values of $f(x)$ at $x = 2$ and $x = 4$ were not used here. It can hardly be the best use of the data when two of the five given values of the function are ignored.

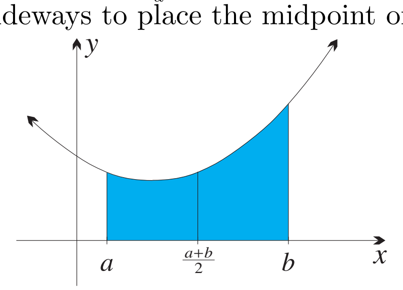
(b) The best use of the data is to apply Simpson's rule separately on each of the intervals $1 \le x \le 3$ and $3 \le x \le 5$ and then add the two results.

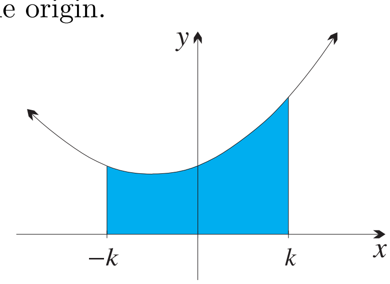
First,
$$\begin{aligned}\int_1^3 f(x)\,dx &\doteqdot \frac{3-1}{6} \times \Big(f(1) + 4f(2) + f(3)\Big)\\ &\doteqdot \tfrac{1}{3} \times (2{\cdot}31 + 4 \times 4{\cdot}56 + 5{\cdot}34)\\ &\doteqdot 8{\cdot}63.\end{aligned}$$

Secondly,
$$\begin{aligned}\int_3^5 f(x)\,dx &\doteqdot \frac{5-3}{6} \times \Big(f(3) + 4f(4) + f(5)\Big)\\ &\doteqdot \tfrac{1}{3} \times (5{\cdot}34 + 4 \times 3{\cdot}02 + 0{\cdot}22)\\ &\doteqdot 5{\cdot}88.\end{aligned}$$

Adding these two results,
$$\begin{aligned}\int_1^5 f(x)\,dx &= \int_1^3 f(x)\,dx + \int_1^3 f(x)\,dx\\ &\doteqdot 8{\cdot}63 + 5{\cdot}88\\ &\doteqdot 14{\cdot}51.\end{aligned}$$

Proof of Simpson's Rule: We need to prove that when $f(x)$ is a quadratic function, Simpson's rule gives the exact value of the integral. The first diagram below shows the integral $\int_a^b f(x)\,dx$. In the second diagram, the function has been shifted sideways to place the midpoint of the interval at the origin.





This shift of origin only moves the region sideways and does not change the value of the integral. Hence the only case to consider is the interval $-k \le x \le k$, whose endpoints are $x = -k$ and $x = k$ and whose midpoint is $x = 0$.

Thus we must prove that if $f(x) = Ax^2 + Bx + C$ is any quadratic function, then

$$\int_{-k}^{k} f(x)\,dx = \tfrac{2k}{6}\Big(f(-k) + 4f(0) + f(k)\Big),$$

that is, $\displaystyle\int_{-k}^{k} f(x)\,dx = \tfrac{k}{3}\Big(f(-k) + 4f(0) + f(k)\Big).$

To prove this, consider the left- and right-hand sides separately:

$$\begin{aligned}\text{LHS} &= \Big[\tfrac{1}{3}Ax^3 + \tfrac{1}{2}Bx^2 + Cx\Big]_{-k}^{k}\\ &= \tfrac{2}{3}Ak^3 + 2Ck\\ \text{RHS} &= \tfrac{k}{3}\Big((Ak^2 - Bk + C) + 4C + (Ak^2 + Bk + C)\Big)\\ &= \tfrac{2}{3}Ak^3 + 2Ck, \quad \text{as required.}\end{aligned}$$

NOTE: Simpson's rule also gives the exact answer for cubic functions. This can be seen from the proof above, if one imagines a term Dx^3 being added to the quadratic. Being an odd function, Dx^3 would not affect the value of the integral on the LHS and would also cancel out of the RHS when k and $-k$ were substituted.

Exercise 1I

TECHNOLOGY: As with the trapezoidal rule, a program can be written or downloaded that will allow the calculations of Simpson's rule to be automated and applied to many examples from this exercise. Comparisons with the trapezoidal rule will easily show that Simpson's rule usually gives a better approximation from the same data.

1. Use Simpson's rule to approximate $\displaystyle\int_3^5 f(x)\,dx$ if the values of $f(x)$ are:

(a)

x	3	4	5
$f(x)$	7	4	10

(b)

x	3	4	5
$f(x)$	2·3	4·1	2·6

(c)

x	3	4	5
$f(x)$	-8	-4	-3

2. Five values of a function $f(x)$ are known:

x	2	5	8	11	14
$f(x)$	12	15	20	25	30

(a) Use Simpson's rule to find approximations to $\displaystyle\int_2^8 f(x)\,dx$ and $\displaystyle\int_8^{14} f(x)\,dx$.

(b) Add these results to find an approximation to $\displaystyle\int_2^{14} f(x)\,dx$.

3. (a) Use Simpson's rule to find approximations to $\displaystyle\int_{-12}^{0} f(x)\,dx$ and $\displaystyle\int_0^{12} f(x)\,dx$ for a function $f(x)$ for which the following values are known:

x	-12	-6	0	6	12
$f(x)$	2·4	2·5	3	3·5	4·4

(b) Add these results to find an approximation to $\displaystyle\int_{-12}^{12} f(x)\,dx$.

4. (a) Complete this table for the function $y = \dfrac{2}{x}$:

x	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y					

(b) Use Simpson's rule with three function values to approximate $\displaystyle\int_1^2 \frac{2}{x}\,dx$.

(c) Use Simpson's rule with three function values to approximate $\displaystyle\int_2^3 \frac{2}{x}\,dx$

(d) Hence use Simpson's rule with five function values to approximate $\displaystyle\int_1^3 \frac{2}{x}\,dx$.

5. (a) Complete this table for the function $y = \sqrt{x+5}$:

x	-4	-3	-2
y			

(b) Use Simpson's rule to approximate $\displaystyle\int_{-4}^{-2} \sqrt{x+5}\,dx$, giving your answer correct to three significant figures.

6. (a) Sketch a graph of the function $y = \sqrt{9-x^2}$.

(b) Hence evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$, giving your answer correct to three decimal places.

(c) Complete this table for $y = \sqrt{9-x^2}$:

x	-3	$-1{\cdot}5$	0	$1{\cdot}5$	3
y					

(d) Using five function values, approximate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$, giving your answer correct to three decimal places:

(i) by the trapezoidal rule, (ii) by Simpson's rule.

7. (a) Complete this table for $y = 3 - 2x - x^2$:

x	-3	-2	-1	0	1
y					

(b) Use Simpson's rule with five function values to approximate $\displaystyle\int_{-3}^{1} (3 - 2x - x^2)\,dx$.

(c) Evaluate $\displaystyle\int_{-3}^{1} (3 - 2x - x^2)\,dx$. How does this compare with the answer obtained in part (b)? Why is this the case?

DEVELOPMENT

8. Use Simpson's rule with three function values to approximate:

(a) $\displaystyle\int_1^3 \frac{dx}{x^2+1}$ (b) $\displaystyle\int_{-1}^{1} 3^{-x}\,dx$

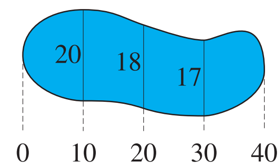
9. Use Simpson's rule with five function values to approximate each of the following integrals. Give each answer correct to four significant figures where necessary.

(a) $\displaystyle\int_3^5 \sqrt{x^2-1}\,dx$ (b) $\displaystyle\int_{-1}^{1} \log_{10}(x+3)\,dx$ (c) $\displaystyle\int_2^6 3^x\,dx$

10. An object is moving along the x-axis with the values of the velocity v in m/s at various times t given in the table to the right. Given that the distance travelled may be found by calculating the area under a velocity/time graph, use Simpson's rule to estimate the distance travelled by the particle during the first 4 seconds.

t	0	1	2	3	4
v	1·5	1·3	1·4	2·0	2·4

11. The diagram to the right shows the width of a lake at 10-metre intervals. Use Simpson's rule to estimate the surface area of the water.



CHALLENGE

12. (a) Use Simpson's rule with five function values to approximate $\int_0^1 \sqrt{1-x^2}\,dx$. Give your answer correct to four decimal places.

(b) Use part (a) and the fact that $y = \sqrt{1-x^2}$ is a semicircle to approximate π. Give your answer correct to three decimal places.

13. The region bounded by the curve $y = 3^{x-1}$ and the x-axis between $x = 1$ and $x = 3$ is rotated about the x-axis.

(a) State the integral that will give the volume of the solid that is formed.

(b) Use Simpson's rule with five function values to approximate the volume of the solid that is formed. Give your answer correct to two decimal places.

1J Chapter Review Exercise

1. Evaluate the following definite integrals, using the fundamental theorem:

(a) $\int_0^1 3x^2\,dx$ (d) $\int_{-1}^1 x^4\,dx$ (g) $\int_0^2 (x+3)\,dx$

(b) $\int_1^2 x\,dx$ (e) $\int_{-4}^{-2} 2x\,dx$ (h) $\int_{-1}^4 (2x-5)\,dx$

(c) $\int_2^5 4x^3\,dx$ (f) $\int_{-3}^{-1} x^2\,dx$ (i) $\int_{-3}^1 (x^2-2x+1)\,dx$

2. By expanding the brackets where necessary, evaluate the following definite integrals:

(a) $\int_1^3 x(x-1)\,dx$ (b) $\int_{-1}^0 (x+1)(x-3)\,dx$ (c) $\int_0^1 (2x-1)^2\,dx$

3. By dividing each numerator through by the denominator, evaluate each integral:

(a) $\int_1^2 \dfrac{x^2-3x}{x}\,dx$ (b) $\int_2^3 \dfrac{3x^4-4x^2}{x^2}\,dx$ (c) $\int_{-2}^{-1} \dfrac{x^3-2x^4}{x^2}\,dx$

4. (a) (i) Show that $\int_4^k 5\,dx = 5k-20$.

(ii) Hence find the value of k if $\int_4^k 5\,dx = 10$.

(b) (i) Show that $\displaystyle\int_0^k (2x-1)\,dx = k^2 - k$.

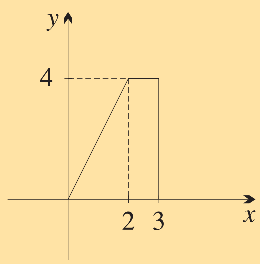
(ii) Hence find the value of k if $k > 0$ and $\displaystyle\int_0^k (2x-1)\,dx = 6$.

5. Without finding a primitive, use the properties of the definite integral to evaluate these integrals, stating reasons:

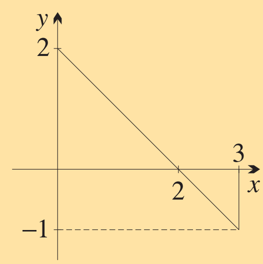
(a) $\displaystyle\int_3^3 (x^3 - 5x + 4)\,dx$ (b) $\displaystyle\int_{-2}^2 x^3\,dx$ (c) $\displaystyle\int_{-3}^3 (x^3 - 9x)\,dx$

6. Use area formulae to find $\displaystyle\int_0^3 f(x)\,dx$, given the following sketches of $f(x)$:

(a)



(b)



7. Find the following indefinite integrals:

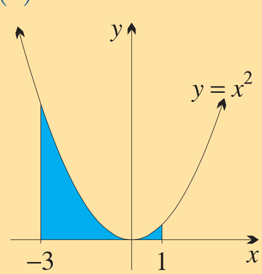
(a) $\displaystyle\int (x+2)\,dx$ (d) $\displaystyle\int (x-3)(2-x)\,dx$ (g) $\displaystyle\int \sqrt{x}\,dx$

(b) $\displaystyle\int (x^3 + 3x^2 - 5x + 1)\,dx$ (e) $\displaystyle\int x^{-2}\,dx$ (h) $\displaystyle\int (x+1)^4\,dx$

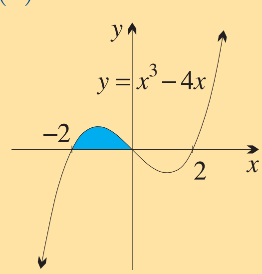
(c) $\displaystyle\int x(x-1)\,dx$ (f) $\displaystyle\int \tfrac{1}{x^7}\,dx$ (i) $\displaystyle\int (2x-3)^5\,dx$

8. Find the area of each shaded region below by evaluating the appropriate integral:

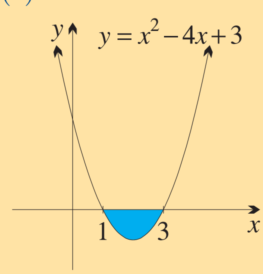
(a)



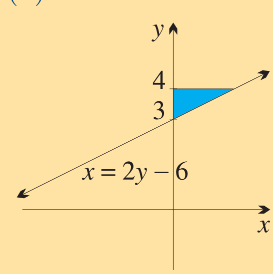
(b)



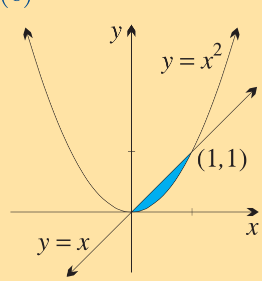
(c)



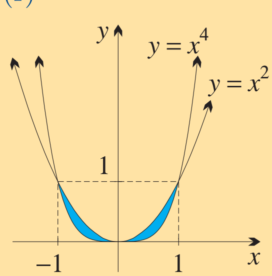
(d)



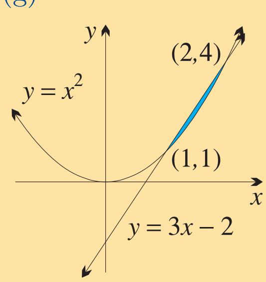
(e)



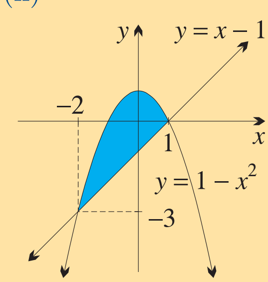
(f)



(g)

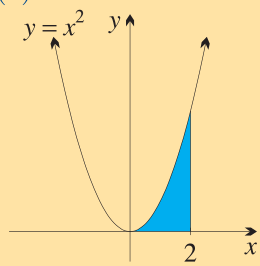


(h)

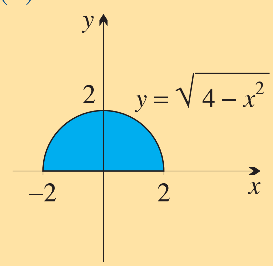


9. Calculate the volume of the solid generated when each shaded region is rotated about the x-axis.

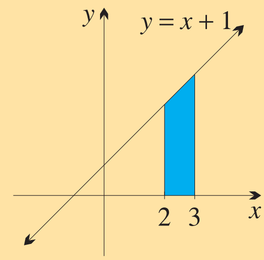
(a)



(b)

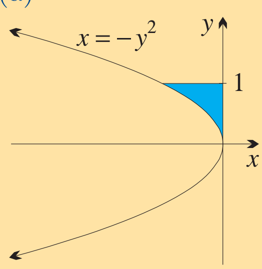


(c)

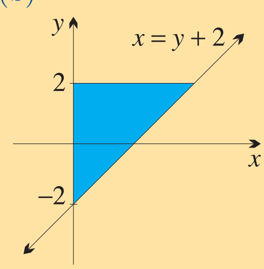


10. Calculate the volume of the solid generated when each shaded region is rotated about the y-axis.

(a)



(b)



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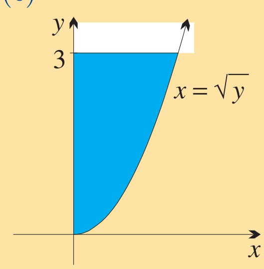


11. (a) By solving the equations simultaneously, show that the curves $y = x^2 - 3x + 5$ and $y = x + 2$ intersect at the points $(1, 3)$ and $(3, 5)$.

(b) Sketch both curves on the same diagram and find the area of the region enclosed between them.

12. The curve $y = x^4$ meets the line $y = x$ at $O(0, 0)$ and $A(1, 1)$.

(a) Sketch the diagram and shade the region between the curve and the line.

(b) Find the volume of the solid generated when this region is rotated about the x-axis.

13. (a) Use the trapezoidal rule with three function values to approximate $\displaystyle\int_1^3 2^x\,dx$.

(b) Use Simpson's rule with five function values to approximate $\displaystyle\int_1^3 \log_{10} x\,dx$. Give your answer correct to three significant figures.

Appendix — The Proof of the Fundamental Theorem

This appendix explains the rather difficult proof of the fundamental theorem stated and used in Section 1B.

Change of Pronumeral: First, notice that the pronumeral in the definite integral notation is a *dummy variable*, meaning that it can be replaced by any other pronumeral. For example, the four integrals

$$\int_0^1 x\,dx = \int_0^1 t\,dt = \int_0^1 y\,dy = \int_0^1 \lambda\,d\lambda$$

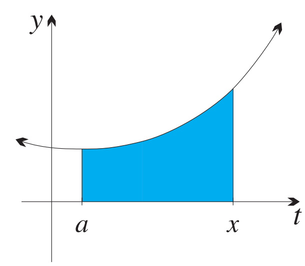
all have the same value $\frac{1}{2}$ — the letter used for the variable has changed, but the function remains the same and so the area involved remains the same.

Similarly, the pronumeral in sigma notation is a dummy variable. For example,

$$\sum_{n=1}^{4} n = \sum_{r=1}^{4} r = \sum_{x=1}^{4} x = \sum_{\lambda=1}^{4} \lambda$$

all have the same value $1 + 2 + 3 + 4 = 10$.

The Definite Integral as a Function of its Upper Bound: The value of the definite integral $\displaystyle\int_a^b f(x)\,dx$ changes when the value of b is changed. This means that it is a function of its upper bound b. To suggest the functional relationship with the upper bound b more closely, the letter b is best replaced by the letter x, which is conventionally the variable of a function.



In turn, the original letter x needs to be replaced by some other letter, most suitably t. Then one can see more clearly that the definite integral

$$A(x) = \int_a^x f(t)\,dt$$

is a function of its upper bound x. This integral is represented in the sketch above.

First Part of the Proof — $A(x)$ is a Primitive of $f(x)$: First, we prove that $A(x)$ is a primitive function of $f(x)$. That is, we shall prove that

$$A'(x) = f(x).$$

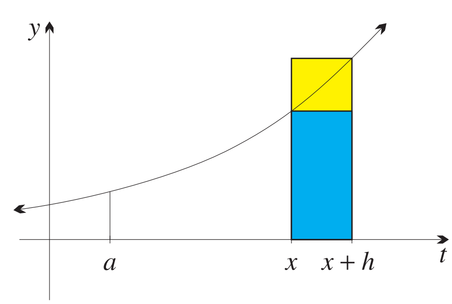
Because the theorem is so fundamental, its proof must begin with the definition of the derivative as a limit:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}.$$

Now $\displaystyle A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

$$= \int_x^{x+h} f(t)\,dt,$$

so $\displaystyle A'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt.$



This limit is handled by means of a clever sandwiching technique.
Suppose that $f(t)$ is increasing in the interval $x \le t \le x+h$, as in the diagram above.
Then the lower rectangle on the interval $x \le t \le x+h$ has height $f(x)$,
and the upper rectangle on the interval $x \le t \le x+h$ has height $f(x+h)$,
so, using areas, $h \times f(x) \le \displaystyle\int_x^{x+h} f(t)\,dt \le h \times f(x+h)$

$\boxed{\div h}$ $$f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x+h). \qquad (1)$$

Thus the middle expression, whose limit is required,
is trapped or 'sandwiched' between $f(x)$ and $f(x+h)$.
Since $f(x)$ is continuous, $f(x+h) \to f(x)$ as $h \to 0$,
and so by (1), $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$, meaning that $A'(x) = f(x)$, as required.
If $f(x)$ is decreasing in $x \le t \le x+h$, the same argument applies,
but with the inequalities reversed.

Second Part of the Proof — A Formula for the Definite Integral: The required formula for the definite integral (given in Box 4 on page 7) now follows reasonably quickly.
We now know that $F(x)$ and $\displaystyle\int_a^x f(t)\,dt$ are both primitives of $f(x)$,
so the primitives $\displaystyle\int_a^x f(t)\,dt$ and $F(x)$ must differ only by a constant,
that is, $$\int_a^x f(t)\,dt = F(x) + C, \quad \text{for some constant } C.$$
Substituting $x = a$, $$\int_a^a f(t)\,dt = F(a) + C,$$
but $\displaystyle\int_a^a f(t)\,dt = 0$, because the area in this definite integral has zero width,
so $$0 = F(a) + C.$$
Hence $C = -F(a)$ and $\displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$.
Changing letters from x to b and from t to x gives
$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{as required.}$$

CHAPTER **TWO**

The Exponential Function

So far, calculus has been developed for algebraic functions like

$$x^3 + 8 \qquad \text{and} \qquad \sqrt{1-4x} \qquad \text{and} \qquad x^2 + \frac{1}{x^2}$$

that involve only powers of x and the four operations of arithmetic. Some of the most important functions used in science, however, cannot be written in this way. This chapter begins to extend calculus to such *non-algebraic* functions.

The first functions to be considered are *exponential functions*, which are functions like $y = 2^x$, where the variable is in the exponent or index. These functions describe natural phenomena like radioactive decay or the noise of a plucked guitar string, where something is dying away, and populations or inflation, where a quantity is rapidly growing.

Calculus with exponential functions turns out to be easiest not when the base is an integer like 2 or 3 or 10, but when it is a particular irrational number called e whose value is approximately $e \doteqdot 2{\cdot}7183$. This number makes its first appearance in this chapter, which mostly deals with the function $y = e^x$.

Logarithms are not mentioned in this chapter — they are the principal topic of Chapter Three, which also covers the calculus of exponential functions with bases different from e. Natural growth and decay have been left until Chapter Six.

2 A Review of Exponential Functions

Indices have already been reviewed in the Year 11 volume. This section presents another brief review of indices and of exponential functions like $y = 2^x$. In this section, the bases a and b must be positive numbers, but the indices can be any real numbers.

Index Laws — Combining Powers with the Same Base: The index laws in the first group below show how to combine powers when the base is fixed.

INDEX LAWS — PRODUCTS, QUOTIENTS AND POWERS OF POWERS:

1

$$a^x \times a^y = a^{x+y}$$

$$a^x \div a^y = a^{x-y} \qquad \left(\text{also written as } \frac{a^x}{a^y} = a^{x-y}\right)$$

$$(a^x)^y = a^{xy}$$

WORKED EXERCISE:

Simplify each expression:

(a) $(5^{2x})^4$ (b) $3^{2x} \times 3^{5x}$ (c) $\dfrac{2^{4x+2}}{2^x}$

SOLUTION:

(a) $(5^{2x})^4 = 5^{8x}$ (b) $3^{2x} \times 3^{5x} = 3^{7x}$ (c) $\dfrac{2^{4x+2}}{2^x} = 2^{3x+2}$

Index Laws — Powers of Products and Quotients: The index laws in the second group show how to work with powers of products and quotients.

2 INDEX LAWS — POWERS OF PRODUCTS AND QUOTIENTS:

$$(ab)^x = a^x \times b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

WORKED EXERCISE:

Expand the brackets in these expressions:

(a) $\left(\dfrac{2^x}{5}\right)^3$ (b) $(3 \times 2^x)^4$

SOLUTION:

(a) $\left(\dfrac{2^x}{5}\right)^3 = \dfrac{(2^x)^3}{5^3}$

$= \dfrac{2^{3x}}{125}$

(b) $(3 \times 2^x)^4 = 3^4 \times (2^x)^4$

$= 81 \times 2^{4x}$

Zero and Negative Indices: Any power with a zero index has value 1. A negative sign in the index is an indication to take the reciprocal.

3 ANY POWER WITH A ZERO INDEX IS 1:

$$a^0 = 1$$

NEGATIVE INDICES MEAN TAKE THE RECIPROCAL:

$$a^{-x} = \frac{1}{a^x}$$

In particular, $a^{-1} = \dfrac{1}{a}$ and $a^{-2} = \dfrac{1}{a^2}$.

WORKED EXERCISE:

Write these expressions using fractions instead of negative indices:

(a) 5×3^{-x} (b) $\frac{1}{7} \times 5^{-x}$

SOLUTION:

(a) $5 \times 3^{-x} = \dfrac{5}{1} \times \dfrac{1}{3^x}$

$= \dfrac{5}{3^x}$

(b) $\frac{1}{7} \times 5^{-x} = \dfrac{1}{7} \times \dfrac{1}{5^x}$

$= \dfrac{1}{7 \times 5^x}$

WORKED EXERCISE:
Write these expressions using negative indices instead of fractions:

(a) $\dfrac{4}{3^{5x}}$ (b) $\dfrac{36}{4 \times 10^x}$

SOLUTION:

(a) $\dfrac{4}{3^{5x}} = 4 \times 3^{-5x}$ (b) $\dfrac{36}{4 \times 10^x} = 9 \times 10^{-x}$

Fractional Indices: A denominator in the index means take the corresponding root.

4 FRACTIONAL INDICES MEAN TAKE THE ROOT:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = \left(\sqrt[n]{a}\right)^m.$$

In particular, $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{-\frac{1}{2}} = \dfrac{1}{\sqrt{a}}$.

WORKED EXERCISE:
Simplify:

(a) $125^{\frac{1}{3}}$ (b) $8^{\frac{4}{3}}$ (c) $16^{-\frac{1}{2}}$ (d) $27^{-\frac{2}{3}}$

SOLUTION:

(a) $125^{\frac{1}{3}} = 5$

(b) $8^{\frac{4}{3}} = 2^4 = 16$

(c) $16^{-\frac{1}{2}} = \dfrac{1}{16^{\frac{1}{2}}} = \frac{1}{4}$

(d) $27^{-\frac{2}{3}} = 3^{-2} = \frac{1}{9}$

Exponential Functions: Familiarity with exponential functions and their graphs is needed before calculus can be applied to them.

WORKED EXERCISE:

(a) Sketch on one set of axes the curves $y = 2^x$ and $y = 2^{-x}$.
(b) Name the asymptotes of these functions.
(c) Describe the behaviour of these functions as $x \to \infty$ and as $x \to -\infty$.

SOLUTION:

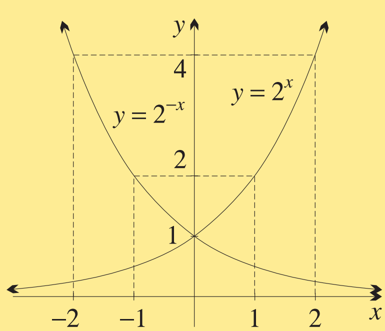
(a) The tables of values make it clear that the two graphs are reflections of each other in the y-axis.

For $y = 2^x$:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

For $y = 2^{-x}$:

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



(b) The x-axis is an asymptote to both curves.

(c) As $x \to \infty$, $2^x \to \infty$ and $2^{-x} \to 0$.
As $x \to -\infty$, $2^x \to 0$ and $2^{-x} \to \infty$.

Exercise 2A

1. (a) Copy and complete the table of values of the function $y = 3^x$:

x	-2	-1	0	1	2
3^x					

(b) Sketch the curve, choosing appropriate scales on the axes.

(c) What are the domain and range of $y = 3^x$?

2. (a) Copy and complete the table of values of the function $y = 3^{-x}$:

x	-2	-1	0	1	2
3^{-x}					

(b) Sketch the curve, choosing appropriate scales on the axes.

(c) What are the domain and range of $y = 3^{-x}$?

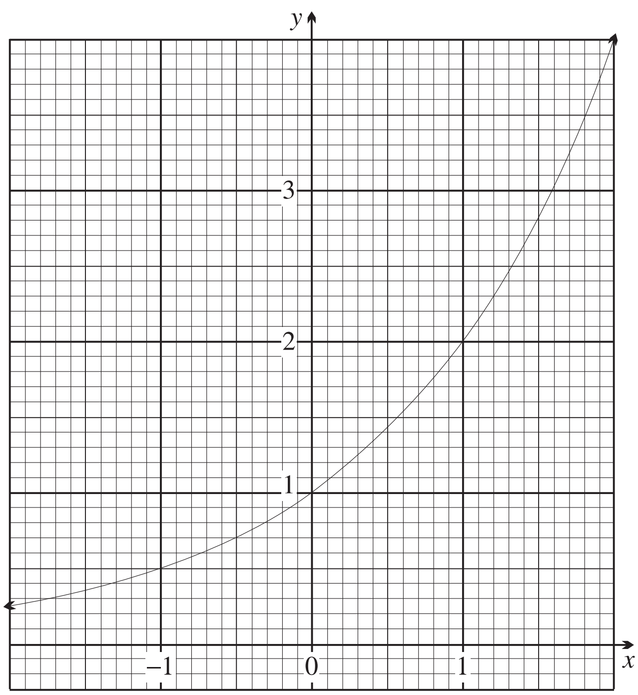
3. (a) Use your calculator to copy and complete the table of values of $y = 10^x$:

x	-2	-1	$-0{\cdot}75$	$-0{\cdot}5$	0	$0{\cdot}25$	$0{\cdot}5$	$0{\cdot}75$	1
10^x									

(b) Sketch the curve, choosing appropriate scales on the axes.

(c) What are the domain and range of $y = 10^x$?

4.



Use the graph of $y = 2^x$ to read off the following values, correct to two decimal places.

(a) 2^0
(b) 2^{-1}
(c) $2^{\frac{1}{2}}$
(d) $2^{1\frac{1}{2}}$
(e) $2^{-1{\cdot}5}$
(f) $2^{0{\cdot}7}$

Check your answers using the function labelled $\boxed{x^y}$ on the calculator.

5. Use the index law $a^x \times a^y = a^{x+y}$ to simplify the following. Leave answers in index form.

(a) $2^4 \times 2^6$
(b) $6^3 \times 6^4$
(c) $(2 \times 3^2) \times (5 \times 3^3)$
(d) $(4 \times 5^3) \times (2 \times 5^7)$

6. Use the index law $\dfrac{a^x}{a^y} = a^{x-y}$ to simplify the following. Leave your answers in index form.

(a) $\dfrac{2^6}{2^4}$ (c) $\dfrac{5^4}{5^{10}}$ (e) $\dfrac{14 \times 6^8}{7 \times 6^5}$

(b) $\dfrac{6^4}{6^3}$ (d) $\dfrac{10^3}{10^6}$ (f) $\dfrac{27 \times 5^3}{9 \times 5^7}$

7. Simplify:

(a) 2^0 (b) 3^0 (c) p^0 (d) x^0

8. Write as fractions without using negative indices:

(a) 2^{-1} (d) x^{-1} (g) p^{-2} (j) 3^{-x}

(b) 3^{-1} (e) 2^{-2} (h) x^{-2} (k) 5^{-2x}

(c) p^{-1} (f) 3^{-2} (i) 2^{-x} (l) p^{-2x}

9. Simplify:

(a) $16^{\frac{1}{2}}$ (c) $9^{\frac{3}{2}}$ (e) $4^{-\frac{1}{2}}$ (g) $8^{-\frac{5}{3}}$

(b) $25^{\frac{1}{2}}$ (d) $16^{\frac{3}{4}}$ (f) $27^{-\frac{1}{3}}$ (h) $81^{-\frac{3}{4}}$

DEVELOPMENT

10. Sketch the graph of $y = 2^x$, then use your knowledge of shifting to graph the following functions. Show the horizontal asymptote and state the range in each case.

(a) $y = 2^x + 1$ (b) $y = 2^x + 2$ (c) $y = 2^x - 1$ (d) $y = 2^x - 2$

11. Sketch the graph of $y = 2^{-x}$, then use your knowledge of transformations to graph the following functions. Show the horizontal asymptote and state the range in each case.

(a) $y = 2^{-x} + 1$ (b) $y = 2^{-x} + 2$ (c) $y = 2^{-x} - 1$ (d) $y = 2^{-x} - 2$

12. Simplify:

(a) $3^{2x} \times 3^x$ (d) $(2 \times 3^x) \times (7 \times 3^{3x})$ (g) $2^{x-1} \times 2^{2x+1}$

(b) $2^{5x} \times 2^{4x}$ (e) $(4 \times 7^{3x}) \times (5 \times 7^{5x})$ (h) $3^{1-2x} \times 3^{2+x}$

(c) $10^x \times 10^{10x}$ (f) $5^{x-3} \times 5^{x+3}$ (i) $6^{2x-4} \times 6^{3-x}$

13. Simplify:

(a) $\dfrac{3^{2x}}{3^x}$ (d) $\dfrac{24 \times 3^{5x}}{6 \times 3^{2x}}$ (g) $\dfrac{4^{2x-1}}{4^{x+2}}$

(b) $\dfrac{2^{5x}}{2^{3x}}$ (e) $\dfrac{40 \times 7^{2x}}{5 \times 7^{5x}}$ (h) $\dfrac{2^{1-2x}}{2^{2-3x}}$

(c) $\dfrac{7^{1+x}}{7^{1-x}}$ (f) $\dfrac{3^{x+2}}{3^{2x+2}}$ (i) $\dfrac{5^{x+4}}{5}$

14. Use the graph of $y = 2^x$ and your knowledge of transformations to graph the following functions:

(a) $y = 2^{x-1}$ (c) $y = 2^{x+1}$ (e) $y = -2^x$

(b) $y = 2^{x-3}$ (d) $y = 2^{x+2}$ (f) $y = 2^{-x}$

CHALLENGE

15. [Technology] Use a graphing program to graph $y = a^x$ for values of a increasing from 2 to 5 (including fractional values of a). Explain what happens to the graph as a increases.

16. [Technology] Use a graphing program to graph transformations of exponential graphs. Start with the graphs in questions 10, 11, 14, 18 and 19, and then experiment with further similar graphs.

17. [Technology] Solve $10^x = 3$ by trial and error using your calculator. Give your answer correct to four decimal places.

18. Use the graphs of $y = 2^x$ and $y = 2^{-x}$ and your knowledge of transformations to graph the following functions. Show the horizontal asymptote and state the range in each case.
(a) $y = 1 - 2^x$ (b) $y = 3 - 2^x$ (c) $y = -2^{-x}$ (d) $y = -2^{x-1}$

19. Write each function as a power of 2. For example, $8^x = (2^3)^x = 2^{3x}$.
(a) 4^x (c) $8^x \times 2^{2x}$ (e) $32^x \div 2^x$
(b) 32^x (d) $16^x \times 2^{3x}$ (f) $16^x \div 2^{2x}$

20. Simplify:
(a) $(64 \times 3^x)^{\frac{1}{3}}$ (b) $(81 \times 2^{12x})^{\frac{3}{4}}$ (c) $(36 \times 5^{4x})^{-\frac{3}{2}}$ (d) $(125 \times 3^{12x})^{-\frac{2}{3}}$

2 B The Exponential Function e^x and the Definition of e

This section introduces an important new number called e. This number e is not a whole number, or even a fraction, but is a real number between 2 and 3 whose value is approximately $e \doteqdot 2{\cdot}7183$.

Differentiating exponential functions whose base is a whole number, like $y = 2^x$ or $y = 10^x$, turns out to be inconvenient. It is much easier to use this new number e as a base and to study the exponential function $y = e^x$. The fundamental result of this section is that the function $y = e^x$ is its own derivative:

$$\frac{d}{dx}\, e^x = e^x.$$

Differentiating $y = 2^x$:

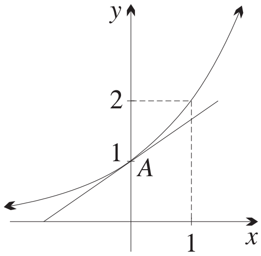
We begin by looking at $y = 2^x$ and trying to differentiate it. Below is a sketch of $y = 2^x$, with the tangent drawn at its y-intercept $A(0, 1)$.

Differentiating 2^x requires first-principles differentiation, because the theory so far hasn't provided any rule for differentiating 2^x.

The formula for first-principles differentiation is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Applying this formula to the function $f(x) = 2^x$,



$$f'(x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$

$$= \lim_{h \to 0} \frac{2^x \times 2^h - 2^x}{h}, \quad \text{since } 2^x \times 2^h = 2^{x+h},$$

and taking out the common factor 2^x in the numerator,

$$f'(x) = 2^x \times \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$= 2^x \times m, \quad \text{where } m = \lim_{h \to 0} \frac{2^h - 1}{h}.$$

This limit $m = \lim_{h \to 0} \dfrac{2^h - 1}{h}$ cannot be found by algebraic methods,
but substituting $x = 0$ gives a very simple geometric interpretation of the limit:

$$\begin{aligned} f'(0) &= 2^0 \times m \\ &= m, \quad \text{since } 2^0 = 1, \end{aligned}$$

so $m = \lim_{h \to 0} \dfrac{2^h - 1}{h}$ is just the gradient of the tangent to $y = 2^x$ at its y-intercept $(0, 1)$.

The conclusion of all this is that

$$\frac{d}{dx}\, 2^x = 2^x \times m, \quad \text{where } m \text{ is the gradient of } y = 2^x \text{ at its } y\text{-intercept.}$$

Exactly the same argument can be applied to any exponential function $y = a^x$ simply by replacing 2 by a in the calculation above:

5 DIFFERENTIATING $y = a^x$: For all positive numbers a,

$$\frac{d}{dx}\, a^x = a^x \times m, \quad \text{where } m \text{ is the gradient of } y = a^x \text{ at its } y\text{-intercept.}$$

That is, the derivative of an exponential function is a multiple of itself.

The Definition of e: It now makes sense to choose the base that will make the gradient exactly 1 at the y-intercept, because the value of m will then be exactly 1. This base is given the symbol e and has the value $e \doteqdot 2{\cdot}7183$.

6 THE DEFINITION OF e: Define e to be the number such that the exponential function $y = e^x$ has gradient exactly 1 at its y-intercept. Then

$$e \doteqdot 2{\cdot}7183.$$

The function $y = e^x$ is called *<u>the</u> exponential function* to distinguish it from all other exponential functions $y = a^x$.

A calculator will provide approximate values of e^x correct to about ten significant figures — use the function labelled $\boxed{e^x}$, which is usually located above the button labelled $\boxed{\ln}$.

Since $e = e^1$, an approximation for the number e itself is found using the function labelled $\boxed{e^x}$, using the input $x = 1$.

WORKED EXERCISE:

Use your calculator to find, correct to four significant figures:

(a) e^2 (b) e (c) $\dfrac{1}{e}$ (d) $\sqrt{e}$

SOLUTION: Using the function labelled $\boxed{e^x}$ on the calculator:

(a) $e^2 \doteqdot 7{\cdot}389$

(b) $e = e^1 \doteqdot 2{\cdot}718$

(c) $\dfrac{1}{e} = e^{-1} \doteqdot 0{\cdot}3679$

(d) $\sqrt{e} = e^{\frac{1}{2}} \doteqdot 1{\cdot}649$

The Derivative of e^x: The fundamental result of this section then follows immediately from the previous boxed results.

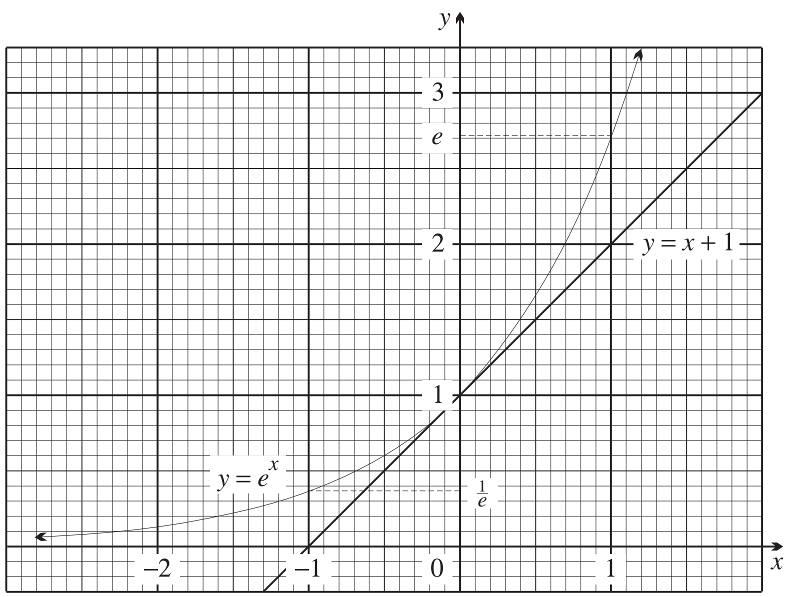
$$\begin{aligned}\frac{d}{dx}\,e^x &= e^x \times m, \quad \text{where } m \text{ is the gradient of } y = e^x \text{ at its } y\text{-intercept,}\\ &= e^x \times 1, \quad \text{by the definition of } e,\\ &= e^x.\end{aligned}$$

7 THE DERIVATIVE OF THE EXPONENTIAL FUNCTION $y = e^x$:
The exponential function $y = e^x$ is its own derivative:

$$\frac{d}{dx}\,e^x = e^x.$$

The graph of $y = e^x$ has been drawn below on graph paper. The tangent has been drawn at the y-intercept $(0, 1)$ — this shows that the gradient of the curve there is exactly 1.

This graph of $y = e^x$ is one of the most important graphs in the whole course and its shape and properties need to be memorised.



- The domain is all values of x.
  The range is $y > 0$.
- There are no zeroes.
  The curve is always above the x-axis.
- As $x \to -\infty$, $y \to 0$.
  This means that the x-axis $y = 0$ is a horizontal asymptote to the curve on the left-hand side.
- As $x \to \infty$, $y \to \infty$.
  On the right-hand side, the curve rises steeply.
- The curve has gradient 1 at its y-intercept $(0, 1)$.
- The curve is always concave up.

Gradient Equals Height: The fact that the derivative of the exponential function e^x is the same function has a striking geometrical interpretation in terms of its graph. If $y = e^x$, then $\dfrac{dy}{dx} = e^x$, which means that for this function

$$\frac{dy}{dx} = y.$$

Thus at each point on the curve $y = e^x$, the gradient $\dfrac{dy}{dx}$ of the curve is equal to the height y of the curve above the x-axis. We have already seen this happening at the y-intercept $(0, 1)$, where the gradient is 1 and the height is also 1.

8 GRADIENT EQUALS HEIGHT:
At each point on the graph of the exponential function $y = e^x$,

$$\frac{dy}{dx} = y.$$

That is, the gradient of the curve is always equal to its height above the x-axis.

This property of the exponential function is the reason why the function is so important in calculus. An important question in the next exercise asks for this to be verified on a graph-paper graph of $y = e^x$.

Transformations of the Exponential Graph: The usual methods of shifting and reflecting graphs can be applied to $y = e^x$. When the graph is shifted vertically, the horizontal asymptote at $y = 0$ will be shifted also.

A small table of approximate values can be a very useful check, particularly when a sequence of transformations is involved.

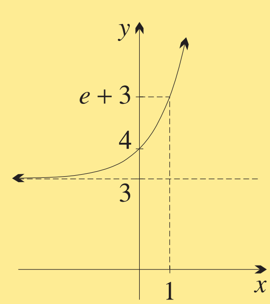
WORKED EXERCISE:
Use transformations of the graph of $y = e^x$, and a table of values, to generate a sketch of each function. Show the y-intercept and the horizontal asymptote and state the range.

(a) $y = e^x + 3$ (b) $y = e^{-x}$ (c) $y = e^{x-2}$

SOLUTION:
(a) Graph $y = e^x + 3$ by shifting $y = e^x$ up 3 units.

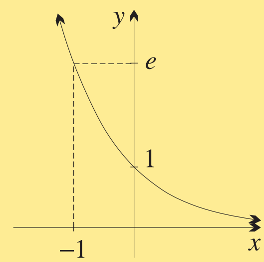
x	-1	0	1
y	$e^{-1} + 3$	4	$e + 3$
approximation	$3{\cdot}37$	4	$5{\cdot}72$

y-intercept: $(0, 4)$
Asymptote: $y = 3$
Range: $y > 3$



(b) Graph $y = e^{-x}$ by reflecting $y = e^{x}$ in the y-axis.

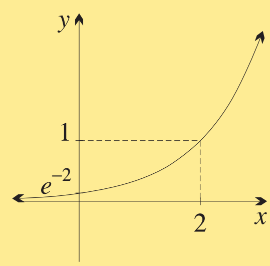
x	-1	0	1
y	e	1	e^{-1}
approximation	$2{\cdot}72$	1	$0{\cdot}37$



y-intercept: $(0, 1)$
Asymptote: $y = 0$ (the x-axis)
Range: $y > 0$

(c) Graph $y = e^{x-2}$ by shifting $y = e^{x}$ to the right by 2 units.

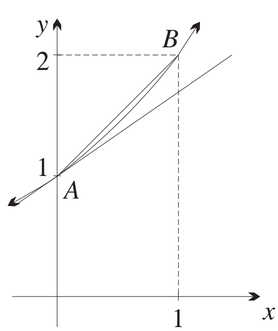
x	0	1	2	3
y	e^{-2}	e^{-1}	1	e
approximation	$0{\cdot}14$	$0{\cdot}37$	1	$2{\cdot}72$

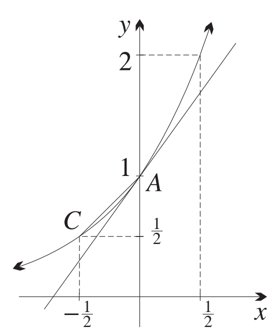


y-intercept: $(0, e^{-2})$
Asymptote: $y = 0$ (the x-axis)
Range: $y > 0$

Bounds for e: It's not appropriate in this course to spend time calculating close approximations to e. The graphs below at least show very quickly that the number e lies between 2 and 4.

Here are tables of values and sketches of $y = 2^{x}$ and $y = 4^{x}$. On each graph, the tangent at the y-intercept $A(0, 1)$ has been drawn.





For $y = 2^{x}$,

x	0	1
y	1	2

Let $B = (1, 2)$.
Then the chord AB has gradient 1,
so the tangent at A must
have gradient less than 1.

For $y = 4^{x}$,

x	$-\frac{1}{2}$	0
y	$\frac{1}{2}$	1

Let $C = (-\frac{1}{2}, \frac{1}{2})$.
Then the chord CA has gradient 1,
so the tangent at A must
have gradient greater than 1.

These two diagrams show that the tangent at the y-intercept $A(0, 1)$ has gradient less than 1 for $y = 2^{x}$ and greater than 1 for $y = 4^{x}$. Since the gradient of $y = e^{x}$ at its y-intercept is exactly 1, it follows that

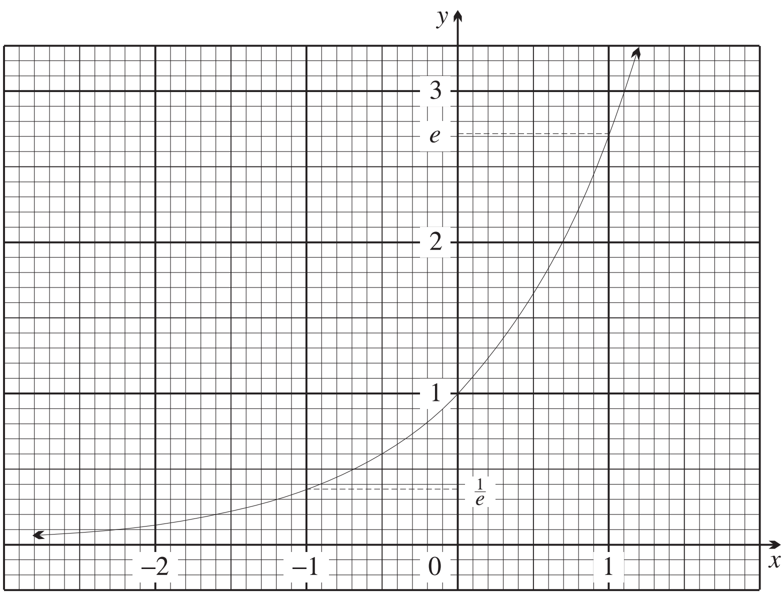
$$2 < e < 4.$$

Exercise 2B

1. Use the function labelled $\boxed{e^x}$ on your calculator to approximate the following correct to four decimal places:

(a) e^2
(b) e^3
(c) e^{10}
(d) e^0
(e) e^1
(f) e^{-1}
(g) e^{-2}
(h) $e^{\frac{1}{2}}$
(i) $e^{-\frac{1}{2}}$

2.



These questions refer to the graph of $y = e^x$ drawn above .

(a) Photocopy the graph of $y = e^x$ above and on it draw the tangent at the point $(0, 1)$ where the height is 1, extending the tangent down to the x-axis.

(b) Measure the gradient of this tangent and confirm that it is equal to the height of the exponential graph at the point of contact.

(c) Copy and complete the table of values below by measuring the gradient y' of the tangent at the points where the height y is $\frac{1}{2}$, 1, 2 and 3.

height y	$\frac{1}{2}$	1	2	3
gradient y'				
$\dfrac{\text{gradient}}{\text{height}}$				

(d) What do you notice about the ratios of gradient to height?

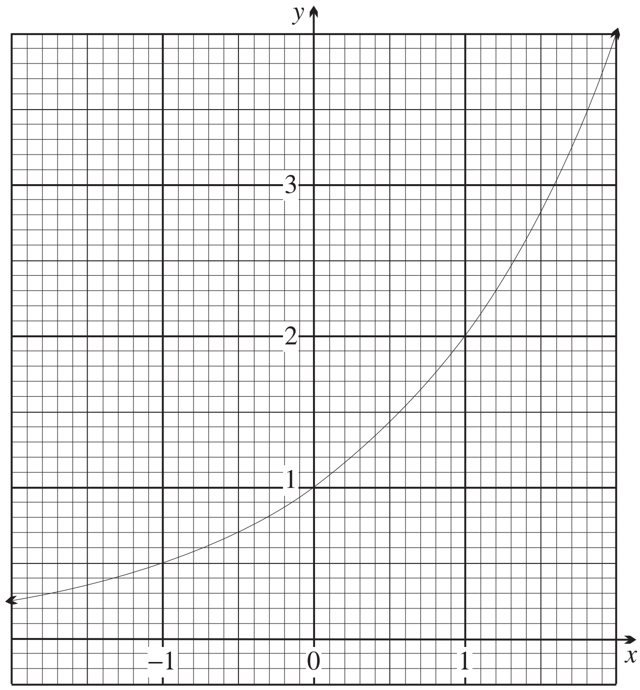
3. (a) Photocopy the graph of $y = e^x$ in question 2 and on it draw the tangent at $(0, 1)$, extending the tangent down to the x-axis.

(b) Measure the gradient of this tangent and confirm that it is equal to the height of the graph at $(0, 1)$.

(c) Draw the tangents at the three points where $x = -2$, $x = -1$ and $x = 1$. Measure the gradient of each of these tangents and confirm that it is equal to the height of each point of contact.

(d) What do you notice about the x-intercepts of all the tangents?

4.



(a) Photocopy the graph of $y = 2^x$ and on it draw tangents at $x = -1$, 0, 1 and 2.

(b) Copy and complete the table of values to the right by measuring the gradient y' of each tangent.

(c) What do you notice about the ratios of gradient to height?

x	-2	-1	0	1	2
height y					
gradient y'					
$\dfrac{\text{gradient}}{\text{height}}$					

DEVELOPMENT

5. Sketch the graph of $y = e^x$, then use your knowledge of transformations to graph the following functions. Show the horizontal asymptote and state the range in each case.

(a) $y = e^x + 1$ (b) $y = e^x + 2$ (c) $y = e^x - 1$ (d) $y = e^x - 2$

6. Sketch the graph of $y = e^{-x}$, then use your knowledge of transformations to graph the following functions. Show the horizontal asymptote and state the range in each case.

(a) $y = e^{-x} + 1$ (b) $y = e^{-x} + 2$ (c) $y = e^{-x} - 1$ (d) $y = e^{-x} - 2$

7. (a) What is the y-coordinate of the point on the curve $y = e^x$ where $x = 0$?

(b) Use the result $\dfrac{d}{dx}\, e^x = e^x$ to find the gradient of the tangent at this point.

(c) Hence write down the equation of the tangent, and find its x-intercept.

(d) Repeat the above steps for the points where $x = -2$, -1 and 1.

(e) Compare the values of the x-intercepts with those found in question 3.

8. (a) Sketch a graph of $y = e^x$ and hence write down its range.

(b) Write down y' and hence explain why the graph always has positive gradient.

(c) Write down y'' and hence explain why the graph is always concave up.

9. (a) Copy and complete the following tables of values for the functions $y = e^x$ and $y = e^{-x}$, giving your answers correct to two decimal places.

x	-2	-1	0	1	2
e^x					

x	-2	-1	0	1	2
e^{-x}					

(b) Sketch both graphs on one number plane, and draw the tangents at each y-intercept.

(c) Find the gradients of the two tangents, and hence explain why they are perpendicular.

10. Use the graph of $y = e^x$ and your knowledge of transformations to graph:

(a) $y = e^{x-1}$ (c) $y = e^{x+1}$ (e) $y = -e^x$

(b) $y = e^{x-3}$ (d) $y = e^{x+2}$ (f) $y = e^{-x}$

11. (a) Photocopy the graph of $y = 2^x$ in question 4 and on it draw the chord from $x = -1$ to $x = 0$, and another from $x = 0$ to $x = 1$. Find the gradients of these chords.

(b) Draw the tangent at $x = 0$ and compare it with the chords. Hence explain why the gradient of the tangent lies between $\frac{1}{2}$ and 1.

(c) Measure the gradient of the tangent to confirm this.

CHALLENGE

12. [Technology]

(a) A graphing program can be used to draw tangents to a graph of $y = e^x$ at various points on the curve and confirm that at each point the gradient equals the height. This exercise was done on graph paper in questions 2 and 3 above.

(b) The transformations in questions 5, 6, 10, 16 and 17 of the graph of the exponential function $y = e^x$ can be confirmed using a graphing program, after which experimentation with further transformed graphs can be carried out.

13. [Technology] On page 65 it was shown from first principles that the gradient of $y = 2^x$ at its y-intercept is given by $\lim\limits_{h \to 0} \dfrac{2^h - 1}{h}$. Use a calculator or computer to evaluate $\dfrac{2^h - 1}{h}$, correct to five decimal places, for the following values of h:

(a) 1 (b) 0·1 (c) 0·01 (d) 0·001 (e) 0·0001 (f) 0·000 01

14. [Technology]

(a) Use a graphing program to graph $y = 2^x$ and $y = x + 1$ on the same number plane, and hence observe that the gradient of $y = 2^x$ at $(0, 1)$ is less than 1.

(b) Similarly, graph $y = 3^x$ and $y = x + 1$ on the same number plane and hence observe that the gradient of $y = 3^x$ at $(0, 1)$ is greater than 1.

(c) Choose a sequence of bases between 2 and 3 so that the gradient at $(0, 1)$ converges to exactly 1. In this way a reasonable approximation for e can be obtained.

15. Use the graph of $y = e^x$ and your knowledge of transformations to graph the following functions. Show the horizontal asymptote and state the range in each case.

(a) $y = 1 - e^x$ (b) $y = 3 - e^x$ (c) $y = -e^{-x}$ (d) $y = -e^{x-1}$

16. The function e^x may be approximated using the power series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4} + \cdots .$$

Use this power series to approximate each of the following, correct to two decimal places. Then compare your answers with those given by your calculator.

(a) e (b) e^{-1} (c) e^2 (d) e^{-2}

2 C Differentiation of Exponential Functions

Now that the new standard form $\dfrac{d}{dx}e^x = e^x$ has been established, the familiar chain, product and quotient rules can now be applied to functions involving e^x.

Using the Basic Standard Form: The example below uses only the basic standard form for differentiation:

$$\frac{d}{dx}e^x = e^x.$$

WORKED EXERCISE:

(a) Differentiate $y = e^x - x$.

(b) Show that the tangent at the y-intercept is horizontal.

SOLUTION:

(a) The function is $y = e^x - x$.

Differentiating, $\dfrac{dy}{dx} = e^x - 1$, since the derivative of e^x is e^x.

(b) To find the gradient of the tangent at the y-intercept, substitute $x = 0$ into $\dfrac{dy}{dx}$.

When $x = 0$,
$$\begin{aligned}\frac{dy}{dx} &= e^0 - 1\\ &= 1 - 1\\ &= 0,\end{aligned}$$

so the tangent at the y-intercept is horizontal.

Standard Forms for Differentiation: We now apply the chain rule to differentiating functions like $y = e^{3x+4}$, where the index $3x + 4$ is a linear function.

WORKED EXERCISE:

Use the chain rule to differentiate:

(a) $y = e^{3x+4}$ (b) $y = e^{ax+b}$

SOLUTION:

(a) Let $y = e^{3x+4}$.

Then using the chain rule,
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= 3\,e^{3x+4}.\end{aligned}$$

Let $u = 3x + 4$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = e^u$.

(b) Let $y = e^{ax+b}$.

Then using the chain rule,
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= a\,e^{ax+b}.\end{aligned}$$

Let $u = ax + b$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = e^u$.

This situation occurs so often that the result should be learnt as a standard form.

STANDARD FORMS FOR DIFFERENTIATING EXPONENTIAL FUNCTIONS:

9
$$\frac{d}{dx}\, e^x = e^x$$
$$\frac{d}{dx}\, e^{ax+b} = a\, e^{ax+b}$$

WORKED EXERCISE:

Use the standard forms above to find the derivatives of:

(a) $y = e^{4x-7}$ (b) $y = 2\,e^{-9x}$ (c) $y = e^{\frac{1}{2}(6-x)}$

SOLUTION:

Each function needs to be written in the form $y = e^{ax+b}$.

(a) $y = e^{4x-7}$

$\dfrac{dy}{dx} = 4\,e^{4x-7}$ (Here $a = 4$ and $b = -7$.)

(b) $y = 2\,e^{-9x}$

$\dfrac{dy}{dx} = -18\,e^{-9x}$ (Here $a = -9$ and $b = 0$.)

(c) $y = e^{\frac{1}{2}(6-x)}$

$y = e^{3-\frac{1}{2}x}$ (Expand the brackets in the index.)

$\dfrac{dy}{dx} = -\frac{1}{2}\,e^{3-\frac{1}{2}x}$ (Here $a = -\frac{1}{2}$ and $b = 3$.)

Differentiating Using the Chain Rule: The chain rule is needed when exponential and algebraic functions are combined.

WORKED EXERCISE:

Use the chain rule to differentiate:

(a) $y = e^{1-x^2}$ (b) $y = (e^{2x} - 3)^4$

SOLUTION:

(a) Here $y = e^{1-x^2}$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -2x\,e^{1-x^2}.$$

Let $u = 1 - x^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = -2x$
and $\dfrac{dy}{du} = e^u$.

(b) Here $y = (e^{2x} - 3)^4$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4(e^{2x} - 3)^3 \times 2\,e^{2x}$$
$$= 8\,e^{2x}(e^{2x} - 3)^3.$$

Let $u = e^{2x} - 3$.
Then $y = u^4$.
Hence $\dfrac{du}{dx} = 2\,e^{2x}$
and $\dfrac{dy}{du} = 4u^3$.

Using the Product Rule: A function like $y = x^3\, e^x$ is the product of the two functions $u = x^3$ and $v = e^x$. Thus it can be differentiated by the product rule.

WORKED EXERCISE:

Find the derivatives of:

(a) $y = x^3\, e^x$ (b) $y = x\, e^{5x-2}$

SOLUTION:

(a) Here $y = x^3\, e^x$.

Applying the product rule,

$$\frac{dy}{dx} = v\,\frac{du}{dx} + u\,\frac{dv}{dx}$$

$$= e^x \times 3x^2 \;+\; x^3 \times e^x,$$

and taking out the common factor $x^2\, e^x$,

$$\frac{dy}{dx} = x^2\, e^x\,(3 + x).$$

Let $u = x^3$
and $v = e^x$.
Then $\dfrac{du}{dx} = 3x^2$
and $\dfrac{dv}{dx} = e^x$.

(b) Here $y = x\, e^{5x-2}$.

Applying the product rule,

$$y' = vu' + uv'$$

$$= e^{5x-2} \times 1 \;+\; x \times 5\, e^{5x-2}$$

and taking out the common factor e^{5x-2},

$$y' = e^{5x-2}(1 + 5x).$$

Let $u = x$
and $v = e^{5x-2}$.
Then $u' = 1$
and $v' = 5\, e^{5x-2}$.

Using the Quotient Rule: A function like $y = \dfrac{e^{5x}}{x}$ is the quotient of the two functions $u = e^{5x}$ and $v = x$. Thus it can be differentiated by the quotient rule.

WORKED EXERCISE:

Find the derivatives of:

(a) $\dfrac{e^{5x}}{x}$ (b) $\dfrac{e^x}{1 - x^2}$

SOLUTION:

(a) Let $y = \dfrac{e^{5x}}{x}$. Then applying the quotient rule,

$$\frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$

$$= \frac{5x\, e^{5x} - e^{5x}}{x^2}$$

and taking out the common factor e^{5x} in the numerator,

$$\frac{dy}{dx} = \frac{e^{5x}\,(5x - 1)}{x^2}\,.$$

Let $u = e^{5x}$
and $v = x$.
Then $\dfrac{du}{dx} = 5\, e^{5x}$
and $\dfrac{dv}{dx} = 1$.

(b) Let $y = \dfrac{e^x}{1-x^2}$. Then applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{(1-x^2)e^x + 2x\,e^x}{(1-x^2)^2}$$

and taking out the common factor e^x in the numerator,

$$y' = \frac{e^x(1+2x-x^2)}{(1-x^2)^2}.$$

Let $u = e^x$
and $v = 1 - x^2$.
Then $u' = e^x$
and $v' = -2x$.

Exercise 2C

TECHNOLOGY: Programs that perform algebraic differentiation can be used to confirm the answers to many of these exercises.

1. Use the standard form $\dfrac{d}{dx}\,e^{ax+b} = a\,e^{ax+b}$ to differentiate:

(a) e^{2x} (b) e^{7x} (c) $4\,e^{3x}$ (d) e^{-x} (e) $-e^{5x}$ (f) $-5\,e^{2x}$ (g) $e^{\frac{1}{2}x}$ (h) $6\,e^{\frac{1}{3}x}$ (i) $-e^{-7x}$ (j) $e^{-\frac{1}{3}x}$ (k) $5\,e^{\frac{1}{5}x}$ (l) $-\frac{1}{2}\,e^{-2x}$

2. Using the standard form $\dfrac{d}{dx}\,e^{ax+b} = a\,e^{ax+b}$ to differentiate:

(a) $y = e^{x+2}$ (b) $y = e^{x-3}$ (c) $y = e^{3x+4}$ (d) $y = e^{5x+1}$ (e) $y = e^{2x-1}$ (f) $y = e^{4x-3}$ (g) $y = e^{-4x+1}$ (h) $y = e^{-3x+4}$ (i) $y = e^{-2x-7}$ (j) $y = e^{-3x-6}$ (k) $y = 2\,e^{\frac{1}{2}x+4}$ (l) $y = \frac{1}{3}\,e^{3x-2}$

3. Differentiate:

(a) $e^x + e^{-x}$ (b) $e^{2x} - e^{-3x}$ (c) $\dfrac{e^x - e^{-x}}{2}$ (d) $\dfrac{e^x + e^{-x}}{3}$ (e) $\dfrac{e^{2x}}{2} + \dfrac{e^{3x}}{3}$ (f) $\dfrac{e^{4x}}{4} + \dfrac{e^{5x}}{5}$

4. Use the index laws to write each expression as a single power of e and then differentiate it.

(a) $y = e^x \times e^{2x}$ (b) $y = e^{3x} \times e^{-x}$ (c) $y = (e^x)^2$ (d) $y = \left(e^{2x}\right)^3$ (e) $y = \dfrac{e^{4x}}{e^x}$ (f) $y = \dfrac{e^x}{e^{2x}}$ (g) $y = \dfrac{1}{e^{3x}}$ (h) $y = \dfrac{1}{e^{5x}}$

5. In each part, first find $\dfrac{dy}{dx}$. Then evaluate $\dfrac{dy}{dx}$ at $x = 2$, giving your answer first in exact form and then correct to two decimal places.

(a) $y = e^x$ (b) $y = e^{3x}$ (c) $y = e^{-x}$ (d) $y = e^{-2x}$ (e) $y = e^{\frac{1}{2}x}$ (f) $y = e^{\frac{3}{2}x}$

6. Consider the function $f(x) = e^{-x}$.

(a) Find: (i) $f'(x)$ (ii) $f''(x)$ (iii) $f'''(x)$ (iv) $f^{(4)}(x)$

(b) What is the pattern in these derivatives?

7. Consider the function $f(x) = e^{2x}$.

(a) Find: (i) $f'(x)$ (ii) $f''(x)$ (iii) $f'''(x)$ (iv) $f^{(4)}(x)$

(b) What is the pattern in these derivatives?

DEVELOPMENT

8. Expand the brackets and then differentiate:

(a) $e^x(e^x+1)$
(b) $e^x(e^x-1)$
(c) $e^{-x}(2\,e^{-x}-1)$
(d) $(e^x+1)^2$
(e) $(e^x+3)^2$
(f) $(e^x-1)^2$
(g) $(e^x-2)^2$
(h) $(e^x+e^{-x})(e^x-e^{-x})$
(i) $(e^{5x}+e^{-5x})(e^{5x}-e^{-5x})$

9. Use the chain rule to differentiate:

(a) e^{2x+1}
(b) e^{3x-5}
(c) e^{1-2x}
(d) e^{3-5x}
(e) e^{x^2}
(f) e^{-x^2}
(g) e^{x^2+1}
(h) e^{1-x^2}
(i) e^{x^2+2x}
(j) e^{6+x-x^2}
(k) $\frac{1}{2}\,e^{3x^2-2x+1}$
(l) $\frac{1}{4}\,e^{2x^2-4x+3}$

10. Use the product rule to differentiate:

(a) $x\,e^x$
(b) $x\,e^{-x}$
(c) $(x-1)\,e^x$
(d) $(x+1)\,e^{3x-4}$
(e) $x^2\,e^{-x}$
(f) $(2x-1)\,e^{2x}$
(g) $(x^2-5)\,e^x$
(h) $x^3\,e^{2x}$

11. Use the quotient rule to differentiate:

(a) $y=\dfrac{e^x}{x}$
(b) $y=\dfrac{x}{e^x}$
(c) $y=\dfrac{e^x}{x^2}$
(d) $y=\dfrac{x^2}{e^x}$
(e) $y=\dfrac{e^x}{x+1}$
(f) $y=\dfrac{x+1}{e^x}$
(g) $y=\dfrac{x-3}{e^{2x}}$
(h) $y=\dfrac{1-x^2}{e^x}$

12. Expand and simplify each expression, then differentiate it:

(a) $(e^x+1)(e^x+2)$
(b) $(e^{2x}+3)(e^{2x}-2)$
(c) $(e^{-x}+2)(e^{-x}+4)$
(d) $(e^{-3x}-1)(e^{-3x}-5)$
(e) $(e^{2x}+1)(e^x+1)$
(f) $(e^{3x}-1)(e^{-x}+4)$

13. Use the chain rule to differentiate:

(a) $(1-e^x)^5$
(b) $(e^{4x}-9)^4$
(c) $\dfrac{1}{e^x-1}$
(d) $\dfrac{1}{(e^{3x}+4)^2}$

14. (a) Show by substitution that the function $y=e^{5x}$ satisfies the equation $\dfrac{dy}{dx}=5y$.

(b) Show by substitution that the function $y=3\,e^{2x}$ satisfies the equation $\dfrac{dy}{dx}=2y$.

(c) Show by substitution that the function $y=5\,e^{-4x}$ satisfies the equation $\dfrac{dy}{dx}=-4y$.

(d) Show by substitution that the function $y=2\,e^{-3x}$ satisfies the equation $\dfrac{dy}{dx}=-3y$.

15. Determine the first and second derivatives of each function below. Then evaluate both derivatives at the value given.

(a) $f(x)=e^{2x+1}$, $x=0$
(b) $f(x)=e^{-3x}$, $x=1$
(c) $f(x)=x\,e^{-x}$, $x=2$
(d) $f(x)=e^{-x^2}$, $x=0$

16. Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y=e^x$ at the points where:

(a) $x=0$
(b) $x=1$
(c) $x=-2$
(d) $x=5$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

17. Use the standard form $\dfrac{d}{dx}\,e^{ax+b} = a\,e^{ax+b}$ to differentiate:

(a) $y = e^{ax}$ (c) $y = Ae^{kx}$ (e) $y = e^{px+q}$ (g) $y = \dfrac{e^{px} + e^{-qx}}{r}$

(b) $y = e^{-kx}$ (d) $y = Be^{-\ell x}$ (f) $y = Ce^{px+q}$ (h) $\dfrac{e^{ax}}{a} + \dfrac{e^{-px}}{p}$

18. Use the product, quotient and chain rules as appropriate to differentiate:

(a) $(e^x + 1)^3$ (c) $(1 + x^2)\,e^{1+x}$ (e) $\dfrac{e^x}{e^x + 1}$

(b) $(e^x + e^{-x})^4$ (d) $(x^2 - x)\,e^{2x-1}$ (f) $\dfrac{e^x + 1}{e^x - 1}$

19. Write each expression as the sum of simple powers of e, and then differentiate it.

(a) $y = \dfrac{e^x + 1}{e^x}$ (c) $y = \dfrac{2 - e^x}{e^{2x}}$ (e) $y = \dfrac{e^x + e^{2x} - 3\,e^{4x}}{e^x}$

(b) $y = \dfrac{e^{2x} + e^x}{e^x}$ (d) $y = \dfrac{3 + e^x}{e^{4x}}$ (f) $y = \dfrac{e^{2x} + 2\,e^x + 1}{e^{2x}}$

CHALLENGE

20. Write each expression as a simple power of e, and then differentiate it.

(a) $y = \sqrt{e^x}$ (b) $y = \sqrt[3]{e^x}$ (c) $y = \dfrac{1}{\sqrt{e^x}}$ (d) $y = \dfrac{1}{\sqrt[3]{e^x}}$

21. Use the chain rule to differentiate:

(a) $e^{\sqrt{x}}$ (c) $e^{\frac{1}{x}}$ (e) $e^{x-\frac{1}{x}}$

(b) $e^{-\sqrt{x}}$ (d) $e^{-\frac{1}{x}}$ (f) $e^{3-\frac{1}{x^2}}$

22. We define two new functions, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

(a) Show that $\dfrac{d}{dx}\cosh x = \sinh x$ and $\dfrac{d}{dx}\sinh x = \cosh x$.

(b) Find the second derivative of each function, and show that they both satisfy $y'' = y$.

(c) Show that $\cosh^2 x - \sinh^2 x = 1$.

23. (a) Show that $y = 2\,e^{3x}$ is a solution of each of the following equations by substituting separately into the LHS and RHS:

(i) $y' = 3y$ (ii) $y'' - 9y = 0$

(b) Show that $y = \frac{1}{2}\,e^{-3x} + 4$ is a solution of $\dfrac{dy}{dx} = -3(y - 4)$ by substituting y into each side of the equation.

(c) Show by substitution that: (i) $y = e^{-x}$ and (ii) $y = x\,e^{-x}$ are solutions of the equation $y'' + 2y' + y = 0$.

24. Find the values of λ that make $y = e^{\lambda x}$ a solution of:

(a) $y'' + 3y' - 10y = 0$ (b) $y'' + y' - y = 0$

2 D Applications of Differentiation

Differentiation can now be applied in the usual ways to functions involving e^x. Sketching of curves whose equations involve e^x is an important application. Some of these sketches require some subtle limits which are beyond the 2 Unit course — they would normally be given if a question needed them.

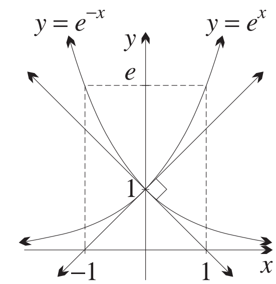
The Graphs of e^x and e^{-x}: The graphs of $y = e^x$ and $y = e^{-x}$ are the fundamental graphs of this chapter. Since x is replaced by $-x$ in the second equation, the two graphs are reflections of each other in the y-axis.

For $y = e^x$:

x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$. The gradient of $y = e^x$ at $(0, 1)$ is 1, and so by reflection, the gradient of $y = e^{-x}$ at $(0, 1)$ must be -1. This means that the curves are perpendicular at their point of intersection.

NOTE: The function $y = e^{-x}$ is as important as $y = e^x$ in applications, or even more important. It describes a great many physical situations where a quantity 'dies away exponentially', like the dying away of the sound of a plucked string.

The Geometry of Tangents and Normals: The following worked exercise shows how to work with tangents and normals of exponential functions.

WORKED EXERCISE:
Let A be the point on the curve $y = 2\,e^x$ where $x = 1$.
(a) Find the equation of the tangent to the curve at the point A.
(b) Show that the tangent at A passes through the origin.
(c) Find the equation of the normal at the point A.
(d) Find the point B where this normal crosses the x-axis.
(e) Find the area of $\triangle AOB$.

SOLUTION:
(a) The given function is $y = 2\,e^x$
and differentiating, $y' = 2\,e^x$.
Substituting $x = 1$, $y = 2\,e^1$
$= 2\,e$,
and $y' = 2\,e^1$
$= 2\,e$,
so A has coordinates $A(1, 2e)$ and the tangent at A has gradient $2e$.
Hence using point–gradient form,
the tangent at A is $y - 2e = 2e(x - 1)$
$y = 2ex$.

(b) Since its y-intercept is zero, the tangent passes through the origin.

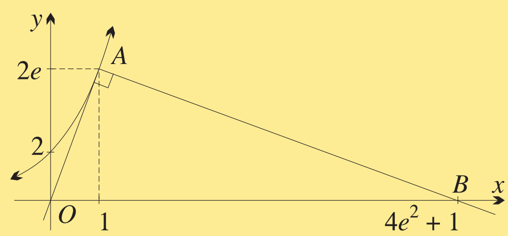
(c) From part (a), the tangent has gradient $2e$,

so the normal has gradient $-\dfrac{1}{2e}$.

Thus the normal is $y - 2e = -\dfrac{1}{2e}(x-1)$

$$2ey - 4e^2 = -x + 1$$

$$x + 2ey = 4e^2 + 1.$$



(d) To find the x-intercept, put $y = 0$,

thus $x = 4e^2 + 1$,

so B has coordinates $(4e^2 + 1,\, 0)$.

(e) Hence
$$\begin{aligned}\text{area } \triangle AOB &= \tfrac{1}{2} \times \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times (4e^2+1) \times 2e\\ &= e(4e^2+1) \text{ square units.}\end{aligned}$$

An Example of Curve Sketching: The following curve-sketching exercise illustrates the use of the six steps of the 'curve-sketching menu' in the context of exponential functions. One special limit is given so that the sketch may be completed.

WORKED EXERCISE:

Sketch the graph of $y = x\,e^{-x}$ after carrying out the following steps:

(a) Write down the domain.

(b) Test whether the function is even or odd or neither.

(c) Find any zeroes of the function and examine its sign.

(d) Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. [HINT: You may assume that $x\,e^{-x} \to 0$ as $x \to \infty$.]

(e) Find any stationary points and examine their nature.

(f) Find any points of inflexion.

SOLUTION:

(a) The domain of $y = x\,e^{-x}$ is the whole real number line.

(b) $f(-x) = -x\,e^{x}$, which is neither $f(x)$ nor $-f(x)$, so the function is neither even nor odd.

(c) The only zero is $x = 0$. Since e^{-x} is always positive,

y is positive for $x > 0$ and negative for $x < 0$.

(d) As given in the hint, $y \to 0$ as $x \to \infty$.

Also, $y \to -\infty$ as $x \to -\infty$.

(e) Differentiating using the product rule,

$$\begin{aligned}f'(x) &= vu' + uv'\\ &= e^{-x} - x\,e^{-x}\\ &= e^{-x}(1-x),\end{aligned}$$

Let $u = x$
and $v = e^{-x}$.
Then $u' = 1$
and $v' = -e^{-x}$.

Hence $f'(x) = 0$ when $x = 1$ (notice that e^{-x} can never be zero),

so $(1, \frac{1}{e})$ is the only stationary point.

Differentiating again by the product rule,

$$f''(x) = vu' + uv'$$
$$= -e^{-x} - (1-x)\,e^{-x}$$
$$= e^{-x}(x-2),$$

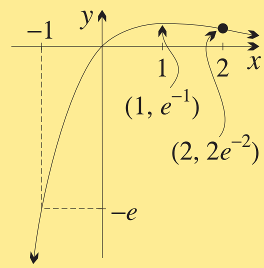
Let $u = 1 - x$
and $v = e^{-x}$.
Then $u' = -1$
and $v' = -e^{-x}$.

so $f''(1) = -e^{-1} < 0$, and so $(1, e^{-1})$ is a maximum turning point.

(f) $f''(x) = e^{-x}(x-2)$ has a zero at $x = 2$,
and taking test points around $x = 2$,

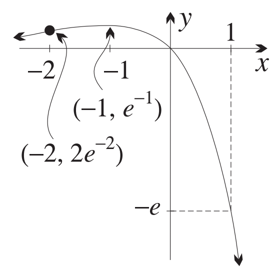
x	0	2	3
$f''(x)$	-2	0	e^{-3}
	$\frown$	$\cdot$	$\smile$

Thus there is an inflexion at $(2\,,\ 2e^{-2}) \doteqdot (2, 0{\cdot}27)$.



WORKED EXERCISE: [Transforming graphs]
Use a suitable transformation of the graph sketched in the previous worked exercise to sketch $y = -x\,e^x$.

SOLUTION:
$y = x\,e^{-x}$ becomes $y = -x\,e^x$ when x is replaced by $-x$.
Graphically, this transformation is a reflection in the y-axis, hence the new graph is as sketched to the right.



A Difficulty with the Limits of $x\,e^x$ and $x\,e^{-x}$: Sketching the graph of $y = x\,e^{-x}$ above required knowing the behaviour of $x\,e^{-x}$ as $x \to \infty$. This limit is puzzling, because when x is a large number, e^{-x} is a small positive number, and the product of a large number and a small number could be large, small, or anything in between.

In fact, e^{-x} gets small as $x \to \infty$ much more quickly than x gets large and the product $x\,e^{-x}$ gets small. The technical term for this is that e^{-x} *dominates* x. The following table of values should make it reasonably clear that $\lim\limits_{x \to \infty} x\,e^{-x} = 0$:

x	0	1	2	3	4	5	6	7	...
xe^{-x}	0	$\dfrac{1}{e}$	$\dfrac{2}{e^2}$	$\dfrac{3}{e^3}$	$\dfrac{4}{e^4}$	$\dfrac{5}{e^5}$	$\dfrac{6}{e^6}$	$\dfrac{7}{e^7}$	...
approx	0	0·37	0·27	0·15	0·073	0·034	0·015	0·006	...

Limits like this are usually not regarded as part of the 2 Unit course and would normally be given in any curve-sketching question where they were required.

Similarly, when x is a large negative number, e^x is a very small number, and so it is unclear whether $x\,e^x$ is large or small. Again, e^x dominates x, meaning that $x\,e^x \to 0$ as $x \to -\infty$. A very similar table should make this reasonably obvious:

x	0	-1	-2	-3	-4	-5	-6	-7	...
xe^x	0	$-\dfrac{1}{e}$	$-\dfrac{2}{e^2}$	$-\dfrac{3}{e^3}$	$-\dfrac{4}{e^4}$	$-\dfrac{5}{e^5}$	$-\dfrac{6}{e^6}$	$-\dfrac{7}{e^7}$	...
approx	0	$-0{\cdot}37$	$-0{\cdot}27$	$-0{\cdot}15$	$-0{\cdot}073$	$-0{\cdot}034$	$-0{\cdot}015$	$-0{\cdot}006$	...

Again, this limit would normally be given in any question where it arose.

Exercise 2D

TECHNOLOGY: Graphing programs can be used in this exercise to sketch the curves and then to investigate the effects on the curve of making small changes in the equations. It is advisable, however, to puzzle out most of the graphs first using the standard methods of the curve-sketching menu.

1. In this question you will need the point–gradient formula $y - y_1 = m(x - x_1)$ for the equation of a straight line.
(a) Use calculus to find the gradient of the tangent to $y = e^x$ at $P(1, e)$.
(b) Hence find the equation of the tangent at P, and prove that it passes through O.

2. (a) Use calculus to find the gradient of the tangent to $y = e^x$ at $Q(0, 1)$.
(b) Hence find the equation of the tangent at Q, and prove that it passes through $A(-1, 0)$.

3. (a) Use calculus to find the gradient of the tangent to $y = e^x$ at $R(-1, \frac{1}{e})$.
(b) Hence find the equation of the tangent at R, and prove that it passes through $B(-2, 0)$.

4. (a) Find the y-coordinate of the point A on the curve $y = e^{2x-1}$ where $x = \frac{1}{2}$.
(b) Find the derivative of $y = e^{2x-1}$, and show that the gradient of the tangent at A is 2.
(c) Hence find the equation of the tangent at A, and prove that it passes through O.

5. (a) Write down the coordinates of the point R on the curve $y = e^{3x+1}$ where $x = -\frac{1}{3}$.
(b) Find $\dfrac{dy}{dx}$ and hence show that the gradient of the tangent at R is 3.
(c) What is the gradient of the normal at R?
(d) Hence find the equation of the normal at R in general form.

6. (a) What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?
(b) Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.
(c) Hence find the equation of the tangent at the point P found in part (a).

7. (a) Find the gradient of the tangent to $y = e^{-x}$ at the point $P(-1, e)$.
(b) Thus write down the gradient of the normal at this point.
(c) Hence determine the equation of this normal.
(d) Find the x- and y-intercepts of the normal.
(e) Find the area of the triangle whose vertices lie at the intercepts and the origin.

8. (a) Use the derivative to find the gradient of the tangent to $y = e^x$ at $B(0, 1)$.
(b) Hence find the equation of this tangent and show that it meets the x-axis at $F(-1, 0)$.
(c) Use the derivative to find the gradient of the tangent to $y = e^{-x}$ at $B(0, 1)$.
(d) Hence find the equation of this tangent and show that it meets the x-axis at $G(1, 0)$.
(e) Sketch $y = e^x$ and $y = e^{-x}$ on the same set of axes, showing the two tangents.
(f) What sort of triangle is $\triangle BFG$, and what is its area?

DEVELOPMENT

9. (a) Find the gradient of the tangent to $y = x - e^x$ at $x = 1$.
(b) Write down the equation of the tangent, and show that it passes through the origin.

10. (a) Find the equation of the tangent to $y = (1 - x)\, e^x$ at $x = -1$.
(b) Hence find the x-intercept of the tangent.

11. (a) Show that the equation of the tangent to $y = (x+1)\,e^{-x}$ at $x = -1$ is $y = e(x+1)$.
(b) Find the x-intercept and y-intercept of the tangent.
(c) Hence find the area of the triangle with its vertices at the two intercepts and the origin.

12. (a) Find the derivative of $y = e^{3x-6}$.
(b) Explain why every tangent to the curve has positive gradient.
(c) Find the point on the curve where the gradient is 3.
(d) Find the gradients of the tangent and normal at the y-intercept.

13. (a) Find the equation of the normal to $y = e^{-x^2}$ at the point where $x = 1$.
(b) Determine the x-intercept of the normal.

14. (a) Find the equation of the tangent to $y = e^x$ at $x = t$.
(b) Hence show that the x-intercept of this tangent is $t - 1$.

15. (a) Show that the equation of the tangent to $y = 1 - e^{-x}$ at the origin is $y = x$.
(b) Deduce the equation of the normal at the origin without further use of calculus.
(c) What is the equation of the asymptote of this curve?
(d) Sketch the curve, showing the points T and N where the tangent and normal respectively cut the asymptote.
(e) Find the area of $\triangle OTN$.

16. (a) Find the first and second derivatives for the curve $y = x - e^x$.
(b) Deduce that the curve is concave down for all values of x.
(c) Find any stationary points and determine their nature.
(d) Sketch the curve and write down its range.
(e) Finally, sketch $y = e^x - x$ by recognising the simple transformation.

17. [Technology]
(a) Use your calculator to complete the table of values for $y = \dfrac{e^x}{x}$ to the right. [NOTE: This table is intended to confirm that $\dfrac{e^x}{x} \to \infty$ as $x \to \infty$.]

x	2	5	10	20	40
y					

(b) Use your calculator to complete the table of values for $y = x\,e^x$ to the right. Then use the table to help you guess the value of $\lim\limits_{x\to-\infty} x\,e^x$.

x	-2	-5	-10	-20	-40
y					

(c) Use your calculator to complete the table of values for $y = \dfrac{e^{-x}}{x}$ to the right. [NOTE: This table is intended to confirm that $\dfrac{e^{-x}}{x} \to -\infty$ as $x \to -\infty$.]

x	-2	-5	-10	-20	-40
y					

(d) Use your calculator to complete the table of values for $y = x\,e^{-x}$ to the right. Then use the table to help you guess the value of $\lim\limits_{x\to\infty} x\,e^{-x}$.

x	2	5	10	20	40
y					

18. Consider the curve $y = x\,e^x$.
(a) Show that $y' = (1+x)\,e^x$ and $y'' = (2+x)\,e^x$.
(b) Show that there is one stationary point, and determine its nature.
(c) Find the coordinates of the lone point of inflexion.
(d) Given that $y \to 0$ as $x \to -\infty$, sketch the curve, then write down its range.
(e) Hence also sketch $y = -x\,e^{-x}$ by recognising the simple transformation.

19. (a) Given that $y = e^{-\frac{1}{2}x^2}$, show that $y' = -xe^{-\frac{1}{2}x^2}$ and $y'' = (x^2-1)\,e^{-\frac{1}{2}x^2}$.
(b) Show that the function is even.
(c) Show that this curve has a maximum turning point at its y-intercept, and has two points of inflexion.
(d) Given that $y \to 0$ as $x \to \infty$, sketch the graph and write down its range.
(e) Hence also sketch $y = 1 + e^{-\frac{1}{2}x^2}$ by recognising the simple transformation.

20. (a) Given that $y = (1-x)\,e^x$, show that $y' = -x\,e^x$ and $y'' = -(x+1)\,e^x$.
(b) Show that this curve has a maximum turning point at its y-intercept, and an inflexion point at $(-1, 2\,e^{-1})$.
(c) Given that $y \to 0$ as $x \to -\infty$, sketch the graph and write down its range.

CHALLENGE

21. We define the new function $\cosh x = \dfrac{e^x + e^{-x}}{2}$.
(a) Show that $y = \cosh x$ is an even function.
(b) Find $\dfrac{dy}{dx}$ and show there is a stationary point at the y-intercept.
(c) Show that the function is always concave up.
(d) Sketch the graph of $y = \cosh x$. You may assume that $y \to \infty$ as $x \to \infty$.

22. (a) Given that $y = x^2\,e^{-x}$, show that $y' = x(2-x)\,e^{-x}$ and $y'' = (2-4x+x^2)\,e^{-x}$.
(b) Show that the function has a minimum turning point at the origin and a maximum turning point at $(2, 4\,e^{-2})$.
(c) (i) Show that $y'' = 0$ at $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$.
(ii) Use a table of values for y'' to show that there are inflexion points at these values.
(d) Given that $y \to 0$ as $x \to \infty$, sketch the graph and write down its range.

2 E Integration of Exponential Functions

Finding primitives is the reverse of differentiation. Thus the new standard forms for differentiating functions involving e^x can now be reversed to provide standard forms for integration.

Standard Forms for Integration: Reversing the standard forms for differentiating exponential funtions will give the standard forms for integrating them.

First, the derivative of e^x is e^x.

That is, $\dfrac{d}{dx}\,e^x = e^x$.

Reversing this, $\displaystyle\int e^x\,dx = e^x + C$, for some constant C.

Secondly, the derivative of e^{ax+b} is $a\,e^{ax+b}$, that is

$$\frac{d}{dx}\,e^{ax+b} = a\,e^{ax+b}.$$

Reversing this gives $\displaystyle\int a\,e^{ax+b} = e^{ax+b},$

and dividing through by a, $\displaystyle\int e^{ax+b} = \frac{1}{a}\,e^{ax+b} + C,$ for some constant C.

These two standard forms for integration need to be memorised:

10 STANDARD FORMS FOR INTEGRATION:

$$\int e^x\,dx = e^x + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + C$$

WORKED EXERCISE:

Find the following indefinite integrals:

(a) $\displaystyle\int e^{3x+2}\,dx$ (b) $\displaystyle\int (1 - x + e^x)\,dx$

SOLUTION:

(a) $\displaystyle\int e^{3x+2}\,dx = \tfrac{1}{3}\,e^{3x+2} + C$ (Here $a = 3$ and $b = 2$.)

(b) $\displaystyle\int (1 - x + e^x)\,dx = x - \tfrac{1}{2}x^2 + e^x + C$ (Integrate each term separately.)

Definite Integrals: Definite integrals are evaluated in the usual way by finding the primitive and substituting.

WORKED EXERCISE:

Evaluate the following definite integrals:

(a) $\displaystyle\int_0^2 e^x\,dx$ (b) $\displaystyle\int_2^3 e^{5-2x}\,dx$

SOLUTION:

(a) $$\begin{aligned}\int_0^2 e^x\,dx &= \Big[e^x\Big]_0^2 \\ &= e^2 - e^0 \\ &= e^2 - 1.\end{aligned}$$

(b) $$\begin{aligned}\int_2^3 e^{5-2x}\,dx &= -\tfrac{1}{2}\Big[e^{5-2x}\Big]_2^3 \qquad \text{(Here } a = -2 \text{ and } b = 5.\text{)} \\ &= -\tfrac{1}{2}(e^{-1} - e) \\ &= -\frac{1}{2}\left(\frac{1}{e} - e\right) \\ &= \frac{e^2 - 1}{2e}\,.\end{aligned}$$

Given the Derivative, Find the Function: As before, if the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be determined.

WORKED EXERCISE:

It is known that $f'(x) = e^x$ and also that $f(1) = 0$.

(a) Find the original function $f(x)$.

(b) Hence find $f(0)$.

SOLUTION:

(a) It is given that $f'(x) = e^x$.

Taking the primitive, $f(x) = e^x + C$, for some constant C.

It is known that $f(1) = 0$, so substituting $x = 1$,

$$0 = e^1 + C$$
$$C = -e.$$

Hence $f(x) = e^x - e$.

(b) Substituting $x = 0$ into this function,

$$\begin{aligned} f(0) &= e^0 - e \\ &= 1 - e. \end{aligned}$$

WORKED EXERCISE:

(a) If $f'(x) = 1 + 2e^{-x}$ and $f(0) = 1$, find $f(x)$.

(b) Hence find $f(1)$.

SOLUTION:

(a) It is given that $f'(x) = 1 + 2e^{-x}$.

Taking the primitive, $f(x) = x - 2e^{-x} + C$, for some constant C.

It is known that $f(0) = 1$, so substituting $x = 0$,

$$1 = 0 - 2\,e^0 + C$$
$$1 = 0 - 2 + C$$
$$C = 3.$$

Hence $f(x) = x - 2e^{-x} + 3$.

(b) Substituting $x = 1$ into this function,

$$\begin{aligned} f(1) &= 1 - 2e^{-1} + 3 \\ &= 4 - 2e^{-1}. \end{aligned}$$

Given a Derivative, Find an Integral: The result of any differentiation can be reversed. This often allows a new primitive to be found.

WORKED EXERCISE: [These questions are always hard.]

(a) Use the chain rule to differentiate e^{x^2}.

(b) Hence find $\displaystyle\int_{-1}^{1} 2x\, e^{x^2}\, dx$.

SOLUTION:

(a) Let $y = e^{x^2}$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2x\, e^{x^2}.$$

Let $u = x^2$.
Then $y = e^u$.
Hence $\frac{du}{dx} = 2x$
and $\frac{dy}{du} = e^u$.

(b) From part (a), $\frac{d}{dx} e^{x^2} = 2x\, e^{x^2}$.

Reversing this to give a primitive,

$$\int 2x\, e^{x^2}\, dx = e^{x^2}.$$

Hence
$$\int_{-1}^{1} 2x\, e^{x^2}\, dx = \Big[e^{x^2}\Big]_{-1}^{1} = e^1 - e^1 = 0.$$

NOTE: The fact that the definite integral is zero could have been discovered without ever finding the primitive. The function $f(x) = x\, e^{x^2}$ is an odd function, because

$$f(-x) = (-x)\, e^{(-x)^2} = -x\, e^{x^2} = -f(x).$$

Hence the definite integral over the interval $-1 \le x \le 1$ must be zero.

Exercise 2E

TECHNOLOGY: Some algebraic programs can display the primitive and evaluate the exact value of an integral. These can be used to check the questions in this exercise and also to investigate the effect of making small changes to the function or to the bounds.

1. Use the standard form $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$ to find each indefinite integral:

(a) $\int e^{2x}\, dx$
(b) $\int e^{3x}\, dx$
(c) $\int e^{\frac{1}{3}x}\, dx$
(d) $\int e^{\frac{1}{2}x}\, dx$
(e) $\int 10\, e^{2x}\, dx$
(f) $\int 12\, e^{3x}\, dx$

2. Use the standard form $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$ to find each indefinite integral:

(a) $\int e^{4x+5}\, dx$
(b) $\int e^{3x+1}\, dx$
(c) $\int e^{4x-2}\, dx$
(d) $\int e^{x-1}\, dx$
(e) $\int 6\, e^{3x+2}\, dx$
(f) $\int 15\, e^{5x+1}\, dx$
(g) $\int 2\, e^{2x-1}\, dx$
(h) $\int 4\, e^{4x+3}\, dx$
(i) $\int e^{3-x}\, dx$
(j) $\int e^{7-2x}\, dx$
(k) $\int 4\, e^{5x-1}\, dx$
(l) $\int \frac{1}{2}\, e^{1-3x}\, dx$

3. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^1 e^x\,dx$ (d) $\displaystyle\int_{-2}^0 e^{-x}\,dx$ (g) $\displaystyle\int_{-1}^2 20\,e^{-5x}\,dx$

(b) $\displaystyle\int_1^2 e^x\,dx$ (e) $\displaystyle\int_0^2 e^{2x}\,dx$ (h) $\displaystyle\int_{-3}^1 8\,e^{-4x}\,dx$

(c) $\displaystyle\int_{-1}^3 e^{-x}\,dx$ (f) $\displaystyle\int_{-1}^1 e^{3x}\,dx$ (i) $\displaystyle\int_{-1}^3 9\,e^{6x}\,dx$

4. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^2 e^{x-1}\,dx$ (d) $\displaystyle\int_{-2}^{-1} e^{3x+2}\,dx$ (g) $\displaystyle\int_1^2 6\,e^{3x+1}\,dx$

(b) $\displaystyle\int_{-1}^1 e^{2x+1}\,dx$ (e) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x}\,dx$ (h) $\displaystyle\int_2^3 12\,e^{4x-5}\,dx$

(c) $\displaystyle\int_{-2}^0 e^{4x-3}\,dx$ (f) $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} e^{2+3x}\,dx$ (i) $\displaystyle\int_1^2 12\,e^{8-3x}\,dx$

5. Express each of the following functions using negative indices instead of fractions, and hence find its primitive.

(a) $\dfrac{1}{e^x}$ (b) $\dfrac{1}{e^{2x}}$ (c) $\dfrac{1}{e^{3x}}$ (d) $-\dfrac{3}{e^{3x}}$ (e) $\dfrac{6}{e^{2x}}$ (f) $\dfrac{8}{e^{-2x}}$

DEVELOPMENT

6. Find $f(x)$ and then find $f(1)$, given that:

(a) $f'(x) = 1 + 2\,e^x$ and $f(0) = 1$

(b) $f'(x) = 1 - 3\,e^x$ and $f(0) = -1$

(c) $f'(x) = 2 + e^{-x}$ and $f(0) = 0$

(d) $f'(x) = 4 - e^{-x}$ and $f(0) = 2$

(e) $f'(x) = e^{2x-1}$ and $f(\frac{1}{2}) = 3$

(f) $f'(x) = e^{1-3x}$ and $f(\frac{1}{3}) = \frac{2}{3}$

(g) $f'(x) = e^{\frac{1}{2}x+1}$ and $f(-2) = -4$

(h) $f'(x) = e^{\frac{1}{3}x+2}$ and $f(-6) = 2$

7. Expand the brackets and then find primitives of:

(a) $e^x(e^x + 1)$ (d) $(e^x + 1)^2$ (g) $(e^x - 2)^2$

(b) $e^x(e^x - 1)$ (e) $(e^x + 3)^2$ (h) $(e^x + e^{-x})(e^x - e^{-x})$

(c) $e^{-x}(2\,e^{-x} - 1)$ (f) $(e^x - 1)^2$ (i) $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

8. Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + C$ to find these indefinite integrals:

(a) $\displaystyle\int e^{2x+b}\,dx$ (e) $\displaystyle\int e^{ax+3}\,dx$ (i) $\displaystyle\int p\,e^{px+q}\,dx$

(b) $\displaystyle\int e^{7x+q}\,dx$ (f) $\displaystyle\int e^{sx+1}\,dx$ (j) $\displaystyle\int m\,e^{mx+k}\,dx$

(c) $\displaystyle\int e^{3x-k}\,dx$ (g) $\displaystyle\int e^{mx-2}\,dx$ (k) $\displaystyle\int A\,e^{sx-t}\,dx$

(d) $\displaystyle\int e^{6x-\lambda}\,dx$ (h) $\displaystyle\int e^{kx-1}\,dx$ (l) $\displaystyle\int B\,e^{kx-\ell}\,dx$

9. Express each function below as a power of e and hence find its primitive.

(a) $\dfrac{1}{e^{x-1}}$ (c) $\dfrac{1}{e^{2x+5}}$ (e) $\dfrac{10}{e^{2-5x}}$

(b) $\dfrac{1}{e^{3x-1}}$ (d) $\dfrac{4}{e^{2x-1}}$ (f) $\dfrac{12}{e^{3x-5}}$

10. By writing each function as the sum of powers of e, find:

(a) $\displaystyle\int \frac{e^x+1}{e^x}\,dx$ (c) $\displaystyle\int \frac{e^x-1}{e^{2x}}\,dx$ (e) $\displaystyle\int \frac{2\,e^{2x}-3\,e^x}{e^{4x}}\,dx$

(b) $\displaystyle\int \frac{e^{2x}+1}{e^x}\,dx$ (d) $\displaystyle\int \frac{e^x-3}{e^{3x}}\,dx$ (f) $\displaystyle\int \frac{2\,e^x-e^{2x}}{e^{3x}}\,dx$

11. (a) Find y as a function of x if $y' = e^{x-1}$, and $y = 1$ when $x = 1$. What is the y-intercept of this curve?

(b) The gradient of a curve is given by $y' = e^{2-x}$, and the curve passes through the point $(0,1)$. What is the equation of this curve? What is its horizontal asymptote?

(c) It is known that $f'(x) = e^x + \dfrac{1}{e}$ and that $f(-1) = -1$. Find $f(0)$.

(d) Given that $f''(x) = e^x - e^{-x}$ and that $y = f(x)$ is horizontal as it passes through the origin, find $f(x)$.

12. By first simplifying each function, find:

(a) $\displaystyle\int_0^1 e^x(2\,e^x-1)\,dx$ (c) $\displaystyle\int_0^1 (e^x-1)(e^{-x}+1)\,dx$ (e) $\displaystyle\int_0^1 \frac{e^{3x}+e^x}{e^{2x}}\,dx$

(b) $\displaystyle\int_{-1}^1 (e^x+2)^2\,dx$ (d) $\displaystyle\int_{-1}^1 (e^{2x}+e^{-x})(e^{2x}-e^{-x})\,dx$ (f) $\displaystyle\int_{-1}^1 \frac{e^x-1}{e^{2x}}\,dx$

CHALLENGE

13. (a) (i) Differentiate e^{x^2+3}. (ii) Hence find $\displaystyle\int 2x\,e^{x^2+3}\,dx$.

(b) (i) Differentiate e^{x^2-2x+3}. (ii) Hence find $\displaystyle\int (x-1)\,e^{x^2-2x+3}\,dx$.

(c) (i) Differentiate e^{3x^2+4x+1}. (ii) Hence find $\displaystyle\int (3x+2)\,e^{3x^2+4x+1}\,dx$.

(d) (i) Find the first derivative of $y = e^{x^3}$. (ii) Hence find $\displaystyle\int_{-1}^0 x^2\,e^{x^3}\,dx$.

14. Write each function as a power of e, and hence find the indefinite integral:

(a) $\displaystyle\int \frac{1}{(e^x)^2}\,dx$ (c) $\displaystyle\int \sqrt{e^x}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{e^x}}\,dx$

(b) $\displaystyle\int \frac{1}{(e^x)^3}\,dx$ (d) $\displaystyle\int \sqrt[3]{e^x}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt[3]{e^x}}\,dx$

15. (a) (i) Differentiate $y = x\,e^x - e^x$. (ii) Hence find $\displaystyle\int_0^2 x\,e^x\,dx$.

(b) (i) Differentiate $y = x\,e^x + e^{-x}$. (ii) Hence find $\displaystyle\int_{-2}^0 x\,e^{-x}\,dx$.

16. By first simplifying each function, determine:

(a) $\displaystyle\int \frac{e^x-e^{-x}}{\sqrt{e^x}}\,dx$ (b) $\displaystyle\int \frac{e^x+e^{-x}}{\sqrt[3]{e^x}}\,dx$

17. (a) Show that $f(x) = x\,e^{-x^2}$ is an odd function.

(b) Hence evaluate $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} x\,e^{-x^2}\,dx$ without finding a primitive.

2 F Applications of Integration

The normal methods of finding areas and volumes by integration can now be applied to functions involving e^x.

Finding the Area Between a Curve and the x-axis: A sketch is essential here, because an area below the x-axis is represented as a negative number by the definite integral.

WORKED EXERCISE:

(a) Use shifting to sketch $y = e^x - e$, showing the intercepts and asymptote.

(b) Find the area of the region between this curve, the x-axis and the y-axis.

SOLUTION:

(a) Move the graph of $y = e^x$ down e units.

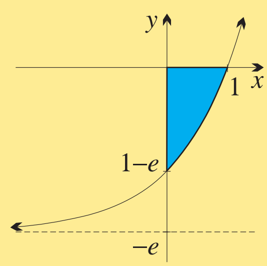
The y-intercept is $y = 1 - e$,

because when $x = 0$, $y = e^0 - e$
$= 1 - e$.

The x-intercept is $x = 1$,

because when $y = 0$, $e^x = e$
$x = 1$.

The horizontal asymptote moves to $y = -e$.



(b) $\displaystyle\int_0^1 (e^x - e)\,dx = \Big[e^x - ex\Big]_0^1$ (Note that e is a constant.)

$$= (e^1 - e) - (e^0 - 0)$$
$$= (e - e) - (1 - 0)$$
$$= -1.$$

This answer is negative because the region is below the x-axis.

Hence the required area is 1 square unit.

Finding Volumes of Revolution: The standard formula still applies. If the region between the curve $y = f(x)$ and the x-axis, from $x = a$ to $x = b$, is rotated about the x-axis, then

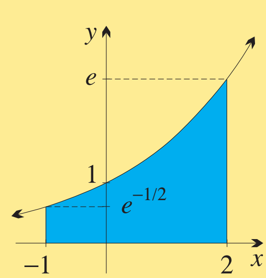
$$\text{volume} = \int_a^b \pi y^2\,dx.$$

WORKED EXERCISE:

Find the volume of the solid generated when the region between the curve $y = e^x$ and the x-axis, from $x = 0$ to $x = 2$, is rotated about the x-axis.

SOLUTION:

$$\text{Volume} = \int_0^2 \pi y^2\,dx$$
$$= \pi\int_0^2 e^{2x}\,dx \qquad (\text{Here } y^2 = (e^x)^2 = e^{2x}.)$$
$$= \pi\left[\tfrac{1}{2}e^{2x}\right]_0^2$$
$$= \pi(\tfrac{1}{2}e^2 - \tfrac{1}{2}e^0)$$
$$= \tfrac{\pi}{2}(e^2 - 1) \text{ cubic units.}$$



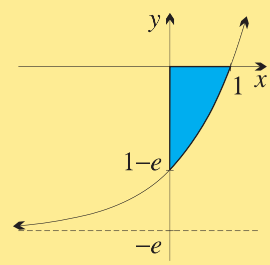
WORKED EXERCISE:

(a) If $y = e^x - e$, find and expand y^2.

(b) The region between the curve $y = e^x - e$, the x-axis and the y-axis was graphed in the first worked exercise in this section. Find the volume of the solid generated when this region is rotated about the x-axis.

SOLUTION:

(a)
$$\begin{aligned} y &= e^x - e \\ y^2 &= (e^x - e)^2 \\ &= e^{2x} - (2e \times e^x) + e^2 \\ &= e^{2x} - 2e^{x+1} + e^2. \end{aligned}$$



(b)
$$\begin{aligned} \text{Volume} &= \int_0^1 \pi y^2 \, dx \\ &= \pi \int_0^1 (e^{2x} - 2e^{x+1} + e^2) \, dx \\ &= \pi \Big[\tfrac{1}{2} e^{2x} - 2e^{x+1} + e^2 x \Big]_0^1 \qquad \text{(Note that } e^2 \text{ is a constant.)} \\ &= \pi \Big((\tfrac{1}{2} e^2 - 2e^2 + e^2) - (\tfrac{1}{2}\, e^0 - 2\, e^1 + 0) \Big) \\ &= \pi \Big(-\tfrac{1}{2} e^2 - (\tfrac{1}{2} - 2e + 0) \Big) \\ &= \pi \Big(-\tfrac{1}{2} e^2 - \tfrac{1}{2} + 2e \Big) \\ &= \tfrac{\pi}{2}(-e^2 + 4e - 1) \text{ cubic units.} \end{aligned}$$

Finding Areas Between Curves: If a curve $y = f(x)$ is always above $y = g(x)$ in an interval $a \le x \le b$, then the area of the region between the curves is

$$\text{area between the curves} = \int_a^b \Big(f(x) - g(x) \Big) \, dx.$$

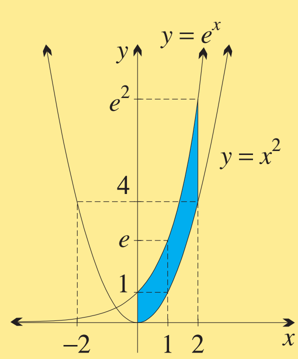
WORKED EXERCISE:

(a) Sketch the curves $y = e^x$ and $y = x^2$ in the interval $-2 \le x \le 2$.

(b) Find the area of the region between the curves, from $x = 0$ to $x = 2$.

SOLUTION:

(a) The graphs are drawn to the right.
Note that for $x > 0$, $y = e^x$ is always above $y = x^2$.



(b) Using the standard formula above,

$$\begin{aligned} \text{area} &= \int_0^2 (e^x - x^2) \, dx \\ &= \Big[e^x - \tfrac{1}{3} x^3 \Big]_0^2 \\ &= (e^2 - \tfrac{8}{3}) - (e^0 - 0) \\ &= e^2 - 3\tfrac{2}{3} \text{ square units.} \end{aligned}$$

Exercise 2F

TECHNOLOGY: Graphing programs that can calculate the areas of specified regions may make the problems in this exercise clearer, particularly when no diagram has been given. Graphing programs that can show solids of revolution would help to visualise the solid involved in such problems.

1. (a) Use the standard form $\int e^x\,dx = e^x + C$ to evaluate each of definite integrals below. Then approximate them correct to two decimal places.

(i) $\int_0^1 e^x\,dx$ (ii) $\int_{-1}^0 e^x\,dx$ (iii) $\int_{-2}^0 e^x\,dx$ (iv) $\int_{-3}^0 e^x\,dx$

(b) The graph below shows $y = e^x$ from $x = -5$ to $x = 1$, with a scale of 10 divisions to 1 unit, so that 100 little squares equal 1 square unit.

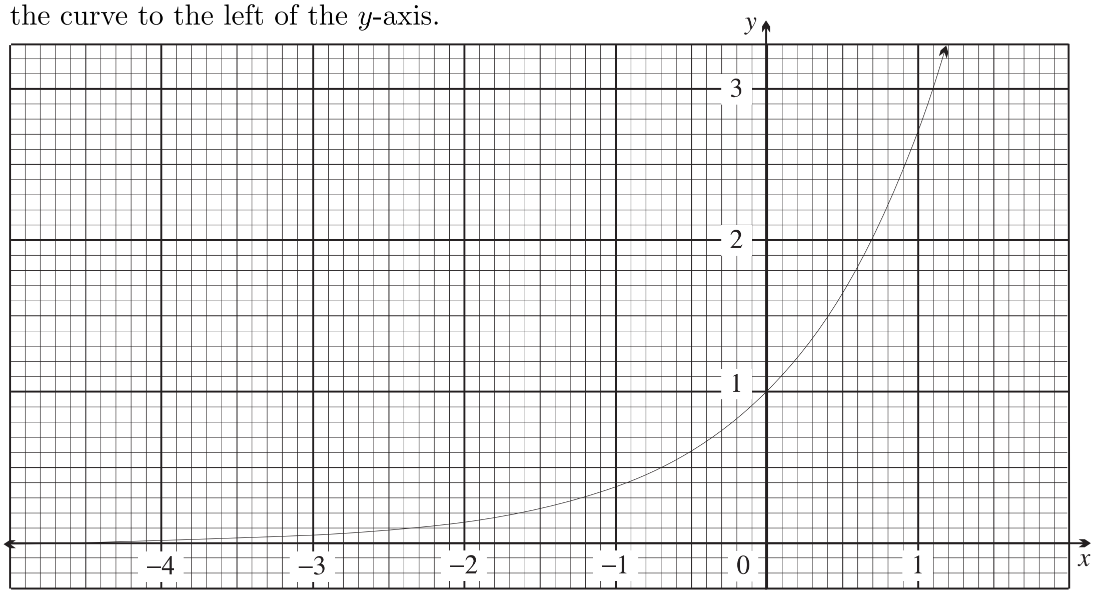
By counting squares under the curve from $x = 0$ to $x = 1$, find an approximation to $\int_0^1 e^x\,dx$, and compare it with the approximation obtained in part (a).

(c) Count squares to the left of the y-axis to obtain approximations to:

(i) $\int_{-1}^0 e^x\,dx$, (ii) $\int_{-2}^0 e^x\,dx$, (iii) $\int_{-3}^0 e^x\,dx$,

and compare the results with the approximations obtained in part (a).

(d) Continue counting squares to the left of $x = -3$, and estimate the total area under the curve to the left of the y-axis.



2. Find the area between $y = e^x$ and the x-axis for:

(a) $-1 \le x \le 0$ (b) $1 \le x \le 3$ (c) $-1 \le x \le 1$ (d) $-2 \le x \le 1$

3. Answer these questions first in exact form, then correct to four significant figures. In each case you will need to use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

(a) Find the area between the curve $y = e^{2x}$ and the x-axis:

(i) from $x = 0$ to $x = 3$ (ii) from $x = -3$ to $x = 0$

(b) Find the area between the curve $y = e^{-x}$ and the x-axis:

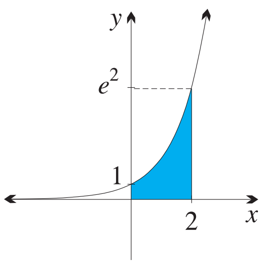
(i) from $x = 0$ to $x = 1$ (ii) from $x = -1$ to $x = 0$

(c) Find the area between the curve $y = e^{\frac{1}{3}x}$ and the x-axis:
 (i) from $x = 0$ to $x = 3$ (ii) from $x = -3$ to $x = 0$

4. In each case find the area between the x-axis and the given curve between the given x-values. You will need to use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

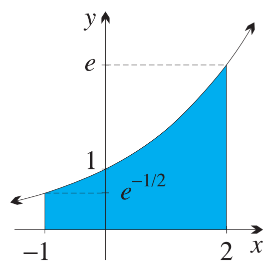
(a) $y = e^{x+1}$, for $0 \le x \le 2$
(b) $y = e^{x+3}$, for $-2 \le x \le 0$
(c) $y = e^{2x-1}$, for $0 \le x \le 1$
(d) $y = e^{3x-5}$, for $1 \le x \le 2$
(e) $y = e^{-x+1}$, for $-1 \le x \le 1$
(f) $y = e^{-2x-1}$, for $-2 \le x \le -1$
(g) $y = e^{\frac{1}{3}x+2}$, for $0 \le x \le 3$
(h) $y = e^{\frac{1}{2}x-1}$, for $-2 \le x \le 2$

5. (a)



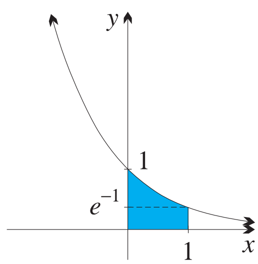
Find the area of the region bounded by the curve $y = e^x$, the x-axis, the y-axis and the line $x = 2$.

(b)



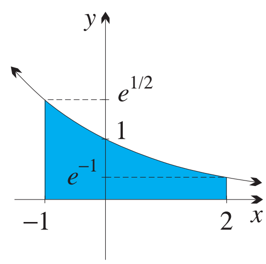
Find the area of the region bounded by the curve $y = e^{\frac{1}{2}x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

(c)



Find the area of the region bounded by the curve $y = e^{-x}$, the x-axis, the y-axis and the line $x = 1$.

(d)

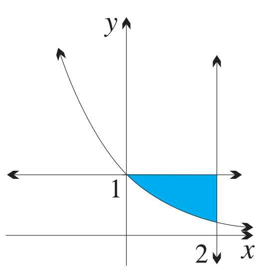


Find the area of the region bounded by the curve $y = e^{-\frac{1}{2}x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

DEVELOPMENT

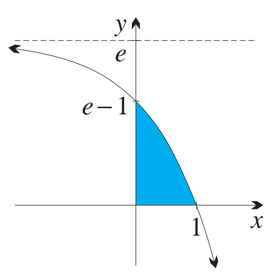
6. (a) Find the area between the curve $y = e^{-x} + 1$ and the x-axis, from $x = 0$ to $x = 2$.
(b) Find the area between the curve $y = 1 - e^x$ and the x-axis, from $x = -1$ to $x = 0$.
(c) Find the area between the curve $y = e^x + e^{-x}$ and the x-axis, from $x = -2$ to $x = 2$.
(d) Find the area between the curve $y = x^2 + e^x$ and the x-axis, from $x = -3$ to $x = 3$.

7. (a)

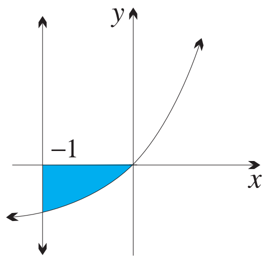


Find the area of the region bounded by the curve $y = e^{-x}$ and the lines $x = 2$ and $y = 1$.

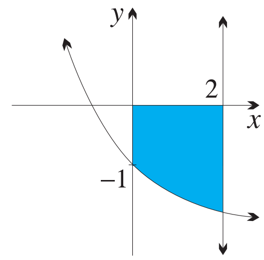
(b)



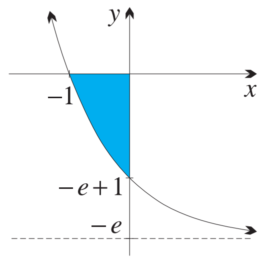
Find the area of the region in the first quadrant bounded by the coordinate axes and the curve $y = e - e^x$.

(c) 

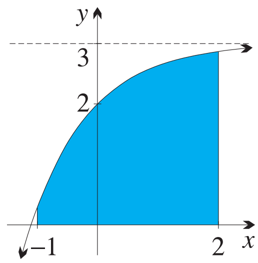
Find the area between the x-axis, the curve $y = e^x - 1$ and the line $x = -1$.

(d) 

What is the area bounded by $x = 2$, $y = e^{-x} - 2$, the x-axis and the y-axis?

(e) 

Find the area of the region bounded by the curve $y = e^{-x} - e$ and the coordinate axes.

(f) 

Find the area of the region bounded by the curve $y = 3 - e^{-x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

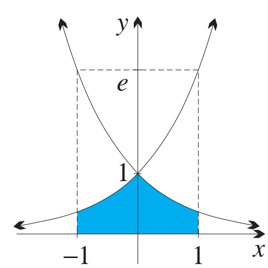
8. (a) Sketch the curves $y = e^x$ and $y = x + 1$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

(b) Sketch the curves $y = e^x$ and $y = 1 - x$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

9. The diagram to the right shows the region above the x-axis, below both $y = e^x$ and $y = e^{-x}$, between $x = -1$ and $x = 1$.

(a) Explain why the area of this shaded region may be written as $2\displaystyle\int_0^1 e^{-x}\,dx$.

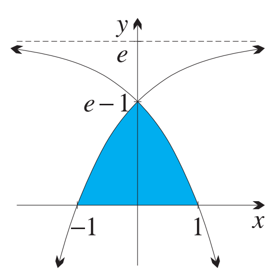
(b) Hence find the area of this region.



10. The diagram to the right shows the region above the x-axis, below both $y = e - e^{-x}$ and $y = e - e^x$.

(a) Explain why the area of this shaded region may be written as $2\displaystyle\int_0^1 (e - e^x)\,dx$.

(b) Hence find the area of this region.



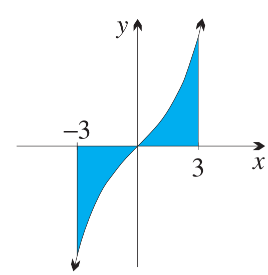
11. The diagram to the right shows the region between the curve $y = e^x - e^{-x}$, the x-axis and the lines $x = -3$ and $x = 3$.

(a) Show that $y = e^x - e^{-x}$ is an odd function.

(b) Hence write down the value of $\displaystyle\int_{-3}^{3} \left(e^x - e^{-x}\right)\,dx$ without finding a primitive.

(c) Explain why the area of the shaded region may be written as $2\displaystyle\int_0^3 \left(e^x - e^{-x}\right)\,dx$.

(d) Hence find the area of this region.



12. (a) Show that the curves $y = x^2$ and $y = e^{x+1}$ intersect at $x = -1$.
(b) Hence sketch the region in the second quadrant between these two curves and the y-axis.
(c) Find its area.

13. The region under $y = e^x$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Write down the volume of the resulting solid as an integral, and evaluate it.

14. The shape of a metal stud is created by rotating the curve $y = e^x - e^{-x}$ about the x-axis between $x = 0$ and $x = \frac{1}{2}$. Find its volume.

15. A horn is generated by rotating the curve $y = 1 + e^{-x}$ about the x-axis between $x = 1$ and $x = 3$. Find its volume, correct to three decimal places.

16. (a) Show that the curves $y = e^x$ and $y = (e-1)x + 1$ meet at $A(0,1)$ and $B(1,e)$.
(b) Sketch the graphs, and find the area contained between the line and the curve.

17. Sketch the region between the graphs of $y = e^x$ and $y = x$, between the y-axis and $x = 2$, then find its area.

18. In this question, give your answers correct to four decimal places whenever you are asked to approximate.
(a) Find the area between the curve $y = e^x$ and the x-axis, for $0 \le x \le 1$, by evaluating an appropriate integral. Then approximate the result.
(b) Estimate the area using the trapezoidal rule with three function values.
(c) Estimate the area using Simpson's rule with three function values.

19. (a) Use the trapezoidal rule with five function values to approximate the area between the curve $y = e^{-x^2}$ and the x-axis, from $x = 0$ to $x = 4$. Give your answer correct to four decimal places.
(b) Use Simpson's rule with five function values to approximate the area in part (a).

20. (a) Use the trapezoidal rule with five function values to approximate the area between the curve $y = e^{\frac{1}{x}}$ and the x-axis, from $x = 1$ to $x = 3$. Give your answer correct to four decimal places.
(b) Use Simpson's rule with five function values to approximate the area in part (a).

CHALLENGE

21. (a) (i) Evaluate the integral $\displaystyle\int_N^0 e^x\,dx$.
(ii) What value does this integral approach in the limit as $N \to -\infty$?
(b) (i) Evaluate the integral $\displaystyle\int_0^N e^{-x}\,dx$.
(ii) What value does this integral approach in the limit as $N \to \infty$?

22. (a) Differentiate e^{-x^2} and hence write down a primitive of $2xe^{-x^2}$.
(b) A certain bulb and capillary tube are generated when the curve $y = \sqrt{2x}\,e^{-\frac{1}{2}x^2}$ is rotated about the x-axis, between $x = 0\,\text{cm}$ and $x = 2\,\text{cm}$. Find the volume of liquid the apparatus could hold. Give your answer correct to four significant figures.

2G Chapter Review Exercise

1. Use the index laws to simplify the following. Leave your answers in index form.

(a) $3^4 \times 3^5$ (b) $(3^6)^2$ (c) $\dfrac{3^{10}}{3^5}$ (d) $2^5 \times 3^5$

2. Write as fractions:

(a) 5^{-1} (b) 10^{-2} (c) x^{-3} (d) 3^{-x}

3. Simplify:

(a) $9^{\frac{1}{2}}$ (b) $27^{\frac{1}{3}}$ (c) $8^{\frac{2}{3}}$ (d) $16^{-\frac{1}{2}}$ (e) $27^{-\frac{2}{3}}$ (f) $100^{-\frac{3}{2}}$

4. Simplify:

(a) $2^x \times 2^{2x}$ (b) $\dfrac{2^{6x}}{2^{2x}}$ (c) $(2^{3x})^2$ (d) $2^x \times 5^x$ (e) $2^{x+1} \times 2^{x+2}$ (f) $\dfrac{2^{3x+2}}{2^{x+3}}$

5. Sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on the same number plane. Then write down the equation of the line that reflects each graph onto the other graph.

6. Use your calculator to approximate the following, correct to four significant figures:

(a) e (b) e^4 (c) e^{-2} (d) $e^{\frac{3}{2}}$

7. Simplify:

(a) $e^{2x} \times e^{3x}$ (b) $e^{7x} \div e^x$ (c) $\dfrac{e^{2x}}{e^{6x}}$ (d) $(e^{3x})^3$

8. Sketch the graph of each function on a separate number plane, and state its range:

(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = e^x + 1$ (d) $y = e^{-x} - 1$

9. Differentiate:

(a) $y = e^x$ (b) $y = e^{3x}$ (c) $y = e^{x+3}$ (d) $y = e^{2x+3}$ (e) $y = e^{-x}$ (f) $y = e^{-3x}$ (g) $y = e^{3-2x}$ (h) $y = 3e^{2x+5}$ (i) $y = 4e^{\frac{1}{2}x}$ (j) $y = e^{x^3}$ (k) $y = e^{x^2-3x}$ (l) $y = \dfrac{2e^{6x-5}}{3}$

10. Write each function as a single power of e, and then differentiate it.

(a) $y = e^{3x} \times e^{2x}$ (b) $y = \dfrac{e^{7x}}{e^{3x}}$ (c) $y = \dfrac{e^x}{e^{4x}}$ (d) $y = (e^{-2x})^3$

11. Differentiate each function using the chain, product and quotient rules as appropriate:

(a) $y = xe^{2x}$ (b) $y = (e^{2x} + 1)^3$ (c) $y = \dfrac{e^{3x}}{x}$ (d) $y = x^2 e^{x^2}$ (e) $y = (e^x - e^{-x})^5$ (f) $y = \dfrac{e^{2x}}{2x+1}$

12. Find the first and second derivatives of:

(a) $y = e^{2x+1}$ (b) $y = e^{x^2+1}$

13. Find the gradient of the tangent to the curve $y = e^{2x}$ at the point $(0, 1)$.

14. Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x-intercept and y-intercept of this tangent.

15. Consider the curve $y = e^{-3x}$.

(a) Find the gradient of the normal to the curve at the point where $x = 0$.

(b) Find y'' and hence determine the concavity of the curve at the point where $x = 0$.

16. Consider the curve $y = e^x - x$.

(a) Find y' and y''.

(b) Show that there is a stationary point at $(0, 1)$, and determine its nature.

(c) Explain why the curve is concave up for all values of x.

(d) Sketch the curve and write down its range.

17. Find the stationary point on the curve $y = xe^{-2x}$ and determine its nature.

18. Find:

(a) $\displaystyle\int e^{5x}\,dx$

(b) $\displaystyle\int e^{5x+3}\,dx$

(c) $\displaystyle\int 5e^{-5x}\,dx$

(d) $\displaystyle\int 10e^{2-5x}\,dx$

(e) $\displaystyle\int e^{\frac{1}{5}x}\,dx$

(f) $\displaystyle\int 3e^{5x-4}\,dx$

19. Find the exact value of:

(a) $\displaystyle\int_0^2 e^x\,dx$

(b) $\displaystyle\int_0^1 e^{2x}\,dx$

(c) $\displaystyle\int_{-1}^0 e^{-x}\,dx$

(d) $\displaystyle\int_{-\frac{2}{3}}^0 e^{3x+2}\,dx$

(e) $\displaystyle\int_0^{\frac{1}{2}} e^{3-2x}\,dx$

(f) $\displaystyle\int_0^2 2e^{\frac{1}{2}x}\,dx$

20. Find the primitive of:

(a) $\dfrac{1}{e^{5x}}$

(b) $e^{3x} \times e^x$

(c) $\dfrac{6}{e^{3x}}$

(d) $(e^{3x})^2$

(e) $\dfrac{e^{3x}}{e^{5x}}$

(f) $\dfrac{e^{3x}+1}{e^{2x}}$

(g) $e^{2x}(e^x + e^{-x})$

(h) $(1 + e^{-x})^2$

21. Find the exact value of:

(a) $\displaystyle\int_0^1 (1 + e^{-x})\,dx$

(b) $\displaystyle\int_0^2 (e^{2x} + x)\,dx$

(c) $\displaystyle\int_0^1 \frac{2}{e^x}\,dx$

(d) $\displaystyle\int_0^{\frac{1}{3}} e^{3x}(1 - e^{-3x})\,dx$

(e) $\displaystyle\int_0^1 \frac{e^{2x}+1}{e^x}\,dx$

(f) $\displaystyle\int_0^1 (e^x + 1)^2\,dx$

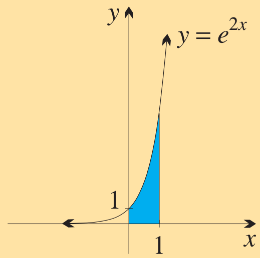
22. If $f'(x) = e^x - e^{-x} - 1$ and $f(0) = 3$, find $f(x)$ and then find $f(1)$.

23. (a) Differentiate e^{x^3}.

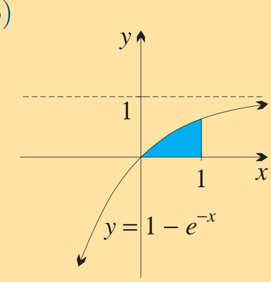
(b) Hence find $\displaystyle\int_0^1 x^2 e^{x^3}\,dx$.

24. Find the area of each region below, correct to three significant figures:

(a)



(b)



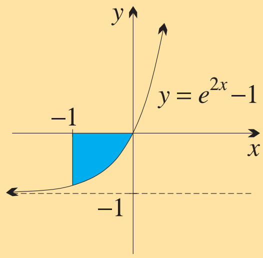
25. Find the exact volume of the solid generated:

(a) when the region bounded by the curve $y = e^{\frac{1}{2}x}$ and the x-axis, from $x = -1$ to $x = 1$, is rotated about the x-axis,

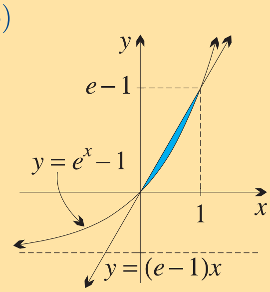
(b) when the region bounded by the curve $y = e^{x} + e^{-x}$ and the x-axis, from $x = 0$ to $x = \frac{1}{2}$, is rotated about the x-axis.

26. Find the exact area of the region shaded in each diagram below:

(a)



(b)



CHAPTER **THREE**

The Logarithmic Function

No study of the exponential function can proceed very far without logarithms. The logarithmic function $y = \log_e x$ is the inverse function of $y = e^x$, so applications of e^x constantly generate equations involving $\log_e x$. The previous chapter developed the calculus of e^x; this chapter will develop the calculus of $\log_e x$.

Calculus with the logarithmic function begins by developing the standard forms for differentiation and integration,

$$\frac{d}{dx} \log_e x = \frac{1}{x} \qquad \text{and} \qquad \int \frac{1}{x}\, dx = \log_e x + C.$$

These standard forms are surprising, because they show a very close relationship between the reciprocal function $y = \dfrac{1}{x}$, which is an algebraic function, and the logarithmic function $y = \log_e x$, which is not algebraic. These standard forms also fill a gap in the earlier development of integration in Chapter One, where primitives were established for every power of x except $x^{-1} = \dfrac{1}{x}$.

The last section of this chapter develops the calculus of exponential and logarithmic functions with bases different from e. This work requires fluency with logarithms, but is not required in subsequent work and could well be left until later.

3 A Review of Logarithmic Functions

Logarithmic functions were developed in the Year 11 volume and are reviewed here. It is important to have a clear picture of the graphs and a clear understanding that the logarithmic and exponential functions with a given base a are mutually inverse.

Logarithmic Functions: The base a must always be positive and not equal to 1.

1 DEFINITION OF LOGARITHMS: The *logarithm* base a of a positive number x is the index, when the number x is expressed as a power of the base a:

$$y = \log_a x \qquad \text{means that} \qquad x = a^y.$$

This definition simply states that the functions $y = a^x$ and $y = \log_a x$ are mutually inverse functions.

WORKED EXERCISE:
Write each statement in index form and hence find x:
(a) $x = \log_{10} 1000$ (b) $x = \log_2 \frac{1}{16}$

SOLUTION:
(a) $x = \log_{10} 1000$
$10^x = 1000$
$x = 3$

(b) $x = \log_2 \frac{1}{16}$
$2^x = \frac{1}{16}$
$x = -4$

WORKED EXERCISE:
Write each statement in logarithmic form, then use the function labelled $\boxed{\text{log}}$ on your calculator to approximate x, correct to four significant figures:
(a) $10^x = 850$ (b) $10^x = 0{\cdot}07$

SOLUTION:
(a) $10^x = 850$
$x = \log_{10} 850$
$\doteqdot 2{\cdot}929$

(b) $10^x = 0{\cdot}07$
$x = \log_{10} 0{\cdot}07$
$\doteqdot -1{\cdot}155$

WORKED EXERCISE:
Without using a calculator, explain why:
(a) $\log_{10} 12\,345$ is between 4 and 5, (b) $\log_{10} 0{\cdot}0035$ is between -3 and -2.

SOLUTION:
(a) We can write $10^4 < 12\,345 < 10^5$.
Taking logarithms, $4 < \log_{10} 12\,345 < 5$.

(b) We can write $10^{-3} < 0{\cdot}0035 < 10^{-2}$.
Taking logarithms, $-3 < \log_{10} 0{\cdot}0035 < -2$.

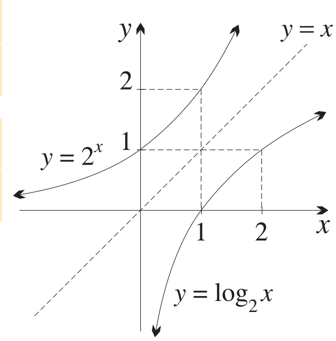
The Graphs of $y = 2^x$ and $y = \log_2 x$: Graphing $y = 2^x$ and $y = \log_2 x$ should help explain the relationship between the graphs of exponential and logarithmic functions. Below are tables of values of $y = 2^x$ and $y = \log_2 x$. Notice that the only difference between the two tables is that the rows are reversed.

For $y = 2^x$:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

For $y = \log_2 x$:

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-3	-2	-1	0	1	2	3



The graphs of $y = 2^x$ and $y = \log_2 x$ are sketched to the right. The only difference between the two functions is that the x- and y-values have been reversed. The two graphs are thus reflections of each other in the diagonal line $y = x$.

The domains and ranges of the two graphs are exchanged.
For $y = 2^x$, the domain is all real x and the range is $y > 0$.
For $y = \log_2 x$, the domain is $x > 0$ and the range is all real y.

Combining the Logarithmic and Exponential Functions: Because the two functions are mutually inverse, if they are applied one after the other to a number, then the number remains the same. For example, applying them in turn to 8,

$$\begin{aligned}\log_2 2^8 &= \log_2 256 \\ &= 8\end{aligned} \qquad \text{and} \qquad \begin{aligned}2^{\log_2 8} &= 2^3 \\ &= 8.\end{aligned}$$

The general statement of this result is:

2 THE FUNCTIONS $y = a^x$ AND $y = \log_a x$ ARE MUTUALLY INVERSE:

$$\log_a a^x = x \qquad \text{and} \qquad a^{\log_a x} = x.$$

NOTE: Because the functions 10^x and $\log_{10} x$ are inverse functions, they are usually located on the same button on the calculator and labelled $\boxed{10^x}$ and $\boxed{\log}$. It is worthwhile taking some time to experiment with these keys. The vital point is that when they are applied one after the other, they cancel each other out.

WORKED EXERCISE:

Use the functions labelled $\boxed{\log}$ and $\boxed{10^x}$ on your calculator to demonstrate that:

(a) $\log_{10} 10^{1\cdot 8} = 1\cdot 8$ (b) $10^{\log_{10} 250} = 250$

SOLUTION:

(a) $\begin{aligned}\log_{10} 10^{1\cdot 8} &= \log_{10} 63\cdot 095\,734\ldots \\ &\doteqdot 1\cdot 8\end{aligned}$

(b) $\begin{aligned}10^{\log_{10} 250} &= 10^{2\cdot 397\,940\ldots} \\ &\doteqdot 250\end{aligned}$

WORKED EXERCISE:

Use your calculator to confirm that:

(a) $\log_{10} 10^3 = 3$ (b) $10^{\log_{10} 1000} = 3$

SOLUTION:

(a) $\begin{aligned}\log_{10} 10^3 &= \log_{10} 1000 \\ &= 3\end{aligned}$

(b) $\begin{aligned}10^{\log_{10} 1000} &= 10^3 \\ &= 1000\end{aligned}$

The Laws for Logarithms: Here again are the basic laws for manipulating the logarithms of products, quotients and powers:

3 THREE LAWS FOR LOGARITHMS:

- The log of a product is the sum of the logs:

  $$\log_a xy = \log_a x + \log_a y$$

- The log of a quotient is the difference of the logs:

  $$\log_a \frac{x}{y} = \log_a x - \log_a y$$

- The log of a power is the multiple of the log:

  $$\log_a x^n = n \log_a x$$

Some particular values and identities of the log function occur very often and are worth committing to memory:

4 SOME PARTICULAR VALUES AND IDENTITIES OF THE LOGARITHMIC FUNCTION:

$$\log_a 1 = 0, \qquad \text{because } 1 = a^0.$$
$$\log_a a = 1, \qquad \text{because } a = a^1.$$
$$\log_a \sqrt{a} = \tfrac{1}{2}, \qquad \text{because } \sqrt{a} = a^{\frac{1}{2}}.$$
$$\log_a \frac{1}{a} = -1, \qquad \text{because } \frac{1}{a} = a^{-1}.$$
$$\log_a \frac{1}{x} = -\log_a x, \qquad \text{because } \log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x.$$

WORKED EXERCISE:

Use the log laws to expand :

(a) $\log_7 \dfrac{7}{x}$ (b) $\log_3 5x^2$

SOLUTION:

(a) $\log_7 \dfrac{7}{x} = \log_7 7 - \log_7 x$ (The log of a quotient is the difference of the logs.)
$= 1 - \log_7 x$ (Use the identity $\log_7 7 = 1$, as in Box 4 above.)

(b) $\log_3 5x^2 = \log_3 5 + \log_3 x^2$ (The log of a product is the sum of the logs.)
$= \log_3 5 + 2\log_3 x$ (The log of a power is the multiple of the log.)

The Change-of-Base Formula:

Suppose that b is some other base.

5 THE CHANGE-OF-BASE FORMULA:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Remember this as 'the log of the number over the log of the base'.

Using this formula, logarithms to any base can be approximated by writing them as logarithms base 10 and using the function labelled $\boxed{\log}$ on the calculator.

WORKED EXERCISE:

Find the value of x, correct to four significant figures, if:

(a) $x = \log_3 100$ (b) $2^x = 80$

Check your results using the function labelled $\boxed{x^y}$ on the calculator.

SOLUTION:

(a) Here
$$x = \log_3 100 = \frac{\log_{10} 100}{\log_{10} 3} = \frac{2}{\log_{10} 3} \doteqdot 4{\cdot}192.$$
Checking, $3^{4\cdot192} \doteqdot 100$.

(b) Here
$$2^x = 80$$
$$x = \log_2 80 = \frac{\log_{10} 80}{\log_{10} 2} \doteqdot 6{\cdot}322.$$
Checking, $2^{6\cdot322} \doteqdot 80$.

Exercise 3A

1. Use the button labelled $\boxed{\log}$ on your calculator to find the following, approximating correct to four decimal places where appropriate.

(a) $\log_{10} 1$ (c) $\log_{10} 10$ (e) $\log_{10} \frac{1}{2}$ (g) $\log_{10} \frac{1}{15}$

(b) $\log_{10} 2$ (d) $\log_{10} 15$ (f) $\log_{10} \frac{1}{10}$ (h) $\log_{10} \frac{1}{100}$

2. Write each log equation in index form and hence find x.

(a) $x = \log_3 9$ (c) $x = \log_6 36$ (e) $x = \log_2 \frac{1}{32}$ (g) $x = \log_7 \frac{1}{49}$

(b) $x = \log_2 8$ (d) $x = \log_3 81$ (f) $x = \log_3 \frac{1}{27}$ (h) $x = \log_{10} \frac{1}{10}$

3. Use the results of Box 4 in the notes to simplify:

(a) $\log_2 1$ (c) $\log_2 \frac{1}{2}$ (e) $\log_2 2$ (g) $\log_2 \sqrt{2}$

(b) $\log_3 1$ (d) $\log_3 \frac{1}{3}$ (f) $\log_3 3$ (h) $\log_3 \sqrt{3}$

4. (a) Copy and complete the table of values of the function $y = \log_2 x$:

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$					

(b) Sketch the curve, choosing appropriate scales on the axes.

(c) What are the domain and range of $y = \log_2 x$?

5. (a) Copy and complete the table of values of the function $y = \log_3 x$:

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$					

(b) Sketch the curve, choosing appropriate scales on the axes.

(c) What are the domain and range of $y = \log_3 x$?

6. (a) Copy and complete the table of values of the function $y = \log_{10} x$, using a calculator to approximate the values where necessary:

x	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
$\log_{10} x$							

(b) Sketch the curve, choosing appropriate scales on the axes.

(c) What are the domain and range of $y = \log_{10} x$?

7. Write each equation in logarithmic form and then use the function labelled $\boxed{\log}$ on your calculator to approximate x correct to four decimal places.

(a) $10^x = 3$ (c) $10^x = 25$ (e) $10^x = 3000$ (g) $10^x = 0{\cdot}05$

(b) $10^x = 8$ (d) $10^x = 150$ (f) $10^x = 0{\cdot}2$ (h) $10^x = 0{\cdot}006\,25$

8. Use the functions labelled $\boxed{\log}$ and $\boxed{10^x}$ on your calculator to demonstrate that:

(a) $\log_{10} 10^{1\cdot4} = 1{\cdot}4$ (c) $\log_{10} 10^{0\cdot5} = 0{\cdot}5$ (e) $\log_{10} 10^{-3\cdot7} = -3{\cdot}7$

(b) $\log_{10} 10^{2\cdot3} = 2{\cdot}3$ (d) $\log_{10} 10^{0\cdot2} = 0{\cdot}2$ (f) $\log_{10} 10^{-0\cdot6} = -0{\cdot}6$

9. Use the functions labelled $\boxed{\log}$ and $\boxed{10^x}$ on your calculator to demonstrate that:

(a) $10^{\log_{10} 1\cdot4} = 1{\cdot}4$ (c) $10^{\log_{10} 0\cdot5} = 0{\cdot}5$ (e) $10^{\log_{10} 55} = 55$

(b) $10^{\log_{10} 2\cdot3} = 2{\cdot}3$ (d) $10^{\log_{10} 0\cdot2} = 0{\cdot}2$ (f) $10^{\log_{10} 127} = 127$

10. Use the log laws to expand the following:

(a) $\log_2 5x$
(b) $\log_2 3x$
(c) $\log_2 2x$
(d) $\log_2 4x$
(e) $\log_2 \frac{x}{7}$
(f) $\log_2 \frac{x}{3}$
(g) $\log_2 \frac{1}{x}$
(h) $\log_2 \frac{12}{x}$
(i) $\log_2 x^2$
(j) $\log_2 5x^2$
(k) $\log_2 \frac{5}{x^2}$
(l) $\log_2 \sqrt{x}$

DEVELOPMENT

11. Sketch the graph of $y = \log_2 x$, then use your knowledge of transformations to graph the following functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

(a) $y = \log_2 x + 1$
(b) $y = \log_2 x + 2$
(c) $y = \log_2 x - 1$
(d) $y = \log_2 x - 2$

12. (a) Copy and complete the following tables of values for the functions $y = \log_2 x$ and $y = -\log_2 x$:

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$					

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$-\log_2 x$					

(b) Sketch both graphs on the same number plane.

13. Use the log laws to expand:

(a) $\log_2 (x+1)(x+2)$
(b) $\log_2 x(x-1)$
(c) $\log_2 \dfrac{x+3}{x-1}$
(d) $\log_2 \dfrac{x-2}{x+5}$

14. Use the change-of-base-formula and the function labelled $\boxed{\text{log}}$ on your calculator to evaluate the following, correct to four decimal places:

(a) $\log_2 5$
(b) $\log_5 2$
(c) $\log_3 47$
(d) $\log_6 112$

15. Use logarithms to solve the following, correct to four significant figures. Then check your results by using the function labelled $\boxed{x^y}$ on your calculator.

(a) $10^x = 3$
(b) $2^x = 10$
(c) $5^x = 150$
(d) $3^x = 90$

16. Use the graph of $y = \log_2 x$ and your knowledge of transformations to graph the following functions. Show the vertical asymptote and state the domain.

(a) $y = \log_2(x-1)$
(b) $y = \log_2(x-3)$
(c) $y = \log_2(x+1)$
(d) $y = \log_2(x+2)$
(e) $y = -\log_2 x$
(f) $y = \log_2(-x)$

17. (a) Use the log laws to show that $\log_{10} 10^4 = 4$.

(b) Now use part (a) to show that $10^{\log_{10} 10\,000} = 10\,000$.

(c) Likewise show that:

(i) $\log_2 2^7 = 7$ and $2^{\log_2 128} = 128$
(ii) $\log_3 3^4 = 4$ and $3^{\log_3 81} = 81$
(iii) $\log_5 5^5 = 5$ and $5^{\log_5 3125} = 3125$

CHALLENGE

18. [Technology] Use a graphing program to graph $y = \log_a x$ for various values of a increasing from 2 to 5 (including fractional values of a). Explain what happens to the graph as a increases.

19. [Technology] Use a graphing program to graph transformations of logarithmic curves. Start with the graphs in questions of this exercise and then experiment with further similar graphs.

20. [Technology]

(a) Why does the calculator give an error when you try to evaluate $\log_{10}(-7)$?

(b) Why does the calculator give a negative number for $\log_{10}\frac{1}{8}$?

(c) Why does the calculator give a number less than 1 when you evaluate $10^{-3\cdot2}$?

(d) For what values of x does the calculator give a negative number when you use the function labelled $\boxed{\log}$?

(e) For what values of x is the output zero when you use the function labelled:

(i) $\boxed{10^x}$? (ii) $\boxed{\log}$?

(f) What should be the input if the function labelled $\boxed{\log}$ is to return:

(i) 5? (ii) -2? (iii) $3\cdot5$? (iv) $-1\cdot7$?

(g) What should be the input if the function labelled $\boxed{10^x}$ is to return:

(i) 100? (ii) $0\cdot001$? (iii) 60? (iv) $0\cdot3$?

21. (a) Simplify $\log_{10} 100$ and $\log_{10} 1000$, and hence explain why $\log_{10} 274$ lies between the numbers 2 and 3.

(b) Explain why $3 < \log_{10} 4783 < 4$.

(c) What two integers does $\log_{10} 516\,287$ lie between?

(d) Now use your calculator to find the number of digits in:

(i) 3^{500} (ii) 2^{2000} (iii) 7^{300} (iv) 11^{432}

22. (a) Write $y = \log_b x$ in index form.

(b) Take logs base a of both sides of this result, and hence show that $\log_a x = y \times \log_a b$.

(c) Hence prove the change-of-base formula $\log_b x = \dfrac{\log_a x}{\log_a b}$.

3 B The Logarithmic Function Base *e*

We have seen that $e \doteqdot 2\cdot7183$ is the most natural base to use for exponential functions in calculus. This number e is similarly the most natural base to use for the calculus of logarithmic functions. This section introduces $\log_e x$ in preparation for the calculus of the logarithmic function developed in the rest of the chapter.

The Logarithmic Function: Because e is the most natural base to use in calculus, the function $y = \log_e x$ is called *the logarithmic function* to distinguish it from other logarithmic functions like $\log_2 x$ and $\log_{10} x$ that have other bases.

6 THE LOGARITHMIC FUNCTION: The logarithmic function with base e,

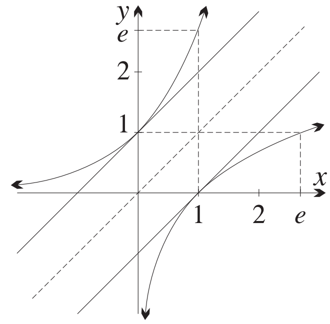
$$y = \log_e x,$$

is called *the logarithmic function* to distinguish it from all other logarithmic functions $y = \log_a x$.

Below are tables of values and the graphs of the mutually inverse functions $y = e^x$ and $y = \log_e x$. These are two of the most important graphs in the course.

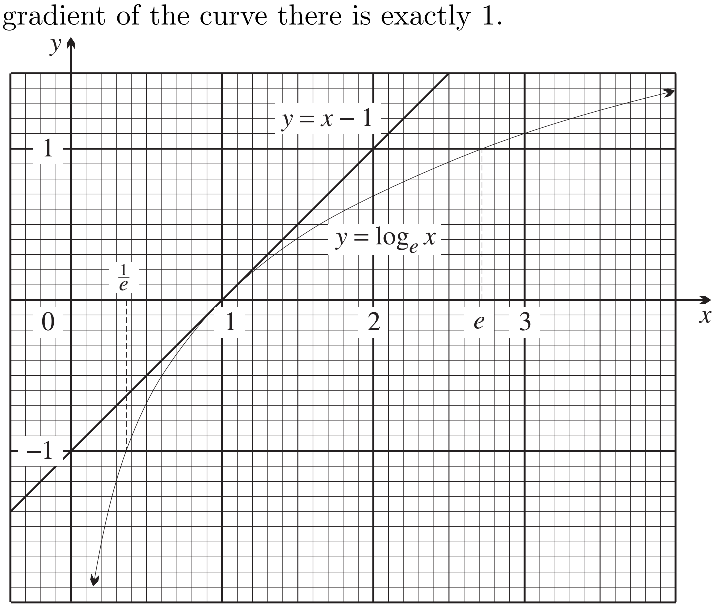
x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
$\log_e x$	-2	-1	0	1	2

x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2



We know already that the tangent to $y = e^x$ at its y-intercept $(0,1)$ has gradient 1. When this graph is reflected in the line $y = x$, this tangent is reflected to a tangent to $y = \log_e x$ at its x-intercept $(1,0)$. Both these tangents must have gradient 1.

Properties of the Graph of $y = \log_e x$: The graph of $y = \log_e x$ alone is drawn below on graph paper. The tangent has been drawn at the x-intercept $(1,0)$ to show that the gradient of the curve there is exactly 1.



The graph of $y = \log_e x$ and its properties must be thoroughly known. Its properties correspond to the properties listed earlier of its inverse function $y = e^x$.

- The domain is $x > 0$.
  The range is all values of y.
- The y-axis $x = 0$ is a vertical asymptote to the curve.
- As $x \to 0^+$, $y \to -\infty$.
  As $x \to \infty$, $y \to \infty$.
- The curve has gradient 1 at its x-intercept $(1,0)$.
- The curve is always concave down.

The Notation for the Logarithmic Function — $\log_e x$, $\log x$ and $\ln x$: In calculus, $\log_e x$ is the only logarithmic function that matters. It is often written simply as $\log x$ and from now on, if no base is given, base e will be understood.

The function $\log_e x$ is also written as $\ln x$, the 'n' standing for 'natural' logarithms. The 'n' also stands for 'Napierian' logarithms, in honour of the Scottish mathematician John Napier (1550–1617), who first developed tables of logarithms base e for calculations (first published in 1614).

NOTE: Be careful of the quite different convention on calculators, where $\boxed{\log}$ means $\log_{10} x$. The function labelled $\boxed{\ln}$ is used to find logarithms base e. Notice that the function $\boxed{e^x}$ is usually located on the same button as $\boxed{\ln}$ because the two functions $y = e^x$ and $y = \log_e x$ are inverses of each other.

7 THE NOTATION FOR THE LOGARITHMIC FUNCTION: In this course,

$$\log_e x = \log x = \ln x$$

all mean the same thing, that is, the logarithm base e of x.

ON CALCULATORS, HOWEVER:

$\boxed{\ln}$ is used to approximate $\log_e x$. It is the inverse function of $\boxed{e^x}$.

$\boxed{\log}$ is used to approximate $\log_{10} x$. It is the inverse function of $\boxed{10^x}$.

WORKED EXERCISE:

(a) Use your calculator to find, correct to four significant figures:
(i) $\log_e 10$ (ii) $\log_e \frac{1}{10}$ (iii) $\log_e 100$

(b) How are the answers to parts (ii) and (iii) related to the answer to part (i)?

SOLUTION:

(a) Using the function labelled $\boxed{\ln}$ on the calculator,
(i) $\log_e 10 \doteqdot 2{\cdot}303$ (ii) $\log_e \frac{1}{10} \doteqdot -2{\cdot}303$ (iii) $\log_e 100 \doteqdot 4{\cdot}605$

(b) Using the log laws,

$$\log_e \tfrac{1}{10} = -\log_e 10 \qquad \text{and} \qquad \log_e 10^2 = 2\log_e 10,$$

and these relationships are clear from the approximations above.

Combining the Logarithmic and Exponential Functions: As with any base, if the logarithmic and exponential functions base e are applied successively to any number, the result is the original number.

8 THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS ARE MUTUALLY INVERSE:

$$\log_e e^x = x \qquad \text{and} \qquad e^{\log_e x} = x.$$

Again, these identities follow immediately from the definition of logarithms, but here is further explanation if it is needed.

First, $\log_e e^x = x \log_e e$, by the log laws,
$= x \times 1$
$= x$.

Secondly, $e^{\log_e x} = x$ can be proven by taking logarithms of both sides:

$$\begin{aligned} \log_e \text{LHS} &= \log_e e^{\log_e x} \\ &= \log_e x, \text{ by the previous identity,} \\ &= \log_e \text{RHS}. \end{aligned}$$

WORKED EXERCISE:

Use the functions labelled $\boxed{\ln}$ and $\boxed{e^x}$ on your calculator to demonstrate that:

(a) $\log_e e^{10} = 10$ (b) $e^{\log_e 10} = 10$

SOLUTION:

(a) $\log_e e^{10} = \log_e 22\,026{\cdot}46\ldots$
$\doteqdot 10$

(b) $e^{\log_e 10} = e^{2{\cdot}302\,585\ldots}$
$\doteqdot 10$

Differentiating the Logarithmic Function: The logarithmic function $y = \log_e x$ can be differentiated quickly using the known derivative of its inverse function e^x.

Let $y = \log_e x$.
Then $x = e^y$, by the definition of logarithms.
Differentiating, $\dfrac{dx}{dy} = e^y$, since the exponential function is its own derivative,
$= x$, since $e^y = x$,
and inverting, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Hence the derivative of the logarithmic function is the reciprocal function.

9 THE DERIVATIVE OF THE LOGARITHMIC FUNCTION IS THE RECIPROCAL FUNCTION:

$$\frac{d}{dx} \log_e x = \frac{1}{x}$$

The following worked exercise uses this derivative to confirm that the tangent at the x-intercept has gradient 1. This was established earlier in the section using the graphical argument that the graphs of $y = e^x$ and $y = \log_e x$ are mutual reflections in $y = x$.

WORKED EXERCISE:

Find the gradient of the tangent to $y = \log_e x$ at its x-intercept.

SOLUTION:

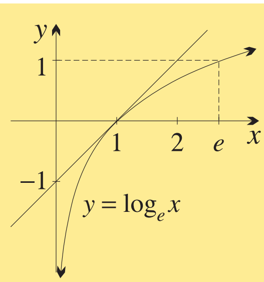
The function is $y = \log_e x$.

Differentiating, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

The graph crosses the x-axis at $A(1, 0)$.

Hence, substituting $x = 1$ into the formula for the derivative,

gradient at x-intercept $= 1$.



Transformations of the Logarithmic Graph: The usual methods of transforming graphs can be applied to $y = \log_e x$. When the graph is shifted sideways, the vertical asymptote at $x = 0$ will also be shifted.

A small table of approximate values can be a very useful check, particulary when a sequence of transformations is involved. Remember that $y = \log_e x$ has domain $x > 0$ and that the vertical asymptote is at $x = 0$.

WORKED EXERCISE:

Use transformations of the graph of $y = \log_e x$, and a table of values, to generate a sketch of each function. State the domain and show the x-intercept and the vertical asymptote.

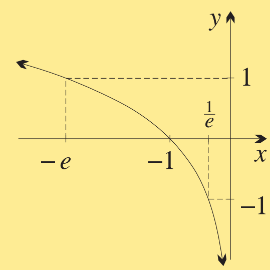
(a) $y = \log_e(-x)$ (b) $y = \log_e x - 2$ (c) $y = \log_e(x + 3)$

SOLUTION:

(a) Graph $y = \log_e(-x)$ by reflecting $y = \log_e x$ in the y-axis.

x	$-e$	-1	$-\frac{1}{e}$
y	1	0	-1

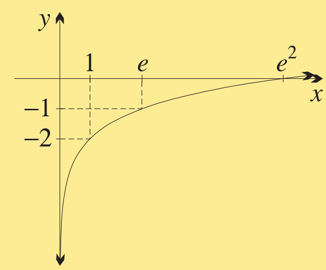
Domain: $x < 0$
x-intercept: $(-1, 0)$
Asymptote: $x = 0$ (the y-axis)



(b) Graph $y = \log_e x - 2$ by shifting $y = \log_e x$ down 2 units.

x	$\frac{1}{e}$	1	e	e^2
y	-3	-2	-1	0

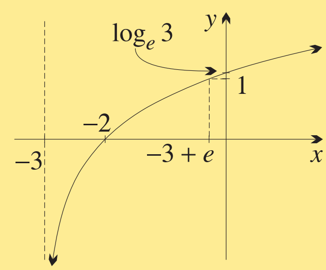
Domain: $x > 0$
x-intercept: $(e^2, 0)$
Asymptote: $x = 0$ (the y-axis)



(c) Graph $y = \log_e(x + 3)$ by shifting $y = \log_e x$ left 3 units.

x	$\frac{1}{e} - 3$	-2	$e - 3$
y	-1	0	1

Domain: $x > -3$
x-intercept: $(-2, 0)$
Asymptote: $x = -3$



Exercise 3B

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

1. Use your calculator to approximate the following, correct to four decimal places where necessary. Read the note above and remember to use the $\boxed{\ln}$ key on the calculator.

(a) $\log_e 1$ (c) $\ln 3$ (e) $\log \frac{1}{2}$ (g) $\ln \frac{1}{8}$
(b) $\log_e 2$ (d) $\ln 8$ (f) $\log \frac{1}{3}$ (h) $\ln \frac{1}{10}$

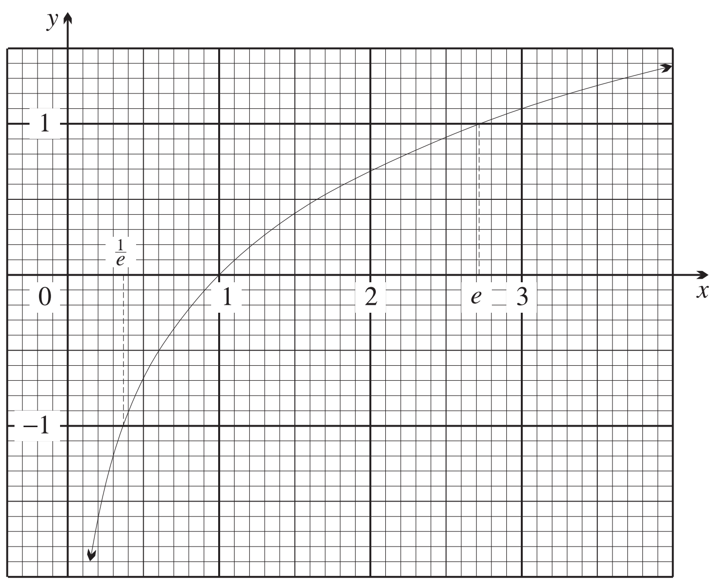
2. Use the functions labelled $\boxed{\ln}$ and $\boxed{e^x}$ on your calculator to demonstrate that:

(a) $\log_e e^2 = 2$ (c) $\log_e e^1 = 1$ (e) $\log_e e^{-3} = -3$
(b) $\log_e e^3 = 3$ (d) $\log_e e^{-2} = -2$ (f) $\log_e e^{-1} = -1$

3. Use the functions labelled $\boxed{\ln}$ and $\boxed{e^x}$ on your calculator to demonstrate that:

(a) $e^{\log_e 2} = 2$ (c) $e^{\log_e 1} = 1$ (e) $e^{\log_e \frac{1}{2}} = \frac{1}{2}$
(b) $e^{\log_e 3} = 3$ (d) $e^{\log_e 10} = 10$ (f) $e^{\log_e \frac{1}{10}} = \frac{1}{10}$

4.



(a) Photocopy the graph of $y = \log_e x$ above, and on it draw the tangent at $(1, 0)$, extending the tangent across to the y-axis.

(b) Measure the gradient of this tangent and confirm that it is equal to the reciprocal of the x-coordinate at the point of contact.

(c) Copy and complete the table of values to the right by measuring the gradient y' of each tangent.

(d) What do you notice about the y-intercepts of the tangents?

x	$\frac{1}{e}$	$\frac{1}{2}$	1	2	e
gradient y'					
$\frac{1}{x}$					

5. (a) Photocopy the graph of $y = \log_e x$ in the previous question, and on it draw the tangent at $y = 1$, extending the tangent across to the y-axis.

(b) Measure the gradient of this tangent and confirm that it is equal to the reciprocal of the x-coordinate at the point of contact.

(c) Repeat for the tangents at $y = 0$, -1, $\frac{1}{2}$ and $-\frac{1}{2}$.

(d) What do you notice about the y-intercepts of the tangents?

DEVELOPMENT

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

6. Sketch the graph of $y = \log_e x$ and use your knowledge of transformations to graph the following functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

(a) $y = \log_e x + 1$ (b) $y = \log_e x + 2$ (c) $y = \ln x - 1$ (d) $y = \ln x - 2$

7. (a) Copy and complete the following tables of values for the functions $y = \log_e x$ and $y = -\log_e x$, giving your answers correct to two decimal places.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_e x$					

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$-\log_e x$					

(b) Sketch both graphs on the same number plane, and draw the tangent to each at the x-intercept.

(c) Find the gradients of the two tangents, and hence explain why they are perpendicular.

8. Use the log laws to simplify:

(a) $e\log_e e$
(b) $\frac{1}{e}\log_e \frac{1}{e}$
(c) $3\log_e e^2$
(d) $\log_e \sqrt{e}$
(e) $e\log_e e^3 - e\log_e e$
(f) $\log_e e + \log_e \frac{1}{e}$
(g) $\log_e e^e$
(h) $\log_e(\log_e e^e)$
(i) $\log_e(\log_e(\log_e e^e))$

9. Express as a single logarithm:

(a) $\log_e 3 + \log_e 2$
(b) $\log_e 100 - \log_e 25$
(c) $\log_e 2 - \log_e 3 + \log_e 6$
(d) $\log_e 54 - \log_e 10 + \log_e 5$

10. (a) What is the x-coordinate of the point on the curve $y = \log x$ where $y = 0$?

(b) Use the result $\dfrac{d}{dx}\log x = \dfrac{1}{x}$ to find the gradient of the tangent at this point.

(c) Hence write down the equation of the tangent, and find its y-intercept.

(d) Repeat the above steps for the points where $y = -1$, 1 and 2.

(e) Compare the values of the y-intercepts with those found in question 5.

11. (a) Sketch a graph of $y = \log x$ and hence write down its domain.

(b) Write down y' and hence explain why the graph always has positive gradient.

(c) Find y'' and hence explain why the graph is always concave down.

12. Use the graph of $y = \log x$ and your knowledge of transformations to graph the following functions. Show the vertical asymptote and state the domain in each case.

(a) $y = \log(x-1)$
(b) $y = \log(x-3)$
(c) $y = \log(x+1)$
(d) $y = \log(x+2)$
(e) $y = -\log x$
(f) $y = \log(-x)$

CHALLENGE

13. Sketch the graph of $y = -\log_e x$ and use your knowledge of transformations to graph the following functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

(a) $y = -\log_e x + 1$ (b) $y = -\log_e x + 2$ (c) $y = -\log x - 1$ (d) $y = -\log x - 2$

14. The function $\log(1+x)$ may be approximated using the power series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad \text{for } 0 \le x \le 1.$$

Use this power series to approximate each of the following, correct to two decimal places. Then compare your answers with those given by your calculator.

(a) $\log 1\frac{1}{2}$ (b) $\log \frac{5}{4}$ (c) $\log \frac{1}{2}$ (d) $\log \frac{1}{3}$

3 C Differentiation of Logarithmic Functions

This section develops the standard forms and procedures for differentiating functions involving $\log_e x$.

Using the Basic Standard Form: The basic standard form for differentiating $\log_e x$ was developed in the previous section:

$$\frac{d}{dx}\log_e x = \frac{1}{x}\,.$$

WORKED EXERCISE:

Differentiate these functions using the standard form above:

(a) $y = x + \log_e x$ (b) $y = 5x^2 - 7\log_e x$

SOLUTION:

(a) $y = x + \log_e x$

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

(b) $y = 5x^2 - 7\log_e x$

$$\frac{dy}{dx} = 10x - \frac{7}{x}$$

Further Standard Forms: The following examples use the chain rule to develop two further standard forms for differentiation.

WORKED EXERCISE:

Differentiate the following functions using the chain rule:

(a) $\log_e(3x+4)$ (b) $\log_e(ax+b)$ (c) $\log_e(x^2+1)$

SOLUTION:

(a) Let $y = \log_e(3x+4)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{3x+4} \times 3$$

$$= \frac{3}{3x+4}\,.$$

Let $u = 3x + 4$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = \dfrac{1}{u}\,.$

(b) Let $y = \log_e(ax+b)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{ax+b} \times a$$

$$= \frac{a}{ax+b}\,.$$

Let $u = ax + b$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{u}\,.$

(c) Let $y = \log_e(x^2+1)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{x^2+1} \times 2x$$

$$= \frac{2x}{x^2+1}\,.$$

Let $u = x^2 + 1$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = \dfrac{1}{u}\,.$

Standard Forms for Differentiation: It is convenient to write down two further standard forms for differentiation based on the chain rule, giving three forms altogether.

10 THREE STANDARD FORMS FOR DIFFERENTIATING LOGARITHMIC FUNCTIONS:

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

$$\frac{d}{dx}\log_e(ax+b) = \frac{a}{ax+b}$$

$$\frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$$

The second of these standard forms was proven in part (b) of the previous worked exercise. Part (a) was an example of it.

The third standard form is a more general chain-rule extension — part (c) of the previous worked exercise was a good example of it. This standard form will be needed later for integration, but for now you can either learn it or apply the chain rule each time.

WORKED EXERCISE:

Using the standard forms developed above, differentiate:

(a) $y = \log_e(4x-9)$ (b) $y = \log_e(1 - \frac{1}{2}x)$ (c) $y = \log_e(4 + x^2)$

SOLUTION:

(a) For $y = \log_e(4x-9)$, use the second standard form with $ax+b = 4x-9$.

Thus $y' = \dfrac{4}{4x-9}$.

(b) For $y = \log_e(1-\frac{1}{2}x)$, use the second standard form with $ax+b = -\frac{1}{2}x+1$.

$$\text{Thus } y' = \frac{-\frac{1}{2}}{-\frac{1}{2}x+1}$$

$$= \frac{1}{x-2}, \quad \text{multiplying top and bottom by } -2.$$

(c) For $y = \log_e(4+x^2)$, use the third standard form with $f(x) = 4+x^2$.

Thus $y' = \dfrac{2x}{4+x^2}$.

Alternatively, use the chain rule, as in the previous worked exercise.

Using the Product and Quotient Rules: These two rules are used in the usual way.

WORKED EXERCISE: Differentiate:

(a) $x^3\log_e x$ by the product rule, (b) $\dfrac{\log_e(1-x)}{x}$ by the quotient rule.

SOLUTION:

(a) Let $y = x^3\log_e x$.

Then $y' = vu' + uv'$

$$= 3x^2\log_e x \;+\; x^3 \times \frac{1}{x}$$

$$= x^2(1 + 3\log_e x).$$

Let $u = x^3$
and $v = \log_e x$.
Then $u' = 3x^2$
and $v' = \dfrac{1}{x}$.

(b) Let $y = \dfrac{\log_e(1-x)}{x}$.

Let $u = \log_e(1-x)$
and $v = x$.
Then $u' = -\dfrac{1}{1-x}$
$= \dfrac{1}{x-1}$
and $v' = 1$.

Then $y' = \dfrac{vu' - uv'}{v^2}$
$= \dfrac{\dfrac{x}{x-1} - \log_e(1-x)}{x^2}$
$= \dfrac{x}{(x-1)x^2} - \dfrac{\log_e(1-x)}{x^2}$
$= \dfrac{1}{x(x-1)} - \dfrac{\log_e(1-x)}{x^2}$.

Using the Log Laws to Make Differentiation Easier: The following examples show the use of the log laws to avoid a combination of the chain and quotient rules.

WORKED EXERCISE:

Use the log laws to simplify each expression, then differentiate it:

(a) $\log_e 7x^2$ (b) $\log_e(3x-7)^5$ (c) $\log_e \dfrac{1+x}{1-x}$.

SOLUTION:

(a) Let $y = \log_e 7x^2$.
Then $y = \log_e 7 + \log_e x^2$ (The log of a product is the sum of the logs.)
$= \log_e 7 + 2\log_e x$. (The log of a power is the multiple of the log.)
Hence $\dfrac{dy}{dx} = \dfrac{2}{x}$. (Note that $\log_e 7$ is a constant, with derivative zero.)

(b) Let $y = \log_e(3x-7)^5$.
Then $y = 5\log_e(3x-7)$. (The log of a power is the multiple of the log.)
Hence $\dfrac{dy}{dx} = \dfrac{15}{3x-7}$.

(c) Let $y = \log_e \dfrac{1+x}{1-x}$.
Then $y = \log_e(1+x) - \log_e(1-x)$, (Log of a quotient is the difference of the logs.)
so $\dfrac{dy}{dx} = \dfrac{1}{1+x} + \dfrac{1}{1-x}$.

Exercise 3C

1. Use the standard form $\dfrac{d}{dx}\log_e(ax+b) = \dfrac{a}{ax+b}$ to differentiate:

(a) $y = \log_e(x+2)$
(b) $y = \log_e(x-3)$
(c) $y = \log_e(3x+4)$
(d) $y = \log_e(5x+1)$
(e) $y = \log_e(2x-1)$
(f) $y = \log_e(4x-3)$
(g) $y = \log_e(-4x+1)$
(h) $y = \log_e(-3x+4)$
(i) $y = \log_e(-2x-7)$
(j) $y = \log_e(-3x-6)$
(k) $y = 3\log_e(2x+4)$
(l) $y = 5\log_e(3x-2)$

2. Differentiate these functions:

(a) $y = \log_e 2x$
(b) $y = \log_e 5x$
(c) $y = \log_e 3x$
(d) $y = \log_e 7x$
(e) $y = 4\log_e 7x$
(f) $y = 3\log_e 5x$
(g) $y = 4\log_e 6x$
(h) $y = 3\log_e 9x$

3. Find $\dfrac{dy}{dx}$ for each function. Then evaluate $\dfrac{dy}{dx}$ at $x = 3$.

(a) $y = \log_e(x+1)$
(b) $y = \log_e(2x-1)$
(c) $y = \log_e(2x-5)$
(d) $y = \log_e(4x+3)$
(e) $y = 5\log_e(x+1)$
(f) $y = 6\log_e(2x+9)$

4. Differentiate these functions:

(a) $2 + \log_e x$
(b) $7 - \log_e x$
(c) $5 - \log_e(x+1)$
(d) $x + 4\log_e x$
(e) $2x + 1 + 3\log_e x$
(f) $\log_e(2x-1) + 3x^2$
(g) $\log_e(3-x) + x^2 + 3x$
(h) $2x^4 - \log_e 2x$
(i) $x^3 - 3x + 4 + \log_e(5x-7)$

DEVELOPMENT

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

5. Use the log laws to simplify each function, then differentiate it.

(a) $y = \log x^3$
(b) $y = \log x^2$
(c) $y = \log x^{-3}$
(d) $y = \log x^{-2}$
(e) $y = \log \sqrt{x}$
(f) $y = \log \sqrt{x+1}$

6. Differentiate these functions:

(a) $y = \log_e \frac{1}{2}x$
(b) $y = \log_e \frac{1}{3}x$
(c) $y = 3\log_e \frac{1}{5}x$
(d) $y = -6\log_e \frac{1}{2}x$
(e) $y = x + \log_e \frac{1}{7}x$
(f) $y = 4x^3 - \log_e \frac{1}{5}x$

7. Use the standard form $\dfrac{d}{dx}\log_e f(x) = \dfrac{f'(x)}{f(x)}$ to differentiate:

(a) $\log_e(x^2+1)$
(b) $\log_e(x^2+3x+2)$
(c) $\log(2-x^2)$
(d) $\log(1+2x^3)$
(e) $\ln(1+e^x)$
(f) $\ln(e^x-2)$
(g) $x + 3 - \log(x^2+x)$
(h) $x^2 + \log(x^3-x)$
(i) $4x^3 - 5x^2 + \log(2x^2-3x+1)$

8. Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = \log x$ at the points where:

(a) $x = 1$
(b) $x = 3$
(c) $x = \frac{1}{2}$
(d) $x = 4$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

9. Differentiate these functions using the product rule:

(a) $x\log x$
(b) $x\log(2x+1)$
(c) $(2x+1)\log x$
(d) $x^4\log x$
(e) $(x+3)\log(x+3)$
(f) $(x-1)\log(2x+7)$
(g) $e^x\log x$
(h) $e^{-x}\log x$

10. Differentiate these functions using the quotient rule:

(a) $y = \dfrac{\log x}{x}$
(b) $y = \dfrac{\log x}{x^2}$
(c) $y = \dfrac{x}{\log x}$
(d) $y = \dfrac{x^2}{\log x}$
(e) $y = \dfrac{\log x}{e^x}$
(f) $y = \dfrac{e^x}{\log x}$

11. Use the log laws to simplify the following, then differentiate them.

(a) $y = \log 5x^3$
(b) $y = \log 3x^4$
(c) $y = \log \sqrt[3]{x}$
(d) $y = \log \sqrt[4]{x}$
(e) $y = \log \dfrac{3}{x}$
(f) $y = \log \dfrac{2}{5x}$
(g) $y = \log \sqrt{2-x}$
(h) $y = \log \sqrt{5x+2}$

12. Find the first and second derivatives of each function, then evaluate both derivatives at the value given.

(a) $f(x) = \log(x-1)$, $x = 3$
(b) $f(x) = \log(2x+1)$, $x = 0$
(c) $f(x) = \log x^2$, $x = 2$
(d) $f(x) = x\log x$, $x = e$

13. Differentiate each of the following using the chain, product and quotient rules. Then find any values of x for which the derivative is zero.

(a) $y = x \log x - x$
(b) $y = x^2 \log x$
(c) $y = \dfrac{\log x}{x}$
(d) $y = (\log x)^2$
(e) $y = (\log x)^4$
(f) $y = \dfrac{1}{1 + \log x}$
(g) $y = (2 \log x - 3)^4$
(h) $y = \dfrac{1}{\log x}$
(i) $y = \log(\log x)$

14. Find the point(s) where the tangent to each of these curves is horizontal:

(a) $y = x \ln x$
(b) $y = \dfrac{1}{x} + \ln x$

CHALLENGE

15. (a) Find the derivative of $y = \dfrac{x}{\log x}$.

(b) Hence show that $y = \dfrac{x}{\log x}$ is a solution of the equation $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2$ by substituting separately into the LHS and the RHS.

16. Use the log laws to simplify the following, then differentiate them.

(a) $y = \log_e(x + 2)(x + 1)$
(b) $y = \log_e(x + 5)(3x - 4)$
(c) $y = \ln \dfrac{1 + x}{1 - x}$
(d) $y = \ln \dfrac{3x - 1}{x + 2}$
(e) $y = \log \dfrac{(x - 4)^2}{3x + 1}$
(f) $y = \log x\sqrt{x + 1}$

17. Use the log laws to simplify the following, then differentiate them.

(a) $y = \log_e 2^x$
(b) $y = \log_e e^x$
(c) $y = \log_e x^x$

3 D Applications of Differentiation of $\log x$

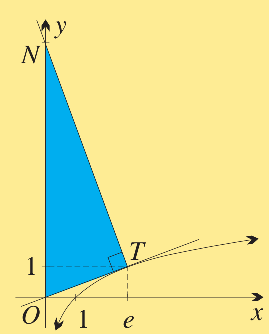
Differentiation can now be applied in the usual way to study the graphs of functions involving $\log_e x$.

The Geometry of Tangents and Normals: The derivative can be used as usual to investigate the geometry of tangents and normals to a curve.

WORKED EXERCISE:

(a) Show that the tangent to $y = \log_e x$ at $T(e, 1)$ has equation $x = ey$.
(b) Find the equation of the normal to $y = \log_e x$ at $T(e, 1)$.
(c) Sketch the curve, the tangent and the normal, and find the area of the triangle formed by the y-axis and the tangent and normal at T.

SOLUTION:



(a) Differentiating, $\dfrac{dy}{dx} = \dfrac{1}{x}$,

so the tangent at $T(e, 1)$ has gradient $\dfrac{1}{e}$,

and the tangent is
$$y - 1 = \frac{1}{e}(x - e)$$
$$ey - e = x - e$$
$$x = ey$$
$$y = \frac{x}{e}.$$

Notice that this tangent has gradient $\dfrac{1}{e}$ and passes through the origin.

(b) The tangent at $T(e, 1)$ has gradient $\dfrac{1}{e}$, so the normal there has gradient $-e$.

Hence the normal has equation $y - 1 = -e(x - e)$

$$y = -ex + (e^2 + 1).$$

(c) Substituting $x = 0$, the normal has y-intercept $N(0, e^2 + 1)$.

Hence the base ON of $\triangle ONT$ is $(e^2 + 1)$ and its altitude is e.

Thus the triangle $\triangle ONT$ has area $\frac{1}{2}e(e^2 + 1)$ square units.

An Example of Curve Sketching: Here are the usual six steps of the 'curve-sketching menu' applied to the function $y = x \log_e x$.

WORKED EXERCISE:

Sketch the graph of $y = x \log x$ after carrying out the following steps:

(a) Write down the domain.

(b) Test whether the function is even or odd or neither.

(c) Find any zeroes of the function and examine its sign.

(d) Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. [HINT: You may assume that $x \log_e x \to 0$ as $x \to 0^+$.]

(e) Find any stationary points and examine their nature.

(f) Find any points of inflexion.

SOLUTION:

(a) The domain is $x > 0$, because $\log_e x$ is undefined for $x \leq 0$.

(b) The function is undefined when x is negative, so it is neither even nor odd.

(c) The only zero is at $x = 1$, and the curve is continuous for $x > 0$.

Taking test points at $x = e$ and at $x = \dfrac{1}{e}$:

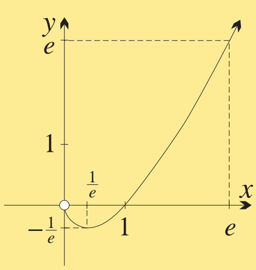
when $x = e$, $y = e \log_e e$
$= e \times 1$
$= e$,

when $x = \dfrac{1}{e}$, $y = e^{-1} \log_e e^{-1}$
$= e^{-1} \times (-1)$
$= -\dfrac{1}{e}$.

This gives the following table of signs:

x	0	$\frac{1}{e}$	1	e
y	$*$	$-\frac{1}{e}$	0	e
sign	$*$	$-$	0	$+$

Hence y is negative for $0 < x < 1$ and positive for $x > 1$.



(d) As given in the hint, $y \to 0$ as $x \to 0^+$.

Also, $y \to \infty$ as $x \to \infty$.

(e) Differentiating by the product rule,

$$f'(x) = vu' + uv'$$
$$= \log_e x + x \times \frac{1}{x}$$
$$= \log_e x + 1,$$

and $f''(x) = \dfrac{1}{x}$.

Let $u = x$
and $v = \log_e x$.
Then $u' = 1$
and $v' = \dfrac{1}{x}$.

Putting $f'(x) = 0$ gives $\log_e x = -1$

$$x = \frac{1}{e}.$$

Substituting, $f''(\tfrac{1}{e}) = e > 0$
and $f(\tfrac{1}{e}) = -\tfrac{1}{e}$, as above,
so $(\tfrac{1}{e}, -\tfrac{1}{e})$ is a minimum turning point.
[A more subtle point: $f'(x) \to -\infty$ as $x \to 0^+$,
so the curve becomes vertical near the origin.]

(f) Since $f''(x)$ is always positive, there are no inflexions, and the curve is always concave up.

A Difficulty with the Limits of $x \log_e x$ and $\dfrac{\log_e x}{x}$: The curve-sketching exercise above involved knowing the behaviour of $x \log_e x$ as $x \to 0^+$. When x is a small positive number, $\log_e x$ is a large negative number, and so it is not immediately clear whether the product $x \log_e x$ becomes large or small as $x \to 0^+$.

In fact, $x \log_e x \to 0$ as $x \to 0^+$, and x is said to *dominate* $\log_e x$, in the same way that e^x dominated x in Section 2D. Here is a table of values that should make it reasonably clear that $\lim\limits_{x \to 0^+} x \log_e x = 0$:

x	$\frac{1}{e}$	$\frac{1}{e^2}$	$\frac{1}{e^3}$	$\frac{1}{e^4}$	$\frac{1}{e^5}$	$\frac{1}{e^6}$	$\frac{1}{e^7}$	…
$x \log_e x$	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	…
approx.	$-0{\cdot}37$	$-0{\cdot}27$	$-0{\cdot}15$	$-0{\cdot}073$	$-0{\cdot}034$	$-0{\cdot}015$	$-0{\cdot}006$	…

Such limits are usually not regarded as part of the 2 Unit course and would be normally given in any curve-sketching question where they were required.

A similar problem arises with the behaviour of $\dfrac{\log_e x}{x}$ as $x \to \infty$, because both top and bottom get large when x is large. Again, x dominates $\log_e x$, meaning that $\dfrac{\log_e x}{x} \to 0$ as $x \to \infty$, as the following table should make reasonably obvious:

x	e	e^2	e^3	e^4	e^5	e^6	e^7	…
$\frac{\log_e x}{x}$	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	…
approx	$0{\cdot}37$	$0{\cdot}27$	$0{\cdot}15$	$0{\cdot}073$	$0{\cdot}034$	$0{\cdot}015$	$0{\cdot}006$	…

Again, such limits would normally be given in any question where they arose.

Exercise 3D

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

1. In this question you will need the point–gradient formula $y - y_1 = m(x - x_1)$ for the equation of a straight line.
(a) Use calculus to find the gradient of the tangent to $y = \log_e x$ at $P(e, 1)$.
(b) Hence find the equation of the tangent at P, and prove that it passes through O.

2. (a) Use calculus to find the gradient of the tangent to $y = \log_e x$ at $Q(1, 0)$.
(b) Hence find the equation of the tangent at Q, and prove that it passes through $A(0, -1)$.

3. (a) Use calculus to find the gradient of the tangent to $y = \log_e x$ at $R(\frac{1}{e}, -1)$.
(b) Hence find the equation of the tangent at R, and prove that it passes through $B(0, -2)$.

4. (a) Find the gradient of the tangent to $y = \log_e x$ at the point $A(1, 0)$.
(b) Show that the gradient of the normal is -1.
(c) Hence find the equation of the normal at A, and its y-intercept.

5. Find, giving answers in the form $y = mx + b$, the equations of the tangent and normal to:
(a) $y = 4\log_e x$ at the point $Q(1, 0)$,
(b) $y = \log_e x + 3$ at the point $R(1, 3)$,
(c) $y = 2\log_e x - 2$ at the point $S(1, -2)$,
(d) $y = 1 - 3\log_e x$ at the point $T(1, 1)$.

6. (a) Show that the point $P(1, 0)$ lies on the curve $y = \log_e(3x - 2)$.
(b) Find the gradients of the tangent and normal at P.
(c) Find the equations of the tangent and the normal at P, and their y-intercepts.
(d) Find the area of the triangle formed by the tangent, the normal and the y-axis.

7. In question 1 you showed that the tangent at $P(e, 1)$ on the curve $y = \log_e x$ passes through the origin. Sketch the graph, showing the tangent, and explain graphically why no other tangent can pass through the origin.

DEVELOPMENT

8. (a) Find the gradient of the tangent to $y = \log x - \dfrac{x}{2} + 1$ at $x = 1$.
(b) Write down the equation of the tangent, and show that it passes through the origin.

9. (a) Find the equation of the tangent to $y = (2 - x)\log x$ at $x = 2$.
(b) Hence find the y-intercept of the tangent.

10. (a) Write down the domain of $y = \log x$ and the derivative of $y = \log x$.
(b) Hence explain why the gradient of a tangent to $y = \log x$ must be positive.
(c) Explain also why the gradient of a normal to $y = \log x$ must be negative.
(d) Draw the graph of $y = \log x$ to confirm your answers to parts (c) and (d).
(e) Find y'' and show that it must always be negative. What aspect of the curve does this describe?

11. (a) Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient $\frac{1}{2}$. Then find the equation of the tangent and normal there, in the form $y = mx + b$.
(b) Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient 2. Then find the equation of the tangent and normal there, in the form $y = mx + b$.

12. (a) Write down the natural domain of $y = x - \log_e x$.
(b) Determine its first two derivatives.
(c) Show that the curve is concave up for all values of x in its natural domain.
(d) Find the minimum turning point.
(e) Sketch the curve and write down its range.
(f) Finally sketch the curve $y = \log_e x - x$ by recognising the simple transformation.

13. (a) Write down the domain of $y = \dfrac{1}{x} + \log x$.
(b) Show that the first and second derivatives may be expressed as single fractions as $y' = \dfrac{x-1}{x^2}$ and $y'' = \dfrac{2-x}{x^3}$.
(c) Show that the curve has a minimum at $(1, 1)$ and an inflexion at $(2, \frac{1}{2} + \log 2)$.
(d) Sketch the graph and write down its range.

14. (a) Use your calculator to complete the table of values for $y = \dfrac{\log x}{x}$ to the right. Then use the table to help you guess the value of $\displaystyle\lim_{x \to \infty} \frac{\log x}{x}$.

x	2	5	10	20	40	4000
y						

(b) Use your calculator to complete the table of values for $y = x \log x$ to the right. Then use the table to help you guess the value of $\displaystyle\lim_{x \to 0^+} x \log x$.

x	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{4000}$
y						

15. Consider the curve $y = x \log x$.
(a) Write down the domain and x-intercept.
(b) Show that $y' = 1 + \log x$ and find y''.
(c) Hence show there is one stationary point and determine its nature.
(d) Given that $y \to 0^-$ as $x \to 0^+$ and that the tangent becomes closer and closer to vertical as $x \to 0^+$, sketch the curve and write down its range.

CHALLENGE

16. Consider the curve $y = x \log x - x$.
(a) Write down the domain and x-intercept.
(b) Show that $y' = \log x$ and find y''.
(c) Hence show there is one stationary point and determine its nature.
(d) Given that $y \to 0^-$ as $x \to 0^+$ and that the tangent approaches the vertical as $x \to 0^+$, sketch the curve and write down its range.

17. (a) Write down the domain of $y = \log(1 + x^2)$.
(b) Show that $y' = \dfrac{2x}{1+x^2}$ and $y'' = \dfrac{2(1-x^2)}{(1+x^2)^2}$.
(c) Hence show that $y = \log(1 + x^2)$ has one stationary point, and determine its nature.
(d) Find the coordinates of the two points of inflexion.
(e) Hence sketch the curve, and then write down its range.

18. (a) Find the domain of $y = (\ln x)^2$.

(b) Find y' and show that $y'' = \dfrac{2(1 - \ln x)}{x^2}$.

(c) Hence show that the curve has an inflexion at $x = e$.

(d) Classify the stationary point at $x = 1$, sketch the curve, and write down the range.

19. (a) Write down the domain of $y = \dfrac{\log x}{x}$.

(b) Show that $y' = \dfrac{1 - \log x}{x^2}$ and $y'' = \dfrac{2\log x - 3}{x^3}$.

(c) Find any stationary points and determine their nature.

(d) Find the exact coordinates of the lone point of inflexion.

(e) Sketch the curve, and write down its range. You may assume that $y \to 0$ as $x \to \infty$, and that $y \to -\infty$ as $x \to 0^+$.

20. (a) Show that the tangent to $y = \log_e x$ at $A(a, \log_e a)$ is $x - ay = a(1 - \log_e a)$.

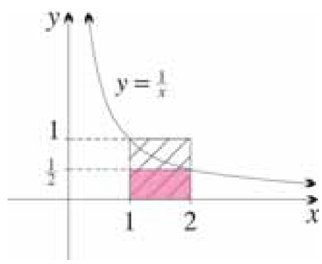
(b) Hence show that the only point on $y = \log_e x$ where the tangent passes through the origin is $(e, 1)$.

3 E Integration of the Reciprocal Function

The reciprocal function $y = \dfrac{1}{x}$ is obviously an important function, because it is required whenever two quantities are inversely proportional to each other. So far, however, it has not been possible to integrate the reciprocal function. The usual rule for integrating powers of x gives nonsense:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{with } n = -1 \text{ gives} \quad \int x^{-1}\,dx = \frac{x^0}{0},$$

which is undefined because of the division by zero.



Yet the graph of $y = \dfrac{1}{x}$ to the right shows that there should be no problem with definite integrals involving $\dfrac{1}{x}$, provided that the integral does not cross the discontinuity at $x = 0$. For example, the diagram shows the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$, which the little rectangles show must have a value between $\frac{1}{2}$ and 1.

Integration of the Reciprocal Function: Reversing the standard forms for differentiating logarithmic functions will give the standard forms needed.

First, $\qquad \dfrac{d}{dx} \log_e x = \dfrac{1}{x}$.

Reversing this, $\qquad \displaystyle\int \frac{1}{x}\,dx = \log_e x + C.$

This gives a new standard form for integrating the reciprocal function.
The only qualification is that $x > 0$, otherwise $\log_e x$ is undefined.

11 THE PRIMITIVE OF THE RECIPROCAL FUNCTION:

$$\int \frac{1}{x}\,dx = \log_e x + C, \quad \text{provided that } x > 0.$$

WORKED EXERCISE:

(a) Find the definite integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$ sketched above.

(b) Approximate the integral correct to three decimal places and verify that

$$\tfrac{1}{2} < \int_1^2 \frac{1}{x}\,dx < 1.$$

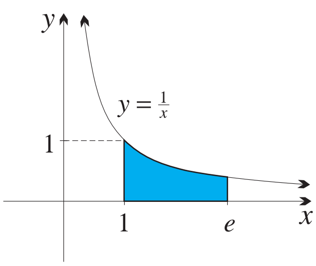
SOLUTION:

(a)
$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx &= \Big[\log_e x\Big]_1^2 \\ &= \log_e 2 - \log_e 1 \\ &= \log_e 2, \quad \text{since } \log_e 1 = 0.\end{aligned}$$

(b) Hence $\displaystyle\int_1^2 \frac{1}{x}\,dx \doteqdot 0{\cdot}693$,

which is indeed between $\frac{1}{2}$ and 1, as the diagram above indicated.

A Characterisation of e: Integrating the reciprocal function from 1 to e gives an amazingly simple result:



$$\begin{aligned}\int_1^e \frac{1}{x}\,dx &= \Big[\log_e x\Big]_1^e \\ &= \log_e e - \log_e 1 \\ &= 1 - 0 \\ &= 1.\end{aligned}$$

The integral is sketched to the right. The example is very important because it characterises e as the number satisfying $\displaystyle\int_1^e \frac{1}{x}\,dx = 1$. This integral can actually be taken as the definition of e, as is done, for example, in the 3 Unit Year 11 volume of this textbook.

Three Standard Forms: Reversing the other standard forms for differentiation gives two more standard forms. To avoid complications, constants of integration have been ignored until the results are summarised in Box 12 on the next page.

First, $$\frac{d}{dx}\log_e(ax+b) = \frac{a}{ax+b}\,.$$

Reversing this, $$\int \frac{a}{ax+b}\,dx = \log_e(ax+b),$$

and dividing by a, $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b).$$

Secondly, $$\frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}\,.$$

Reversing this, $$\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x).$$

STANDARD FORMS FOR INTEGRATING RECIPROCAL FUNCTIONS:

12

$$\int \frac{1}{x}\,dx = \log_e x + C, \qquad \text{provided that } x > 0.$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + C, \qquad \text{provided that } ax+b > 0.$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + C, \qquad \text{provided that } f(x) > 0.$$

WORKED EXERCISE:

Evaluate these definite integrals using the first two standard forms above:

(a) $\displaystyle\int_e^{e^2} \frac{5}{x}\,dx$ (b) $\displaystyle\int_0^1 \frac{1}{2x+1}\,dx$

SOLUTION:

(a) $$\begin{aligned}\int_e^{e^2} \frac{5}{x}\,dx &= 5\Big[\log_e x\Big]_e^{e^2}\\ &= 5(\log_e e^2 - \log_e e)\\ &= 5(2-1)\\ &= 5\end{aligned}$$

(b) $$\begin{aligned}\int_0^1 \frac{1}{2x+1}\,dx &= \tfrac{1}{2}\Big[\log_e(2x+1)\Big]_0^1 \qquad \text{(Here } a=2 \text{ and } b=1.)\\ &= \tfrac{1}{2}(\log_e 3 - \log_e 1)\\ &= \tfrac{1}{2}(\log_e 3 - 0)\\ &= \tfrac{1}{2}\log_e 3\end{aligned}$$

Using the Third Standard Form: The vital point in using the third standard form

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x)$$

is that the top must be the derivative of the bottom.

WORKED EXERCISE:

Evaluate these definite integrals using the third standard form above:

(a) $\displaystyle\int_0^1 \frac{2x}{x^2+2}\,dx$ (b) $\displaystyle\int_0^2 \frac{x}{9-x^2}\,dx$

SOLUTION:

(a) Let $f(x) = x^2 + 2.$

Then $f'(x) = 2x.$

Hence in the fraction $\dfrac{2x}{x^2+1}$, the top is the derivative of the bottom.

Thus, using the third standard form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x)$,

$$\begin{aligned}\int_0^1 \frac{2x}{x^2+2}\,dx &= \Big[\log_e(x^2+2)\Big]_0^1\\ &= \log_e 3 - \log_e 2.\end{aligned}$$

(b) Let $f(x) = 9 - x^2$.

Then $f'(x) = -2x$.

The first step is to make the top the derivative of the bottom:

$$\begin{aligned}\int_0^2 \frac{x}{9-x^2}\,dx &= -\tfrac{1}{2}\int_0^2 \frac{-2x}{9-x^2}\,dx, \quad \text{which has the form } \int \frac{f'(x)}{f(x)}\,dx,\\ &= -\tfrac{1}{2}\Big[\log_e(9-x^2)\Big]_0^2\\ &= -\tfrac{1}{2}(\log_e 5 - \log_e 9)\\ &= -\tfrac{1}{2}(\log_e 5 - 2\log_e 3)\\ &= \log_e 3 - \tfrac{1}{2}\log_e 5.\end{aligned}$$

Given the Derivative, Find the Function: Finding the function from the derivative involves a constant that can be found if the value of y is known for some value of x.

WORKED EXERCISE:

(a) Find $f(x)$, if $f'(x) = \dfrac{2}{3-x}$ and the graph passes through the origin.

(b) Hence find $f(2)$.

SOLUTION:

(a) Here $f'(x) = \dfrac{2}{3-x}$.

Taking the primitive, $f(x) = -2\log_e(3-x) + C$, for some constant C.

Since $f(0) = 0$, $0 = -2\log_e 3 + C$

$C = 2\log_e 3$.

Hence $f(x) = 2\log_e 3 - 2\log_e(3-x)$.

(b) Substituting $x = 2$ gives

$$\begin{aligned}f(2) &= 2\log_e 3 - 2\log_e 1\\ &= 2\log_e 3.\end{aligned}$$

A Primitive of $\log_e x$: The following exercise is difficult, but it is important because it produces a primitive of $\log_e x$, which the theory has not yielded so far. There is no need to memorise the result.

WORKED EXERCISE: [These questions are always difficult.]

(a) Differentiate $x\log_e x$ by the product rule.

(b) Show by differentiation that $x\log_e x - x$ is a primitive of $\log_e x$.

(c) Use this result to evaluate $\displaystyle\int_1^e \log_e x\,dx$.

SOLUTION:

(a) Differentiating by the product rule,

$$\begin{aligned}\frac{d}{dx}(x\log_e x) &= vu' + uv'\\ &= \log_e x + x \times \frac{1}{x},\\ &= 1 + \log_e x.\end{aligned}$$

Let $u = x$

and $v = \log_e x$.

Then $u' = 1$

and $v' = \dfrac{1}{x}$.

(b) Let $y = x\log_e x - x$.

Then $y' = (1 + \log_e x) - 1$, using the result of part (a),

$= \log_e x$.

Reversing this result gives the primitive of $\log_e x$,

$$\int \log_e x\,dx = x\log_e x - x + C.$$

(c) Part (b) can now be used to find the definite integral:

$$\begin{aligned}\int_1^e \log_e x\,dx &= \Big[x\log_e x - x\Big]_1^e \\ &= (e\log_e e - e) - (1\log_e 1 - 1) \\ &= (e\log_e e - e) - (0 - 1) \\ &= (e - e) + 1 \\ &= 1.\end{aligned}$$

Exercise **3E**

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

1. First rewrite each integral using the result $\int \frac{k}{x}\,dx = k\int \frac{1}{x}\,dx$, where k is a constant. Then use the standard form $\int \frac{1}{x}\,dx = \log_e x + C$ to integrate it.

(a) $\int \frac{2}{x}\,dx$ (c) $\int \frac{1}{2x}\,dx$ (e) $\int \frac{4}{5x}\,dx$

(b) $\int \frac{5}{x}\,dx$ (d) $\int \frac{1}{3x}\,dx$ (f) $\int \frac{3}{2x}\,dx$

2. Use the standard form $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + C$ to find the following indefinite integrals:

(a) $\int \frac{1}{4x+1}\,dx$ (e) $\int \frac{6}{3x+2}\,dx$ (i) $\int \frac{dx}{3-x}$

(b) $\int \frac{1}{2x+1}\,dx$ (f) $\int \frac{15}{5x+1}\,dx$ (j) $\int \frac{dx}{7-2x}$

(c) $\int \frac{1}{5x-3}\,dx$ (g) $\int \frac{2}{2x-1}\,dx$ (k) $\int \frac{4\,dx}{5x-1}$

(d) $\int \frac{1}{7x-2}\,dx$ (h) $\int \frac{4}{4x+3}\,dx$ (l) $\int \frac{12\,dx}{1-3x}$

3. Evaluate the following definite integrals. Simplify your answers where possible.

(a) $\int_1^5 \frac{1}{x}\,dx$ (c) $\int_2^8 \frac{1}{x}\,dx$ (e) $\int_1^4 \frac{dx}{2x}$

(b) $\int_1^3 \frac{1}{x}\,dx$ (d) $\int_3^9 \frac{1}{x}\,dx$ (f) $\int_5^{15} \frac{dx}{5x}$

4. Evaluate the following definite integrals, then use the function labelled $\boxed{\ln}$ on your calculator to approximate each integral correct to four significant figures.

(a) $\displaystyle\int_0^1 \frac{dx}{x+1}$ (d) $\displaystyle\int_1^3 \frac{dx}{3x-1}$ (g) $\displaystyle\int_{-1}^1 \frac{3}{7-3x}\,dx$

(b) $\displaystyle\int_1^3 \frac{dx}{x+2}$ (e) $\displaystyle\int_{-1}^2 \frac{dx}{2x+3}$ (h) $\displaystyle\int_1^4 \frac{6}{4x-1}\,dx$

(c) $\displaystyle\int_4^{18} \frac{dx}{x-2}$ (f) $\displaystyle\int_1^2 \frac{3}{5-2x}\,dx$ (i) $\displaystyle\int_0^{11} \frac{5}{2x+11}\,dx$

5. Evaluate the following definite integrals. Simplify your answers where possible.

(a) $\displaystyle\int_1^e \frac{dx}{x}$ (b) $\displaystyle\int_1^{e^2} \frac{dx}{x}$ (c) $\displaystyle\int_e^{e^4} \frac{dx}{x}$ (d) $\displaystyle\int_{\sqrt{e}}^e \frac{dx}{x}$

6. Find primitives of the following by first writing them as separate fractions:

(a) $\dfrac{x+1}{x}$ (c) $\dfrac{2-x}{3x}$ (e) $\dfrac{3x^2-2x}{x^2}$ (g) $\dfrac{3x^3+4x-1}{x^2}$

(b) $\dfrac{x+3}{5x}$ (d) $\dfrac{1-8x}{9x}$ (f) $\dfrac{2x^2+x-4}{x}$ (h) $\dfrac{x^4-x+2}{x^2}$

DEVELOPMENT

7. In each case show that the numerator is the derivative of the denominator. Then use the result $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + C$ to integrate the expression.

(a) $\dfrac{2x}{x^2-9}$ (d) $\dfrac{5-6x}{2+5x-3x^2}$ (g) $\dfrac{e^x}{1+e^x}$

(b) $\dfrac{6x+1}{3x^2+x}$ (e) $\dfrac{x+3}{x^2+6x-1}$ (h) $\dfrac{e^{-x}}{1+e^{-x}}$

(c) $\dfrac{2x+1}{x^2+x-3}$ (f) $\dfrac{3-x}{12x-3-2x^2}$ (i) $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

8. Find $f(x)$, and then find $f(2)$, given that:

(a) $f'(x) = 1 + \dfrac{2}{x}$ and $f(1) = 1$ (c) $f'(x) = 3 + \dfrac{5}{2x-1}$ and $f(1) = 0$

(b) $f'(x) = 2x + \dfrac{1}{3x}$ and $f(1) = 2$ (d) $f'(x) = 6x^2 + \dfrac{15}{3x+2}$ and $f(1) = 5\log 5$

9. (a) Find y as a function of x if $y' = \dfrac{1}{4x}$ and $y = 1$ when $x = e^2$. What is the x-intercept of this curve?

(b) The gradient of a curve is given by $y' = \dfrac{2}{x+1}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve?

(c) Find $y(x)$, given that $y' = \dfrac{2x+5}{x^2+5x+4}$ and $y = 1$ when $x = 1$. Hence evaluate $y(0)$.

(d) Write down the equation of the family of curves with the property $y' = \dfrac{2+x}{x}$. Hence find the curve that passes through $(1, 1)$ and evaluate y at $x = 2$ for this curve.

(e) Given that $f''(x) = \dfrac{1}{x^2}$, $f'(1) = 0$ and $f(1) = 3$, find $f(x)$ and hence evaluate $f(e)$.

10. Use the standard form $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log(ax+b) + C$ to find these integrals:

(a) $\displaystyle\int \frac{1}{2x+b}\,dx$ (c) $\displaystyle\int \frac{1}{ax+3}\,dx$ (e) $\displaystyle\int \frac{p}{px+q}\,dx$

(b) $\displaystyle\int \frac{1}{3x-k}\,dx$ (d) $\displaystyle\int \frac{1}{mx-2}\,dx$ (f) $\displaystyle\int \frac{A}{sx-t}\,dx$

11. Use the result $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + C$ to find:

(a) $\displaystyle\int \frac{3x^2}{x^3-5}\,dx$ (c) $\displaystyle\int \frac{x^3-3x}{x^4-6x^2}\,dx$ (e) $\displaystyle\int_2^3 \frac{3x^2-1}{x^3-x}\,dx$

(b) $\displaystyle\int \frac{4x^3+1}{x^4+x-5}\,dx$ (d) $\displaystyle\int \frac{10x^3-7x}{5x^4-7x^2+8}\,dx$ (f) $\displaystyle\int_e^{2e} \frac{2x+2}{x^2+2x}\,dx$

12. (a) Given that the derivative of $f(x)$ is $\dfrac{x^2+x+1}{x}$ and $f(1) = 1\tfrac{1}{2}$, find $f(x)$.

(b) Given that the derivative of $g(x)$ is $\dfrac{2x^3-3x-4}{x^2}$ and $g(2) = -3\log 2$, find $g(x)$.

CHALLENGE

13. Find the value of a if: (a) $\displaystyle\int_1^a \frac{1}{x}\,dx = 5$, (b) $\displaystyle\int_a^e \frac{1}{x}\,dx = 5$.

14. Find $\displaystyle\int_1^e \left(x + \frac{1}{x^2}\right)^2 dx$.

15. (a) Differentiate $y = x\log_e x - x$.

(b) Hence find: (i) $\displaystyle\int \log_e x\,dx$, (ii) $\displaystyle\int_{\sqrt{e}}^{e} \log_e x\,dx$.

16. (a) Show that the derivative of $y = 2x^2\log x - x^2$ is $y' = 4x\log x$.

(b) Hence write down a primitive of $x\log x$.

(c) Use this result to evaluate $\displaystyle\int_e^2 x\log x\,dx$.

17. (a) Differentiate $(\log_e x)^2$ using the chain rule.

(b) Hence determine $\displaystyle\int_{\sqrt{e}}^{e} \frac{\log_e x}{x}\,dx$.

18. Differentiate $\log(\log x)$ and hence determine the family of primitives of $\dfrac{1}{x\log x}$.

19. There appear to be two primitives of the function $\dfrac{1}{5x}$. Taking out a factor of $\tfrac{1}{5}$,

$$\int \frac{1}{5x}\,dx = \frac{1}{5}\int \frac{1}{x}\,dx = \tfrac{1}{5}\log x + C_1, \quad \text{for some constant } C_1.$$

Alternatively, using the second formula in Box 12 on page 123, with $a = 5$ and $b = 0$,

$$\int \frac{1}{5x}\,dx = \tfrac{1}{5}\log 5x + C_2, \quad \text{for some constant } C_2.$$

How can these two answers be reconciled?

20. Although it is not required in this course, it can be shown that:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log_e\left(x+\sqrt{x^2+a^2}\right)+C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log_e\left(x+\sqrt{x^2-a^2}\right)+C$$

Use these results to find:

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx$ (b) $\displaystyle\int_4^8 \frac{1}{\sqrt{x^2-16}}\,dx$

3 F Applications of Integration of $1/x$

The usual applications of integration can now be applied to the reciprocal function, whose primitive was previously unavailable.

Finding Areas by Integration: The following worked exercise involves finding the area between two given curves.

WORKED EXERCISE:

(a) Show that the hyperbola $xy = 2$ and the line $x + y = 3$ meet at the points $A(1,2)$ and $B(2,1)$.

(b) Sketch the situation.

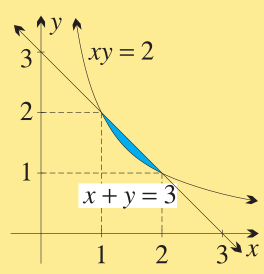
(c) Find the area of the region between the two curves, correct to three decimal places.

SOLUTION:

(a) Substituting $A(1,2)$ into the hyperbola $xy = 2$,
$\text{LHS} = 1\times 2 = 2 = \text{RHS}$,
and substituting $A(1,2)$ into the line $x + y = 3$,
$\text{LHS} = 1 + 2 = 3 = \text{RHS}$,
so $A(1,2)$ lies on both curves.
Similarly, $B(2,1)$ lies on both curves.

(b) The hyperbola $xy = 2$ has both axes as aymptotes.
The line $x + y = 3$ has x-intercept $(3,0)$ and y-intercept $(0,3)$.

(c)
$$\begin{aligned}
\text{Area} &= \int_1^2 (\text{top curve} - \text{bottom curve})\,dx \\
&= \int_1^2 \left((3-x)-\frac{2}{x}\right)dx \\
&= \Big[3x-\tfrac12 x^2 - 2\log_e x\Big]_1^2 \\
&= (6-2-2\log_e 2)-(3-\tfrac12-2\log_e 1) \\
&= (4-2\log_e 2)-(2\tfrac12-0) \\
&= (1\tfrac12-2\log_e 2)\text{ square units} \\
&\doteqdot 0{\cdot}114\text{ square units.}
\end{aligned}$$



Finding Volumes by Integration: The next worked exercise finds the volume of a solid of revolution of a region about the y-axis. The required formula is

$$\text{volume} = \int_a^b \pi x^2\, dy.$$

WORKED EXERCISE:

Find the volume of the solid generated when the region contained between the curve $y = \dfrac{1}{x^2}$ and the lines $y = 1$ and $y = 2$ is rotated about the y-axis.

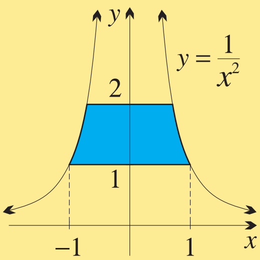
SOLUTION:

The equation is $\qquad y = \dfrac{1}{x^2}$.

Solving for x^2 gives $\qquad x^2 = \dfrac{1}{y}$.

Using the formula above for a volume of revolution about the y-axis,

$$\begin{aligned}\text{volume} &= \int_1^2 \pi x^2\, dy \\ &= \pi \int_1^2 \frac{1}{y}\, dy \\ &= \pi \Big[\log_e y\Big]_1^2 \\ &= \pi(\log_e 2 - \log_e 1) \\ &= \pi \log_e 2 \text{ cubic units.}\end{aligned}$$



Exercise 3F

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

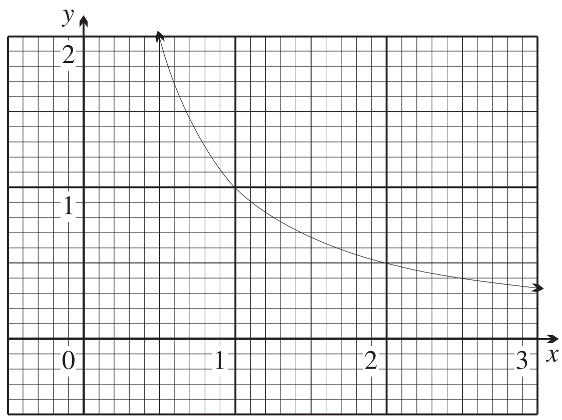
1. (a) Prove that $\displaystyle\int_1^e \frac{1}{x}\, dx = 1$.

(b) This question uses the above result to estimate e from a graph of $y = \dfrac{1}{x}$.

The diagram to the right shows the graph of $y = \dfrac{1}{x}$ from $x = 0$ to $x = 3$, drawn with a scale of 10 little divisions to 1 unit, so that 100 of the little squares make 1 square unit.

Count the number of squares in the column from $x = 1{\cdot}0$ to $1{\cdot}1$, then the squares in the column from $x = 1{\cdot}1$ to $1{\cdot}2$, and so on.

Continue until the number of squares equals 100 — the x-value at this point will be an estimate of e.



2. Answer each question by first giving your answer in exact form and then finding an approximation correct to four significant figures.

(a) Find the area between the curve $y = \dfrac{1}{x}$ and the x-axis for:

(i) $1 \le x \le e$ (ii) $1 \le x \le 5$

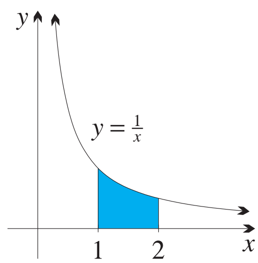
(b) Find the area between the curve $y = \dfrac{1}{x}$ and the x-axis for:

(i) $e \le x \le e^2$ (ii) $2 \le x \le 8$

(c) Find the area between the curve $y = \dfrac{7}{x}$ and the x-axis for:

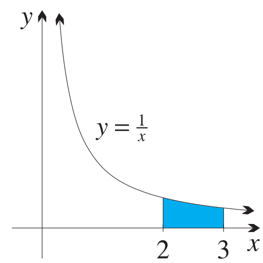
(i) $1 \le x \le e^2$ (ii) $1 \le x \le 25$

3. (a)



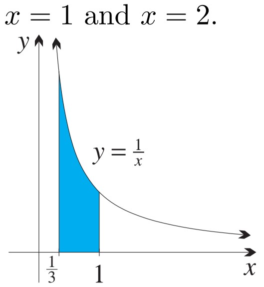
Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = 2$.

(b)



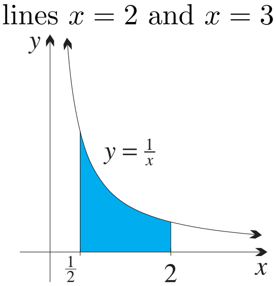
Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = 2$ and $x = 3$.

(c)



Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = \frac{1}{3}$ and $x = 1$.

(d)



Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = \frac{1}{2}$ and $x = 2$.

4. Answer each question by first giving your answer in exact form and then finding an approximation correct to four significant figures. In each case you will need to use the standard form

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + C.$$

(a) Find the area between $y = \dfrac{1}{2x+1}$ and the x-axis for: (i) $2 \le x \le 5$ (ii) $1 \le x \le 4$

(b) Find the area between $y = \dfrac{1}{3x+2}$ and the x-axis for: (i) $0 \le x \le 1$ (ii) $0 \le x \le 6$

(c) Find the area between $y = \dfrac{1}{2x-5}$ and the x-axis for: (i) $3 \le x \le 4$ (ii) $4 \le x \le 16$

(d) Find the area between $y = \dfrac{3}{x-1}$ and the x-axis for: (i) $2 \le x \le e^3+1$ (ii) $3 \le x \le 12$

DEVELOPMENT

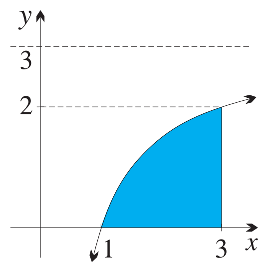
5. (a) Find the area between the graph of $y = \dfrac{1}{x} + 1$ and the x-axis, from $x = 1$ to $x = 2$.

(b) Find the area between the graph of $y = \dfrac{1}{x} + x$ and the x-axis, from $x = \frac{1}{2}$ to $x = 2$.

(c) Find the area between the graph of $y = \dfrac{1}{x} + x^2$ and the x-axis, from $x = 1$ to $x = 3$.

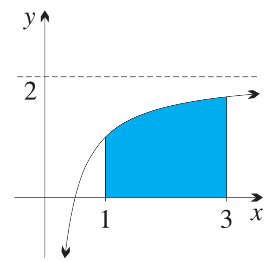
6. Give your answers to each question below in exact form.

(a)



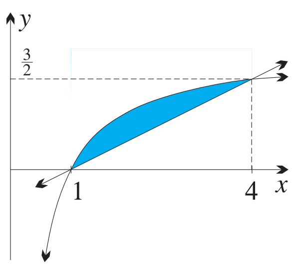
Find the area of the region bounded by $y = 3 - \dfrac{3}{x}$, the x-axis and $x = 3$.

(b)



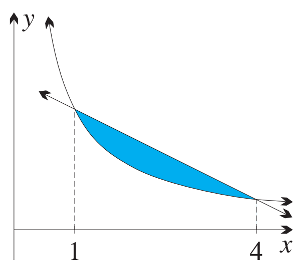
Find the area of the region bounded by $y = 2 - \dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 3$.

(c)



Find the area of the region bounded by $y = 2 - \dfrac{2}{x}$ and the line $y = \frac{1}{2}(x - 1)$.

(d)

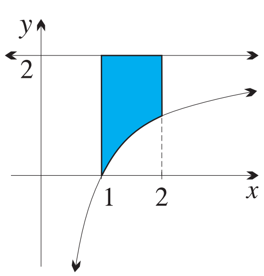


Find the area of the region between $y = \dfrac{2}{x}$ and the line $x + 2y - 5 = 0$.

7. (a) Sketch the region bounded by $y = 1$, $x = 8$ and the curve $y = \dfrac{4}{x}$.

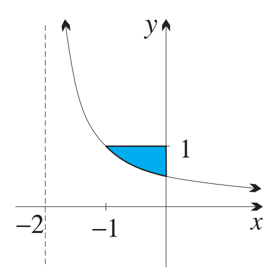
(b) Determine the area of this region with the aid of an appropriate integral.

8. (a)



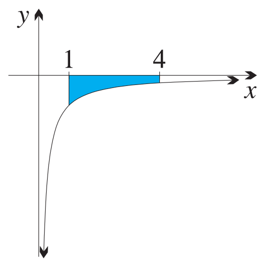
Find the area of the region in the first quadrant bounded by $y = 2 - \dfrac{2}{x}$ and $y = 2$, and lying between $x = 1$ and $x = 2$.

(b)



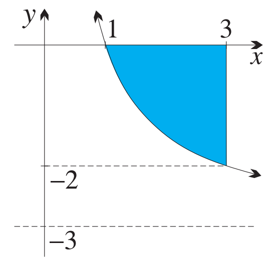
Find the area of the region bounded by the curve $y = \dfrac{1}{x+2}$, the y-axis and the horizontal line $y = 1$.

9. (a)



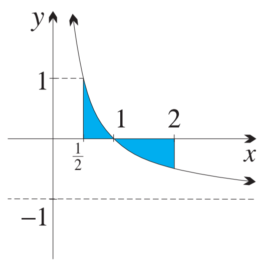
Find the area of the region bounded by $y = -\dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 4$.

(b)



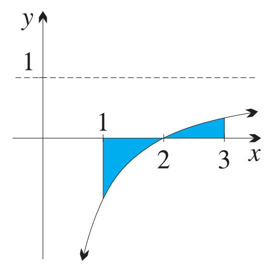
Find the area of the region bounded by $y = \dfrac{3}{x} - 3$, the x-axis and $x = 3$.

(c)



Find the area of the region bounded by $y = \dfrac{1}{x} - 1$, the x-axis, $x = \frac{1}{2}$ and $x = 2$.

(d)



Find the area of the region bounded by $y = 1 - \dfrac{1}{2x}$, the x-axis, $x = 1$ and $x = 3$.

10. (a) Find the two intersection points of the curve $y = \dfrac{1}{x}$ with the line $y = 4 - 3x$.

(b) Determine the area between these two curves.

11. (a) What is the derivative of $x^2 + 1$?

(b) Find the area under the graph $y = \dfrac{x}{x^2 + 1}$, between $x = 0$ and $x = 2$.

12. (a) Find the derivative of $x^2 + 2x + 3$.

(b) Find the area under the graph $y = \dfrac{x + 1}{x^2 + 2x + 3}$, between $x = 0$ and $x = 1$.

13. The curve $y = \dfrac{1}{x^2}$ is rotated about the y-axis, between $y = 1$ and $y = 6$. Evaluate the resulting volume.

14. (a) Sketch the region bounded by the x-axis, $y = x$, $y = \dfrac{1}{x}$ and $x = e$.

(b) Hence find the area of this region by using two appropriate integrals.

15. (a) Find the exact value of $\displaystyle\int_1^2 \frac{1}{x}\,dx$, then approximate it correct to three decimal places.

(b) Use the trapezoidal rule with function values at $x = 1$, $\frac{3}{2}$ and 2 to approximate the area found in part (a).

(c) Now use Simpson's rule to obtain another approximation.

16. (a) Find the volume of the solid generated when the curve $y = \dfrac{1}{\sqrt{x}}$ is rotated about the x-axis between $x = 2$ and $x = 4$.

(b) A horn is created by rotating the curve $y = \dfrac{1}{\sqrt{4-x}}$ about the x-axis between $x = 0$ and $x = 3\frac{3}{4}$. Find the volume of the horn.

(c) Another horn is generated by rotating the curve $y = 1 + \dfrac{1}{x}$ about the x-axis between $x = \frac{1}{2}$ and $x = 3$. Find its volume.

17. In this question, give your answers correct to four decimal places when asked to approximate.

(a) Find the area between the curve $y = \dfrac{1}{x}$ and the x-axis, for $1 \le x \le 3$, by evaluating an appropriate integral. Then approximate the result.

(b) Estimate the area using the trapezoidal rule with three function values.

(c) Estimate the area using Simpson's rule with three function values.

18. (a) Use the trapezoidal rule with five function values to approximate the area between the curve $y = \log x$ and the x-axis, between $x = 1$ and $x = 5$. Answer correct to four decimal places.

(b) Use Simpson's rule with five function values to approximate the area in part (a).

CHALLENGE

19. (a) Sketch $y = \log x$, for $0 \le x \le e$.

(b) Evaluate the area between the curve and the y-axis, between $y = 0$ and $y = 1$.

(c) Hence find the area between the curve and the x-axis, between $x = 1$ and $x = e$.

20. (a) The area between $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$, and between $x = 1$ and $x = 4$, is rotated about the x-axis. Find the volume of the resulting solid.

(b) Compare the volume found in part (a) with the volume of the solid generated when the area below $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$, also between $x = 1$ and $x = 4$, is rotated about the x-axis.

21. Consider the two curves $y = 6e^{-x}$ and $y = e^x - 1$.

(a) Let $u = e^x$. Show that the x-coordinate of the point of intersection of these two curves satisfies $u^2 - u - 6 = 0$.

(b) Hence find the coordinates of the point of intersection.

(c) Sketch the curves on the same number plane, and shade the region bounded by them and the y-axis.

(d) Find the area of the shaded region.

3 G Calculus with Other Bases

In applications of exponential functions where calculus is required, the base e can generally be used. For example, the treatment of natural growth in Chapter Six is done entirely using base e. Nevertheless, calculus can be applied to exponential and logarithmic functions with other bases. The general principle is to express these functions in terms of the functions e^x and $\log_e x$ that have base e.

This section is a little more difficult than the other work on exponential and logarithmic functions and could well be left until later.

Logarithmic Functions to Other Bases: Any logarithmic functions can be expressed easily in terms of $\log_e x$ by using the change-of-base formula. For example,

$$\log_2 x = \frac{\log_e x}{\log_e 2}\,.$$

Thus every other logarithmic function is just a constant multiple of $\log_e x$. This allows any other logarithmic function to be differentiated easily.

WORKED EXERCISE:

(a) Express the function $y = \log_5 x$ in terms of the function $\log_e x$.

(b) Hence use the calculator function labelled $\boxed{\ln}$ to approximate, correct to four decimal places:
(i) $\log_5 30$ (ii) $\log_5 2$ (iii) $\log_5 0{\cdot}07$

(c) Check the results of part (b) using the function labelled $\boxed{x^y}$.

SOLUTION:

(a) $\log_5 x = \dfrac{\log_e x}{\log_e 5}$

(b) (i) $\log_5 30 = \dfrac{\log_e 30}{\log_e 5} \doteqdot 2{\cdot}1133$ (ii) $\log_5 2 = \dfrac{\log_e 2}{\log_e 5} \doteqdot 0{\cdot}4307$ (iii) $\log_5 0{\cdot}07 = \dfrac{\log_e 0{\cdot}07}{\log_e 5} \doteqdot -1{\cdot}6523$

(c) Checking these results using the function labelled $\boxed{x^y}$:
(i) $5^{2{\cdot}1133} \doteqdot 30$ (ii) $5^{0{\cdot}4307} \doteqdot 2$ (iii) $5^{-1{\cdot}6523} \doteqdot 0{\cdot}07$

NOTE: Either of the calculator functions $\boxed{\log}$ or $\boxed{\ln}$ can be used to approximate a number like $\log_5 30$, because the base can be changed to either 10 or e:

$$\log_5 30 = \frac{\log_{10} 30}{\log_{10} 5} \qquad \text{or} \qquad \log_5 30 = \frac{\log_e 30}{\log_e 5}$$

WORKED EXERCISE:

Use the change-of-base formula to differentiate:

(a) $y = \log_2 x$ (b) $y = \log_a x$

SOLUTION:

(a) Here $y = \log_2 x$.
Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e 2}\,.$$

Since $\log_e 2$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e 2} = \frac{1}{x \log_e 2}\,.$$

(b) Here $y = \log_a x$.
Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e a}\,.$$

Since $\log_e a$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e a} = \frac{1}{x \log_e a}\,.$$

Part (b) above gives the formula in the general case:

13 **DIFFERENTIATING A LOGARITHMIC FUNCTION WITH ANOTHER BASE:**
Use the change-of-base formula to write $\log_a x$ as a multiple of $\log_e x$:

$$\log_a x = \frac{\log_e x}{\log_e a}\,.$$

Alternatively, remember the result as a standard form:

$$\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}\,.$$

For example,

$$\frac{d}{dx}\log_{10} x = \frac{1}{x\log_e 10} \qquad \text{and} \qquad \frac{d}{dx}\log_{1\cdot 05} x = \frac{1}{x\log_e 1\cdot 05}\,.$$

A Characterisation of the Logarithmic Function: We have already seen in Section 3B that the tangent to $y = \log_e x$ at the x-intercept has gradient exactly 1.

The following worked exercise shows that this property distinguishes the logarithmic function base e from all other logarithmic functions.

WORKED EXERCISE:

(a) Show that the tangent to $y = \log_a x$ at the x-intercept has gradient $\dfrac{1}{\log_e a}$.

(b) Show that the function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

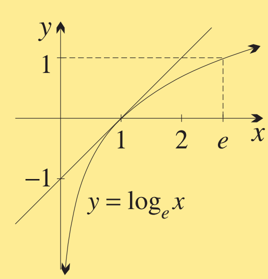
SOLUTION:

(a) Here $y = \log_a x$.
When $y = 0$, $\log_a x = 0$
$x = 1$,
and so the x-intercept is $(1, 0)$.
Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e a}$$

and differentiating, $y' = \dfrac{1}{x\log_e a}\,.$

Hence when $x = 1$, $y' = \dfrac{1}{\log_e a}$, as required.



(b) The gradient at the x-intercept is 1 if and only if

$$\begin{aligned}\log_e a &= 1,\\ a &= e^1\\ &= e,\end{aligned}$$

that is, if and only if the original base a is equal to e.

14 **THE GRADIENT AT THE x-INTERCEPT:** The function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

Exponential Functions with Other Bases: Before calculus can be applied to an exponential function $y = a^x$ with base a different from e, it must be written as an exponential function with base e. The important identity used to do this is

$$e^{\log_e a} = a,$$

which simply expresses the fact that the functions e^x and $\log_e x$ are inverse functions. Now a^x can be written as

$$\begin{aligned} a^x &= (e^{\log_e a})^x, && \text{replacing } a \text{ by } e^{\log_e a}, \\ &= e^{x \log_e a}, && \text{using the index law } (e^k)^x = e^{kx}. \end{aligned}$$

Thus a^x has been expressed in the form e^{kx}, where $k = \log_e a$ is a constant.

15 EXPONENTIAL FUNCTIONS WITH OTHER BASES:

Every number can be written as a power of e:

$$a = e^{\log_e a}$$

Every exponential function can be written as an exponential function base e:

$$a^x = e^{x \log_e a}$$

WORKED EXERCISE:

Express these numbers and functions as power of e:

(a) 2 (b) 2^x (c) 5^{-x}

SOLUTION:

(a) $2 = e^{\log_e 2}$

(b) $2^x = \left(e^{\log_e 2}\right)^x = e^{x \log_e 2}$

(c) $5^{-x} = \left(e^{\log_e 5}\right)^{-x} = e^{-x \log_e 5}$

Differentiating and Integrating Exponential Functions with Other Bases: Write the function as a power of e. It can then be differentiated and integrated.

First,
$$\begin{aligned} a^x &= e^{\log_e a^x} \\ &= e^{x \log_e a}. \end{aligned}$$

Differentiating,
$$\begin{aligned} \frac{d}{dx}\, a^x &= \frac{d}{dx}\, e^{x \log_e a} \\ &= e^{x \log_e a} \times \log_e a, \quad \text{since } \frac{d}{dx} e^{kx} = k\, e^{kx}, \\ &= a^x \, \log_e a, \quad \text{since } e^{x \log_e a} = a^x. \end{aligned}$$

Integrating,
$$\begin{aligned} \int a^x \, dx &= \int e^{x \log_e a} \, dx \\ &= \frac{e^{x \log_e a}}{\log_e a}, \quad \text{since } \int e^{kx} = \frac{1}{k}\, e^{kx}, \\ &= \frac{a^x}{\log_e a}\,, \quad \text{since } e^{x \log_e a} = a^x. \end{aligned}$$

This process can be carried through every time, or the results can be remembered as standard forms.

DIFFERENTIATION AND INTEGRATION WITH OTHER BASES:

16
$$\frac{d}{dx}\,a^x = a^x \log_e a$$
$$\int a^x\,dx = \frac{a^x}{\log_e a} + C$$

NOTE: The formulae for differentiating and integrating a^x both involve the constant $\log_e a$. This constant $\log_e a = 1$ when $a = e$, so the formulae are simplest when the base is e. Again, this indicates that e is the appropriate base to use for the calculus of exponential functions.

WORKED EXERCISE:

Differentiate $y = 2^x$. Hence find the gradient of $y = 2^x$ at the y-intercept, correct to three significant figures.

SOLUTION:

Here $y = 2^x$.

Using the standard form, $y' = 2^x \log_e 2$.

Hence when $x = 0$, $y' = 2^0 \times \log_e 2$

$= \log_e 2$

$\doteqdot 0{\cdot}693$.

NOTE: This result should be compared with the results of physically measuring this gradient in question 4 of Exercise 3B.

WORKED EXERCISE:

(a) Show that the line $y = x + 1$ meets the curve $y = 2^x$ at $A(0, 1)$ and $B(1, 2)$.

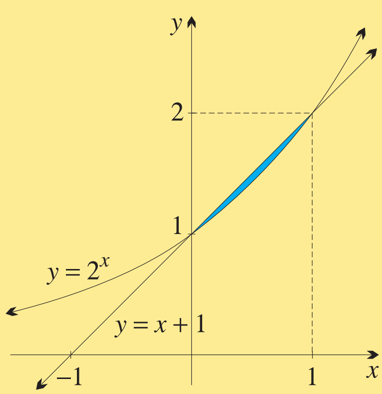
(b) Sketch the two curves and shade the region contained between them.

(c) Find the area of this shaded region, correct to four significant figures.

SOLUTION:

(a) Simple substitution of $x = 0$ and $x = 1$ into both functions verifies the result.

(b) The graph is drawn to the right.



(c)
$$\begin{aligned}\text{Area} &= \int_0^1 (\text{upper curve} - \text{lower curve})\,dx\\ &= \int_0^1 (x + 1 - 2^x)\,dx\\ &= \left[\tfrac{1}{2}x^2 + x - \frac{2^x}{\log_e 2}\right]_0^1\\ &= \left(\tfrac{1}{2} + 1 - \frac{2}{\log_e 2}\right) - \left(0 + 0 - \frac{1}{\log_e 2}\right)\\ &= 1\tfrac{1}{2} - \frac{1}{\log_e 2} \text{ square units}\\ &\doteqdot 0{\cdot}057\,30 \text{ square units.}\end{aligned}$$

Exercise 3G

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\boxed{\log}$ means $\log_{10} x$ and $\boxed{\ln}$ means $\log_e x$).

1. Use the change-of-base formula $\log_a x = \dfrac{\log_e x}{\log_e a}$ and the function labelled $\boxed{\ln}$ on your calculator to evaluate the following, correct to three significant figures. Then check your answers using the function labelled $\boxed{x^y}$.

(a) $\log_2 3$ (b) $\log_2 5$ (c) $\log_2 10$ (d) $\log_5 16$ (e) $\log_5 26$ (f) $\log_3 8$ (g) $\log_3 47$ (h) $\log_6 112$

2. Use the change-of-base formula to express these to base e, then differentiate them:

(a) $\log_2 x$ (b) $\log_{10} x$ (c) $\log_5 x$ (d) $\log_3 x$

3. Express these functions as powers of e, then differentiate them:

(a) 3^x (b) 4^x (c) 2^x (d) 10^x

4. Use the result $\displaystyle\int a^x\,dx = \frac{a^x}{\log_e a} + C$ to determine the following indefinite integrals:

(a) $\displaystyle\int 2^x\,dx$ (b) $\displaystyle\int 6^x\,dx$ (c) $\displaystyle\int 7^x\,dx$ (d) $\displaystyle\int 3^x\,dx$

5. Evaluate the following definite integrals, then approximate your answers correct to four significant figures.

(a) $\displaystyle\int_0^1 2^x\,dx$ (b) $\displaystyle\int_0^1 3^x\,dx$ (c) $\displaystyle\int_{-1}^1 5^x\,dx$ (d) $\displaystyle\int_0^2 4^x\,dx$

6. (a) Complete the following table of values, giving your answers correct to two decimal places where necessary.

(b) Use this table of values to sketch the three curves $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ on the same set of axes.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$					
$\log_e x$					
$\log_4 x$					

DEVELOPMENT

7. (a) Differentiate $y = \log_2 x$. Hence find the gradient of the tangent to the curve at $x = 1$.

(b) Hence find the equation of the tangent there.

(c) Do likewise for: (i) $y = \log_3 x$, (ii) $y = \log_5 x$.

8. Give the exact value of each integral, then evaluate it correct to four decimal places.

(a) $\displaystyle\int_0^1 2^x\,dx$ (b) $\displaystyle\int_{-1}^1 (3^x + 1)\,dx$ (c) $\displaystyle\int_0^2 (10^x - 10x)\,dx$

9. Use the change-of-base formula to express $y = \log_{10} x$ with base e, and hence find y'.

(a) Find the gradient of the tangent to this curve at the point $(10, 1)$.

(b) Thus determine the equation of this tangent in general form.

(c) At what value of x will the tangent have gradient 1?

10. (a) Find the equations of the tangents to each of $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ at the points where $x = 3$.

(b) Show that the three tangents all meet at the same point on the x-axis.

11. (a) Show that the curves $y = 2^x$ and $y = 1 + 2x - x^2$ intersect at $A(0,1)$ and $B(1,2)$.
(b) Sketch the curves and find the area between them.

12. Find the intercepts of the curve $y = 8 - 2^x$, and hence find the area of the region bounded by this curve and the coordinate axes.

13. (a) Sketch the curve $y = 3 - 3^x$, showing the intercepts and asymptote.
(b) Find the area contained between the curve and the axes.
(c) What is the volume of the solid generated when this area is rotated about the x-axis?

14. (a) Show that the curves $y = x + 1$ and $y = 4^x$ intersect at the y-intercept and at $(-\frac{1}{2}, \frac{1}{2})$.
(b) Write the area of the region enclosed between these two curves as an integral.
(c) Evalute the integral found in part (b).

CHALLENGE

15. (a) Show that the tangent to $y = \log_3 x$ at $x = e$ passes through the origin.
(b) Show that the tangent to $y = \log_5 x$ at $x = e$ passes through the origin.
(c) Show that the same is true for $y = \log_a x$, for any positive value of the base a.

16. (a) Differentiate $x \log_e x - x$ and hence find $\int \log_e x \, dx$.
(b) Use the change-of-base formula and the integral in part (a) to evaluate $\int_1^{10} \log_{10} x \, dx$.

3H Chapter Review Exercise

1. Use the appropriate button on your calculator to approximate the following, correct to four decimal places:
(a) $\log_{10} 27$ (b) $\log_{10} \frac{1}{2}$ (c) $\log_e 2$ (d) $\log_e 14$

2. Write each equation below in logarithmic form. Then use the appropriate button on your calculator to approximate x correct to four decimal places.
(a) $10^x = 15$ (b) $10^x = 3$ (c) $e^x = 7$ (d) $e^x = \frac{1}{3}$

3. Use logarithms to solve the following, correct to four significant figures. You will need to use the change-of-base formula before using your calculator.
(a) $3^x = 14$ (b) $2^x = 51$ (c) $4^x = 1345$ (d) $5^x = 132$

4. Sketch graphs of the following, clearly indicating the vertical asymptote in each case.
(a) $y = \log_2 x$ (b) $y = -\log_2 x$ (c) $y = \log_2(x - 1)$ (d) $y = \log_2(x + 3)$

5. Sketch graphs of the following, clearly indicating the vertical asymptote in each case.
(a) $y = \log_e x$ (b) $y = \log_e(-x)$ (c) $y = \log_e(x - 2)$ (d) $y = \log_e x + 1$

6. Use the log laws to simplify:
(a) $e \log_e e$ (b) $\log_e e^3$ (c) $\log_e \dfrac{1}{e}$ (d) $2e \log_e \sqrt{e}$

7. Differentiate the following functions:
(a) $\log_e x$
(b) $\log_e 2x$
(c) $\log_e(x + 4)$
(d) $\log_e(2x - 5)$
(e) $2 \log_e(5x - 1)$
(f) $x + \log_e x$
(g) $\log_e(x^2 - 5x + 2)$
(h) $\log_e(1 + 3x^5)$
(i) $4x^2 - 8x^3 + \log_e(x^2 - 2)$

8. Use the log laws to simplify each function and then find its derivative.

(a) $\log_e x^3$ (b) $\log_e \sqrt{x}$ (c) $\log_e x(x+2)$ (d) $\log_e \dfrac{x}{x-1}$

9. Differentiate the following functions by using the product or quotient rule.

(a) $x \log x$ (b) $e^x \log x$ (c) $\dfrac{x}{\log x}$ (d) $\dfrac{\log x}{x^2}$

10. Find the equation of the tangent to the curve $y = 3\log_e x + 4$ at the point $(1, 4)$.

11. Consider the function $y = x - \log_e x$.

(a) Show that $y' = \dfrac{x-1}{x}$.

(b) Hence show that the graph of $y = x - \log_e x$ has a minimum turning point at $(1, 1)$.

12. Find the following indefinite integrals:

(a) $\displaystyle\int \frac{1}{x}\,dx$ (c) $\displaystyle\int \frac{1}{5x}\,dx$ (e) $\displaystyle\int \frac{1}{2x-1}\,dx$ (g) $\displaystyle\int \frac{2}{2x+9}\,dx$

(b) $\displaystyle\int \frac{3}{x}\,dx$ (d) $\displaystyle\int \frac{1}{x+7}\,dx$ (f) $\displaystyle\int \frac{1}{2-3x}\,dx$ (h) $\displaystyle\int \frac{8}{1-4x}\,dx$

13. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^1 \frac{1}{x+2}\,dx$ (b) $\displaystyle\int_1^4 \frac{1}{4x-3}\,dx$ (c) $\displaystyle\int_1^e \frac{1}{x}\,dx$ (d) $\displaystyle\int_{e^2}^{e^3} \frac{1}{x}\,dx$

14. Use the standard form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + C$ to find:

(a) $\displaystyle\int \frac{2x}{x^2+4}\,dx$ (b) $\displaystyle\int \frac{3x^2-5}{x^3-5x+7}\,dx$ (c) $\displaystyle\int \frac{x}{x^2-3}\,dx$ (d) $\displaystyle\int \frac{x^3-1}{x^4-4x}\,dx$

15. Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 4$.

16. (a) By solving the equations simultaneously, show that the curve $y = \dfrac{5}{x}$ and the line $y = 6 - x$ intersect at the points $(1, 5)$ and $(5, 1)$.

(b) By sketching both graphs on the same number plane, find the area of the region enclosed between them.

17. The region bounded by the curve $y = \dfrac{1}{\sqrt{x}}$, the x-axis and the lines $x = e$ and $x = 2e$ is rotated about the x-axis. Find the volume of the solid that is formed.

18. Find the derivatives of the following:

(a) e^x (b) 2^x (c) 3^x (d) 5^x

19. Find the following indefinite integrals:

(a) $\displaystyle\int e^x\,dx$ (b) $\displaystyle\int 2^x\,dx$ (c) $\displaystyle\int 3^x\,dx$ (d) $\displaystyle\int 5^x\,dx$

CHAPTER **FOUR**

The Trigonometric Functions

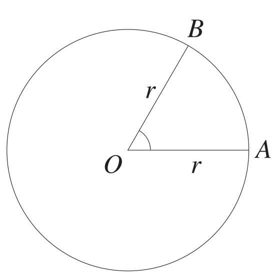
This chapter will extend calculus to the trigonometric functions. The sine and cosine functions are extremely important because their graphs are waves. They are therefore essential in the modelling of all the many wave-like phenomena such as sound waves, light and radio waves, vibrating strings, tides and economic cycles. Most of the attention in this chapter is given to these two functions.

4 A Radian Measure of Angle Size

This section will introduce a new way of measuring angle size in *radians*, based on the number π. The new way of measuring angle size is needed for the calculus of the trigonometric functions.

The use of degrees to measure angle size is based on astronomy, not on mathematics. There are 360 days in the year, to the nearest convenient number, so 1° is the angle through which the sun moves against the fixed stars each day, or (after Copernicus) the angle swept out by the Earth each day in its orbit around the sun. Mathematics is far too general a discipline to be tied to the particularities of the solar system, so it is quite natural to develop a new system for measuring angles based on mathematics alone.

Radian Measure of Angle Size: The size of an angle in radians is defined as the ratio of two lengths. Given an angle with vertex O, construct a circle with centre O meeting the two arms of the angle at A and B. Then:

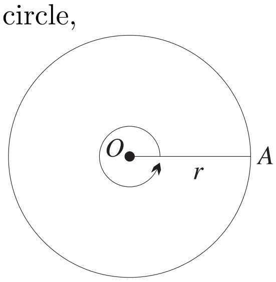


1 RADIAN MEASURE: Size of $\angle AOB = \dfrac{\text{arc length } AB}{\text{radius } OA}$

This definition gives the same angle size, no matter what the radius of the circle, because all circles are similar to one another.

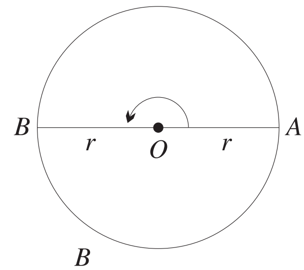
The arc subtended by a revolution is the whole circumference of the circle,

$$\text{so} \quad 1 \text{ revolution} = \frac{\text{arc length}}{\text{radius}} = \frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi.$$



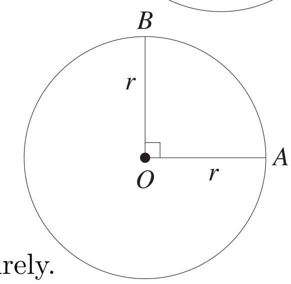
Similarly, a straight angle subtends a semicircle,

$$\text{so} \quad 1 \text{ straight angle} = \frac{\text{arc length of semicircle}}{\text{radius}} = \frac{\pi r}{r} = \pi,$$



and a right angle subtends a quarter-circle,

$$\text{so} \quad 1 \text{ right angle} = \frac{\text{arc length of quarter-circle}}{\text{radius}} = \frac{\frac{1}{2}\pi r}{r} = \frac{\pi}{2}.$$



These three basic conversions should be memorised very securely.

2 CONVERSIONS BETWEEN DEGREES AND RADIAN MEASURE:

$$360^\circ = 2\pi \qquad 180^\circ = \pi \qquad 90^\circ = \tfrac{\pi}{2}$$

Because $180^\circ = \pi$ radians, an angle size in radians can be converted to an angle size in degrees by multiplying by $\dfrac{180^\circ}{\pi}$.

Conversely, degrees are converted to radians by multiplying by $\dfrac{\pi}{180}$.

3 CONVERTING RADIANS TO DEGREES:

To convert radians to degrees, multiply by $\dfrac{180^\circ}{\pi}$.

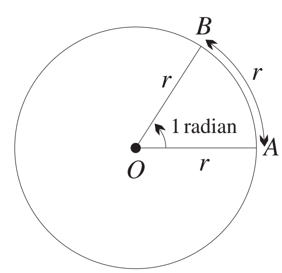
CONVERTING DEGREES TO RADIANS:

To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.

ONE RADIAN AND ONE DEGREE: In particular,

$$1 \text{ radian} = \frac{180^\circ}{\pi} \doteqdot 57^\circ 18' \qquad \text{and} \qquad 1 \text{ degree} = \frac{\pi}{180} \doteqdot 0{\cdot}0175.$$

One radian is the angle subtended at the centre of a circle by an arc of length equal to the radius. Notice that the sector OAB in the diagram to the right is almost an equilateral triangle and so 1 radian is about 60°. This makes sense of the value given above, that 1 radian is about 57°.



NOTE: The size of an angle in radians is a ratio of lengths and so is a dimensionless real number. It is therefore unnecessary to mention radians. For example, 'an angle of size 1·3' means an angle of 1·3 radians.

This new definition of angle size is very similar to the definitions of the six trigonometric functions, which are also ratios of lengths and so are also pure numbers.

WORKED EXERCISE:

Express these angle sizes in radians:

(a) 60° (b) 270° (c) 495° (d) 37°

SOLUTION:

(a) $60^\circ = 60 \times \dfrac{\pi}{180}$

$= \dfrac{\pi}{3}$

(b) $270^\circ = 270 \times \dfrac{\pi}{180}$

$= \dfrac{3\pi}{2}$

(c) $495^\circ = 495 \times \dfrac{\pi}{180}$

$= \dfrac{11\pi}{4}$

(d) $37^\circ = 37 \times \dfrac{\pi}{180}$

$= \dfrac{37\pi}{180}$

WORKED EXERCISE:

Express these angles sizes in degrees. Give exact answers, and then where appropriate give answers correct to the nearest degree:

(a) $\frac{\pi}{6}$ (b) $0{\cdot}3$ (c) $\frac{3\pi}{4}$ (d) 20

SOLUTION:

(a) $\dfrac{\pi}{6} = \dfrac{\pi}{6} \times \dfrac{180^\circ}{\pi}$

$= 30^\circ$

(b) $0{\cdot}3 = \dfrac{3}{10} \times \dfrac{180^\circ}{\pi}$

$= \dfrac{54^\circ}{\pi}$

$\doteqdot 17^\circ$

(c) $\dfrac{3\pi}{4} = \dfrac{3\pi}{4} \times \dfrac{180^\circ}{\pi}$

$= 135^\circ$

(d) $20 = 20 \times \dfrac{180^\circ}{\pi}$

$= \dfrac{3600^\circ}{\pi}$

$\doteqdot 1146^\circ$

Evaluating Trigonometric Functions of Special Angles: The value of a trigonometric function of an angle is the same whether the angle size is given in degrees or radians.

With angles whose related angle is one of the three special angles, it is a matter of recognising the special angles

$$\tfrac{\pi}{6} = 30^\circ \qquad \text{and} \qquad \tfrac{\pi}{4} = 45^\circ \qquad \text{and} \qquad \tfrac{\pi}{3} = 60^\circ.$$

WORKED EXERCISE:

Evaluate these trigonometric functions, using the special triangles sketched below:

(a) $\sin \frac{\pi}{6}$ (b) $\operatorname{cosec} \frac{\pi}{4}$

SOLUTION:

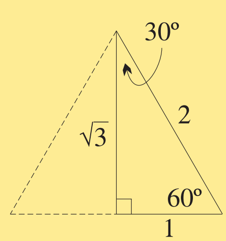
(a) $\sin \frac{\pi}{6} = \sin 30^\circ$

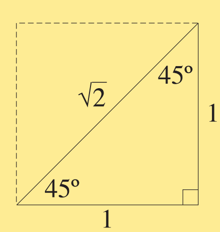
$= \frac{1}{2}$

(b) $\operatorname{cosec} \frac{\pi}{4} = \dfrac{1}{\sin \frac{\pi}{4}}$

$= \dfrac{1}{\sin 45^\circ}$

$= \sqrt{2}$

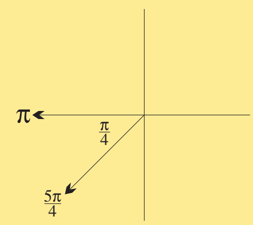




WORKED EXERCISE: [Angles whose related angle is a special angle]
Use special angles to evaluate these trigonometric functions:
(a) $\sin \frac{5\pi}{4}$ (b) $\sec \frac{11\pi}{6}$

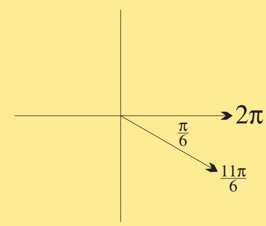
SOLUTION:
(a) Since $\frac{5\pi}{4}$ is in the third quadrant, with related angle $\frac{\pi}{4}$,

$$\begin{aligned}\sin \tfrac{5\pi}{4} &= -\sin \tfrac{\pi}{4}\\ &= -\sin 45^\circ\\ &= -\frac{1}{\sqrt{2}}\,.\end{aligned}$$



(b) Since $\frac{11\pi}{6}$ is in the fourth quadrant, with related angle $\frac{\pi}{6}$,

$$\begin{aligned}\sec \tfrac{11\pi}{6} &= +\sec \tfrac{\pi}{6}\\ &= \frac{1}{\cos \frac{\pi}{6}}\\ &= \frac{1}{\cos 30^\circ}\\ &= \frac{2}{\sqrt{3}}\,.\end{aligned}$$



Approximating Trigonometric Functions of Other Angles: For other angles, exact values of the trigonometric functions cannot normally be found. When calculator approximations are required, *it is vital to set the calculator to radians mode first*.

Your calculator has a key labelled $\boxed{\text{mode}}$ or something similar to make the change — calculators set to the wrong mode routinely cause havoc at this point!

4 SETTING THE CALCULATOR TO RADIANS MODE OR DEGREES MODE:
From now on, always decide whether the calculator should be in radians mode or degrees mode before using any of the trigonometric functions.

WORKED EXERCISE:
Evaluate correct to four decimal places:
(a) $\cos 1$ (b) $\cot 1{\cdot}3$

SOLUTION:
Here the calculator must be set to radians mode.

(a) $\cos 1 \doteqdot 0{\cdot}5403$

(b) $$\begin{aligned}\cot 1{\cdot}3 &= \frac{1}{\tan 1{\cdot}3}\\ &\doteqdot 0{\cdot}2776\end{aligned}$$

Solving Trigonometric Equations in Radians: Solving a trigonometric equation is done exactly the same way whether the solution is to be given in radians or degrees.

5 SOLVING A TRIGONOMETRIC EQUATION IN RADIANS:
First, establish the quadrants in which the angle can lie.
Secondly, find the related angle — but use radian measure, not degrees.

WORKED EXERCISE: [Acute angles]
Solve each trigonometric equation in radians, where the angle x is an acute angle. Give each answer in exact form if possible, otherwise give an approximation correct to five significant figures.
(a) $\sin x = \frac{1}{2}$ (b) $\tan x = 3$

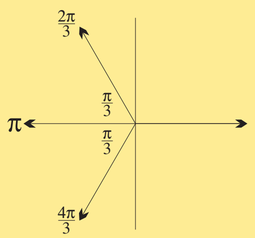
SOLUTION:
(a) $\sin x = \frac{1}{2}$
$x = \frac{\pi}{6}$ (The special angle 30°, which is $\frac{\pi}{6}$ radians.)

(b) $\tan x = 3$
$x \doteqdot 1{\cdot}2490$ (Use the calculator here.)

WORKED EXERCISE: [General angles]
Solve these trigonometric equations. Give answers in exact form if possible, otherwise correct to five significant figures.
(a) $\cos x = -\frac{1}{2}$, where $0 \le x \le 2\pi$
(b) $\sin x = -\frac{1}{3}$, where $0 \le x \le 2\pi$

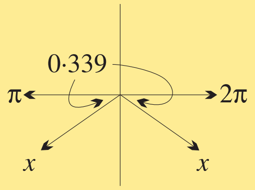
NOTE: When using the calculator's inverse trigonometric functions, *never work with a negative number.* Always enter the absolute value of the number in order to find the related angle.

SOLUTION:
(a) $\cos x = -\frac{1}{2}$, where $0 \le x \le 2\pi$.
Since $\cos x$ is negative, x is in quadrant 2 or 3.
The acute angle whose cosine is $+\frac{1}{2}$ is the special angle 60°, which in radian measure is $\frac{\pi}{3}$.
Hence, from the diagram to the right,



$$x = \pi - \tfrac{\pi}{3} \text{ or } \pi + \tfrac{\pi}{3}$$
$$= \tfrac{2\pi}{3} \text{ or } \tfrac{4\pi}{3}.$$

(b) $\sin x = -\frac{1}{3}$, where $0 \le x \le 2\pi$.
Since $\sin x$ is negative, x is in quadrant 3 or 4.
With the calculator in radians mode, enter $+\frac{1}{3}$, then the related angle $0{\cdot}339\,836\ldots$ (hold this in memory).
Hence, from the diagram to the right,



$$x = \pi + 0{\cdot}339\,836\ldots \text{ or } 2\pi - 0{\cdot}339\,836\ldots$$
$$\doteqdot 3{\cdot}4814 \text{ or } 5{\cdot}9433.$$

Exercise 4A

NOTE: Be very careful throughout this chapter whether your calculator is set in radians mode or degrees mode. The button used to make the change is usually labelled $\boxed{\text{mode}}$.

1. Express the following angles in radians by multiplying by $\dfrac{\pi}{180}$:

(a) 90°	(d) 60°	(g) 135°	(j) 300°
(b) 45°	(e) 120°	(h) 225°	(k) 270°
(c) 30°	(f) 150°	(i) 360°	(l) 210°

2. Express the following angles in degrees by multiplying by $\dfrac{180^\circ}{\pi}$:

(a) π
(b) 2π
(c) 4π
(d) $\dfrac{\pi}{2}$
(e) $\dfrac{\pi}{3}$
(f) $\dfrac{\pi}{4}$
(g) $\dfrac{2\pi}{3}$
(h) $\dfrac{5\pi}{6}$
(i) $\dfrac{3\pi}{4}$
(j) $\dfrac{3\pi}{2}$
(k) $\dfrac{4\pi}{3}$
(l) $\dfrac{7\pi}{4}$
(m) $\dfrac{11\pi}{6}$

3. Use your calculator to express in radians, correct to three decimal places:
(a) 73°
(b) 14°
(c) 168°
(d) $21^\circ 36'$
(e) $95^\circ 17'$
(f) $211^\circ 12'$

4. Use your calculator to express in degrees and minutes, correct to the nearest minute:
(a) 2 radians
(b) 0·3 radians
(c) 1·44 radians
(d) 0·123 radians
(e) 3·1985 radians
(f) 5·64792 radians

5. Use your calculator in radian mode to evaluate, correct to two decimal places:
(a) $\sin 2$
(b) $\cos 2{\cdot}5$
(c) $\tan 3{\cdot}21$
(d) $\operatorname{cosec} 0{\cdot}7$
(e) $\sec 1{\cdot}23$
(f) $\cot 5{\cdot}482$

6. Using the two special triangles, find the exact value of:
(a) $\sin \frac{\pi}{6}$
(b) $\sin \frac{\pi}{4}$
(c) $\cos \frac{\pi}{6}$
(d) $\tan \frac{\pi}{3}$
(e) $\tan \frac{\pi}{4}$
(f) $\cos \frac{\pi}{3}$
(g) $\sec \frac{\pi}{4}$
(h) $\cot \frac{\pi}{3}$

DEVELOPMENT

7. Solve for x over the domain $0 \le x \le 2\pi$:
(a) $\sin x = \frac{1}{2}$
(b) $\cos x = -\frac{1}{2}$
(c) $\tan x = -1$
(d) $\sin x = 1$
(e) $2\cos x = \sqrt{3}$
(f) $\sqrt{3}\tan x = 1$
(g) $\cos x + 1 = 0$
(h) $\sqrt{2}\sin x + 1 = 0$
(i) $\cot x = 1$

8. Express in radians in terms of π:
(a) 20°
(b) $22{\cdot}5^\circ$
(c) 36°
(d) 100°
(e) $112{\cdot}5^\circ$
(f) 252°

9. Express in degrees:
(a) $\dfrac{\pi}{12}$
(b) $\dfrac{2\pi}{5}$
(c) $\dfrac{20\pi}{9}$
(d) $\dfrac{11\pi}{8}$
(e) $\dfrac{17\pi}{10}$
(f) $\dfrac{23\pi}{15}$

10. (a) Find the complement of $\frac{\pi}{6}$.
(b) Find the supplement of $\frac{\pi}{6}$.

11. Two angles of a triangle are $\frac{\pi}{3}$ and $\frac{2\pi}{9}$. Find, in radians, the third angle.

12. Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:
(a) $\sin \frac{2\pi}{3}$
(b) $\cos \frac{2\pi}{3}$
(c) $\cos \frac{5\pi}{6}$
(d) $\tan \frac{4\pi}{3}$
(e) $\tan \frac{3\pi}{4}$
(f) $\cos \frac{5\pi}{3}$
(g) $\sin \frac{5\pi}{4}$
(h) $\tan \frac{7\pi}{6}$

13. If $f(x) = \sin x$, $g(x) = \cos 2x$ and $h(x) = \tan 3x$, find, correct to three significant figures:
(a) $f(1) + g(1) + h(1)$
(b) $f(g(h(1)))$

14. (a) Copy and complete the table, giving values correct to three decimal places.

(b) Hence use Simpson's rule with three function values to find and approximation of $\int_1^2 \sin x\, dx$. Give your answer correct to one decimal place.

x	1	1·5	2
$\sin x$			

CHALLENGE

15. [Technology] Graphing programs provide an excellent way to see what is happening when an equation has many solutions. The equations in question 7 are quite simple to graph, because $y =$ LHS is a single trigonometric function and $y =$ RHS is a horizontal line. Every one of the infinitely many points of intersection corresponds to a solution.

16. Find, correct to three decimal places, the angle in radians through which:

(a) the second hand of a clock turns in 7 seconds,

(b) the hour hand of a clock turns between 6 am and 6:40 am.

17. (a) What is the interior angle sum, in radians, of a pentagon?

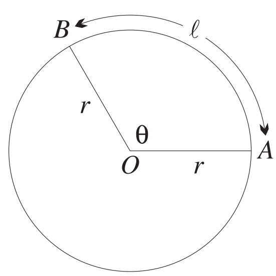
(b) The angles of a pentagon are in arithmetic progression, and the largest angle is twice the smallest. Show that the common difference is $\frac{\pi}{10}$.

(c) Find, in radians, the size of each angle of the pentagon.

4 B Mensuration of Arcs, Sectors and Segments

The lengths of arcs and the areas of sectors and segments can already be calculated using fractions of circles, but radian measure allows the three formulae to be expressed in more elegant forms.

Arc Length: In the diagram to the right, the arc AB has length ℓ and subtends an angle θ at the centre O of a circle with radius r.



The definition of angle size in radians is $\theta = \dfrac{\ell}{r}$.

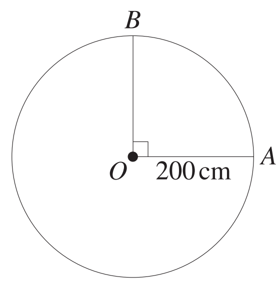
Multiplying through by r, $\ell = r\theta$.

This is the standard formula for arc length:

6 ARC LENGTH: $\ell = r\theta$

WORKED EXERCISE:

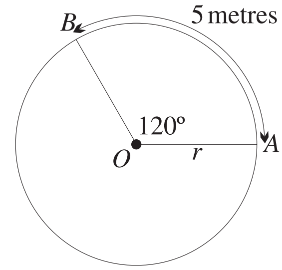
What is the length of an arc subtending a right angle at the centre of a circle of radius 200 cm?



SOLUTION:

$$\begin{aligned}\text{Arc length} &= r\theta \\ &= 200 \times \frac{\pi}{2} \qquad \text{(A right angle has size } \tfrac{\pi}{2}.) \\ &= 100\pi \text{ cm.}\end{aligned}$$

WORKED EXERCISE:
What is radius of a circle in which an arc of length 5 metres subtends an angle of 120° at the centre? Answer correct to the nearest centimetre.



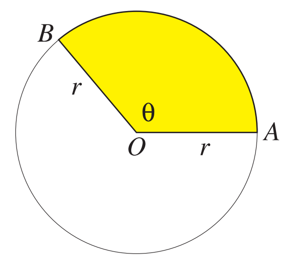
SOLUTION:
Substituting into $\ell = r\theta$,

$$5 = r \times \frac{2\pi}{3} \qquad (120° \text{ in radians is } \tfrac{2\pi}{3}.)$$

$$\boxed{\times \frac{3}{2\pi}} \qquad r = \frac{15}{2\pi}$$

$$\doteqdot 2{\cdot}39 \text{ metres.}$$

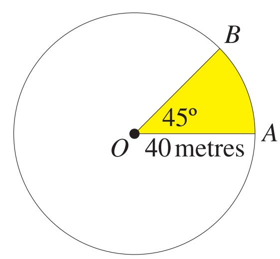
Area of a Sector: In the diagram to the right, the *sector AOB* is the shaded area bounded by the arc AB and the two radii OA and OB. Its area can be calculated as a fraction of the total area:



$$\begin{aligned} \text{area of sector} &= \frac{\theta}{2\pi} \times \text{area of circle} \\ &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \tfrac{1}{2} r^2 \theta. \end{aligned}$$

7 AREA OF SECTOR: Area $= \frac{1}{2} r^2 \theta$

WORKED EXERCISE:
What is the area of a sector subtending an angle of 45° at the centre of a circle of radius 40 metres?



SOLUTION:

$$\begin{aligned} \text{Area of sector} &= \tfrac{1}{2} r^2 \theta \\ &= \frac{1}{2} \times 1600 \times \frac{\pi}{4} \qquad (45° \text{ in radians is } \tfrac{\pi}{4}.) \\ &= 200\pi \text{ square metres.} \end{aligned}$$

WORKED EXERCISE:
A circular cake has radius 12 cm. What angle at the centre is subtended by a sector of area 100 cm^2? Answer correct to the nearest degree.

SOLUTION:

$$\text{Area of sector} = \tfrac{1}{2} r^2 \theta.$$

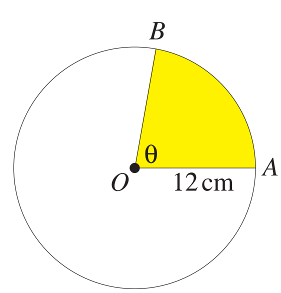
$$100 = \frac{1}{2} \times 144 \times \theta$$

$$\boxed{\div 72} \qquad \theta = \frac{100}{72} \qquad \text{(This answer is in radians.)}$$

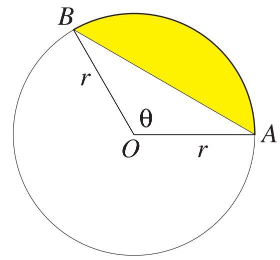
$$= \frac{25}{18} \times \frac{180°}{\pi} \qquad \text{(Converting to degrees.)}$$

$$= \frac{250°}{\pi}$$

$$\doteqdot 80°.$$



Area of a Segment: In the diagram to the right, the *segment* is the shaded area between the arc AB and the chord AB. To find its area, the area of the isosceles triangle must be subtracted from the area of the sector.



First, area of sector $AOB = \frac{1}{2}r^2\theta$.

Secondly, using the formula for the area of a triangle,

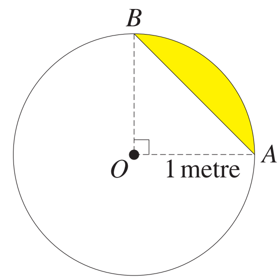
area of isosceles triangle $\triangle AOB = \frac{1}{2}r^2 \sin\theta$

Hence
$$\begin{aligned}\text{area of segment} &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta \\ &= \tfrac{1}{2}r^2(\theta - \sin\theta).\end{aligned}$$

8 AREA OF SEGMENT: Area $= \frac{1}{2}r^2(\theta - \sin\theta)$

WORKED EXERCISE:

In a circle of radius 1 metre, what is the area of a segment subtending a right angle at the centre? Answer correct to the nearest cm^2.



SOLUTION:

$$\begin{aligned}\text{Using the formula, area} &= \tfrac{1}{2}r^2(\theta - \sin\theta) \\ &= \tfrac{1}{2} \times 10\,000 \times \left(\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right) \\ &= 5000(\tfrac{\pi}{2} - 1) \\ &\doteqdot 2854\,\text{cm}^2.\end{aligned}$$

WORKED EXERCISE:

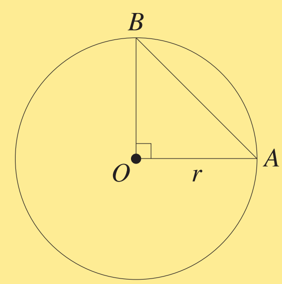
Find, correct to the nearest mm, the radius of a circle in which:

(a) a sector, (b) a segment,

of area 1 square metre subtends an angle of 90° at the centre of the circle.

SOLUTION:

(a) Substituting into the sector area formula,



$$\begin{aligned}\text{area of sector} &= \tfrac{1}{2}r^2\theta \\ 1 &= \frac{1}{2} \times r^2 \times \frac{\pi}{2} \\ 1 &= \frac{\pi}{4} \times r^2 \\ \boxed{\times\frac{4}{\pi}} \qquad r^2 &= \frac{4}{\pi} \\ r &\doteqdot 1{\cdot}128 \text{ metres.}\end{aligned}$$

(b) Substituting into the segment area formula,

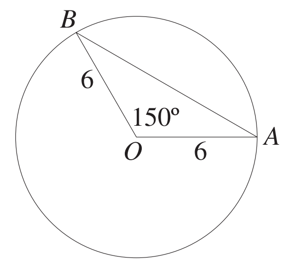
$$\begin{aligned}\text{area of segment} &= \tfrac{1}{2}r^2(\theta - \sin\theta) \\ 1 &= \frac{1}{2} \times r^2 \times \left(\frac{\pi}{2} - 1\right) \\ 1 &= \frac{1}{2} \times r^2 \times \frac{\pi - 2}{2} \\ \boxed{\times\frac{4}{\pi - 2}} \qquad r^2 &= \frac{4}{\pi - 2} \\ r &\doteqdot 1{\cdot}872 \text{ metres.}\end{aligned}$$

Major and Minor Arcs, Sectors and Segments: The arc AB marked in the diagram of the worked exercise below is a *minor arc*, because it is less than half the circumference, and the angle θ that it subtends is less than a straight angle. The *opposite arc* (that is, the rest of the circumference) is called a *major arc*, being more than half the circumference. It subtends an angle of $2\pi - \theta$ at the centre, which is more than a straight angle.

The words 'major' and 'minor' simply mean 'greater' and 'lesser', and they apply also to sectors and segments in the obvious way.

WORKED EXERCISE:

(a) Find the lengths of the minor and major arcs formed by two radii of a circle of radius 6 metres meeting at 150°.
(b) Find the areas of the minor and major sectors.
(c) Find the areas of the major and minor segments.
(d) Find the length AB, correct to the nearest centimetre.



SOLUTION:

The minor arc subtends 150° at the centre, which in radians is $\frac{5\pi}{6}$,
and the major arc subtends 210° at the centre, which in radians is $\frac{7\pi}{6}$.

(a) Minor arc $= r\theta$
$= 6 \times \frac{5\pi}{6}$
$= 5\pi$ metres.

Major arc $= r\theta$
$= 6 \times \frac{7\pi}{6}$
$= 7\pi$ metres.

(b) Minor sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{5\pi}{6}$
$= 15\pi\ \text{m}^2$.

Major sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{7\pi}{6}$
$= 21\pi\ \text{m}^2$.

(c) Minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \times 36(\frac{5\pi}{6} - \sin\frac{5\pi}{6})$
$= 18(\frac{5\pi}{6} - \frac{1}{2})$
$= 3(5\pi - 3)\ \text{m}^2$.

Major segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \times 36(\frac{7\pi}{6} - \sin\frac{7\pi}{6})$
$= 18(\frac{7\pi}{6} + \frac{1}{2})$
$= 3(7\pi + 3)\ \text{m}^2$.

(d) Using the cosine rule in $\triangle AOB$,
$AB^2 = 6^2 + 6^2 - 2 \times 6 \times 6 \times \cos\frac{5\pi}{6}$
$= 72\left(1 + \frac{1}{2}\sqrt{3}\right)$
$= 36\left(2 + \sqrt{3}\right)$,
$AB \doteqdot 11{\cdot}59$ metres.

WORKED EXERCISE:

An athlete runs at a steady 4 m/s around a circular track of radius 300 metres. She runs clockwise, starting at the southernmost point.

(a) How far has she run after 3 minutes?
(b) What angle does this distance then subtend at the centre?
(c) How far, in a direct line across the field, is she from her start?
(d) What is her bearing from the centre then?

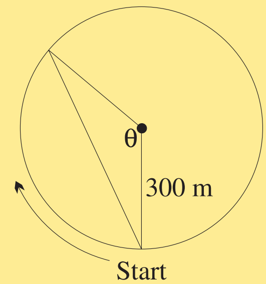
SOLUTION:

(a) She has run for $3 \times 60 = 180$ seconds.

$$\begin{aligned}\text{Hence distance run} &= 4 \times 180 \\ &= 720 \text{ metres.}\end{aligned}$$

(b) Substituting into $\ell = r\theta$,

$$\begin{aligned}720 &= 300\,\theta \\ \theta &= 2{\cdot}4 \quad \text{(which is about } 137^\circ 31'\text{).}\end{aligned}$$



(c) Using the cosine rule,

$$\begin{aligned}(\text{distance from start})^2 &= 300^2 + 300^2 - 2 \times 300^2 \times \cos 2{\cdot}4 \\ &= 300^2(2 - 2\cos 2{\cdot}4), \\ \text{distance} &= 300\sqrt{2 - 2\cos 2{\cdot}4} \\ &\doteqdot 559{\cdot}22 \text{ metres.}\end{aligned}$$

(d) The original bearing was 180°T,

$$\begin{aligned}\text{so final bearing} &\doteqdot 180^\circ + 137^\circ 31' \\ &\doteqdot 317^\circ 31'\text{T.}\end{aligned}$$

Exercise 4B

NOTE: Are you working with radians or degrees? Remember the button labelled $\boxed{\text{mode}}$.

1. In the formula $\ell = r\theta$:
(a) find ℓ, if $r = 18$ and $\theta = \frac{\pi}{6}$,
(b) find ℓ, if $r = 10$ and $\theta = \frac{\pi}{4}$,
(c) find r, if $\ell = 15$ and $\theta = 2$,
(d) find r, if $\ell = 6\pi$ and $\theta = \frac{\pi}{4}$,
(e) find θ, if $\ell = 2\pi$ and $r = 8$,
(f) find θ, if $\ell = 3\pi$ and $r = 1{\cdot}5$.

2. In the formula $A = \frac{1}{2}r^2\theta$:
(a) find A, if $r = 4$ and $\theta = \frac{\pi}{4}$,
(b) find A, if $r = 2$ and $\theta = \frac{2\pi}{3}$,
(c) find θ, if $A = 16$ and $r = 4$,
(d) find θ, if $A = 12\pi$ and $r = 6$,
(e) find r, if $A = 54$ and $\theta = 3$,
(f) find r, if $A = 40\pi$ and $\theta = \frac{\pi}{5}$.

3. In the formula $A = \frac{1}{2}r^2(\theta - \sin\theta)$:
(a) find A, if $r = 1$ and $\theta = \frac{\pi}{2}$,
(b) find A, if $r = 6$ and $\theta = \frac{\pi}{6}$.

4. A circle has radius 6 cm. Find the length of an arc of this circle that subtends an angle at the centre of:
(a) 2 radians (b) $0{\cdot}5$ radians (c) $\frac{\pi}{3}$ radians (d) $\frac{\pi}{4}$ radians

5. A circle has radius 8 cm. Find the area of a sector of this circle that subtends an angle at the centre of:
(a) 1 radian (b) 3 radians (c) $\frac{\pi}{4}$ radians (d) $\frac{3\pi}{8}$ radians

6. What is the radius of the circle in which an arc of length 10 cm subtends an angle of $2{\cdot}5$ radians at the centre?

7. If a sector of a circle of radius 4 cm has area 12 cm^2, find the angle at the centre in radians.

DEVELOPMENT

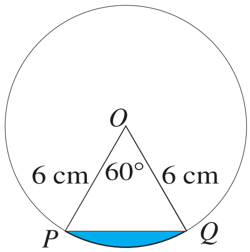
8. A circle has radius 3·4 cm. Find, correct to the nearest millimetre, the length of an arc of this circle that subtends an angle at the centre of:
(a) 40° (b) $73^\circ 38'$
[HINT: Remember that θ must be in radians.]

9. Find, correct to the nearest square metre, the area of a sector of a circle of radius 100 metres if the angle at the centre is 100°.

10. A circle has radius 12 cm. Find, in exact form:
(a) the length of an arc that subtends an angle of 120°at the centre,
(b) the area of a sector in which the angle at the centre is 40°.

11. An arc of a circle of radius 7·2 cm is 10·6 cm in length. Find the angle subtended at the centre by this arc, correct to the nearest degree.

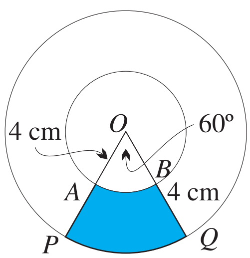
12. A sector of a circle has area 52 cm^2 and subtends an angle of $44^\circ 16'$. Find the radius in cm, correct to one decimal place.



13. In the diagram to the right:
(a) find the exact area of sector OPQ,
(b) find the exact area of $\triangle OPQ$,
(c) hence find the exact area of the shaded minor segment.

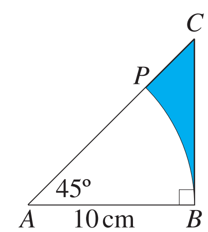
14. A chord of a circle of radius 4 cm subtends an angle of 150° at the centre. Use the formula $A = \frac{1}{2}r^2(\theta - \sin\theta)$ to find:
(a) the area of the minor segment cut off by the chord,
(b) the area of the major segment cut off by the chord.

15. A circle has centre C and radius 5 cm, and an arc AB of this circle has length 6 cm. Find the area of the sector ACB.



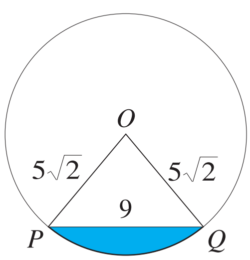
16. The diagram to the right shows two concentric circles with common centre O.
(a) Find the exact perimeter of the region $APQB$.
(b) Find the exact area of the region $APQB$.

17. In the diagram to the right, $\triangle ABC$ is a triangle that is right-angled at B, $AB = 10$ cm and $\angle A = 45^\circ$. The circular arc BP has centre A and radius AB. It meets the hypotenuse AC at P.
(a) Find the exact area of sector ABP.
(b) Hence find the exact area of the shaded portion BCP.



18. (a) Through how many radians does the minute hand of a watch turn between 7:10 am and 7:50 am?
(b) If the minute hand is 1·2 cm long, find, correct to the nearest cm, the distance travelled by its tip in that time.

19. In a circle with centre O and radius $5\sqrt{2}$ cm, a chord of length 9 cm is drawn.
(a) Use the cosine rule to find $\angle POQ$ in radians, correct to two decimal places.
(b) Hence find, correct to the nearest cm^2, the area of the minor segment cut off by the chord.



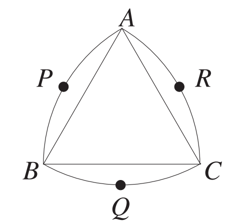
20. Triangle ABC is equilateral with side length 2 cm. Circular arcs AB, BC and CA have centres C, A and B respectively.

(a) Find the length of the arc AB.

(b) Find the area of the sector $CAPBC$.

(c) Find the length of the perimeter $APBQCRA$.

(d) Find the area of $\triangle ABC$ and hence find the area enclosed by the perimeter $APBQCRA$. (Give all answers in exact form.)



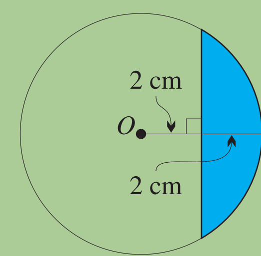
CHALLENGE

21. [Technology] Use a graphing program to sketch on one set of axes the sector area formula and the segment area formula as a function of θ for a circle of radius 1.

(a) For what values of θ do the two formulae have the same value?

(b) For what values of θ does the segment area formula give a value greater than the sector area formula?

(c) By examining the relative sizes of sectors and segments subtending the same angle at the centre of a circle, explain these results geometrically.



22. Find the exact area of the shaded region shown to the right.

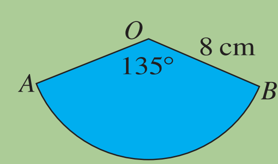
23. A piece of paper is cut in the shape of a sector of a circle. The radius is 8 cm and the angle at the centre is 135°. The straight edges of the sector are placed together so that a cone is formed.

(a) Show that the base of the cone has radius 3 cm.

(b) Show that the cone has perpendicular height $\sqrt{55}$ cm.

(c) Hence find, in exact form, the volume of the cone.

(d) Find the curved surface area of the cone.



24. The radii OP and OQ of a circle centred at O have length r cm. The arc PQ of the circle subtends an angle of θ radians at O, and the perimeter of the sector OPQ is 12 cm.

(a) Show that the area $A\,\text{cm}^2$ of the sector is given by $A = \dfrac{72\theta}{(2+\theta)^2}$.

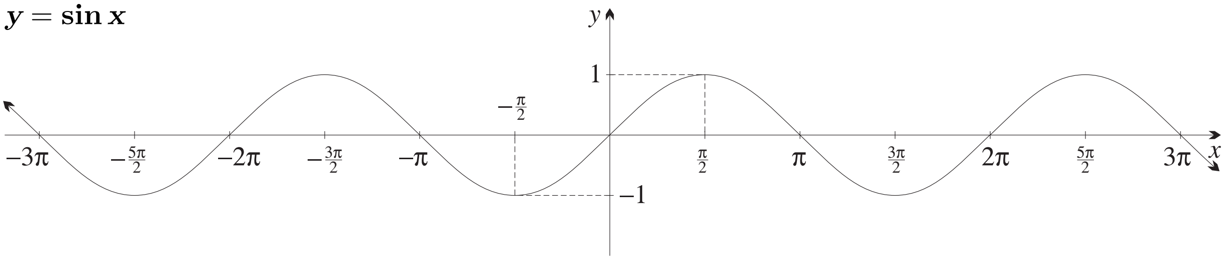
(b) Hence find the maximum area of the sector.

4 C Graphs of the Trigonometric Functions in Radians

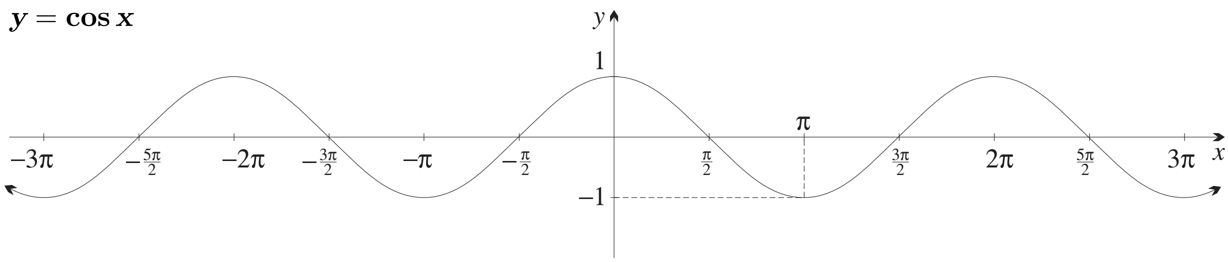
Now that angle size has been defined as a ratio, that is, as a pure number, the trigonometric functions can be drawn in their true shapes. On the next full page, the graphs of the six functions have been drawn using the same scale on the x-axis and y-axis. This means that the gradient of the tangent at each point now equals the true value of the derivative there, and the areas under the graphs faithfully represent the appropriate definite integrals.

For example, place a ruler on the graph of $y = \sin x$ so that it makes a tangent to the curve at the origin. The ruler should lie along the line $y = x$, indicating that the tangent at the origin has gradient 1. In the language of calculus, this means that the derivative of $\sin x$ has value 1 when $x = 0$.

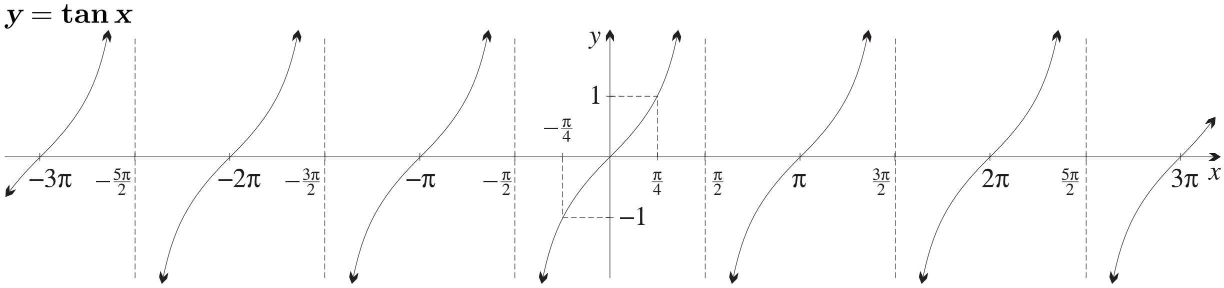
$y = \sin x$



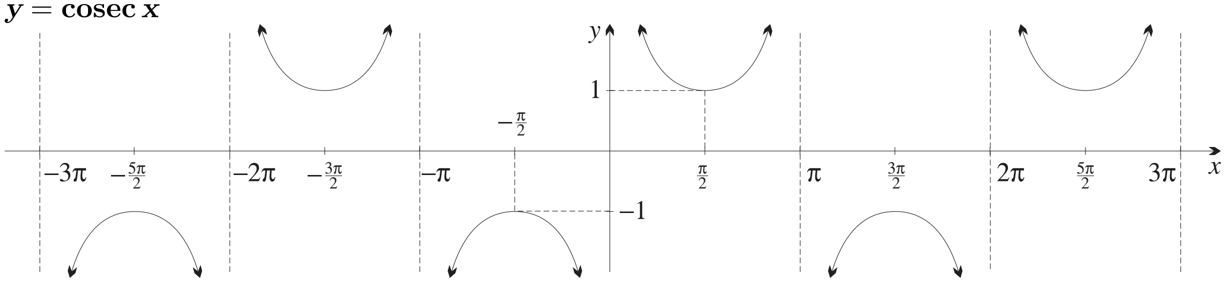
$y = \cos x$



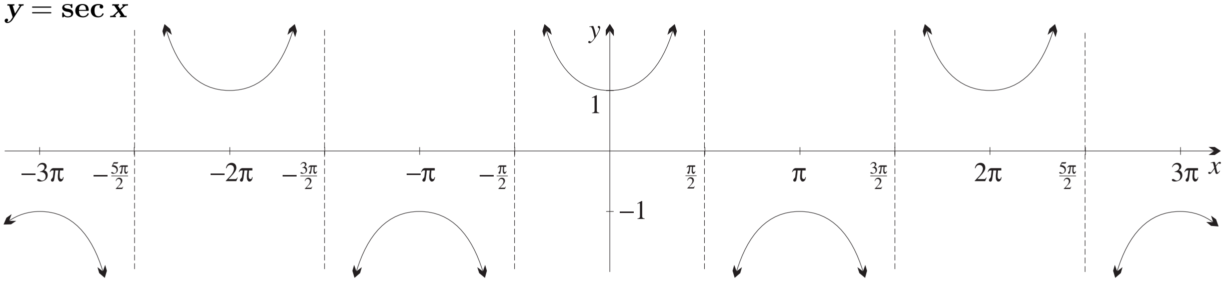
$y = \tan x$



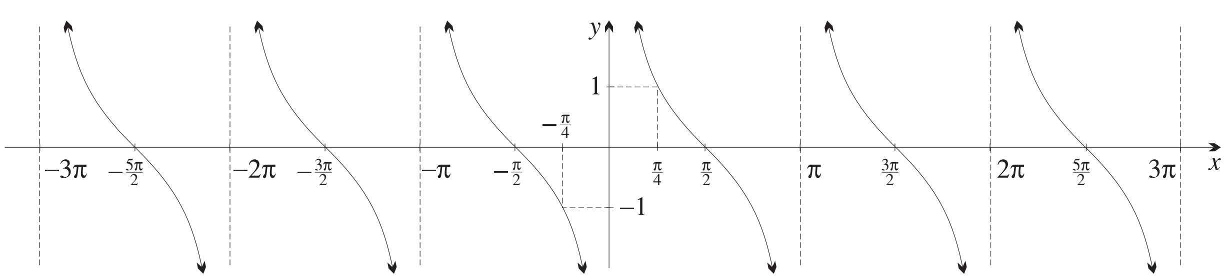
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



Amplitude of the Sine and Cosine Functions: The *amplitude* of a wave is the maximum height of the wave above the mean position. Both $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0. Thus both have amplitude 1.

More generally, for positive constants a and b, the functions $y = a \sin bx$ and $y = a \cos bx$ have maximum value a and minimum value $-a$. Thus both have amplitude a.

9 AMPLITUDE OF THE SINE AND COSINE FUNCTIONS:
- $y = \sin x$ and $y = \cos x$ have amplitude 1.
- $y = a \sin bx$ and $y = a \cos bx$ have amplitude a.

The other four trigonometric functions increase without bound near their asymptotes and so the idea of amplitude makes no sense. We can conveniently tie down the vertical scale of $y = a \tan bx$, however, by using the fact that $\tan \frac{\pi}{4} = 1$.

The Periods of the Trigonometric Functions: The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the previous page are waves, with a pattern that repeats every revolution. Thus they both have period 2π.

The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π.

10 THE PERIODS OF THE SINE, COSINE AND TANGENT FUNCTIONS:
- $y = \sin x$ and $y = \cos x$ have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

For any positive constants a and b, the function $y = a \sin bx$ is also periodic. Because the function $y = \sin x$ repeats over an interval of 2π, the period of $y = a \sin bx$ is the value T for which $bT = 2\pi$, and solving for T, the period is $\dfrac{2\pi}{b}$.

Similarly, $y = a \cos bx$ has the same period $\dfrac{2\pi}{b}$, and $y = a \tan bx$ has period $\dfrac{\pi}{b}$.

11 THE PERIODS OF $y = a \sin bx$, $y = a \cos bx$ AND $y = a \tan bx$:
- $y = a \sin bx$ and $y = a \cos bx$ have period $\dfrac{2\pi}{b}$.
- $y = a \tan bx$ has period $\dfrac{\pi}{b}$.

The secant and cosecant functions are reciprocals of the cosine and sine functions and so have the same periods as they do. Similarly, the cotangent function has the same period as the tangent function.

WORKED EXERCISE:

Sketch one period of:

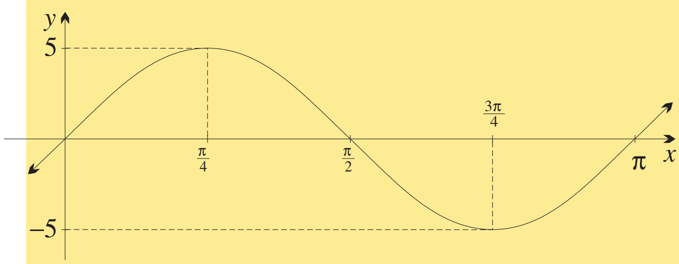
(a) $y = 5 \sin 2x$, (b) $y = 2 \tan \frac{1}{3}x$,

showing all intercepts, turning points and asymptotes.

SOLUTION:

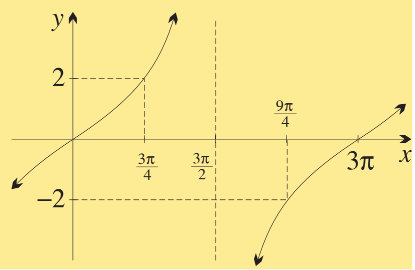
(a) $y = 5\sin 2x$ has an amplitude of 5,

and a period of $\dfrac{2\pi}{2} = \pi$.



(b) $y = 2\tan \frac{1}{3}x$ has period $\dfrac{\pi}{1/3} = 3\pi$,

and when $x = \frac{3\pi}{4}$, $y = 2\tan\frac{\pi}{4} = 2$.



Oddness and Evenness of the Trigonometric Functions: The graphs of $y = \sin x$ and $y = \tan x$ have point symmetry in the origin, as can easily be seen from their graphs on the previous page. This means that the functions $\sin x$ and $\tan x$ are odd functions. Algebraically, $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$.

The graph of $y = \cos x$, however, has line symmetry in the y-axis. This means that the function $\cos x$ is an even function. Algebraically, $\cos(-x) = \cos x$.

12

ODDNESS AND EVENNESS OF THE TRIGONOMETRIC FUNCTIONS:

- The functions $\sin x$ and $\tan x$ are odd functions. Thus for all x,

$$\sin(-x) = -\sin x \qquad \text{and} \qquad \tan(-x) = -\tan x.$$

- The function $\cos x$ is an even function. Thus for all x,

$$\cos(-x) = \cos x.$$

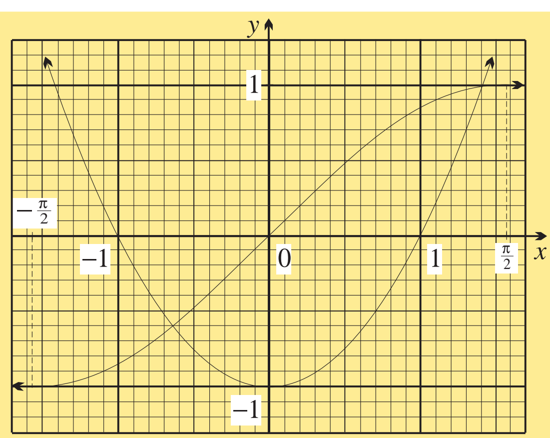
Graphical Solutions of Trigonometric Equations: Many trigonometric equations cannot be solved by algebraic methods. Approximation methods using the graphs can usually be used instead and a graph-paper sketch will show:

- how many solutions there are, and
- the approximate values of the solutions.

WORKED EXERCISE:

(a) Find, by drawing a graph, the number of solutions to $\sin x = x^2 - 1$.

(b) Then use the graph to find approximations correct to one decimal place.



SOLUTION:

(a) Here are $y = \sin x$ and $y = x^2 - 1$.

Clearly the equation has two solutions.

(b) The positive solution is $x \doteqdot 1{\cdot}4$,

and the negative solution is $x \doteqdot -0{\cdot}6$.

NOTE: Graphics calculators and computer packages are particularly useful here. They allow sketches to be drawn quickly, and many programs will give the approximate coordinates of the intersections.

Exercise 4C

TECHNOLOGY: Computer sketching can provide experience of a large number of graphs similar to the ones listed in this exercise. In particular, it is very useful in making clear the importance of period and amplitude and in reinforcing the formulae for them.

Graphical solution of equations is an approximation method that can be done to a much greater degree of accuracy with a graphing program. Questions 12, 13, 16, 17 and 19 are interesting questions to graph and solve in this way.

1. Sketch on separate diagrams the graphs of the following functions, for $-2\pi \le x \le 2\pi$. State the period in each case.
(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

2. Sketch on one diagram the following three functions, for $0 \le \theta \le 2\pi$:
(a) $y = \sin\theta$ (b) $y = 2\sin\theta$ (c) $y = 4\sin\theta$

3. Sketch on one diagram the following three functions, for $0 \le \alpha \le 2\pi$:
(a) $y = \cos\alpha$ (b) $y = \cos 2\alpha$ (c) $y = \cos 4\alpha$

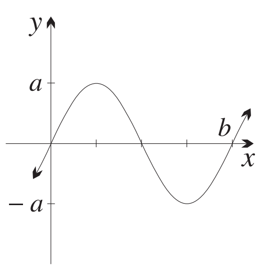
4. Sketch both these functions on the one diagram, for $-\pi \le x \le \pi$:
(a) $y = 2\sin x$ (b) $y = \sin 2x$

5. Sketch both these functions on the one diagram, for $-\pi \le x \le \pi$:
(a) $y = 2\cos x$ (b) $y = \cos 2x$

6. Sketch on one diagram the following three functions, for $0 \le t \le 2\pi$:
(a) $y = \cos t$ (b) $y = \cos(t - \pi)$ (c) $y = \cos(t - \frac{\pi}{2})$

7. State the periods and amplitudes of these functions, then sketch them on separate diagrams, for $0 \le x \le 2\pi$:
(a) $y = \sin 2x$ (c) $y = 4\sin 3x$ (e) $y = \tan 2x$
(b) $y = 2\cos 2x$ (d) $y = 3\cos\frac{1}{2}x$ (f) $y = 3\tan\frac{1}{2}x$

8. State the periods and amplitudes of these functions, then sketch them on separate diagrams, for $-\pi \le x \le \pi$:
(a) $y = 3\cos 2x$ (c) $y = \tan\frac{3x}{2}$
(b) $y = 2\sin\frac{1}{2}x$ (d) $y = 2\cos 3x$



9. The graph to the right shows the sketch of a sine curve. What are the values of a and b if the equation of the curve in the sketch is:
(a) $y = \sin x$ (c) $y = 2\sin 3x$
(b) $y = 3\sin 2x$ (d) $y = 4\sin 4x$

DEVELOPMENT

10. (a) Write down the amplitude and period of $y = \sin 2\pi x$.
(b) Hence sketch $y = \sin 2\pi x$, for $-1 \le x \le 2$.

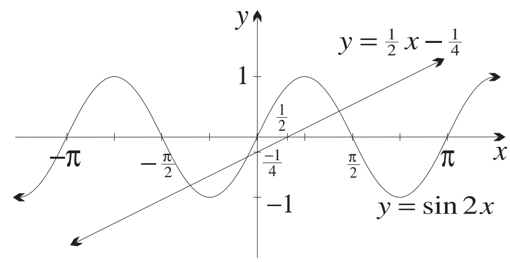
11. (a) Sketch $y = \sin x$, for $-\pi \le x \le \pi$.
(b) On the same diagram sketch $y = \sin(x + \frac{\pi}{2})$, for $-\pi \le x \le \pi$.
(c) Hence simplify $\sin(x + \frac{\pi}{2})$.

12. In the diagram to the right, the curve $y = \sin 2x$ is graphed in the interval $-\pi \le x \le \pi$, and the line $y = \frac{1}{2}x - \frac{1}{4}$ is graphed.

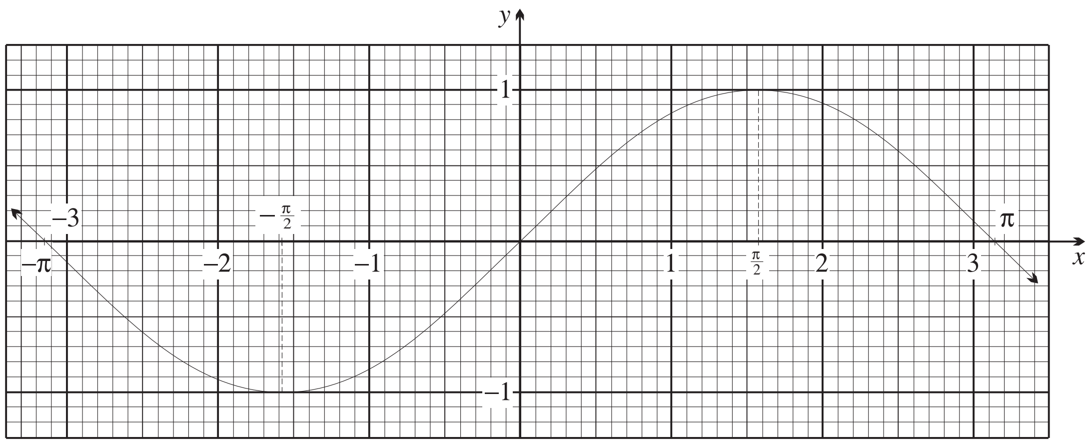
(a) At how many points does the line $y = \frac{1}{2}x - \frac{1}{4}$ meet the curve $y = \sin 2x$?

(b) State the number of solutions of the equation $\sin 2x = \frac{1}{2}x - \frac{1}{4}$. How many of these solutions are positive?

(c) Briefly explain why the line $y = \frac{1}{2}x - \frac{1}{4}$ will not meet the curve $y = \sin 2x$ outside the domain $-\pi \le x \le \pi$.



13.



Photocopy the above graph of $y = \sin x$, for $-\pi \le x \le \pi$, and on it graph the line $y = \frac{1}{2}x$. Hence find the three solutions of the equation $\sin x = \frac{1}{2}x$, giving your answers correct to one decimal place where necessary.

14. (a) Sketch, for $0 \le x \le 2\pi$: (i) $y = \cos 2x$, (ii) $y = -\cos 2x$.

(b) Hence sketch $y = 3 - \cos 2x$, for $0 \le x \le 2\pi$.

15. (a) Carefully sketch the curve $y = \sin^2 x$, for $0 \le x \le 2\pi$, after completing the table to the right.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y					

(b) Explain why $y = \sin^2 x$ has range $0 \le y \le 1$.

(c) Write down the period and amplitude of $y = \sin^2 x$.

CHALLENGE

16. (a) Sketch the graph of $y = 2\cos x$, for $-2\pi \le x \le 2\pi$.

(b) On the same diagram, carefully sketch the line $y = 1 - \frac{1}{2}x$, showing its x- and y-intercepts.

(c) How many solutions does the equation $2\cos x = 1 - \frac{1}{2}x$ have?

(d) Mark with the letter P the point on the diagram from which the negative solution of the equation in part (c) is obtained.

(e) Prove algebraically that if n is a solution of the equation in part (c), then $-2 \le n \le 6$.

17. (a) Sketch the curve $y = 2\cos 2x$, for $0 \le x \le 2\pi$.

(b) Sketch the line $y = 1$ on the same diagram.

(c) How many solutions does the equation $\cos 2x = \frac{1}{2}$ have in the domain $0 \le x \le 2\pi$?

(d) What is the first positive solution to $\cos 2x = \frac{1}{2}$?

(e) Use your diagram to find the values of x in $0 \le x \le 2\pi$ for which $\cos 2x < \frac{1}{2}$.

18. Sketch $y = \cos x$ and then answer the following questions.

(a) Give the equations of all axes of symmetry. ($x = -2\pi$ to $x = 2\pi$ will do.)

(b) Around which points does the graph have rotational symmetry?

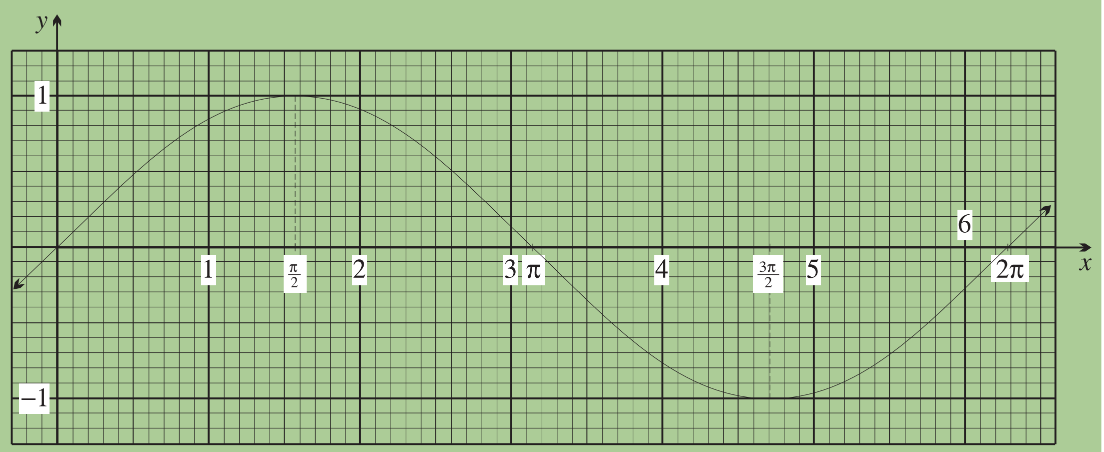
(c) What translations will leave the graph unchanged?

(d) Describe two translations that will move the graph of $y = \cos x$ to $y = -\cos x$.

(e) Describe two translations that will move the graph of $y = \cos x$ to the graph of $y = \sin x$.

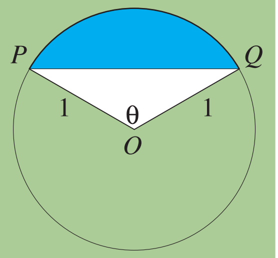
(f) Name two vertical lines such that reflection of $y = \cos x$ in either of these lines will reflect the graph into the graph of $y = \sin x$.

19.



(a) (i) Photocopy the above graph of $y = \sin x$, for $0 \le x \le 2\pi$, and on it carefully graph the line $y = x - 2$.

(ii) Use your graph to estimate, correct to two decimal places, the value of x for which $\sin x = x - 2$.



(b) The diagram shows points P and Q on a circle with centre O whose radius is 1 unit. $\angle POQ = \theta$. If the area of the shaded segment is 1 square unit, use part (a) to find θ, correct to the nearest degree.

(c) Suppose instead that the area of the segment is 2 square units.

(i) Show that $\sin\theta = \theta - 4$.

(ii) By drawing a suitable line on the graph in part (a), find θ, correct to the nearest degree.

4 D The Behaviour of sin x Near the Origin

This section proves an important limit that is a crucial step in finding the derivative of $\sin x$ in the next section. This limit establishes that the curve $y = \sin x$ has gradient 1 when it passes through the origin. Geometrically, this means that the line $y = x$ is the tangent to $y = \sin x$ at the origin.

NOTE: This section is needed for the derivatives of the trigonometric functions to be established, but the material is not easy and the section could well be left until after the calculus of the trigonometric functions has been covered.

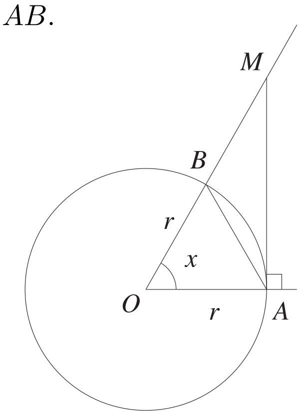
A Fundamental Inequality: First, an appeal to geometry is needed to establish an inequality concerning x, $\sin x$ and $\tan x$.

13 AN INEQUALITY FOR $\sin x$ AND $\tan x$ NEAR THE ORIGIN:

A. For all acute angles x,
$$\sin x < x < \tan x.$$
B. For angles x in the interval $-\frac{\pi}{2} < x < 0$,
$$\sin x > x > \tan x.$$

PROOF:

A. Suppose that x is an acute angle.
Construct a circle of centre O and any radius r,
and a sector AOB subtending the angle x at the centre O.
Let the tangent at A meet the radius OB at M
(the radius OB will need to be produced) and join the chord AB.



In $\triangle OAM$, $\dfrac{AM}{r} = \tan x$,

so $AM = r\tan x$.

It is clear from the diagram that
$$\text{area } \triangle OAB < \text{area sector } OAB < \text{area } \triangle OAM$$
and using area formulae for triangles and sectors,
$$\tfrac{1}{2}r^2 \sin x < \tfrac{1}{2}r^2 x < \tfrac{1}{2}r^2 \tan x$$
$\boxed{\div \tfrac{1}{2}r^2}$ $\sin x < x < \tan x$.

B. Since x, $\sin x$ and $\tan x$ are all odd functions,
$$\sin x > x > \tan x, \quad \text{for } -\tfrac{\pi}{2} < x < 0.$$

The Main Theorem:

This inequality now allows two fundamental limits to be proven:

14 THE FUNDAMENTAL LIMITS:
$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

PROOF:

Suppose first that x is acute, so that $\sin x < x < \tan x$.

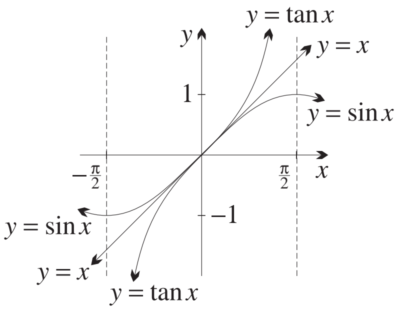
Dividing through by $\sin x$ gives $\quad 1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$.

But $\cos x \to 1$ as $x \to 0^+$, so $\quad \dfrac{x}{\sin x} \to 1$ as $x \to 0^+$, as required.

Since $\dfrac{x}{\sin x}$ is even, it follows also that $\dfrac{x}{\sin x} \to 1$ as $x \to 0^-$.

Finally,
$$\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$$
$$\to 1 \times 1, \text{ as } x \to 0.$$

The diagram to the right shows what has been proven about the graphs of $y = x$, $y = \sin x$ and $y = \tan x$ near the origin. The line $y = x$ is a common tangent at the origin to both $y = \sin x$ and $y = \tan x$. On both sides of the origin, $y = \sin x$ curves away from the tangent towards the x-axis, and $y = \tan x$ curves away from the tangent in the opposite direction.



15 THE BEHAVIOUR OF $\sin x$ AND $\tan x$ NEAR THE ORIGIN:

- The line $y = x$ is a tangent to both $y = \sin x$ and $y = \tan x$ at the origin.
- When $x = 0$, the derivatives of both $\sin x$ and $\tan x$ are exactly 1.

Approximations to the Trigonometric Functions for Small Angles: For 'small' angles, positive or negative, the limits above yield good approximations for the three trigonometric functions (the angle must, of course, be expressed in radians).

16 SMALL-ANGLE APPROXIMATIONS: Suppose that x is a 'small' angle. Then:

$$\sin x \doteqdot x$$
$$\cos x \doteqdot 1$$
$$\tan x \doteqdot x$$

In order to use these approximations, one needs to get some idea about how good the approximations are. Two questions in the following exercise ask for tables of values for $\sin x$, $\cos x$ and $\tan x$ for progressively smaller angles.

WORKED EXERCISE:

Give approximate values of: (a) $\sin 1°$, (b) $\cos 1°$, (c) $\tan 1°$.

SOLUTION:

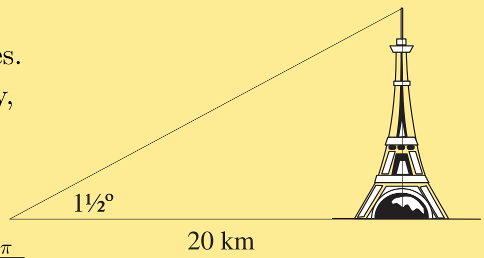
The 'small angle' of $1°$ is $\frac{\pi}{180}$ radians. Hence, using the approximations above:

(a) $\sin 1° \doteqdot \frac{\pi}{180}$ (b) $\cos 1° \doteqdot 1$ (c) $\tan 1° \doteqdot \frac{\pi}{180}$

WORKED EXERCISE:

Approximately how high is a tower that subtends an angle of $1\frac{1}{2}°$ when it is 20 km away?

SOLUTION:



First, 20 km needs to be converted to 20 000 metres.

Then from the diagram, using simple trigonometry,

$$\frac{\text{height}}{20\,000} = \tan 1\tfrac{1}{2}°$$
$$\text{height} = 20\,000 \times \tan 1\tfrac{1}{2}°.$$

But the 'small' angle $1\frac{1}{2}°$ expressed in radians is $\frac{\pi}{120}$,

so $\tan 1\frac{1}{2}° \doteqdot \frac{\pi}{120}$.

Hence, approximately, $\text{height} \doteqdot 20\,000 \times \dfrac{\pi}{120}$

$$\doteqdot \frac{500\pi}{3} \text{ metres}$$
$$\doteqdot 524 \text{ metres.}$$

WORKED EXERCISE:
The sun subtends an angle of $0°31'$ at the Earth, which is 150 000 000 km away. What is the sun's approximate diameter?

NOTE: This problem can be done similarly to the previous problem, but like many small-angle problems, it can also be done by approximating the diameter to an arc of the circle.

SOLUTION:

First, $0°31' = \dfrac{31}{60}^{\circ}$

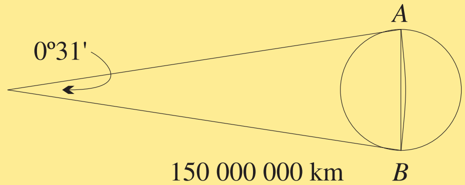
$= \dfrac{31}{60} \times \dfrac{\pi}{180}$ radians.

Since the diameter AB is approximately equal to the arc length AB,

diameter $\doteqdot r\theta$

$\doteqdot 150\,000\,000 \times \dfrac{31}{60} \times \dfrac{\pi}{180}$

$\doteqdot 1\,353\,000$ km.



Exercise 4D

1. (a) Copy and complete the following table of values, giving entries correct to six decimal places. (Your calculator must be in radian mode.)

angle size in radians	1	0·5	0·2	0·1	0·08	0·05	0·02	0·01	0·005	0·002
$\sin x$										
$\dfrac{\sin x}{x}$										
$\tan x$										
$\dfrac{\tan x}{x}$										
$\cos x$										

(b) What limits do $\dfrac{\sin x}{x}$ and $\dfrac{\tan x}{x}$ approach as $x \to 0$?

2. [Technology] The previous question is perfect for a spreadsheet approach. The spreadsheet columns can be identical to the rows above. Various graphs can then be drawn using the data from the spreadsheet.

3. (a) Express $2°$ in radians.

(b) Explain why $\sin 2° \doteqdot \dfrac{\pi}{90}$.

(c) Taking π as $3{\cdot}142$, find $\sin 2°$, correct to four decimal places, *without* using a calculator.

DEVELOPMENT

4. (a) Copy and complete the following table of values, giving entries correct to four significant figures. For each column, hold x in the calculator's memory until the column is complete:

angle size in degrees	60°	30°	10°	5°	2°	1°	$20'$	$5'$	$1'$	$30''$	$10''$
angle size x in radians											
$\sin x$											
$\dfrac{\sin x}{x}$											
$\tan x$											
$\dfrac{\tan x}{x}$											
$\cos x$											

(b) Write x, $\sin x$ and $\tan x$ in ascending order, for acute angles x.

(c) Although $\sin x \to 0$ and $\tan x \to 0$ as $x \to 0$, what are the limits, as $x \to 0$, of:

(i) $\dfrac{\sin x}{x}$? (ii) $\dfrac{\tan x}{x}$?

(d) Experiment with your calculator, or a spreadsheet, to find how small x must be in order for $\dfrac{\sin x}{x} > 0{\cdot}999$ to be true.

5. [Technology] A properly prepared spreadsheet makes it easy to ask a sequence of questions like part (d) of the previous question. One can ask how small x must be for each of the following three functions to be closer to 1 than 0·1, 0·001, 0·0001, 0·00001, ...

$$\frac{\sin x}{x} \qquad \text{and} \qquad \frac{\tan x}{x} \qquad \text{and} \qquad \cos x$$

6. A car travels 1 km up a road that is inclined at 5° to the horizontal. Through what vertical distance has the car climbed? (Use the fact that $\sin x \doteqdot x$ for small angles, and give your answer correct to the nearest metre.)

7. A tower is 30 metres high. What angle, correct to the nearest minute, does it subtend at a point 4 km away? (Use the fact that when x is small, $\tan x \doteqdot x$.)

CHALLENGE

8. [Technology] Draw on one screen the graphs $y = \sin x$, $y = \tan x$ and $y = x$, noting how the two trigonometric graphs curl away from $y = x$ in opposite directions. Zoom in on the origin until the three graphs are indistinguishable.

9. [Technology] Draw the graph of $y = \dfrac{\sin x}{x}$. It is undefined at the y-intercept, but the curve around this point is flat and clearly has limit 1 as $x \to 0$. Other features of the graph can be explained, and the exercise can be repeated with the function $y = \dfrac{\tan x}{x}$.

10. The moon subtends an angle of $31'$ at an observation point on Earth, 400 000 km away. Use the fact that the diameter of the moon is approximately equal to an arc of a circle whose centre is the point of observation to show that the diameter of the moon is approximately 3600 km. [HINT: Use a diagram like that in the last worked exercise in the notes above.]

11. A regular polygon of 300 sides is inscribed in a circle of radius 60 cm. Show that each side is approximately 1·26 cm.

12. [A better approximation for $\cos x$ when x is small] The chord AB of a circle of radius r subtends an angle x at the centre O.

(a) Find AB^2 by the cosine rule, and find the length of the arc AB.

(b) By equating arc and chord, show that for small angles, $\cos x \doteqdot 1 - \dfrac{x^2}{2}$.

Explain whether the approximation is bigger or smaller than $\cos x$.

(c) Check the accuracy of the approximation for angles of 1°, 10°, 20° and 30°.

13. [Technology] Sketch on one screen the graphs of $y = \cos x$ and $y = 1 - \frac{1}{2}x^2$ as discussed in the previous question. Which one is larger, and why? A spreadsheet may help you to identify the size of the error for different values of x.

4 E The Derivatives of the Trigonometric Functions

Finally we can establish the derivatives of the three trigonometric functions $\sin x$, $\cos x$ and $\tan x$. The proofs of these standard forms are quite difficult and they have therefore been placed in an appendix at the end of the chapter. Using them to differentiate further trigonometric functions, however, is reasonably straightforward and is the subject of this section.

Standard Forms: Here are the formulae for the derivatives of the first three trigonometric functions.

STANDARD DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

17

$$\frac{d}{dx}\sin x = \cos x$$

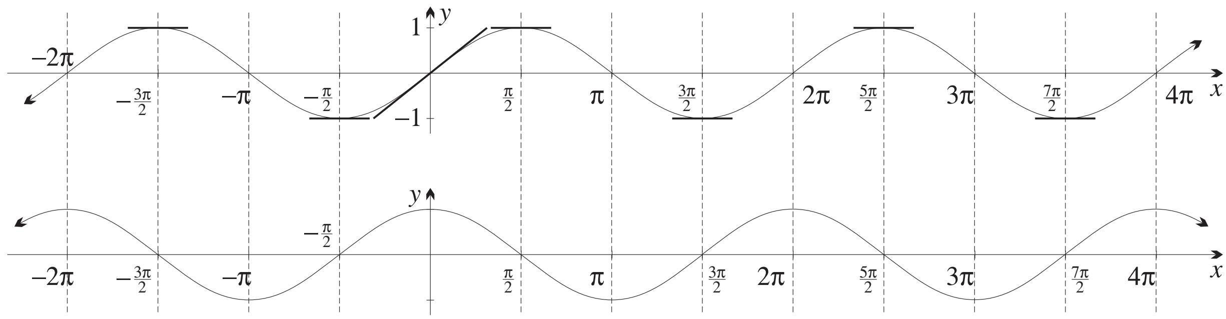
$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

The exercises ask for derivatives of the secant, cosecant and cotangent functions.

A Graphical Demonstration that the Derivative of $\sin x$ is $\cos x$: The upper graph in the sketch below is $y = \sin x$. The lower graph is a rough sketch of the derivative of $y = \sin x$. This second graph is straightforward to construct simply by paying attention to where the gradients of tangents to $y = \sin x$ are zero, maximum and minimum. The lower graph is periodic, with period 2π, and has a shape unmistakably like a cosine graph.

Moreover, it was proven in the previous section that the gradient of $y = \sin x$ at the origin is exactly 1. This means that the lower graph has a maximum of 1 when $x = 0$. By symmetry, all its maxima must be 1 and all its minima must be -1. Thus the lower graph not only has the distinctive shape of the cosine curve, but has the correct amplitude as well. This doesn't prove conclusively that the derivative of $\sin x$ is $\cos x$, but it is very convincing.



Differentiating using the Three standard Forms: These worked examples use the standard forms to differentiate functions involving $\sin x$, $\cos x$ and $\tan x$.

Worked Exercise:

Differentiate the following functions:

(a) $y = \sin x + \cos x$ (b) $y = x - \tan x$

Hence find the gradient of each curve when $x = \frac{\pi}{4}$.

Solution:

(a) The function is $y = \sin x + \cos x$.

Differentiating, $y' = \cos x - \sin x$.

When $x = \frac{\pi}{4}$,
$$\begin{aligned} y' &= \cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= 0. \end{aligned}$$

(b) The function is $y = x - \tan x$.

Differentiating, $y' = 1 - \sec^2 x$.

When $x = \frac{\pi}{4}$,
$$\begin{aligned} y' &= 1 - \sec^2\tfrac{\pi}{4} \\ &= 1 - (\sqrt{2}\,)^2 \\ &= -1. \end{aligned}$$

Worked Exercise:

If $f(x) = \sin x$, find $f'(0)$. Hence find the equation of the tangent to $y = \sin x$ at the origin, and then sketch the curve and the tangent.

Solution:

Here $f(x) = \sin x$,

and substituting $x = 0$, $f(0) = 0$,

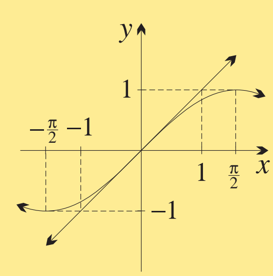
confirming that the curve passes through the origin.

Differentiating, $f'(x) = \cos x$,

and substituting $x = 0$,
$$\begin{aligned} f'(0) &= \cos 0 \\ &= 1, \end{aligned}$$

so the tangent to $y = \sin x$ at the origin has gradient 1.

Hence its equation is
$$\begin{aligned} y - 0 &= 1(x - 0) \\ y &= x. \end{aligned}$$



NOTE: This result was already clear from the limit $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ proven in the previous section. The simplicity of the result confirms that radian measure is the correct measure to use for angles when doing calculus.

Using the Chain Rule to Generate More Standard Forms: A simple pattern emerges when the chain rule is used to differentiate functions like $\cos(3x + 4)$, where the angle $3x + 4$ is a linear function.

WORKED EXERCISE:

Use the chain rule to differentiate:

(a) $y = \cos(3x + 4)$ (b) $y = \tan(5x - 1)$ (c) $y = \sin(ax + b)$

SOLUTION:

(a) Here $y = \cos(3x + 4)$.

Applying the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -\sin(3x + 4) \times 3 \\ &= -3\sin(3x + 4). \end{aligned}$$

Let $u = 3x + 4$.
Then $y = \cos u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = -\sin u$.

(b) Here $y = \tan(5x - 1)$.

Applying the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \sec^2(5x - 1) \times 5 \\ &= 5\sec^2(5x - 1). \end{aligned}$$

Let $u = 5x - 1$.
Then $y = \tan u$.
Hence $\dfrac{du}{dx} = 5$
and $\dfrac{dy}{du} = \sec^2 u$.

(c) Here $y = \sin(ax + b)$.

Applying the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos(ax + b) \times a \\ &= a\cos(ax + b). \end{aligned}$$

Let $u = ax + b$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \cos u$.

The last result in the previous worked exercise can be extended to the other trigonometric functions, giving the following standard forms:

STANDARD DERIVATIVES OF FUNCTIONS OF $ax + b$:

18

$$\frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$$

$$\frac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$$

$$\frac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$$

WORKED EXERCISE:
Use the extended standard forms given in Box 18 above to differentiate the following functions:
(a) $y = \cos 7x$ (b) $y = 4\sin(3x - \frac{\pi}{3})$ (c) $y = \tan \frac{3}{2}x$

SOLUTION:
(a) The function is $y = \cos 7x$.
Here $a = 7$ and $b = 0$,
so $$\frac{dy}{dx} = -7\sin 7x.$$
(b) The function is $y = 4\sin(3x - \frac{\pi}{3})$.
Here $a = 3$ and $b = -\frac{\pi}{3}$,
so $$\frac{dy}{dx} = 12\cos(3x - \frac{\pi}{3}).$$
(c) The function is $y = \tan \frac{3}{2}x$.
Here $a = \frac{3}{2}$ and $b = 0$,
so $$\frac{dy}{dx} = \frac{3}{2}\sec^2 . \frac{3}{2}x$$

Using the Chain Rule with Trigonometric Functions: The chain rule can also be applied in the usual way to differentiate compound functions.

WORKED EXERCISE:
Use the chain rule to differentiate:
(a) $y = \tan^2 x$ (b) $y = \sin(x^2 - \frac{\pi}{4})$

SOLUTION:
(a) Here $y = \tan^2 x$.
Applying the chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2\tan x \sec^2 x.$$

Let $u = \tan x$.
Then $y = u^2$.
Hence $\frac{du}{dx} = \sec^2 x$
and $\frac{dy}{du} = 2u$.

(b) Here $y = \sin(x^2 - \frac{\pi}{4})$.
Applying the chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2x\cos(x^2 - \frac{\pi}{4}).$$

Let $u = x^2 - \frac{\pi}{4}$.
Then $y = \sin u$.
Hence $\frac{du}{dx} = 2x$
and $\frac{dy}{du} = \cos u$.

Using the Product Rule with Trigonometric Functions: A function like $y = e^x \cos x$ is the product of the two functions $u = e^x$ and $v = \cos x$. It can therefore be differentiated by using the product rule.

WORKED EXERCISE:

Use the product rule to differentiate:

(a) $y = e^x \cos x$ (b) $y = 5\cos 2x \cos \frac{1}{2}x$

SOLUTION:

(a) Here $y = e^x \cos x$.

Applying the product rule,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^x \cos x - e^x \sin x$$
$$= e^x(\cos x - \sin x).$$

Let $u = e^x$
and $v = \cos x$.
Then $\frac{du}{dx} = e^x$
and $\frac{dv}{dx} = -\sin x$.

(b) Here $y = 5\cos 2x \cos \frac{1}{2}x$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= -10\sin 2x \cos \tfrac{1}{2}x - \tfrac{5}{2}\cos 2x \sin \tfrac{1}{2}x.$$

Let $u = 5\cos 2x$
and $v = \cos \frac{1}{2}x$.
Then $u' = -10\sin 2x$
and $v' = -\frac{1}{2}\sin \frac{1}{2}x$.

Using the Quotient Rule with Trigonometric Functions: A function like $y = \dfrac{\sin x}{x}$ is the quotient of the two functions $u = \sin x$ and $v = x$. Thus it can be differentiated by using the quotient rule.

WORKED EXERCISE:

Use the quotient rule to differentiate:

(a) $y = \dfrac{\sin x}{x}$ (b) $y = \dfrac{\cos 2x}{\cos 5x}$

SOLUTION:

(a) Here $y = \dfrac{\sin x}{x}$.

Applying the quotient rule,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$
$$= \frac{x\cos x - \sin x}{x^2}.$$

Let $u = \sin x$
and $v = x$.
Then $\frac{du}{dx} = \cos x$
and $\frac{dv}{dx} = 1$.

(b) Here $y = \dfrac{\cos 2x}{\cos 5x}$.

Applying the quotient rule,

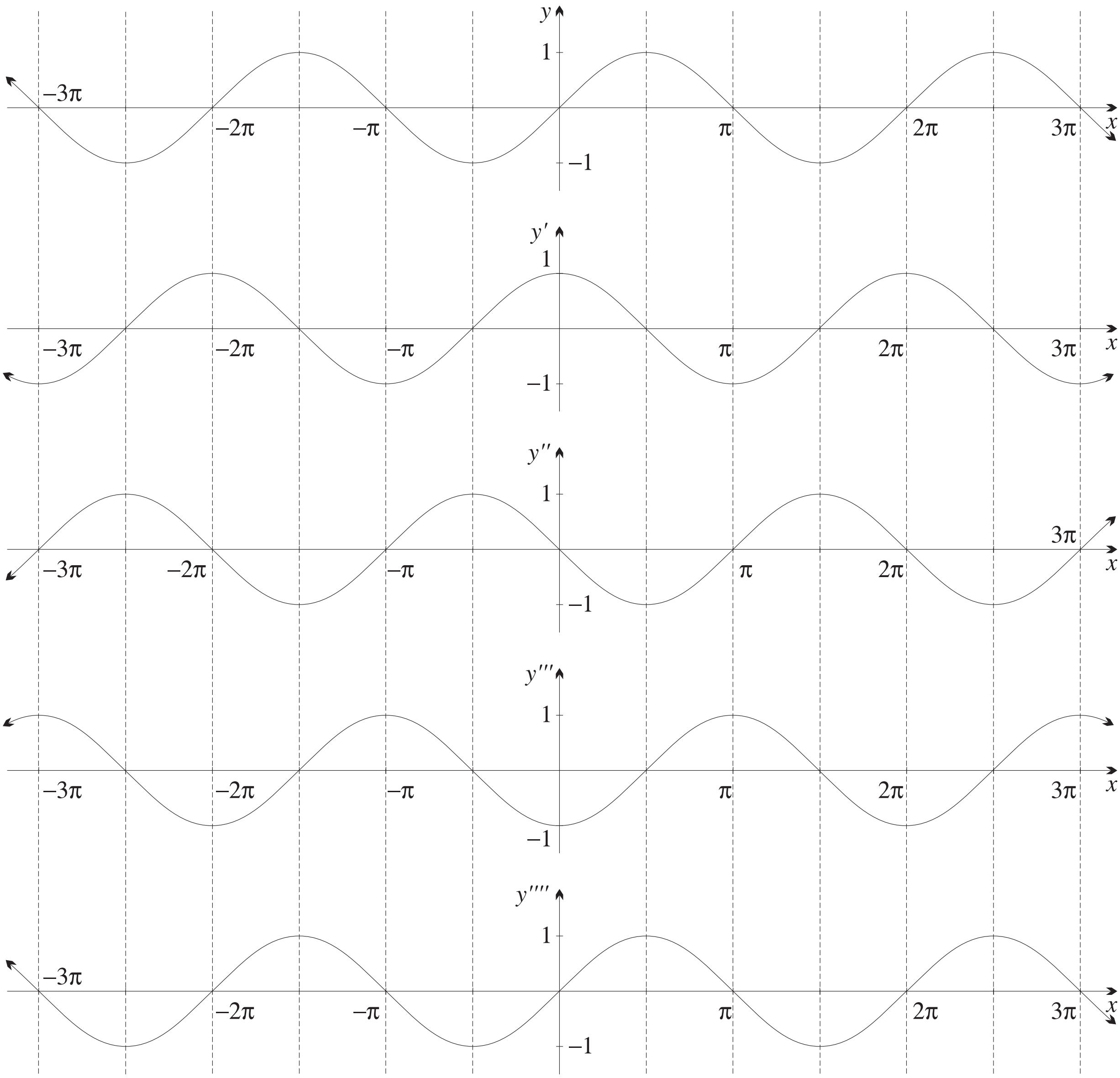
$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{-2\sin 2x \cos 5x + 5\cos 2x \sin 5x}{\cos^2 5x}.$$

Let $u = \cos 2x$
and $v = \cos 5x$.
Then $u' = -2\sin 2x$
and $v' = -5\sin 5x$.

Successive Differentiation of Sine and Cosine: Differentiating $y = \sin x$ repeatedly,

$$\frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x.$$

Thus differentiation is an order 4 operation on the sine function, which means that when differentiation is applied four times, the original function returns. Sketched below are the graphs of $y = \sin x$ and its first four derivatives.



Each application of differentiation shifts the wave backwards a quarter-revolution, so four applications shift it backwards one revolution, where it coincides with itself again.

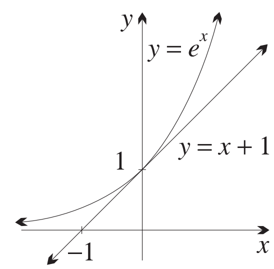
Notice that double differentiation exchanges $y = \sin x$ with its opposite function $y = -\sin x$, with each graph being the reflection of the other in the x-axis. It has a similar effect on the cosine function. Thus both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'' = -y$.

The properties of the exponential function $y = e^x$ are quite similar. The first derivative of $y = e^x$ is $y' = e^x$ and the second derivative of $y = e^{-x}$ is $y'' = e^{-x}$. This means there are now four functions whose fourth derivatives are equal to themselves:

$$y = \sin x, \qquad y = \cos x, \qquad y = e^x, \qquad y = e^{-x}.$$

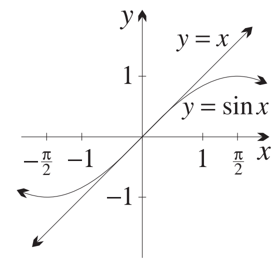
This is one clue amongst many others in the course that the trigonometric functions and the exponential functions are closely related.

Some Analogies between π and e: In the previous chapter, choosing the base of the exponential function to be the special number e meant that the derivative of $y = e^x$ was exactly $y' = e^x$.



In particular, the tangent to $y = e^x$ at the y-intercept has gradient exactly 1.

The choice of radian measure, based on the special number π, was motivated in exactly the same way. As has just been explained, the derivative of $y = \sin x$ using radian measure is exactly $y' = \cos x$.



In particular, the tangent to $y = \sin x$ at the origin has gradient exactly 1.

Both numbers $\pi = 3{\cdot}141\,592\ldots$ and $e = 2{\cdot}718\,281\ldots$ are irrational. The number π is associated with the area of a circle and e is associated with areas under the rectangular hyperbola. These things are further hints of connections between trigonometric and exponential functions.

Exercise 4E

1. Use the standard forms to differentiate with respect to x:

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$ (d) $y = 2\sin x$ (e) $y = \sin 2x$ (f) $y = 3\cos x$ (g) $y = \cos 3x$

(h) $y = \tan 4x$ (i) $y = 4\tan x$ (j) $y = 2\sin 3x$ (k) $y = 2\tan 2x$ (l) $y = 4\cos 2x$ (m) $y = -\sin 2x$ (n) $y = -\cos 2x$

(o) $y = -\tan 2x$ (p) $y = \tan \frac{1}{2}x$ (q) $y = \cos \frac{1}{2}x$ (r) $y = \sin \frac{x}{2}$ (s) $y = 5\tan \frac{1}{5}x$ (t) $y = 6\cos \frac{x}{3}$ (u) $y = 12\sin \frac{x}{4}$

2. Find the first, second, third and fourth derivatives of:

(a) $y = \sin 2x$ (b) $y = \cos 10x$ (c) $y = \sin \frac{1}{2}x$ (d) $y = \cos \frac{1}{3}x$

3. If $f(x) = \cos 2x$, find $f'(x)$ and then find:

(a) $f'(0)$ (b) $f'(\frac{\pi}{12})$ (c) $f'(\frac{\pi}{6})$ (d) $f'(\frac{\pi}{4})$

4. If $f(x) = \sin(\frac{1}{4}x + \frac{\pi}{2})$, find $f'(x)$ and then find:

(a) $f'(0)$ (b) $f'(2\pi)$ (c) $f'(-\pi)$ (d) $f'(\pi)$

5. Find $\dfrac{dy}{dx}$ using the product rule in each case:

(a) $y = x\sin x$ (b) $y = 2x\tan 2x$ (c) $y = x^2\cos 2x$ (d) $y = x^3\sin 3x$

6. Find $\dfrac{dy}{dx}$ using the quotient rule in each case:

(a) $y = \dfrac{\sin x}{x}$ (b) $y = \dfrac{\cos x}{x}$ (c) $y = \dfrac{x^2}{\cos x}$ (d) $y = \dfrac{x}{1 + \sin x}$

7. Find $\dfrac{dy}{dx}$ using the chain rule in each case. [HINT: Remember that $\cos^2 x$ means $(\cos x)^2$.]

(a) $y = \sin(x^2)$

(b) $y = \sin(1 - x^2)$

(c) $y = \cos(x^3 + 1)$

(d) $y = \sin \dfrac{1}{x}$

(e) $y = \cos^2 x$

(f) $y = \sin^3 x$

(g) $y = \tan^2 x$

(h) $y = \tan \sqrt{x}$

DEVELOPMENT

8. Differentiate with respect to x:

(a) $\sin 2\pi x$

(b) $\tan \frac{\pi}{2} x$

(c) $3 \sin x + \cos 5x$

(d) $4 \sin \pi x + 3 \cos \pi x$

(e) $\sin(2x - 1)$

(f) $\tan(1 + 3x)$

(g) $2\cos(1 - x)$

(h) $\cos(5x + 4)$

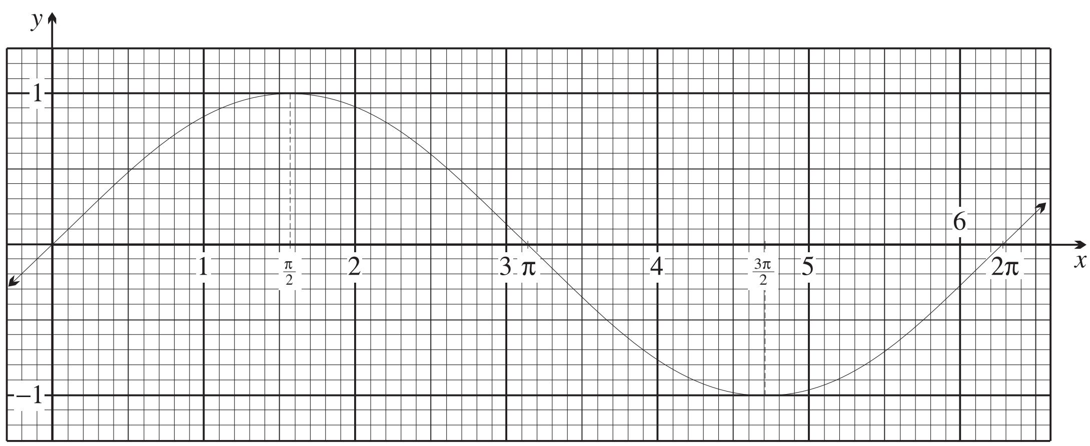
(i) $7\sin(2 - 3x)$

(j) $10\tan(10 - x)$

(k) $6 \sin\left(\frac{x+1}{2}\right)$

(l) $15 \cos\left(\frac{2x+1}{5}\right)$

9.



(a) Photocopy the sketch above of $f(x) = \sin x$. Carefully draw tangents at the points where $x = 0$, $0{\cdot}5$, 1, $1{\cdot}5$, $\ldots$, 3, and also at $x = \frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π.

(b) Measure the gradient of each tangent correct to two decimal places, and copy and complete the following table.

x	0	0·5	1	1·5	$\frac{\pi}{2}$	2	2·5	3	π	3·5	4	4·5	$\frac{3\pi}{2}$	5	5·5	6	2π
$f'(x)$																	

(c) Use these values to plot the graph of $y = f'(x)$.

(d) What is the equation of this graph?

10. [Technology] Most graphing programs can graph the derivative of a function. Start with $y = \sin x$, as in the previous question, then graph y', y'', y''' and y'''', and compare your results with the graphs printed in the theory introducing this exercise.

11. Differentiate these functions using the chain rule:

(a) $f(x) = e^{\tan x}$

(b) $f(x) = e^{\sin 2x}$

(c) $f(x) = \sin(e^{2x})$

(d) $f(x) = \log_e(\cos x)$

(e) $f(x) = \log_e(\sin x)$

(f) $f(x) = \log_e(\cos 4x)$

12. Differentiate these functions:

(a) $y = \sin x \cos x$

(b) $y = \sin^2 7x$

(c) $y = \cos^5 3x$

(d) $y = (1 - \cos 3x)^3$

(e) $y = \sin 2x \sin 4x$

(f) $y = \tan^3(5x - 4)$

13. Find $f'(x)$, given that:

(a) $f(x) = \dfrac{1}{1+\sin x}$

(b) $f(x) = \dfrac{\sin x}{1+\cos x}$

(c) $f(x) = \dfrac{1-\sin x}{\cos x}$

(d) $f(x) = \dfrac{\cos x}{\cos x+\sin x}$

14. (a) Sketch $y = \cos x$, for $-3\pi \le x \le 3\pi$.

(b) Find y', y'', y''' and y'''', and sketch them underneath the first graph.

(c) What geometrical interpretations can be given of the facts that:

(i) $y'' = -y$?

(ii) $y'''' = y$?

15. [Technology] The previous question is well suited to a graphing program, and the results should be compared with those of successive differentiation of $\sin x$.

16. (a) If $y = e^x \sin x$, find y' and y'', and show that $y'' - 2y' + 2y = 0$.

(b) If $y = e^{-x} \cos x$, find y' and y'', and show that $y'' + 2y' + 2y = 0$.

17. Consider the function $y = \frac{1}{3}\tan^3 x - \tan x + x$.

(a) Show that $\dfrac{dy}{dx} = \tan^2 x \sec^2 x - \sec^2 x + 1$.

(b) Hence use the identity $\sec^2 x = 1 + \tan^2 x$ to show that $\dfrac{dy}{dx} = \tan^4 x$.

CHALLENGE

18. (a) Copy and complete: $\log_b\left(\frac{P}{Q}\right) = \ldots$.

(b) If $f(x) = \log_e\left(\dfrac{1+\sin x}{\cos x}\right)$, show that $f'(x) = \sec x$.

19. (a) By writing $\sec x$ as $(\cos x)^{-1}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(b) Similarly, show that $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

(c) Similarly, show that $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

20. (a) If $y = \ln(\tan 2x)$, show that $\dfrac{dy}{dx} = 2\sec 2x \operatorname{cosec} 2x$.

(b) Show that $\dfrac{d}{dx}\left(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right) = \sin^4 x \cos^3 x$.

4 F Applications of Differentiation

The differentiation of the trigonometric functions can be applied in the usual way to the analysis of a number of functions that are very significant in the practical application of calculus.

Tangents and Normals: As always, the derivative is used to find the gradients of the relevant tangents, then point–gradient form is used to find their equations.

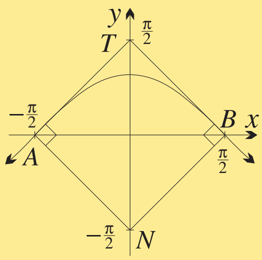
WORKED EXERCISE:
Find the equation of the tangent to $y = 2\sin x$ at the point P where $x = \frac{\pi}{6}$.

SOLUTION:
When $x = \frac{\pi}{6}$, $\quad y = 2\sin\frac{\pi}{6}$
$= 1 \qquad$ (since $\sin\frac{\pi}{6} = \frac{1}{2}$),
so the point P has coordinates $(\frac{\pi}{6}, 1)$.
Differentiating, $\dfrac{dy}{dx} = 2\cos x$.
When $x = \frac{\pi}{6}$, $\quad \dfrac{dy}{dx} = 2\cos\frac{\pi}{6}$
$= \sqrt{3} \qquad$ (since $\cos\frac{\pi}{6} = \frac{1}{2}\sqrt{3}$).
so the tangent at $P(\frac{\pi}{6}, 1)$ has gradient $\sqrt{3}$.
Hence its equation is $y - y_1 = m(x - x_1) \qquad$ (point–gradient form)

$$y - 1 = \sqrt{3}(x - \tfrac{\pi}{6})$$
$$y = x\sqrt{3} + 1 - \tfrac{\pi}{6}\sqrt{3}.$$

WORKED EXERCISE:
(a) Find the equations of the tangents and normals to the curve $y = \cos x$ at $A(-\frac{\pi}{2}, 0)$ and $B(\frac{\pi}{2}, 0)$.
(b) Show that these four lines form a square and find the other two vertices.

SOLUTION:
(a) The function is $\quad y = \cos x$,
and the derivative is $\quad y' = -\sin x$.
Hence gradient of tangent at $A(-\frac{\pi}{2}, 0) = -\sin(-\frac{\pi}{2})$
$= 1$,
and gradient of normal at $A(-\frac{\pi}{2}, 0) = -1$.
Similarly, gradient of tangent at $B(\frac{\pi}{2}, 0) = -\sin\frac{\pi}{2}$
$= -1$,
and gradient of normal at $B(\frac{\pi}{2}, 0) = 1$.
Hence the tangent at A is $\quad y - 0 = 1 \times (x + \frac{\pi}{2})$
$y = x + \frac{\pi}{2}$,
and the normal at A is $\quad y - 0 = -1 \times (x + \frac{\pi}{2})$
$y = -x - \frac{\pi}{2}$.
Similarly, the tangent at B is $\quad y - 0 = -1 \times (x - \frac{\pi}{2})$
$y = -x + \frac{\pi}{2}$,
and the normal at B is $\quad y - 0 = 1 \times (x - \frac{\pi}{2})$
$y = x - \frac{\pi}{2}$.



(b) Hence the two tangents meet on the y-axis at $T(0, \frac{\pi}{2})$,
and the two normals meet on the y-axis at $N(0, -\frac{\pi}{2})$.
Since adjacent sides are perpendicular, $ANBT$ is a rectangle,
and since the diagonals are perpendicular, it is also a rhombus,
so the quadrilateral $ANBT$ is a square.

WORKED EXERCISE:

(a) Find the equation of the tangent to $y = \tan 2x$ at the point on the curve where $x = \frac{\pi}{8}$.

(b) Find the x-intercept and y-intercept of this tangent.

(c) Sketch the situation.

(d) Find the area of the triangle formed by this tangent and the coordinate axes.

SOLUTION:

(a) The function is $y = \tan 2x$,

and differentiating, $y' = 2\sec^2 2x$.

When $x = \frac{\pi}{8}$, $y = \tan\frac{\pi}{4}$

$= 1$

and $y' = 2\sec^2\frac{\pi}{4}$

$= 2 \times \left(\sqrt{2}\right)^2$

$= 4,$

so the tangent is $y - 1 = 4(x - \frac{\pi}{8})$.

$y = 4x - \frac{\pi}{2} + 1$

(b) When $x = 0$, $y = 1 - \frac{\pi}{2}$

$= \dfrac{2-\pi}{2},$

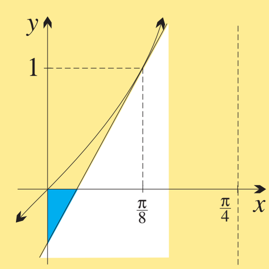
and when $y = 0$, $0 = 4x - \frac{\pi}{2} + 1$

$4x = \frac{\pi}{2} - 1$

$4x = \dfrac{\pi - 2}{2}$

$\boxed{\div 4}$ $x = \dfrac{\pi - 2}{8}.$

(c) The sketch is drawn opposite.



(d) Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times \dfrac{\pi - 2}{2} \times \dfrac{\pi - 2}{8}$

$= \dfrac{(\pi - 2)^2}{32}$ square units.

Curve Sketching: Curve-sketching problems involving trigonometric functions can be long, with difficult details. Nevertheless, the usual steps of the 'curve-sketching menu' still apply and the working of each step is done exactly the same as usual.

Sketching these curves using either a computer package or a graphics calculator would greatly aid understanding of the relationships between the equations of the curves and their graphs.

NOTE: With trigonometric functions, it is often easier to determine the nature of stationary points from an examination of the second derivative than from a table of values of the first derivative.

WORKED EXERCISE:

Consider the curve $y = \sin x + \cos x$ in the interval $0 \le x \le 2\pi$.

(a) Find the values of the function at the endpoints of the domain.

(b) Find the x-intercepts of the graph.

(c) Find any stationary points and determine their nature.

(d) Find any points of inflexion and sketch the curve.

SOLUTION:

(a) When $x = 0$, $y = \sin 0 + \cos 0 = 1$,
and when $x = 2\pi$, $y = \sin 2\pi + \cos 2\pi = 1$.

(b) To find the x-intercepts, put $y = 0$.

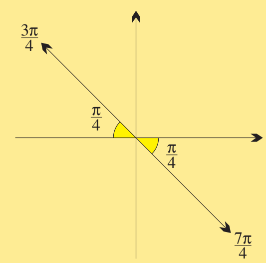
Then $\sin x + \cos x = 0$

$$\sin x = -\cos x$$

$$\tan x = -1 \quad \text{(dividing through by } \cos x\text{)}.$$

Hence x is in quadrant 2 or 4, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.



(c) Differentiating, $\quad y' = \cos x - \sin x$,

so y' has zeroes when $\sin x = \cos x$,

that is, $\quad \tan x = 1 \quad$ (dividing through by $\cos x$).

Hence x is in quadrant 1 or 3, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

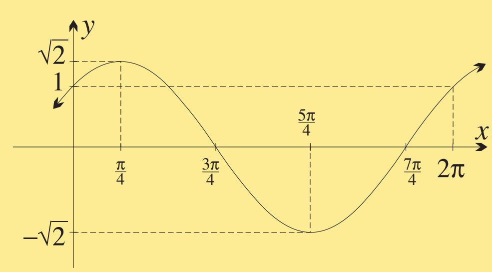
When $x = \frac{\pi}{4}$, $\quad y = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$

$$= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= \sqrt{2},$$

and when $x = \frac{5\pi}{4}$, $\quad y = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}$

$$= -\sqrt{2}.$$



Differentiating again, $\quad y'' = -\sin x - \cos x$,

so when $x = \frac{\pi}{4}$, $\quad y'' = -\sqrt{2}$,

and when $x = \frac{5\pi}{4}$, $\quad y'' = \sqrt{2}$.

Hence $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is a maximum turning point,

and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a minimum turning point.

(d) The second derivative y'' has zeroes when $-\sin x - \cos x = 0$,
that is, at the zeroes of y, which are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

x	0	$\frac{3\pi}{4}$	π	$\frac{7\pi}{4}$	2π
y''	-1	0	1	0	-1
	$\frown$	$\cdot$	$\smile$	$\cdot$	$\frown$

Hence the x-intercepts $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$ are also inflexions.

NOTE: Notice that the final graph is simply a wave with the same period 2π as $\sin x$ and $\cos x$, but with amplitude $\sqrt{2}$. It is actually $y = \sqrt{2}\cos x$ shifted right by $\frac{\pi}{4}$. Any function of the form $y = a\sin x + b\cos x$ has a similar graph.

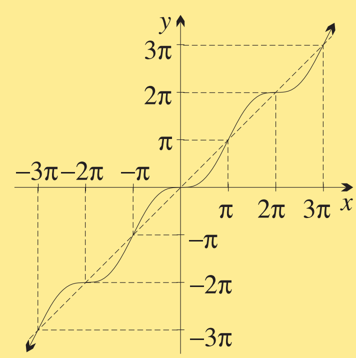
WORKED EXERCISE: [A harder example]
Sketch the graph of $f(x) = x - \sin x$ after carrying out the following steps:
(a) Write down the domain.
(b) Test whether the function is even or odd or neither.
(c) Find any zeroes of the function and examine its sign.
(d) Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$.
(e) Find any stationary points and examine their nature.
(f) Find any points of inflexion.

NOTE: This function is essentially the function describing the area of a segment, if the radius in the formula $A = \frac{1}{2}r^2(x - \sin x)$ is held constant while the angle x at the centre varies.

SOLUTION:
(a) The domain of $f(x) = x - \sin x$ is the set of all real numbers.

(b) $f(x)$ is odd, since both $\sin x$ and x are odd.

(c) The function is zero at $x = 0$ and nowhere else,
since $\sin x < x$, for $x > 0$,
and $\sin x > x$, for $x < 0$.



(d) The value of $\sin x$ always remains between -1 and 1,
so for $f(x) = x - \sin x$, $f(x) \to \infty$ as $x \to \infty$,
and $f(x) \to -\infty$ as $x \to -\infty$.

(e) Differentiating, $f'(x) = 1 - \cos x$,
so $f'(x)$ has zeroes whenever $\cos x = 1$,
that is, for $x = \ldots, -2\pi, 0, 2\pi, 4\pi, \ldots$.
But $f'(x) = 1 - \cos x$ is never negative, since $\cos x$ is never greater than 1,
thus the curve $f(x)$ is always increasing except at its stationary points.
Hence each stationary point is a stationary inflexion,
and these points are $\ldots, (-2\pi, -2\pi), (0, 0), (2\pi, 2\pi), (4\pi, 4\pi), \ldots$.

(f) Differentiating again, $f''(x) = \sin x$,
which is zero for $x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$.
We know that $\sin x$ changes sign around each of these points,
so $\ldots, (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi), \ldots$ are also inflexions.
Since $f'(\pi) = 1 - (-1) = 2$, the gradient at these other inflexions is 2.

Exercise 4F

TECHNOLOGY: The large number of sketches in this exercise should allow many of the graphs to be drawn first on a computer. Such sketching should be followed by an algebraic explanation of the features.

Many graphing packages allow tangents and normals to be drawn at specific points so that diagrams can be drawn of the earlier questions in the exercise.

1. Find the gradient of the tangent to each of the following curves at the point indicated:

(a) $y = \sin x$ at $x = 0$
(b) $y = \cos x$ at $x = \frac{\pi}{2}$
(c) $y = \sin x$ at $x = \frac{\pi}{3}$
(d) $y = \cos x$ at $x = \frac{\pi}{6}$
(e) $y = \sin x$ at $x = \frac{\pi}{4}$
(f) $y = \tan x$ at $x = 0$
(g) $y = \tan x$ at $x = \frac{\pi}{4}$
(h) $y = \cos 2x$ at $x = \frac{\pi}{4}$
(i) $y = -\cos \frac{1}{2}x$ at $x = \frac{2\pi}{3}$
(j) $y = \sin \frac{x}{2}$ at $x = \frac{2\pi}{3}$
(k) $y = \tan 2x$ at $x = \frac{\pi}{6}$
(l) $y = \sin 2x$ at $x = \frac{\pi}{12}$

2. (a) Show that the line $y = x$ is the tangent to the curve $y = \sin x$ at $(0, 0)$.
(b) Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.
(c) Show that the line $y = \frac{\pi}{2} - x$ is the tangent to the curve $y = \cos x$ at $(\frac{\pi}{2}, 0)$.

3. Find the equation of the tangent at the given point on each of the following curves:

(a) $y = \sin x$ at $(\pi, 0)$
(b) $y = \tan x$ at $(\frac{\pi}{4}, 1)$
(c) $y = \cos x$ at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$
(d) $y = \cos 2x$ at $(\frac{\pi}{4}, 0)$
(e) $y = \sin 2x$ at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$
(f) $y = x \sin x$ at $(\pi, 0)$

4. Find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on each of the following curves where the gradient of the tangent is zero.

(a) $y = 2\sin x$
(b) $y = 2\sin x - x$
(c) $y = 2\cos x + x$
(d) $y = 2\sin x + \sqrt{3}\,x$

DEVELOPMENT

5. (a) Find the equations of the tangent and normal to the curve $y = 2\sin x - \cos 2x$ at $(\frac{\pi}{6}, \frac{1}{2})$.
(b) Show that the tangent meets the x-axis at $\left(\frac{\pi}{6} - \frac{1}{12}\sqrt{3}\,, 0\right)$.
(c) Show that the normal meets the x-axis at $\left(\frac{\pi}{6} + \sqrt{3}\,, 0\right)$.

6. (a) Show that $y = \sin^2 x$ has derivative $y' = 2\sin x \cos x$.
(b) Find the gradients of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.
(c) Find the equations of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.
(d) If the tangent meets the x-axis at P and the normal meets the y-axis at Q, find the area of $\triangle OPQ$, where O is the origin.

7. (a) Use the chain rule to show that $y = e^{\sin x}$ has derivative $y' = \cos x\, e^{\sin x}$.
(b) Hence find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on the curve $y = e^{\sin x}$ where the tangent is horizontal.

8. (a) Show that $y = e^{\cos x}$ has derivative $y' = -\sin x\, e^{\cos x}$.
(b) Hence find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on the curve $y = e^{\cos x}$ where the tangent is horizontal.

9. (a) Find the first and second derivatives of $y = \cos x + \sqrt{3}\sin x$.
(b) Find the stationary points in the domain $0 \le x \le 2\pi$, and use the second derivative to determine their nature.
(c) Find the points of inflexion.
(d) Hence sketch the curve, for $0 \le x \le 2\pi$.

10. (a) Repeat the previous question for $y = \cos x - \sin x$.
(b) Verify your results by sketching $y = \cos x$ and $y = -\sin x$ on the same diagram, and then sketching $y = \cos x - \sin x$ by addition of heights.

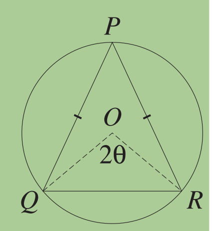
11. (a) Find the derivative of $y = x + \sin x$, and show that $y'' = -\sin x$.
(b) Find the stationary points in the domain $-2\pi < x < 2\pi$, and determine their nature.
(c) Find the points of inflexion.
(d) Hence sketch the curve, for $-2\pi \le x \le 2\pi$.

12. Repeat the steps of the previous question for $y = x - \cos x$.

CHALLENGE

13. Find any stationary points and inflexions of the curve $y = 2\sin x + x$ in the interval $0 \le x \le 2\pi$, then sketch the curve.

14. An isosceles triangle PQR is inscribed in a circle with centre O of radius 1 unit, as shown in the diagram to the right. Let $\angle QOR = 2\theta$, where θ is acute.



(a) Join PO and extend it to meet QR at M. Then prove that $QM = \sin\theta$ and $OM = \cos\theta$.
(b) Show that the area A of $\triangle PQR$ is $A = \sin\theta(\cos\theta + 1)$.
(c) Hence show that, as θ varies, $\triangle PQR$ has its maximum possible area when it is equilateral.

15. (a) Show that $\dfrac{d}{d\theta}\left(\dfrac{2-\sin\theta}{\cos\theta}\right) = \dfrac{2\sin\theta - 1}{\cos^2\theta}$.
(b) Hence find the maximum and minimum values of the expression $\dfrac{2-\sin\theta}{\cos\theta}$ in the interval $0 \le \theta \le \frac{\pi}{4}$, and state the values of θ for which they occur.

4 G Integration of the Trigonometric Functions

As always, the standard forms for differentiation can be reversed to give standard forms for integration.

The Standard Forms for Integrating the Trigonometric Functions: When the standard forms for differentiating $\sin x$, $\cos x$ and $\tan x$ are reversed, they give three new standard integrals. To avoid complications, the constants of integration have been ignored until the results are summarised in Box 19 on the next page.

First, $\dfrac{d}{dx}\sin x = \cos x$,

and reversing this, $\displaystyle\int \cos x\,dx = \sin x$.

Secondly, $\dfrac{d}{dx}\cos x = -\sin x$,

and reversing this, $\displaystyle\int -\sin x\,dx = \cos x$

$\boxed{\times(-1)}$ $\displaystyle\int \sin x\,dx = -\cos x$.

Thirdly, $\dfrac{d}{dx}\tan x = \sec^2 x$,

and reversing this, $\displaystyle\int \sec^2 x\,dx = \tan x$.

This gives three new standard integrals. These three standard forms should be carefully memorised — pay attention to the signs in the first two standard forms.

STANDARD TRIGONOMETRIC INTEGRALS:

19

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

WORKED EXERCISE:

The curve $y = \sin x$ is sketched below. Show that the first arch of the curve, as shaded in the diagram, has area 2 square units.

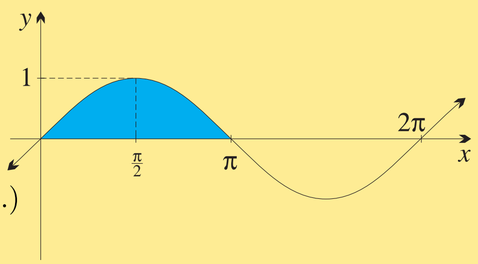
SOLUTION:

Since the region is entirely above the x-axis,

$$\begin{aligned} \text{area} &= \int_0^\pi \sin x\,dx \\ &= \Big[-\cos x\Big]_0^\pi \\ &= -\cos\pi + \cos 0 \\ &= -(-1) + 1 \end{aligned}$$

(The graph of $y = \cos x$ shows that $\cos\pi = -1$.)

$$= 2 \text{ square units.}$$



NOTE: The fact that this area is such a simple number is another confirmation that radians are the correct angle units to use for the calculus of the trigonometric functions. Comparably simple results were obtained earlier when e was used as the base for logarithms and powers. For example,

$$\int_0^1 e^x\,dx = e - 1 \qquad \text{and} \qquad \int_1^e \log_e x\,dx = 1.$$

WORKED EXERCISE:

Evaluate the following definite integrals:

(a) $\displaystyle\int_0^\pi \cos x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$

SOLUTION:

(a)
$$\begin{aligned} \int_0^\pi \cos x\,dx &= \Big[\sin x\Big]_0^\pi \\ &= \sin\pi - \sin 0 \\ &= 0 \end{aligned}$$
(The graph shows that $\sin\pi = 0$ and $\sin 0 = 0$.)

(b)
$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sec^2 x\,dx &= \Big[\tan x\Big]_0^{\frac{\pi}{3}} \\ &= \tan\tfrac{\pi}{3} - \tan 0 \\ &= \sqrt{3} \end{aligned}$$
(Here $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan 0 = 0$.)

Replacing x by $ax + b$: Reversing the standard forms for derivatives in Section 4E gives a further set of standard forms. Again, the constants of integration have been ignored until the boxed statement of the standard forms.

First, $$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b),$$

so $$\int a\cos(ax+b)\,dx = \sin(ax+b)$$

and dividing by a, $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b).$$

Secondly, $$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b),$$

so $$\int -a\sin(ax+b)\,dx = \cos(ax+b)$$

and dividing by $-a$, $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b).$$

Thirdly, $$\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b),$$

so $$\int a\sec^2(ax+b)\,dx = \tan(ax+b).$$

and dividing by a, $$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b).$$

The result is extended forms of the three standard integrals. These extended standard forms should also be carefully memorised.

20 **STANDARD INTEGRALS FOR FUNCTIONS OF $ax + b$:**

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

WORKED EXERCISE:

Evaluate the following definite integrals:

(a) $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x\,dx$ (b) $\displaystyle\int_\pi^{2\pi} \sin \tfrac{1}{4}x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx$

SOLUTION:

(a)
$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos 3x\,dx &= \tfrac{1}{3}\Big[\sin 3x\Big]_0^{\frac{\pi}{6}}\\ &= \tfrac{1}{3}(\sin\tfrac{\pi}{2} - 3\sin 0)\\ &= \tfrac{1}{3} && \text{(The graph shows that } \sin\tfrac{\pi}{2} = 1 \text{ and } \sin 0 = 0.)\end{aligned}$$

(b)
$$\begin{aligned}\int_\pi^{2\pi} \sin\tfrac{1}{4}x\,dx &= -4\Big[\cos\tfrac{1}{4}x\Big]_\pi^{2\pi} && \text{(The reciprocal of } \tfrac{1}{4} \text{ is 4.)}\\ &= -4\cos\tfrac{\pi}{2} + 4\cos\tfrac{\pi}{4}\\ &= 0 + 4\times\frac{\sqrt{2}}{2} && \text{(Look at the graph to see that } \cos\tfrac{\pi}{2} = 0.)\\ &= 2\sqrt{2}\end{aligned}$$

(c) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx = \tfrac{1}{2}\Big[\tan(2x+\pi)\Big]_0^{\frac{\pi}{8}}$

$= \tfrac{1}{2}(\tan\tfrac{5\pi}{4} - \tan\pi)$

$= \tfrac{1}{2}(1-0)$

(The angle $\frac{5\pi}{4}$ is in the third quadrant and has related angle $\frac{\pi}{4}$.)

$= \tfrac{1}{2}$

The Primitives of $\tan x$ and $\cot x$: The primitives of $\tan x$ and $\cot x$ can be found by using the ratio formulae $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$ and then applying the standard form from the previous chapter,

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + C.$$

WORKED EXERCISE:

Find primitives of the functions:

(a) $\cot x$ (b) $\tan x$

SOLUTION:

(a) $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$= \log_e(\sin x) + C$, since $\cos x = \dfrac{d}{dx}\sin x$.

(b) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$\displaystyle = -\int \frac{-\sin x}{\cos x}\,dx$

$= -\log_e(\cos x) + C$, since $-\sin x = \dfrac{d}{dx}\cos x$.

Finding a Function whose Derivative is Known: If the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be found.

WORKED EXERCISE:

The derivative of a certain function is $y' = \cos x$ and the graph of the function has y-intercept $(0,3)$. Find the original function $f(x)$ and then find $f(\frac{\pi}{2})$.

SOLUTION:

Here $y' = \cos x$,

and taking the primitive, $y = \sin x + C$, for some constant C.

When $x=0$, $y=3$, so substituting $x=0$,

$3 = \sin 0 + C$

$C = 3$.

Hence $y = \sin x + 3$.

When $x = \frac{\pi}{2}$, $y = \sin\frac{\pi}{2} + 3$

$= 4$, since $\sin\frac{\pi}{2} = 1$.

WORKED EXERCISE:

Given that $f'(x) = \sin 2x$ and $f(\pi) = 1$:

(a) find the function $f(x)$,

(b) find $f(\frac{\pi}{4})$.

SOLUTION:

(a) Here $\quad f'(x) = \sin 2x,$

and taking the primitive, $\quad f(x) = -\frac{1}{2}\cos 2x + C,\;$ for some constant C.

It is known that $f(\pi) = 1$, so substituting $x = \pi$,

$$
\begin{aligned}
1 &= -\tfrac{1}{2}\cos 2\pi + C \\
1 &= -\tfrac{1}{2} \times 1 \;+\; C \\
C &= 1\tfrac{1}{2}.
\end{aligned}
$$

Hence $\quad f(x) = -\frac{1}{2}\cos 2x + 1\frac{1}{2}.$

(b) Substituting $x = \frac{\pi}{4}$,

$$
\begin{aligned}
f(\tfrac{\pi}{4}) &= -\tfrac{1}{2} \times \cos\tfrac{\pi}{2} + 1\tfrac{1}{2} \\
&= 1\tfrac{1}{2},\quad \text{since } \cos\tfrac{\pi}{2} = 0.
\end{aligned}
$$

Given a Chain-Rule Derivative, Find an Integral: As always, the results of a chain-rule differentiation can be reversed to give a primitive.

WORKED EXERCISE:

(a) Use the chain rule to differentiate $\cos^5 x$.

(b) Hence find $\displaystyle\int_0^{\pi} \sin x \cos^4 x \, dx$.

SOLUTION:

(a) Let $\quad y = \cos^5 x.$

By the chain rule,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= -5\sin x \cos^4 x.
\end{aligned}
$$

Let $\quad u = \cos x.$

Then $\quad y = u^5.$

Hence $\quad \dfrac{du}{dx} = -\sin x$

and $\quad \dfrac{dy}{du} = 5u^4.$

(b) From part (a), $\quad \dfrac{d}{dx}(\cos^5 x) = -5\sin x \cos^4 x.$

Reversing this, $\quad \displaystyle\int (-5\sin x \cos^4 x)\, dx = \cos^5 x.$

Dividing both sides by -5 gives

$$\int \sin x \cos^4 x \, dx = -\tfrac{1}{5}\cos^5 x.$$

Hence

$$
\begin{aligned}
\int_0^{\pi} \sin x \cos^4 x \, dx &= -\tfrac{1}{5}\Big[\cos^5 x\Big]_0^{\pi} \\
&= -\tfrac{1}{5}(-1 - 1) \\
&= \tfrac{2}{5}.
\end{aligned}
$$

Exercise 4G

1. Find the following indefinite integrals:

(a) $\int \sec^2 x\,dx$

(b) $\int \cos x\,dx$

(c) $\int \sin x\,dx$

(d) $\int -\sin x\,dx$

(e) $\int 2\cos x\,dx$

(f) $\int \cos 2x\,dx$

(g) $\int \frac{1}{2}\cos x\,dx$

(h) $\int \cos\frac{1}{2}x\,dx$

(i) $\int \sin 2x\,dx$

(j) $\int \sec^2 5x\,dx$

(k) $\int \cos 3x\,dx$

(l) $\int \sec^2\frac{1}{3}x\,dx$

(m) $\int \sin\frac{x}{2}\,dx$

(n) $\int -\cos\frac{1}{5}x\,dx$

(o) $\int -4\sin 2x\,dx$

(p) $\int \frac{1}{4}\sin\frac{1}{4}x\,dx$

(q) $\int 12\sec^2\frac{1}{3}x\,dx$

(r) $\int 2\cos\frac{x}{3}\,dx$

2. Find the value of:

(a) $\int_0^{\frac{\pi}{2}} \cos x\,dx$

(b) $\int_0^{\frac{\pi}{6}} \cos x\,dx$

(c) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$

(d) $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$

(e) $\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx$

(f) $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$

(g) $\int_0^{\frac{\pi}{2}} \sec^2(\frac{1}{2}x)\,dx$

(h) $\int_{\frac{\pi}{3}}^{\pi} \cos(\frac{1}{2}x)\,dx$

(i) $\int_0^{\pi} (2\sin x - \sin 2x)\,dx$

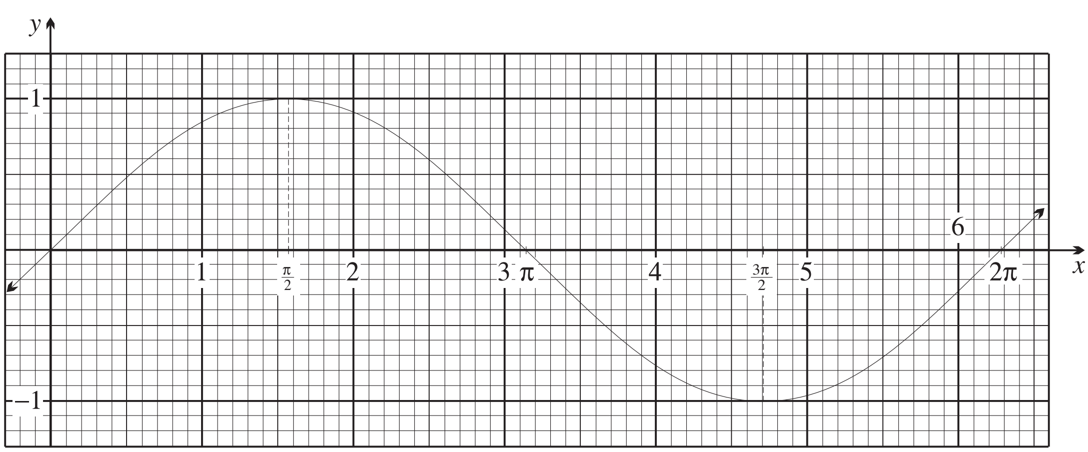
3. (a) The gradient function of a certain curve is given by $\dfrac{dy}{dx} = \sin x$. If the curve passes through the origin, find its equation.

(b) Another curve passing through the origin has gradient function $y' = \cos x - 2\sin 2x$. Find its equation.

(c) If $\dfrac{dy}{dx} = \sin x + \cos x$, and $y = -2$ when $x = \pi$, find y as a function of x.

DEVELOPMENT

4.



The graph of $y = \sin x$ is sketched above.

(a) The first worked exercise in the notes for this section proved that $\int_0^{\pi} \sin x\,dx = 2$. Count squares on the graph of $y = \sin x$ above to confirm this result.

(b) On the same graph of $y = \sin x$, count squares and use symmetry to find:

(i) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x\,dx$ (iv) $\displaystyle\int_0^{\frac{5\pi}{4}} \sin x\,dx$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$ (v) $\displaystyle\int_0^{\frac{3\pi}{2}} \sin x\,dx$

(iii) $\displaystyle\int_0^{\frac{3\pi}{4}} \sin x\,dx$ (vi) $\displaystyle\int_0^{\frac{7\pi}{4}} \sin x\,dx$

(c) Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$, and confirm the results of part (b).

5. [Technology] Programs that sketch the graph and then approximate definite integrals would help reinforce the previous very important investigation. The investigation could then be continued past $x = \pi$, after which the definite integral decreases again.

Similar investigation with the graphs of $\cos x$ and $\sec^2 x$ would also be helpful, comparing the results of computer integration with the exact results obtained by integration using the standard primitives.

6. Find the following indefinite integrals:

(a) $\displaystyle\int \cos(x+2)\,dx$ (d) $\displaystyle\int \sin(2x+1)\,dx$ (g) $\displaystyle\int \sec^2(4-x)\,dx$

(b) $\displaystyle\int \cos(2x+1)\,dx$ (e) $\displaystyle\int \cos(3x-2)\,dx$ (h) $\displaystyle\int \sec^2\left(\tfrac{1-x}{3}\right)\,dx$

(c) $\displaystyle\int \sin(x+2)\,dx$ (f) $\displaystyle\int \sin(7-5x)\,dx$ (i) $\displaystyle\int \sin\left(\tfrac{1-x}{3}\right)\,dx$

7. (a) Find $\displaystyle\int (6\cos 3x - 4\sin \tfrac{1}{2}x)\,dx$.

(b) Find $\displaystyle\int (8\sec^2 2x - 10\cos \tfrac{1}{4}x + 12\sin \tfrac{1}{3}x)\,dx$.

8. (a) If $f'(x) = \pi\cos \pi x$ and $f(0) = 0$, find $f(x)$ and $f(\tfrac{1}{3})$.

(b) If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(x)$ and $f(\tfrac{1}{6})$.

(c) If $f''(x) = 18\cos 3x$ and $f'(0) = f(\tfrac{\pi}{2}) = 1$, find $f(x)$.

9. Find the following indefinite integrals, where a, b, u and v are constants:

(a) $\displaystyle\int a\sin(ax+b)\,dx$ (c) $\displaystyle\int \frac{1}{u}\sec^2(v+ux)\,dx$

(b) $\displaystyle\int \pi^2\cos \pi x\,dx$ (d) $\displaystyle\int \frac{a}{\cos^2 ax}\,dx$

10. (a) Copy and complete $1 + \tan^2 x = \ldots$, and hence find $\displaystyle\int \tan^2 x\,dx$.

(b) Simplify $1 - \sin^2 x$, and hence find the value of $\displaystyle\int_0^{\frac{\pi}{3}} \frac{2}{1-\sin^2 x}\,dx$.

11. (a) Copy and complete $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ldots$.

(b) Hence show that $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx \doteqdot 0{\cdot}4$.

12. (a) Use the fact that $\tan x = \dfrac{\sin x}{\cos x}$ to show that $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\, dx = \tfrac{1}{2}\ln 2$.

(b) Use the fact that $\cot x = \dfrac{\cos x}{\sin x}$ to find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\, dx$.

13. (a) (i) Find $\dfrac{d}{dx}\big(\sin x^2\big)$.

(ii) Hence find $\displaystyle\int 2x\cos x^2\, dx$.

(b) (i) Find $\dfrac{d}{dx}\big(\cos x^3\big)$.

(ii) Hence find $\displaystyle\int x^2 \sin x^3\, dx$.

(c) (i) Find $\dfrac{d}{dx}\big(\tan\sqrt{x}\big)$.

(ii) Hence find $\displaystyle\int \frac{1}{\sqrt{x}}\sec^2\sqrt{x}\, dx$.

CHALLENGE

14. (a) Show that $\dfrac{d}{dx}(\sin x - x\cos x) = x\sin x$, and hence find $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\, dx$.

(b) Show that $\dfrac{d}{dx}(\tfrac{1}{3}\cos^3 x - \cos x) = \sin^3 x$, and hence find $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\, dx$.

15. (a) Find $\dfrac{d}{dx}(\sin^5 x)$, and hence find $\displaystyle\int \sin^4 x\cos x\, dx$.

(b) Find $\dfrac{d}{dx}(\tan^3 x)$, and hence find $\displaystyle\int \tan^2 x\sec^2 x\, dx$.

16. (a) Differentiate $e^{\sin x}$, and hence find the value of $\displaystyle\int_0^{\frac{\pi}{2}} \cos x e^{\sin x}\, dx$.

(b) Differentiate $e^{\tan x}$, and hence find the value of $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x}\, dx$.

17. Find $\dfrac{d}{dx}(\tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x)$, and hence find $\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\, dx$.

18. Here are two standard forms that are not in the 2 Unit course:

$$\int \cos^2 x\, dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C \qquad \text{and} \qquad \int \sec ax\, dx = \frac{1}{a}\log_e(\sec ax + \tan ax) + C.$$

Use these standard forms to find:

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{12}} \sec 3x\, dx$

4 H Applications of Integration

The trigonometric integrals can now be used to find areas and volumes in exactly the same way that has been done previously.

Finding Areas by Integration: As always, *a sketch is essential*, because areas below the x-axis are represented as a negative number by the definite integral.

It is best to evaluate the separate integrals first and then make a conclusion about areas.

WORKED EXERCISE:

(a) Sketch $y = \cos \frac{1}{2}x$ in the interval $0 \le x \le 4\pi$, marking both x-intercepts.

(b) Hence find the area between the curve and the x-axis, for $0 \le x \le 4\pi$.

SOLUTION:

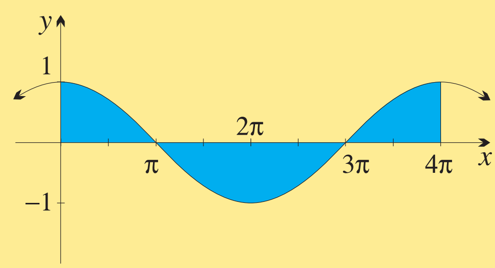
(a) The curve $y = \cos \frac{1}{2}x$ has amplitude 1,

and the period is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.

The two x-intercepts in the interval are $x = \pi$ and $x = 3\pi$.

(b) We must integrate separately over the three intervals

$0 \le x \le \pi$ and $\pi \le x \le 3\pi$ and $3\pi \le x \le 4\pi$.



First, $\displaystyle\int_0^{\pi} \cos \tfrac{1}{2}x \, dx = \Big[2 \sin \tfrac{1}{2}x\Big]_0^{\pi}$

$= 2\sin \frac{\pi}{2} - 2 \sin 0$

$= 2 - 0$

$= 2,$

which is positive, because the curve is above the x-axis for $0 \le x \le \pi$.

Secondly, $\displaystyle\int_{\pi}^{3\pi} \cos \tfrac{1}{2}x \, dx = \Big[2 \sin \tfrac{1}{2}x\Big]_{\pi}^{3\pi}$

$= 2\sin \frac{3\pi}{2} - 2 \sin \frac{\pi}{2}$

$= -2 - 2$

$= -4,$

which is negative, because the curve is below the x-axis for $\pi \le x \le 3\pi$.

Thirdly , $\displaystyle\int_{3\pi}^{4\pi} \cos \tfrac{1}{2}x \, dx = \Big[2 \sin \tfrac{1}{2}x\Big]_{3\pi}^{4\pi}$

$= 2\sin 2\pi - 2 \sin \frac{3\pi}{2}$

$= 0 - (-2)$

$= 2,$

which is positive, because the curve is above the x-axis for $3\pi \le x \le 4\pi$.

Hence total area $= 2 + 4 + 2$

$= 8$ square units.

Finding Areas between Curves: The following worked exercises use the principle that if $y = f(x)$ is above $y = g(x)$ throughout some interval $a \le x \le b$, then the area between the curves is given by the formula

$$\text{area between the curves} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

WORKED EXERCISE:

(a) Show that the curves $y = \sin x$ and $y = \sin 2x$ intersect when $x = \frac{\pi}{3}$.

(b) Sketch these curves in the interval $0 \le x \le \pi$.

(c) Find the area contained between the curves in the interval $0 \le x \le \frac{\pi}{3}$.

SOLUTION:

(a) The curves intersect at $x = \frac{\pi}{3}$ because $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{1}{2}\sqrt{3}$.

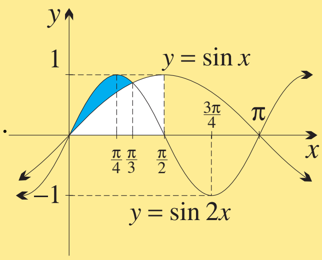
(b) The curves are sketched to the right below.

(c) In the interval $0 \le x \le \frac{\pi}{3}$, the curve $y = \sin 2x$ is always above $y = \sin x$,

$$\text{so area between} = \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx$$

$$= \Big[-\tfrac{1}{2}\cos 2x + \cos x\Big]_0^{\frac{\pi}{3}}$$

$$= (-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}) - (-\tfrac{1}{2}\cos 0 + \cos 0).$$



Since $\cos 0 = 1$ and $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{2\pi}{3} = -\frac{1}{2}$,

$$\text{area} = (\tfrac{1}{4} + \tfrac{1}{2}) - (-\tfrac{1}{2} + 1)$$

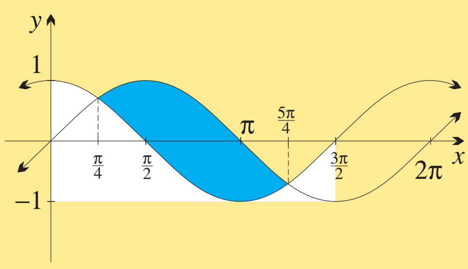
$$= \tfrac{1}{4} \text{ square units.}$$

WORKED EXERCISE:

(a) Show that in the interval $0 \le x \le 2\pi$, the curves $y = \sin x$ and $y = \cos x$ intersect when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$.

(b) Sketch the curves in this interval and find the area contained between them.

SOLUTION:



(a) Put $\sin x = \cos x$.

Then $\tan x = 1$

$$x = \tfrac{\pi}{4} \text{ or } \tfrac{5\pi}{4},$$

so the curves intersect at the points $\left(\frac{\pi}{4}, \frac{1}{2}\sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{2}\sqrt{2}\right)$.

(b)
$$\text{Area between} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\cos\tfrac{5\pi}{4} - \sin\tfrac{5\pi}{4} + \cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}$$

$$= -\Big(-\tfrac{1}{2}\sqrt{2}\Big) - \Big(-\tfrac{1}{2}\sqrt{2}\Big) + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= 2\sqrt{2} \text{ square units.}$$

Finding Volumes by Integration: As always, volumes of revolution about the x-axis can be found using the standard formula

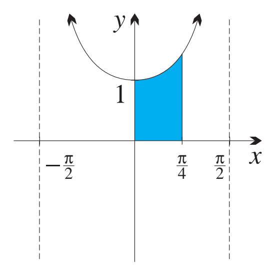
$$\text{volume} = \int_a^b \pi y^2 \, dx.$$

WORKED EXERCISE:
Find the volume of the solid generated when the shaded area under the curve $y = \sec x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis.

SOLUTION:

Here $y = \sec x$,
and $y^2 = \sec^2 x$.

$$\begin{aligned}\text{Hence volume} &= \int_0^{\frac{\pi}{4}} \pi y^2 \, dx \\ &= \int_0^{\frac{\pi}{4}} \pi \sec^2 x \, dx \\ &= \pi \Big[\tan x\Big]_0^{\frac{\pi}{4}} \\ &= \pi(\tan \tfrac{\pi}{4} - \tan 0) \\ &= \pi(1 - 0) \\ &= \pi \text{ cubic units.}\end{aligned}$$



WORKED EXERCISE: [A much harder example]

(a) Use the Pythagorean identity $1 + \tan^2 x = \sec^2 x$ to find $\displaystyle\int_0^{\frac{\pi}{4}} (1 + \tan^2 x) \, dx$.

(b) Differentiate $y = \log_e(\cos x)$ and hence find $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \, dx$.

(c) Sketch the curve $y = 1 + \tan x$, from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

(d) Using the integrals in parts (a) and (b), find the volume of the solid generated when the region under this curve from $x = 0$ to $x = \frac{\pi}{4}$ is rotated about the x-axis.

SOLUTION:

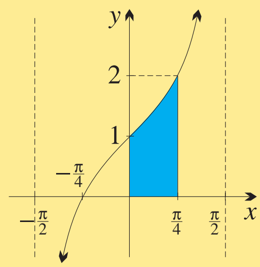
(a) $$\begin{aligned}\int_0^{\frac{\pi}{4}} (1 + \tan^2 x) \, dx &= \int_0^{\frac{\pi}{4}} \sec^2 x \, dx \\ &= \Big[\tan x\Big]_0^{\frac{\pi}{4}} \\ &= \tan \tfrac{\pi}{4} - \tan 0 \\ &= 1\end{aligned}$$

(b) By the chain rule, $$\begin{aligned}\frac{d}{dx} \log_e(\cos x) &= \frac{1}{\cos x} \times (-\sin x), \\ &= -\tan x.\end{aligned}$$

Reversing this, $\displaystyle\int \tan x \, dx = -\log_e(\cos x) + C$, for some constant C.

$$\text{Hence } \int_0^{\frac{\pi}{4}} \tan x\, dx = \Big[-\log_e(\cos x)\Big]_0^{\frac{\pi}{4}}$$
$$= -\log_e(\cos \tfrac{\pi}{4}) + \log_e(\cos 0)$$
$$= -\log_e \frac{1}{\sqrt{2}} + \log_e 1$$
$$= \log_e \sqrt{2} + 0$$
$$= \tfrac{1}{2}\log_e 2.$$

(c) The curve is sketched opposite.



(d) $$\text{Volume} = \int_0^{\frac{\pi}{4}} \pi y^2\, dx$$
$$= \int_0^{\frac{\pi}{4}} \pi(1+\tan x)^2\, dx$$
$$= \pi \int_0^{\frac{\pi}{4}} (1 + 2\tan x + \tan^2 x)\, dx$$
$$= \pi \int_0^{\frac{\pi}{4}} (1+\tan^2 x)\, dx \;+\; 2\pi \int_0^{\frac{\pi}{4}} \tan x\, dx.$$

The first integral was found in part (a),
and the second integral in part (b), so

$$\text{volume} = \pi \times 1 \;+\; 2\pi \times \tfrac{1}{2}\log_e 2$$
$$= \pi(1 + \log_e 2) \text{ cubic units.}$$

Exercise 4H

TECHNOLOGY: Some graphing programs can perform numerical integration on specified regions. Such programs would help to confirm the integrals in this exercise and to investigate quickly further integrals associated with these curves.

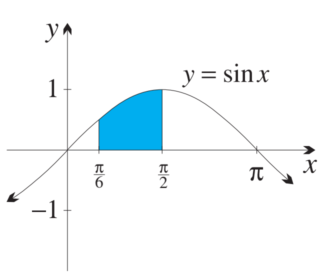
1. Find the exact area between the curve $y = \cos x$ and the x-axis:
(a) from $x = 0$ to $x = \frac{\pi}{2}$, (b) from $x = 0$ to $x = \frac{\pi}{6}$.

2. Find the exact area between the curve $y = \sec^2 x$ and the x-axis:
(a) from $x = 0$ to $x = \frac{\pi}{4}$, (b) from $x = 0$ to $x = \frac{\pi}{3}$.

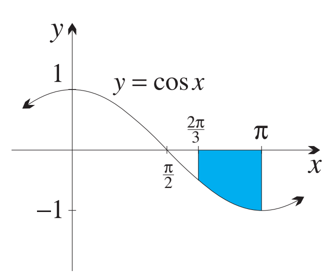
3. Find the exact area between the curve $y = \sin x$ and the x-axis:
(a) from $x = 0$ to $x = \frac{\pi}{4}$, (b) from $x = 0$ to $x = \frac{\pi}{6}$.

4. Calculate the area of the shaded region in each diagram below (and then observe that the two regions have equal area):

(a)



(b)



5. Find the area enclosed between each curve and the x-axis over the specified domain:

(a) $y = \sin x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$

(b) $y = \sin 2x$, from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$

(c) $y = \cos x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$

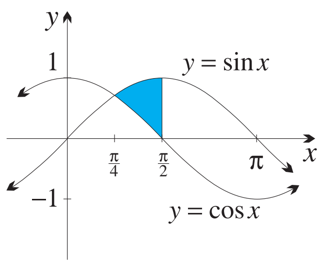
(d) $y = \cos 3x$, from $x = \frac{\pi}{12}$ to $x = \frac{\pi}{6}$

(e) $y = \sec^2 x$, from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$

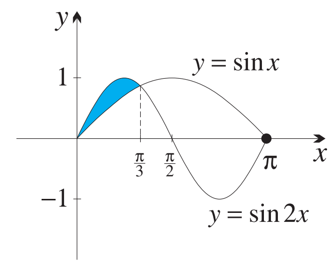
(f) $y = \sec^2 \frac{1}{2}x$, from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

6. Calculate the area of the shaded region in each diagram below:

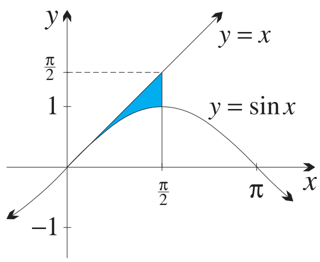
(a)



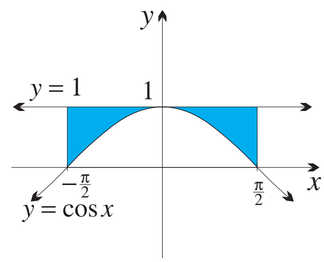
(b)



(c)

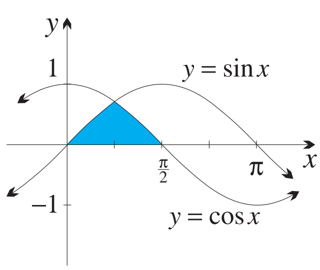


(d)

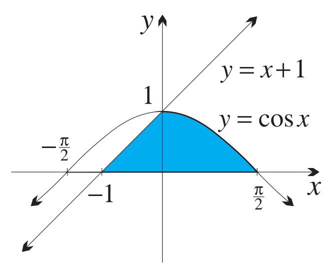


7. Calculate the area of the shaded region in each diagram below:

(a)



(b)



8. Find the exact volume of the solid of revolution formed when each region described below is rotated about the x-axis:

(a) the region bounded by the curve $y = \sec x$ and the x-axis, from $x = 0$ to $x = \frac{\pi}{3}$,

(b) the region bounded by the curve $y = \sqrt{\cos 4x}$ and the x-axis, from $x = 0$ to $x = \frac{\pi}{8}$,

(c) the region bounded by the curve $y = \sqrt{1 + \sin 2x}$ and the x-axis, from $x = 0$ to $x = \frac{\pi}{4}$.

DEVELOPMENT

9. Find, using a diagram, the area bounded by one arch of each curve and the x-axis:

(a) $y = \sin x$

(b) $y = \cos 2x$

10. Sketch the area enclosed between each curve and the x-axis over the specified domain, and then find the exact value of the area. (Make use of symmetry wherever possible.)

(a) $y = \cos x$, from $x = 0$ to $x = \pi$

(b) $y = \sin x$, from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$

(c) $y = \cos 2x$, from $x = 0$ to $x = \pi$

(d) $y = \sin 2x$, from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$

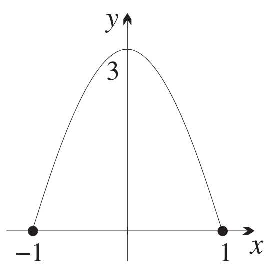
(e) $y = \sin x$, from $x = -\frac{5\pi}{6}$ to $x = \frac{7\pi}{6}$

(f) $y = \cos 3x$, from $x = \frac{\pi}{6}$ to $x = \frac{2\pi}{3}$

11. (a) Sketch the curve $y = 2\cos \pi x$, for $-1 \le x \le 1$, clearly marking the two x-intercepts.

(b) Find the exact area bounded by the curve $y = 2\cos \pi x$ and the x-axis, between the two x-intercepts.

12. An arch window 3 metres high and 2 metres wide is made in the shape of the curve $y = 3\cos(\frac{\pi}{2}x)$, as shown to the right. Find the area of the window in square metres, correct to one decimal place.



13. The graphs of $y = x - \sin x$ and $y = x$ are sketched together in a worked exercise in Section 4F. Find the total area enclosed between these graphs, from $x = 0$ to $x = 2\pi$.

14. The region R is bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$.

(a) Sketch R and then find its area.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

15. (a) Sketch the region bounded by the graphs of $y = \sin x$ and $y = \cos x$, and by the vertical lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

(b) Find the area of the region in part (a).

16. (a) Show by substitution that $y = \sin x$ and $y = \cos 2x$ meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

(b) On the same number plane, sketch $y = \sin x$ and $y = \cos 2x$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{6}$.

(c) Hence find the area of the region bounded by the two curves.

17. (a) Show that $\displaystyle\int_0^1 \sin \pi x \, dx = \frac{2}{\pi}$.

(b) Use Simpson's rule with five function values to approximate $\displaystyle\int_0^1 \sin \pi x \, dx$.

(c) Hence show that $\pi\left(1 + 2\sqrt{2}\right) \doteqdot 12$.

CHALLENGE

18. (a) Show that for all positive integers n:

(i) $\displaystyle\int_0^{2\pi} \sin nx \, dx = 0$ (ii) $\displaystyle\int_0^{2\pi} \cos nx \, dx = 0$

(b) Sketch each of the following graphs, and then find the area between the curve and the x-axis, from $x = 0$ to $x = 2\pi$:

(i) $y = \sin x$ (ii) $y = \sin 2x$ (iii) $y = \sin 3x$ (iv) $y = \sin nx$ (v) $y = \cos nx$

19. (a) Show that $\displaystyle\int_0^n (1 + \sin 2\pi x) \, dx = n$, for all positive integers n.

(b) Sketch $y = 1 + \sin 2\pi x$, and interpret the result geometrically.

20. (a) Sketch $y = 1 - \tan x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes.

(b) Find the area of R.

(c) Find the volume of the solid generated when R is rotated about the x-axis.

4I Chapter Review Exercise

1. Express in radians in terms of π:
(a) $180°$ (b) $20°$ (c) $240°$ (d) $315°$

2. Express in degrees:
(a) $\dfrac{\pi}{6}$ (b) $\dfrac{3\pi}{5}$ (c) 3π (d) $\dfrac{5\pi}{3}$

3. Find the exact value of:
(a) $\sin\frac{\pi}{3}$ (b) $\tan\frac{5\pi}{6}$

4. Solve, for $0 \le x \le 2\pi$:
(a) $\cos x = \frac{1}{\sqrt{2}}$ (b) $\tan x = -\sqrt{3}$

5. A circle has radius 12 cm. Find, in exact form:
(a) the length of an arc that subtends an angle at the centre of $45°$,
(b) the area of a sector in which the angle at the centre is $60°$.

6. A chord of a circle of radius 8 cm subtends an angle at the centre of $90°$. Find, correct to three significant figures, the area of the minor segment cut off by the chord.

7. Find, correct to the nearest minute, the angle subtended at the centre of a circle of radius 5 cm by an arc of length 13 cm.

8. State the period and amplitude in each case below, then sketch the graph of the function for $0 \le x \le 2\pi$:
(a) $y = \cos x$ (b) $y = 4\sin 2x$ (c) $y = \tan\frac{1}{2}x$

9. Sketch $y = 2\cos \pi x$, for $0 \le x \le 1$.

10. Differentiate with respect to x:
(a) $y = 5\sin x$
(b) $y = \sin 5x$
(c) $y = 5\cos 5x$
(d) $y = \tan(5x - 4)$
(e) $y = x\sin 5x$
(f) $y = \dfrac{\cos 5x}{x}$
(g) $y = \sin^5 x$
(h) $y = \tan(x^5)$
(i) $y = e^{\cos 5x}$
(j) $y = \log_e(\sin 5x)$

11. Find the gradient of the tangent to $y = \cos 2x$ at the point on the curve where $x = \frac{\pi}{3}$.

12. (a) Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{3}$.
(b) Find the equation of the tangent to $y = x\cos x$ at the point where $x = \frac{\pi}{2}$.

13. Find:
(a) $\displaystyle\int 4\cos x\,dx$ (b) $\displaystyle\int \sin 4x\,dx$ (c) $\displaystyle\int \sec^2 \tfrac{1}{4}x\,dx$

14. Find the value of:
(a) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x\,dx$ (c) $\displaystyle\int_0^{\frac{1}{3}} \pi\sin \pi x\,dx$

15. Find the value of $\displaystyle\int_0^{\frac{1}{4}} \sin 3x\,dx$, correct to three decimal places.

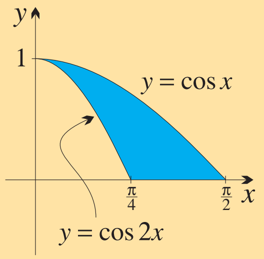
16. A curve has gradient function $y' = \cos\frac{1}{2}x$ and passes through the point $(\pi, 1)$. Find its equation.

17. (a) Sketch the curve $y = 2\sin 2x$, for $0 \le x \le \pi$, and then shade the area between the curve and the x-axis from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

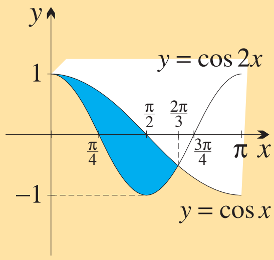
(b) Calculate the shaded area in part (a).

18. Find the area of the shaded region in the diagrams below.

(a)



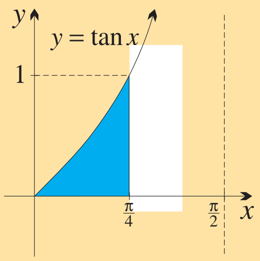
(b)



19. (a) Make $\tan^2 x$ the subject of the identity

$$1 + \tan^2 x = \sec^2 x.$$

(b) Hence find the volume of the solid formed when the shaded region in the diagram to the right is rotated one revolution about the x-axis.



Appendix — Differentiating the Trigonometric Functions

Proving the formulae given in Section 4F for the derivatives of the trigonometric functions is not easy at all. The first step is to prove that the derivative of $\sin x$ is $\cos x$. This will require first-principles differentiation using the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Applying this formula to the function $f(x) = \sin x$ gives the formula

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

In order to work with this definition, a formula is needed for the expansion of $\sin(x+h)$.

The Expansion of sin($x + h$): This formula is a standard part of the 3 Unit course, but it will play no further part in the 2 Unit course and has therefore not been boxed as a formula to remember.

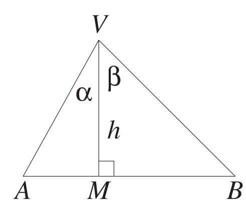
LEMMA: $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, for all angles α and β.

PROOF: We shall only prove this formula in the case where both α and β are acute angles. The more general proof is 3 Unit work — see, for example, the last chapter in the Year 11 volume of *Cambridge Mathematics 3 Unit*.

Construct an interval VM of any length h.
Construct the line AMB through M perpendicular to VM,
such that the points A and B lie on alternate sides of M,
and $\angle AVM = \alpha$ and $\angle BVM = \beta$.

Then $\dfrac{AV}{h} = \dfrac{1}{\cos\alpha}$ and $\dfrac{BV}{h} = \dfrac{1}{\cos\beta}$,

that is, $AV = \dfrac{h}{\cos\alpha}$ and $BV = \dfrac{h}{\cos\beta}$.



Now area $\triangle AVB$ = area $\triangle AVM$ + area $\triangle BVM$,

and applying the area formula in each of the three triangles,

$$\tfrac{1}{2}\left(\frac{h}{\cos\alpha}\right)\left(\frac{h}{\cos\beta}\right)\sin(\alpha+\beta) = \tfrac{1}{2}h\left(\frac{h}{\cos\alpha}\right)\sin\alpha + \tfrac{1}{2}h\left(\frac{h}{\cos\beta}\right)\sin\beta$$

$\boxed{\div \tfrac{1}{2}h^2}$ $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}.$

Multiplying through by $\cos\alpha\cos\beta$,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \text{ as required.}$$

The Derivative of sin x is cos x: The proof that the derivative of $\sin x$ is $\cos x$ depends on the fundamental limit proven in Section 4D,

$$\lim_{u \to 0} \frac{\sin u}{u} = 1.$$

THEOREM: The derivative of $\sin x$ is $\cos x$:

$$\frac{d}{dx}\sin x = \cos x$$

Proof: Going back to the definition of the derivative as a limit,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

that is, $$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}.$$

Now
$$\begin{aligned}\frac{\sin(x+h)-\sin x}{h} &= \frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\\ &= \frac{\cos x \sin h}{h} + \frac{\sin x(\cos h - 1)}{h}\\ &= \cos x \times \frac{\sin h}{h} + \sin x \times \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}\\ &= \cos x \times \frac{\sin h}{h} + \sin x \times \frac{\cos^2 h - 1}{h(\cos h + 1)}\\ &= \cos x \times \frac{\sin h}{h} - \sin x \times \frac{\sin^2 h}{h(\cos h + 1)}\\ &= \cos x \times \frac{\sin h}{h} - \sin x \times \frac{\sin h}{h} \times \frac{\sin h}{\cos h + 1}.\end{aligned}$$

As $h \to 0$, the first term has limit $\cos x$, because $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$,

and the second term has limit 0, because $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$, and $\lim\limits_{h\to 0} \dfrac{\sin h}{\cos h + 1} = \dfrac{0}{2}$.

Hence $$\frac{d}{dx}(\sin x) = \cos x - 0, \quad \text{as required.}$$

The Derivatives of $\cos x$ and $\tan x$: Once the derivative of $\sin x$ is proven, it is straightforward to differentiate the next two trigonometric functions.

Lemma: $\dfrac{d}{dx}\cos x = -\sin x$ and $\dfrac{d}{dx}\tan x = \sec^2 x$

Proof:

A. Let $y = \cos x$.

Then $y = \sin(\frac{\pi}{2} - x)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \quad \text{(chain rule)}\\ &= -\cos(\tfrac{\pi}{2} - x)\\ &= -\sin x.\end{aligned}$$

Let $u = \frac{\pi}{2} - x$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = -1$
and $\dfrac{dy}{du} = \cos u$.

B. Let $y = \tan x$.

Then $y = \dfrac{\sin x}{\cos x}$.

$$\begin{aligned}y' &= \frac{vu' - uv'}{v^2} \quad \text{(quotient rule)}\\ &= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}, \quad \text{because } \cos^2 x + \sin^2 x = 1,\\ &= \sec^2 x.\end{aligned}$$

Let $u = \sin x$
and $v = \cos x$.
Then $u' = \cos x$
and $v' = -\sin x$.

CHAPTER **FIVE**

Motion

Anyone watching objects in motion can see that they often make patterns with a striking simplicity and predictability. These patterns are related to the simplest objects in geometry and arithmetic. A thrown ball traces out a parabolic path. A cork bobbing in flowing water traces out a sine wave. A rolling billiard ball moves in a straight line, rebounding symmetrically off the table edge. The stars and planets move in more complicated, but highly predictable, paths across the sky. The relationship between physics and mathematics, logically and historically, begins with these and many similar observations.

This short chapter, however, does little more than introduce the relationship between calculus and motion. This is a mathematics course, not a physics course. Thus the attention will not be on the nature of space, time and motion, but on the new insights that the physical world brings to the mathematical objects already developed earlier in the course.

The principal goal will be to produce a striking alternative interpretation of the first and second derivatives as the physical notions of velocity and acceleration so well known to our senses. The examples of motion in this chapter also provide models of the familiar linear, quadratic, exponential and trigonometric functions.

5 A Average Velocity and Speed

This first section sets up the mathematical description of motion in one dimension, using a function to describe the relationship between time and the position of an object in motion. Average velocity is described as the gradient of the chord on this displacement–time graph. This will lead, in the next section, to the description of instantaneous velocity as the gradient of a tangent.

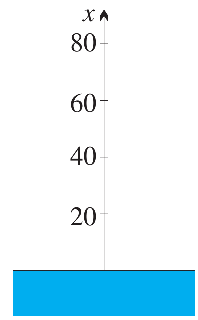
Motion in One Dimension: When a particle is moving in one dimension (along a line) its position is varying over time. That position can be specified at any time t by a single number x, called the *displacement*, and the whole motion can be described by giving x as a function of the *time* t.

For example, suppose that a ball is hit vertically upwards from ground level and lands 8 seconds later in the same place. Its motion can be described approximately by the following quadratic equation and table of values:

$$x = 5t(8 - t)$$

t	0	2	4	6	8
x	0	60	80	60	0

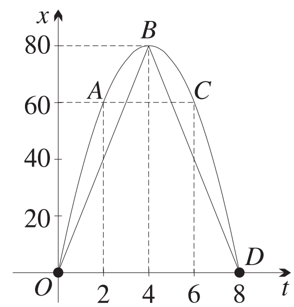
Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line.



This vertical line has been made into a number line, with the ground as the origin, upwards as the positive direction, and metres as the units of distance.

Time has also become a number line. The origin of time is when the ball is thrown, and the units of time are seconds.

The graph to the right is the resulting graph of the equation of motion $x = 5t(8-t)$. The horizontal axis is time and the vertical axis is displacement — the graph must not be mistaken as a picture of the ball's path.



The graph is a section of a parabola with vertex at $(4, 80)$, which means that the ball achieves a maximum height of 80 metres after 4 seconds. When $t = 8$, the height is zero, and the ball strikes the ground again. The equation of motion therefore has quite restricted domain and range:

$$0 \le t \le 8 \qquad \text{and} \qquad 0 \le x \le 80.$$

Most equations of motion have this sort of restriction on the domain of t. In particular, *it is a convention of this course that negative values of time are excluded unless the question specifically allows it.*

1

MOTION IN ONE DIMENSION:

- Motion in one dimension is specified by giving the displacement x on the number line as a function of time t after time zero.
- *Negative values of time are excluded unless otherwise stated.*

WORKED EXERCISE:

Consider the example above, where $x = 5t(8-t)$.

(a) Find the height of the ball after 1 second.

(b) At what other time is the ball at this same height above the ground?

SOLUTION:

(a) When $t = 1$, $x = 5 \times 1 \times 7$

$$= 35.$$

Hence the ball is 35 metres above the ground after 1 second.

(b) To find when the height is 35 metres, solve the equation $x = 35$.

Substituting into $x = 5t(8-t)$ gives

$$\begin{aligned} 5t(8-t) &= 35 \\ \boxed{\div 5} \qquad t(8-t) &= 7 \\ 8t - t^2 - 7 &= 0 \\ \boxed{\times(-1)} \qquad t^2 - 8t + 7 &= 0 \\ (t-1)(t-7) &= 0 \\ t &= 1 \text{ or } 7. \end{aligned}$$

Hence the ball is 35 metres high after 1 second and again after 7 seconds.

Average Velocity: During its ascent, the ball in the example above moved 80 metres upwards. This is a change in displacement of +80 metres in 4 seconds, giving an average velocity of 20 metres per second.

Average velocity thus equals the gradient of the chord OB on the displacement–time graph (be careful, because there are different scales on the two axes). Hence the formula for average velocity is the familiar gradient formula.

2 AVERAGE VELOCITY:

Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$. Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

That is, on the displacement–time graph,

$$\text{average velocity} = \text{gradient of the chord.}$$

During its descent, the ball moved 80 metres downwards in 4 seconds, which is a change in displacement of $0 - 80 = -80$ metres. The average velocity is therefore -20 metres per second, which is equal to the gradient of the chord BD.

WORKED EXERCISE:

Consider again the example $x = 5t(8 - t)$. Find the average velocities of the ball:

(a) during the first second, (b) during the fifth second.

SOLUTION:

The first second stretches from $t = 0$ to $t = 1$ and the fifth second stretches from $t = 4$ to $t = 5$. The displacements at these times are given in the table to the right.

t	0	1	4	5
x	0	35	80	75

(a) Average velocity during 1st second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{35 - 0}{1 - 0}$$
$$= 35\,\text{m/s}.$$

(b) Average velocity during 5th second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{75 - 80}{5 - 4}$$
$$= -5\,\text{m/s}.$$

Distance Travelled: The change in displacement can be positive, negative or zero. *Distance*, however, is always positive or zero. In the previous example, the change in displacement during the 4 seconds from $t = 4$ to $t = 8$ is -80 metres, but the distance travelled is 80 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the total distance travelled by the ball is $80 + 80 = 160$ metres, even though the ball's change in displacement over the first 8 seconds is zero because the ball is back at its original position on the ground.

3 DISTANCE TRAVELLED:

- The *distance travelled* takes into account any journey and return.
- Distance travelled can never be negative.

Average Speed: The *average speed* is the distance travelled divided by the time taken. Average speed, unlike average velocity, can never be negative.

4 AVERAGE SPEED: $\text{average speed} = \dfrac{\text{distance travelled}}{\text{time taken}}$

Average speed can never be negative.

During the 8 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 160 metres, so

$$\begin{aligned}\text{average velocity} &= \frac{0-0}{8-0}\\ &= 0\,\text{m/s},\end{aligned} \qquad \begin{aligned}\text{average speed} &= \frac{160}{8}\\ &= 20\,\text{m/s}.\end{aligned}$$

WORKED EXERCISE:

Find the average velocity and the average speed of the ball:

(a) during the eighth second, (b) during the last six seconds.

SOLUTION:

The eighth second stretches from $t = 7$ to $t = 8$ and the last six seconds stretch from $t = 2$ to $t = 8$. The displacements at these times are given in the table to the right.

t	0	2	7	8
x	0	60	35	0

(a) During the eighth second, the ball moves 35 metres down from $x = 35$ to $x = 0$.

Hence $\quad \text{average velocity} = \dfrac{0-35}{8-7}$

$= -35\,\text{m/s}.$

Also $\quad$ distance travelled $= 35$ metres,

so $\quad$ average speed $= 35\,\text{m/s}$.

(b) During the last six seconds, the ball rises 20 metres from $x = 60$ to $x = 80$, and then falls 80 metres from $x = 80$ to $x = 0$.

Hence $\quad \text{average velocity} = \dfrac{0-60}{8-2}$

$= -10\,\text{m/s}.$

Also $\quad$ distance travelled $= 20 + 80$

$= 100$ metres,

so $\quad \text{average speed} = \dfrac{100}{6}$

$= 16\frac{2}{3}\,\text{m/s}.$

Exercise 5A

1. A particle is moving with displacement function $x = t^2 + 2$, where time t is in seconds and displacement x is in metres.

(a) Find the position when $t = 0$.

(b) Find the position when $t = 4$.

(c) Find the average velocity during the first 4 seconds, using the definition

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}}.$$

2. For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity during the first 2 seconds. The units in each part are seconds and centimetres.

t	0	2
x		

(a) $x = 12t - t^2$ (c) $x = t^3 - 4t + 3$

(b) $x = (t-2)^2$ (d) $x = 2^t$

3. A particle moves according to the equation $x = t^3 - 4t$, where x is the displacement in centimetres from the origin O at time t seconds after time zero.

(a) Copy and complete the table of values to the right.

t	0	1	2	3	4
x					

(b) Hence find the average velocity:

(i) during the first second (that is, from $t = 0$ to $t = 1$),

(ii) during the second second (that is, from $t = 1$ to $t = 2$),

(iii) during the third second (that is, from $t = 2$ to $t = 3$),

(iv) during the fourth second (that is, from $t = 3$ to $t = 4$).

4. A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

(a) Copy and complete the table of values to the right.

t	0	1	2	3
x				

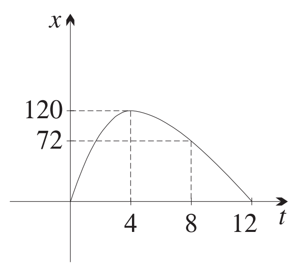
(b) Hence find the average velocity:

(i) during the first second, (iii) during the first three seconds,

(ii) during the first two seconds, (iv) during the third second.

(c) Use the table of values above to sketch the displacement–time graph. Then add the chords corresponding to the average velocities calculated in part (b).

5. A piece of cardboard is shot 120 metres vertically into the air by an explosion and floats back to the ground, landing at the same place. The graph to the right gives its height x metres above the ground t seconds after the explosion.



(a) Copy and complete the following table of values.

t	0	4	8	12
x				

(b) What is the total distance travelled by the cardboard?

(c) Find the average speed of the cardboard during its travels, using the formula

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

(d) Find the average velocity during:

(i) the ascent, (ii) the descent, (iii) the full 12 seconds.

6. A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

(a) Copy and complete the table of values to the right.

t	0	1	2	3	4
x					

(b) Hence sketch the displacement–time graph.

(c) Find the total distance travelled during the first 4 seconds. Then find the average speed during this time.

(d) Find the average velocity during the time:

(i) from $t = 0$ to $t = 2$, (ii) from $t = 2$ to $t = 4$, (iii) from $t = 0$ to $t = 4$.

(e) Add to your graph the chords corresponding to the average velocities in part (d).

DEVELOPMENT

7. Michael the mailman rides his bicycle 1 km up a hill at a constant speed of 10 km/hr, and then rides 1 km down the other side of the hill at a constant speed of 30 km/hr.

(a) How many minutes does he take to travel:

(i) the first kilometre, when he is riding up the hill,

(ii) the second kilometre, when he is riding down the other side?

(b) Use these values to draw a displacement–time graph, with the time axis in minutes.

(c) What is his average speed over the total 2 km journey?

(d) What is the average of his speeds up and down the hill?

8. Sadie the snail is crawling up a 6-metre-high wall. She takes an hour to crawl up 3 metres, then falls asleep for an hour and slides down 2 metres, repeating the cycle until she reaches the top of the wall. Let x be Sadie's height in metres after t hours.

(a) Copy and complete the table of values of Sadie's height up the wall.

t	0	1	2	3	4	5	6	7
x								

(b) Hence sketch the displacement–time graph.

(c) How long does Sadie take to reach the top?

(d) What total distance does she travel, and what is her average speed?

(e) What is her average velocity over this whole time?

(f) Which places on the wall does she visit exactly three times?

9. A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance x is in centimetres and time t is in seconds.

(a) Copy and complete the table of values to the right.

t					
x	0	2	4	6	8

(b) Hence find the average velocity as the particle moves:

(i) from $x = 0$ to $x = 2$,

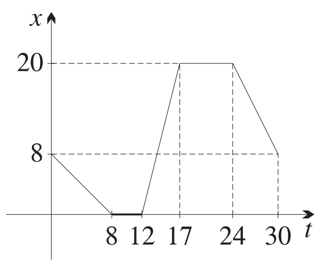
(ii) from $x = 2$ to $x = 4$,

(iii) from $x = 4$ to $x = 6$,

(iv) from $x = 0$ to $x = 6$.

(c) What does the equality of the answers to parts (ii) and (iv) of part (b) tell you about the corresponding chords in part (c)?

10. Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x in metres from the front gate after t seconds.



(a) What is her average velocity:

(i) during the first 8 seconds,

(ii) while she is driving forwards,

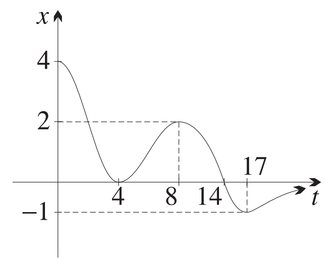
(iii) while she is reversing the second time?

(b) Find the total distance she travelled, and her average speed, over the 30 seconds.

(c) Find her change in displacement, and her average velocity, over the 30 seconds.

(d) What would her average speed have been if she had not paused at the gate and at the garage?

11. A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The graph shows the height x in metres of the weight above the water as a function of time t seconds after she first drops it.



(a) How many times is the weight:

(i) at $x = 3$, (ii) at $x = 1$, (iii) at $x = -\frac{1}{2}$?

(b) At what times is the weight:

(i) at the water surface, (ii) above the water surface?

(c) How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?

(d) What is the weight's greatest depth under the water and when does it occur?

(e) What happens to the weight eventually?

(f) What is its average velocity:

(i) during the first 4 seconds, (ii) from $t = 4$ to $t = 8$, (iii) from $t = 8$ to $t = 17$?

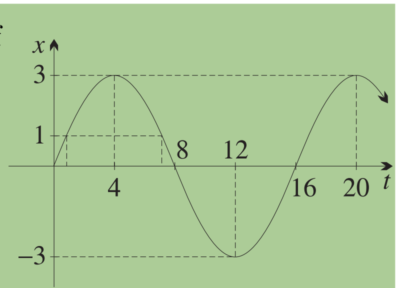
(g) What distance does it travel:

(i) over the first 4 seconds,
(ii) over the first 8 seconds,
(iii) over the first 17 seconds,
(iv) eventually?

(h) What is its average speed over the first: (i) 4, (ii) 8, (iii) 17 seconds?

CHALLENGE

12. A particle is moving according to $x = 3 \sin \frac{\pi}{8} t$, in units of centimetres and seconds. Its displacement–time graph is sketched to the right.



(a) Use $T = \dfrac{2\pi}{n}$ to confirm that the period is 16 seconds.

(b) Find the maximum and minimum values of the displacement.

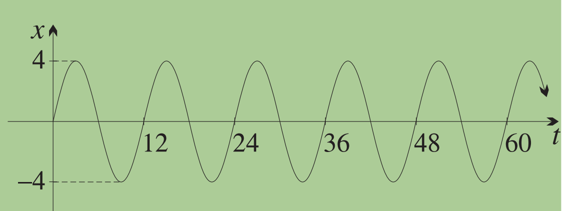
(c) Find the first two times when the displacement is maximum.

(d) Find the first two times when the particle returns to its initial position.

(e) When, during the first 20 seconds, is the particle on the negative side of the origin?

(f) Find the total distance travelled during the first 16 seconds, and the average speed.

13. A particle is moving according to the equation $x = 4 \sin \frac{\pi}{6} t$, in units of metres and seconds. The graph of its displacement for the first minute is sketched to the right.



(a) Find the amplitude and period.

(b) How many times does the particle return to the origin by the end of the first minute?

(c) Find at what times it visits $x = 4$ during the first minute.

(d) Find how far it travels during the first 12 seconds, and its average speed in that time.

(e) Find the values of x when $t = 0$, $t = 1$ and $t = 3$. Hence show that the average speed during the first second is twice the average speed during the next 2 seconds.

14. A balloon rises so that its height h in metres after t minutes is $h = 8000(1 - e^{-0 \cdot 06t})$.

(a) What height does it start from, and what happens to the height as $t \to \infty$?

(b) Copy and complete the table to the right, correct to the nearest metre.

t	0	10	20	30
h				

(c) Sketch the displacement–time graph of the motion.

(d) Find the balloon's average velocity during the first 10 minutes, the second 10 minutes and the third 10 minutes, correct to the nearest metre per minute.

(e) Use your calculator to show that the balloon has reached 99% of its final height after 77 minutes, but not after 76 minutes.

5 B Velocity as a Derivative

If I drive the 160 km from Sydney to Newcastle in 2 hours, my average velocity is 80 km per hour. But my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. Just as an average velocity corresponds to the gradient of a chord on the displacement–time graph, so an instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous Velocity and Speed: From now on, the words *velocity* and *speed* alone will mean instantaneous velocity and instantaneous speed.

5 INSTANTANEOUS VELOCITY AND INSTANTANEOUS SPEED:

- The *instantaneous velocity* v is the derivative of the displacement with respect to time:

$$v = \frac{dx}{dt} \qquad \text{(This derivative } \frac{dx}{dt} \text{ can also be written as } \dot{x}.)$$

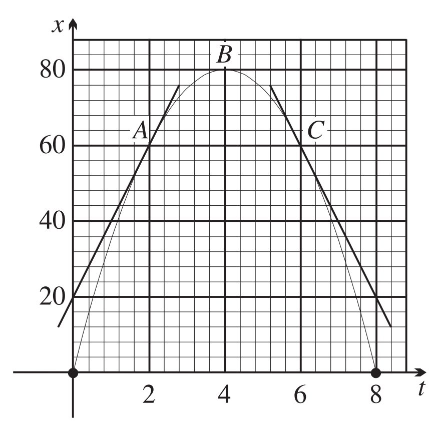
That is, $v =$ gradient of the tangent on the displacement–time graph.

- The *instantaneous speed* is the absolute value $|v|$ of the velocity.

The notation $\dot{x}$ is yet another way of writing the derivative. The dot over the x, or over any symbol, stands for differentiation with respect to time t. Thus the symbols v, $\dfrac{dx}{dt}$ and $\dot{x}$ are alternative notations for velocity.

WORKED EXERCISE:

Here again is the displacement–time graph of the ball moving with equation $x = 5t(8 - t)$.



(a) Differentiate to find the equation for the velocity v, draw up a table of values at 2-second intervals and sketch the velocity–time graph.

(b) Measure the gradients of the tangents that have been drawn at A, B and C on the displacement–time graph and compare your answers with the table of values in part (a).

(c) With what velocity was the ball originally hit?

(d) What is its impact speed when it hits the ground?

SOLUTION:

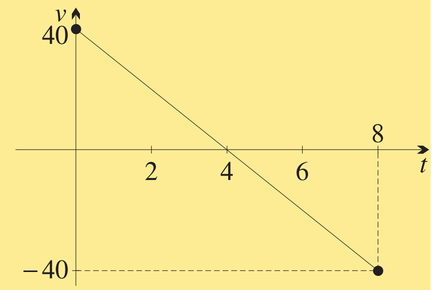
(a) The equation of motion is $x = 5t(8 - t)$

$$x = 40t - 5t^2.$$

Differentiating, $v = 40 - 10t$.

The graph of velocity is a straight line, with v-intercept 40 and gradient -10.

t	0	2	4	6	8
v	40	20	0	-20	-40



(b) These values agree with the measurements of the gradients of the tangents at A where $t = 2$, at B where $t = 4$, and at C where $t = 6$.
(Be careful to take account of the different scales on the two axes.)

(c) When $t = 0$, $v = 40$, so the ball was originally hit upwards at 40 m/s.

(d) When $t = 8$, $v = -40$, so the ball hits the ground again at 40 m/s.

Vector and Scalar Quantities: Displacement and velocity are *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards and a negative displacement would mean it was below ground level. Distance and speed, however, are called *scalar quantities* — they measure only the magnitude of displacement and velocity respectively so they cannot be negative.

WORKED EXERCISE:

A particle moves with displacement $x = t^3 - 6t - 2$, in units of metres and seconds.

(a) Differentiate to find the velocity function.

(b) Find the displacement, distance from the origin, velocity and speed when:
(i) $t = 0$, (ii) $t = 3$.

SOLUTION:

(a) The displacement equation is $x = t^3 - 6t - 2$.
Differentiating, $v = 3t^2 - 6$.

(b) (i) When $t = 0$, $x = -2$
and $v = -6$.
Thus when $t = 0$, the displacement is -2 metres, the particle is 2 metres from the origin, the velocity is -6 m/s and the speed is 6 m/s.

(ii) When $t = 3$,
$$x = 27 - 18 - 2 = 7,$$
and
$$v = 27 - 6 = 21.$$
Thus when $t = 3$, the displacement is 7 metres, the particle is 7 metres from the origin, the velocity is 21 m/s and the speed is 21 m/s.

Finding when a Particle is Stationary: A particle is said to be *stationary* when its velocity v is zero, that is, when $\dfrac{dx}{dt} = 0$. This is the origin of the word 'stationary point', introduced in the last chapter of the Year 11 volume to describe a point on a graph where the derivative is zero. For example, the ball in the first example was stationary for an instant at the top of its flight when $t = 4$, because the velocity was zero at the instant when its motion changed from upwards to downwards.

6 FINDING WHEN A PARTICLE IS STATIONARY:
- A particle is *stationary* when its velocity is zero.
- To find when a particle is stationary, put $v = 0$ and solve for t.

WORKED EXERCISE:
A particle moves so that its distance in metres from the origin at time t seconds is given by $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.
(a) Find the times when the particle is stationary.
(b) Find its distance from the origin at these times.

SOLUTION:
(a) The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.
Differentiating, $v = t^2 - 12t + 27$
$= (t-3)(t-9)$,
so the particle is stationary after 3 seconds and after 9 seconds.

(b) When $t = 3$, $x = 9 - 54 + 81 - 18$
$= 18$,
and when $t = 9$, $x = 243 - 486 + 243 - 18$
$= -18$.
Thus the particle is 18 metres from the origin on both occasions.

Acceleration as the Second Derivative: A particle is said to be *accelerating* if its velocity is changing. The *acceleration* of an object is defined to be the rate at which the velocity is changing. Thus the acceleration a is the derivative $\dfrac{dv}{dt} = \dot{v}$ of the velocity with respect to time.

Since velocity is the derivative of displacement, the acceleration is the second derivative $\dfrac{d^2x}{dt^2} = \ddot{x}$ of displacement.

7 ACCELERATION AS A DERIVATIVE:
- Acceleration is the first derivative of velocity with respect to time:
$$a = \frac{dv}{dt} = \dot{v}.$$
- Acceleration is the second derivative of displacement with respect to time:
$$a = \frac{d^2x}{dt^2} = \ddot{x}.$$

Again, the dot stands for differentiation with respect to time t. Thus

$$\dot{x} \text{ means } \frac{dx}{dt}, \qquad \ddot{x} \text{ means } \frac{d^2x}{dt^2}, \qquad \dot{v} \text{ means } \frac{dv}{dt}.$$

The symbols a, $\ddot{x}$, $\dot{v}$, $\dfrac{d^2x}{dt^2}$ and $\dfrac{dv}{dt}$ all mean the acceleration. Be careful with the symbol a, because here acceleration is a function, whereas elsewhere the pronumeral a is usually used for constants.

WORKED EXERCISE:

Consider again the ball moving with displacement function $x = 5t(8 - t)$.

(a) Find the velocity function v and the acceleration function a.

(b) Sketch the graph of the acceleration function.

(c) Find and describe the displacement, velocity and acceleration when $t = 2$.

(d) State when the ball is speeding up and when it is slowing down, explaining why this can happen when the acceleration is constant.

SOLUTION:

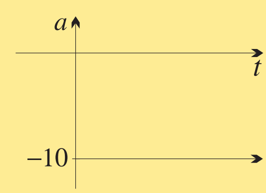
(a) The function is $\qquad x = 40t - 5t^2$,

Differentiating, $\qquad v = 40 - 10t$

Differentiating again, $a = -10$, which is a constant.

(b) Hence the acceleration is always $10\,\text{m/s}^2$ downwards.

The graph is drawn to the right.



(c) Substitute $t = 2$ into the functions x, v and a.

When $t = 2$, $x = 60$,

$v = 20$,

$a = -10$.

Thus when $t = 2$, the displacement is 60 metres above the ground, the velocity is $20\,\text{m/s}$ upwards and the acceleration is $10\,\text{m/s}^2$ downwards.

(d) During the first 4 seconds, the ball has positive velocity, meaning that it is rising, and the ball is slowing down by $10\,\text{m/s}$ every second.

During the last 4 seconds, however, the ball has negative velocity, meaning that it is falling, and the ball is speeding up by $10\,\text{m/s}$ every second.

Units of Acceleration: In the previous example, the particle's velocity was decreasing by $10\,\text{m/s}$ every second. The particle is said to be 'accelerating at -10 metres per second per second', written shorthand as $-10\,\text{m/s}^2$ or as $-10\,\text{ms}^{-2}$. The units of acceleration correspond with the indices of the second derivative $\dfrac{d^2x}{dt^2}$.

Acceleration should normally be regarded as a vector quantity, that is, with a direction built into it. This is why the particle's acceleration is written with a minus sign as $-10\,\text{m/s}^2$. Alternatively, one can omit the minus sign and specify the direction instead, writing '$10\,\text{m/s}^2$ in the downwards direction'.

WORKED EXERCISE:

In an earlier worked exercise, we examined the function $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

(a) Find the acceleration function and find when the acceleration is zero.

(b) Where is the particle at this time and what is its velocity?

SOLUTION:

(a) The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

Differentiating, $v = t^2 - 12t + 27$

and differentiating again, $a = 2t - 12$

$= 2(t - 6)$.

Thus the acceleration is zero when $t = 6$.

(b) When $t = 6$, $v = 36 - 72 + 27$

$= -9$,

and $x = 72 - 216 + 162 - 18$

$= 0$.

Thus when $t = 6$, the particle is at the origin, moving with velocity $-9\,\text{m/s}$.

Trigonometric Equations of Motion: When a particle's motion is described by a sine or cosine function, it moves backwards and forwards and is therefore stationary over and over again.

The wavy graphs of x, v and a are very helpful in interpreting the particle's motion. In fact, in the next worked exercise, it is possible to solve all the trigonometric equations simply by looking at these three graphs.

WORKED EXERCISE:

A particle's displacement function is $x = 2\sin \pi t$.

(a) Find its velocity and acceleration functions.

(b) Graph all three functions in the time interval $0 \le t \le 2$.

(c) Find the times within the time interval $0 \le t \le 2$ when the particle is at the origin, and find its speed and acceleration at those times.

(d) Find the times within the time interval $0 \le t \le 2$ when the particle is stationary, and find its displacement and acceleration at those times.

(e) Briefly describe the motion.

SOLUTION:

(a) The displacement function is $x = 2\sin \pi t$,

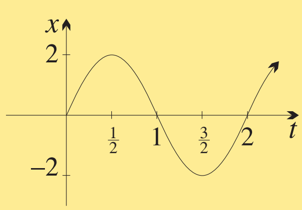
which has amplitude 2 and period $\dfrac{2\pi}{\pi} = 2$.

Differentiating, $v = 2\pi \cos \pi t$,

which has amplitude 2π and period 2.

Differentiating again, $a = -2\pi^2 \sin \pi t$.

which has amplitude $2\pi^2$ and period 2.

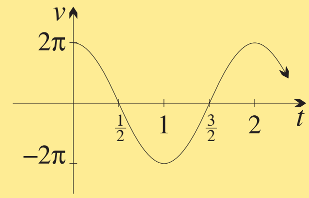


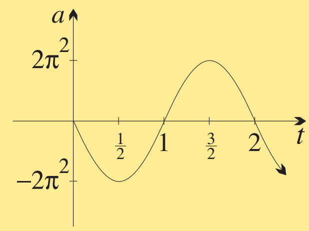
(b) The three graphs are drawn opposite.

(c) The condition for the particle to be at the origin is

$$x = 0.$$

and reading from the displacement graph on page 207,
this occurs when $\quad t = 0, 1$ or 2.
Reading now from the velocity graph opposite,
when $t = 0$ or 2, $\quad v = 2\pi$,
and when $t = 1$, $\quad v = -2\pi$,
so in all cases the speed is 2π.
Reading finally from the acceleration graph,
when $t = 0$, 1 or 2, $\quad a = 0$,
so in all cases the acceleration is zero.





(d) The condition for the particle to be stationary is

$$v = 0$$

and reading from the velocity graph above,
this occurs when $\quad t = \frac{1}{2}$ or $1\frac{1}{2}$.
Reading from the displacement graph on page 207,
when $t = \frac{1}{2}$, $\quad x = 2$,
and when $t = 1\frac{1}{2}$, $\quad x = -2$.
Reading from the acceleration graph above,
when $t = \frac{1}{2}$, $\quad a = -2\pi^2$,
and when $t = 1\frac{1}{2}$, $\quad a = 2\pi^2$.

(e) The particle oscillates forever between $x = -2$ and $x = 2$, with period 2, beginning at the origin and moving first to $x = 2$.

Motion with Exponential Functions — Limiting Values of Displacement and Velocity:

Sometimes a question will ask what happens to the particle 'eventually', or 'as time goes on'. This simply means taking the limit of the displacement and the velocity as $t \to \infty$. Particles whose motion is described by an exponential function are the most usual examples of this. Remember that $e^{-x} \to 0$ as $x \to \infty$.

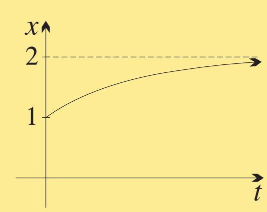
WORKED EXERCISE:

A particle is moving so that its height x metres above the ground at time t seconds after time zero is $x = 2 - e^{-3t}$.

(a) Find the velocity and acceleration functions.
(b) Sketch the three graphs of displacement, velocity and acceleration.
(c) Find the initial values of displacement, velocity and acceleration.
(d) What happens to the displacement, velocity and acceleration eventually?
(e) Briefly describe the motion.

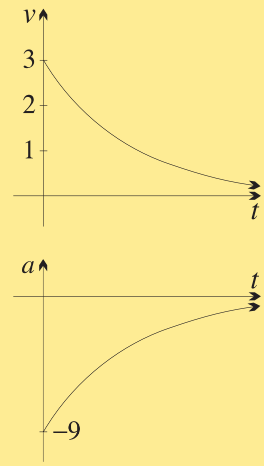
SOLUTION:

(a) The displacement function is $\quad x = 2 - e^{-3t}$.
Differentiating, $\quad v = 3e^{-3t}$,
and differentiating again, $\quad a = -9e^{-3t}$.



(b) The three graphs are drawn to the right.

(c) Substitute $t = 0$ and use the fact that $e^0 = 1$.
Thus initially, $x = 1$,
$v = 3$,
$a = -9$.



(d) As t increases, that is, as $t \to \infty$, $e^{-3t} \to 0$.
Hence *eventually* (meaning as $t \to \infty$), $x \to 2$
$v \to 0$,
$a \to 0$.

(e) The particle starts 1 metre above the ground with initial velocity of 3 m/s upwards. It is constantly slowing down and it moves towards a limiting position at height 2 metres.

Extension — Newton's Second Law of Motion: Newton's second law of motion — a law of physics, not of mathematics — says that when a force is applied to a body that is free to move, the body accelerates with an acceleration proportional to the force and inversely proportional to the mass of the body. Written symbolically,

$$F = ma,$$

where m is the mass of the body, F is the force applied and a is the acceleration. (The units of force are chosen to make the constant of proportionality 1 — in units of kilograms, metres and seconds, the units of force are, appropriately, called *newtons*.)

This means that acceleration is felt in our bodies as a force, as we all know when a car we are in accelerates away from the lights, or comes to a stop quickly. In this way, the second derivative becomes directly observable to our senses as a force, just as the first derivative, velocity, is observable to our sight.

Although these things are not treated in the 2 Unit course, it is helpful to have an intuitive idea that force and acceleration are closely related.

Exercise 5B

NOTE: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The mathematics should be well-known, but the physical interpretations can be confusing.

1. A particle is moving with displacement function $x = 20 - t^2$, in units of metres and seconds.
(a) Differentiate to find the velocity v as a function of time t.
(b) Differentiate again to find the acceleration a.
(c) Find the displacement, velocity and acceleration when $t = 3$.
(d) What are the distance from the origin and the speed when $t = 3$?

2. For each displacement function below, differentiate to find the velocity function v and differentiate again to find the acceleration function a. Then find the displacement, velocity and acceleration when $t = 1$. The units are metres and seconds.
(a) $x = 5t^2 - 10t$ (b) $x = 3t - 2t^3$ (c) $x = t^4 - t^2 + 4$

3. A particle's displacement function is $x = t^2 - 10t$, in units of centimetres and seconds.

(a) Differentiate to find v as a function of t.

(b) What are the displacement, the distance from the origin, the velocity and the speed after 3 seconds?

(c) When is the particle stationary and where is it then?

4. A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2$.

(a) Differentiate to find v as a function of t, and differentiate again to find a.

(b) Where is the particle initially and what are its speed and acceleration then?

(c) At time $t = 3$, is the particle to the left or to the right of the origin?

(d) At time $t = 3$, is the particle travelling to the left or to the right?

(e) At time $t = 3$, is the particle accelerating to the left or to the right?

(f) Show that the particle is stationary when $t = 4$ and find where it is at this time.

(g) Show that the particle is at the origin when $t = 6$ and find its velocity and speed at this time.

5. For each displacement function below, differentiate to find the velocity function v, and differentiate again to find the acceleration function a. Then find the displacement, velocity and acceleration when $t = \frac{\pi}{2}$. The units are centimetres and seconds.

(a) $x = \sin t$ (b) $x = \cos t$

6. For each displacement function below, find by differentiation the velocity function v and the acceleration function a. Then find the displacement, velocity and acceleration when $t = 1$. The units are metres and seconds.

(a) $x = e^t$ (b) $x = e^{-t}$

7. A cricket ball is thrown vertically upwards. Its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.

(a) Find v and a as functions of t, and show that the ball is always accelerating downwards. Then sketch graphs of x, v and a against t.

(b) Find the speed at which the ball was thrown.

(c) Find when it returns to the ground (that is, when $x = 0$) and show that its speed then is equal to the initial speed.

(d) Find its maximum height above the ground and the time taken to reach this height.

(e) Find the acceleration at the top of the flight, and explain why the acceleration can be nonzero when the ball is stationary.

DEVELOPMENT

8. If $x = e^{-4t}$, find the velocity function $\dot{x}$ and the acceleration function $\ddot{x}$.

(a) Explain why none of the functions x, $\dot{x}$ and $\ddot{x}$ can ever change sign, and state their signs.

(b) Using the displacement function, find where the particle is:

(i) initially (substitute $t = 0$),

(ii) eventually (take the limit as $t \to \infty$).

(c) What are the particle's velocity and acceleration:

(i) initially,

(ii) eventually?

9. Find the velocity function v and the acceleration function a for a particle P moving according to $x = 2\sin \pi t$.

(a) Show that P is at the origin when $t = 1$ and find its velocity and acceleration then.

(b) In what direction is the particle: (i) moving, (ii) accelerating, when $t = \frac{1}{3}$?

10. A particle moves according to $x = t^2 - 8t + 7$, in units of metres and seconds.

(a) Find the velocity $\dot{x}$ and the acceleration $\ddot{x}$ as functions of time t.

(b) Sketch the graphs of the displacement x, velocity $\dot{x}$ and acceleration $\ddot{x}$.

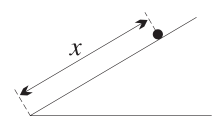
(c) When is the particle: (i) at the origin, (ii) stationary?

(d) What is the maximum distance from the origin, and when does it occur: (i) during the first 2 seconds, (ii) during the first 6 seconds, (iii) during the first 10 seconds?

(e) What is the particle's average velocity during the first 7 seconds? When and where is its instantaneous velocity equal to this average?

(f) How far does it travel during the first 7 seconds, and what is its average speed?

11. A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.



(a) Find the functions for velocity v and acceleration a.

(b) Sketch the graphs of displacement x and velocity v.

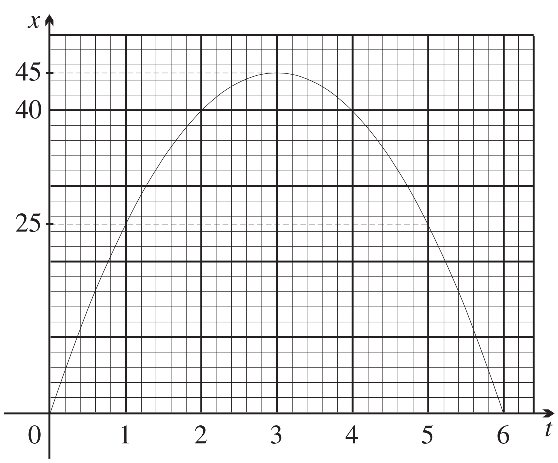
(c) In which direction is the ice moving, and in which direction is it accelerating: (i) when $t = 2$? (ii) when $t = 4$?

(d) When is the ice stationary, for how long is it stationary, where is it then, and is it accelerating then?

(e) Show that the average velocity over the first 2 seconds is 4 m/s. Then find the time and place at which the instantaneous velocity equals this average velocity.

(f) Show that the average speed during the first 3 seconds, the next 3 seconds and the first 6 seconds are all the same.

12. A stone was thrown vertically upwards. The graph to the right shows its height x metres at time t seconds after it was thrown.



(a) What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?

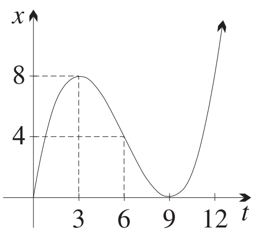
(b) Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0$, 1, 2, 3, 4, 5 and 6.

(c) For what length of time was the stone stationary at the top of its flight?

(d) The graph is concave down everywhere. How is this relevant to the motion?

(e) Draw a graph of the instantaneous velocity of the stone from $t = 0$ to $t = 6$. What does this velocity–time graph tell you about what happened to the velocity during these 6 seconds?

13. A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.



(a) In the first 10 seconds, what is its maximum distance from the origin and when does it occur?

(b) By examining the gradient, find when the particle is:
(i) stationary, (ii) moving to the right, (iii) moving to the left.

(c) When does it return to the origin, what is its velocity then, and in which direction is it accelerating?

(d) When is its acceleration zero, where is it then, and in what direction is it moving?

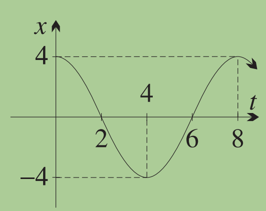
(e) By examining the concavity, find the time interval during which the particle's acceleration is negative.

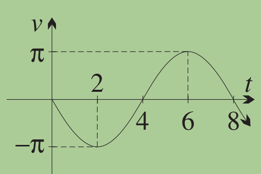
(f) At about what times are: (i) the displacement, (ii) the velocity, about the same as those at $t = 2$?

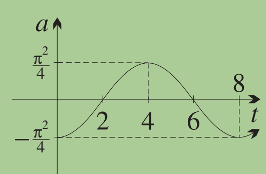
(g) Sketch (roughly) the graphs of velocity v and acceleration a.

CHALLENGE

14. A particle is moving according to $x = 4\cos\frac{\pi}{4}t$, where the units are metres and seconds. The displacement, velocity and acceleration graphs are drawn below, for $0 \le t \le 8$.







(a) Differentiate to find the functions for velocity v and the acceleration a.

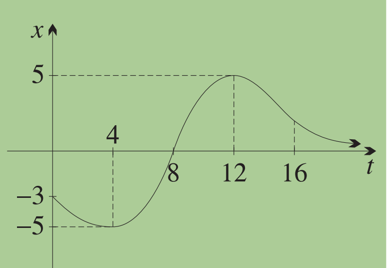
(b) What are the particle's maximum displacement, velocity and acceleration, and when, during the first 8 seconds, do they occur?

(c) How far does it travel during the first 20 seconds, and what is its average speed?

(d) Show by substitution that $x = 2$ when $t = 1\frac{1}{3}$ and when $t = 6\frac{2}{3}$. Hence use the graph to find when $x < 2$ during the first 8 seconds.

(e) When, during the first 8 seconds, is: (i) $v = 0$, (ii) $v > 0$?

15. A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



(a) At what times is this particle:
(i) below the origin,
(ii) moving downwards,
(iii) accelerating downwards?

(b) At about what time is its speed greatest?

(c) At about what times are: (i) the distance from the origin, (ii) the velocity, about the same as those at $t = 3$?

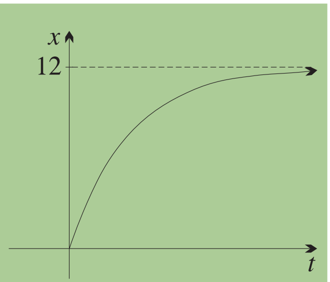
(d) How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?

(e) How far will the particle eventually travel?

(f) Draw an approximate sketch of the graph of v as a function of time.

16. A large stone is falling through a layer of mud. Its depth x metres below ground level at time t minutes is

$$x = 12 - 12e^{-0\cdot5t}.$$



Its displacement–time graph is drawn to the right.

(a) Show that the velocity and acceleration functions are

$$\dot{x} = 6e^{-0\cdot5t} \quad \text{and} \quad \ddot{x} = -3e^{-0\cdot5t}.$$

(b) In which direction is the stone always:

(i) travelling,

(ii) accelerating?

(c) What happens to the position, velocity and acceleration of the particle as $t \to \infty$?

(d) Show that the stone is halfway between the origin and its final position at the time when $e^{-0\cdot5t} = \frac{1}{2}$, and solve this equation for t. Show that its speed is then half its initial speed, and its acceleration is half its initial acceleration.

(e) Use your calculator to show that the stone is within 1 mm of its final position after 19 minutes, but not after 18 minutes.

5 C Integrating with Respect to Time

The inverse process of differentiation is integration. Thus if the acceleration function is known, integration will generate the velocity function. In the same way, if the velocity function is known, integration will generate the displacement function.

Using Initial Conditions: Taking the primitive of a function always involves a constant of integration. Determining such a constant requires an *initial condition* to be known. For example, the problem may tell us the velocity when $t = 0$, or give us the displacement when $t = 3$. *In this chapter, the constants of integration cannot be omitted.*

8 INTEGRATING WITH RESPECT TO TIME:

- Given the acceleration function a, integrate to find the velocity function v.
- Given the velocity function v, integrate to find the displacement function x.
- An initial condition is needed to evaluate each constant of integration.

In the first worked exercise below, the velocity function is given. Integration, using the initial condition, gives the displacement function. Then differentiation gives the acceleration function.

WORKED EXERCISE:

A particle is moving so that its velocity t seconds after time zero is $v = 2t - 2$ m/s. Initially it is at $x = 1$.

(a) Integrate, substituting the initial condition, to show that $x = (t-1)^2$.

(b) Find when the particle is at the origin and its velocity then.

(c) Explain why the particle is never on the negative side of the origin.

(d) Differentiate to find the acceleration, and show that it is constant.

SOLUTION:

(a) The given velocity function is $v = 2t - 2$. (1)

Integrating, $x = t^2 - 2t + C$, for some constant C.

When $t = 0$, $x = 1$, so $1 = 0 - 0 + C$,

so $C = 1$ and $x = t^2 - 2t + 1$

$x = (t-1)^2$. (2)

(b) Put $x = 0$.

Then from (2), $(t-1)^2 = 0$

$t = 1$.

Hence the particle is at the origin when $t = 1$,

and substituting $t = 1$ into (1), $v = 2 - 2 = 0\,\text{m/s}$.

(c) Since $x = (t-1)^2$ is a square, the value of x can never be negative, so the particle is never on the negative side of the origin.

(d) Differentiating the velocity function $v = 2t - 2$

gives $a = 2$, (3)

so the acceleration is a constant $2\,\text{m/s}^2$.

WORKED EXERCISE:

A particle's acceleration function is $a = 24t$. Initially it is at the origin, moving with velocity $-12\,\text{cm/s}$.

(a) Integrate, substituting the initial condition, to find the velocity function.

(b) Integrate again to find the displacement function.

(c) Find when the particle is stationary and find the displacement then.

(d) Find when the particle returns to the origin and the acceleration then.

SOLUTION:

(a) The given acceleration function is $a = 24t$. (1)

Integrating, $v = 12t^2 + C$, for some constant C.

When $t = 0$, $v = -12$, so $-12 = 0 + C$,

so $C = -12$ and $v = 12t^2 - 12$. (2)

(b) Integrating again, $x = 4t^3 - 12t + D$, for some constant D.

When $t = 0$, $x = 0$, so $0 = 0 - 0 + D$,

so $D = 0$ and $x = 4t^3 - 12t$. (3)

(c) Put $v = 0$. Then from (2), $12t^2 - 12 = 0$

$t^2 = 1$

$t = 1$. (Remember that $t \geq 0$.)

Hence the particle is stationary after 1 second.

When $t = 1$, $x = -8$, so at this time its displacement is $x = -8\,\text{cm}$.

(d) Put $x = 0$. Then using (3), $4t^3 - 12t = 0$

$4t(t^2 - 3) = 0$,

so $t = 0$ or $t = \sqrt{3}$. (Again, $t \geq 0$.)

Hence the particle returns to the origin after $\sqrt{3}$ seconds,

and at this time, $a = 24\sqrt{3}\,\text{cm/s}^2$.

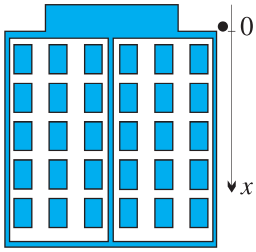
The Acceleration Due to Gravity: Since the time of Galileo, it has been known that on the surface of the Earth, a body that is free to fall accelerates downwards at a constant rate, whatever its mass and whatever its velocity, provided that air resistance is ignored. This acceleration is called the *acceleration due to gravity* and is conventionally given the symbol g. The value of this acceleration is about $9{\cdot}8\,\text{m/s}^2$, or in rounder figures, $10\,\text{m/s}^2$.

The acceleration is downwards. Thus if upwards is taken as positive, the acceleration is $-g$, but if downwards is taken as positive, the acceleration is g.

9 THE ACCELERATION DUE TO GRAVITY:

- A body that is falling accelerates downwards at a constant rate $g \doteqdot 9{\cdot}8\,\text{m/s}^2$, provided that air resistance is ignored.
- If upwards is taken as positive, start with the function $a = -g$ and integrate.
- If downwards is taken as positive, start with the function $a = g$ and integrate.

WORKED EXERCISE:
A stone is dropped from the top of a high building. How far has it travelled, and how fast is it going, after 5 seconds? Take $g = 9{\cdot}8\,\text{m/s}^2$.



SOLUTION:
Let x be the distance travelled t seconds after the stone is dropped. This puts the origin of space at the top of the building and the origin of time at the instant when the stone is dropped and makes downwards positive.

Then $\quad a = 9{\cdot}8$ (given). (1)

Integrating, $\quad v = 9{\cdot}8t + C$, for some constant C.
Since the stone was dropped, its initial speed was zero,
and substituting, $\quad 0 = 0 + C$,
so $C = 0$, and $\quad v = 9{\cdot}8t$. (2)

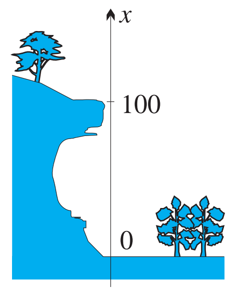
Integrating again, $x = 4{\cdot}9t^2 + D$, for some constant D.
Since the initial displacement of the stone was zero,
$\quad 0 = 0 + D$,
so $D = 0$, and $\quad x = 4{\cdot}9t^2$. (3)

When $t = 5$, $\quad v = 49$ (substituting into (2) above)
and $\quad x = 122{\cdot}5$ (substituting into (3) above).
Hence the stone has fallen $122{\cdot}5$ metres and is moving downwards at $49\,\text{m/s}$.

Making a Convenient Choice of the Origin and the Positive Direction: Physical problems do not come with origins and directions attached. Thus it is up to us to choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible.

The previous worked exercise made reasonable choices, but the following worked exercise makes quite different choices. In all such problems, the physical interpretation of negatives and displacements is the mathematician's responsibility, and the final answer should be given in ordinary language.

WORKED EXERCISE:
A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of 40 m/s and it falls back past the lookout onto the valley floor below.



How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10\,\text{m/s}^2$.)

SOLUTION:
Let x be the distance above the valley floor t seconds after the stone is thrown. This puts the origin of space at the valley floor and the origin of time at the instant when the stone is thrown. It also makes upwards positive, so that $a = -10$, because the acceleration is downwards.

As discussed, $a = -10$. (1)

Integrating, $v = -10t + C$, for some constant C.
Since $v = 40$ when $t = 0$, $40 = 0 + C$,
so $C = 40$, and $v = -10t + 40$. (2)

Integrating again, $x = -5t^2 + 40t + D$, for some constant D.
Since $x = 100$ when $t = 0$, $100 = 0 + 0 + D$,
so $D = 100$, and $x = -5t^2 + 40t + 100$. (3)

The stone hits the ground when $x = 0$, that is, using (3) above,
$$-5t^2 + 40t + 100 = 0$$
$$t^2 - 8t - 20 = 0$$
$$(t-10)(t+2) = 0$$
$$t = 10 \text{ or } -2.$$
Since the ball was not in flight at $t = -2$, the ball hits the ground after 10 seconds.
Substituting $t = 10$ into equation (2), $v = -100 + 40 = -60$,
so the ball hits the ground at 60 m/s.

Formulae from Physics Cannot be Used: This course requires that even problems where the acceleration is constant, such as the two above, must be solved by integrating the acceleration function. Many readers will know of three very useful equations for motion with constant acceleration a:
$$v = u + at \qquad \text{and} \qquad s = ut + \tfrac{1}{2}at^2 \qquad \text{and} \qquad v^2 = u^2 + 2as.$$
These equations automate the integration process, and so cannot be used in this course. A question in Exercise 5C develops a proper proof of these results.

Integrating Trigonometric Functions: The following worked exercise applies the same methods of integration to motion involving trigonometric functions.

WORKED EXERCISE:
The velocity of a particle initially at the origin is $v = \sin \frac{1}{4}t$, in units of metres and seconds.
(a) Find the displacement function. (b) Find the acceleration function.
(c) Find the values of displacement, velocity and acceleration when $t = 4\pi$.
(d) Briefly describe the motion, and sketch the displacement–time graph.

SOLUTION:

(a) The velocity is $v = \sin \frac{1}{4}t$. (1)

Integrating (1), $x = -4\cos\frac{1}{4}t + C$, for some constant C.

Substituting $x = 0$ when $t = 0$:

$$0 = -4 \times 1 + C,$$
$$C = 4.$$

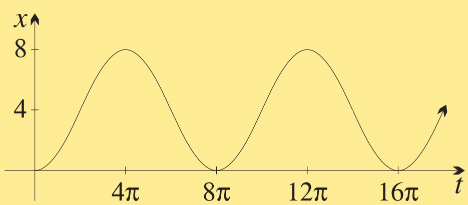
Thus $x = 4 - 4\cos\frac{1}{4}t$. (2)

(b) Differentiating (1), $a = \frac{1}{4}\cos\frac{1}{4}t$. (3)

(c) When $t = 4\pi$, $x = 4 - 4 \times \cos\pi$, using (2),
$= 8$ metres.

Also $v = \sin\pi$, using (1),
$= 0\,\text{m/s}$,

and $a = \frac{1}{4}\cos\pi$, using (3),
$= -\frac{1}{4}\,\text{m/s}^2$.



(d) The particle oscillates between $x = 0$ and $x = 8$ with period 8π seconds.

Integrating Exponential Functions: The following worked exercise involves exponential functions. The velocity function approaches a limit 'as time goes on'.

WORKED EXERCISE:

The acceleration of a particle is given by $a = e^{-2t}$ (in units of metres and seconds), and the particle is initially stationary at the origin.

(a) Find the velocity and displacement functions.

(b) Find the displacement when $t = 10$.

(c) Sketch the velocity–time graph and describe briefly what happens to the velocity of the particle as time goes on.

SOLUTION:

(a) The acceleration is $a = e^{-2t}$. (1)

Integrating, $v = -\frac{1}{2}e^{-2t} + C$.

It is given that when $t = 0$, $v = 0$,

so $0 = -\frac{1}{2} + C$

$C = \frac{1}{2}$,

and $v = -\frac{1}{2}e^{-2t} + \frac{1}{2}$. (2)

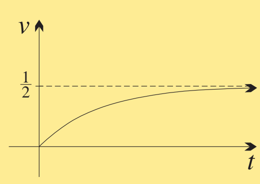
Integrating again, $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t + D$.

It is given that when $t = 0$, $x = 0$,

so $0 = \frac{1}{4} + D$

$D = -\frac{1}{4}$,

and $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$. (3)



(b) When $t = 10$, $x = \frac{1}{4}e^{-20} + 5 - \frac{1}{4}$
$= 4\frac{3}{4} + \frac{1}{4}e^{-20}$ metres.

(c) Using equation (2), the velocity is initially zero, and increases so that the limiting velocity as time goes on is $\frac{1}{2}$ m/s.

Exercise 5C

1. A particle is moving with velocity function $v = 3t^2 - 6t$, in units of metres and seconds. At time $t = 0$, its displacement is $x = 4$.

(a) Integrate, substituting the initial condition, to find the displacement function.

(b) Show that the particle is at the origin when $t = 2$ and find its velocity then.

(c) Differentiate the given velocity function to find the acceleration function.

(d) Show that the acceleration is zero when $t = 1$ and find the displacement then.

2. A particle is moving with acceleration $a = -6t$, in units of centimetres and seconds. Initially it is at rest at $x = 8$.

(a) Integrate, substituting the initial condition, to find the velocity function.

(b) Find the particle's velocity and speed when $t = 5$.

(c) Integrate again to find the displacement function.

(d) Show that the particle is at the origin when $t = 2$ and find its acceleration then.

3. A particle is moving with acceleration function $a = 8$. Two seconds after time zero, it is stationary at the origin.

(a) Integrate to find the velocity function.

(b) Integrate again to find the displacement function.

(c) Where was the particle initially, and what were its velocity and speed?

4. The velocity of a particle is the constant function $v = 6\,\text{m/s}$. At time $t = 0$, the particle is at $x = -30$.

(a) Integrate to find the displacement function.

(b) By solving $x = 0$, find how long it takes the particle to reach the origin.

(c) What is the acceleration function of the particle?

5. A particle is moving with acceleration function $a = 2$, in units of metres and seconds. Initially, it is at the origin, moving with velocity $-20\,\text{m/s}$.

(a) Find the velocity function.

(b) Find the displacement function.

(c) By solving $v = 0$, find when the particle is stationary and find where it is then.

(d) By solving $x = 0$, find when it returns to the origin, and show that its speed then is equal to its initial speed.

6. A stone is dropped from a lookout 80 metres high. Take $g = 10\,\text{m/s}^2$ and downwards as positive, so that the acceleration function is $a = 10$.

(a) Using the lookout as the origin, find the velocity and displacement as functions of t. [HINT: When $t = 0$, $v = 0$ and $x = 0$.]

(b) Show that the stone takes 4 seconds to fall, and find its impact speed.

(c) Where is it, and what is its speed, halfway through its flight time?

(d) Show that it takes $2\sqrt{2}$ seconds to go halfway down, and find its speed then.

7. A stone is thrown downwards from the top of a 120-metre building, with an initial speed of $25\,\text{m/s}$. Take $g = 10\,\text{m/s}^2$ and take upwards as positive, so that $a = -10$.

(a) Using the ground as the origin, find the acceleration, velocity and height x of the stone t seconds after it is thrown. [HINT: When $t = 0$, $v = -25$ and $x = 120$.]

(b) By solving $x = 0$, find the time it takes to reach the ground.

(c) Find the impact speed.

(d) What is the average speed of the stone during its descent?

DEVELOPMENT

8. Find the velocity function $\dot{x}$ and the displacement function x of a particle whose initial velocity and displacement are zero if:

(a) $\ddot{x} = -4$ (b) $\ddot{x} = 6t$ (c) $\ddot{x} = e^{\frac{1}{2}t}$ (d) $\ddot{x} = e^{-3t}$ (e) $\ddot{x} = 8\sin 2t$ (f) $\ddot{x} = \cos \pi t$ (g) $\ddot{x} = \sqrt{t}$ (h) $\ddot{x} = 12(t+1)^{-2}$

9. Find the acceleration function a and the displacement function x of a particle whose initial displacement is -2 if:

(a) $v = -4$ (b) $v = 6t$ (c) $v = e^{\frac{1}{2}t}$ (d) $v = e^{-3t}$ (e) $v = 8\sin 2t$ (f) $v = \cos \pi t$ (g) $v = \sqrt{t}$ (h) $v = 12(t+1)^{-2}$

10. A particle is moving with acceleration $\ddot{x} = 12t$. Initially, it has velocity -24 m/s and is 20 metres on the positive side of the origin.

(a) Find the velocity function $\dot{x}$ and the displacement function $\ddot{x}$.

(b) When does the particle return to its initial position, and what is its speed then?

(c) What is the minimum displacement, and when does it occur?

(d) Find x when $t = 0$, 1, 2, 3 and 4, and sketch the displacement–time graph.

11. A body is moving with its acceleration proportional to the time elapsed. That is,

$$a = kt,$$

where k is a constant of proportionality. When $t = 1$, $v = -6$, and when $t = 2$, $v = 3$.

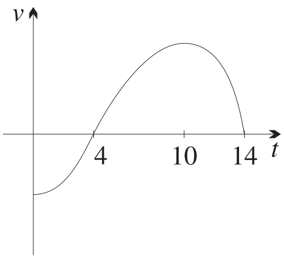
(a) Integrate the given acceleration function, adding the constant C of integration. Then substitute the two given conditions to find the values of k and C.

(b) Suppose now that the particle is initially at $x = 2$.

(i) Integrate again to find the displacement function.

(ii) When does the body return to its original position?

12. The graph to the right shows a particle's velocity–time graph.



(a) When is the particle moving forwards?

(b) When is the acceleration positive?

(c) When is it furthest from its starting point?

(d) When is it furthest in the negative direction?

(e) About when does it return to its starting point?

(f) Sketch the graphs of acceleration and displacement, assuming that the particle is initially at the origin.

13. A car moves along a straight road from its front gate, where it is initially stationary. During the first 10 seconds, it has a constant acceleration of $2\ \text{m/s}^2$, it has zero acceleration during the next 30 seconds, and it decelerates at $1\ \text{m/s}^2$ for the final 20 seconds until it stops.

(a) What is the car's speed after 20 seconds?

(b) Show that the car travels:

(i) 100 metres during the first 10 seconds,

(ii) 600 metres during the next 30 seconds,

(iii) 200 metres during the last 20 seconds.

(c) Sketch the graphs of acceleration, velocity and distance from the gate.

14. A particle is moving with velocity $\dot{x} = 16 - 4t$ cm/s on a horizontal number line.

(a) Find $\ddot{x}$ and x. (The function x will have a constant of integration.)

(b) When does it return to its original position, and what is its speed then?

(c) When is the particle stationary? Find the maximum distances right and left of the initial position during the first 10 seconds, and the corresponding times and accelerations.

(d) How far does it travel in the first 10 seconds, and what is its average speed?

CHALLENGE

15. A particle moves from $x = -1$ with velocity $v = \dfrac{1}{t+1}$.

(a) Find its displacement and acceleration functions.

(b) Find how long it takes to reach the origin, and its speed and acceleration then.

(c) Describe its subsequent motion.

16. A body moving vertically through air experiences an acceleration $\ddot{x} = -40e^{-2t}$ m/s^2 (we are taking upwards as positive). Initially, it is thrown upwards with speed 15 m/s.

(a) Taking the origin at the point where it is thrown, find the velocity function $\dot{x}$ and the displacement function x, and find when the body is stationary.

(b) Find its maximum height and its acceleration then.

(c) Describe the velocity of the body as $t \to \infty$.

17. A moving particle is subject to an acceleration of $a = -2\cos t$ m/s^2. Initially it is at $x = 2$, moving with velocity 1 m/s, and it travels for 2π seconds.

(a) Find the velocity function v and the displacement function x.

(b) When is the acceleration positive?

(c) When and where is the particle stationary?

(d) What are the maximum and minimum velocities, and when do they occur?

18. [A proof of three constant-acceleration formulae from physics — not to be used elsewhere.] A particle moves with constant acceleration a. Its initial velocity is u, and at time t it is moving with velocity v and its distance from its initial position is s. Show that:

(a) $v = u + at$ (b) $s = ut + \frac{1}{2}at^2$ (c) $v^2 = u^2 + 2as$

5D Chapter Review Exercise

1. For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity from $t = 2$ to $t = 4$. The units in each part are centimetres and seconds.

t	2	4
x		

(a) $x = 20 + t^2$

(b) $x = (t+2)^2$

(c) $x = t^2 - 6t$

(d) $x = 3^t$

2. For each displacement function below, find the velocity function and the acceleration function. Then find the displacement, the velocity and the acceleration of the particle when $t = 5$. All units are metres and seconds.

(a) $x = 40t - t^2$

(b) $x = t^3 - 25t$

(c) $x = 4(t-3)^2$

(d) $x = 50 - t^4$

(e) $x = 4\sin \pi t$

(f) $x = 7\,e^{3t-15}$

3. A ball rolls up an inclined plane and back down again. Its distance x metres up the plane after t seconds is given by $x = 16t - t^2$.

(a) Find the velocity function v and the acceleration function a.

(b) What are the ball's position, velocity, speed and acceleration after 10 seconds?

(c) When does the ball return to its starting point, and what is its velocity then?

(d) When is the ball farthest up the plane, and where is it then?

(e) Sketch the displacement–time graph, the velocity–time graph and the acceleration–time graph.

4. Differentiate each velocity function below to find the acceleration function a. Then integrate v to find the displacement function x, given that the particle is initially at $x = 4$.

(a) $v = 7$ (c) $v = (t-1)^2$ (e) $v = 12\cos 2t$

(b) $v = 4 - 9t^2$ (d) $v = 0$ (f) $v = 12\,e^{-3t}$

5. For each acceleration function below, find the velocity function v and the displacement function x, given that the particle is initially stationary at $x = 2$.

(a) $a = 6t + 2$ (c) $a = 36t^2 - 4$ (e) $a = 5\cos t$

(b) $a = -8$ (d) $a = 0$ (f) $a = 7\,e^t$

6. A particle is moving with acceleration function $\ddot{x} = 6t$, in units of centimetres and seconds. Initially it is at the origin and has velocity $12\,\text{m/s}$ in the negative direction.

(a) Find the velocity function $\dot{x}$ and the displacement function x.

(b) Show that the particle is stationary when $t = 2$.

(c) Hence find its maximum distance on the negative side of the origin.

(d) When does it return to the origin, and what are its velocity and acceleration then?

(e) What happens to the particle's position and velocity as time goes on?

7. A stone is thrown vertically upwards with velocity $40\,\text{m/s}$ from a fixed point B situated 45 metres above the ground. Take $g = 10\,\text{m/s}^2$.

(a) Taking upwards as positive, explain why the acceleration function is $a = -10$.

(b) Using the ground as the origin, find the velocity function v and the displacement function x.

(c) Hence find how long the stone takes to reach its maximum height, and find that maximum height.

(d) Show that the time of flight of the stone until it strikes the ground is 9 seconds.

(e) With what speed does the stone strike the ground?

(f) Find the height of the stone after 1 second and after 2 seconds.

(g) Hence find the average velocity of the stone during the 2nd second.

8. The acceleration of a body moving along a line is given by $\ddot{x} = \sin t$, where x is the distance from the origin O at time t seconds.

(a) Sketch the acceleration–time graph.

(b) From your graph, state the first two times after $t = 0$ when the acceleration is zero.

(c) Integrate to find the velocity function, given that the initial velocity is -1 m/s.

(d) What is the first time when the body stops moving?

(e) The body is initially at $x = 5$.

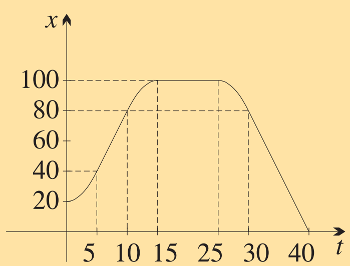
(i) Find the displacement function x.

(ii) Find where the body is when $t = \frac{\pi}{2}$.

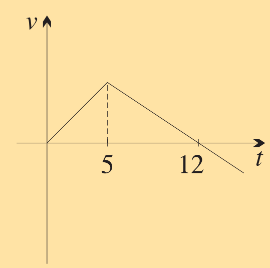
9. The velocity of a particle is given by $v = 20\,e^{-t}$, in units of metres and seconds.
(a) What is the velocity when $t = 0$?
(b) Why is the particle always moving in a positive direction?
(c) Find the acceleration function a.
(d) What is the acceleration at time $t = 0$?
(e) The particle is initially at the origin. Find the displacement function x.
(f) What happens to the acceleration, the velocity and the displacement as t increases?

10. The stud farm at Benromach sold a prize bull to a grazier at Dalmore, 300 kilometres west. The truck delivering the bull left Benromach at 9:00 am, driving over the dirt roads at a constant speed of 50 km/hr. At 10:00 am, the driver realised that he had left the sale documents behind, so he drove back to Benromach at the same speed. He then drove the bull and the documents straight to Dalmore at 60 km/hr.
(a) Draw the displacement–time graph of his displacement x kilometres west of Benromach at time t hours after 9:00 am.
(b) What total distance did he travel?
(c) What was his average road speed for the whole journey?

11. Crispin was trying out his bicycle in Abigail Street. The graph below shows his displacement in metres north of the oak tree after t seconds.
(a) Where did he start from, and what was his initial speed?
(b) What his velocity:
(i) from $t = 5$ to $t = 10$,
(ii) from $t = 15$ to $t = 25$,
(iii) from $t = 30$ to $t = 40$?
(c) In what direction was he accelerating:
(i) from $t = 0$ to $t = 5$,
(ii) from $t = 10$ to $t = 15$,
(iii) from $t = 25$ to $t = 30$?
(d) Draw a possible sketch of the velocity–time graph.



12. A small rocket was launched vertically from the ground. The graph to the right shows its velocity–time graph. After a few seconds the motor cut out. A few seconds later the rocket reached its maximum height and then began to fall back towards the ground.
(a) When did the motor cut out?
(b) When was the rocket stationary, when was it moving upwards and when was it moving downwards?
(c) When was the rocket accelerating upwards, and when was it accelerating downwards?
(d) When was the rocket at its maximum height?
(e) Sketch the acceleration–time graph.
(f) Sketch the displacement–time graph.



CHAPTER **SIX**

Rates and Finance

This chapter covers several topics that could seem unrelated, but are in fact closely linked in various ways. First, exponential functions are the mathematical basis of natural growth, compound interest, geometric sequences and housing loans. Secondly, the rate of change of a quantity can be described either by sequences or by continuous functions, depending on the situation. Thirdly, many of the applications throughout the chapter are financial.

Spreadsheets are useful in this chapter in providing experience of how superannuation funds and housing loans behave over time, and computer programs may be helpful in modelling rates of change of some quantities. The intention of the course, however, is to establish the relationships between these phenomena and the known theories of sequences, exponential functions and calculus.

6 A Applications of APs and GPs

Arithmetic and geometric sequences were studied in the Year 11 volume. This section will review the main results about APs and GPs, and apply them to problems, many of them financial, in preparation for later sections.

Formulae for Arithmetic Sequences: Here are the essential definitions and formulae that are needed when working with APs.

1

ARITHMETIC SEQUENCES:

- A sequence T_n is called an *arithmetic sequence* if

$$T_n - T_{n-1} = d, \quad \text{for } n \geq 2,$$

where d is a constant, called the *common difference*.

- The nth term of an AP with first term a and common difference d is

$$T_n = a + (n-1)d.$$

- Three numbers a, x and b are in AP if

$$b - x = x - a, \qquad \text{that is,} \qquad x = \frac{a+b}{2}.$$

- The sum S_n of the first n terms of an AP is

$$S_n = \tfrac{1}{2}n(a + \ell) \qquad \text{(use when the last term } \ell = T_n \text{ is known)},$$

$$\text{or } S_n = \tfrac{1}{2}n\Big(2a + (n-1)d\Big) \qquad \text{(use when the difference } d \text{ is known)}.$$

WORKED EXERCISE: [Salaries and APs]
Georgia earned \$25 000 in her first year at Information Holdings, and her salary then increased every year by \$3000. She worked at the company for 12 years.

(a) What was her annual salary in her final year?

(b) What were her total earnings over the 12 years?

SOLUTION:
Her annual salaries form a series, $25\,000 + 28\,000 + 31\,000 + \cdots$, with 12 terms.
This is an AP with $a = 25\,000$, $d = 3000$ and $n = 12$.

(a) Her final salary is the twelfth term T_{12} of the series.

$$\begin{aligned}\text{Final salary} &= a + 11d \qquad \text{(This is the formula for } T_{12}.\text{)}\\ &= 25\,000 + 33\,000\\ &= \$58\,000.\end{aligned}$$

(b) Her total earnings are the sum S_{12} of the first twelve terms of the series.
Because a and d are known, we use the second of the two formulae for S_n.

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2} \times 12 \times (2a + 11d)\\ &= 6 \times (50\,000 + 33\,000)\\ &= \$498\,000.\end{aligned}$$

WORKED EXERCISE:
The Roxanne Cinema has a concession for groups. It charges \$12 for the first ticket and then \$8 for each additional ticket.

(a) How much would 20 tickets cost?

(b) Find a formula for the cost of n tickets.

(c) How many people are in a group whose tickets cost \$300?

SOLUTION:
The costs of $1,\ 2,\ 3,\ \ldots$ tickets form the sequence $\$12,\ \$20,\ \$28,\ \ldots$.
This is an AP with $a = 12$ and $d = 8$.

(a) The cost of 20 tickets is the 20th term T_{20} of the sequence.

$$\begin{aligned}\text{Cost of 20 tickets} &= a + 19d \qquad \text{(This is the formula for } T_{20}.\text{)}\\ &= 12 + 19 \times 8\\ &= \$164.\end{aligned}$$

(b)
$$\begin{aligned}\text{Cost of } n \text{ tickets} &= a + (n-1)d \qquad \text{(This is the formula for } T_n.\text{)}\\ &= 12 + (n-1)8\\ &= 12 + 8n - 8\\ &= 8n + 4 \text{ dollars.}\end{aligned}$$

(c) Put cost of n tickets $= 300$ dollars.

$$\begin{aligned}\text{Then} \qquad 8n + 4 &= 300 \qquad \text{(The formula for } n \text{ tickets was found in part (b).)}\\ 8n &= 296\\ n &= 37 \text{ tickets.}\end{aligned}$$

Counting When the Years are Named: Problems in which events happen in particular named years are notoriously tricky. The following problem becomes clearer when the years are stated in terms of 'years after 1990'.

WORKED EXERCISE:

Gulgarindi Council is very happy. It had 2870 complaints in 1991, but only 2170 in 2001. The number of complaints decreased by the same amount each year.

(a) What was the total number of complaints during these years?

(b) By how much did the number of complaints decrease each year?

(c) How many complaints were there in 1993?

(d) Find a formula for the number of complaints in the nth year.

(e) If the trend continued, in what year would there be no complaints at all?

SOLUTION:

The first year is 1991, the second year is 1992 and the 11th year is 2001.
In general, the nth year of the problem is the nth year after 1990.
The successive numbers of complaints form an AP with $a = 2870$, $\ell = 2170$ and $n = 11$.

(a) The total number of complaints is the sum S_{11} of the first 11 terms of the series.

$$\begin{aligned} \text{Total number of complaints} &= \tfrac{1}{2} \times 11 \times (a + \ell) \qquad \text{(This is the first formula for } S_n.) \\ &= \tfrac{1}{2} \times 11 \times (2870 + 2170) \\ &= \tfrac{1}{2} \times 11 \times 5040 \\ &= 27\,720. \end{aligned}$$

(b) This question is asking for the common difference d, which is negative here.

$$\begin{aligned} \text{Put} \qquad T_{11} &= 2170 \\ a + 10d &= 2170 \qquad \text{(This is the formula for } T_{11}.) \\ 2870 + 10d &= 2170 \\ 10d &= -700 \\ d &= -70. \end{aligned}$$

Hence the number of complaints decreased by 70 each year.

(c) The year 1993 is the third year, so we find the third term T_3 of the series.

$$\begin{aligned} \text{Number of complaints in 1993} &= a + 2d \qquad \text{(This is the formula for } T_3.) \\ &= 2870 + 2 \times (-70) \\ &= 2730. \end{aligned}$$

(d) The number of complaints in the nth year is the nth term T_n of the series.

$$\begin{aligned} \text{Number of complaints} &= a + (n - 1)d \qquad \text{(This is the formula for } T_n.) \\ &= 2870 - 70(n - 1) \\ &= 2870 - 70n + 70 \\ &= 2940 - 70n. \end{aligned}$$

(e) To find the year in which there are no complaints at all,

$$\begin{aligned} \text{put} \qquad T_n &= 0. \\ \text{Then } 2940 - 70n &= 0 \qquad \text{(This formula was found in part (d).)} \\ 70n &= 2940 \\ n &= 42. \end{aligned}$$

Thus there would be no complaints at all in the year $1990 + 42 = 2032$.

Formulae for Geometric Sequences: The formulae for GPs correspond roughly to the formulae for APs, except that the formula for the limiting sum of a GP has no analogy for arithmetic sequences.

GEOMETRIC SEQUENCES:

- A sequence T_n is called a *geometric sequence* if
$$\frac{T_n}{T_{n-1}} = r, \quad \text{for } n \ge 2,$$
where r is a constant, called the *common ratio*.
- The nth term of a GP with first term a and common ratio r is
$$T_n = a\,r^{n-1}.$$
- Three numbers a, x and b are in GP if

2
$$\frac{b}{x} = \frac{x}{a}, \qquad \text{that is,} \qquad x = \sqrt{ab} \text{ or } -\sqrt{ab}\,.$$
- The sum S_n of the first n terms of a GP is
$$S_n = \frac{a(r^n - 1)}{r-1} \qquad \text{(easier when } r > 1\text{)},$$
$$\text{or } S_n = \frac{a(1 - r^n)}{1-r} \qquad \text{(easier when } r < 1\text{)}.$$
- The limiting sum S_∞ exists if and only if $-1 < r < 1$, in which case
$$S_\infty = \frac{a}{1-r}\,.$$

WORKED EXERCISE: [An example with $r > 1$]
The town of Elgin grew quite fast in the eight years after the new distillery was opened. In the first year afterwards, there were just 15 car accidents, but over these eight years, the number of accidents doubled every year.

(a) Find the number of accidents in the eighth year after the distillery opened.

(b) Find the total number of accidents over these eight years.

(c) What percentage of the total accidents occurred in the final year?

SOLUTION:

The numbers of accidents per year form the sequence 15, 30, 60, ...
This is a GP with $a = 15$ and $r = 2$.

(a) The number of accidents during the eighth year is the eighth term T_8.

$$\begin{aligned} \text{Hence number of accidents} &= a\,r^7 \qquad \text{(This is the formula for } T_8.\text{)} \\ &= 15 \times 2^7 \\ &= 15 \times 128 \\ &= 1920. \end{aligned}$$

(b) The total accidents during these eight years is the sum S_8 of the first eight terms.

$$\text{Hence number of accidents} = \frac{a(r^8-1)}{r-1} \qquad \text{(Since } r>1\text{, use the first formula for } S_8.)$$
$$= \frac{15 \times (2^8-1)}{2-1}$$
$$= 15 \times 255$$
$$= 3825.$$

(c) $$\frac{\text{Accidents in the final year}}{\text{Total accidents}} = \frac{1920}{3825}$$
$$\doteqdot 50{\cdot}20\% \qquad \text{(correct to the nearest } 0{\cdot}01\%\text{).}$$

WORKED EXERCISE: [An example with $r < 1$]
Sales from the Gumnut Softdrinks Factory in the mountain town of Wadelbri are declining by 6% every year. In 2001, 50 000 bottles were sold.

(a) How many bottles will be sold in 2010?

(b) How many bottles will be sold in total in the years 2001–2010?

SOLUTION:
Here 2001 is the first year, 2002 is the second year and 2010 is the 10th year.
The annual sales form a GP with $a = 50\,000$ and $r = 0{\cdot}94$.

(a) The sales in 2010 are the 10th term T_{10}, because 2001–2010 consists of 10 years.

$$\text{Sales in 2010} = a\,r^9$$
$$= 50\,000 \times 0{\cdot}94^9$$
$$\doteqdot 28\,650 \quad \text{(correct to the nearest bottle).}$$

(b) The total sales in 2001–2010 are the sum S_{10} of the first 10 terms of the series.

$$\text{Total sales} = \frac{a(1-r^{10})}{1-r} \qquad \text{(Since } r<1\text{, use the second formula for } S_8.)$$
$$= \frac{50\,000 \times (1-0{\cdot}94^{10})}{0{\cdot}06}$$
$$\doteqdot 384\,487 \quad \text{(correct to the nearest bottle).}$$

Limiting Sums: Provided that the ratio of a GP is between -1 and 1, the sum S_n of the first n terms of the GP converges to the limit $S_\infty = \dfrac{a}{1-r}$ as $n \to \infty$. In applications, this allows us to speak about the sum of the terms 'eventually', or 'as time goes on'.

WORKED EXERCISE:
Consider again the Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year and 50 000 bottles were sold in 2001. Suppose now that the company continues in business indefinitely.

(a) What would the total sales from 2001 onwards be eventually?

(b) What proportion of those sales would occur by the end of 2010?

SOLUTION:
The sales form a GP with $a = 50\,000$ and $r = 0{\cdot}94$.
Since $-1 < r < 1$, the limiting sum exists.

(a) Eventual sales $= S_\infty$

$$= \frac{a}{1-r}$$

$$= \frac{50\,000}{0{\cdot}06}$$

$$\doteqdot 833\,333 \quad \text{(correct to the nearest bottle).}$$

(b) Using the results from part (a) and from the previous worked exercise,

$$\frac{\text{Sales in 2001–2010}}{\text{Eventual sales}} = \frac{50\,000 \times (1 - 0{\cdot}94^{10})}{0{\cdot}06} \times \frac{0{\cdot}06}{50\,000} \quad \text{(Use the exact values.)}$$

$$= 1 - 0{\cdot}94^{10}$$

$$\doteqdot 46{\cdot}14\% \quad \text{(correct to the nearest } 0{\cdot}01\%\text{).}$$

WORKED EXERCISE: [A harder trigonometric application]

Consider the series $1 - \tan^2 x + \tan^4 x - \cdots$, where x is an acute angle.

(a) For what values of x does the series have a limiting sum?

(b) What is this limiting sum when it exists?

SOLUTION:

(a) The series is a GP with $a = 1$ and $r = -\tan^2 x$.
Hence the condition for the series to converge is $\tan^2 x < 1$,
that is, $-1 < \tan x < 1$.
But $\tan 45^\circ = 1$ and $\tan 0^\circ = 0$, and the angle x is acute,
so from the graph of $\tan x$, $0^\circ < x < 45^\circ$.

(b) When the series converges, $S_\infty = \dfrac{a}{1-r}$

$$= \frac{1}{1 + \tan^2 x}$$

$$= \frac{1}{\sec^2 x}$$

$$= \cos^2 x.$$

Exercise 6A

NOTE: The theory for this exercise was covered in the Year 11 volume. This exercise is therefore a medley of problems on APs and GPs, with six introductory questions to revise the formulae for APs and GPs.

1. (a) Show that 8, 15, 22, … is an arithmetic sequence.
(b) State the first term a and the common difference d.
(c) Use the formula $T_n = a + (n-1)d$ to find the 51st term T_{51}.
(d) Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum S_{20} of the first 20 terms.

2. Consider the sum $2 + 4 + 6 + \cdots + 1000$ of the first 500 even numbers.
(a) Show that this is an AP and write down the first term a and common difference d.
(b) Use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum.

3. (a) Show that 5, 10, 20, 40, ... is a geometric sequence.
(b) State the first term a and the common ratio r.
(c) Use the formula $T_n = ar^{n-1}$ to find the seventh term T_7.
(d) Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum S_7 of the first seven terms.

4. (a) Show that the sequence 96, 48, 24, ... is a GP.
(b) Write down the first term a and the common ratio r.
(c) Use the formula $T_n = ar^{n-1}$ to find the eighth term T_8.
(d) Use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$ to find the sum S_8 of the first eight terms.
(e) Give a reason why this series has a limiting sum, and use the formula $S_\infty = \dfrac{a}{1 - r}$ to find it.

5. (a) Consider the series $52 + 58 + 64 + \cdots + 130$.
(i) Show that it is an AP and write down the first term and common difference.
(ii) How many terms are there in this series?
(iii) Find the sum.
(b) In a particular arithmetic series, the first term is 15 and the fiftieth term is -10.
(i) What is the sum of all the terms?
(ii) What is the common difference?
(c) Consider the series $100 + 97 + 94 + \cdots$.
(i) Show that the series is an AP and find the common difference.
(ii) Show that the nth term is $T_n = 103 - 3n$ and find the first negative term.
(iii) Find an expression for the sum S_n of the first n terms, and show that 68 is the minimum number of terms for which S_n is negative.

6. (a) Consider the sequence 100, 101, 102·01, ...
(i) Show that it is a geometric sequence and find the common ratio.
(ii) Write down the twentieth term and evaluate it, correct to two decimal places.
(iii) Find the sum of the first 20 terms, correct to two decimal places.
(b) The first few terms of a particular series are $2000 + 3000 + 4500 + \cdots$.
(i) Show that it is a geometric series and find the common ratio.
(ii) What is the sum of the first five terms?
(iii) Explain why the series does not have a limiting sum.
(c) Consider the series $18 + 6 + 2 + \cdots$.
(i) Show that it is a geometric series and find the common ratio.
(ii) Explain why this geometric series has a limiting sum and find its value.
(iii) Find the limiting sum and the sum of the first ten terms, and show that they are approximately equal, correct to the first three decimal places.

7. A secretary starts on an annual salary of \$30 000, with annual increments of \$2000.
(a) Use the AP formulae to find his annual salary, and his total earnings, at the end of 10 years.
(b) In which year will his salary be \$42 000?

8. An accountant receives an annual salary of \$40 000, with 5% increments each year.

(a) Show that her annual salary forms a GP and find the common ratio.

(b) Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.

DEVELOPMENT

9. Lawrence and Julian start their first jobs on low wages. Lawrence starts at \$25 000 per annum, with annual increases of \$2500. Julian starts at the lower wage of \$20 000 per annum, with annual increases of 15%.

(a) Find Lawrence's annual wages in each of the first three years and explain why they form an arithmetic sequence.

(b) Find Julian's annual wages in each of the first three years and explain why they form a geometric sequence.

(c) Show that the first year in which Julian's annual wage is the greater of the two will be the sixth year, and find the difference, correct to the nearest dollar.

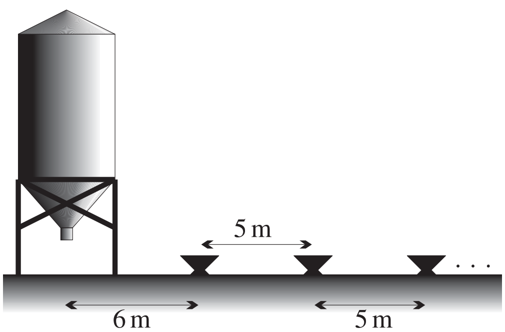
10. (a) An initial salary of \$50 000 increases by \$3000 each year.

(i) Find a formula for T_n, the salary in the nth year.

(ii) In which year will the salary first be at least twice the original salary?

(b) An initial salary of \$50 000 increases by 4% each year. What will the salary be in the tenth year, correct to the nearest dollar?

11. A farmhand is filling a row of feed troughs with grain. The distance between adjacent troughs is 5 metres, and the silo that stores the grain is 6 metres from the closest trough. He decides that he will fill the closest trough first and work his way to the far end.



(a) How far will the farmhand walk to fill the 1st trough and return to the silo? How far for the 2nd trough? How far for the 3rd trough?

(b) How far will the farmhand walk to fill the nth trough and return to the silo?

(c) If he walks a total of 62 metres to fill the furthest trough:

(i) how many feed troughs are there,

(ii) what is the total distance he will walk to fill all the troughs?

12. One Sunday, 120 days before Christmas, Franksworth store publishes an advertisement saying '120 shopping days until Christmas'. Franksworth subsequently publishes similar advertisements every Sunday until Christmas.

(a) How many times does Franksworth advertise?

(b) Find the sum of the numbers of days published in all the advertisements.

(c) On which day of the week is Christmas?

13. On a certain day at the start of a drought, 900 litres of water flowed from the Neverfail Well. The next day, only 870 litres flowed from the well, and each day, the volume of water flowing from the well was $\frac{29}{30}$ of the previous day's volume. Find the total volume of water that would have flowed from the well if the drought had continued indefinitely.

14. The number of infections in an epidemic rose from 10 000 on 1st July to 160 000 on 1st of September.

(a) If the number of infections increased by a constant difference each month, what was the number of infections on 1st August?

(b) If the number of infections increased by a constant ratio each month, what was the number of infections on 1st August?

15. Theodor earns \$30 000 in his first year of work, and his salary increases each year by a fixed amount $\$D$.

(a) Find D if his salary in his tenth year is \$58 800.

(b) Find D if his total earnings in the first ten years are \$471 000.

(c) If $D = 2200$, in which year will his salary first exceed \$60 000?

(d) If $D = 2000$, show that his total earnings first exceed \$600 000 during his 14th year.

16. A line of four cones is used in a fitness test. John starts at the first cone. He runs 20 metres to the last cone and runs back again. Then he runs 10 metres to the third cone and runs back again. Finally he runs 5 metres to the 2nd cone and runs back.

(a) Write down the distances that John travels on each run. Show that they form a GP and write down the first term and the common ratio.

(b) Suppose that more and more cones are added to continue this pattern of runs. What distance will John eventually travel?

(c) The coach asks Stewart to run the original course in reverse, which he does. Explain why Stewart does not want more and more cones to be added to continue with his pattern.

CHALLENGE

17. Margaret opens a hardware store. Sales in successive years form a GP, and sales in the fifth year are half the sales in the first year. Let sales in the first year be $\$F$.

(a) Find, in exact form, the ratio of the GP.

(b) Find the total sales of the company as time goes on, as a multiple of the first year's sales, correct to two decimal places.

18. [Limiting sums of trigonometric series]

(a) Consider the series $1 + \cos^2 x + \cos^4 x + \cdots$.

(i) Show that the series is a GP and find its common ratio.

(ii) For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?

(iii) Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \operatorname{cosec}^2 x$.

(b) Consider the series $1 + \sin^2 x + \sin^4 x + \cdots$.

(i) Show that the series is a GP and find its common ratio.

(ii) For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?

(iii) Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \sec^2 x$.

19. 

Two bulldozers are sitting in a construction site facing each other. Bulldozer A is at $x = 0$ and bulldozer B is 36 metres away at $x = 36$. A bee is sitting on the scoop at the very front of bulldozer A.

At 7:00 am the workers start up both bulldozers and start them moving towards each other at the same speed V m/s. The bee is disturbed by the commotion and flies at twice the speed of the bulldozers to land on the scoop of bulldozer B.

(a) Show that the bee reaches bulldozer B when it is at $x = 24$.

(b) Immediately the bee lands, it takes off again and flies back to bulldozer A. Where is bulldozer A when the two meet?

(c) Assume that the bulldozers keep moving towards each other and the bee keeps flying between the two, so that the bee will eventually be squashed.

(i) Where will this happen?

(ii) How far will the bee have flown?

6 B The Use of Logarithms with GPs

None of the exercises in the previous section asked about the number of terms in a given GP. Such questions require either trial-and-error or logarithms.

Trial-and-error may be easier to understand, but it is a clumsy method when the numbers are larger. Logarithms provide a better approach, but require understanding of the relationship between logarithms and indices.

Solving Exponential Equations Using Trial-and-Error: Questions about the number of terms in a GP involve solving an equation with n in the index. The following worked exercise shows how to solve such an equation using trial-and-error.

WORKED EXERCISE:

Use trial-and-error on your calculator to find the smallest integer n such that:

(a) $3^n > 400\,000$

(b) $0{\cdot}95^n < 0{\cdot}01$

SOLUTION:

(a) Using the function labelled $\boxed{x^y}$

$3^{11} = 177\,147$

and $3^{12} = 531\,441$,

so the smallest such integer is 12.

(b) Using the function labelled $\boxed{x^y}$

$0{\cdot}95^{89} = 0{\cdot}010\,408\ldots$

and $0{\cdot}95^{90} = 0{\cdot}009\,888\ldots$,

so the smallest such integer is 90.

NOTE: In practice, quite a few more trial calculations are usually needed in order to trap the given number between two integer powers.

Notice too how the powers of 3 get bigger because the base 3 is greater than 1. The powers of 0·95, however, get smaller because the base 0·95 is less than 1.

WORKED EXERCISE: [Inflation and GPs]
The General Widget Company has sold 2000 widgets per year since its foundation in 1991 when the company charged \$300 per widget. Each year, the company lifts its prices by 5% because of cost increases.

(a) Find the value of the sales in the nth year after 1990.

(b) Using trial-and-error, find the first year in which sales exceeded \$900 000.

(c) Find the total sales from the foundation of the company to the end of the nth year.

(d) Using trial-and-error, find the year during which the total sales of the company since its foundation will first exceed \$20 000 000.

SOLUTION:
The value of the annual sales in 1991 were $\$300 \times 2000 = \$600\,000$.
Hence the annual sales form a GP with $a = 600\,000$ and $r = 1{\cdot}05$.

(a) The sales in the nth year after 1990 constitute the nth term T_n of the series,

$$\text{and } T_n = a\,r^{n-1}$$
$$= 600\,000 \times 1{\cdot}05^{n-1}.$$

(b)
$$\begin{aligned}\text{Sales in 1999} &= T_9\\ &= 600\,000 \times 1{\cdot}05^8\\ &\doteqdot \$886\,473.\end{aligned}$$
$$\begin{aligned}\text{Sales in 2000} &= T_{10}\\ &= 600\,000 \times 1{\cdot}05^9\\ &\doteqdot \$930\,797.\end{aligned}$$
Hence the sales first exceeded \$900 000 in the year 2000.

(c) The total sales since the foundation of the company constitute the sum S_n of the first n terms of the series,

$$\begin{aligned}\text{and } S_n &= \frac{a(r^n - 1)}{r - 1} \qquad \text{(Use this formula because } r > 1.)\\ &= \frac{600\,000 \times (1{\cdot}05^n - 1)}{0{\cdot}05}\\ &= 12\,000\,000 \times (1{\cdot}05^n - 1).\end{aligned}$$

(d)
$$\begin{aligned}\text{Total sales to 2010} &= S_{20}\\ &= 12\,000\,000 \times (1{\cdot}05^{20} - 1) \qquad \text{(This is the formula from part (c).)}\\ &\doteqdot \$19\,839\,572.\end{aligned}$$
$$\begin{aligned}\text{Total sales to 2011} &= S_{21}\\ &= 12\,000\,000 \times (1{\cdot}05^{21} - 1)\\ &\doteqdot \$21\,431\,551.\end{aligned}$$
Hence cumulative sales will first exceed \$20 000 000 during 2011.

The Use of Logarithms with GPs: To solve an exponential inequation using logarithms, first solve the corresponding equation. To do this, convert the equation to a logarithmic equation, then convert to logarithms base 10 using the change-of-base formula

$$\log_b x = \frac{\log_{10} x}{\log_{10} b} \qquad \text{(log of the number over log of the base).}$$

WORKED EXERCISE:

Use logarithms to find the smallest integer n such that:

(a) $3^n > 400\,000$ (b) $1{\cdot}04^n > 2$

SOLUTION:

(a) Put $3^n = 400\,000.$ (Begin with the corresponding equation.)

Then $n = \log_3 400\,000$ (Convert to a logarithmic equation.)

$= \dfrac{\log_{10} 400\,000}{\log_{10} 3}$ (Use the change-of-base formula.)

$= 11{\cdot}741\ \ldots.$

Thus the smallest such integer is 12, because $3^{11} < 400\,000$ and $3^{12} > 400\,000$.

(b) Put $1{\cdot}04^n = 2.$

Then $n = \log_{1\cdot04} 2$

$= \dfrac{\log_{10} 2}{\log_{10} 1{\cdot}04}$

$= 17{\cdot}672\ \ldots.$

Thus the smallest such integer is 18, because $1{\cdot}02^{17} < 2$ and $1{\cdot}02^{18} > 2$.

An Alternative Approach — Taking Logarithms of Both Sides: There is an alternative and equally effective approach — take logarithms base 10 of both sides and then use the logarithmic law

$$\log_{10} a^n = n \log_{10} a \qquad \text{(log of the power is the multiple of the base).}$$

Below is the previous worked exercise done again using this alternative approach. The working takes one more line and involves, in effect, a proof of the change-of-base formula. Although the method has not been illustrated again, students may prefer to adopt it.

WORKED EXERCISE:

Use logarithms to find the smallest integer n such that:

(a) $3^n > 400\,000$ (b) $1{\cdot}04^n > 2$

SOLUTION:

(a) Put $3^n = 400\,000.$ (Begin with the corresponding equation.)

Then $\log_{10} 3^n = \log_{10} 400\,000$ (Take logarithms base 10 of both sides.)

$n \log_{10} 3 = \log_{10} 400\,000$ (The log of a power is the multiple of the log.)

$n = \dfrac{\log_{10} 400\,000}{\log_{10} 3}$ (Divide both sides by $\log_{10} 3$.)

$= 11{\cdot}741\ \ldots.$

Thus the smallest such integer is 12, because $3^{11} < 400\,000$ and $3^{12} > 400\,000$.

(b) Put $1{\cdot}04^n = 2.$

Then $\log_{10} 1{\cdot}04^n = \log_{10} 2$

$n \log_{10} 1{\cdot}04 = \log_{10} 2$

$n = \dfrac{\log_{10} 2}{\log_{10} 1{\cdot}04}$

$= 17{\cdot}672\ \ldots.$

Thus the smallest such integer is 18, because $1{\cdot}02^{17} < 2$ and $1{\cdot}02^{18} > 2$.

Applying Logarithms to Problems: The following worked exercise is a typical example where logarithms are used to solve a problem involving a GP.

WORKED EXERCISE:

The profits of the Extreme Sports Adventure Company have been increasing by 15% every year since its formation, when its profit was \$60 000 in the first year.

(a) Find a formula for its profit in the nth year.

(b) During which year did its profit first exceed \$1 200 000?

(c) Find a formula for its total profit during the first n years.

(d) During which year did its total profit since foundation first exceed \$4 000 000?

SOLUTION:

The successive profits form a GP with $a = 60\,000$ and $r = 1{\cdot}15$.

(a) Profit in the nth year $= T_n$

$$= a\,r^{n-1}$$
$$= 60\,000 \times 1{\cdot}15^{n-1}.$$

(b) Put $T_n = 1\,200\,000$. (This is the corresponding equation.)

Then $60\,000 \times 1{\cdot}15^{n-1} = 1\,200\,000$

$\boxed{\div 60\,000}$ $1{\cdot}15^{n-1} = 20$

$$n - 1 = \log_{1\cdot15} 20$$
$$n - 1 = \frac{\log_{10} 20}{\log_{10} 1{\cdot}15}$$
$$n - 1 \doteqdot 21{\cdot}43$$

$\boxed{+1}$ $n \doteqdot 22{\cdot}43$

So the profit first exceeds \$1 200 000 during the 23rd year.

(c) Total profit in the first n years $= S_n$

$$= \frac{a(r^n - 1)}{r - 1} \qquad \text{(Use this form because } r > 1.\text{)}$$
$$= \frac{60\,000 \times (1{\cdot}15^n - 1)}{0{\cdot}15}$$
$$= 400\,000 \times (1{\cdot}15^n - 1)$$

(d) Put $S_n = 4\,000\,000$. (This is the corresponding equation.)

Then $400\,000 \times (1{\cdot}15^n - 1) = 4\,000\,000$

$\boxed{\div 400\,000}$ $1{\cdot}15^n - 1 = 10$

$$1{\cdot}15^n = 11$$
$$n = \log_{1\cdot15} 11$$
$$= \frac{\log_{10} 11}{\log_{10} 1{\cdot}15}$$
$$\doteqdot 17{\cdot}16$$

So the total profit since foundation first exceeds \$4 000 000 during the 18th year.

Using Logarithms when the Base is Less than 1: The successive powers of a base greater than 1 form an increasing sequence. For example, the powers of 2 are

$$2^1 = 2, \qquad 2^2 = 4, \qquad 2^3 = 8, \qquad 2^4 = 16, \qquad \ldots$$

But when the base is less than 1, the successive powers form a decreasing sequence. For example, the powers of $\frac{1}{2}$ are

$$(\tfrac{1}{2})^1 = \tfrac{1}{2}, \qquad (\tfrac{1}{2})^2 = \tfrac{1}{4}, \qquad (\tfrac{1}{2})^3 = \tfrac{1}{8}, \qquad (\tfrac{1}{2})^4 = \tfrac{1}{16}, \qquad \ldots$$

Thus when the base is less than one, questions will be asking for the smallest value of the index that makes the power *less* than some small number.

WORKED EXERCISE:

Use logarithms to find the smallest value of n such that:

(a) $(\frac{1}{3})^n < 0{\cdot}000\,001$ (b) $0{\cdot}95^n < 0{\cdot}01$

SOLUTION:

(a) Put $(\frac{1}{3})^n = 0{\cdot}000\,001$. (Begin with the corresponding equation.)

Then $n = \log_{\frac{1}{3}} 0{\cdot}000\,001$ (Convert to a logarithmic equation.)

$= \dfrac{\log_{10} 0{\cdot}000\,001}{\log_{10} \frac{1}{3}}$ (Use the change-of-base formula.)

$= 12{\cdot}575\ldots.$

Thus the smallest such integer is 13, because $(\frac{1}{3})^{12} > 0{\cdot}000\,001$ and $(\frac{1}{3})^{13} < 0{\cdot}000\,001$.

(b) Put $0{\cdot}95^n = 0{\cdot}01$.

Then $n = \log_{0{\cdot}95} 0{\cdot}01$

$= \dfrac{\log_{10} 0{\cdot}01}{\log_{10} 0{\cdot}95}$

$= 89{\cdot}781\ldots.$

Thus the smallest such integer is 90, because $0{\cdot}95^{89} > 0{\cdot}01$ and $0{\cdot}95^{90} < 0{\cdot}01$.

WORKED EXERCISE:

Consider again the slowly failing Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year, with 50 000 bottles sold in 2001.

During which year will sales first fall below 20 000?

SOLUTION:

The sales form a GP with $a = 50\,000$ and $r = 0{\cdot}94$.

Put $T_n = 20\,000$. (This is the corresponding equation.)

Then $a\,r^{n-1} = 20\,000$

$50\,000 \times 0{\cdot}94^{n-1} = 20\,000$

$\boxed{\div\, 50\,000}$ $0{\cdot}94^{n-1} = 0{\cdot}4$

$n - 1 = \log_{0{\cdot}94} 0{\cdot}4$ (Convert to a logarithmic equation.)

$n - 1 = \dfrac{\log_{10} 0{\cdot}4}{\log_{10} 0{\cdot}94}$ (Use the change-of-base formula.)

$n - 1 \doteqdot 14{\cdot}81$

$\boxed{+1}$ $n \doteqdot 15{\cdot}81.$

Hence sales will first fall below 20 000 when $n = 16$, that is, in 2016.

Exercise 6B

1. Use trial-and-error (and your calculator, where necessary) to find the smallest integer n such that:

(a) $2^n > 30$
(b) $2^n > 15\,000$
(c) $2^n > 7\,000\,000$
(d) $3^n > 10$
(e) $3^n > 16\,000$
(f) $3^n > 5\,000\,000$
(g) $(\frac{1}{2})^n < 0{\cdot}1$
(h) $(\frac{1}{2})^n < 0{\cdot}005$
(i) $(\frac{1}{2})^n < 0{\cdot}0001$
(j) $(\frac{1}{3})^n < 0{\cdot}2$
(k) $(\frac{1}{3})^n < 0{\cdot}01$
(l) $(\frac{1}{3})^n < 0{\cdot}00001$

2. (a) Show that 10, 11, 12·1, … is a geometric sequence.
(b) State the first term and the common ratio.
(c) Use the formula $T_n = ar^{n-1}$ to write down the fifteenth term.
(d) Find the number of terms less than 60 using trial-and-error on your calculator.

3. An accountant receives an annual salary of \$40 000, with 5% increments each year.
(a) Show that her annual salary forms a GP and find the common ratio.
(b) Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.
(c) In which year will her salary first exceed \$70 000?

4. An initial salary of \$50 000 increases by 4% each year. In which year will the salary first be at least twice the original salary?

DEVELOPMENT

5. Confirm your answers to question 1 by solving each equation using logarithms.

6. Confirm your answers to the last part of each of questions 2, 3 and 4 by using logarithms.

7. A certain company manufactures three types of shade cloth. The product with code SC50 cuts out 50% of harmful UV rays, SC75 cuts out 75% and SC90 cuts out 90% of UV rays. In the following questions, you will need to consider the amount of UV light let through.
(a) What percentage of UV light does each cloth let through?
(b) Show that two layers of SC50 would be equivalent to one layer of SC75 shade cloth.
(c) Use logarithms to find the minimum number of layers of SC50 that would be required to cut out at least as much UV light as one layer of SC90.
(d) Similarly find how many layers of SC50 would be required to cut out 99% of UV rays.

8. Yesterday, a tennis ball used in a game of cricket in the playground was hit onto the science block roof. Luckily it rolled off the roof. After bouncing on the playground it reached a height of 3 metres. After the next bounce it reached 2 metres, then $1\frac{1}{3}$ metres and so on.
(a) What was T_n, the height in metres reached after the nth bounce?
(b) What was the height of the roof the ball fell from?
(c) The last time the ball bounced, its height was below 1 cm for the first time. After that it rolled away across the playground.
  (i) Show that if $T_n < 0{\cdot}01$, then $(\frac{3}{2})^{n-1} > 300$.
  (ii) Use logarithms, or trial and error on the calculator, to find how many times it bounced.

9. Madeleine opens a business selling computer stationery. In its first year, the business has sales of \$200 000, and each year sales are 20% more than the previous year's sales.
(a) In which year do annual sales first exceed \$1 000 000?
(b) In which year do total sales since foundation first exceed \$2 000 000?

CHALLENGE

10. Consider the geometric series $3, 2, \frac{4}{3}, \cdots$.
(a) Write down a formula for the sum S_n of the first n terms of the series.
(b) Explain why the geometric series has a limiting sum, and determine its value S.
(c) Find the smallest value of n for which $S - S_n < 0{\cdot}01$.

6 C Simple and Compound Interest

This section will review the formulae for simple and compound interest. Simple interest is both an arithmetic sequence and a linear function. Compound interest is both a geometric sequence and an exponential function.

Simple Interest, Arithmetic Sequences and Linear Functions: The formula for simple interest I should be well known from earlier years:

$$I = PRn,$$

where P is the principal invested, R is the interest rate per unit time, and n is the number of units of time. This is a linear function of n and the interest at the end of 1 year, 2 years, 3 years, ... forms an AP

$$PR, \quad 2PR, \quad 3PR, \quad 4PR, \quad \ldots$$

The simple interest formula gives the interest alone. To find the total amount at the end of n units of time, add the original principal P to the interest.

3 SIMPLE INTEREST:
Suppose that a principal $\$P$ earns simple interest at a rate R per unit time.
Then the simple interest $\$I$ earned in n units of time is

$$I = PRn.$$

To find the total amount at the end of n units of time, add the principal P.

Note that the interest rate here is a number, not a percentage. For example, if the interest rate is 7% pa, then $R = 0{\cdot}07$. (The initials 'pa' stand for *per annum*, which is Latin for 'per year'.)

WORKED EXERCISE:
A principal $\$P$ is invested at 6% pa simple interest.
(a) If the principal $\$P$ is \$3000, how much money will there be after seven years?
(b) Find the principal $\$P$, if the total at the end of five years is \$6500.

SOLUTION:
(a) Using the formula, interest $= PRn$
$= 3000 \times 0{\cdot}06 \times 7$
$= \$1260.$
Hence final amount $= 3000 + 1260$ (principal + interest)
$= \$4260.$

(b) Final amount after 5 years $= P + PRn$ (principal + interest)
$= P + P \times 0{\cdot}06 \times 5$
$= P + P \times 0{\cdot}3$
$= P \times 1{\cdot}3.$
Hence $P \times 1{\cdot}3 = 6500$
$\boxed{\div 1{\cdot}3}$ $P = \$5000.$

Compound Interest, Geometric Sequences and Exponential Functions: The formula for compound interest should also be well known from earlier years:

$$A_n = P(1+R)^n,$$

where P is the principal, R is the interest rate per unit time, n is the number of units of time, and A_n is the final amount.

This is an exponential function of n, with base $1 + R$. On the other hand, substituting $n = 1, 2, 3, \ldots$ gives the sequence

$$A_1 = P(1+R)^1, \quad A_2 = P(1+R)^2, \quad A_3 = P(1+R)^3, \quad \ldots$$

which is a GP with first term $P(1 + R)$ and common ratio $1 + R$.

Sometimes a question will ask what interest was earned on the principal. To find the interest, subtract the principal from the final amount.

4 COMPOUND INTEREST:
Suppose that a principal $\$P$ earns compound interest at a rate R per unit time for n units of time, compounded every unit of time. Then the total amount A_n after n units of time is

$$A_n = P(1+R)^n.$$

This forms a GP with first term $P(1 + R)$ and common ratio $1 + R$.
To find the interest, subtract the principal from the final amount.

NOTE: The formula only works when compounding occurs after every unit of time. For example, if the interest rate is 24% per year with interest compounded monthly, then the units of time must be months and the interest rate per month is $R = 0{\cdot}24 \div 12 = 0{\cdot}02$.

PROOF: Although the formula was developed in earlier years, it is important to understand how it arises and how the process of compounding generates a GP.

The initial principal is P and the interest rate is R per unit time.
Hence the amount A_1 at the end of one unit of time is

$$A_1 = \text{principal} + \text{interest} = P + PR = P(1+R).$$

This means that adding the interest is effected by multiplying by $1 + R$.
Similarly, the amount A_2 is obtained by multiplying A_1 by $1 + R$:

$$A_2 = A_1(1+R) = P(1+R)^2.$$

Then, continuing the process,

$$A_3 = A_2(1+R) = P(1+R)^3,$$
$$A_4 = A_3(1+R) = P(1+R)^4,$$

so that when the money has been invested for n units of time,

$$A_n = A_{n-1}(1+R) = P(1+R)^n.$$

WORKED EXERCISE:
Amelda takes out a loan of \$5000 at a rate of 12% pa, compounded monthly. She makes no repayments.

(a) Find the total amount owing after five years and after ten years.

(b) Hence find the interest alone after five years and after ten years.

(c) Use logarithms to find when the amount owing doubles, giving your answer correct to the nearest month.

SOLUTION:
Because the interest is compounded every month, the units of time must be months. The interest rate is therefore 1% per month and so $R = 0{\cdot}01$.

(a)
$$\begin{aligned} A_{60} &= P(1+R)^{60} && \text{(5 years has been converted to 60 months.)} \\ &= 5000 \times 1{\cdot}01^{60} \\ &\doteqdot \$9083. \\ A_{120} &= P(1+R)^{120} && \text{(10 years has been converted to 120 months.)} \\ &= 5000 \times 1{\cdot}01^{120} \\ &\doteqdot \$16\,502. \end{aligned}$$

(b)
$$\begin{aligned} \text{After five years, interest} &= \$9083 - \$5000 && \text{(Subtract the principal.)} \\ &= \$4083. \\ \text{After ten years, interest} &= \$16\,502 - \$5000 && \text{(Subtract the principal.)} \\ &= \$11\,502. \end{aligned}$$

(c) The formula is $A_n = P(1+R)^n$.
Substitute $P = 5000$, $1 + R = 1{\cdot}01$ and $A_n = 10\,000$.

$$\begin{aligned} \text{Then} \quad 10\,000 &= 5000 \times 1{\cdot}01^n \\ \boxed{\div 5000} \quad 1{\cdot}01^n &= 2 \\ n &= \log_{1\cdot 01} 2 && \text{(Convert to a logarithmic equation.)} \\ &= \frac{\log_{10} 2}{\log_{10} 1{\cdot}01} && \text{(Use the change-of-base formula.)} \\ &\doteqdot 70 \text{ months.} && \text{(Trial-and-error could have been used here.)} \end{aligned}$$

Depreciation: *Depreciation* is important when a business buys equipment because the equipment becomes worn or obsolete over time and loses its value. Depreciation is usually expressed as the loss per unit time of a percentage of the value of an item. The formula for depreciation is therefore the same as the formula for compound interest, except that the rate is negative.

5

DEPRECIATION:
Suppose that goods originally costing $\$P$ depreciate at a rate R per unit time.
Then the value A_n of the goods after n units of time is

$$A_n = P(1-R)^n.$$

This forms a GP with first term $P(1-R)$ and common ratio $1-R$.
To find the loss of value, subtract the final value from the initial value.

The loss of value is important for a business because it must be regarded as an expense when the profit is being calculated.

WORKED EXERCISE:

An espresso machine bought for \$15 000 on 1st January 2001 depreciates at a rate of $12\frac{1}{2}\%$ pa.

(a) What will the depreciated value be on 1st January 2010?

(b) What is the loss of value over those nine years?

(c) During which year will the value drop below 10% of the original cost?

SOLUTION:

This is depreciation with $R = 0{\cdot}125$, so $1 - R = 0{\cdot}875$.

(a) Depreciated value $= A_9$ (There are 9 years from 1/1/2001 to 1/1/2010.)

$$\begin{aligned} &= P(1-R)^n, \\ &= 15\,000 \times 0{\cdot}875^9 \\ &\doteqdot \$4510. \end{aligned}$$

(b) Loss of value $\doteqdot 15\,000 - 4510$ (Subtract the depreciated value.)

$$\doteqdot \$10\,490.$$

(c) The formula is $A_n = P(1-R)^n$.

Substituting $P = 15\,000$, $1 - R = 0{\cdot}875$ and $A_n = 1500$,

$$1500 = 15\,000 \times 0{\cdot}875^n$$

$\boxed{\div 15\,000}$ $\quad 0{\cdot}875^n = 0{\cdot}1$

$n = \log_{0\cdot 875} 0{\cdot}1$ (Convert to a logarithmic equation.)

$= \dfrac{\log_{10} 0{\cdot}1}{\log_{10} 0{\cdot}875}$ (Use the change-of-base formula.)

$\doteqdot 17{\cdot}24.$

Hence the depreciated value will drop below 10% during 2018.

(There are 17 years from 1/1/2001 to 1/1/2018, so the drop occurs during 2018.)

Exercise 6C

1. Use the formula $I = PRn$ to find: (i) the simple interest, (ii) the total amount, when:

(a) \$5000 is invested at 6% per annum for three years,

(b) \$300 is invested at 5% per annum for eight years,

(c) \$10 000 is invested at $7\frac{1}{2}\%$ per annum for five years,

(d) \$12 000 is invested at $6{\cdot}15\%$ per annum for seven years.

2. Use the formula $A = P(1+R)^n$ to find: (i) the total value, (ii) the interest alone, correct to the nearest cent, of:

(a) \$5000 invested at 6% per annum, compounded annually, for three years,

(b) \$300 invested at 5% per annum, compounded annually, for eight years,

(c) \$10 000 invested at $7\frac{1}{2}\%$ per annum, compounded annually, for five years,

(d) \$12 000 invested at $6{\cdot}15\%$ per annum, compounded annually, for seven years.

3. Use the formula $A = P(1-R)^n$ to find: (i) the final value, (ii) the loss of value, correct to the nearest cent, of:

(a) \$5000 depreciating at 6% per annum for three years,

(b) \$300 depreciating at 5% per annum for eight years,

(c) \$10 000 depreciating at $7\frac{1}{2}\%$ per annum for five years,

(d) \$12 000 depreciating at 6·15% per annum for seven years.

4. First convert the interest rate to the appropriate unit of time, then find the value, correct to the nearest cent, when:

(a) \$400 is invested at 12% per annum, compounded monthly, for two years,

(b) \$1000 is invested at 8% per annum, compounded quarterly, for five years,

(c) \$750 is invested at 10% per annum, compounded six-monthly, for three years,

(d) \$10 000 is invested at 7·28% per annum, compounded weekly, for one year.

5. (a) Find the total value of an investment of \$5000 earning 7% per annum simple interest for three years.

(b) A woman invested an amount for nine years at a rate of 6% per annum. She earned a total of \$13 824 in simple interest. What was the initial amount she invested?

(c) A man invested \$23 000 at 3·25% per annum simple interest. At the end of the investment period he withdrew all the funds from the bank, a total of \$31 222.50. How many years did the investment last?

(d) The total value of an investment earning simple interest after six years is \$22 610. If the original investment was \$17 000, what was the interest rate?

6. A man invested \$10 000 at 6·5% per annum simple interest.

(a) Write down a formula for A_n, the total value of the investment at the end of the nth year.

(b) Show that the investment exceeds \$20 000 at the end of 16 years, but not at the end of 15 years.

7. A company has just bought several cars for a total of \$229 000. The depreciation rate on these cars is 15% per annum.

(a) What will be the net worth of the fleet of cars five years from now?

(b) What will be the loss in value then?

8. Howard is arguing with Juno over who has the better investment. Each invested \$20 000 for one year. Howard has his invested at 6·75% per annum simple interest, while Juno has hers invested at 6·6% per annum compound interest.

(a) If Juno's investment is compounded annually, who has the better investment, and what are the final values of the two investments?

(b) Juno then points out that her interest is compounded monthly, not yearly. Now who has the better investment, and by how much?

DEVELOPMENT

9. To what value does \$1000 grow if invested for a year at 12% per annum compound interest, compounded:

(a) annually, (b) six-monthly, (c) quarterly, (d) monthly.

10. (a) The final value of an investment, after ten years earning 15% per annum, compounded yearly, was \$32 364. Find the amount invested, correct to the nearest dollar.

(b) The final value of an investment that earned 7% compound interest per annum for 18 years was \$40 559.20. What was the original amount, correct to the nearest dollar?

(c) A sum of money is invested at 4·5% interest per annum, compounded monthly. At the end of three years the value is \$22 884.96. Find the amount of the original investment, correct to the nearest dollar.

11. An insurance company recently valued my car at \$14 235. The car is three years old and the depreciation rate used by the insurance company was 10·7% per annum. What was the cost of the car, correct to the nearest dollar, when I bought it?

12. (a) What does \$6000 grow to at 8·25% per annum for three years, compounded monthly?

(b) How much interest is earned over the three years?

(c) What rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

13. An amount of \$10 000 is invested for five years at 4% pa interest, compounded monthly.

(a) Find the final value of the investment.

(b) What rate of simple interest, correct to two significant figures, would be needed to yield the same final balance.

14. The present value of a company asset is \$350 000. If it has been depreciating at $17\frac{1}{2}\%$ per annum for the last six years, what was the original value of the asset, correct to the nearest \$1000?

15. Find the smallest integer n for which:

(a) $8000 \times (1{\cdot}07)^n > 40\,000$

(b) $10\,000 \times (1{\cdot}1)^n > 35\,000$

(c) $20\,000 \times (0{\cdot}8)^n < 5000$

(d) $100\,000 \times (0{\cdot}75)^n < 10\,000$

16. Write down the formula for the total value A_n when a principal of \$6000 is invested at 12% pa compound interest for n years. Hence find the smallest number of years required for the investment to:

(a) double, (b) treble, (c) quadruple, (d) increase by a factor of 10.

17. Xiao and Mai win a prize in the lottery and decide to put \$100 000 into a retirement fund offering 8·25% per annum interest, compounded monthly. How long will it be before their money has doubled? Give your answer correct to the nearest month.

CHALLENGE

18. After six years of compound interest, the final value of a \$30 000 investment was \$45 108.91. What was the rate of interest, correct to two significant figures, if it was compounded annually?

19. (a) Find the interest on \$15 000 invested at 7% per annum simple interest for five years.

(b) Hence write down the total value of the investment.

(c) What rate of compound interest would yield the same amount if compounded annually? Give your answer correct to three significant figures.

20. A student was asked to find the original value, correct to the nearest dollar, of an investment earning 9% per annum, compounded annually for three years, given its current value of \$54 391.22.

(a) She incorrectly thought that since she was working in reverse, she should use the depreciation formula. What value did she get?

(b) What is the correct answer?

21. A bank customer earned \$7824.73 in interest on a \$40 000 investment at 6% per annum, compounded quarterly.
(a) Show that $1{\cdot}015^n \doteqdot 1{\cdot}1956$, where n is the number of quarters.
(b) Hence find the period of the investment, correct to the nearest quarter.

6 D Investing Money by Regular Instalments

Investment schemes such as superannuation schemes require money to be invested at regular intervals, for example every month or every year. This complicates things because each individual instalment earns compound interest for a different length of time. Calculating the value of these investments at some future time requires adding the terms of a GP.

This section and the next are applications of GPs. Learning new formulae is not recommended, because they will all need to be derived within each question.

Developing the GP and Summing It: The most straightforward way to solve these problems is to find what each instalment grows to as it accrues compound interest. These final amounts form a GP, which can then be summed.

6 **FINDING THE FUTURE VALUE OF AN INVESTMENT SCHEME:**
- Find what each instalment will amount to as it earns compound interest.
- Add up all these amounts using the formula for the sum of a GP.

WORKED EXERCISE:
Rawen's parents invested \$1000 in his name on the day that he was born. They continued to invest \$1000 for him on each birthday until his 20th birthday. On his 21st birthday they gave him the value of the investment.

If all the money earned interest of 7% compounded annually, what was the final value of the scheme, correct to the nearest dollar?

SOLUTION:
The 1st instalment is invested for 21 years and so amounts to $1000 \times 1{\cdot}07^{21}$.
The 2nd instalment is invested for 20 years and so amounts to $1000 \times 1{\cdot}07^{20}$.
The 20th instalment is invested for 2 years and so amounts to $1000 \times 1{\cdot}07^{2}$.
The 21st and last instalment is invested for 1 year and so amounts to $1000 \times 1{\cdot}07^{1}$.
Thus the total amount A_{21} at the end of 21 years is the sum

$$\begin{aligned} A_{21} &= \text{instalments plus interest} \\ &= (1000 \times 1{\cdot}07^1) + (1000 \times 1{\cdot}07^2) + \cdots + (1000 \times 1{\cdot}07^{21}). \end{aligned}$$

This is a GP with first term $a = 1000 \times 1{\cdot}07$, ratio $r = 1{\cdot}07$ and 21 terms.

$$\text{Hence } A_{21} = \frac{a(r^{21}-1)}{r-1} \qquad \text{(This is the formula for } S_n \text{ for a GP with } r > 1.)$$

$$= \frac{1000 \times 1{\cdot}07 \times (1{\cdot}07^{21} - 1)}{0{\cdot}07}$$

$$\doteqdot \$48\,006.$$

WORKED EXERCISE:

Robin and Robyn are investing \$10 000 in a superannuation scheme on 1st July each year, beginning in the year 2000. The money earns compound interest at 8% pa, compounded annually.

(a) How much will the fund amount to by 30th June 2020?

(b) How much will the fund amount to by the end of n years?

(c) Show that 2021 is the year when the fund first exceeds \$500 000 on 30th June.

(d) What annual instalment would have produced \$1 000 000 by 2020?

SOLUTION:

(a) The 1st instalment is invested for 20 years and so amounts to $10\,000 \times 1{\cdot}08^{20}$.

The 2nd instalment is invested for 19 years and so amounts to $10\,000 \times 1{\cdot}08^{19}$.

The 19th instalment is invested for 2 years and so amounts to $10\,000 \times 1{\cdot}08^{2}$.

The 20th and last is invested for 1 year and so amounts to $10\,000 \times 1{\cdot}08^{1}$.

Thus the total amount A_{20} at the end of 20 years is the sum

$$\begin{aligned} A_{20} &= \text{instalments plus interest} \\ &= (10\,000 \times 1{\cdot}08^1) + (10\,000 \times 1{\cdot}08^2) + \cdots + (10\,000 \times 1{\cdot}08^{20}). \end{aligned}$$

This is a GP with first term $a = 10\,000 \times 1{\cdot}08$, ratio $r = 1{\cdot}08$ and 20 terms.

$$\begin{aligned} \text{Hence } A_{20} &= \frac{a(r^{20}-1)}{r-1} \qquad \text{(This is the GP formula for } S_n \text{ when } r>1.) \\ &= \frac{10\,000 \times 1{\cdot}08 \times (1{\cdot}08^{20}-1)}{0{\cdot}08} \\ &\doteqdot \$494\,229 \quad \text{(correct to the nearest dollar).} \end{aligned}$$

(b) The 1st instalment is invested for n years and so amounts to $10\,000 \times 1{\cdot}08^{n}$.

The 2nd instalment is invested for $n-1$ years and so amounts to $10\,000 \times 1{\cdot}08^{n-1}$.

The nth and last is invested for 1 year and so amounts to $10\,000 \times 1{\cdot}08^{1}$.

Thus the total amount A_n at the end of n years is the sum

$$\begin{aligned} A_n &= \text{instalments plus interest} \\ &= (10\,000 \times 1{\cdot}08^1) + (10\,000 \times 1{\cdot}08^2) + \cdots + (10\,000 \times 1{\cdot}08^{n}). \end{aligned}$$

This is a GP with first term $a = 10\,000 \times 1{\cdot}08$, ratio $r = 1{\cdot}08$ and n terms.

$$\begin{aligned} \text{Hence } A_n &= \frac{a(r^{n}-1)}{r-1} \qquad \text{(This is the GP formula for } S_n \text{ when } r>1.) \\ &= \frac{10\,000 \times 1{\cdot}08 \times (1{\cdot}08^{n}-1)}{0{\cdot}08} \\ &= 135\,000 \times (1{\cdot}08^n - 1). \end{aligned}$$

(c) From part(a), the total after 20 years is just under \$500 000.

Substituting $n = 21$ into the formula in part (b),

$$\begin{aligned} A_{21} &= 135\,000 \times (1{\cdot}08^{21} - 1) \\ &\doteqdot \$544\,568. \end{aligned}$$

Hence 2021 is the year when the fund first exceeds \$500 000 on 30th June.

(d) Reworking part (b) with an instalment $\$M$ instead of \$10 000 gives the formula

$$A_n = 13{\cdot}5 \times M \times (1{\cdot}08^n - 1)\,.$$

Substituting $n = 20$ and $A_{20} = 1\,000\,000$ into this formula,

$$1\,000\,000 = 13{\cdot}5 \times M \times (1{\cdot}08^{20} - 1)$$

$$M = \frac{1\,000\,000}{13{\cdot}5 \times (1{\cdot}08^{20} - 1)} \qquad \text{(Make } M \text{ the subject.)}$$

$$\doteqdot \$20\,234 \quad \text{(correct to the nearest dollar).}$$

WORKED EXERCISE: [Using logarithms to find n]
Continuing with the previous example, use logarithms to find the year in which the fund first exceeds \$700 000 on 30th June.

SOLUTION:

Substituting $M = 10\,000$ and $A_n = 700\,000$ into the formula found in part (b),

$$700\,000 = 135\,000 \times (1{\cdot}08^n - 1)$$

$$\boxed{\div\, 135\,000} \quad 1{\cdot}08^n - 1 = \tfrac{700}{135}$$

$$\boxed{+\,1} \quad 1{\cdot}08^n = \tfrac{835}{135}$$

$$n = \log_{1{\cdot}08} \tfrac{835}{135} \qquad \text{(Convert to a logarithmic equation.)}$$

$$= \frac{\log_{10} \frac{835}{135}}{\log_{10} 1{\cdot}08} \qquad \text{(This is the change-of-base formula.)}$$

$$\doteqdot 23{\cdot}68.$$

Hence the fund first exceeds \$700 000 on 30th June when $n = 24$, that is, in 2024.

WORKED EXERCISE: [Monthly and weekly compounding]

(a) Charmaine has a superannuation scheme with monthly instalments of \$600 for 10 years and an interest rate of 7·8% pa, compounded monthly. What will the final value of her investment be?

(b) Charmaine was offered an alternative scheme with interest of 7·8% pa, compounded weekly, and weekly repayments. What weekly instalments would have yielded the same final value as the scheme in part (a)?

(c) Which scheme would have cost her more per year?

SOLUTION:

(a) The monthly interest rate is $0{\cdot}078 \div 12 = 0{\cdot}0065$.
There are 120 months in 10 years.
The 1st instalment is invested for 120 months and so amounts to $600 \times 1{\cdot}0065^{120}$.
The 2nd instalment is invested for 119 months and so amounts to $600 \times 1{\cdot}0065^{119}$.
The 120th and last is invested for 1 month and so amounts to $600 \times 1{\cdot}0065^{1}$.
Thus the total amount A_{120} at the end of 120 months is the sum

$$A_{120} = \text{instalments plus interest}$$

$$= (600 \times 1{\cdot}0065^1) + (600 \times 1{\cdot}0065^2) + \cdots + (600 \times 1{\cdot}0065^{120}).$$

This is a GP with first term $a = 600 \times 1{\cdot}0065$, ratio $r = 1{\cdot}0065$ and 120 terms.

Hence $A_{120} = \dfrac{a(r^{120} - 1)}{r - 1}$ (This is the GP formula for S_n when $r > 1$.)

$= \dfrac{600 \times 1{\cdot}0065 \times (1{\cdot}0065^{10} - 1)}{0{\cdot}0065}$

$\doteqdot \$109\,257.$ (Retain this in the memory for part(b).)

(b) The weekly interest rate is $0{\cdot}078 \div 52 = 0{\cdot}0015$.
Let $\$M$ be the weekly instalment. There are 520 weeks in 10 years.
The 1st instalment is invested for 520 weeks and so amounts to $M \times 0{\cdot}0015^{520}$.
The 2nd instalment is invested for 519 weeks and so amounts to $M \times 0{\cdot}0015^{519}$.
The 520th and last is invested for 1 week and so amounts to $M \times 0{\cdot}0015^{1}$.
Thus the total amount A_{520} at the end of 520 weeks is the sum

$$A_{520} = \text{instalments plus interest}$$
$$= M \times 0{\cdot}0015 \;+\; M \times 0{\cdot}0015^{2} \;+\; \cdots \;+\; M \times 0{\cdot}0015^{520}.$$

This is a GP with first term $a = M \times 0{\cdot}0015$, ratio $r = 0{\cdot}0015$ and 520 terms.

Hence $A_{520} = \dfrac{a(r^{520} - 1)}{r - 1}$ (This is the GP formula for S_n when $r > 1$.)

$$A_{520} = \frac{M \times 1{\cdot}0015 \times \left(1{\cdot}0015^{520} - 1\right)}{0{\cdot}0015}$$

$$A_{520} = \frac{M \times 10\,015 \times \left(1{\cdot}0015^{520} - 1\right)}{15}$$

Writing this formula with M as the subject,

$$M = \frac{15 \times A_{520}}{10\,015 \times \left(1{\cdot}0015^{520} - 1\right)}$$

But the final value A_{520} is to be the same as the final value in part (a),
so substituting the answer to part (a) for A_{520} gives

$$M \doteqdot \$138.65 \ \big(\text{retain in the memory for part(c)}\big).$$

(c) The weekly scheme in part (b) therefore costs about \$7210.04 per year, compared with \$7200 per year for the monthly scheme in part (a).

An Alternative Approach Using Recursion: There is an alternative approach, using recursion, to developing the GPs involved in these calculations. Because the working is slightly longer, we have chosen not to display this method in the notes. It has, however, the advantage that its steps follow the progress of a banking statement. For those who are interested in the recursive method, it is developed in two structured questions in the Challenge section of the following exercise.

Exercise 6D

NOTE: Questions 1–5 of this exercise have been heavily structured to follow the approach given in the worked exercises above. There are several other satisfactory approaches, including the recursive method outlined in questions 16 and 17. If a different approach is chosen, the structuring in the first five questions below can be ignored.

1. Suppose that an instalment of \$500 is invested in a superannuation scheme on 1st January each year for four years, beginning in 2005. The money earns interest at 10% pa, compounded annually.

(a) (i) What is the value of the first instalment on 31st December 2008?
(ii) What is the value of the second instalment on 31st December 2008?
(iii) What is the value of the third instalment on this date?
(iv) What is the value of the fourth instalment?
(v) What is the total value of the superannuation on 31st December 2008?

(b) (i) Write down the four answers to parts (i) to (iv) above in increasing order, and notice that they form a GP.
(ii) Write down the first term, common ratio and number of terms.
(iii) Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP and hence check your answer to part (a) (v).

2. Suppose that an instalment of \$1200 is invested in a superannuation scheme on 1st April each year for five years, beginning in 2005. The money earns interest at 5% pa, compounded annually.

(a) In each part round your answer correct to the nearest cent.
(i) What is the value of the first instalment on 31st March 2010?
(ii) What is the value of the second instalment on this date?
(iii) Do the same for the third, fourth and fifth instalments.
(iv) What is the total value of the superannuation on 31st March 2010?

(b) (i) Write down the answers to parts (i) to (iii) above in increasing order, and notice that they form a GP.
(ii) Write down the first term, common ratio and number of terms.
(iii) Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part (a) (iv).

3. Joshua makes contributions of \$1500 to his superannuation scheme on 1st April each year. The money earns compound interest at 7% per annum.

(a) Let A_{15} be the total value of the fund at the end of 15 years.
(i) How much does the first instalment amount to at the end of 15 years?
(ii) How much does the second instalment amount to at the end of 14 years?
(iii) How much does the last contribution amount to at the end of just one year?
(iv) Hence write down a series for A_{15}.

(b) Hence show that the final value of the fund is $A_{15} = \dfrac{1500 \times 1{\cdot}07 \times (1{\cdot}07^{15} - 1)}{0{\cdot}07}$, and evaluate this correct to the nearest dollar.

4. Laura makes contributions of \$250 to her superannuation scheme on the first day of each month. The money earns interest at 6% per annum, compounded monthly (that is, at 0·5% per month).

(a) Let A_{24} be the total value of the fund at the end of 24 months.
(i) How much does the first instalment amount to at the end of 24 months?
(ii) How much does the second instalment amount to at the end of 23 months?
(iii) What is the value of the last contribution, invested for just one month?
(iv) Hence write down a series for A_{24}.

(b) Hence show that the total value of the fund after contributions have been made for two years is $A_{24} = \dfrac{250 \times 1{\cdot}005 \times (1{\cdot}005^{24} - 1)}{0{\cdot}005}$, and evaluate this correct to the nearest dollar.

5. A company makes contributions of \$3000 to the superannuation fund of one of its employees on 1st July each year. The money earns compound interest at 6·5% per annum. In the following parts, round all currency amounts correct to the nearest dollar.

(a) Let A_{25} be the value of the fund at the end of 25 years.

(i) How much does the first instalment amount to at the end of 25 years?

(ii) How much does the second instalment amount to at the end of 24 years?

(iii) How much does the last instalment amount to at the end of just one year?

(iv) Hence write down a series for A_{25}.

(b) Hence show that $A_{25} = \dfrac{3000 \times 1{\cdot}065 \times (1{\cdot}065^{25} - 1)}{0{\cdot}065}$.

(c) What will be the value of the fund after 25 years, and what will be the total amount of the contributions?

DEVELOPMENT

6. Finster and Finster Superannuation offer a superannuation scheme with annual contributions of \$12 000 invested at an interest rate of 9% pa, compounded annually. Contributions are paid on 1st of January each year.

(a) Zoya decides to invest in the fund for the next 20 years. Show that the final value of her investment is given by $A_{20} = \dfrac{12000 \times 1{\cdot}09 \times (1{\cdot}09^{20} - 1)}{0{\cdot}09}$.

(b) Evaluate A_{20}.

(c) By how much does this exceed the total contributions Zoya made?

(d) The company agrees to let Zoya make a higher contribution to the scheme. Let this instalment be M. Show that in this case $A_{20} = \dfrac{M \times 1{\cdot}09 \times (1{\cdot}09^{20} - 1)}{0{\cdot}09}$.

(e) What would Zoya's annual contribution have to be in order for her superannuation to have a total value of \$1 000 000 at the end of the 20 years?

7. The company that Itsushi works for makes contributions to his superannuation scheme on 1st January each year. Any amount invested in this scheme earns interest at the rate of 7·5% pa.

(a) Let M be the annual contribution. Show that the value of the investment at the end of the nth year is $A_n = \dfrac{M \times 1{\cdot}075 \times (1{\cdot}075^n - 1)}{0{\cdot}075}$.

(b) Itsushi plans to have \$1 500 000 in superannuation when he retires in 25 years time. Show that the company must contribute \$20 526.52 each year, correct to the nearest dollar.

(c) The first year that Itsushi's superannuation is worth more than \$750 000, he decides to change jobs. Let this year be n.

(i) Show that n is the smallest integer solution of $(1{\cdot}075)^n > \dfrac{750\,000 \times 0{\cdot}075}{20\,526{\cdot}52 \times 1{\cdot}075} + 1$.

(ii) Evaluate the right-hand side and hence show that $(1{\cdot}075)^n > 3{\cdot}5492$.

(iii) Use logarithms or trial-and-error to find the value of n.

8. A person invests \$10 000 each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first payment is made on 1st January 2001 and the last payment is made on 1st January 2020.

(a) How much did the person invest over the life of the fund?

(b) Calculate, correct to the nearest dollar, the amount to which the 2001 payment has grown by the beginning of 2021.

(c) Find the total value of the fund when it is paid out on 1st January 2021.

(d) The person wants to reach a total value of \$1 000 000 in superannuation.

(i) Find a formula for A_n, the value of the investment after n years.

(ii) Show that the target is reached when $1{\cdot}1^n > \dfrac{10}{1{\cdot}1} + 1$.

(iii) At the end of which year will the superannuation be worth \$1 000 000?

(e) Suppose instead that the person wanted to achieve the same total investment of \$1 000 000 after only 20 years. What annual contribution would produce this amount? [HINT: Let M be the amount of each contribution.]

9. Each year on her birthday, Jane's parents put \$20 into an investment account earning $9\frac{1}{2}\%$ per annum compound interest. The first deposit took place on the day of her birth. On her 18th birthday, Jane's parents gave her the account and \$20 cash in hand.

(a) How much money had Jane's parents deposited in the account?

(b) How much money did she receive from her parents on her 18th birthday?

10. A man about to turn 25 is getting married. He has decided to pay \$5000 each year on his birthday into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. If he dies before age 65, his wife will inherit the value of the insurance to that point. If he lives to age 65, the insurance company will pay out the value of the policy in full. Answer the following correct to the nearest dollar.

(a) The man is in a dangerous job. What will be the payout if he dies just before he turns 30?

(b) The man's father died of a heart attack just before age 50. Suppose that the man also dies of a heart attack just before age 50. How much will his wife inherit?

(c) What will the insurance company pay the man if he survives to his 65th birthday?

11. In 2001, the school fees at a private girls' school are \$10 000 per year. Each year the fees rise by $4\frac{1}{2}\%$ due to inflation.

(a) Susan is sent to the school, starting in Year 7 in 2001. If she continues through to her HSC year, how much will her parents have paid the school over the six years?

(b) Susan's younger sister is starting in Year 1 in 2001. How much will they spend on her school fees over the next 12 years if she goes through to her HSC?

CHALLENGE

12. A woman has just retired with a payment of \$500 000, having contributed for 25 years to a superannuation fund that pays compound interest at the rate of $12\frac{1}{2}\%$ per annum. What was the size of her annual premium, correct to the nearest dollar?

13. At age 20, a woman takes out a life insurance policy under which she agrees to pay premiums of \$500 per year until she turns 65, when she is to be paid a lump sum. The insurance company invests the money and gives a return of 9% per annum, compounded annually. If she dies before age 65, the company pays out the current value of the fund plus 25% of the difference had she lived until 65.

(a) What is the value of the payout, correct to the nearest dollar, at age 65?

(b) Unfortunately she dies at age 53, just before her 35th premium is due.

(i) What is the current value of the life insurance?

(ii) How much does the life insurance company pay her family?

14. A person pays \$2000 into an investment fund every year, and it earns compound interest at a rate of 6% pa.

(a) How much is the fund worth at the end of 10 years?

(b) In which year will the fund reach \$70 000?

15. [Technology] In the first column of a spreadsheet, enter the numbers from 1 to 30 on separate rows. In the first 30 rows of the second column, enter the formula

$$\frac{20\,256{\cdot}52 \times 1{\cdot}075 \times (1{\cdot}075^n - 1)}{0{\cdot}075}$$

for the value of a superannuation investment, where n is the value given in the first column.

(a) Which value of n is the first to give a superannuation amount greater than \$750 000?

(b) Compare this answer with your answer to question 7(c).

(c) Try to do question 8(d) in the same way.

16. [Technology] If you have access to a program like Excel™, try checking your answers to questions 3 to 11 using the built-in financial functions. In particular, the built-in Excel™ function `FV(rate, nper, pmt, pv, type)`, which calculates the future value of an investment, seems to produce an answer different from what might be expected. Investigate this and explain the difference.

NOTE: The following two questions illustrate an alternative approach to superannuation questions, using a recursive method to generate the appropriate GP. The advantage of the method is that its steps follow the progress of a banking statement.

17. [The recursive method] At the start of each month, Cecilia deposits $\$M$ into a savings scheme paying 1% per month, compounded monthly. Let A_n be the amount in her account at the end of the nth month.

(a) Explain why $A_1 = 1{\cdot}01\,M$.

(b) Explain why $A_2 = 1{\cdot}01(M + A_1)$, and why $A_{n+1} = 1{\cdot}01(M + A_n)$, for $n \geq 2$.

(c) Use the recursive formulae in part (b), together with the value of A_1 in part (a), to obtain expressions for $A_2, A_3, \ldots, A_n$.

(d) Using the formula for the sum of n terms of a GP, show that $A_n = 101M(1{\cdot}01^n - 1)$.

(e) If each deposit is \$100, how much will be in the fund after three years?

(f) Hence find, correct to the nearest cent, how much each deposit M must be if Cecilia wants the fund to amount to \$30 000 at the end of five years.

18. [The recursive method] A couple saves \$100 at the start of each week in an account paying 10·4% pa interest, compounded weekly. Let A_n be the amount in the account at the end of the nth week.

(a) Explain why $A_1 = 1{\cdot}002 \times 100$, and why $A_{n+1} = 1{\cdot}002 \times (100 + A_n)$, for $n \geq 2$.

(b) Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.

(c) Using GP formulae, show that $A_n = 50\,100 \times (1{\cdot}01^n - 1)$.

(d) Hence find how many weeks it will be before the couple has \$100 000.

6 E Paying Off a Loan

Long-term loans such as housing loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time. The calculations associated with paying off a loan are therefore similar to the investment calculations of the previous section.

Developing the GP and Summing It: As with superannuation, the most straightforward method is to calculate the final value of each instalment as it earns compound interest, and then add these final values up as before, using the theory of GPs. But there is an extra complication — these instalments must be balanced against the initial loan, which is growing with compound interest. The loan is finally paid off when the amount owing is zero.

7

CALCULATIONS ASSOCIATED WITH PAYING OFF A LOAN:
To find the amount A_n still owing after n units of time:
- Find what the principal, earning compound interest, would amount to if no instalments were paid.
- Find what each instalment will amount to as it earns compound interest, then add up all these amounts, using the formula for the sum of a GP.
- The amount A_n still owing at the end of n units of time is

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest}).$$

The loan is paid off when the amount A_n still owing is zero.

WORKED EXERCISE:
Yianni and Eleni borrow \$20 000 from the Town and Country Bank to go on a trip to Constantinople. Interest is charged at 12% per annum, compounded monthly. They start repaying the loan one month after taking it out, and their monthly instalments are \$300.

(a) How much will they still owe the bank at the end of six years?

(b) How much interest will they have paid in these six years?

NOTE: When paying off a loan, the first payment is usually made one unit of time after the loan is taken out. But read the question carefully!

SOLUTION:

(a) The monthly interest rate is 1%, so $1 + R = 1{\cdot}01$.
The initial loan of \$20 000, after 72 months, amounts to $20\,000 \times 1{\cdot}01^{72}$.
The 1st instalment is invested for 71 months and so amounts to $300 \times 1{\cdot}01^{71}$.
The 2nd instalment is invested for 70 months and so amounts to $300 \times 1{\cdot}01^{70}$.
The 71st instalment is invested for 1 month and so amounts to $300 \times 1{\cdot}01^{1}$.
The 72nd and last instalment is invested for no time at all and so amounts to 300.
Hence the amount A_{72} still owing at the end of 72 months is

$$\begin{aligned} A_{72} &= (\text{principal plus interest}) - (\text{instalments plus interest}) \\ &= 20\,000 \times 1{\cdot}01^{72} - (300 + 300 \times 1{\cdot}01 + \cdots + 300 \times 1{\cdot}01^{71}). \end{aligned}$$

The bit in brackets is a GP with first term $a = 300$, ratio $r = 1{\cdot}01$ and 72 terms.

$$\text{Hence}\quad A_{72} = 20\,000 \times 1{\cdot}01^{72} - \frac{a(r^{72}-1)}{r-1} \qquad \text{(Find the sum of the GP.)}$$
$$= 20\,000 \times 1{\cdot}01^{72} - \frac{300 \times (1{\cdot}01^{72}-1)}{0{\cdot}01}$$
$$= 20\,000 \times 1{\cdot}01^{72} - 30\,000 \times (1{\cdot}01^{72}-1)$$
$$\doteqdot \$9529 \quad \text{(correct to the nearest dollar).}$$

(b) Total instalments over six years $= 300 \times 72$
$= \$21\,600.$

Reduction in loan over six years $= 20\,000 - 9529$
$= \$10\,471.$

Hence interest charged $= 21\,600 - 10\,471$
$= \$11\,129$ (more than half the original loan).

WORKED EXERCISE: [Finding what instalments should be paid]
Ali takes out a loan of \$10 000 to buy a car. He will repay the loan in five years, paying 60 equal monthly instalments, beginning one month after he takes out the loan. Interest is charged at 6% pa, compounded monthly.

Find how much the monthly instalment should be, correct to the nearest cent.

SOLUTION:
The monthly interest rate is 0·5%, so $1 + R = 1{\cdot}005$. Let each instalment be $\$M$.
First calculate the amount A_{60} still owing at the end of 60 months.
Then find M by setting A_{60} equal to zero.

The initial loan of \$10 000, after 60 months, amounts to $10\,000 \times 1{\cdot}005^{60}$.
The 1st instalment is invested for 59 months and so amounts to $M \times 1{\cdot}005^{59}$.
The 2nd instalment is invested for 58 months and so amounts to $M \times 1{\cdot}005^{58}$.
The 59th instalment is invested for 1 month and so amounts to $M \times 1{\cdot}005^{1}$.
The 60th and last instalment is invested for no time at all and so amounts to M.
Hence the amount A_{60} still owing at the end of 60 months is

$$A_{60} = \text{(principal plus interest)} - \text{(instalments plus interest)}$$
$$= 10\,000 \times 1{\cdot}005^{60} - (M + M \times 1{\cdot}005 + \cdots + M \times 1{\cdot}005^{59}).$$

The bit in brackets is a GP with first term $a = M$, ratio $r = 1{\cdot}005$ and 60 terms.

$$\text{Hence}\quad A_{60} = 10\,000 \times 1{\cdot}005^{60} - \frac{a(r^{60}-1)}{r-1}$$
$$= 10\,000 \times 1{\cdot}005^{60} - \frac{M(1{\cdot}005^{60}-1)}{0{\cdot}005}$$
$$= 10\,000 \times 1{\cdot}005^{60} - 200M(1{\cdot}005^{60}-1). \qquad \text{(Notice that } 0{\cdot}005 = \tfrac{1}{200}.\text{)}$$

But the loan is exactly paid off in these 5 years, so $A_{60} = 0$.

$$\text{Hence}\quad 10\,000 \times 1{\cdot}005^{60} - 200M(1{\cdot}005^{60}-1) = 0$$
$$200M(1{\cdot}005^{60}-1) = 10\,000 \times 1{\cdot}005^{60}$$
$$\boxed{\div 200} \qquad M(1{\cdot}005^{60}-1) = 50 \times 1{\cdot}005^{60}$$
$$M = \frac{50 \times 1{\cdot}005^{60}}{1{\cdot}005^{60}-1}$$
$$\doteqdot \$193.33.$$

Finding the Length of the Loan: A loan is fully repaid when the amount A_n still owing is zero. Thus finding the length of a loan means solving an equation for the index n, a process that requires logarithms.

WORKED EXERCISE:

Natasha and Richard take out a loan of \$200 000 on 1st January 2002 to buy a house. They will repay the loan in monthly instalments of \$2200. Interest is charged at 12% pa, compounded monthly.

(a) Find a formula for the amount owing at the end of n months.

(b) How much is owing after five years?

(c) How long does it takes to repay:
  (i) the full loan?
  (ii) half the loan?

(d) Why would instalments of \$1900 per month never repay the loan?

SOLUTION:

(a) The monthly interest rate is 1%, so $1 + R = 1{\cdot}01$.
The initial loan, after n months, amounts to $200\,000 \times 1{\cdot}01^n$.
The 1st instalment is invested for $n-1$ months and so amounts to $2200 \times 1{\cdot}01^{n-1}$.
The 2nd instalment is invested for $n-2$ months and so amounts to $2200 \times 1{\cdot}01^{n-2}$.
The nth and last instalment is invested for no time at all and so amounts to 2200.
Hence the amount A_n still owing at the end of n months is

$$\begin{aligned} A_n &= \text{(principal plus interest)} - \text{(instalments plus interest)} \\ &= 200\,000 \times 1{\cdot}01^n - (2200 + 2200 \times 1{\cdot}01 + \cdots + 2200 \times 1{\cdot}01^{n-1}). \end{aligned}$$

The bit in brackets is a GP with first term $a = 2200$, ratio $r = 1{\cdot}01$ and n terms.

$$\begin{aligned} \text{Hence } A_n &= 200\,000 \times 1{\cdot}01^n - \frac{a(r^n-1)}{r-1} \\ &= 200\,000 \times 1{\cdot}01^n - \frac{2200 \times (1{\cdot}01^n - 1)}{0{\cdot}01} \\ &= 200\,000 \times 1{\cdot}01^n - 220\,000 \times (1{\cdot}01^n - 1) \\ &= 220\,000 - 20\,000 \times 1{\cdot}01^n. \end{aligned}$$

(b) To find the amount owing after 5 years, substitute $n = 60$:

$$\begin{aligned} A_{60} &= 220\,000 - 20\,000 \times 1{\cdot}01^{60} \\ &\doteqdot \$183\,666 \qquad \text{(This is still almost as much as the original loan!)} \end{aligned}$$

(c) (i) To find when the loan is repaid, put $A_n = 0$:

$$\begin{aligned} 220\,000 - 20\,000 \times 1{\cdot}01^n &= 0 \\ 20\,000 \times 1{\cdot}01^n &= 220\,000 \\ \boxed{\div 20\,000} \qquad 1{\cdot}01^n &= 11 \\ n &= \log_{1\cdot01} 11 \qquad \text{(Convert to a logarithmic equation.)} \\ n &= \frac{\log_{10} 11}{\log_{10} 1{\cdot}01} \qquad \text{(Use the change-of-base formula.)} \\ &\doteqdot 20 \text{ years and 1 month.} \end{aligned}$$

(ii) To find when the loan is half repaid, put $A_n = 100\,000$:

$$220\,000 - 20\,000 \times 1{\cdot}01^n = 100\,000$$
$$20\,000 \times 1{\cdot}01^n = 120\,000$$
$$\boxed{\div 20\,000} \qquad 1{\cdot}01^n = 6$$
$$n = \log_{1\cdot 01} 6 \qquad \text{(Convert to a logarithmic equation.)}$$
$$n = \frac{\log_{10} 6}{\log_{10} 1{\cdot}01} \qquad \text{(Use the change-of-base formula.)}$$
$$\doteqdot 15 \text{ years.}$$

Notice that this is about three-quarters, not half, the total time of the loan.

(d) With a loan of \$200 000 at an interest rate of 1% per month,

$$\text{initial interest per month} = 200\,000 \times 0{\cdot}01$$
$$= \$2000.$$

This means that at the start of the loan, \$2000 of the instalment is required just to pay the interest. Hence with repayments of only \$1900, the debt would increase rather than decrease.

The Alternative Approach Using Recursion: As with superannuation, the GP involved in a loan-repayment calculation can also be developed using a recursive method, whose steps follow the progress of a banking statement. Again, this method is developed in two structured questions at the end of the Challenge section in the following exercise.

Exercise 6E

NOTE: As in the previous exercise, questions 1 and 2 have been heavily structured to follow the approach given in the two worked exercises above. There are several other satisfactory approaches, including the recursive method outlined in questions 17 and 18. If a different approach is chosen, the structuring in the first three questions below can be ignored.

1. On 1st January 2005, Lizbet borrows \$501 from a bank for four years at an interest rate of 10% pa. She repays the loan with four equal instalments of \$158.05 at the end of each year.

(a) Use the compound interest formula to show that the initial loan amounts to \$733.51 at the end of four years.

(b) (i) What is the value of the first instalment on 31st December 2008, having been invested for three years?

(ii) What is the value of the second instalment on this date?

(iii) What is the value of the third instalment?

(iv) What is the value of the fourth (and last) instalment?

(v) Find the total value of all the instalments on 31st December 2008 and hence show that Lizbet has now repaid the loan.

(c) (i) Write down the four answers to parts (i) to (iv) above in increasing order and notice that they form a GP.

(ii) Write down the first term, common ratio and number of terms.

(iii) Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part (b) (v).

2. Suppose that on 1st April 2005 a loan of \$5600 is made, which is repayed with equal instalments of \$1293.46 made on 31st March each year for five years, beginning in 2006. The loan attracts interest at 5% pa, compounded annually.

(a) Use the compound interest formula to show that the initial loan amounts to \$7147.18 by 31st March 2010.

(b) In each part, round your answer correct to the nearest cent.

(i) What is the value of the first instalment on 31st March 2010?

(ii) What is the value of the second instalment on this date?

(iii) Do the same for the third, fourth and fifth instalments.

(iv) Find the total value of the instalments on 31st March 2010 and hence show that the loan has been repaid.

(c) (i) Write down your answers to parts (i) to (iii) above in increasing order and notice that they form a GP.

(ii) Write down the first term, common ratio and number of terms.

(iii) Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part (b) (iv).

3. Lome took out a loan with Tornado Credit Union for \$15 000, to be repaid in 15 equal annual instalments of \$1646.92 on 1st April each year. Compound interest is charged at 7% per annum.

(a) Let A_{15} be the amount owed at the end of 15 years.

(i) Use the compound interest formula to show that $15\,000 \times (1{\cdot}07)^{15}$ is owed on the initial loan after 15 years.

(ii) How much does the first instalment amount to at the end of the loan, having been invested for 14 years?

(iii) How much does the second instalment amount to at the end of 13 years?

(iv) What is the value of the second-last instalment?

(v) What is the worth of the last contribution, invested for no time at all?

(vi) Hence write down an expression involving a series for A_{15}.

(b) Show that the final amount owed is

$$A_{15} = 15\,000 \times (1{\cdot}07)^{15} - \frac{1646{\cdot}92 \times (1{\cdot}07^{15} - 1)}{0{\cdot}07}.$$

(c) Evaluate A_{15} and hence show that the loan has been repaid.

4. Matts signed a mortgage agreement for \$100 000 with a bank for 20 years at an interest rate of 6% per annum, compounded monthly (that is, at 0·5% per month).

(a) Let M be the size of each repayment to the bank, and let A_{240} be the amount owing on the loan after 20 years.

(i) What does the initial loan amount to after 20 years?

(ii) Write down the amount that the first repayment grows to by the end of the 240th month.

(iii) Do the same for the second repayment and for the last repayment.

(iv) Hence write down a series expression for A_{240}.

(b) Hence show that $A_{240} = 100\,000 \times 1{\cdot}005^{240} - 200 \times M(1{\cdot}005^{240} - 1)$.

(c) Explain why the bank puts $A_{240} = 0$.

(d) Hence find M, correct to the nearest cent.

(e) How much will Matts have paid the bank over the period of the loan?

5. I took out a personal loan of \$10 000 with a bank for five years at an interest rate of 18% per annum, compounded monthly (that is, at 1·5% per month).

(a) Let M be the size of each instalment to the bank, and let A_{60} be the amount owing on the loan after 60 months.

(i) What does the initial loan amount to after 60 months?

(ii) Write down the amount that the first instalment grows to by the end of the 60th month.

(iii) Do the same for the second instalment and for the last instalment.

(iv) Hence write down a series expression for A_{60}.

(b) Hence show that $0 = 10\,000 \times 1{\cdot}015^{60} - \dfrac{M(1{\cdot}015^{60} - 1)}{0{\cdot}015}$.

(c) Hence find M, correct to the nearest dollar.

DEVELOPMENT

6. A couple take out a \$165 000 mortgage on a house, and they agree to pay the bank \$1700 per month. The interest rate on the loan is 9% per annum, compounded monthly, and the contract requires that the loan be paid off within 15 years.

(a) Let A_{180} be the balance on the loan after 15 years. Find a series expression for A_{180}.

(b) Show that $A_{180} = 165\,000 \times 1{\cdot}0075^{180} - \dfrac{1700(1{\cdot}0075^{180} - 1)}{0{\cdot}0075}$.

(c) Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

7. A couple take out a \$250 000 mortgage on a house, and they agree to pay the bank \$2000 per month. The interest rate on the loan is 7·2% per annum, compounded monthly, and the contract requires that the loan be paid off within 20 years.

(a) Let A_n be the balance on the loan after n months. Find a series expression for A_n.

(b) Hence show that $A_n = 250\,000 \times 1{\cdot}006^n - \dfrac{2000(1{\cdot}006^n - 1)}{0{\cdot}006}$.

(c) Find the amount owing on the loan at the end of the tenth year, and state whether this is more or less than half the amount borrowed.

(d) Find A_{240}, and hence show that the loan is actually paid out in less than twenty years.

(e) If it is paid out after n months, show that $1{\cdot}006^n = 4$, and hence that $n = \dfrac{\log 4}{\log 1{\cdot}006}$.

(f) Find how many months early the loan is paid off.

8. A company borrows \$500 000 from the bank at an interest rate of 5·25% per annum, to be repaid in monthly instalments. The company repays the loan at the rate of \$10 000 per month.

(a) Let A_n be the amount owing at the end of the nth month. Show that

$$A_n = 500\,000 \times 1{\cdot}004\,375^n - \frac{10\,000(1{\cdot}004\,375^n - 1)}{0{\cdot}004\,375}.$$

(b) Given that the loan is paid off, use the result in part (a) show that $1{\cdot}004\,375^n = 1{\cdot}28$.

(c) Use logarithms or trial-and-error to find how long it will take to pay off the loan. Give your answer in whole months.

9. As can be seen from these questions, the calculations involved with reducible loans are reasonably complex. For that reason, it is sometimes convenient to convert the reducible interest rate into a simple interest rate. Suppose that a mortgage is taken out on a \$180 000 house at 6·6% reducible interest per annum for a period of 25 years, with payments of amount M made monthly.

(a) Using the usual pronumerals, explain why $A_{300} = 0$.

(b) Show that $A_{300} = 180\,000 \times 1{\cdot}0055^{300} - \dfrac{M(1{\cdot}0055^{300} - 1)}{0{\cdot}0055}$.

(c) Find the size of each repayment to the bank.

(d) Hence find the total paid to the bank, correct to the nearest dollar, over the life of the loan.

(e) What amount is therefore paid in interest? Use this amount and the simple interest formula to calculate the simple interest rate per annum over the life of the loan, correct to two significant figures.

10. A personal loan of \$15 000 is borrowed from the Min Hua Finance Company at a rate of $13\frac{1}{2}\%$ per annum over five years, compounded monthly. Let M be the amount of each monthly instalment.

(a) Show that $15\,000(1{\cdot}011\,25)^{60} - \dfrac{M(1{\cdot}011\,25^{60} - 1)}{0{\cdot}011\,25} = 0$.

(b) What is the monthly instalment necessary to pay back the loan? Give your answer correct to the nearest dollar.

11. [Problems with rounding] Most questions so far have asked you to round monetary amounts correct to the nearest dollar. This is not always wise, as this question demonstrates. A personal loan for \$30 000 is approved with the following conditions. The reducible interest rate is 13·3% per annum, with payments to be made at six-monthly intervals over five years.

(a) Find the size of each instalment, correct to the nearest dollar.

(b) Using this amount, show that $A_{10} \neq 0$, that is, the loan is not paid off in five years.

(c) Explain why this has happened.

12. A couple have worked out that they can afford to pay \$19 200 each year in mortgage payments. The current home loan rate is 7·5% per annum, with equal payments made monthly over a period of 25 years.

(a) Let P be the principal borrowed and A_{300} the amount owing after 25 years. Show that $A_{300} = P \times 1{\cdot}006\,25^{300} - \dfrac{1600(1{\cdot}006\,25^{300} - 1)}{0{\cdot}006\,25}$.

(b) Hence determine the maximum amount that the couple can borrow and still pay off the loan? Round your answer down to the nearest dollar.

13. The current credit card rate of interest on Bankerscard is 23% per annum, compounded monthly.

(a) If a cardholder can afford to repay \$1500 per month on the card, what is the maximum value of purchases that can be made in one day if the debt is to be paid off in two months?

(b) How much would be saved in interest payments if the cardholder instead saved up the money for two months before making the purchase?

CHALLENGE

14. Some banks offer a 'honeymoon' period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed \$170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay \$1650 per month to the bank. The standard rate of interest is $8\frac{1}{2}\%$ pa, but the bank also offers a special rate of 6% pa for one year to people buying their first home.

(a) Calculate the amount the couple would owe at the end of the first year, using the special rate of interest.

(b) Use this value as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the loan contract?

15. Over the course of years, a couple have saved \$300 000 in a superannuation fund. Now that they have retired, they are going to draw on that fund in equal monthly pension payments for the next 20 years. The first payment is at the beginning of the first month. At the same time, any balance will be earning interest at $5\frac{1}{2}\%$ per annum, compounded monthly. Let B_n be the balance left immediately after the nth payment, and let M be the amount of the pension instalment. Also, let $P = 300\,000$ and R be the monthly interest rate.

(a) Show that $B_n = P \times (1+R)^{n-1} - \dfrac{M\big((1+R)^n - 1\big)}{R}$.

(b) Why is $B_{240} = 0$?

(c) What is the value of M?

16. A company buys machinery for \$500 000 and pays it off by 20 equal six-monthly instalments, the first payment being made six months after the loan is taken out. If the interest rate is 12% pa, compounded monthly, how much will each instalment be?

17. [Technology] In the first column of a spreadsheet, enter the numbers from 1 to 60 on separate rows. In the first 60 rows of the second column, enter the formula

$$500\,000 \times 1{\cdot}004\,375^n - \frac{10\,000 \times (1{\cdot}004\,375^n - 1)}{0{\cdot}004\,375}$$

for the balance of a loan repayment, where n is the value given in the first column.

(a) Observe the pattern of figures in the second column. Notice that the balance decreases more slowly at first and more quickly towards the end of the loan.

(b) Which value of n is the first to give a balance less than or equal to zero?

(c) Compare this answer with your answer to question 8.

(d) Try to do question 7(f) in the same way.

NOTE: The following two questions illustrate the alternative approach to loan repayment questions, using a recursive method to generate the appropriate GP.

18. [The recursive method] A couple buying a house borrow $\$P = \$150\,000$ at an interest rate of 6% pa, compounded monthly. They borrow the money at the beginning of January, and at the end of every month, they pay an instalment of $\$M$. Let A_n be the amount owing at the end of n months.

(a) Explain why $A_1 = 1{\cdot}005\,P - M$.

(b) Explain why $A_2 = 1{\cdot}005\,A_1 - M$, and why $A_{n+1} = 1{\cdot}005\,A_n - M$, for $n \geq 2$.

(c) Use the recursive formulae in part (b), together with the value of A_1 in part (a), to obtain expressions for $A_2, A_3, \ldots, A_n$.

(d) Using GP formulae, show that $A_n = 1{\cdot}005^n\,P - 200M(1{\cdot}005^n - 1)$.

(e) Hence find, correct to the nearest cent, what each instalment should be if the loan is to be paid off in 20 years?

(f) If each instalment is \$1000, how much is still owing after 20 years?

19. [The recursive method] Eric and Enid borrow $\$P$ to buy a house at an interest rate of 9·6% pa, compounded monthly. They borrow the money on 15th September, and on the 14th day of every subsequent month, they pay an instalment of $\$M$. Let A_n be the amount owing after n months have passed.

(a) Explain why $A_1 = 1{\cdot}008\,P - M$, and why $A_{n+1} = 1{\cdot}008\,A_n - M$, for $n \geq 2$.

(b) Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.

(c) Using GP formulae, show that $A_n = 1{\cdot}008^n\,P - 125M(1{\cdot}008^n - 1)$.

(d) If the maximum instalment they can afford is \$1200, what is the maximum they can borrow, if the loan is to be paid off in 25 years? (Answer correct to the nearest dollar.)

(e) Put $A_n = 0$ in part (c), and solve for n. Hence find how long will it take to pay off the loan of \$100 000 if each instalment is \$1000. (Round up to the next month.)

6 F Rates of Change

A *rate of change* is the instantaneous rate at which some quantity Q is changing. Thus the rate of change is the derivative $\dfrac{dQ}{dt}$ of Q with respect to time t. Graphically, it is the gradient of the tangent to the graph of Q.

Differentiating to Find the Rate: When a quantity Q is given as a function of time t, differentiation will give a formula for the rate of change of Q.

NOTE: In problems, the pronumeral Q is usually replaced by a more convenient pronumeral, such as P for population or V for volume or M for mass.

WORKED EXERCISE:

A cockroach plague hit the suburb of Berrawong last year, but was gradually brought under control. The council estimated that the cockroach population P, in millions, t months after 1st January, was given by

$$P = 7 + 6t - t^2.$$

(a) Differentiate to find the rate of change $\dfrac{dP}{dT}$ of the cockroach population.

(b) Find the cockroach population on 1st January and the rate at which the population was increasing at that time.

(c) When did the council manage to stop the cockroach population increasing any further, and what was the population then?

(d) When were the cockroaches finally eliminated?

(e) What was the average rate of increase in the population from 1st January to 1st April?

(f) Draw the graphs of P and $\dfrac{dP}{dt}$ against time. Add to your graph of P the tangents and chords corresponding to parts (b), (c) and (e).

SOLUTION:

(a) The population function is $P = 7 + 6t - t^2$.

Differentiating, the rate of change is $\dfrac{dP}{dt} = 6 - 2t$.

(b) When $t = 0$, $P = 7$,

and when $t = 0$, $\dfrac{dP}{dt} = 6$.

Thus on 1st January, the population is 7 million,
and the population is increasing at 6 million per month.

(c) Put $\dfrac{dP}{dt} = 0$ (to find the time when the population stopped increasing).

Then
$$\begin{aligned} 6 - 2t &= 0 \\ 2t &= 6 \\ t &= 3. \end{aligned}$$

When $t = 3$,
$$\begin{aligned} P &= 7 + 18 - 9 \\ &= 16. \end{aligned}$$

Thus the population stopped increasing when $t = 3$, that is, on 1st April,
and the population then was 16 million.

(d) Put $P = 0$ (to find the time when the population was zero).

Then
$$\begin{aligned} 7 + 6t - t^2 &= 0 \\ t^2 - 6t - 7 &= 0 \\ (t-7)(t+1) &= 0 \\ t &= 7 \text{ or } -1. \end{aligned}$$
(The negative solution $t = -1$ is irrelevant.)

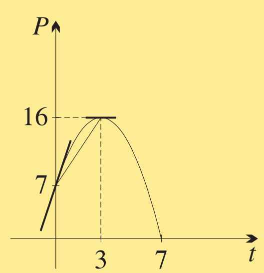
Hence $t = 7$, and the cockroaches were finally eliminated on 1st August.

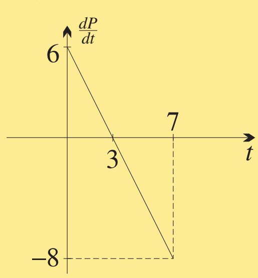
(e) From part (b), the population was 7 million on 1st January,
and from part(c), the population was 16 million on 1st April.

Hence average rate of increase $= \dfrac{16 - 7}{3}$

$= 3$ million per month.

(f)





NOTE: A rate of change is always instantaneous unless otherwise stated, and is the gradient of a tangent. Part (e) of the worked exercise above specifically asked for an *average rate of change*, which is the gradient of a chord.

Integrating to Find the Function: In many situations, what is given is the rate $\dfrac{dQ}{dt}$ at which a quantity Q is changing. The original function Q can then be found by integration. To evaluate the constant of integration, the value of Q at some particular time t is needed.

WORKED EXERCISE:
A tank contains 40 000 litres of water. When the draining valve is opened, the volume V in litres of water in the tank decreases at a variable rate given by $\dfrac{dV}{dt} = -1500 + 30t$, where t is the time in seconds after opening the valve. Once the water stops flowing, the valve shuts off.

(a) When does the water stop flowing?

(b) Give a common-sense reason why the rate $\dfrac{dV}{dt}$ is negative up to this time.

(c) Integrate to find the volume of water in the tank at time t.

(d) How much water has flowed out of the tank and how much remains?

SOLUTION:

(a) Put $\dfrac{dV}{dt} = 0$.

Then $-1500 + 30t = 0$

$t = 50$,

so it takes 50 seconds for the flow to stop.

(b) During this 50 seconds, the water is flowing out of the tank.

Hence the volume V is decreasing and so the derivative $\dfrac{dV}{dt}$ is negative.

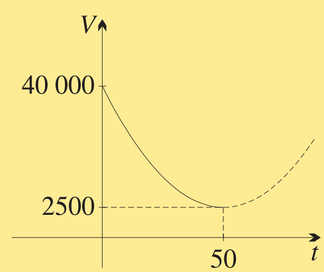
(c) Integrating, $V = -1500t + 15t^2 + C$.

It is given that when $t = 0$, $V = 40\,000$,

and substituting, $40\,000 = 0 + 0 + C$

$C = 40\,000$.

Hence $V = 40\,000 - 1500t + 15t^2$.



(d) When $t = 50$, $V = 40\,000 - (1500 \times 50) + (15 \times 2500)$

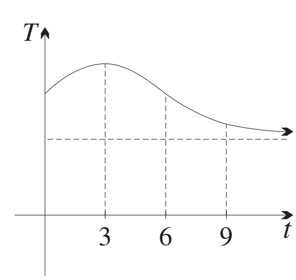
$= 2500$.

Hence the tank still holds 2500 litres when the valve closes,

so $40\,000 - 2500 = 37\,500$ litres has flowed out during the 50 seconds.

Questions with a Diagram or a Graph Instead of an Equation: In some problems about rates, a graph of some function is known, but its equation is unknown. Such problems require careful attention to zeroes and turning points and inflexions. An approximate sketch of another graph often needs to be drawn.

WORKED EXERCISE:
The graph to the right shows the temperature T of a patient suffering from Symond's syndrome at time t hours after her admission to hospital at midnight.



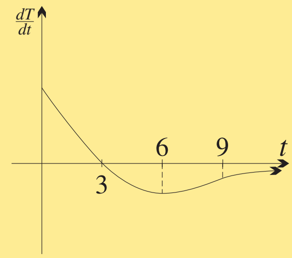
(a) When did her temperature reach its maximum?

(b) When was her temperature increasing most rapidly, and when was it decreasing most rapidly?

(c) What happened to her temperature eventually?

(d) Sketch the graph of the rate $\dfrac{dT}{dt}$ at which the temperature is changing.

SOLUTION:



(a) The maximum temperature occurred when $t = 3$, that is, at 3:00 pm.

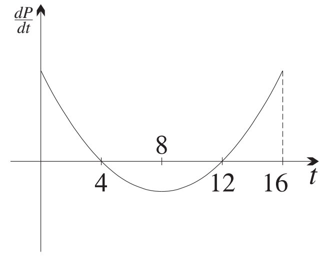
(b) The temperature was increasing most rapidly when $t = 0$, that is, at midnight. It was decreasing most rapidly when $t = 6$, that is, at 6:00 pm.

(c) The patient's temperature eventually stabilised.

(d) The graph of $\dfrac{dT}{dt}$ has a zero at $t = 3$ and a minimum turning point at $t = 6$.

As $t \to \infty$, $\dfrac{dT}{dt} \to 0$, so the t-axis is a horizontal asymptote.

WORKED EXERCISE:



Frog numbers were increasing in the Ranavilla district, but during a long drought, the rate of increase fell and actually became negative for a few years. The rate $\dfrac{dP}{dt}$ of population growth of the frogs has been graphed to the right as a function of the time t years after careful observations began.

(a) When was the frog population neither increasing nor decreasing?

(b) When was the frog population decreasing and when was it increasing?

(c) When was the frog population decreasing most rapidly?

(d) When, during the first 12 years, was the frog population at a maximum?

(e) When, during the years $4 \le t \le 16$, was the frog population at a minimum?

(f) Draw a possible graph of the frog population P against time t.

SOLUTION:

(a) The graph shows that $\dfrac{dP}{dt}$ is zero when $t = 4$ and again when $t = 12$.

These are the times when the frog population was neither increasing nor decreasing.

(b) The graph shows that $\dfrac{dP}{dt}$ is negative when $4 < t < 12$,

so the population was decreasing during the years $4 < t < 12$.

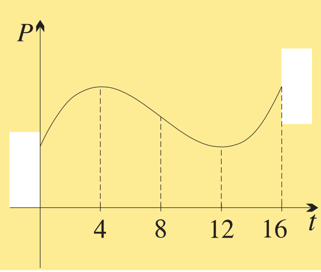
The graph shows that $\dfrac{dP}{dt}$ is positive when $0 < t < 4$ and when $12 < t < 16$,

so the population was increasing during the years $0 < t < 4$ and during $12 < t < 16$.

(c) The frog population was decreasing most rapidly when $t = 8$.

(d) The population was at a maximum when $x = 4$, because from parts (a) and (b), the population was rising before this and falling afterwards.

(e) Similarly, the population was minimum when $x = 12$.



(f) All that matters is to draw the possible graph of P so that its gradients are consistent with the graph of $\dfrac{dP}{dT}$.

These things were discussed above in parts (a)–(d). Also, the frog population must never fall below zero.

Rates Involving the Exponential Function: Many natural events involve a quantity that dies away gradually, with an equation that involves the exponential function. The following worked exercise uses the standard form $\int e^{ax+b} = \dfrac{1}{a}\, e^{ax+b} + C$ to evaluate the primitive of $3\,e^{-0{\cdot}02t}$. The full working is

$$\begin{aligned}\int 3\,e^{-0{\cdot}02t}\,dt &= 3 \times \frac{1}{-0{\cdot}02} \times e^{-0{\cdot}02t} + C\\ &= -3 \times \frac{100}{2} \times e^{-0{\cdot}02t} + C\\ &= -150\,e^{-0{\cdot}02t} + C.\end{aligned}$$

WORKED EXERCISE:

During a drought, the flow rate $\dfrac{dV}{dt}$ of water from Welcome Well gradually diminishes according to the formula $\dfrac{dV}{dt} = 3\,e^{-0{\cdot}02t}$, where V is the volume in megalitres of water that has flowed out during the first t days after time zero.

(a) Show that $\dfrac{dV}{dt}$ is always positive, and explain the physical significance of this.
(b) Find the volume V as a function of time t.
(c) How much water will flow from the well during the first 100 days?
(d) Describe the behaviour of V as $t \to \infty$, and find what percentage of the total flow comes in the first 100 days. Then sketch the function.

SOLUTION:

(a) Since $e^x > 0$ for all x, $\dfrac{dV}{dt} = 3\,e^{-0{\cdot}02t}$ is always positive.

The volume V is always increasing, because V is the volume that has flowed out of the well, and the water doesn't flow backwards into the well.

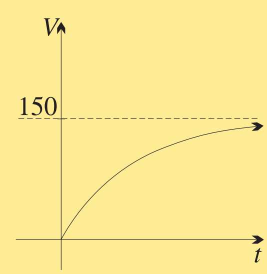
(b) The given rate is $\dfrac{dV}{dt} = 3\,e^{-0{\cdot}02t}$.

Integrating, $V = -150\,e^{-0{\cdot}02t} + C$. (See the calculation above.)
When $t = 0$, no water has flowed out, so $V = 0$,
and substituting, $0 = -150 \times e^0 + C$
$C = 150.$
Hence $V = -150\,e^{-0{\cdot}02t} + 150$
$= 150(1 - e^{-0{\cdot}02t}).$

(c) When $t = 100$, $V = 150(1 - e^{-2})$
$\doteqdot 129{\cdot}7$ megalitres.

(d) As $t \to \infty$, $e^{-0{\cdot}02t} \to 0$, and so $V \to 150$.

$$\text{Hence}\quad \frac{\text{flow in first 100 days}}{\text{total flow}} = \frac{150(1 - e^{-2})}{150}$$
$$= 1 - e^{-2}$$
$$= 0{\cdot}864\,66\ldots$$
$$\doteqdot 86{\cdot}5\%.$$



Exercise 6F

1. Find y as a function of t if:

(a) $\dfrac{dy}{dt} = 3$, and $y = -1$ when $t = 0$,

(b) $\dfrac{dy}{dt} = 1 - 2t$, and $y = 2$ when $t = 0$,

(c) $\dfrac{dy}{dt} = \cos t$, and $y = 1$ when $t = 0$,

(d) $\dfrac{dy}{dt} = e^t$, and $y = 0$ when $t = 0$.

2. Orange juice is being poured into a glass. After t seconds there are V ml of juice in the glass, where $V = 60t$.

(a) How much juice is in the glass after 3 seconds?

(b) Show that the glass was empty to begin with.

(c) If the glass takes 5 seconds to fill, what is its capacity?

(d) At what rate is the glass being filled?

3. Water is being pumped into a tank at the rate of $\dfrac{dV}{dt} = 300$ litres per minute, where V litres is the volume of water in the tank after t minutes of pumping. The tank had 1500 litres of water in it at time $t = 0$.

(a) Show that $V = 300t + 1500$.

(b) How long will the pump take to fill the tank if the tank holds 6000 litres?

4. The quantity of fuel, Q litres, in a tanker t minutes after it has started to empty is given by $Q = 200(400 - t^2)$. Initially the tanker was full.

(a) Find the initial quantity of fuel in the tanker.

(b) Find the quantity of fuel in the tanker after 15 minutes.

(c) Find the time taken for the tanker to empty.

(d) Show that $\dfrac{dQ}{dt} = -400t$, and hence find the rate at which the tanker is emptying after 5 minutes.

5. Water is flowing out of a tank at the rate of $\dfrac{dV}{dt} = 10t - 250$, where V is the volume in litres remaining in the tank at time t minutes after time zero.

(a) When does the water stop flowing?

(b) Given that the tank still has 20 litres left in it when the water flow stops, show that the volume V at any time is given by $V = 5t^2 - 250t + 3145$.

(c) How much water was initially in the tank?

6. A colony of ants is building a nest. The rate at which the ants are moving the earth is given by $\dfrac{dE}{dt} = t + 3$ cubic centimetres per minute.

(a) At what rate are the ants moving the earth:

(i) initially,

(ii) after 10 minutes?

(b) Integrate to find E as a function of t. [HINT: Find the constant of integration by assuming that when $t = 0$, $E = 0$.]

(c) How much earth is moved by the ants in:

(i) the first 10 minutes,

(ii) the next 10 minutes?

DEVELOPMENT

7. The share price $\$P$ of the Eastcom Bank t years after it opened on 1st January 1970 was $P = -0{\cdot}4t^2 + 4t + 2$.

(a) What was the initial share price?

(b) What was the share price after one year?

(c) At what rate was the share price increasing after two years?

(d) By letting $\dfrac{dP}{dt} = 0$, show that the maximum share price was \$12, at the start of 1975.

(e) The directors decided to close the bank when the share price fell back to its initial value. When did this happen?

8. Twenty-five wallabies are released on Wombat Island and the population is observed over the next six years. It is found that the rate of increase in the wallaby population is given by $\dfrac{dP}{dt} = 12t - 3t^2$, where time t is measured in years.

(a) Show that $P = 25 + 6t^2 - t^3$.

(b) After how many years does the population reach a maximum? [HINT: Let $\dfrac{dP}{dt} = 0$.]

(c) What is the maximum population?

(d) When does the population increase most rapidly? [HINT: Let $\dfrac{d^2P}{dt^2} = 0$.]

9. For a certain brand of medicine, the amount M present in the blood after t hours is given by $M = 3t^2 - t^3$, for $0 \le t \le 3$.

(a) Sketch a graph of M against t, showing any stationary points and points of inflexion.

(b) When is the amount of medicine in the blood a maximum?

(c) When is the amount of medicine increasing most rapidly?

10. When a jet engine starts operating, the rate of fuel burn, R kg per minute, t minutes after startup is given by $R = 10 + \dfrac{10}{1 + 2t}$.

(a) What is the rate of fuel burn after:

(i) 2 minutes, (ii) 7 minutes?

(b) What limiting value does R approach as t increases?

(c) Draw a sketch of R as a function of t.

(d) Show that approximately 83·5 kg of fuel is burned in the first 7 minutes.

11. The rate at which a perfume ball loses its scent over time is $\dfrac{dP}{dt} = -\dfrac{2}{t+1}$, where t is measured in days.

(a) Find P as a function of t if the initial perfume content is 6·8.

(b) How long will it be before the perfume in the ball has run out and it needs to be replaced? (Answer correct to the nearest day.)

12. A certain brand of medicine tablet is in the shape of a sphere with diameter 5 mm. The rate at which the pill dissolves is $\dfrac{dr}{dt} = -k$, where r is the radius of the sphere at time t hours, and k is a positive constant.

(a) Show that $r = \frac{5}{2} - kt$.

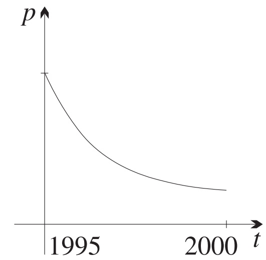
(b) If the pill dissolves completely in 12 hours, find k.

13. A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time t, is given by $\dfrac{dx}{dt} = 250(e^{-0\cdot 2t} - 1)$, where x is the height above the ground.

(a) What is its initial speed? (b) What is its eventual speed?

(c) Find x as a function of t, if the ball is initially 200 metres above the ground.

14. The graph shows the level of pollution in a river between 1995 and 2000. In 1995, the local council implemented a scheme to reduce the level of pollution in the lake. Comment briefly on whether this scheme worked and how the level of pollution changed. Include mention of the rate of change.



15. The graph to the right shows the share price P in Penn & Penn Stationery Suppliers t months after 1st January.

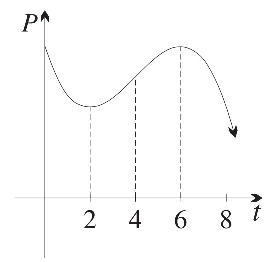
(a) When is the price maximum and when is it minimum?

(b) When is the price increasing and when is it decreasing?

(c) When is the share price increasing most rapidly?

(d) When is the share price increasing at an increasing rate?

(e) Sketch a possible graph of $\dfrac{dP}{dt}$ as a function of time t.



16. The graph to the right shows the rate $\dfrac{dW}{dt}$ at which the average weight W of bullocks at St Vidgeon station was changing t months after a drought was officially proclaimed.

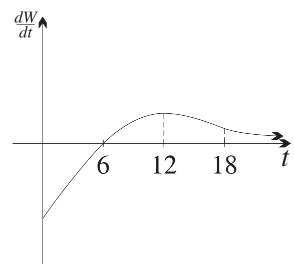
(a) When was the average weight decreasing and when was it increasing?

(b) When was the average weight at a minimum?

(c) When was the average weight increasing most rapidly?

(d) What appears to have happened to the average weight as time went on?

(e) Sketch a possible graph of the average weight W.



17. The number U of unemployed people at time t was studied over a period of time. At the start of this period, the number of unemployed was 600 000.

(a) Throughout the study, $\dfrac{dU}{dt} > 0$. What can be deduced about U over this period?

(b) The study also found that $\dfrac{d^2U}{dt^2} < 0$. What does this indicate about the changing unemployment level?

(c) Sketch a graph of U against t, showing this information.

18. A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by $\dfrac{dV}{dt} = -2 + \frac{1}{10}t\ \text{m}^3/\text{s}$.

(a) What is the initial flow rate, when the tap is fully on?

(b) How long does it take to turn the tap off?

(c) Given that when the tap has been turned off there are still $500\ \text{m}^3$ of water left in the tank, find V as a function of t.

(d) Hence find how much water is released during the time it takes to turn the tap off.

(e) Suppose that it is necessary to let out a total of $300\,\text{m}^3$ from the tank. How long should the tap be left fully on before gradually turning it off?

19. A scientist studying an insect colony estimates the number $N(t)$ of insects after t months to be $N(t) = \dfrac{A}{2 + e^{-t}}$.

(a) When the scientist begins measuring, the number of insects in the colony is estimated to be 3×10^5. Find A.

(b) What is the population of the colony one month later?

(c) How many insects would you expect to find in the nest after a long time?

(d) Find an expression for the rate at which the population increases with time.

CHALLENGE

20. James had a full drink bottle containing 500 ml of Gatorade™. He drank from it so that the volume V ml of Gatorade™ in the bottle changed at a rate given by $\dfrac{dV}{dt} = (\frac{2}{5}t - 20)$ ml/s.

(a) Find a formula for V.

(b) Show that it took James 50 seconds to drink the contents of the bottle.

(c) How long, correct to the nearest second, did it take James to drink half the contents of the bottle?

21. Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\dfrac{dI}{dt} = -5 + 4\cos\frac{\pi}{12}t$, where I is the tonnage of ice on the mountain at time t in hours since 2:00 am on 20th October.

(a) It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find I as a function of t.

(b) Explain, from the given rate, why the ice is always melting.

(c) The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.

6 G Natural Growth and Decay

The most important result in Chapter Two was that the derivative of the exponential function $y = e^x$ is $\dfrac{dy}{dx} = e^x$, the same function. When dealing with rates, the result looks more familiar if the pronumerals are replaced by Q and t:

$$\text{If } Q = e^t, \text{ then } \frac{dQ}{dt} = e^t. \quad \text{That is, } \frac{dQ}{dt} = Q.$$

Geometrically, this means that the gradient $\dfrac{dQ}{dt}$ at any point on the graph is equal to the height Q of the graph at that point.

Replace t by kt. The derivative of the function $Q = e^{kt}$ is k times its derivative:

$$\text{If } Q = e^{kt}, \text{ then } \frac{dQ}{dt} = k\,e^{kt}. \quad \text{That is, } \frac{dQ}{dt} = kQ.$$

Geometrically, this means that the gradient $\dfrac{dQ}{dt}$ at any point on the graph is equal to k times the height Q of the graph at that point.

These functions $Q = e^{kt}$ are the functions used in this section to model events.

Natural Growth: Consider a growing population P, of people in some country, or rabbits on an island, or bacteria in a laboratory culture. The more individuals there are in the population, the more new individuals are born in each unit of time. Thus the rate at which the population is growing at any time t should be roughly proportional to the number of individuals in the population at that time. Writing this in symbols,

$$\frac{dP}{dt} = kP, \quad \text{where } k \text{ is a constant of proportionality.}$$

Such a situation is called *natural growth*, and a population growing in this way is said to obey the *law of natural growth*.

Geometrically, this means that the gradient of the population graph at any point is proportional to the height of the graph at that point.

The Natural Growth Theorem: As explained above, exponential functions are exactly what is needed to model such a situation.

8 NATURAL GROWTH: Suppose that the rate of change of Q is proportional to Q:

$$\frac{dQ}{dt} = kQ, \quad \text{where } k \text{ is the constant of proportionality.}$$

Then $Q = Q_0\, e^{kt}$, where Q_0 is the value of Q at time $t = 0$.

PROOF:

A. Substituting the function $Q = Q_0\, e^{kt}$ into the equation $\dfrac{dQ}{dt} = kQ$,

$$\begin{aligned} \text{LHS} &= \frac{dQ}{dt} \\ &= \frac{d}{dt}\left(Q_0\, e^{kt}\right) \\ &= kQ_0\, e^{kt} \\ &= kQ \\ &= \text{RHS}, \end{aligned}$$

so the function satisfies the differential equation, as required.

B. Secondly, substituting $t = 0$, into the function $Q = Q_0\, e^{kt}$,

$$\begin{aligned} Q &= Q_0\, e^{k\times 0} \\ &= Q_0 \times 1, \quad \text{since } e^0 = 1, \\ &= Q_0, \end{aligned}$$

so the initial value of Q is Q_0, as required.

(It is assumed without further proof that there are no other such functions.)

NOTE: Questions often require a proof that a given function is a solution of the given differential equation. This should be done by substitution of the function into the differential equation, fully set out as in the proof above and in the worked exercises below.

Problems Involving Natural Growth: The constant k can usually be calculated from the data in the problem. The approximate value of k should then be held in the memory of the calculator for later use.

WORKED EXERCISE:

The rabbit population P on Goat Island was estimated to be 1000 at the start of the year 1995 and 3000 at the start of the year 2000. The population is growing according to the law of natural growth. That is, $\dfrac{dP}{dt} = kP$, for some constant k, where P is the rabbit population t years after the start of 1995.

(a) Show that $P = P_0\, e^{kt}$ satisfies the differential equation $\dfrac{dP}{dt} = kP$.

(b) Find the values of P_0 and k, then sketch the graph of P as a function of t.

(c) How many rabbits are there at the start of 2003? (Answer correct to the nearest ten rabbits.)

(d) When will the population be 10 000 (correct to the nearest month)?

(e) Find, correct to the nearest 10 rabbits per year, the rate at which the population is increasing:
(i) when there are 8000 rabbits, (ii) at the start of 1997.

SOLUTION:

(a) Substituting the function $P = P_0\, e^{kt}$ into the equation $\dfrac{dP}{dt} = kP$,

$$\begin{aligned}\text{LHS} &= \frac{dP}{dt}\\ &= \frac{d}{dt}(P_0\, e^{kt})\\ &= kP_0\, e^{kt},\\ &= kP\\ &= \text{RHS},\end{aligned}$$

so the function satisfies the differential equation, as required.

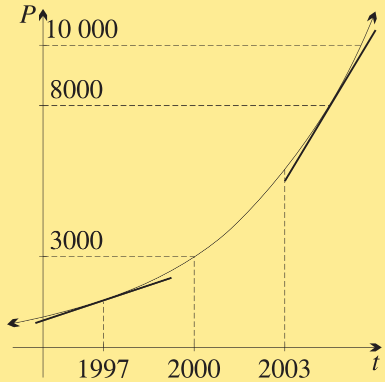
(b) When $t = 0$, $P = 1000$, so

$$\begin{aligned}1000 &= P_0 \times e^0\\ P_0 &= 1000.\end{aligned}$$

When $t = 5$, $P = 3000$, so

$$\begin{aligned}3000 &= 1000\, e^{5k}\\ e^{5k} &= 3\\ 5k &= \log_e 3\\ k &= \tfrac{1}{5}\log_e 3.\end{aligned}$$

(Approximate k and store it in the memory:
$k = 0{\cdot}219\,722\ldots$)



(c) At the start of 2003, when $t = 8$,

$$\begin{aligned}P &= 1000\, e^{8k}\\ &\doteqdot 5800 \text{ rabbits.}\end{aligned}$$

(d) Substituting $P = 10\,000$,

$$\begin{aligned}10\,000 &= 1000\, e^{kt}\\ e^{kt} &= 10\\ kt &= \log_e 10\\ t &= \frac{\log_e 10}{k}\\ &= 10{\cdot}479\,516\ldots\\ &\doteqdot 10 \text{ years and 6 months,}\end{aligned}$$

so the population will reach 10 000 about 6 months into 2005.

(e) (i) Substituting $P = 8000$ into $\dfrac{dP}{dt} = kP$,

$$\frac{dP}{dt} = 8000k$$
$$\doteqdot 1760 \text{ rabbits per year.}$$

(ii) Differentiating, $\dfrac{dP}{dt} = 1000k\, e^{kt}$,

so at the start of 1997, when $t = 2$, $\dfrac{dP}{dt} = 1000k\, e^{2k}$

$$\doteqdot 340 \text{ rabbits per year.}$$

WORKED EXERCISE:

The price P of a pair of shoes rises with inflation so that

$$\frac{dP}{dt} = kP, \text{ for some constant } k,$$

where t is the time in years since records were kept.

(a) Show that $P = P_0\, e^{kt}$, where P_0 is the price at time zero, satisfies the given differential equation.

(b) If the price doubles every 10 years, find k, sketch the curve, and find how long it takes for the price to rise to 10 times its original price.

SOLUTION:

(a) Substituting $P = P_0\, e^{kt}$ into the differential equation $\dfrac{dP}{dt} = kP$,

$$\begin{aligned} \text{LHS} &= \frac{dP}{dt} \\ &= \frac{d}{dt}\left(P_0\, e^{kt}\right) \\ &= kP_0\, e^{kt}, \\ &= kP \\ &= \text{RHS}, \text{ so the function satisfies the differential equation, as required.} \end{aligned}$$

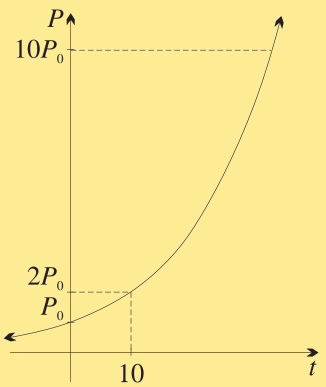
Also, when $t = 0$, $P = P_0\, e^0 = P_0 \times 1$, so P_0 is the price at time zero.

(b) When $t = 10$, we know that $P = 2P_0$,

so
$$\begin{aligned} 2P_0 &= P_0\, e^{10k} \\ e^{10k} &= 2 \\ 10k &= \log_e 2 \\ k &= \tfrac{1}{10} \log_e 2. \\ k &= 0{\cdot}069\,314\ldots \quad \text{(Store } k \text{ in the memory.)} \end{aligned}$$

Now substituting $P = 10P_0$,

$$\begin{aligned} 10P_0 &= P_0\, e^{kt} \\ e^{kt} &= 10 \\ kt &= \log_e 10 \\ t &= \frac{\log_e 10}{k} \\ &\doteqdot 33{\cdot}219, \end{aligned}$$



so it takes about 33·2 years for the price of the shoes to rise tenfold.

Natural Decay: The same method can deal with situations in which some quantity is decreasing at a rate proportional to the quantity. Radioactive substances, for example, decay in this manner. Let M be the mass of the substance, regarded as a function of time t. Because M is decreasing, the derivative $\dfrac{dM}{dt}$ is negative, so

$$\frac{dM}{dt} = -kM, \quad \text{where } k \text{ is a positive constant.}$$

Then applying the theorem, $M = M_0\, e^{-kt}$, where M_0 is the mass at time $t = 0$.

9

NATURAL DECAY:

In situations of natural decay, let the constant of proportionality be $-k$, where k is a positive constant. Then, if M is the quantity that is changing,

$$\frac{dM}{dt} = -kM \qquad \text{and} \qquad M = M_0\, e^{-kt},$$

where M_0 is the value of M at time $t = 0$.

NOTE: It is perfectly acceptable to omit the minus sign and use a negative constant k. Then $\dfrac{dM}{dt} = kM$, where k is a negative constant, and $M = M_0 e^{kt}$. The arithmetic of logarithms, however, is easier if the minus sign is built in.

WORKED EXERCISE:

A paddock has been contaminated with strontium-90, which has a half-life of 28 years. (This means that exactly half of any quantity of the isotope will decay in 28 years.) Let M_0 be the original mass present.

(a) Find the mass of strontium-90 as a function of time, then sketch the graph.

(b) Find what proportion of the radioactivity will remain after 100 years (answer correct to the nearest 0·1%).

(c) How long will it take for the radioactivity to drop to 0·001% of its original value? (Answer correct to the nearest year.)

SOLUTION:

(a) Let M be the quantity of the isotope at time t years.

Then $\dfrac{dM}{dt} = -kM$, for some positive constant k of proportionality,

so $\quad M = M_0\, e^{-kt}$.

After 28 years, only half of the original mass remains,

that is, $M = \frac{1}{2}M_0$ when $t = 28$,

and substituting, $\frac{1}{2}M_0 = M_0\, e^{-28k}$

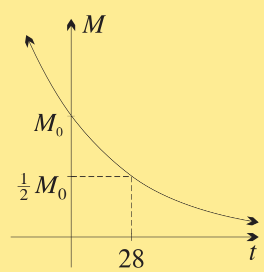
$$e^{-28k} = \tfrac{1}{2}.$$

Taking reciprocals (the reciprocal of e^{-28k} is e^{28k}),

$$e^{28k} = 2$$
$$28k = \log_e 2$$
$$k = \tfrac{1}{28}\log_e 2.$$

(Approximate k and store it in the memory:

$$k = 0{\cdot}024\,755\,\ldots)$$



(b) When $t = 100$, $M = M_0\, e^{-100k}$

$\fallingdotseq 0{\cdot}084M_0$,

so the radioactivity has dropped to about $8{\cdot}4\%$ of its original value.

(c) The radioactivity has dropped to $0{\cdot}001\%$ when $M = 10^{-5} M_0$.

Notice that
$$0{\cdot}001\% = \frac{0{\cdot}001}{100} = 0{\cdot}000\,01 = 10^{-5}.$$

Substituting $M = 10^{-5} M_0$ into the equation from part (a),

$$\begin{aligned} 10^{-5} M_0 &= M_0\, e^{-kt} \\ e^{-kt} &= 10^{-5} \\ -kt &= -5 \log_e 10 \\ t &= \frac{5 \log_e 10}{k} \\ &\fallingdotseq 465 \text{ years.} \end{aligned}$$

Exercise **6G**

1. Use the equation $Q = e^{2t}$ to find Q correct to two decimal places when:

(a) $t = 1$ (b) $t = 0{\cdot}2$ (c) $t = 1{\cdot}76$

2. (a) Given that $P = e^{0{\cdot}6t}$, show that $t = \dfrac{\log_e P}{0{\cdot}6}$.

(b) Hence find t correct to two decimal places when:

(i) $P = 5$ (ii) $P = 1234$

3. Consider the equation $C = 10\, e^{3t}$.

(a) Find C to the nearest whole number when $t = 2$.

(b) Find t to one decimal place when $C = 10\,000$.

(c) Show that $\dfrac{dC}{dt} = 3C$.

(d) Find $\dfrac{dC}{dt}$ when $C = 37{\cdot}8$.

(e) Find $\dfrac{dC}{dt}$ to the nearest whole number when $t = 1$. [HINT: Find C first.]

4. Consider the equation $M = 40\, e^{-\frac{1}{2}t}$.

(a) Find M to four decimal places when $t = 10$.

(b) Find t to two decimal places when $M = 10$.

(c) Show that $\dfrac{dM}{dt} = -\frac{1}{2}M$.

(d) Find $\dfrac{dM}{dt}$ when $M = 25$.

(e) Find $\dfrac{dM}{dt}$ to one decimal place when $t = 4$.

5. Some rabbits were released on Paradise Island. The number R of rabbits after t months can be calculated from the formula $R = 20\,e^{0{\cdot}1t}$.

(a) How many rabbits were released onto the island?

(b) How many rabbits were on the island after 12 months? (Answer correct to the nearest rabbit.)

(c) In which month did the rabbit population reach 200?

(d) Show that $\dfrac{dR}{dt} = 0{\cdot}1R$, and hence find the rate at which the number of rabbits was increasing when there were 50 rabbits.

6. The mass M kg of a certain radioactive substance is decreasing exponentially according to the formula $M = 100\,e^{-0{\cdot}04t}$, where t is measured in years.

(a) What was the initial mass?

(b) What was the mass after 10 years, to the nearest kilogram?

(c) What was the mass after a further 10 years, to the nearest kilogram?

(d) After how many years was the mass 5 kg?

(e) Show that $\dfrac{dM}{dt} = -0{\cdot}04M$, and hence find the rate at which the mass was decreasing when the mass was 20 kg.

(f) Find the rate of decrease of the mass after 18 years, correct to the nearest kg/year. [HINT: First find the mass after 18 years.]

7. The population P of a town rose from 1000 at the beginning of 1975 to 2500 at the beginning of 1985. Assume natural growth, that is, $P = 1000 \times e^{kt}$, where t is the time in years since the beginning of 1975.

(a) Find the value of the positive constant k by using the fact that when $t = 10$, $P = 2500$.

(b) Sketch the graph of $P = 1000 \times e^{kt}$.

(c) What was the population of the town at the beginning of 1998, correct to the nearest 10 people?

(d) In what year does the population reach 10 000?

(e) Find the rate $\dfrac{dP}{dt}$ at which the population is increasing at the beginning of that year. Give your answer correct to the nearest whole number.

8. It is found that under certain conditions, the number of bacteria in a sample grows exponentially with time according to the equation $B = B_0\,e^{\frac{1}{10}t}$, where t is measured in hours.

(a) Show that B satisfies the differential equation $\dfrac{dB}{dt} = \frac{1}{10}B$.

(b) Initially, the number of bacteria is estimated to be 1000. Find how many bacteria there are after three hours. Answer correct to the nearest bacterium.

(c) Use your answers to parts (a) and (b) to find how fast the number of bacteria is growing after three hours.

(d) By solving $1000\,e^{\frac{1}{10}t} = 10\,000$, find, correct to the nearest hour, when there will be 10 000 bacteria.

DEVELOPMENT

9. Twenty grams of salt is gradually dissolved in hot water. Assume that the amount S left undissolved after t minutes satisfies the law of natural decay, that is, $\dfrac{dS}{dt} = -kS$, for some positive constant k.

(a) Show that $S = 20\, e^{-kt}$ satisfies the differential equation.

(b) Given that only half the salt is left after three minutes, show that

$$k = -\tfrac{1}{3} \log \tfrac{1}{2} = -\tfrac{1}{3} \log 2^{-1} = \tfrac{1}{3} \log 2.$$

(c) Find how much salt is left after five minutes, and how fast the salt is dissolving then. (Answer correct to two decimal places.)

(d) After how long, correct to the nearest second, will there be four grams of salt left undissolved?

(e) Find the amounts of undissolved salt when $t = 0$, 1, 2 and 3, correct to the nearest 0·01 g, show that these values form a GP, and find the common ratio.

10. The population P of a rural town has been declining over the last few years. Five years ago the population was estimated at 30 000, and today it is estimated at 21 000.

(a) Assume that the population obeys the law of natural decay $\dfrac{dP}{dt} = -kP$, for some positive constant k, where t is time in years from the first estimate, and show that $P = 30\,000\, e^{-kt}$ satisfies this differential equation.

(b) Find the value of the positive constant k.

(c) Estimate the population 10 years from now.

(d) The local bank has estimated that it will not be profitable to stay open once the population falls below 16 000. When will the bank close?

11. A tank is filled to a depth of 25 metres. The liquid inside is leaking through a small hole in the bottom of the tank, and it is found that the change in depth at any instant t hours after the tank starts leaking is proportional to the depth h metres, that is $\dfrac{dh}{dt} = -kh$.

(a) Show that $h = h_0\, e^{-kt}$ is a solution of this equation. (b) What is the value of h_0?

(c) Given that the depth in the tank is 15 metres after 2 hours, find k.

(d) How long will it take to empty to a depth of just 5 metres? Answer correct to the nearest minute.

12. When a liquid is placed in a refrigerator kept at 0° C, the rate at which it cools is proportional to its temperature h at time t, thus $\dfrac{dh}{dt} = -kh$, where k is a positive constant.

(a) Show that $h = h_0\, e^{-kt}$ is a solution of the differential equation.

(b) Find h_0, given that the liquid is initially at 100° C.

(c) After 5 minutes the temperature has dropped to 40° C. Find the value of k.

(d) Find the temperature of the liquid after 15 minutes.

13. The amount A in grams of carbon-14 isotope in a dead tree trunk after t years is given by $A = A_0\, e^{-kt}$, where A_0 and k are positive constants.

(a) Show that A satisfies the equation $\dfrac{dA}{dt} = -kA$.

(b) The amount of isotope is halved every 5750 years. Find the value of k.

(c) For a certain dead tree trunk, the amount of isotope is only 15% of the original amount in the living tree. How long ago, correct to the nearest 1000 years, did the tree die?

14. Current research into Alzheimer's disease suggests that the rate of loss of percentage brain function is proportional to the percentage brain function already lost. That is, if L is the percentage brain function lost, then $\dfrac{dL}{dt} = kL$, for some constant $k > 0$.

(a) Two years ago a patient was initially diagnosed with Alzheimer's disease, with a 15% loss of brain function. This year the patient was diagnosed with 20% loss of brain function. Show that $L = 15\, e^{kt}$, where $k = \frac{1}{2} \log \frac{4}{3}$.

(b) The nearby care centre will admit patients to 24-hour nursing care when a patient reaches 60% loss of brain function. In how many more years will that be? Answer correct to the nearest year.

15. A chamber is divided into two identical parts by a porous membrane. The left part of the chamber is initially more full of a liquid than the right. The liquid is let through at a rate proportional to the difference in the levels x, measured in centimetres. Thus $\dfrac{dx}{dt} = -kx$.

(a) Show that $x = A\, e^{-kt}$ is a solution of this equation.

(b) Given that the initial difference in heights is 30 cm, find the value of A.

(c) The level in the right compartment has risen 2 cm in five minutes, and the level in the left has fallen correspondingly by 2 cm.

(i) What is the value of x at this time?

(ii) Hence find the value of k.

16. A radioactive substance decays with a half-life of 1 hour. The initial mass is 80 g.

(a) Write down the mass when $t = 0$, 1, 2 and 3 hours (no need for calculus here).

(b) Write down the average loss of mass during the 1st, 2nd and 3rd hour, then show that the percentage loss of mass per hour during each of these hours is the same.

(c) The mass M at any time satisfies the usual equation of natural decay $M = M_0\, e^{-kt}$, where k is a constant. Find the values of M_0 and k.

(d) Show that $\dfrac{dM}{dt} = -kM$, and find the instantaneous rate of mass loss when $t = 0$, $t = 1$, $t = 2$ and $t = 3$.

(e) Sketch the M–t graph, for $0 \le t \le 1$, and add the relevant chords and tangents.

17. [The formulae for compound interest and for natural growth are essentially the same.] The cost C of an article is rising with inflation in such a way that at the start of every month, the cost is 1% more than it was a month before. Let C_0 be the cost at time zero.

(a) Use the compound interest formula of Section 6C to construct a formula for the cost C after t months. Hence find, in exact form and then correct to four significant figures:

(i) the percentage increase in the cost over twelve months,

(ii) the time required for the cost to double.

(b) The natural growth formula $C = C_0\, e^{kt}$ also models the cost after t months. Use the fact that when $t = 1$, $C = 1{\cdot}01\, C_0$ to find the value of k. Hence find, in exact form and then correct to four significant figures:

(i) the percentage increase in the cost over twelve months,

(ii) the time required for the cost to double.

18. The height H of a wave decays such that $H = H_0\, e^{-\frac{1}{3}t}$, where H_0 is the initial height of the wave. Giving your answer correct to the nearest whole percent, what percentage of the initial height is the height of the wave when:

(a) $t = 1$? (b) $t = 3$? (c) $t = 8$?

19. A quantity Q of radium at time t years is given by $Q = Q_0\, e^{-kt}$, where k is a positive constant and Q_0 is the amount of radium at time $t = 0$.

(a) Given that $Q = \frac{1}{2}Q_0$ when $t = 1690$ years, calculate k.

(b) After how many years does only 20% of the initial amount of radium remain? Give your answer correct to the nearest year.

20. Air pressure P in millibars is a function of the altitude a in metres, with $\dfrac{dP}{da} = -\mu P$. The pressure at sea level is 1013·25 millibars.

(a) Show that $P = 1013{\cdot}25\, e^{-\mu a}$ is a solution to this problem.

(b) One reference book quotes the pressure at 1500 metres to be 845·6 millibars. Find the value of μ for the data in that book.

(c) Another reference book quotes the pressure at 6000 metres to be half that at sea level. Find the value of μ in this case.

(d) Are the data in the two books consistent?

(e) Assuming the first book to be correct:

(i) What is the pressure at 4000 metres?

(ii) What is the pressure 1 km down a mine shaft?

(iii) At what altitude is the pressure 100 millibars?

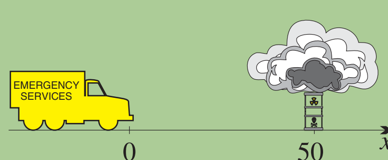
CHALLENGE

21. A certain radioactive isotope decays at such a rate that after 68 minutes only a quarter of the initial amount remains.

(a) Find the half-life of this isotope.

(b) What proportion of the initial amount will remain after 3 hours? Give your answer as a percentage, correct to one decimal place.

22. The emergency services are dealing with a toxic gas cloud around a leaking gas cylinder 50 metres away. The prevailing conditions mean that the concentration C in parts per million (ppm) of the gas increases proportionally to the concentration as one moves towards the cylinder. That is, $\dfrac{dC}{dx} = kC$, where x is the distance in metres towards the cylinder from their current position.



(a) Show that $C = C_0\, e^{kx}$ is a solution of the above equation.

(b) At the truck, where $x = 0$, the concentration is $C = 20\,000$ ppm. Five metres closer, the concentration is $C = 22\,500$ ppm. Use this information to find the values of the constants C_0 and k. (Give k exactly, then correct to three decimal places.)

(c) Find the gas concentration at the cylinder, correct to the nearest part per million.

(d) The accepted safe level for this gas is 30 parts per million. The emergency services calculate how far back from the cylinder they should keep the public, rounding their answer up to the nearest 10 metres.

(i) How far back do they keep the public?

(ii) Why do they round their answer up and not round it in the normal way?

23. In 1980, the population of Bedsworth was $B = 25\,000$ and the population of Yarra was $Y = 12\,500$. That year the mine in Bedsworth was closed, and the population began falling, while the population of Yarra continued to grow, so that

$$B = 25\,000\,e^{-pt} \quad \text{and} \quad Y = 12\,500\,e^{qt}.$$

(a) Ten years later it was found the populations of the two towns were $B = 20\,000$ and $Y = 15\,000$. Find the values of p and q.

(b) In what year were the populations of the two towns equal?

6H Chapter Review Exercise

1. Consider the series $31 + 44 + 57 + \cdots + 226$.

(a) Show that it is an AP and write down the first term and the common difference.

(b) How many terms are there in this series?

(c) Find the sum.

2. Consider the series $24 + 12 + 6 + \cdots$.

(a) Show that it is a geometric series and find the common ratio.

(b) Explain why this geometric series has a limiting sum.

(c) Find the limiting sum and the sum of the first 10 terms, and show that they are approximately equal, correct to the first three significant figures.

3. A chef receives an annual salary of \$35 000, with 4% increments each year.

(a) Show that her annual salaries form a GP and find the common ratio.

(b) Find her annual salary, and her total earnings, at the end of 10 years, each correct to the nearest dollar.

4. Darko's salary is \$47 000 at the beginning of 2004, and it will increase by \$4000 each year.

(a) Find a formula for T_n, his salary in the nth year.

(b) In which year will Darko's salary first be at least twice what it was in 2004?

5. Miss Yamada begins her new job in 2005 on a salary of \$53 000, and it is increased by 3% each year. In which year will her salary be at least twice her original salary?

6. Izadoor invests \$15 000 at 4·7% per annum simple interest.

(a) Write down a formula for the total value A_n of the investment at the end of n years.

(b) Show that the investment exceeds \$30 000 at the end of 22 years, but not at the end of 21 years.

7. Indira has just bought a car worth \$23 000 and has been told that it will depreciate at 19% per annum. What will the car be worth when she sells it in three years time? Round your answer down to the nearest dollar.

8. (a) Find the value of a \$12 000 investment that has earned 5·25% per annum, compounded monthly, for five years.

(b) How much interest was earned over the five years?

(c) What annual rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

9. Katarina has entered a superannuation scheme into which she makes annual contributions of \$8000. The investment earns interest of 7·5% per annum, compounded annually, with contributions made on 1st October each year.

(a) Show that after 15 years of contributions, the value of Katarina's investment is given by $A_{15} = \dfrac{8000 \times 1{\cdot}075 \times (1{\cdot}075^{15} - 1)}{0{\cdot}075}$.

(b) Evaluate A_{15}.

(c) By how much does A_{15} exceed the total contributions Katarina made over these years?

(d) Show that after 17 years of contributions, the value A_{17} of the superannuation is more than double Katarina's contributions over the 17 years.

10. Ahmed wishes to retire with superannuation worth half a million dollars in 25 years time. On 1st August each year he pays a contribution to a scheme that gives interest of 6·6% per annum, compounded annually.

(a) Let M be the annual contribution. Show that the value of the investment at the end of the nth year is $A_n = \dfrac{M \times 1{\cdot}066 \times (1{\cdot}066^n - 1)}{0{\cdot}066}$.

(b) Hence show that the amount of each contribution is \$7852.46 .

11. Alonso takes out a mortgage on a flat for \$159 000, at an interest rate of 6·75% per annum, compounded monthly. He agrees to pay the bank \$1410 each month for 15 years.

(a) Let A_{180} be the balance of the loan after 15 years. Find a series expression for A_{180}.

(b) Show that $A_{180} = 159\,000 \times 1{\cdot}005625^{180} \;-\; \dfrac{1410(1{\cdot}005625^{180} - 1)}{0{\cdot}005625}$.

(c) Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

(d) What monthly payment, correct to the nearest cent, is needed in order to pay off the loan in 15 years?

12. May-Eliane borrowed \$1·7 million from the bank to buy some machinery for her farm. She agreed to pay the bank \$18 000 per month. The interest rate is 4·5% per annum, compounded monthly, and the loan is to be repaid in 10 years.

(a) Let A_n be the balance of the loan after n months. Find a series expression for n.

(b) Hence show that $A_n = 1\,700\,000 \times 1{\cdot}00375^n \;-\; \dfrac{18\,000(1{\cdot}00375^n - 1)}{0{\cdot}00375}$.

(c) Find the amount owing on the loan at the end of the fifth year, and state whether this is more or less than half the amount borrowed.

(d) Find A_{120}, and hence show that the loan is actually paid out in less than 10 years.

(e) If it is paid out after n months (that is, put $A_n = 0$), show that $1{\cdot}00375^n = 1{\cdot}5484$, and hence that

$$n = \frac{\log_{10} 1{\cdot}5484}{\log_{10} 1{\cdot}00375}.$$

(f) Find how many months early the loan is paid off.

13. A quantity Q varies with time t according to the formula $Q = 40\,e^{\frac{1}{5}t}$.

(a) Differentiate to find the rate $\dfrac{dQ}{dt}$ as a function of time t.

(b) Answer these questions correct to four significant figures:

(i) Find Q when $t = 7$.

(ii) Find $\dfrac{dQ}{dt}$ when $t = 10$.

(iii) Find t when $Q = 400$.

(iv) Find t when $\dfrac{dQ}{dt} = 20$.

14. A quantity M varies with time t according to $M = 500\,e^{-kt}$, where k is a constant.

(a) Differentiate to find the rate $\dfrac{dM}{dt}$ as a function of time t.

(b) Answer these questions correct to four significant figures:

(i) Find M when $t = 6$ and $k = 0{\cdot}5$.

(ii) Find k if $M = 10$ when $t = 12$.

(iii) Find t when $M = 100$ and $k = 0{\cdot}3$.

(iv) Find $\dfrac{dM}{dt}$ when $k = 0{\cdot}1$ and $t = 5$.

(v) Find t if $\dfrac{dM}{dt} = -200$ and $k = 2$.

15. It is found that under certain conditions, the number of bacteria in a sample grows exponentially with time according to the equation $B = B_0\,e^{\frac{1}{20}t}$, where t is measured in hours.

(a) Show that B satisfies the differential equation $\dfrac{dB}{dt} = \frac{1}{20}B$.

(b) Initially, the number of bacteria is estimated to be 3000. Find how many bacteria there are after four hours. Answer correct to the nearest bacterium.

(c) Use your answers to parts (a) and (b) to find how fast the number of bacteria is growing after four hours.

(d) By solving $3000\,e^{\frac{1}{20}t} = 6000$, find, correct to the nearest hour, when there will be 6000 bacteria.

16. The population P of a rural town has been declining over the last few years. Five years ago the population was estimated at 8000, and today it is estimated at 5000.

(a) Assume that the population obeys the law of natural decay $\dfrac{dP}{dt} = -kP$, for some positive constant k, where t is time in years from the first estimate, and show that $P = 8000\,e^{-kt}$ satisfies this differential equation.

(b) Find the value of the positive constant k.

(c) Estimate the population five years from now.

(d) Shaw's Department Store has estimated that it will not be profitable to stay open once the population falls below 2700. When will the store have to close?

17. Water is flowing out of a tank at the rate $\dfrac{dV}{dt} = 10t - 250$, where V is the volume in litres remaining in the tank at time t minutes after time zero.

(a) When does the water stop flowing?

(b) Suppose now that the tank still has 10 litres left in it when the water flow stops.

(i) Show that the volume V at any time is given by $V = 5t^2 - 250t + 3135$.

(ii) How much water was initially in the tank?

18. A scientist studying a colony of seals finds the number $N(t)$ of seals after t years to be

$$N(t) = \frac{A}{4 + e^{-t}}\,, \quad \text{where } A \text{ is a constant.}$$

(a) When the scientist begins measuring, the number of seals in the colony is estimated at 1400. Find the value of the constant A.

(b) What is the population of the colony one year later? Round your answer down to a whole number of seals.

(c) Show that the rate at which the population increases with time is $\dfrac{dN}{dt} = \dfrac{7000\,e^{-t}}{(4 + e^{-t})^2}$.

(d) What is the rate of increase in the population after one year? Round your answer down to a whole number of seals per year.

(e) How many seals would you expect to find in the colony after a long time?

CHAPTER **SEVEN**

Euclidean Geometry

The methods and structures of modern mathematics were established first by the ancient Greeks in their studies of geometry and arithmetic. They understood that mathematics must proceed by rigorous proof and argument, that all definitions must be stated with absolute precision, and that any hidden assumptions, called *axioms*, must be brought out into the open and examined. Many Greeks, like the mathematician Pythagoras and the philosopher Plato, spoke of mathematics in mystical terms as the highest form of knowledge. They called their results *theorems* — the Greek word *theorem* means 'a thing to be gazed upon' or 'a thing contemplated by the mind' and comes from $\theta\varepsilon\omega\rho\acute{\varepsilon}\omega$ 'behold'. (The English word *theatre* comes from the same root.)

Of all the Greek books, Euclid's *Elements* has been the most influential and was still used as a textbook in nineteenth-century schools. The geometry presented in this chapter is only introductory and is nothing like as rigorous as Euclid's book, but it is called *Euclidean geometry* because it is based on the Greek methods.

Most of the material here will have been covered in earlier years. In fact, many standard geometrical results have already been assumed throughout this textbook. The only entirely new work is the final section on intercepts. The emphasis of the chapter is therefore more on the logic of the proofs and on the logical sequence established in the chain of theorems. It is intended to provide quite a different insight into mathematical thinking from the rest of the book.

Constructions with straight edge and compasses are central to Euclid's arguments and so proofs of some construction problems have been included. They were drawn in earlier years, but they need to be proven, and perhaps drawn again. Their importance does not lie in their practical use, but in their logic.

A Note on the Exercises: The basis of this topic is a chain of theorems, each of which is marked as a 'Course Theorem'. Some are proven in the notes. Proofs of the others are presented in the following exercise as structured questions, marked 'Course Theorem' and mostly placed near the start of the Development section. Working through these proofs is an essential part of the course.

All theorems marked Course Theorem may be used in later questions, except where the intention of the question is to provide a proof of that particular theorem. Students should note carefully that *the large number of further theorems proven in the exercises cannot be used in subsequent questions*.

Some excellent geometrical programs are readily available. Almost every theorem in this chapter can be illustrated with such programs and specific questions involving technology have therefore not been included in this chapter.

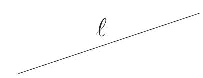
7 A Points, Lines, Parallels and Angles

The elementary objects of geometry are points, lines and planes. It is possible, but very difficult, to give rigorous definitions of them. The approach in this course will therefore be the same as the earlier approach to the real numbers — describe some of their properties and list some of the assumptions that are needed about them.

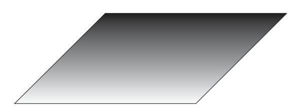
Points, Lines and Planes: These remarks are not definitions of these objects, but simple descriptions of some of their important properties.

POINTS:
A *point* can be described as having a position but no size. The mark opposite has a definite width and so is not a point, but it represents a point in our imagination.

LINES:
A *line* has no breadth, but extends infinitely in both directions. The drawing opposite has width and has ends, but it represents a line in our imagination.

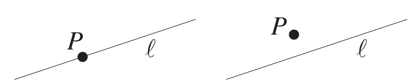


PLANES:
A *plane* has no thickness and extends infinitely in all directions. Almost all the work in this course is two-dimensional and takes place entirely in a fixed plane.

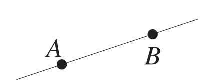


Points and Lines in a Plane: Here are some of the assumptions that will be made about the relationships between points and lines in a plane.

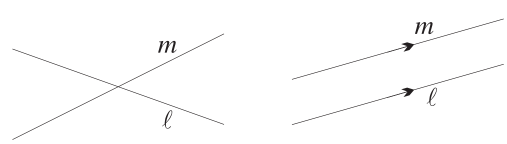
POINT AND LINE:
Given a point P and a line ℓ, the point P may or may not lie on the line ℓ.



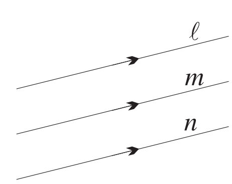
TWO POINTS:
Two distinct points A and B lie on one and only one line. The line can be named either AB or BA.



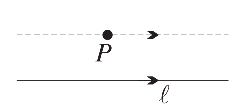
TWO LINES:
Given two distinct lines ℓ and m in a plane, either the lines intersect in a single point, or the two lines have no point in common and are called *parallel lines*, written as $\ell \parallel m$.



THREE PARALLEL LINES:
If two lines are each parallel to a third line, then they are parallel to one another.

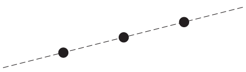


THE PARALLEL LINE THROUGH A GIVEN POINT:
Given a line ℓ and a point P not on ℓ, there is one and only one line through P parallel to ℓ.

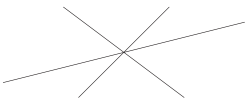


Collinear Points and Concurrent Lines: A third point may or may not lie on the line determined by two other points. Similarly, a third line may or may not pass through the point of intersection of two other lines.

COLLINEAR POINTS:
Three or more distinct points are called *collinear* if they all lie on a single line.

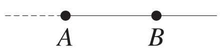


CONCURRENT LINES:
Three or more distinct lines are called *concurrent* if they all pass through a single point.

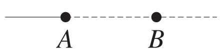


Intervals and Rays: These definitions rely on the idea that a point on a line divides the rest of the line into two parts. Let A and B be two distinct points on a line ℓ.

RAYS:
The *ray* AB consists of the *endpoint* A together with B and all the other points of ℓ on the same side of A as B is.



OPPOSITE RAY:
The ray that starts at this same endpoint A, but goes in the opposite direction, is called the *opposite ray*.

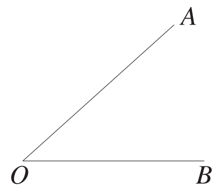


INTERVALS:
The *interval* AB consists of all the points lying on ℓ between A and B, including these two endpoints.

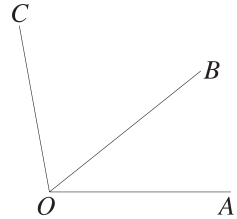
LENGTHS OF INTERVALS:
The idea that an interval has length is based on the assumption that the lengths of intervals can be compared and added and subtracted with compasses.

Angles: There is a distinction between an angle and the size of an angle.

ANGLES:
An *angle* consists of two rays with a common endpoint. The two rays OA and OB in the diagram form an angle named either $\angle AOB$ or $\angle BOA$. The common endpoint O is called the *vertex* of the angle and the rays OA and OB are called the *arms* of the angle.



ADJACENT ANGLES:
Two angles are called *adjacent angles* if they have a common vertex and a common arm. In the diagram opposite, $\angle AOB$ and $\angle BOC$ are adjacent angles with common vertex O and common arm OB. Also, the overlapping angles $\angle AOC$ and $\angle AOB$ are adjacent angles, having common vertex O and common arm OA.

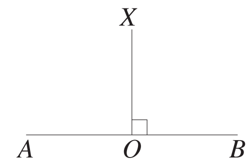


MEASURING ANGLES:
The size of an angle is the amount of turning as one arm is rotated about the vertex onto the other arm. The units of degrees are based on the ancient Babylonian system of dividing a revolution into 360 equal parts. There are about 360 days in a year and so the sun moves about 1° against the fixed stars every day. The measurement of angles is based on the obvious assumption that the sizes of adjacent angles can be added and subtracted.

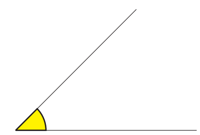
REVOLUTIONS:
A *revolution* is the angle formed by rotating a ray about its endpoint once until it comes back onto itself. A revolution is defined to measure 360°.

STRAIGHT ANGLES:
A *straight angle* is the angle formed by a ray and its opposite ray. A straight angle is half a revolution and so measures 180°.

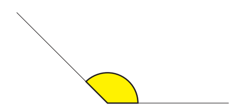
RIGHT ANGLES:
Suppose that AOB is a line and that OX is a ray such that $\angle XOA$ is equal to $\angle XOB$. Then $\angle XOA$ is called a *right angle*. A right angle is half a straight angle and so measures 90°.



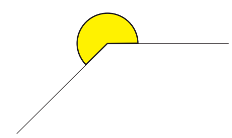
ACUTE ANGLES:
An *acute angle* is an angle greater than 0° and less than a right angle.



OBTUSE ANGLES:
An *obtuse angle* is an angle greater than a right angle and less than a straight angle.



REFLEX ANGLES:
A *reflex angle* is an angle greater than a straight angle and less than a revolution.



Complementary and Supplementary Angles: There are special names for a pair of angles that add to 90° and for a pair of angles that add to 180°.

1

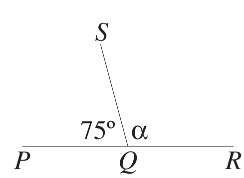
COMPLEMENTARY AND SUPPLEMENTARY ANGLES:
- Two angles are called *complementary* if they add to 90°. For example, 15° is the *complement* of 75°.
- Two angles are called *supplementary* if they add to 180°. For example, 105° is the *supplement* of 75°.

Angles at a Point: Because adjacent angles can be added, several remarks can be drawn about adjacent angles on a straight line and in a revolution.

2

COURSE THEOREM — ANGLES IN A STRAIGHT LINE AND IN A REVOLUTION:
- Two adjacent angles in a straight angle are supplementary.
- Conversely, if adjacent angles are supplementary, they form a straight line.
- Adjacent angles in a revolution add to 360°.

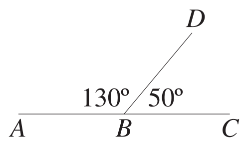
WORKED EXERCISE:
Given that PQR is a line in the diagram to the right, find α.



SOLUTION:

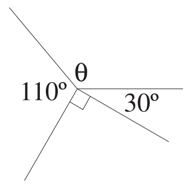
$$\alpha + 75^\circ = 180^\circ \quad \text{(angles in a straight angle)}$$
$$\alpha = 105^\circ$$

WORKED EXERCISE:
Give a reason why A, B and C are collinear.



SOLUTION:
The adjacent angles $\angle ABD$ and $\angle CBD$ are supplementary, so A, B and C are collinear.

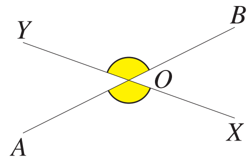
WORKED EXERCISE:
Find θ in the diagram to the right.



SOLUTION:

$$\begin{aligned} \theta + 110^\circ + 90^\circ + 30^\circ &= 360^\circ \quad \text{(angles in a revolution)} \\ \theta &= 130^\circ \end{aligned}$$

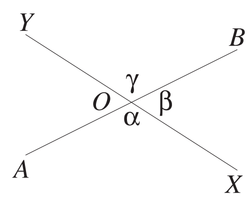
Vertically Opposite Angles: Each pair of opposite angles formed when two lines intersect are called *vertically opposite angles*. In the diagram to the right, AB and XY intersect at O. The marked angles $\angle AOX$ and $\angle BOY$ are vertically opposite. The unmarked angles $\angle AOY$ and $\angle BOX$ are also vertically opposite.



3 COURSE THEOREM — VERTICALLY OPPOSITE ANGLES:
- Vertically opposite angles are equal.

This result can be proven from the assumptions stated so far.

GIVEN:
Let the lines AB and XY intersect at O.
Let $\alpha = \angle AOX$, let $\beta = \angle BOX$, and let $\gamma = \angle BOY$.

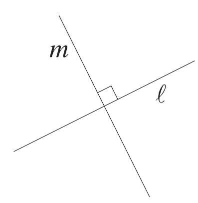


AIM:
To prove that $\alpha = \gamma$.

PROOF:

$$\begin{aligned} &\alpha + \beta = 180^\circ \quad \text{(straight angle } \angle AOB\text{)}, \\ \text{and } &\gamma + \beta = 180^\circ \quad \text{(straight angle } \angle XOY\text{)}, \\ \text{so } &\alpha = \gamma. \end{aligned}$$

Perpendicular Lines: Two lines ℓ and m are called *perpendicular*, written as $\ell \perp m$, if they intersect so that one of the angles between them is a right angle. Because adjacent angles on a straight line are supplementary, all four angles must be right angles.

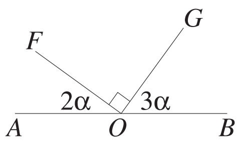


Using Reasons in Arguments: Geometrical arguments require reasons to be given for each statement — the whole topic is traditionally regarded as providing training in the writing of mathematical proofs. These reasons can be expressed in ordinary prose, or each reason can be given in brackets after the statement it justifies. The authors of this book have boxed the theorems and assumptions that can be quoted as reasons.

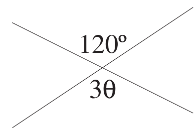
All reasons should, wherever possible, give the names of the angles or lines or triangles referred to, otherwise there can be ambiguities about exactly what argument has been used.

WORKED EXERCISE:
Find α or θ in each diagram below.

(a)



(b)



SOLUTION:

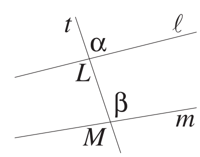
(a) $2\alpha + 90^\circ + 3\alpha = 180^\circ$ (straight angle $\angle AOB$),
$5\alpha = 90^\circ$
$\alpha = 18^\circ$.

(b) $3\theta = 120^\circ$ (vertically opposite angles),
$\theta = 40^\circ$.

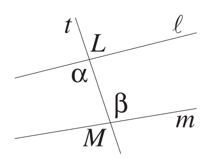
Angles and Parallel Lines: The standard results about alternate, corresponding and co-interior angles are taken as assumptions.

TRANSVERSALS:
A *transversal* is a line that crosses two other lines (the two other lines may or may not be parallel). In each of the three diagrams below, t is a transversal to the lines ℓ and m, meeting them at L and M respectively.

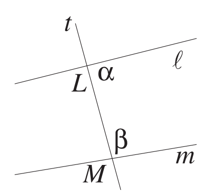
CORRESPONDING ANGLES:
In the diagram to the right, the two angles marked α and β are called *corresponding angles*, because they are in corresponding positions around the two vertices L and M.



ALTERNATE ANGLES:
In the diagram to the right, the two angles marked α and β are called *alternate angles*, because they are on alternate sides of the transversal t (they must also be inside the region between the lines ℓ and m).



CO-INTERIOR ANGLES:
In the diagram to the right, the two angles marked α and β are called *co-interior angles*, because they are inside the two lines ℓ and m and on the same side of the transversal t.

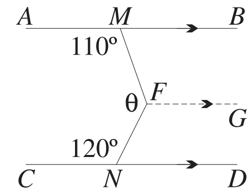


The assumptions about corresponding, alternate and co-interior angles fall into two groups. The first group are consequences arising when the lines are parallel.

ASSUMPTIONS ABOUT TRANSVERSALS ACROSS PARALLEL LINES:
Suppose that a transversal crosses two lines.

4
- If the lines are parallel, then any two corresponding angles are equal.
- If the lines are parallel, then any two alternate angles are equal.
- If the lines are parallel, then any two co-interior angles are supplementary.

WORKED EXERCISE: [A problem requiring a construction]
Find θ in the diagram opposite.



SOLUTION:

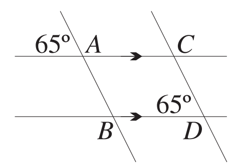
Construct $FG \parallel AB$.

Then	$\angle MFG = 110^\circ$	(alternate angles, $FG \parallel AB$),
and	$\angle NFG = 120^\circ$	(alternate angles, $FG \parallel CD$),
so	$\theta + 110^\circ + 120^\circ = 360^\circ$	(angles in a revolution at F),
	$\theta = 130^\circ$.	

Tests for Parallel Lines: The second group are the converse statements of the first group. They give conditions for the two lines to be parallel.

5 ASSUMPTIONS ABOUT TRANSVERSALS — TESTS FOR PARALLEL LINES:
Suppose that a transversal crosses two lines.
- If any pair of corresponding angles are equal, then the lines are parallel.
- If any pair of alternate angles are equal, then the lines are parallel.
- If any two co-interior angles are supplementary, then the lines are parallel.

WORKED EXERCISE:
Given that $AC \parallel BD$, prove that $AB \parallel CD$.



SOLUTION:

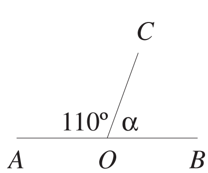
	$\angle CAB = 65^\circ$	(vertically opposite at A),
so	$\angle ABD = 115^\circ$	(co-interior angles, $AC \parallel BD$),
so	$AB \parallel CD$	(co-interior angles are supplementary).

Exercise 7A

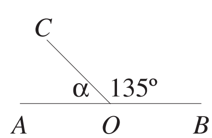
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find the angles α and β in the diagrams below, giving reasons.

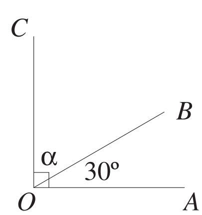
(a)



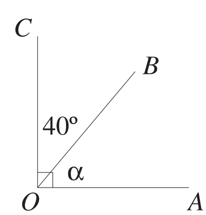
(b)



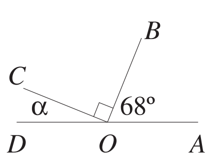
(c)



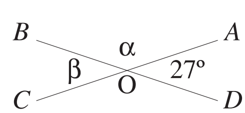
(d)



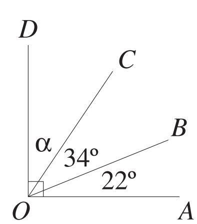
(e)



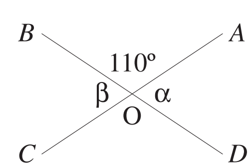
(f)



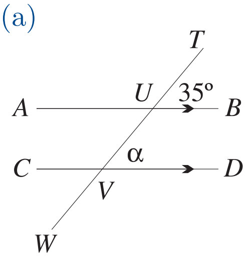
(g)

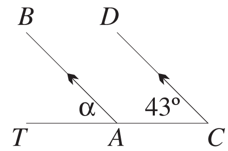


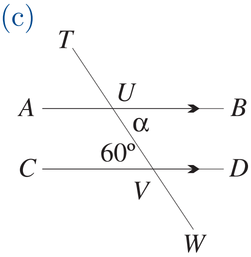
(h)

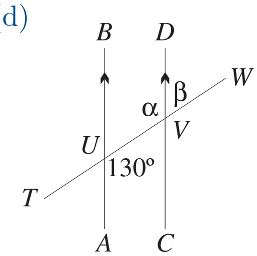


2. Find the angles α and β in each figure below, giving reasons.

(a)


(b)


(c)


(d)


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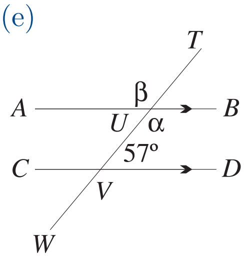
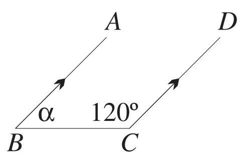
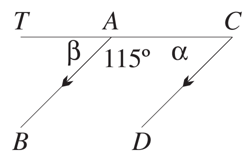
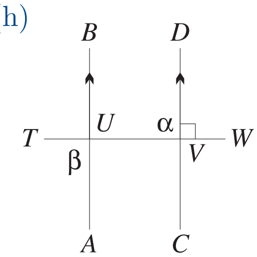
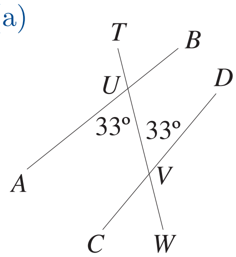


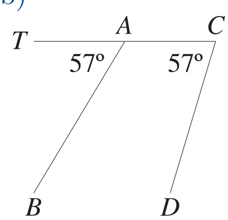
(f)


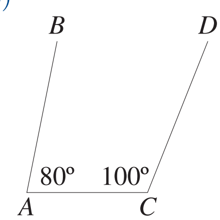
(g)


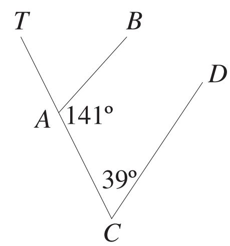
(h)


3. In each diagram below, give a reason why $AB \parallel CD$.

(a)


(b)


(c)


(d)


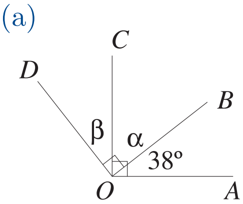
4. These questions are intended to show that when lines are not parallel, the alternate and corresponding angles are not equal and the co-interior angles are not supplementary.

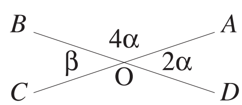
(a) Sketch a transversal crossing two non-parallel lines so that a pair of alternate angles formed by the transversal are about 45° and 65°.

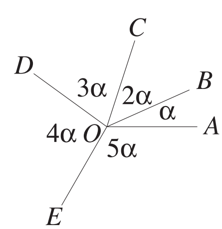
(b) Repeat part (a) so that a pair of corresponding angles are about 90° and 120°.

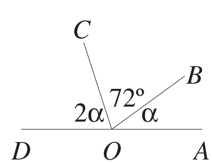
(c) Repeat part (a) so that a pair of co-interior angles are both about 80°.

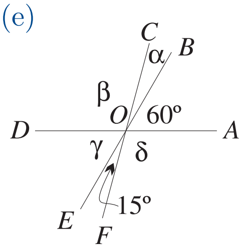
5. Find the angles α, β, γ and δ in the diagrams below, giving reasons.

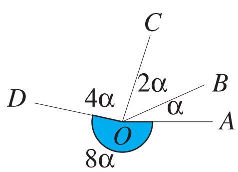
(a)


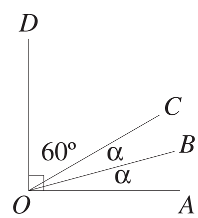
(b)


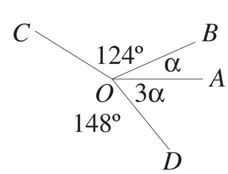
(c)


(d)


(e)


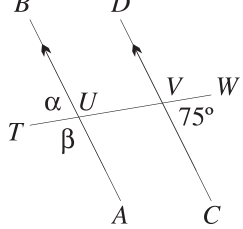
(f)


(g)


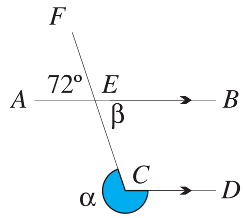
(h)


6. Find the angles α and β in each diagram below. Give reasons for each step in your arguments.

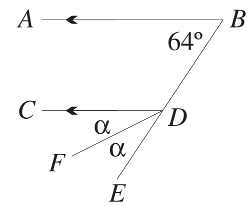
(a)



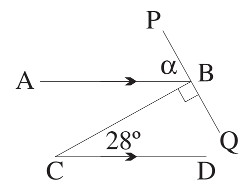
(b)



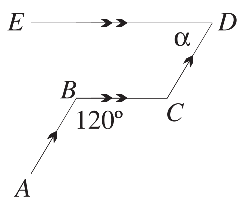
(c)



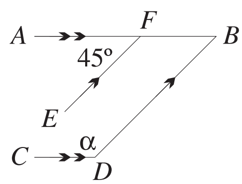
(d)



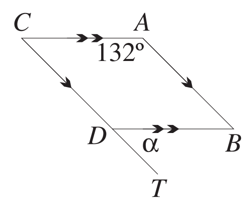
(e)



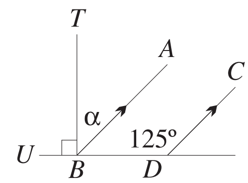
(f)



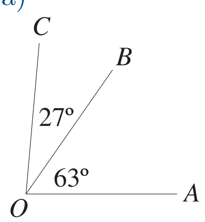
(g)



(h)

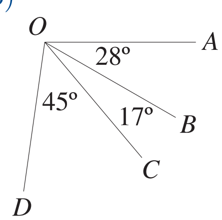


7. (a)



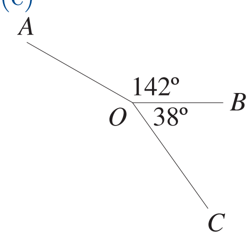
Show that $OC \perp OA$.

(b)



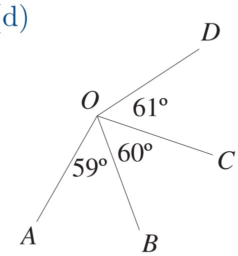
Show that $OD \perp OA$.

(c)



Show that A, O and C are collinear.

(d)

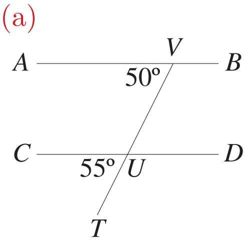


Show that A, O and D are collinear.

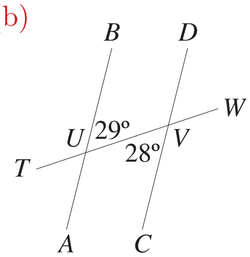
DEVELOPMENT

8. Show that AB is not parallel to CD in the diagrams below, giving all reasons.

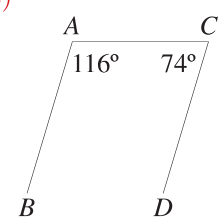
(a)



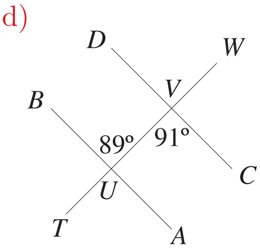
(b)



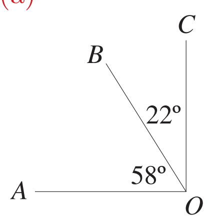
(c)



(d)

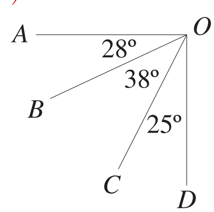


9. (a)



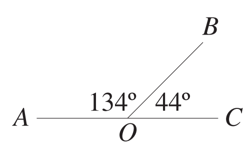
Show that OC is not perpendicular to OA.

(b)



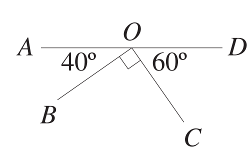
Show that OD is not perpendicular to OA.

(c)



Show that A, O and C are not collinear.

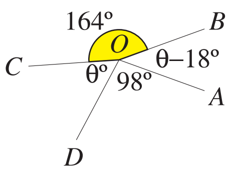
(d)



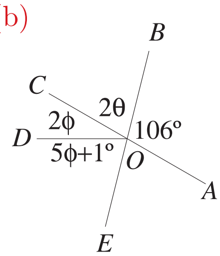
Show that A, O and D are not collinear.

10. Find the angles θ and ϕ in the diagrams below, giving reasons.

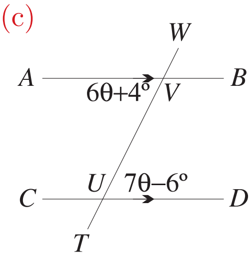
(a)



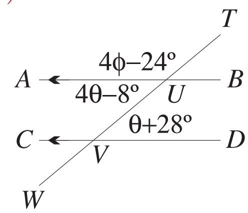
(b)



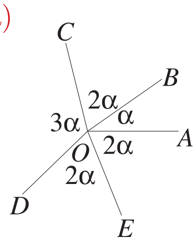
(c)



(d)

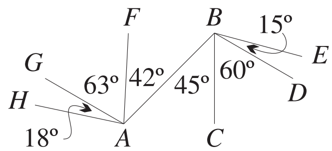


11. (a)



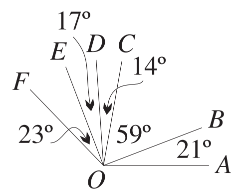
Name all straight angles and vertically opposite angles in the diagram.

(b)



Which two lines in the diagram above are parallel?

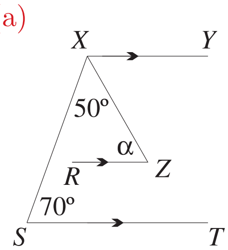
(c)



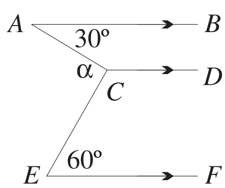
Which two lines in the diagram above form a right angle?

12. Find the angle α in each diagram below.

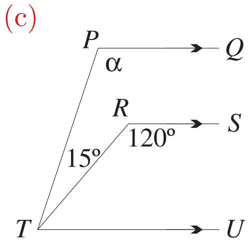
(a)



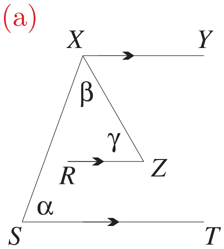
(b)



(c)

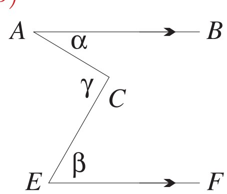


13. (a)



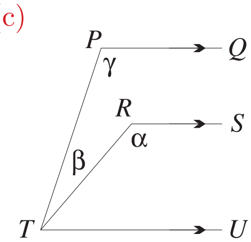
Show that $\gamma = 180° - (\alpha + \beta)$.

(b)



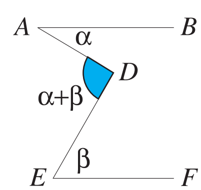
Show that $\gamma = \alpha + \beta$.

(c)



Show that $\gamma = \alpha - \beta$.

(d)



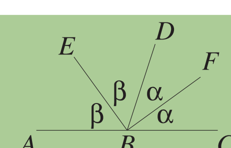
Show that $EF \parallel AB$.

CHALLENGE

14. THEOREM: *The bisectors of adjacent angles on a straight line form a right angle.*

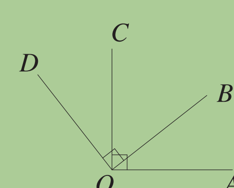
Let ABC be a straight line.
Let $\angle ABD$ and $\angle CBD$ be adjacent angles on the line ABC.
Let the line BE bisect $\angle ABD$ and the line BF bisect $\angle CBD$.

Prove that $\angle EBF = 90°$.



15. In the diagram to the right, $CO \perp AO$ and $DO \perp BO$.

Show that the angles $\angle AOD$ and $\angle BOC$ are supplementary.
[HINT: Let $\angle BOC = \theta$.]



16. Give concrete, everyday examples of the following:

(a) three planes meeting at a point,

(b) three planes meeting at a line,

(c) three parallel planes,

(d) three planes intersecting in three lines,

(e) two parallel planes intersecting with a third plane,

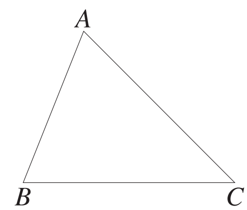
(f) a line parallel to a plane,

(g) a line intersecting a plane.

7 B Angles in Triangles and Polygons

This section deals with angles in triangles, quadrilaterals and other polygons.

Triangles: A *triangle* is formed by taking any three non-collinear points A, B and C and constructing the three intervals AB, BC and CA.



The three intervals are called the *sides* of the triangle and the three points are called its *vertices* (the singular is *vertex*).

Interior Angles of a Triangle:

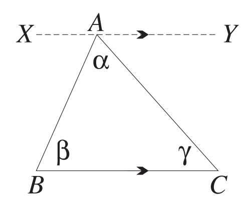
The three angles inside the triangle at the vertices are called the *interior angles*. Their sum is always 180°.

6 COURSE THEOREM — INTERIOR ANGLES OF A TRIANGLE:

- The sum of the interior angles of a triangle is a straight angle.

GIVEN:
Let ABC be a triangle.
Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$.

AIM:
To prove that $\alpha + \beta + \gamma = 180°$.



CONSTRUCTION:
Construct XAY through the vertex A parallel to BC.

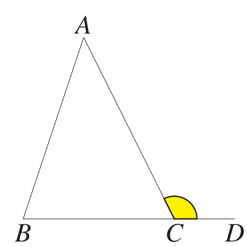
PROOF:

$\angle XAB = \beta$ (alternate angles, $XAY \parallel BC$),

and $\angle YAC = \gamma$ (alternate angles, $XAY \parallel BC$).

Hence $\alpha + \beta + \gamma = 180°$ (straight angle).

Exterior Angles of a Triangle: Suppose that ABC is a triangle and suppose that the side BC is produced to D (the word 'produced' simply means 'extended in the direction BC').



Then the angle $\angle ACD$ between the side AC and the extended side CD is called an *exterior angle* of the triangle.

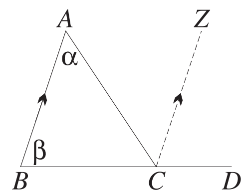
There are two exterior angles at each vertex, and being vertically opposite, they must be equal in size. Also, an exterior angle and the interior angle adjacent to it are adjacent angles on a straight line, so they must be supplementary.

The exterior and interior angles of a triangle are related by the following theorem.

7 COURSE THEOREM — EXTERIOR ANGLES OF A TRIANGLE:
- An exterior angle of a triangle equals the sum of the interior opposite angles.

GIVEN:
Let ABC be a triangle with BC produced to D.
Let $\angle A = \alpha$ and $\angle B = \beta$.

AIM:
To prove that $\angle ACD = \alpha + \beta$.



CONSTRUCTION:
Construct the ray CZ through the vertex C parallel to BA.

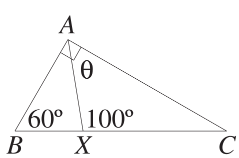
PROOF:

$$\begin{aligned} &\angle ZCD = \beta \quad \text{(corresponding angles, } BA \parallel CZ\text{),} \\ \text{and}\quad &\angle ACZ = \alpha \quad \text{(alternate angles, } BA \parallel CZ\text{).} \end{aligned}$$

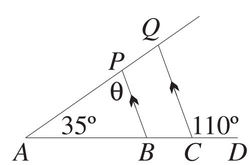
Hence $\angle ACD = \alpha + \beta$ (adjacent angles).

WORKED EXERCISE:
Find θ in each diagram below.

(a)



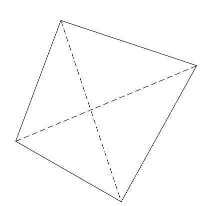
(b)



SOLUTION:
(a) $\angle C = 30^\circ$ (angle sum of $\triangle ABC$),
so $\theta = 50^\circ$ (angle sum of $\triangle ACX$).

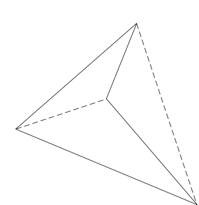
(b) $\angle PBC = 110^\circ$ (corresponding angles, $BP \parallel CQ$),
so $\theta = 75^\circ$ (exterior angle of $\triangle ABP$).

Quadrilaterals: A *quadrilateral* is a closed plane figure bounded by four intervals. As with triangles, the intervals are called *sides* and their four endpoints are called *vertices*. (The sides can't cross each other and no vertex angle can be 180°.)



A quadrilateral may be *convex*, meaning that all its interior angles are less than 180°, or *non-convex*, meaning that one interior angle is greater than 180°.

The intervals joining pairs of opposite vertices are called *diagonals*. Notice that both diagonals of a convex quadrilateral lie inside the figure, but only one diagonal of a non-convex quadrilateral lies inside it.

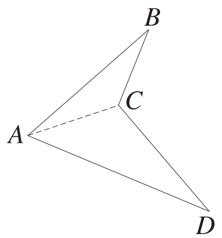


8 COURSE THEOREM — INTERIOR ANGLES OF A QUADRILATERAL:

- The sum of the interior angles of a quadrilateral is two straight angles.

GIVEN:
Let $ABCD$ be a quadrilateral,
labelled so that the diagonal AC lies inside the figure.

AIM:
To prove that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$.

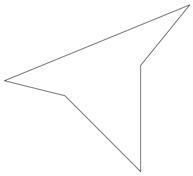


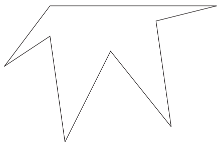
CONSTRUCTION:
Join the diagonal AC.

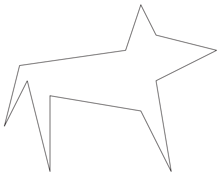
PROOF:
The interior angles of $\triangle ABC$ have sum $180°$,
and the interior angles of $\triangle ADC$ have sum $180°$.
But the interior angles of quadrilateral $ABCD$
are the sums of the interior angles of $\triangle ABC$ and $\triangle ADC$.
Hence the sum of the interior angles of $ABCD$ is $360°$.

Polygons: A *polygon* is a closed figure bounded by any number of straight sides (*polygon* is a Greek word meaning 'many-angled'). A polygon is named according to the number of sides it has — there must be at least three sides or else there would be no enclosed region. Here are some of the names given to polygons:

3 sides: triangle	6 sides: hexagon	9 sides: nonagon
4 sides: quadrilateral	7 sides: heptagon	10 sides: decagon
5 sides: pentagon	8 sides: octagon	12 sides: dodecagon


A pentagon


An octagon


A dodecagon

Like quadrilaterals, polygons can be *convex*, meaning that every interior angle is less than $180°$, or *non-convex*, meaning that at least one interior angle is greater than $180°$. A polygon is convex if and only if every one of its diagonals lies inside the figure. Notice that even a non-convex polygon must have at least one diagonal completely inside the figure.

The following theorem generalises the theorems about the interior angles of triangles and quadrilaterals to polygons with any number of sides.

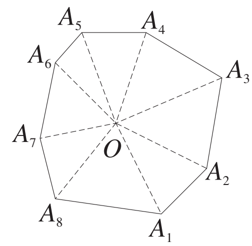
9 COURSE THEOREM — INTERIOR ANGLES OF A POLYGON:

- The interior angles of an n-sided polygon have sum $180(n-2)°$.

The proof of this theorem is complicated when the polygon is non-convex, because it requires repeatedly chopping off a triangle whose angle sum is $180°$. The situation is far easier when the polygon is convex and the following proof is restricted to that case.

GIVEN:
Let $A_1A_2 \ldots A_n$ be a convex polygon.

AIM:
To prove that $\angle A_1 + \angle A_2 + \ldots + \angle A_n = 180(n-2)°$.

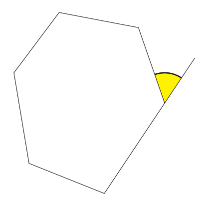


CONSTRUCTION:
Choose any point O inside the polygon,
and construct the intervals OA_1, OA_2, ..., OA_n,
giving n triangles A_1OA_2, A_2OA_3, ..., A_nOA_1.

PROOF:
The angle sum of the n triangles is $180n°$.
But the angles at O form a revolution, with size $360°$.
Hence for the interior angles of the polygon,

$$\begin{aligned} \text{sum} &= 180n° - 360° \\ &= 180(n-2)°. \end{aligned}$$

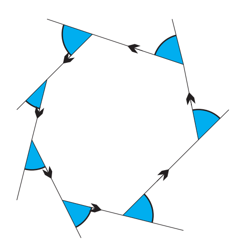
The Exterior Angles of a Polygon: An exterior angle of a convex polygon at any vertex is the angle between one side produced and the other side, just as in a triangle. There is a very simple formula for the sum of the exterior angles.



10 COURSE THEOREM — EXTERIOR ANGLES OF A POLYGON:
- The sum of the exterior angles of any convex polygon is $360°$.

PROOF:
At each vertex, the interior and exterior angles add to $180°$,
so the sum of all the interior and exterior angles is $180n°$.
But the interior angles add to $180(n-2)° = 180n° - 360°$.
Hence the exterior angles must add to $360°$.

Exterior Angles as the Amount of Turning: If one walks around a polygon, the exterior angle at each vertex is the angle one turns at that vertex. Thus the sum of all the exterior angles is the amount of turning when one walks right around the polygon. Walking around a polygon clearly involves a total turning of $360°$, and the previous theorem can be interpreted as saying just that.



Regular Polygons: A *regular polygon* is a polygon in which all sides are equal and all interior angles are equal. Simple division gives:

11 COURSE THEOREM — REGULAR POLYGONS:
- Each exterior angle of an n-sided regular polygon is $\dfrac{360°}{n}$.
- Each interior angle of an n-sided regular polygon is $\dfrac{180(n-2)°}{n}$.

Substitution of $n = 3$ and $n = 4$ gives the familiar results that each angle of an equilateral triangle is $60°$ and each angle of a square is $90°$.

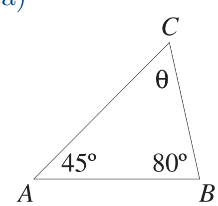
Worked Exercise:
Find the sizes of each exterior angle and each interior angle in a regular 12-sided polygon.

Solution:
The exterior angles have sum 360°, so each exterior angle is $360^\circ \div 12 = 30^\circ$.
Hence each interior angle is 150° (angles in a straight angle).
Alternatively, using the formula, each interior angle is $\left(\dfrac{180 \times (12-2)}{12}\right)^\circ = 150^\circ$.

Exercise 7B

Note: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Use the angle sum of a triangle to find θ in each of the diagrams below, giving reasons.

(a)


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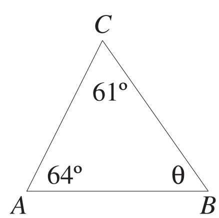

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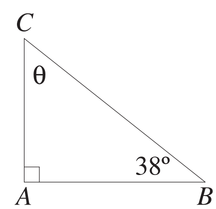

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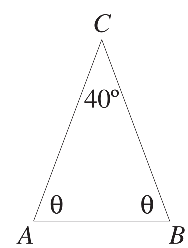

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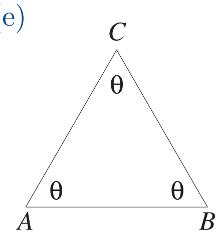

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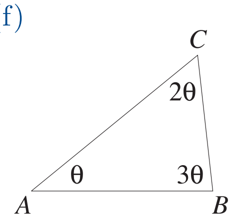

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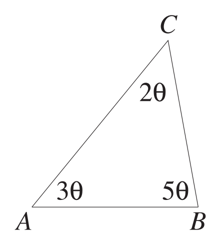

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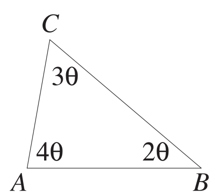
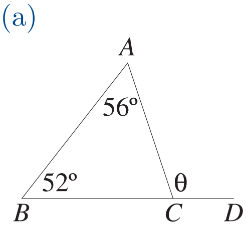


2. Use the exterior angle of a triangle theorem to find θ in each case, giving reasons.

(a)


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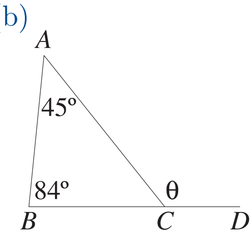

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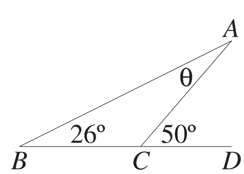

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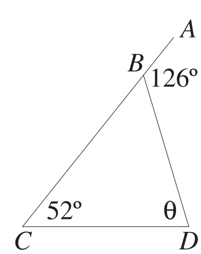
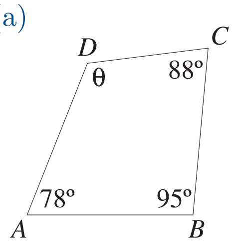


3. Use the angle sum of a quadrilateral to find θ in the diagrams below, giving reasons.

(a)


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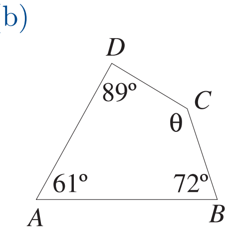

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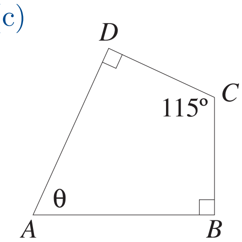

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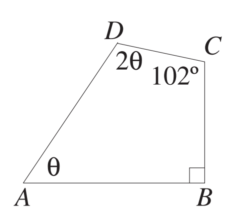

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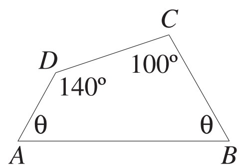

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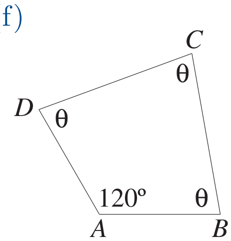

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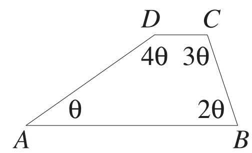

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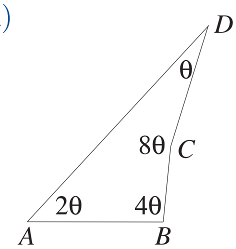
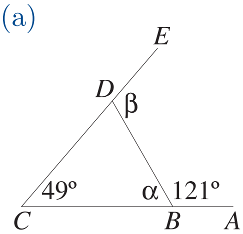


4. Find the angles α and β in the diagrams below. Give all steps in your argument.

(a)



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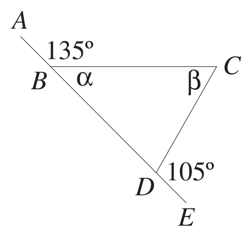

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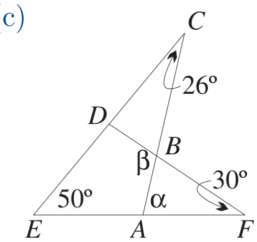

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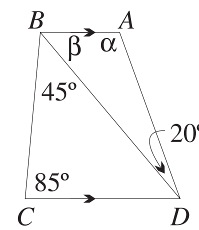

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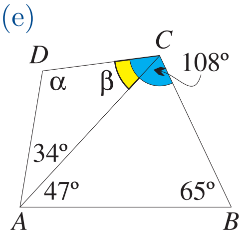

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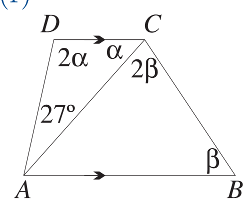

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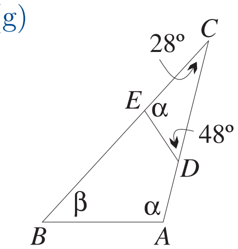

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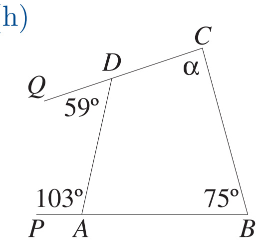
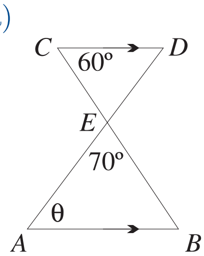


5. Find the angles θ and ϕ in the diagrams below, giving all reasons.

(a)



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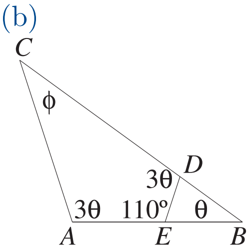

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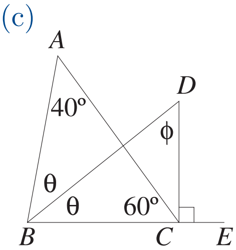

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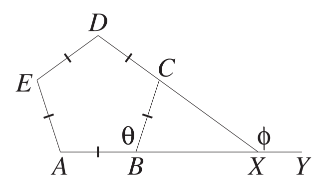


6. Find the size of each (i) interior angle, (ii) exterior angle, of a regular polygon with:

(a) 5 sides, (b) 6 sides, (c) 8 sides, (d) 9 sides, (e) 10 sides, (f) 12 sides.

7. (a) Find the number of sides of a regular polygon if each interior angle is:

(i) 135° (ii) 144° (iii) 172° (iv) 178°

(b) Find the number of sides of a regular polygon if its exterior angle is:

(i) 72° (ii) 40° (iii) 18° (iv) $\frac{1}{2}^\circ$

(c) Why is it not possible for a regular polygon to have an interior angle equal to 123°?

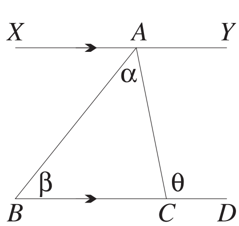
(d) Why is it not possible for a regular polygon to have an exterior angle equal to 71°?

DEVELOPMENT

8. COURSE THEOREM: *An alternative proof of the exterior angle theorem.*

Let ABC be a triangle with the side BC produced to D.
Construct the line XY through A parallel to the side BD.
Let $\angle CAB = \alpha$ and $\angle ABC = \beta$.

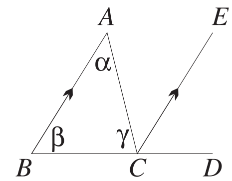
Use alternate angles twice to prove that $\angle ACD = \alpha + \beta$.



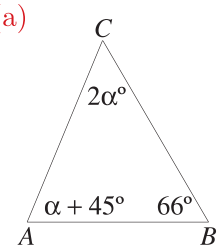
9. COURSE THEOREM: *An alternative proof that the angle sum of a triangle is 180°.*

Let ABC be a triangle. Let the side BC be produced to D. Construct the line CE through C parallel to the side BA. Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$.

Prove that $\alpha + \beta + \gamma = 180°$.



10. Find the value of α in each of the diagrams below, giving all reasons.

(a)


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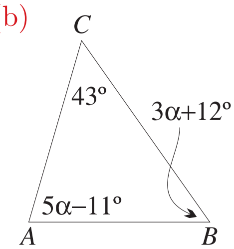

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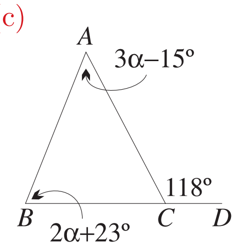

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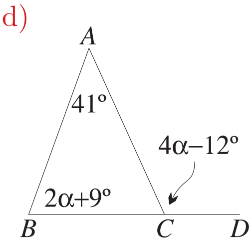

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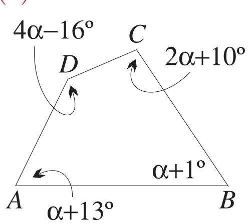

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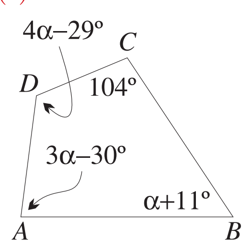

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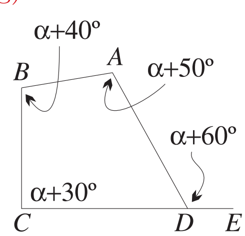

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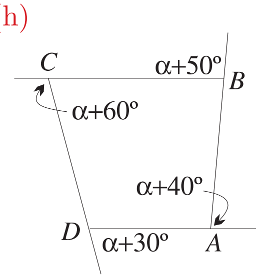
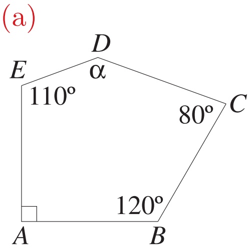


11. Find the value of α in each of the diagrams below, giving all reasons.

(a)


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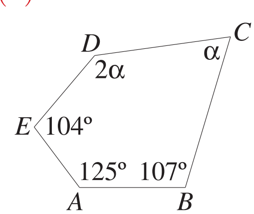

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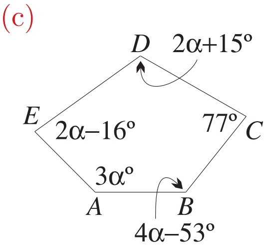
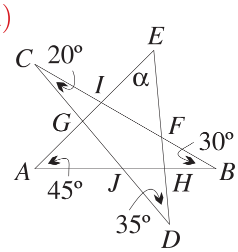
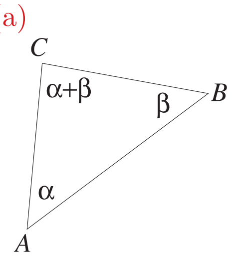
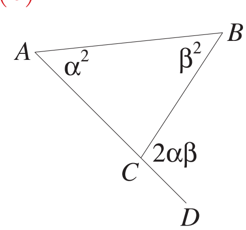


(d)


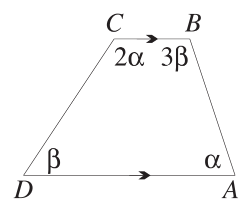
12. Prove the given relationships in the diagrams below.

(a)


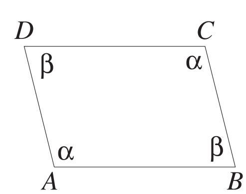
Show that $\alpha + \beta = 90°$.

(b)


Show that $\alpha = \beta$.

(c)


Show that $\alpha = 72°$ and $\beta = 36°$.

(d)


Show that $AB||CD$ and $AD||BC$.

13. COUNTING IN GEOMETRY:

By drawing a diagram, find the number of diagonals of:

(a) a convex pentagon, (b) a convex hexagon, (c) a convex octagon,

and verify in each case that the number of diagonals of the polygon is $\frac{1}{2}n(n-3)$.

14. THE INTERIOR ANGLES OF A NON-CONVEX POLYGON:
For non-convex polygons, the formula $180(n-2)^\circ$ for the sum of the interior angles of an n-sided polygon is proven by dissecting the polygon into $n-2$ triangles. Demonstrate the proof by drawing the following diagrams:

(a) Draw a non-convex pentagon and dissect it into three triangles.

(b) Draw a non-convex hexagon and dissect it into four triangles.

(c) Draw a non-convex octagon and dissect it into six triangles.

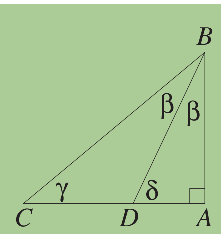
(d) Draw a non-convex dodecagon and dissect it into ten triangles.

15. (a) Show that in a polygon with n sides, $\dfrac{\text{sum of the interior angles}}{\text{sum of the exterior angles}} = \dfrac{n-2}{2}$.

(b) Hence determine if it is possible to have these angles in the ratio:
(i) $\frac{8}{3}$ (ii) $\frac{7}{2}$

CHALLENGE

16. In the right-angled triangle ABC opposite, $\angle CAB = 90^\circ$.
The bisector of $\angle ABC$ meets AC at D.
Let $\angle ABD = \beta$, $\angle ACB = \gamma$ and $\angle ADB = \delta$.



(a) Give reasons why $2\beta = 90^\circ - \gamma$ and $\delta = \beta + \gamma$.

(b) Hence show that $\delta = 45^\circ + \frac{1}{2}\gamma$.

17. SEQUENCES AND GEOMETRY:

(a) The three angles of a triangle ABC form an arithmetic sequence. Show that the middle-sized angle is 60°. [HINT: Let the angles be $\theta - \alpha$, θ and $\theta + \alpha$.]

(b) The five angles of a pentagon $ABCDE$ form an arithmetic sequence. Find the size of the middle-sized angle. [HINT: Let the angles be $\theta - 2\alpha$, $\theta - \alpha$, θ, $\theta + \alpha$ and $\theta + 2\alpha$.]

18. POLYGONS WITH ALL ANGLES EQUAL:

(a) A quadrilateral in which all angles are equal need not have all sides equal (it is in fact a rectangle). Prove, nevertheless, that opposite sides are parallel.

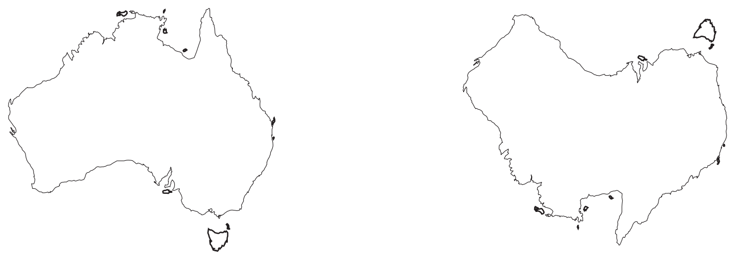
(b) Prove that if all angles of a hexagon are equal, then opposite sides are parallel.

7 C Congruence and Special Triangles

As in all branches of mathematics, symmetry is a vital part of geometry. In Euclidean geometry, symmetry is handled by means of congruence and later through the more general idea of similarity. It is only by these methods that relationships between lengths and angles can be established.

Congruence: Two figures are called *congruent* if one figure can be picked up and placed so that it fits exactly on top of the other figure. More precisely, using the language of transformations:

12 THE DEFINITION OF CONGRUENCE:
Two figures $\mathcal{S}$ and $\mathcal{T}$ are called *congruent*, written as $\mathcal{S} \equiv \mathcal{T}$, if one figure can be moved to coincide with the other figure by means of a sequence of rotations, reflections and translations.



The congruence sets up a correspondence between the elements of the two figures. In this correspondence, angles, lengths and areas are preserved.

13 PROPERTIES OF CONGRUENT FIGURES:

If two figures are congruent:

- matching angles have the same size,
- matching intervals have the same length,
- matching regions have the same area.

Congruent Triangles: In practice, almost all congruence arguments concern congruent triangles. Euclid's geometry book proves four tests for the congruence of two triangles, but in this course they are taken as assumptions.

14 ASSUMPTIONS — THE STANDARD CONGRUENCE TESTS FOR TRIANGLES:

Two triangles are congruent if:

SSS the three sides of one triangle are respectively equal to the three sides of another triangle, or

SAS two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, or

AAS two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, or

RHS the hypotenuse and one side of one right triangle are respectively equal to the hypotenuse and one side of another right triangle.

These standard tests are known from earlier years and have already been discussed in the trigonometry chapter of the Year 11 volume, where they were related to the sine and cosine rules.

As mentioned in those sections, there is no 'ASS' test — two sides and a non-included angle — and two non-congruent triangles with the same 'ASS' specifications were constructed.

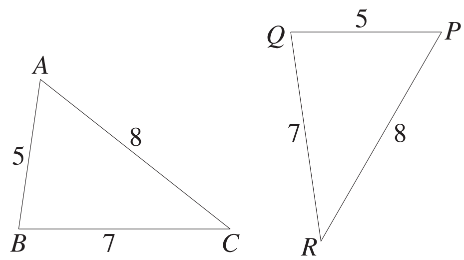
Here are examples of the four tests.

The SSS Congruence Test:

In the diagram to the right,

$\triangle ABC \equiv \triangle PQR$ (SSS).

Hence $\angle P = \angle A$, $\angle Q = \angle B$ and $\angle R = \angle C$ (matching angles of congruent triangles).

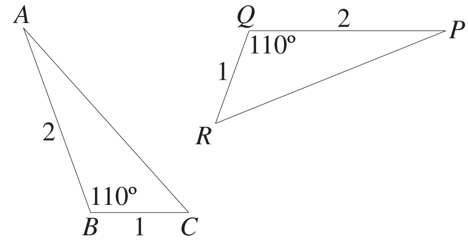


The SAS Congruence Test:

In the diagram to the right,

$\triangle ABC \equiv \triangle PQR$ (SAS).

Hence $\angle P = \angle A$ and $\angle R = \angle C$
(matching angles of congruent triangles).
and $PR = AC$
(matching sides of congruent triangles).

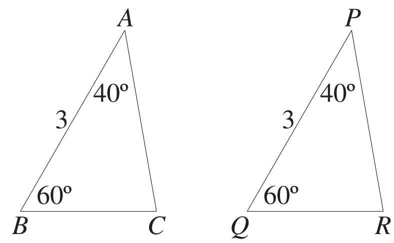


The AAS Congruence Test:

In the diagram to the right,

$\triangle ABC \equiv \triangle PQR$ (AAS).

Hence $QR = BC$ and $RP = CA$
(matching sides of congruent triangles),
and $\angle R = \angle C$
(angle sums of triangles).

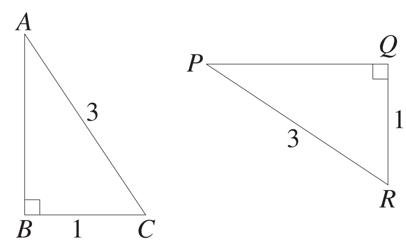


The RHS Congruence Test:

In the diagram to the right,

$\triangle ABC \equiv \triangle PQR$ (RHS).

Hence $\angle P = \angle A$ and $\angle R = \angle C$
(matching angles of congruent triangles).
and $PQ = AB$
(matching sides of congruent triangles).



Using the Congruence Tests: A fully set-out congruence proof has five lines:

- The first line introduces the triangles.
- The next three set out the three pairs of equal sides or angles.
- The final line is the conclusion, naming the congruence test.

Throughout the five lines of the proof, all vertices should be named in corresponding order. Then subsequent conclusions about equal angles or equal sides can easily be written down from the statement of the congruence.

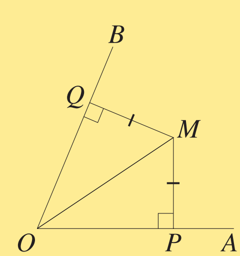
Each of the four standard congruence tests is used in one of the next four proofs.

WORKED EXERCISE:

The point M lies inside the arms of the acute angle $\angle AOB$.
The perpendiculars MP and MQ to OA and OB respectively have equal lengths.

(a) Prove that $\triangle POM \equiv \triangle QOM$.

(b) Hence prove that OM bisects $\angle AOB$.



SOLUTION:

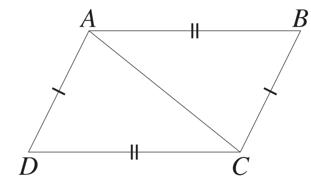
(a) In the triangles POM and QOM:

1. $OM = OM$ (common),
2. $PM = QM$ (given),
3. $\angle OPM = \angle OQM = 90^\circ$ (given),

so $\triangle POM \equiv \triangle QOM$ (RHS).

(b) Hence $\angle POM = \angle QOM$ (matching angles of congruent triangles).

WORKED EXERCISE:



In the diagram to the right:

(a) Prove that $\triangle ABC \equiv \triangle CDA$.

(b) Hence prove that $AD \parallel BC$.

SOLUTION:

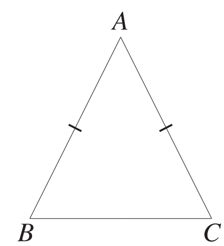
(a) In the triangles ABC and CDA:

1. $AC = CA$ (common),
2. $AB = CD$ (given),
3. $BC = DA$ (given),

so $\triangle ABC \equiv \triangle CDA$ (SSS).

(b) Hence $\angle BCA = \angle DAC$ (matching angles of congruent triangles),
and so $AD \parallel BC$ (alternate angles are equal).

Isosceles Triangles: An *isosceles triangle* is a triangle in which two sides are equal.



The two equal sides are called the *legs* of the triangle (the Greek word 'isosceles' literally means 'equal legs'), their intersection is called the *apex* and the side opposite the apex is called the *base*.

15 THE DEFINITION OF AN ISOSCELES TRIANGLE:

- An *isosceles triangle* is a triangle in which two sides are equal.

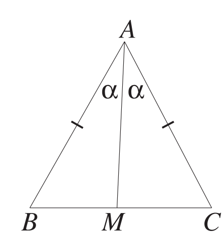
It is well known that the base angles of an isosceles triangle are equal.

16 COURSE THEOREM — A PROPERTY OF ISOSCELES TRIANGLES:

- If two sides of a triangle are equal, then the angles opposite those sides are equal.

GIVEN:
Let ABC be an isosceles triangle with $AB = AC$.

AIM:
To prove that $\angle B = \angle C$.



CONSTRUCTION:
Let the bisector of $\angle A$ meet BC at M.

PROOF:
In the triangles ABM and ACM:

1. $AM = AM$ (common),
2. $AB = AC$ (given),
3. $\angle BAM = \angle CAM$ (construction),

so $\triangle ABM \equiv \triangle ACM$ (SAS).

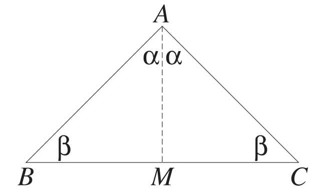
Hence $\angle ABM = \angle ACM$ (matching angles of congruent triangles).

A Test for a Triangle to be Isosceles: The converse of this result is also true, giving a test for a triangle to be isosceles.

17 COURSE THEOREM — A TEST FOR AN ISOSCELES TRIANGLE:
- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.

GIVEN:
Let ABC be a triangle in which $\angle B = \angle C = \beta$.

AIM:
To prove that $AB = AC$.



CONSTRUCTION:
Let the bisector of $\angle A$ meet BC at M.

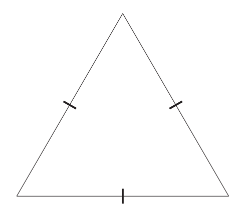
PROOF:
In the triangles ABM and ACM:

1. $AM = AM$ (common),
2. $\angle B = \angle C$ (given),
3. $\angle BAM = \angle CAM$ (construction),

so $\triangle ABM \equiv \triangle ACM$ (AAS).

Hence $AB = AC$ (matching sides of congruent triangles).

Equilateral Triangles: An *equilateral triangle* is a triangle in which all three sides are equal. An equilateral triangle is therefore an isosceles triangle in three different ways — the following property and test thus follow easily from the previous theorem and its converse.



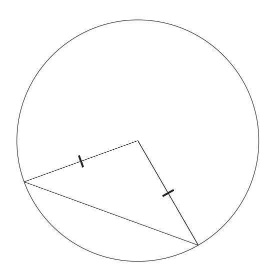
18 COURSE THEOREM — EQUILATERAL TRIANGLES:
- All angles of an equilateral triangle are equal to $60°$.
- Conversely, if all angles of a triangle are equal, then it is equilateral.

PROOF:

A. Suppose that the triangle is equilateral, that is, all three sides are equal. Then by the isosceles triangle theorems, all three angles are equal, and since their sum is $180°$, they must each be $60°$.

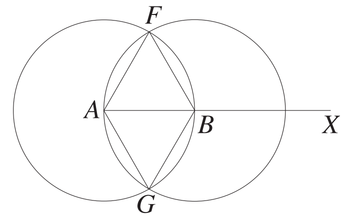
B. Conversely, suppose that all three angles are equal. Then by the isosceles triangle theorems, all three sides are equal, meaning that the triangle is equilateral.

Circles and Isosceles Triangles: A *circle* is the set of all points that are a fixed distance (called the *radius*) from a fixed point (called the *centre*). Compasses are used for drawing circles, because the pencil is held at a fixed distance from the centre, where the compass-point is fixed in the paper.



If two points on the circumference are joined to the centre and to each other, then the equal radii mean that the triangle is isosceles. The following worked exercise shows how to construct an angle of $60°$ using straight edge and compasses.

WORKED EXERCISE:
Construct a circle with centre on the end A of an interval AX, meeting the ray AX at B. With centre B and the same radius, construct a circle meeting the first circle at F and G. Prove that $\angle FAB = \angle GAB = 60°$.



SOLUTION:
Because they are all radii of congruent circles,

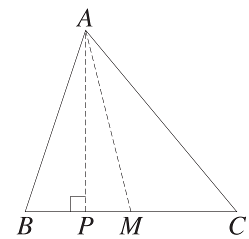
$$AF = AB = AG = BF = BG.$$

Hence $\triangle AFB$ and $\triangle AGB$ are both equilateral triangles, and so $\angle FAB = \angle GAB = 60°$.

Medians and Altitudes: A *median* of a triangle joins a vertex to the midpoint of the opposite side.

An *altitude* of a triangle is the interval from a vertex meeting the opposite side at right angles.

In the diagram to the right, AP is one of the three *altitudes* in $\triangle ABC$. The point M is the midpoint of BC and AM is one of the three *medians* in $\triangle ABC$.

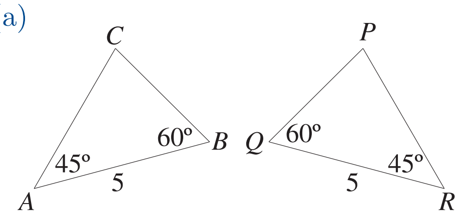


Exercise 7C

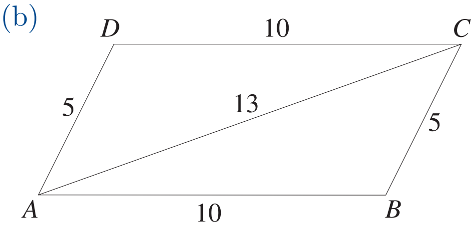
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. The two triangles in each pair below are congruent. Name the congruent triangles in the correct order and state which test justifies the congruence.

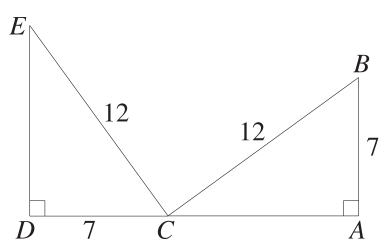
(a)



(b)



(c)



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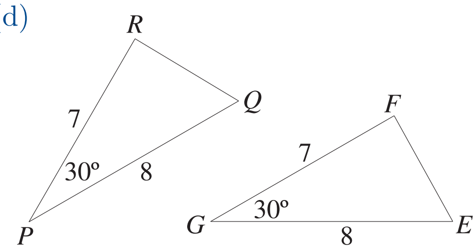
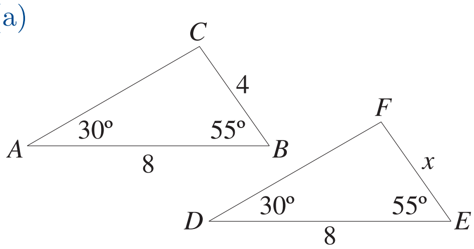


2. In each part, identify the congruent triangles, naming the vertices in matching order and giving a reason. Hence deduce the length of the side x.

(a)



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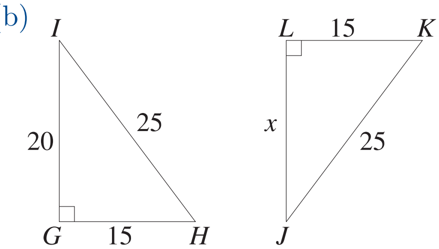

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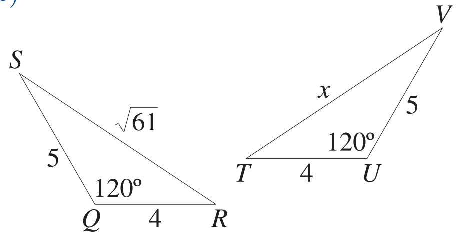

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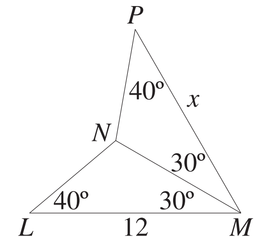
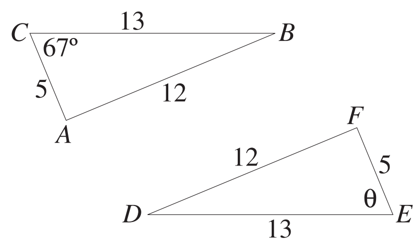


3. In each part, identify the congruent triangles, naming the vertices in matching order and giving a reason. Hence deduce the size of the angle θ.

(a)



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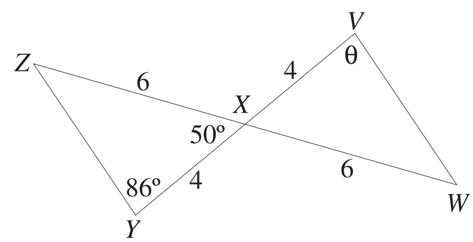

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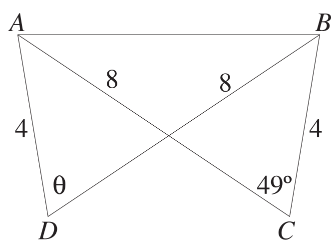

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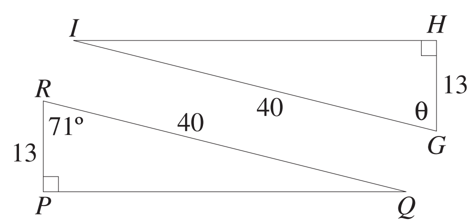
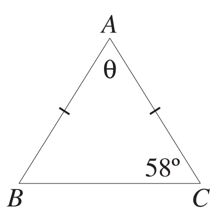


4. Find the size of angle θ in each diagram below, giving reasons.

(a)



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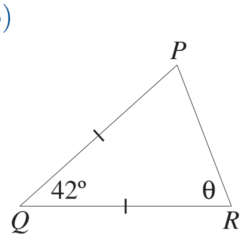

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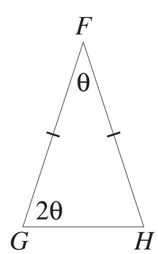

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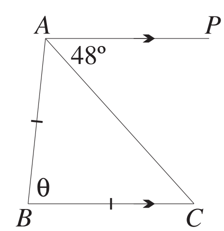
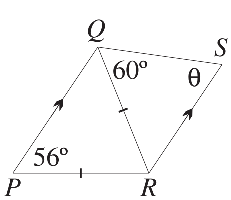
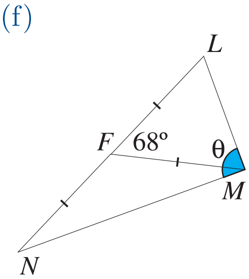


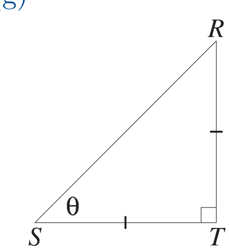
(e)



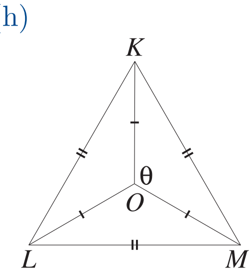
(f)



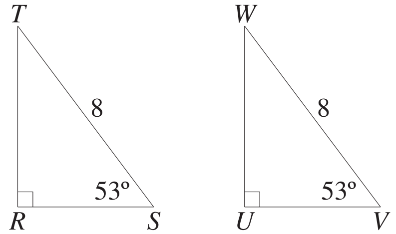
(g)



(h)

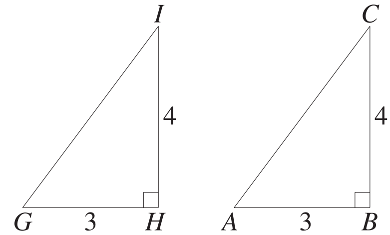


5. (a)



When asked to show that the two triangles above were congruent, a student wrote $\triangle RST \equiv \triangle UVW$ (RHS). Although both triangles are indeed right-angled, explain why the reason given is incorrect.
What is the correct reason?

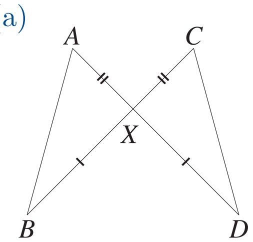
(b)



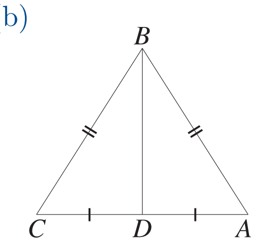
When asked to show that the two triangles above were congruent, another student wrote $\triangle GHI \equiv \triangle ABC$ (RHS). Again, although both triangles are right-angled, explain why the reason given is wrong.
What is the correct reason?

6. In each part, prove that the two triangles in the diagram are congruent.

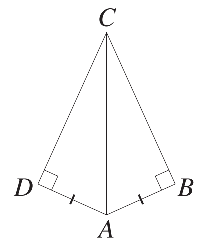
(a)



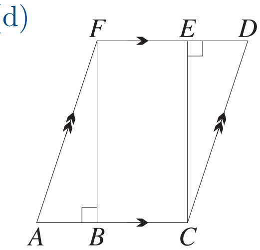
(b)



(c)

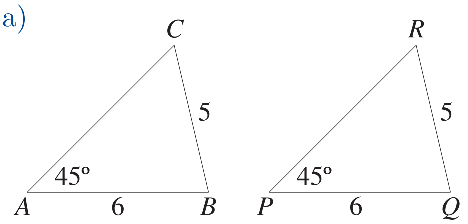


(d)

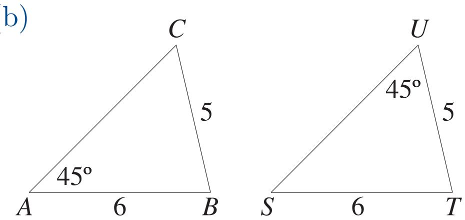


7. Explain why the given pairs of triangles cannot be proven to be congruent.

(a)



(b)



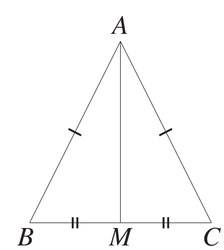
DEVELOPMENT

8. COURSE THEOREM: *A second proof that the base angles of an isosceles triangle are equal.*

Let $\triangle ABC$ be isosceles with $AB = AC$.
Construct the midpoint M of BC. Join the median AM.

(a) Prove that $\triangle AMB \equiv \triangle AMC$.

(b) Hence prove that $\angle B = \angle C$.

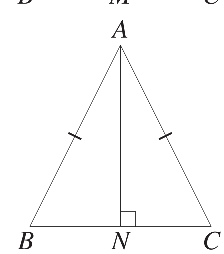


9. COURSE THEOREM: *A third proof that the base angles of an isosceles triangle are equal.*

Let $\triangle ABC$ be isosceles with $AB = AC$.
Construct the altitude AN from A to the base BC.

(a) Prove that $\triangle ANB \equiv \triangle ANC$.

(b) Hence prove that $\angle B = \angle C$.

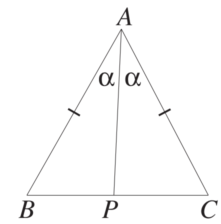


10. Let $\triangle ABC$ be isosceles with $AB = AC$.
Let the *angle bisector* of $\angle A$ meet the base BC at P.

(a) Prove that $\triangle ABP \equiv \triangle ACP$.

(b) Hence show that $BP = CP$ and $AP \perp BC$.

NOTE: You have proven that the bisector of the apex of an isosceles triangle is the perpendicular bisector of the base.

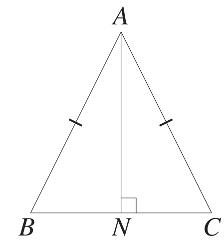


11. Let $\triangle ABC$ be isosceles with $AB = AC$.
Let AN be the *altitude* to the base BC.

(a) Prove that $\triangle ABN \equiv \triangle ACN$.

(b) Hence show that $BN = CN$ and $\angle BAN = \angle CAN$.

NOTE: You have proven that the altitude to the base of an isosceles triangle bisects the base and the apex angle.

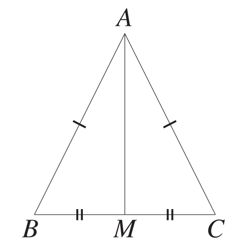


12. Let $\triangle ABC$ be isosceles with $AB = AC$.
Let M be the midpoint of BC. Join the *median* AM.

(a) Prove that $\triangle ABM \equiv \triangle ACM$.

(b) Hence show that $\angle BAM = \angle CAM$ and $AM \perp BC$.

NOTE: You have proven that the median to the base of an isosceles triangle is perpendicular to the base and bisects the apex angle.

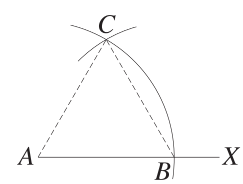


13. CONSTRUCTION: *Constructing an angle of 60°.*

Let an arc with centre A meet an interval AX at B.
Let another arc with centre B and the same radius meet the first arc at C.

(a) Explain why $\triangle ABC$ is equilateral.

(b) Hence explain why $\angle BAC = 60°$.

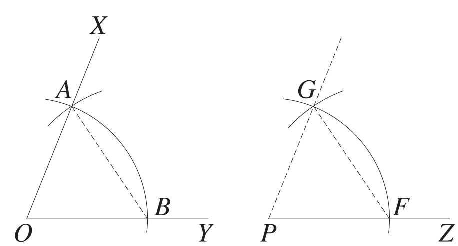


14. CONSTRUCTION: *Copying an angle.*

Let $\angle XOY$ be an angle and PZ an interval.
Let an arc, centre O, meet OX at A and OY at B.
Let a second arc with centre P and the same radius meet PZ at F.
Let a third arc with centre F and radius AB meet the second arc at G.

(a) Prove that $\triangle AOB \equiv \triangle FPG$.

(b) Hence prove that $\angle AOB = \angle FPG$.

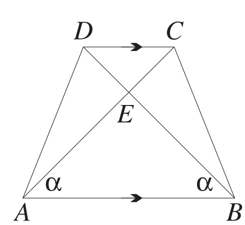


15. Let $ABCD$ be a trapezium with $AB \parallel DC$, and let the diagonals meet at E. Suppose that $\angle CAB = \angle ABD = \alpha$.

(a) Show that $CE = DE$.

(b) Prove that $\triangle ABC \equiv \triangle BAD$.

(c) Hence show that $\angle DAC = \angle CBD$.

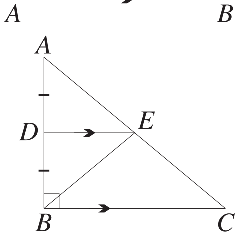


16. In the diagram to the right, $\triangle ABC$ is right-angled at B.
Let D be the midpoint of AB. Let DE be parallel to BC.

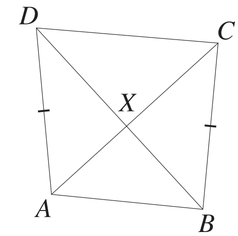
(a) Prove that $\angle ADE$ is a right angle.

(b) Prove that $\triangle AED \equiv \triangle BED$.

(c) Use the base angles of $\triangle BEC$ to prove that $BE = EC$.



17. The diagram to the right shows a quadrilateral $ABCD$. The diagonals AC and BD are equal and intersect at X. The sides AD and BC are equal.



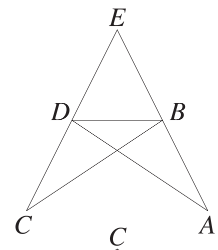
(a) Show that $\triangle ABC \equiv \triangle BAD$.

(b) Hence show that $\triangle ABX$ is isosceles.

(c) Thus show that $\triangle CDX$ is also isosceles.

(d) Show that $AB \parallel DC$.

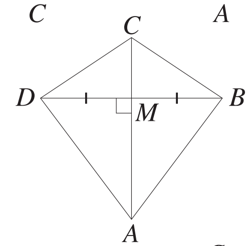
18. In the diagram to the right, $\triangle ABD \equiv \triangle CDB$.



(a) Prove that $\angle EDB = \angle EBD$.
[HINT: You will need to show that these two angles have equal supplements.]

(b) Prove that $\triangle BDE$ is isosceles.

19. In the diagram to the right, $DM = MB$ and $AC \perp DB$.



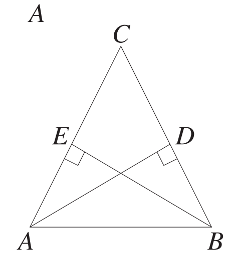
(a) Prove that $\triangle AMB \equiv \triangle AMD$.

(b) Prove that $\triangle ABD$ is isosceles.

(c) Prove that $\triangle CBD$ is isosceles.

20. A TEST FOR AN ISOSCELES TRIANGLE: *If two altitudes of a triangle are equal, then the triangle is isosceles.*

Let AD and BE be two altitudes of a triangle ABC. Suppose that $AD = BE$.

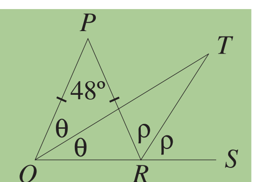


(a) Prove that $\triangle ABE \equiv \triangle BAD$.

(b) Hence prove that $\triangle ABC$ is isosceles with $AC = BC$.

CHALLENGE

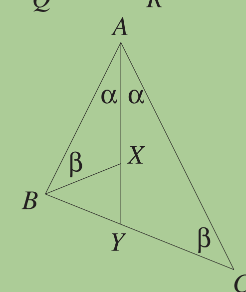
21. Let $\triangle PQR$ be isosceles with $PQ = PR$ and $\angle QPR = 48°$. Let the interval QR be produced to S. Let the angle bisectors of $\angle PQR$ and $\angle PRS$ meet at T.



(a) Find $\angle PQR$.

(b) Find $\angle QTR$.

22. In the diagram, the bisector of $\angle BAC$ meets BC at Y. Construct the point X on AY so that $\angle ABX = \angle ACB$.

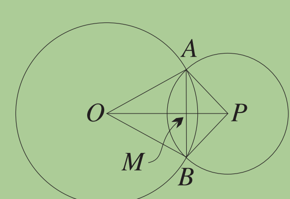


(a) Use exterior angles of triangles to prove that
$\angle BXY = \angle BYX = \alpha + \beta$.

(b) Hence prove that $\triangle BXY$ is isosceles.

23. THEOREM: *The line of centres of two intersecting circles is the perpendicular bisector of the common chord.*

In the diagram, two circles intersect at A and B. The line of centres OP meets the common chord AB at M.



(a) Explain why $\triangle ABO$ and $\triangle ABP$ are isosceles.

(b) Show that $\triangle AOP \equiv \triangle BOP$.

(c) Show that $\triangle AMO \equiv \triangle BMO$.

(d) Hence show that $AM = BM$ and $AB \perp OP$.

7 D Trapeziums and Parallelograms

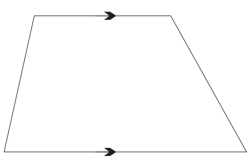
There are a series of important theorems concerning the sides and angles of quadrilaterals. If careful definitions are first given of five special quadrilaterals, these theorems can then be stated elegantly as properties of these special quadrilaterals and tests for them.

This section deals with trapeziums and parallelograms; the following Section 7E deals with rhombuses, rectangles and squares.

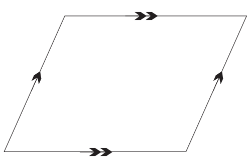
These theorems have been treated in earlier years and most proofs have been given as structured questions in the following exercise. The proofs, however, are an essential part of the course and should be carefully studied.

Definitions of Trapeziums and Parallelograms: These figures are defined in terms of parallel sides. Notice that a parallelogram is a special sort of trapezium.

19 THE DEFINITIONS OF A TRAPEZIUM AND A PARALLELOGRAM:
- A *trapezium* is a quadrilateral with at least one pair of opposite sides parallel.
- A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.



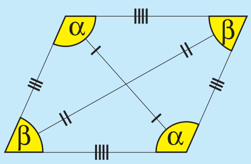
A trapezium



A parallelogram

Properties of Parallelograms: The four standard properties of the parallelogram concern the angles, the sides and the diagonals. The proofs are presented as structured questions in the exercises.

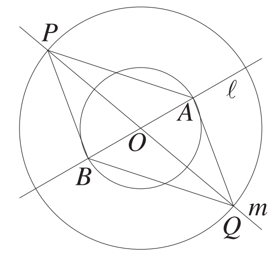
20 COURSE THEOREM — PROPERTIES OF A PARALLELOGRAM:
If a quadrilateral is a parallelogram, then:
- adjacent angles are supplementary, and
- opposite angles are equal, and
- opposite sides are equal, and
- the diagonals bisect each other.



Tests for Parallelograms: The four standard tests for a parallelogram also concern the angles, the sides and the diagonals. Again, the proofs are developed as structured questions in the exercises.

21 COURSE THEOREM — TESTS FOR A PARALLELOGRAM:
Conversely, a quadrilateral is a parallelogram if:
- the opposite angles are equal, or
- the opposite sides are equal, or
- one pair of opposite sides are equal and parallel, or
- the diagonals bisect each other.

WORKED EXERCISE: [A construction of a parallelogram]



Two lines ℓ and m intersect at O,
and concentric circles are constructed with centre O.
Let ℓ meet the inner circle at A and B,
and let m meet the outer circle at P and Q.
Prove that $APBQ$ is a parallelogram.

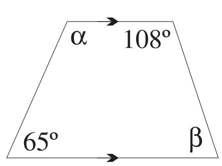
SOLUTION:
Since the point O is the midpoint of AB and of PQ,
the diagonals of $APBQ$ bisect each other.
Hence using the last test above for a parallelogram,
the figure $APBQ$ is a parallelogram.

Exercise 7D

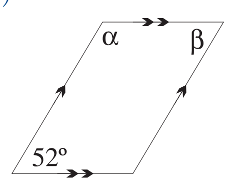
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find the angles α and β in the diagrams below, giving reasons.

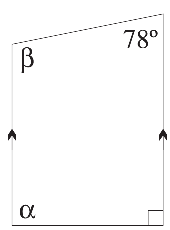
(a)



(b)



(c)



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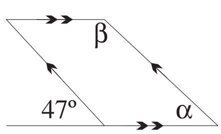
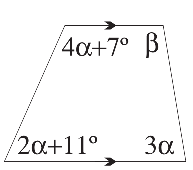
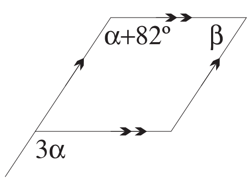


2. Write down an equation for α in each diagram below, giving reasons. Solve this equation to find the angles α and β, giving reasons.

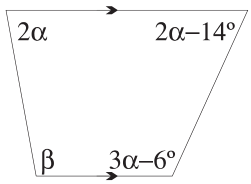
(a)



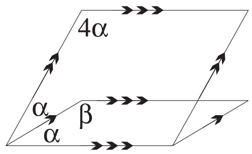
(b)



(c)



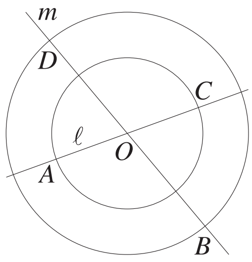
(d)



3. CONSTRUCTION: *Constructing a parallelogram from two equal parallel intervals.*
Place a ruler with two parallel edges flat on the page and draw 4 cm intervals AB and PQ on each side of the ruler. The two intervals can be offset from each other.

What theorem tells us that $ABQP$ is a parallelogram?

4. CONSTRUCTION: *Constructing a parallelogram from its diagonals.*

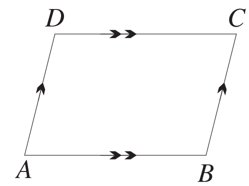


Let ℓ and m be two lines meeting at point O.
Let two circles be drawn with common centre O.
Let ℓ meet the inner circle at A and C and let m meet the outer circle at B and D.

Use the tests for a parallelogram to explain why the quadrilateral $ABCD$ is a parallelogram.

DEVELOPMENT

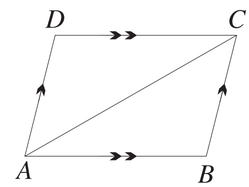
5. PROPERTIES OF A PARALLELOGRAM: In this question, use the definition of a parallelogram as a quadrilateral in which the opposite sides are parallel.



(a) COURSE THEOREM: *Adjacent angles of a parallelogram are supplementary and opposite angles are equal.*

Let $ABCD$ be a parallelogram.

Explain why $\angle A + \angle B = 180^\circ$ and $\angle A = \angle C$.

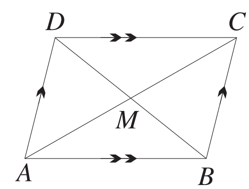


(b) COURSE THEOREM: *Opposite sides of a parallelogram are equal.*

Let $ABCD$ be a parallelogram with diagonal AC.

(i) Prove that $\triangle ACB \equiv \triangle CAD$.

(ii) Hence show that $AB = DC$ and $BC = AD$.



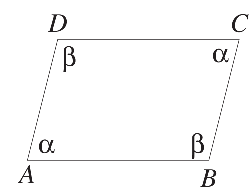
(c) COURSE THEOREM: *The diagonals of a parallelogram bisect each other.*

Let the diagonals of parallelogram $ABCD$ meet at M.

(i) Using part (b), prove that $\triangle ABM \equiv \triangle CDM$.

(ii) Hence show that $AM = CM$ and $BM = DM$.

6. TESTS FOR A PARALLELOGRAM: These four theorems give the standard tests for a quadrilateral to be a parallelogram.

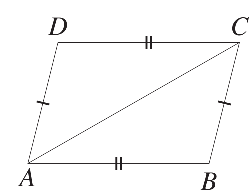


(a) COURSE THEOREM: *If the opposite angles of a quadrilateral are equal, then it is a parallelogram.*

Let $ABCD$ be a quadrilateral.
Suppose that $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$.

(i) Prove that $\alpha + \beta = 180^\circ$.

(ii) Hence show that $AB \parallel DC$ and $AD \parallel BC$.



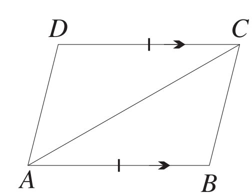
(b) COURSE THEOREM: *If the opposite sides of a quadrilateral are equal, then it is a parallelogram.*

Let $ABCD$ be a quadrilateral.
Suppose that $AB = DC$ and $AD = BC$.
Join the diagonal AC.

(i) Prove that $\triangle ACB \equiv \triangle CAD$.

(ii) Thus prove that $\angle CAB = \angle ACD$ and also that $\angle ACB = \angle CAD$.

(iii) Hence show that $AB \parallel DC$ and $AD \parallel BC$.



(c) COURSE THEOREM: *If one pair of opposite sides of a quadrilateral are equal and parallel, then it is a parallelogram.*

Let $ABCD$ be a quadrilateral.
Suppose that $AB = DC$ and $AB \parallel DC$.
Join the diagonal AC.

(i) Prove that $\triangle ACB \equiv \triangle CAD$.

(ii) Prove that $\angle BCA = \angle DAC$.

(iii) Hence show that $AD \parallel BC$.

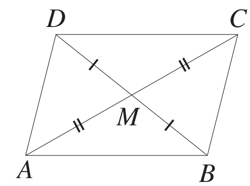
(d) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

Let the diagonals of a quadrilateral $ABCD$ meet at M. Suppose that $AM = MC$ and $BM = MD$.

(i) Prove that $\triangle ABM \equiv \triangle CDM$.

(ii) Prove that $AB = DC$ and $AB \parallel DC$.

(iii) Using part (c), prove that $ABCD$ is a parallelogram.



7. Is it true that if one pair of opposite sides of a quadrilateral are parallel and the other pair of opposite sides are equal, then the quadrilateral must be a parallelogram?

CHALLENGE

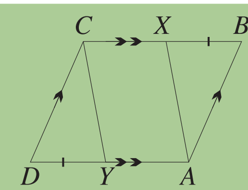
8. In the diagram to the right, $ABCD$ is a parallelogram. Choose X on BC and Y on AD such that $BX = DY$.

(a) Explain why $\angle ABX = \angle CDY$.

(b) Explain why $AB = CD$.

(c) Show that $\triangle ABX \equiv \triangle CDY$.

(d) Hence prove that $AYCX$ is a parallelogram.

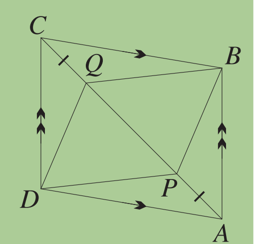


9. In the diagram to the right, $ABCD$ is a parallelogram. Choose P and Q on the diagonal AC such that $AP = CQ$.

(a) Prove that $\triangle ABP \equiv \triangle CDQ$.

(b) Prove that $\triangle ADP \equiv \triangle CBQ$.

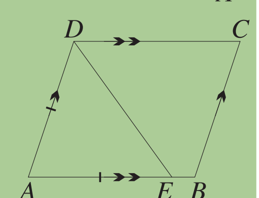
(c) Hence prove that $BQDP$ is a parallelogram.



10. In the diagram to the right, $ABCD$ is a parallelogram. Construct the point E on the side AB such that $AD = AE$.

Prove that the interval DE bisects the angle $\angle ADC$.

[HINT: Begin by letting $\angle ADE = \theta$.]



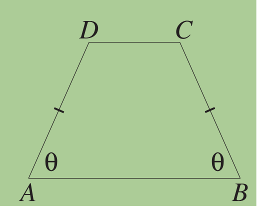
11. In the diagram to the right, $ABCD$ is a parallelogram. Let $\angle BAD = \angle ABC$ and $AD = BC$.

(a) Prove that $\triangle BAD \equiv \triangle ABC$.

(b) Why does $\angle ABD = \angle CAB$?

(c) Show that $\angle DAC = \angle DBC$.

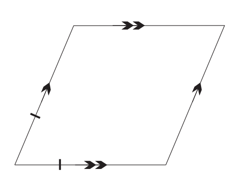
(d) Prove that $ABCD$ is a trapezium.



7 E Rhombuses, Rectangles and Squares

Rhombuses, rectangles and squares are particular types of parallelograms, and their definitions in this course reflect that understanding. Again, most of the proofs have been encountered in earlier years and are given as exercises.

Rhombuses and their Properties: A rhombus is often described as a 'pushed-over square', but it is formally defined as a special sort of parallelogram:



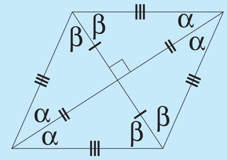
THE DEFINITION OF A RHOMBUS:

22 • A *rhombus* is a parallelogram in which a pair of adjacent sides are equal.

As with parallelograms, the standard properties of rhombuses concern the sides, the vertex angles and the diagonals. The first property is proven below and the other proofs are developed in the exercises.

23 COURSE THEOREM — PROPERTIES OF A RHOMBUS:

If a quadrilateral is a rhombus, then:
- all four sides are equal, and
- the diagonals bisect each other at right angles, and
- the diagonals bisect each vertex angle.



PROOF OF THE FIRST PROPERTY:

Since a rhombus is a parallelogram, its opposite sides are equal.

Since also two adjacent sides are equal, all four sides must be equal.

Tests for Rhombuses: There are three standard tests for rhombuses, corresponding to the three standard properties above. The first test is proven below and the proofs of the other two tests are developed in the exercises.

24 COURSE THEOREM — TESTS FOR A RHOMBUS:

Conversely, a quadrilateral is a rhombus if:
- all sides are equal, or
- the diagonals bisect each other at right angles, or
- the diagonals bisect each vertex angle.

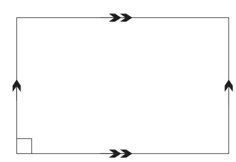
PROOF OF THE FIRST TEST:

Suppose that all four sides of a quadrilateral are equal.

Since opposite sides are equal, it must be a parallelogram,

and since two adjacent sides are equal, it is therefore a rhombus.

Rectangles and their Properties: A rectangle is formally defined, like a rhombus, as a special type of parallelogram.

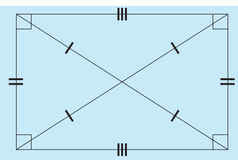


25 THE DEFINITION OF A RECTANGLE:
- A *rectangle* is a parallelogram in which one angle is a right angle.

The standard properties of rectangles are given below. The second property is proven in the exercises.

26 COURSE THEOREM — PROPERTIES OF A RECTANGLE:

If a quadrilateral is a rectangle, then:
- all four angles are right angles, and
- the diagonals are equal and bisect each other.



PROOF OF THE FIRST PROPERTY:

Since a rectangle is a parallelogram, its opposite angles are equal and add to $360°$.

Since one angle is $90°$, it follows that all angles are $90°$.

Tests for a Rectangle: The standard tests for rectangles are the converse of the two standard properties above.

27 COURSE THEOREM — TESTS FOR A RECTANGLE:
Conversely, a quadrilateral is a rectangle if:
- all four angles are equal, or
- the diagonals are equal and bisect each other.

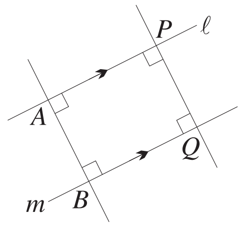
PROOF OF THE FIRST TEST:

Suppose that all angles of a quadrilateral are equal.

Then since they add to 360°, they must each be 90°.

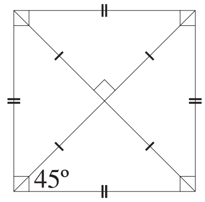
Hence the opposite angles are equal, so the quadrilateral must be a parallelogram, and since one angle is 90°, it is a rectangle.

The Distance Between Parallel Lines: Suppose that AB and PQ are two transversals perpendicular to two parallel lines ℓ and m. Then $ABQP$ forms a rectangle, because all its vertex angles are right angles. Hence the opposite sides AB and PQ are equal. This allows a formal definition of the distance between two parallel lines.



28 THE DEFINITION OF DISTANCE BETWEEN PARALLEL LINES:
- The distance between two parallel lines is the length of any perpendicular transversal.

Squares: Rhombuses and rectangles are different special sorts of parallelograms. A square is simply a quadrilateral that is both a rhombus and a rectangle.

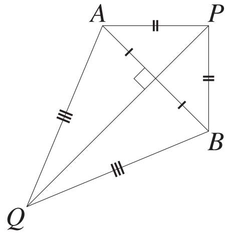


29 THE DEFINITION OF A SQUARE:
- A *square* is a quadrilateral that is both a rhombus and a rectangle.

It follows then that all sides of a square are equal, that all angles are right angles, and that the diagonals bisect each other at right angles and meet each side at 45°.

Conversely, to prove that a quadrilateral is a square, one needs to prove that it is both a rhombus and a rectangle.

A Note on Kites: Kites are not part of the course, but they occur frequently in problems. A *kite* is usually defined as a quadrilateral in which two pairs of adjacent sides are equal, as in the diagram to the right, where $AP = BP$ and $AQ = BQ$.



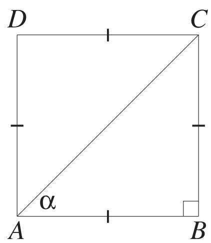
The last question in the following exercise develops the proof that the diagonal PQ is the perpendicular bisector of the diagonal AB and bisects the vertex angles at P and at Q. The second part of the question develops a test for kites.

Theorems about kites, however, are not part of the course and should not be quoted as reasons unless they have been developed earlier in the same question.

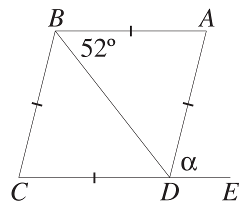
Exercise 7E

1. Find α in each of the figures below, giving reasons.

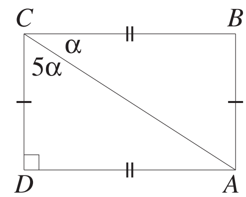
(a)



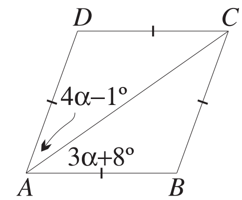
(b)



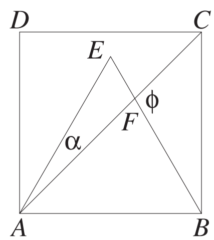
(c)



(d)

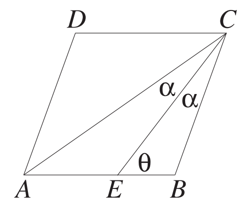


2. (a)



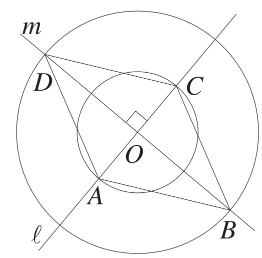
Inside the square $ABCD$ is an equilateral $\triangle ABE$. The diagonal AC intersects BE at F. Find the sizes of angles α and ϕ.

(b)



$ABCD$ is a rhombus with the diagonal AC shown. The line CE bisects $\angle ACB$. Show that $\theta = 3\alpha$.

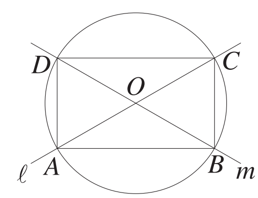
3. CONSTRUCTION: *A rhombus from its diagonals.*



Let ℓ and m be two perpendicular lines meeting at O.
Let two circles be drawn with common centre O.
Let ℓ meet the inner circle at A and C, and let m meet the outer circle at B and D.

Use the standard tests for a rhombus to explain why the quadrilateral $ABCD$ is a rhombus.

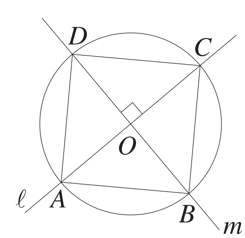
4. CONSTRUCTION: *A rectangle from its diagonals.*



Let ℓ and m be two lines meeting at O.
Let a circle be drawn with centre O and any radius.
Let ℓ meet the circle at A and C, and m meet it at B and D.

Use the standard tests for a rectangle to explain why the quadrilateral $ABCD$ is a rectangle.

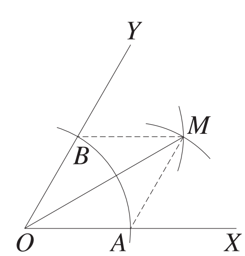
5. CONSTRUCTION: *A square from its diagonals.*



Let ℓ and m be two perpendicular lines meeting at O.
Let a circle be drawn with centre O and any radius.
Let ℓ meet the circle at A and C, and m meet it at B and D.

Use the standard tests for a square to explain why the quadrilateral $ABCD$ is a square.

6. CONSTRUCTION: *The bisector of a given angle.*



Let $\angle XOY$ be any angle.
Let an arc with centre O meet OX at A and OY at B.
Let two further arcs be drawn with centres A and B and the same radius, and let them meet at M. Join the interval OM.

(a) Why is the quadrilateral $OAMB$ a rhombus?

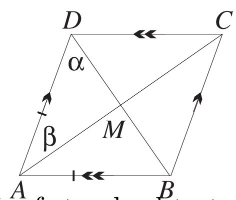
(b) Hence explain why OM bisects $\angle XOY$.

DEVELOPMENT

7. COURSE THEOREM: *The diagonals of a rhombus bisect each other at right angles and bisect the vertex angles.*

Let $ABCD$ be a rhombus with diagonals meeting at M.
Let $\alpha = \angle ADM$ and $\beta = \angle DAM$.

Since a rhombus is a parallelogram, we already know that the diagonals bisect each other.

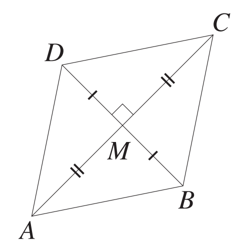


(a) Explain why $\angle ABM = \alpha$.
(b) Hence prove that $\angle CDM = \alpha$.
(c) Prove similarly that $\angle DCM = \beta$.
(d) Hence prove that $AC \perp BD$. (Use the angle sum of $\triangle ADC$ — there is no need for congruence.)

8. TESTS FOR A RHOMBUS: The following two parts are structured proofs of standard tests for a rhombus listed in the notes above.

(a) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.*

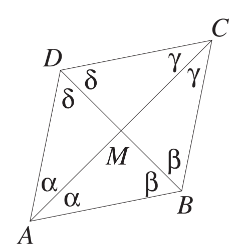
Let the diagonals of a quadrilateral $ABCD$ meet at M.
Suppose that $AM = CM$ and $BM = DM$.
Suppose also that $AC \perp BD$.



(i) What previous theorem proves that the quadrilateral $ABCD$ is a parallelogram?
(ii) Prove that $\triangle AMD \equiv \triangle AMB$.
(iii) Hence prove that $AD = AB$. The quadrilateral $ABCD$ is then a rhombus by definition.

(b) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other vertex angle, then the quadrilateral is a rhombus.*

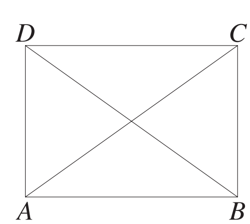
Let $ABCD$ be a quadrilateral in which the diagonals bisect each vertex angle.
Let α, β, γ and δ be as shown.



(i) Prove that $\alpha + \beta + \gamma + \delta = 180°$.
(ii) By taking the sum of the angles in $\triangle ABC$ and $\triangle ADC$, prove that $\beta = \delta$.
(iii) Similarly, prove that $\alpha = \gamma$, and state why $ABCD$ is a parallelogram.
(iv) Finally, prove that $AB = AD$.

9. COURSE THEOREM: *The diagonals of a rectangle are equal and bisect each other.*

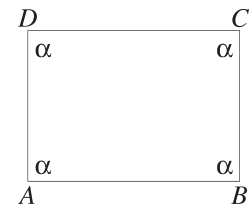
Let $ABCD$ be a rectangle. Join the diagonals AC and BD.



(a) Use the properties of a parallelogram to show that the diagonals bisect each other.
(b) Prove that $\triangle ABC \equiv \triangle BAD$.
(c) Hence prove that $AC = BD$.

10. TESTS FOR A RECTANGLE: The following two parts are structured proofs of the two standard tests for a rectangle given in the notes above.

(a) COURSE THEOREM: *If all angles of a quadrilateral are equal, then the quadrilateral is a rectangle.*

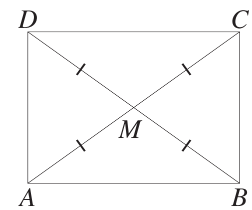


Let $ABCD$ be a quadrilateral with all angles equal.
Let $\angle A = \angle B = \angle C = \angle D = \alpha$.

(i) Prove that all angles are right angles.

(ii) Hence prove that $ABCD$ is a rectangle.

(b) COURSE THEOREM: *If the diagonals of a quadrilateral are equal and bisect each other, then it is a rectangle.*



Let the diagonals of a quadrilateral $ABCD$ meet at M.
Suppose that $AM = BM = CM = DM$.

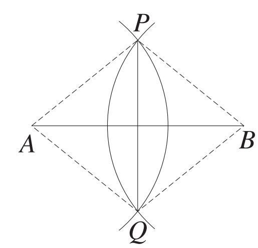
(i) Explain why $ABCD$ is a parallelogram.

(ii) Let $\alpha = \angle BAM$, and explain why $\angle ABM = \alpha$.

(iii) Let $\beta = \angle MBC$, and explain why $\angle MCB = \beta$.

(iv) Using the angle sum of $\triangle ABC$, prove that $\angle ABC = 90^\circ$.

11. CONSTRUCTION: *Perpendicular bisector of an interval.*

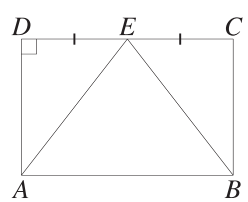


Let AB be an interval. Let arcs of equal radius, with centres at A and B, meet at P and Q.

(a) Why is the quadrilateral $APBQ$ a rhombus?

(b) Hence prove that PQ bisects AB and $PQ \perp AB$.

12. (a)

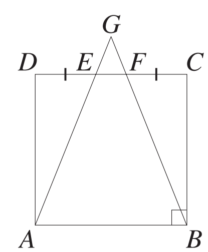


Let $ABCD$ be a rectangle.
Let E be the midpoint of the side CD.
Join the intervals EA and EB.

(i) Prove that $\triangle BCE \equiv \triangle ADE$.

(ii) Hence show that $\triangle ABE$ is isosceles.

(b)

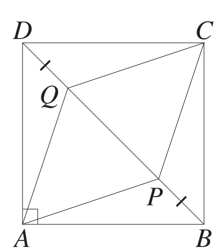


The points E and F lie on the side CD of the square $ABCD$, with $CF = DE$. Produce AE and BF to meet at G.

(i) Prove that $\triangle BCF \equiv \triangle ADE$.

(ii) Hence show that $\triangle ABG$ is isosceles.

13. (a)

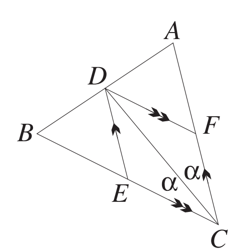


The points P and Q lie on the diagonal BD of the square $ABCD$, and $BP = DQ$.

(i) Prove that
$\triangle ABP \equiv \triangle CBP \equiv \triangle ADQ \equiv \triangle CDQ$.

(ii) Hence show that $APCQ$ is a rhombus.

(b)



In the triangle ABC, DC bisects $\angle ACB$, $DE \parallel AC$ and $DF \parallel BC$.

(i) Explain why the quadrilateral $DECF$ is a parallelogram.

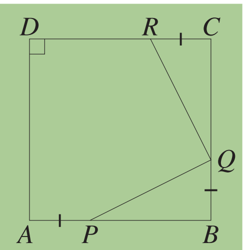
(ii) Show that $DECF$ is a rhombus.

CHALLENGE

14. Let $ABCD$ be a square.
Let P lie on AB, Q on BC, and R on CD.
Suppose that $AP = BQ = CR$.

(a) Prove that $\triangle PBQ \equiv \triangle QCR$.

(b) Prove that $\angle PQR$ is a right angle.



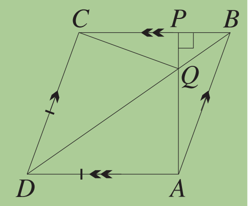
15. In the rhombus $ABCD$, AP is constructed perpendicular to BC and intersects the diagonal BD at Q.

(a) State why $\angle ADB = \angle CDB$.

(b) Prove that $\triangle AQD \equiv \triangle CQD$.

(c) Show that $\angle DAQ$ is a right angle.

(d) Hence find $\angle QCD$.



16. The triangles ABC and APR are both right-angled at the vertices marked in the diagram.
Let Q be the midpoint of PR.
Suppose that $PQ = QR = AB$.

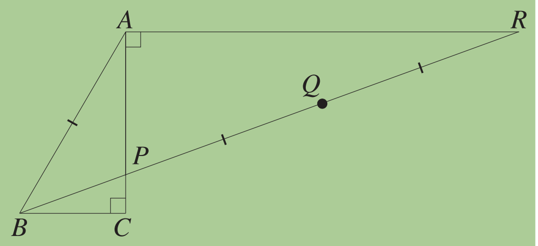
(a) Explain why $\angle PBC = \angle PRA$.

(b) Construct the point S that completes the rectangle $APSR$. Explain why:

(i) Q is also the midpoint of AS,

(ii) $PQ = AQ$.

(c) Hence prove that $\angle PBA = 2 \times \angle PBC$.



17. TWO THEOREMS ABOUT KITES: A *kite* is defined to be a quadrilateral in which two pairs of adjacent sides are equal. [NOTE: This definition is not part of the course.]

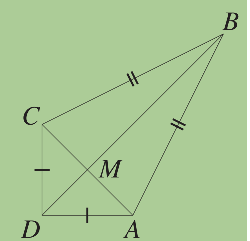
(a) THEOREM: *The diagonals of a kite are perpendicular and one bisects the other.*

Let $ABCD$ be a kite with $AB = BC$ and $AD = DC$.
Let the diagonals AC and BD intersect at M.

(i) Prove that $\triangle BAD \equiv \triangle BCD$.

(ii) Hence prove that $\triangle BMC \equiv \triangle BMA$.

(iii) Hence prove that DB bisects AC at right angles.



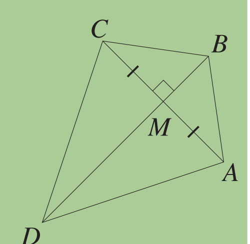
(b) THEOREM: *If the diagonals of a quadrilateral are perpendicular and one is bisected by the other, then the quadrilateral is a kite.*

Let the diagonals of a quadrilateral $ABCD$ meet at M.
Suppose that $AC \perp BD$ and $AM = MC$.

(i) Prove that $\triangle BAM \equiv \triangle BCM$.

(ii) Hence prove that $BA = BC$.

(iii) Similarly, prove that $DA = DC$.



7 F Areas of Plane Figures

The standard area formulae are well known. Some of them were used in the development of the definite integral, which extended the idea of area to regions with curved boundaries. The formulae below apply to figures with straight edges, and their proofs by dissection are reviewed below.

Course Theorem — Area Formulae for Quadrilaterals and Triangles: The various area formulae are based on the definition of the area of a rectangle as length times breadth. The first two formulae below simply restate this definition.

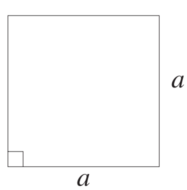
The last four formulae below are proven by dissection. Their proofs are given in the diagrams below, which need to be studied until the logic of each dissection becomes clear.

COURSE THEOREM — STANDARD AREA FORMULAE:

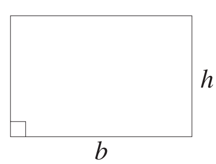
- SQUARE: Area $=$ (side length)^2
- RECTANGLE: Area $=$ (length) $\times$ (breadth)

30
- PARALLELOGRAM: Area $=$ (base) $\times$ (perpendicular height)
- TRIANGLE: Area $= \frac{1}{2} \times$ (base) $\times$ (perpendicular height)
- RHOMBUS: Area $= \frac{1}{2} \times$ (product of the diagonals)
- TRAPEZIUM: Area $=$ (average of parallel sides) $\times$ (perpendicular height)

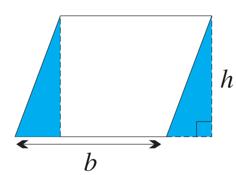
PROOF:



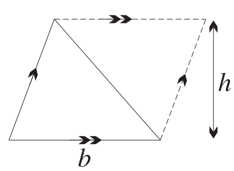
Square: area $= a^2$



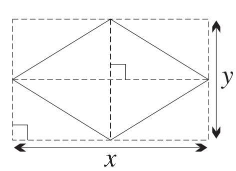
Rectangle: area $= bh$



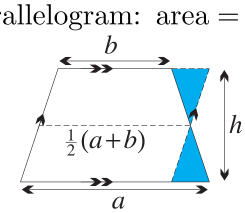
Parallelogram: area $= bh$



Triangle: area $= \frac{1}{2}bh$



Rhombus: area $= \frac{1}{2}xy$



Trapezium: area $= \frac{1}{2}h(a+b)$

Rhombuses and Parallelograms: Because rhombuses are parallelograms, their areas can also be calculated using the formula

$$\text{area} = (\text{base}) \times (\text{perpendicular height}).$$

Rhombuses and Squares: Squares are rhombuses, so their area can also be calculated from their diagonals. Since the diagonals of a square are equal,

$$\text{area of square} = \tfrac{1}{2} \times (\text{square of the diagonal}).$$

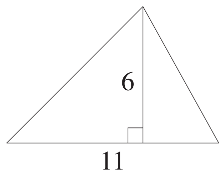
Trigonometry and the Area of a Triangle: Trigonometry yielded a further area formula which should be reviewed here. In any triangle ABC,

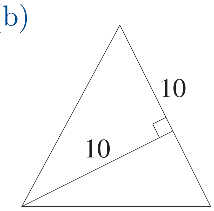
$$\text{area of triangle} = \tfrac{1}{2}ab\sin C.$$

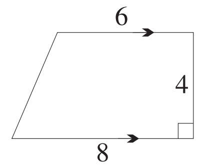
Exercise 7F

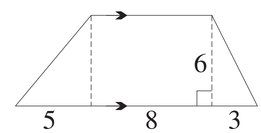
NOTE: The calculation of areas is so linked with Pythagoras' theorem that it is inconvenient to separate them in exercises. Pythagoras' theorem has therefore been used freely in the questions of this exercise, although its formal review is in the next section.

1. Find the areas of the following figures:

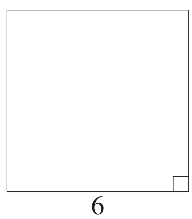
(a)


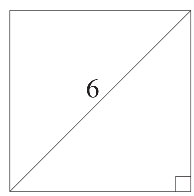
(b)


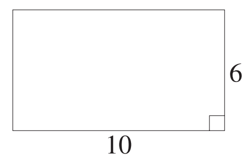
(c)


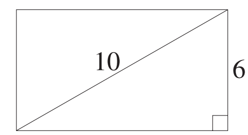
(d)


2. Find the area A and the perimeter P of the squares in parts (a) and (b) and the rectangles in parts (c) and (d). Use Pythagoras' theorem to find missing lengths where necessary.

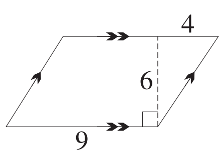
(a)


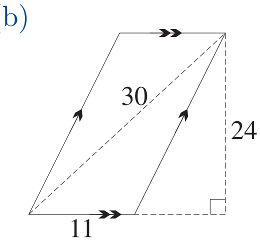
(b)


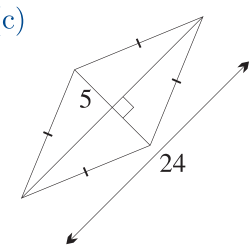
(c)


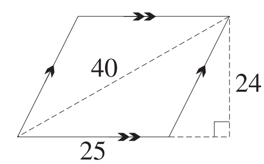
(d)


3. Find the area A and the perimeter P of the following figures, using Pythagoras' theorem where necessary. Then find the lengths of any missing diagonals. [HINT: The second diagonal in part (d) is most easily obtained from the area of the rhombus.]

(a)


(b)


(c)


(d)


4. Find the areas of the following figures — you will need the six formulae boxed in the notes, plus the trigonometric formula for the area of a triangle:

(a) a parallelogram with base 8 metres and perpendicular height 3 metres,

(b) a square of side length $4\frac{1}{2}$ metres,

(c) a square whose diagonal is 5 cm,

(d) a rhombus whose diagonals are 12 cm and 7 cm,

(e) a trapezium whose parallel sides are 12 cm apart and have lengths 10 cm and 18 cm,

(f) a triangle with base 12 kilometres and perpendicular height 1 kilometre,

(g) a rhombus whose sides are 10 cm, in which the distance between parallel sides is 6 cm,

(h) a triangle with an angle of 30° and including sides of length 6 cm and 10 cm,

(i) a parallelogram with an angle of 60° and including sides of length 6 cm and 4 cm.

5. (a) Find the height of a rectangle with area $60\,\text{cm}^2$ and base 8 cm.

(b) Find the lengths of the side and diagonal of a square with area $32\,\text{km}^2$.

(c) Find the distance between the parallel sides of a trapezium with area 60 cm whose parallel sides are 15 cm and 9 cm.

(d) Find the perpendicular height of a triangle with area $50\,\text{cm}^2$ and base $40\,\text{cm}$.

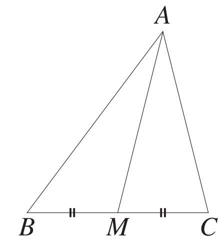
(e) Find the other diagonal of a rhombus with area $36\,\text{m}^2$ and diagonal 12 metres.

6. (a) Explain why the area of a square is half the square of the diagonal.

(b) Show that the area of a rectangle with side lengths a and b is the same as the area of a square with side length $\sqrt{ab}$.

DEVELOPMENT

7. THEOREM: *A median of a triangle divides the triangle into two triangles of equal area.*

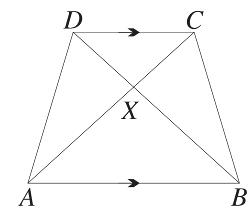


Let ABC be a triangle.
Let M be the midpoint of BC and join the median AM.

(a) Explain why $\triangle ABM$ and $\triangle ACM$ have the same perpendicular height.

(b) Hence explain why these triangles have the same area.

8. THEOREM: *The two triangles formed by the diagonals and the non-parallel sides of a trapezium have the same area.*

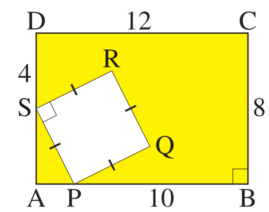


Let $ABCD$ be a trapezium with $AB \parallel DC$.
Let the diagonals AC meet BD at X.

(a) Explain why area $\triangle ABC$ = area $\triangle ABD$.

(b) Hence explain why area $\triangle BCX$ = area $\triangle ADX$.

9. The diagram shows a rectangle with a square offset in one corner. All dimensions shown are in metres.



(a) Use Pythagoras' theorem to find the length SP, and hence find the area of the square.

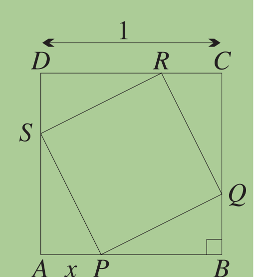
(b) Hence find the shaded area outside the square and inside the rectangle.

CHALLENGE

10. (a) Prove that the four small triangles formed by the two diagonals of a parallelogram all have the same area.

(b) Under what circumstances are they all congruent?

11. MINIMISATION OF AREAS:
In the diagram to the right, $ABCD$ and $PQRS$ are squares and $AB = 1$ metre. Let $AP = x$.



(a) Use Pythagoras' theorem and the fact that $SA = 1 - x$ to show that the area A of the square $PQRS$ is given by $A = 2x^2 - 2x + 1$.

(b) What is the minimum area of $PQRS$, and what value of x gives this minimum?

(c) Explain why the result is the same if the total area of the four triangles is maximised.

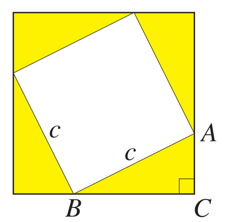
7 G Pythagoras' Theorem and its Converse

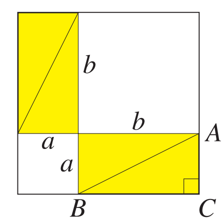
Pythagoras' theorem hardly needs introduction, having been the basis of so much of the course. But its proof needs attention, and the converse theorem and its interesting proof by congruence will be new for many students.

Pythagoras' Theorem: The following proof by dissection of Pythagoras' theorem is very quick and is one of hundreds of known proofs.

31 COURSE THEOREM — PYTHAGORAS' THEOREM:

- In a right-angled triangle, the square on the hypotenuse equals the sum of the squares on the other two sides.





GIVEN:
Let $\triangle ABC$ be a right-angled triangle with $\angle C = 90^\circ$.

AIM:
To prove that $AC^2 + BC^2 = AB^2$.

CONSTRUCTION:
As shown.

PROOF:
Behold! (To quote an Indian text — is anything further required?)

Pythagorean Triads: A *Pythagorean triad* consists of three positive integers a, b and c such that $a^2 + b^2 = c^2$. For example,

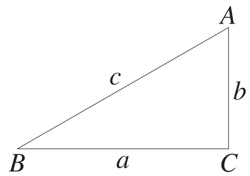
$$3^2 + 4^2 = 5^2 \qquad \text{and} \qquad 5^2 + 12^2 = 13^2,$$

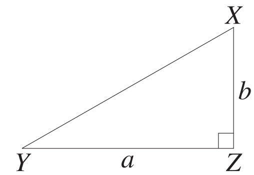
so 3, 4, 5 and 5, 12, 13 are Pythagorean triads. Such triads are very convenient, because they can be the side lengths of a right-angled triangle. Question 10 below shows two methods of constructing families of Pythagorean triads.

Converse of Pythagoras' Theorem: The converse of Pythagoras' theorem is also true, Its proof is an application of congruence. The proof uses the forward theorem and is consequently rather subtle.

32 COURSE THEOREM — CONVERSE OF PYTHAGORAS' THEOREM:

- If the sum of the squares on two sides of a triangle equals the square on the third side, then the angle included by the two sides is a right angle.





GIVEN:
Let ABC be a triangle whose sides satisfy the relation $a^2 + b^2 = c^2$.

AIM:
To prove that $\angle C = 90^\circ$.

CONSTRUCTION:
Construct $\triangle XYZ$ in which $\angle Z = 90^\circ$, $YZ = a$ and $XZ = b$.

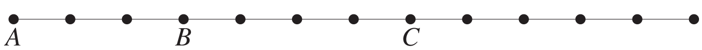
PROOF:
Using Pythagoras' theorem in $\triangle XYZ$,

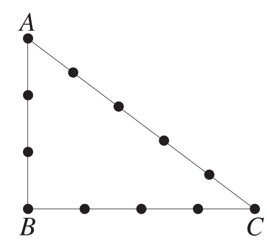
$$\begin{aligned} XY^2 &= a^2 + b^2 \quad \text{(because } XY \text{ is the hypotenuse)} \\ &= c^2 \quad \text{(given)}, \end{aligned}$$

so $\quad XY = c$.

Hence the triangles ABC and XYZ are congruent by the SSS test,
and so $\quad \angle C = \angle Z = 90^\circ \quad$ (matching angles of congruent triangles).

WORKED EXERCISE:
A long rope is divided into twelve equal sections by knots along its length. Explain how it can be used to construct a right angle.





SOLUTION:
Let A be one end of the rope. Let B be the point 3 units along, and let C be the point a further 4 units along.
Join the two ends of the rope and stretch the rope into a triangle with vertices A, B and C.

Then since 3, 4, 5 is a Pythagorean triad, the triangle will be right-angled at B.

Exercise 7G

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Which of the following triplets are the sides of a right-angled triangle?
(a) 15, 20, 25 (b) 15, 24, 28 (c) 10, 24, 26 (d) 7, 24, 25 (e) 13, 20, 24

2. Find the unknown side of each of the following right-angled triangles with base b, altitude a and hypotenuse c. Leave your answer in surd form where necessary.
(a) $a = 12$ and $b = 5$
(b) $a = 4$ and $b = 5$
(c) $b = 15$ and $c = 20$
(d) $a = 3$ and $c = 7$

3. Two vertical poles of heights 5 metres and 8 metres are 4 metres apart.
(a) What is the distance between the tops of the two poles?
(b) What are the distances between the top of each pole and the base of the other?

4. A paddock on level ground is 4 km long and 3 km wide. Answer these questions, correct to the nearest second.
(a) If a farmer walks from one corner to the opposite corner along the fences in 84 minutes, how long will it take him if he walks across the diagonal?
(b) If his assistant jogs along the diagonal in 40 minutes, how long will it take him if he jogs along the fences?

5. (a) The diagonals of a rhombus are 16 cm and 30 cm. Use Pythagoras' theorem, and the fact that the diagonals of a rhombus bisect each other at right angles, to find the lengths of the sides.

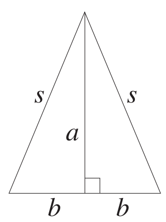
(b) One diagonal of a rhombus is 20 cm and its sides have length 12 cm. How long is the other diagonal?

(c) One diagonal of a rhombus is 20 cm and its area is 100 cm^2.

(i) How long is the other diagonal? (ii) How long are its sides?

DEVELOPMENT

6. (a) Use Pythagoras' theorem to find an equation for the altitude a of an isosceles triangle with base $2b$ and equal legs s.



(b) Hence find the area of an isosceles triangle with:

(i) equal legs 15 cm and base 24 cm,

(ii) equal legs 18 cm and base 20 cm.

(c) Write down the altitude in the special case where $s = 2b$. What type of triangle is this and what is its area?

7. The sides of a rhombus are 5 cm and its area is 24 cm^2.

(a) Let the diagonals have lengths $2x$ and $2y$, and show that $xy = 12$ and $x^2 + y^2 = 25$.

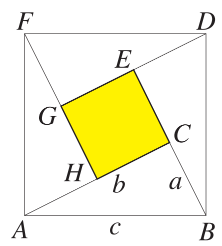
(b) Solve for x (by inspection) and hence find the lengths of the diagonals.

8. PYTHAGORAS' THEOREM AND THE COSINE RULE:
Let $\triangle ABC$ be a triangle right-angled at C. Then the cosine rule, with c^2 as subject, is

$$c^2 = a^2 + b^2 - 2ab\cos C$$

But Pythagoras' theorem says that $c^2 = a^2 + b^2$. Why are the two formulae identical?

9. COURSE THEOREM: *An alternative proof of Pythagoras' theorem.*
Let $\triangle ABC$ be a triangle right-angled at C.
Let the sides be $AB = c$, $BC = a$ and $CA = b$, with $b > a$.
Let $\triangle BDE$, $\triangle DFG$ and $\triangle FAH$ be three copies of $\triangle ABC$, arranged as in the diagram to the right.



(a) Explain why $HC = b - a$.

(b) Find, in terms of the sides a, b and c, the areas of:

(i) the square $ABDF$, (ii) the square $CEGH$,

(iii) the four triangles.

(c) Hence show that $a^2 + b^2 = c^2$.

10. (a) Show that if a and b are integers with $b < a$, then $a^2 - b^2$, $2ab$, $a^2 + b^2$ is a Pythagorean triad. Then generate and check the Pythagorean triads given by:

(i) $a = 2$, $b = 1$ (ii) $a = 3$, $b = 2$ (iii) $a = 4$, $b = 3$ (iv) $a = 7$, $b = 4$

(b) Two sides of a right-angled triangle are $2t$ and $t^2 - 1$.

(i) Show that the hypotenuse is $t^2 + 1$. (ii) What are the two possible lengths of the hypotenuse if another side of the triangle is 8 cm?

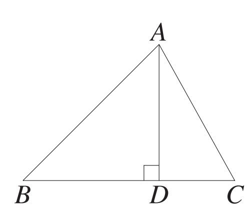
11. Let AD be the altitude to the side BC of a triangle ABC.

(a) Using Pythagoras' theorem in $\triangle ADB$ and $\triangle ADC$:

(i) show that $AD^2 = AB^2 - BD^2$,

(ii) show that $AD^2 = AC^2 - CD^2$.

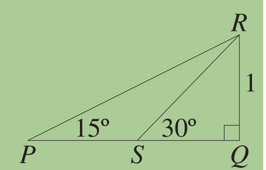
(b) Hence show that $AB^2 + DC^2 = AC^2 + BD^2$.



CHALLENGE

12. SEQUENCES AND GEOMETRY:
The sides of a right-angled triangle add to 36 and are in arithmetic progression. Find the three sides.
[HINT: Let the three sides be $12 - d$, 12 and $12 + d$, then apply Pythagoras' theorem.]

13. In the diagram, $\triangle PQR$ and $\triangle QRS$ are right-angled at Q, with $\angle RPQ = 15^\circ$ and $\angle RSQ = 30^\circ$.

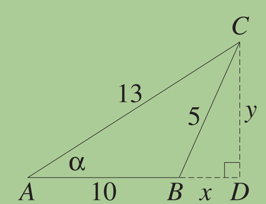


(a) Find $\angle PRS$ and hence show that $PS = RS$.

(b) Given that $QR = 1$ unit, show that $QS = \sqrt{3}$ and that $RS = 2$.

(c) Hence deduce that $\tan 15^\circ = 2 - \sqrt{3}$.

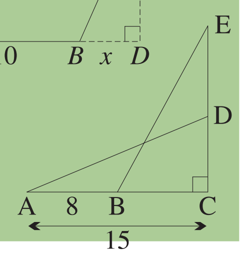
14. In triangle ABC, $AB = 10$, $BC = 5$ and $AC = 13$.
The altitude is $CD = y$.
Let $BD = x$ and $\angle A = \alpha$.



(a) Using Pythagoras' theorem in the triangles BDC and ADC, write down a pair of equations for x and y.

(b) Solve for x, and hence find $\cos\alpha$ without finding α.

15. In the diagram to the right, $AD = BE = 25$.
Also, $\angle C$ is a right angle, $AB = 8$ and $AC = 15$.
Find the length of DE.

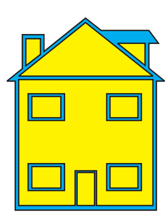


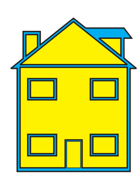
7 H Similarity

Similarity generalises the study of congruence to figures that have the same shape but not necessarily the same size. Its formal definition requires the idea of an *enlargement*, which is a stretching in all directions by the same factor.

33

DEFINITION OF SIMILARITY:
- Two figures $\mathcal{S}$ and $\mathcal{T}$ are called *similar*, written as $\mathcal{S} \parallel\!| \mathcal{T}$, if one figure can be moved to coincide with the other figure by means of a sequence of rotations, reflections, translations and enlargements.
- The enlargement ratio involved in these transformations is called the *similarity ratio* of the two figures.





Like congruence, similarity sets up a correspondence between the elements of the two figures. In this correspondence, angles are preserved, and the ratio of any two matching lengths equals the similarity ratio.

Since an area is the product of two lengths, the ratio of the areas of matching regions is the square of the similarity ratio. Likewise, if the idea is extended into three-dimensional space, then the ratio of the volumes of matching solids is the cube of the similarity ratio.

34 **COURSE THEOREM — SIMILARITY RATIO:**
If two similar figures have similarity ratio $1 : k$, then
- matching angles have the same size,
- matching intervals have lengths in the ratio $1 : k$,
- matching regions have areas in the ratio $1 : k^2$,
- matching solids have volumes in the ratio $1 : k^3$.

Similar Triangles: As with congruence, most similarity arguments concern triangles. The four standard tests for similarity of triangles will be assumptions.

35 **ASSUMPTIONS — THE STANDARD SIMILARITY TESTS FOR TRIANGLES:**
Two triangles are similar if:

SSS the three sides of one triangle are respectively proportional to the three sides of another triangle, or

SAS two sides of one triangle are respectively proportional to two sides of another triangle, and the included angles are equal, or

AA two angles of one triangle are respectively equal to two angles of another triangle, or

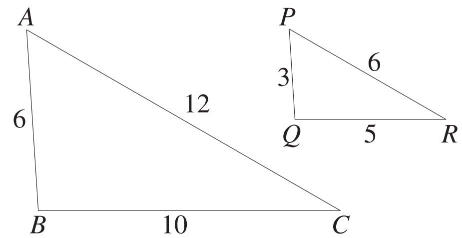
RHS the hypotenuse and one side of a right-angled triangle are respectively proportional to the hypotenuse and one side of another right-angled triangle.

These four tests correspond exactly to the four standard congruence tests, except that equal sides are replaced by proportional sides (and so the AAS congruence test corresponds to the AA similarity test).

The SSS Similarity Test:
In the diagram to the right,

$\triangle ABC \,|||\, \triangle PQR$ (SSS similarity test),

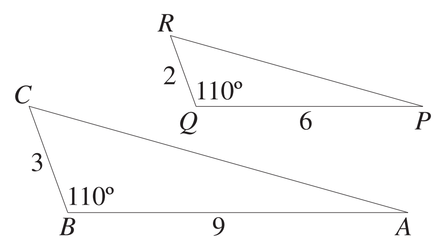
with similarity ratio $2 : 1$.
Hence $\angle P = \angle A$, $\angle Q = \angle B$ and $\angle R = \angle C$ (matching angles of similar triangles).



The SAS Similarity Test:
In the diagram to the right,

$\triangle ABC \,|||\, \triangle PQR$ (SAS similarity test),

with similarity ratio $3 : 2$.
Hence $\angle P = \angle A$, $\angle R = \angle C$ and $PR = \frac{2}{3}AC$ (matching sides and angles of similar triangles).

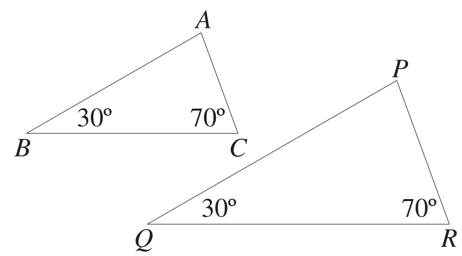


The AA Similarity Test:
In the diagram to the right,

$\triangle ABC \,|||\, \triangle PQR$ (AA similarity test).

Hence $\dfrac{PQ}{AB} = \dfrac{QR}{BC} = \dfrac{RP}{CA}$

(matching sides of similar triangles),
and $\angle P = \angle A$ (angle sums of triangles).



The RHS Similarity Test:

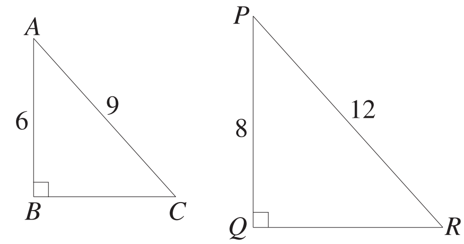
In the diagram to the right,

$\triangle ABC \ ||| \ \triangle PQR$ (RHS similarity test),

with similarity ratio 3 : 4.

Hence $\angle P = \angle A$, $\angle R = \angle C$

and $QR = \frac{4}{3}BC$ (matching sides and angles of similar triangles).



Using the Similarity Tests: Similarity tests should be set out in exactly the same way as congruence tests — the AA similarity test, however, will need only four lines. The similarity ratio should be mentioned if it is known. Keeping vertices in corresponding order is even more important with similarity, because the corresponding order is needed when writing down the proportionality of sides.

WORKED EXERCISE:

A tower TC casts a 300-metre shadow CN, and a man RA 2 metres tall casts a 2·4-metre shadow AY.

(a) Show that $\triangle TCN \ ||| \ \triangle RAY$.

(b) Hence find the height of the tower.

SOLUTION:

(a) In the triangles TCN and RAY:

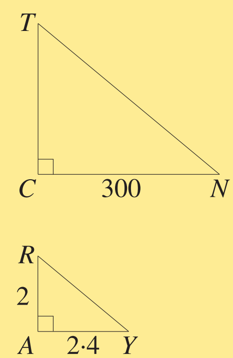
1. $\angle TCN = \angle RAY = 90^\circ$ (given),
2. $\angle CNT = \angle AYR =$ angle of elevation of the sun,

so $\triangle TCN \ ||| \ \triangle RAY$ (AA similarity test).

(b) Hence $\dfrac{TC}{CN} = \dfrac{RA}{AY}$ (matching sides of similar triangles)

$\dfrac{TC}{300} = \dfrac{2}{2{\cdot}4}$

$TC = 250$ metres.



WORKED EXERCISE:

Prove that the interval PQ joining the midpoints of two adjacent sides AB and BC of a parallelogram $ABCD$ is parallel to the diagonal AC. Prove also that PQ cuts off a triangle of area one-eighth the area of the parallelogram.

SOLUTION:

In the triangles BPQ and BAC:

1. $\angle PBQ = \angle ABC$ (common),
2. $PB = \frac{1}{2}AB$ (given),
3. $QB = \frac{1}{2}CB$ (given),

so $\triangle BPQ \ ||| \ \triangle BAC$ (SAS similarity test)

and the similarity ratio is 1 : 2.

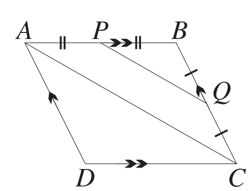
Hence $\angle BPQ = \angle BAC$ (matching angles of similar triangles),

so $PQ \parallel AC$ (corresponding angles are equal).

Also, area $\triangle BPQ = \frac{1}{4} \times$ area $\triangle BAC$ (matching regions),

and area $\triangle ABC =$ area $\triangle CDA$ (congruent triangles),

so area $\triangle BPQ = \frac{1}{8} \times$ area of parallelogram $ABCD$.



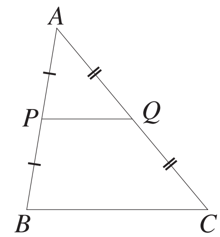
Midpoints of Sides of Triangles: Similarity can be applied to configurations involving the midpoints of sides of triangles. The following theorem and its converse are standard results and will be generalised in Section 7I.

36 COURSE THEOREM — INTERCEPTS:

- The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

GIVEN:
Let P and Q be the midpoints of the sides AB and AC of $\triangle ABC$.

AIM:
To prove that $PQ \parallel BC$ and $PQ = \frac{1}{2}BC$.



PROOF:
In the triangles APQ and ABC:

1. $AP = \frac{1}{2}AB$ (given),
2. $AQ = \frac{1}{2}AC$ (given),
3. $\angle A = \angle A$ (common),

so $\triangle APQ \;|||\; \triangle ABC$ (SAS similarity test)
and the similarity ratio is $1 : 2$.

Hence $\angle APQ = \angle ABC$ (matching angles of similar triangles),
so $PQ \parallel BC$ (corresponding angles are equal).
Also, $PQ = \frac{1}{2}BC$ (matching sides of similar triangles).

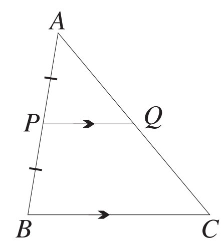
The Converse Theorem: The following converse theorem is standard, and very useful.

37 COURSE THEOREM — INTERCEPTS (THE CONVERSE):

- Conversely, the line through the midpoint of one side of a triangle and parallel to another side bisects the third side.

GIVEN:
Let P be the midpoint of the side AB of $\triangle ABC$.
Let the line though P parallel to BC meet AC at Q.

AIM:
To prove that $AQ = \frac{1}{2}AC$.



PROOF:
In the triangles APQ and ABC:

1. $\angle PAQ = \angle BAC$ (common),
2. $\angle APQ = \angle ABC$ (corresponding angles, $PQ \parallel BC$),

so $\triangle APQ \;|||\; \triangle ABC$ (AA similarity test)
and the similarity ratio is $AP : AB = 1 : 2$.
Hence $AQ = \frac{1}{2}AC$ (matching sides of similar triangles).

Equal Ratios of Intervals and Equal Products of Intervals: The fact that the ratios of two pairs of intervals are equal can be just as well expressed by saying that the products of two pairs of intervals are equal:

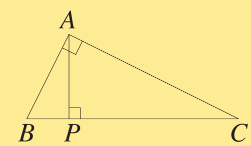
$$\frac{AB}{BC} = \frac{XY}{YZ} \qquad \text{is the same as} \qquad AB \times YZ = BC \times XY.$$

The following worked exercise is one of the best-known examples of this.

WORKED EXERCISE:
Prove that the square on the altitude to the hypotenuse of a right-angled triangle equals the product of the intercepts on the hypotenuse cut off by the altitude.

SOLUTION:

GIVEN:
Let $\triangle ABC$ be a triangle right-angled at A.
Let AP be the altitude to the hypotenuse BC.



AIM:
To prove that $AP^2 = BP \times CP$.

PROOF:

A. Let $\angle B = \beta$.
Then $\angle BAP = 90^\circ - \beta$ (angle sum of $\triangle BAP$),
so $\angle CAP = \beta$ (adjacent angles in the right angle $\angle BAC$).

B. In the triangles PAB and PCA:
1. $\angle APB = \angle CPA = 90^\circ$ (given),
2. $\angle ABP = \angle CAP$ (proven above),

so $\triangle PAB \,|||\, \triangle PCA$ (AA similarity test).

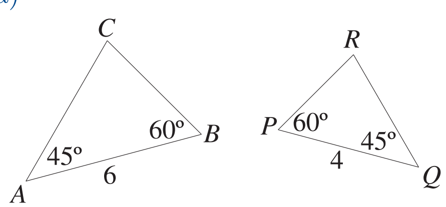
C. Hence $\dfrac{BP}{AP} = \dfrac{AP}{CP}$ (matching sides of similar triangles),
and multiplying out the fractions,
$BP \times CP = AP^2$, as required.

Exercise 7H

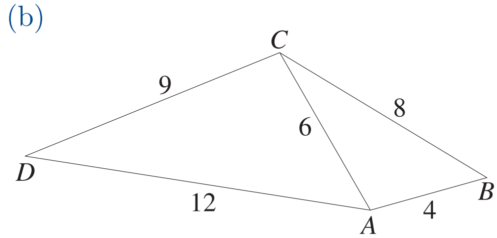
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Both triangles in each pair are similar. Name the similar triangles in the correct order and state which test is used.

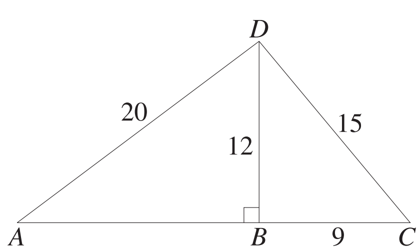
(a)



(b)



(c)



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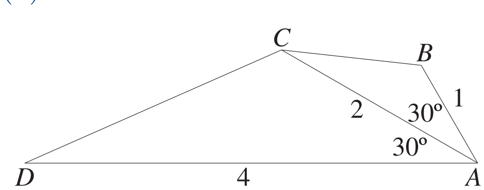
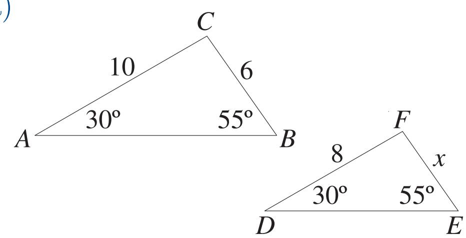
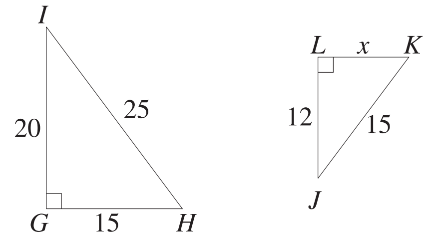


2. Identify the similar triangles, giving a reason, and hence deduce the length of the side x.

(a)



(b)



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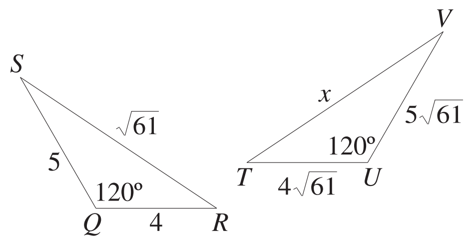
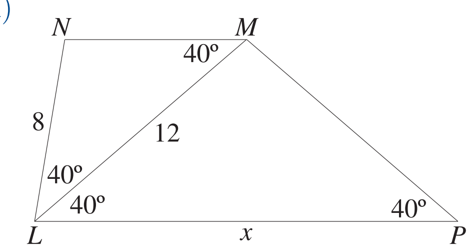
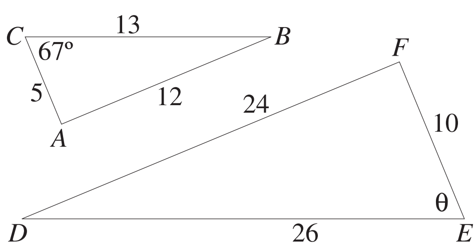


(d)

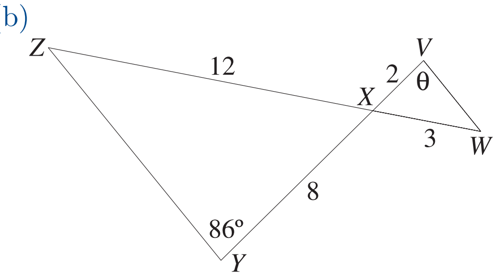


3. Identify the similar triangles, giving a reason, and hence deduce the size of the angle θ. In part (b), prove that $VW \parallel ZY$.

(a)



(b)



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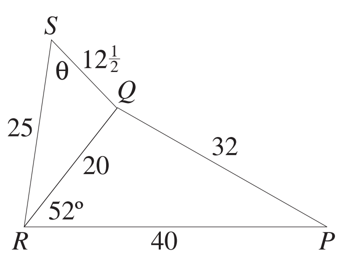

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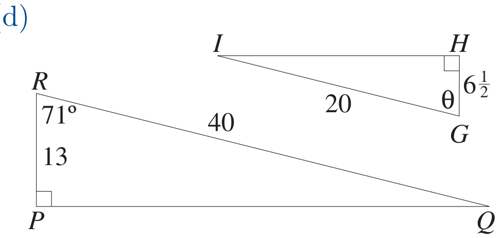
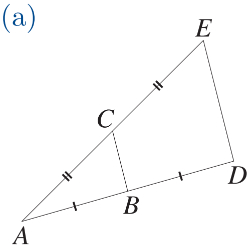
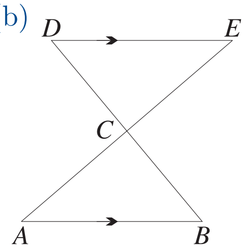


4. Prove that the triangles in each pair are similar. The four questions involve all four similarity tests.

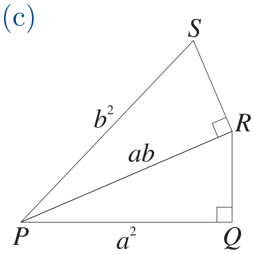
(a)



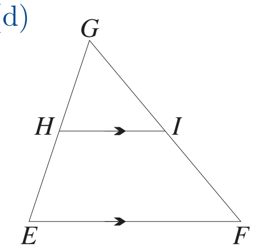
(b)



(c)



(d)



5. (a) A building casts a shadow 24 metres long, while a man 1·6 metres tall casts a 0·6-metre shadow. Draw a diagram and use similarity to find the height of the building.

(b) A map has a scale of 1 : 100 000.

(i) How many kilometres does 1 cm on the map represent?

(ii) How long is a lake that measures 15 cm on the map?

6. For these questions, you need to know that if two figures have similarity ratio $1 : k$, then lengths are in the ratio $1 : k$, areas in the ratio $1 : k^2$ and volumes in the ratio $1 : k^3$.

(a) An architect builds a model of a house to a scale of $1 : 200$. The house is to have a swimming pool of length 10 metres, surface area 60 m^2 and volume 120 m^3. What will the length, area and volume of the architect's model pool be?

(b) An adult is three times taller than a toddler, but the same shape. Find the ratio of:

(i) their arm lengths, (iii) their shoe lengths, (v) the areas of their hands,
(ii) their skin areas, (iv) their masses, (vi) the masses of their t-shirts.

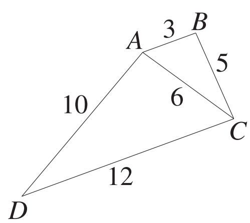
(c) One square has twice the area of another square. What is their similarity ratio?

(d) One cube has twice the volume of another cube. What is their similarity ratio?

(e) Two coins of the same shape and material but different in size weigh 5 grams and 40 grams. If the larger coin has diameter 2 cm, what is the diameter of the smaller coin?

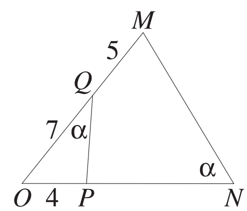
DEVELOPMENT

7. (a)



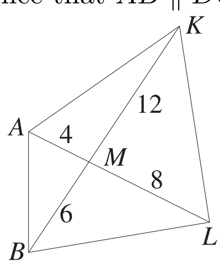
Show that $\triangle ADC \;|||\; \triangle BCA$, and hence that $AB \parallel DC$.

(b)



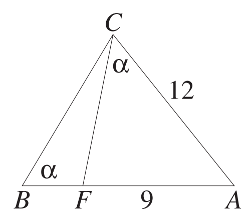
Show that $\triangle OPQ \;|||\; \triangle OMN$, and hence find ON and PN.

(c)



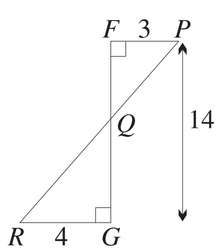
Show that $\triangle AMB \;|||\; \triangle LMK$. What type of quadrilateral is $ABLK$?

(d)



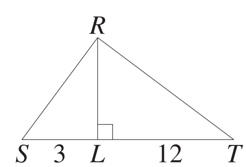
Show that $\triangle ABC \;|||\; \triangle ACF$, and hence find AB and FB.

(e)



Show that $\triangle FPQ \;|||\; \triangle GRQ$, and hence find FQ, GQ, PQ and RQ.

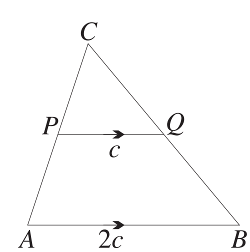
(f)



Show that $\triangle LSR \;|||\; \triangle LRT$, and hence find RL.

8. THEOREM: *The interval parallel to one side of a triangle and half its length bisects the other two sides.*

Let ABC be a triangle, with P on AC and Q on BC. Suppose that $PQ \parallel AB$ and $PQ = \frac{1}{2}AB$.



(a) Prove that $\triangle ABC \;|||\; \triangle PQC$.

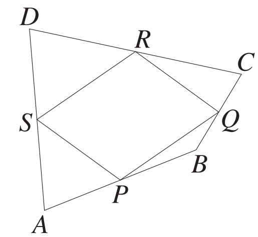
(b) Hence show that $CP = \frac{1}{2}CA$ and $CQ = \frac{1}{2}CB$.

9. THEOREM: *The quadrilateral formed by the midpoints of the sides of a quadrilateral is a parallelogram.*

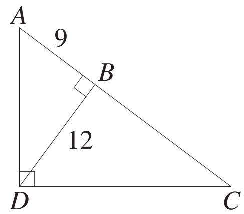
Let $ABCD$ be a quadrilateral. Let P, Q, R and S be the midpoints of the sides AB, BC, CD and DA respectively.

(a) Prove that $\triangle PBQ \mathbin{|||} \triangle ABC$, and hence that $PQ \parallel AC$.

(b) Similarly, prove that $PQ \parallel SR$ and $PS \parallel QR$.

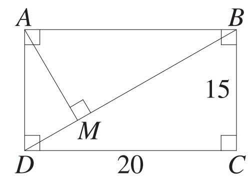


10. (a)



Show that $\triangle ABD \mathbin{|||} \triangle ADC$, and hence find AD, DC and BC.

(b)

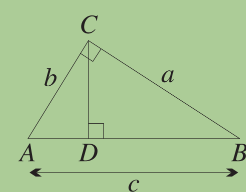


Use Pythagoras' theorem and similarity to find AM, BM and DM.

CHALLENGE

11. AN ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM:

Let $\triangle ABC$ be a triangle right-angled at C.
Let $BC = a$, $CA = b$ and $AB = c$.
Construct the altitude CD from C to the hypotenuse AB.



(a) Prove that $\triangle CBD \mathbin{|||} \triangle ABC$.

(b) Hence show that $BD = \dfrac{a^2}{c}$.

(c) Similarly, prove that $\triangle ACD \mathbin{|||} \triangle ABC$.

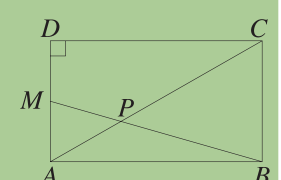
(d) Hence show that $AD = \dfrac{b^2}{c}$.

(e) Use the fact that $c = BD + AD$ to prove that $a^2 + b^2 = c^2$.

12. Explain why the members of the following pairs of figures are, or are not, similar:

(a) two squares,
(b) two rectangles,
(c) two rhombuses,
(d) two parallelograms,
(e) two trapeziums,
(f) two equilateral triangles,
(g) two isosceles triangles,
(h) two circles,
(i) two regular hexagons,
(j) two hexagons.

13. Let $ABCD$ be a rectangle with $AB = 2 \times AD$.
Let M be the midpoint of AD, and let $AM = MD = x$.
Let BM meet the diagonal AC at P.



(a) Show that $\triangle APM \mathbin{|||} \triangle CPB$.

(b) Show that $CP = \frac{2}{3} \times CA$.

(c) Use Pythagoras' theorem to find AC^2 in terms of x.

(d) Hence show that $CP^2 = \frac{80}{9}$.

14. THEOREM: Prove that the intervals joining the midpoints of the sides of any triangle dissect the triangle into four congruent triangles, each similar to the original triangle. [HINT: Draw your own diagram and use the last theorem on intercepts given in the notes.]

7 I Intercepts on Transversals

This section, unlike the previous sections of Chapter Seven, will be entirely new for most students.

The last two theorems of Section 7H concerned the midpoints of the sides of a triangle. They will be generalised in two ways in this section.

- First, the midpoint of a side can be replaced by a point dividing the side in any given ratio.
- Secondly, the situation can be generalised to the intercepts cut off on a transversal to three parallel lines.

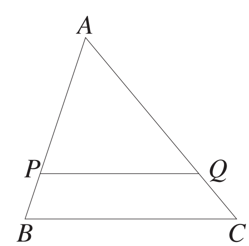
Intercepts: The word *intercepts* used above needs clarification. It means simply the two pieces that an interval is broken into by a point on it.

38 INTERCEPTS:
- A point P on an interval AB divides the interval into two *intercepts* AP and PB.

Points on the Sides of Triangles: The following theorem generalises the earlier theorem about the interval midpoints of the sides of triangles — the point on the side of a triangle can now divide the side in any ratio. The proof is similar to the proofs of the previous two theorems and is developed in the following exercise.

39 COURSE THEOREM — INTERCEPTS:
- If two points P and Q divide two sides AB and AC respectively of a triangle in the same ratio $k : \ell$, then the interval PQ is parallel to the third side BC, and $PQ : BC = k : k + \ell$.

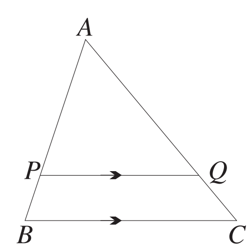
The diagram to the right illustrates the theorem.
Let $\triangle ABC$ be a triangle, with P on AB and Q on AC.
Suppose that $AP : PB = AQ : QC = k : \ell$.
Then it follows that $PQ \parallel BC$
and that $PQ : BC = k : k + \ell$ (ratios of intercepts are equal).



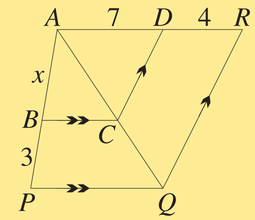
The Converse Theorem: The converse of this theorem also holds. Again, the proof is presented in a structured question in the following exercise.

40 COURSE THEOREM — INTERCEPTS (THE CONVERSE):
- A line parallel to one side of a triangle divides the other two sides in the same ratio.

The diagram to the right illustrates the converse theorem.
Let $\triangle ABC$ be a triangle, with P on AB and Q on AC.
Suppose that $PQ \parallel BC$.
Then $AP : PB = AQ : QC$ (intercepts cut off by parallel lines).



WORKED EXERCISE:
Find x in the diagram opposite.



SOLUTION:
Using intercepts in $\triangle APQ$,

$$\frac{x}{3} = \frac{AC}{CQ} \quad \text{(intercepts cut off by parallel lines)}.$$

Using intercepts in $\triangle AQR$,

$$\frac{AC}{CQ} = \frac{7}{4} \quad \text{(intercepts cut off by parallel lines)}.$$

Hence $\quad \dfrac{x}{3} = \dfrac{7}{4}$

$\boxed{\times 3} \quad x = \dfrac{21}{4}$

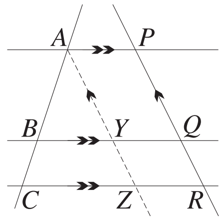
$\quad = 5\frac{1}{4}$.

Transversals to Three Parallel Lines: The situation in the previous theorem in Box 40 can be generalised to the intercepts cut off when two transversals cross three parallel lines.

41 COURSE THEOREM — INTERCEPTS (ON THREE PARALLEL LINES):
- If two transversals cross three parallel lines, then the ratio of the intercepts on one transversal equals the ratio of the intercepts on the other transversal.
- In particular, if three parallel lines cut off equal intercepts on one transversal, then they cut off equal intercepts on all transversals.

The second part follows from the first part with $k : \ell = 1 : 1$, so it will be sufficient to prove only the first part.

GIVEN:
Let two transversals ABC and PQR cross three parallel lines.
Let $AB : BC = k : \ell$.



AIM:
To prove that $PQ : QR = k : \ell$.

CONSTRUCTION:
Construct the line through A parallel to the line PQR.
Let it meet the other two parallel lines at Y and Z respectively.

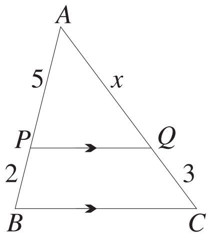
PROOF:
The configuration in $\triangle ACZ$ is the converse theorem stated in Box 40,
and so $AY : YZ = k : \ell$ (intercepts cut off by parallel lines).
But the opposite sides of the parallelograms $APQY$ and $YQRZ$ are equal,
so $\quad AY = PQ$ and $YZ = QR$ (opposites sides of parallelograms).
Hence $PQ : QR = k : \ell$.

Exercise 7I

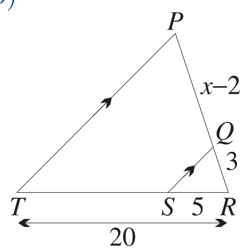
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find the values of x, y and z in the following diagrams, giving reasons.

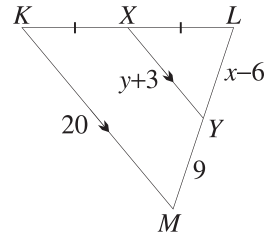
(a)



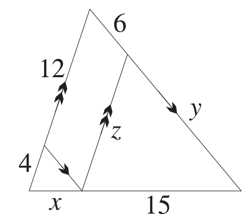
(b)



(c)

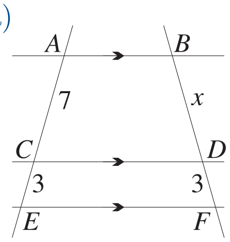


(d)



2. Find the value of x in each diagram below, giving reasons.

(a)



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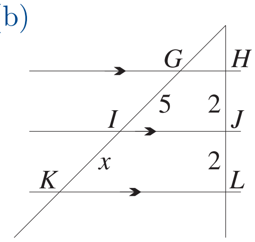
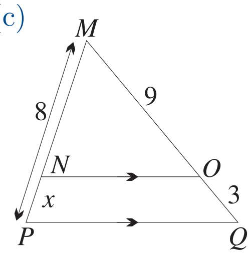
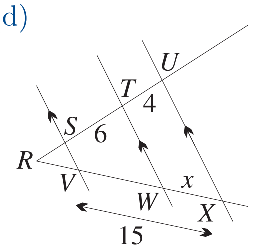


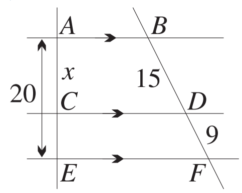
(c)



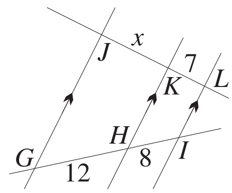
(d)



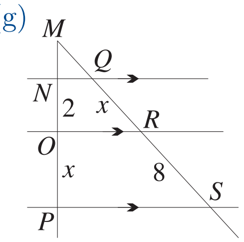
(e)



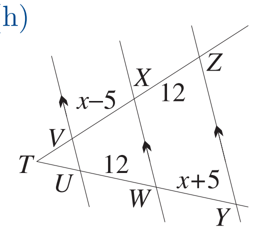
(f)



(g)

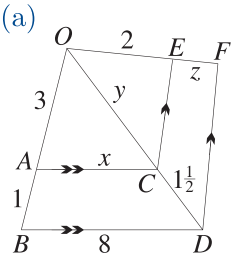


(h)



3. Find the values of x, y and z in the diagrams below, giving reasons.

(a)



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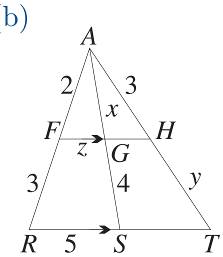

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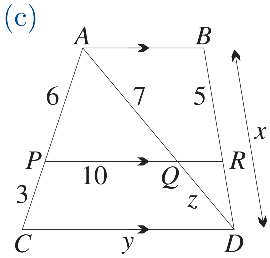

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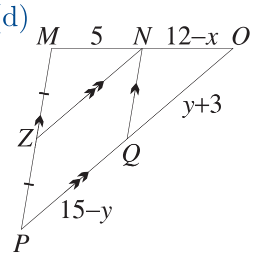
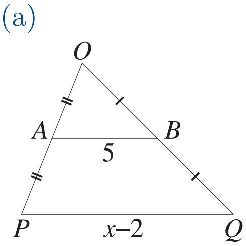
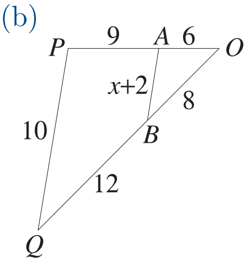


4. Give a reason why $AB \parallel PQ \parallel XY$ as appropriate, then find x, y and z.

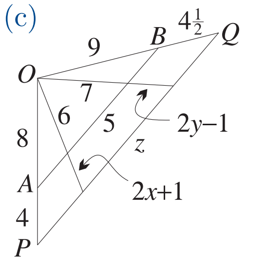
(a)



(b)



(c)



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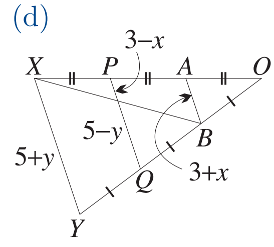
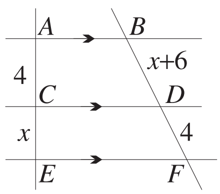


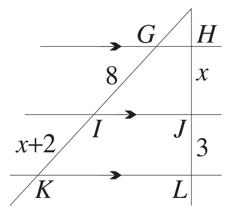
DEVELOPMENT

5. For each diagram, write down a quadratic equation for x, giving reasons. Then solve the equation to find the value of x in each case.

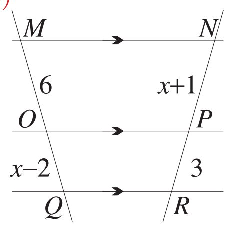
(a)



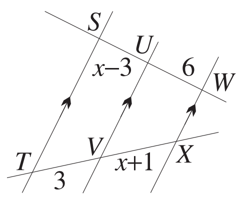
(b)



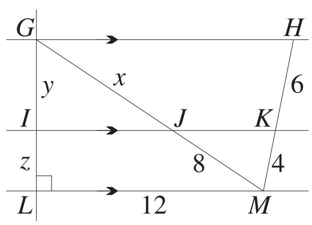
(c)



(d)

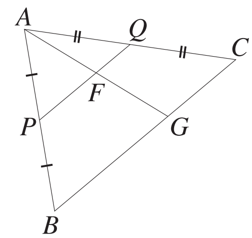


6. (a)



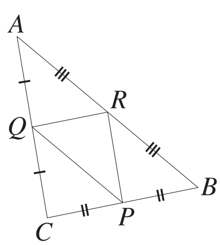
Use Pythagoras' theorem to find x, y and z.

(b)



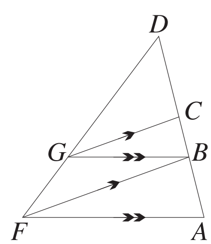
Find $AF : AG$.

(c)



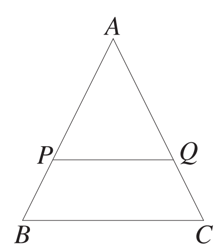
State, with reasons, what sort of quadrilateral $ARPQ$ is. Then find the ratio of the areas of $ARPQ$ and $\triangle ABC$.

(d)



In the diagram above, show, with reasons, that $FG : GD = BC : CD$ and that $AF : BG = BF : CG$.

7. COURSE THEOREM: *If two points P and Q divide two sides AB and AC respectively of a triangle in the same ratio $k : \ell$, then the interval PQ is parallel to the third side BC and $PQ : BC = k : k + \ell$.*

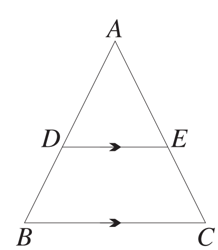


Let ABC be a triangle.
Let P and Q be points on AB and AC respectively.
Suppose that $AP : PB = AQ : QB = k : \ell$.

(a) Prove that $\triangle ABC \mathbin{|||} \triangle APQ$.

(b) Hence prove that $PQ \parallel BC$ and $PQ : BC = k : k + \ell$.

8. COURSE THEOREM: *Conversely, a line parallel to one side of a triangle divides the other two sides in the same ratio.*



Let ABC be a triangle.
Let D and E be points on AB and AC respectively.
Suppose that $DE \parallel BC$.

(a) Prove that $\triangle ABC \mathbin{|||} \triangle ADE$.

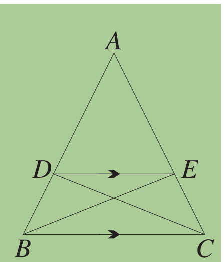
(b) Show that $AD : DB = AE : EC$.

CHALLENGE

9. In the diagram to the right, the triangle ABC is isosceles, with $AB = AC$, and DE is parallel to BC.

(a) Use the intercepts theorem to prove that $DB = EC$.

(b) Show that $\triangle BCD \equiv \triangle CBE$.



10. Two vertical poles AB and PQ (A and$B$ are the tops) have heights 10 metres and 15 metres respectively, and stand 8 metres apart. Wires stretching from the top of each pole to the foot of the other meet at M.

(a) Draw a diagram of the situation and show that the two opposite triangles $\triangle AMB$ and $\triangle QMP$ formed by the poles and the wires are similar, with similarity ratio 2 : 3.

(b) Use horizontal intercepts in $\triangle AQB$ to find how high above the ground the wires cross.

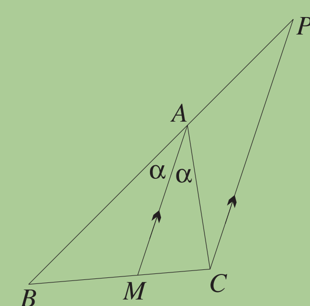
(c) How would this height change if the poles were 11 metres apart?

11. THEOREM: *The bisector of an angle of a triangle divides the opposite side in the ratio of the including sides.*

Let the bisector of $\angle BAC$ in $\triangle ABC$ meet BC at M.
Let $\angle BAM = \angle CAM = \alpha$.
Construct the line through C parallel to MA, meeting BA produced at P.

(a) Prove that $\triangle APC$ is isosceles, with $AP = AC$.

(b) Hence show that $\dfrac{BM}{MC} = \dfrac{BA}{AC}$.

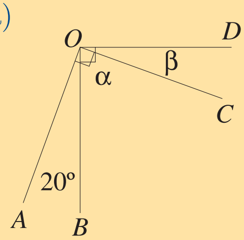


7J Chapter Review Exercise

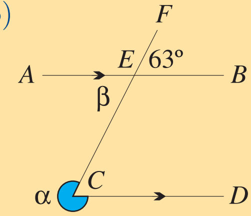
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

1. Find the angles α and β in the diagrams below, giving reasons.

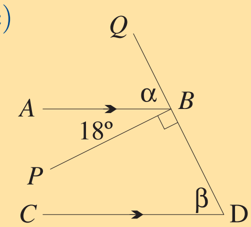
(a)



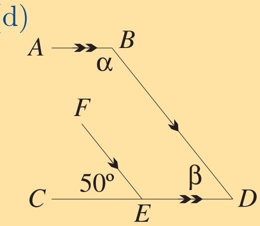
(b)



(c)

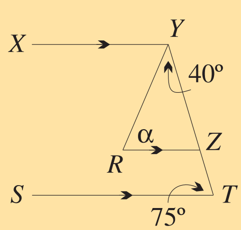


(d)

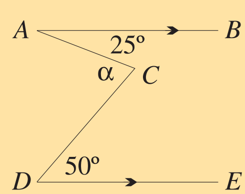


2. Find the angle α in each diagram below, giving reasons.

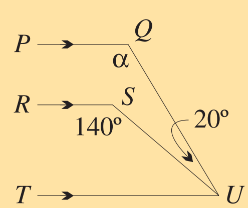
(a)



(b)

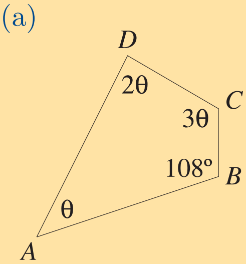


(c)

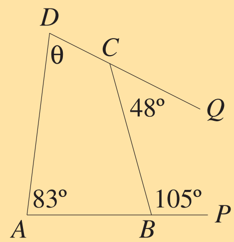


3. Find the angles marked θ and ϕ in the diagrams below, giving all reasons.

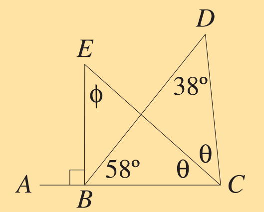
(a)



(b)



(c)



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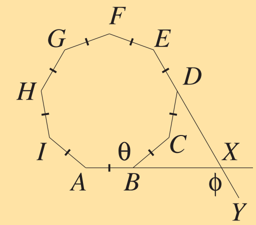
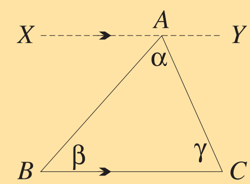


4. COURSE THEOREM: *The angle sum of a triangle is 180°.*

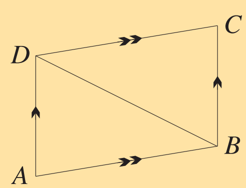
Let ABC be a triangle.
Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$.
Construct the line XAY through the vertex A parallel to BC.

Prove that $\alpha + \beta + \gamma = 180°$.

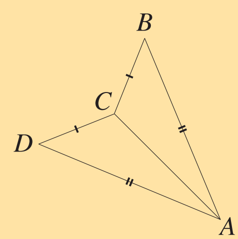


5. In each diagram, state the congruence of the two triangles, giving the congruence test.

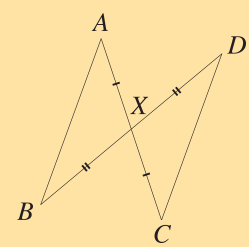
(a)



(b)



(c)

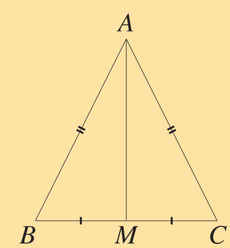


6. COURSE THEOREM: *The base angles of an isosceles triangle are equal.*

Let $\triangle ABC$ be isosceles with $AB = AC$.
Let M be the midpoint of BC.
Join the median AM.

(a) Prove that $\triangle AMB \equiv \triangle AMC$.

(b) Hence prove that $\angle B = \angle C$.

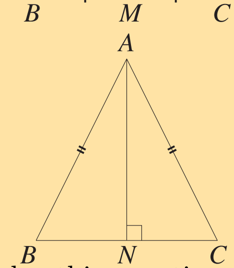


7. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$.
Let AN be the altitude from the apex A to the base BC.

(a) Prove that $\triangle ABN \equiv \triangle ACN$.

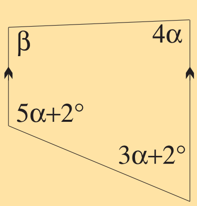
(b) Hence show that $BN = CN$ and $\angle BAN = \angle CAN$.

NOTE: You have proven that the altitude to the base of an isosceles triangle bisects the base and the apex angle.

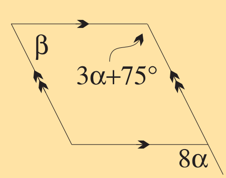


8. Write down an equation for α in each diagram below, giving reasons. Solve this equation to find the angles α and β.

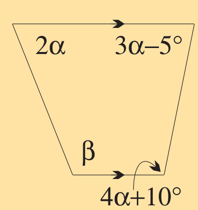
(a)



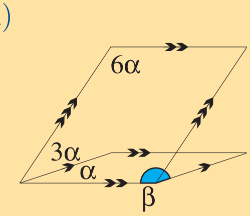
(b)



(c)



(d)

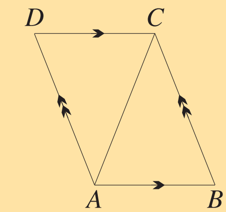


9. COURSE THEOREM: *The opposite sides of a parallelogram are equal.*

Let $ABCD$ be a parallelogram with diagonal AC.

(a) Prove that $\triangle ACB \equiv \triangle CAD$.

(b) Hence show that $AB = DC$ and $BC = AD$.

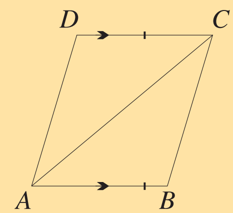


10. COURSE THEOREM: *If one pair of opposite sides of a quadrilateral are equal and parallel, then it is a parallelogram.*

Let $ABCD$ be a quadrilateral with $AB = DC$ and $AB \parallel DC$. Construct the diagonal AC.

(a) Prove that $\triangle ACB \equiv \triangle CAD$.

(b) Hence show that $AD \parallel BC$.



11. COURSE THEOREM: *The diagonals of a rhombus bisect each other at right angles and bisect the vertex angles.*

Let the diagonals of the rhombus $ABCD$ meet at M.
Let $\angle ADM = \alpha$ and $\angle DAM = \beta$.

Since a rhombus is a parallelogram, we already know that the diagonals bisect each other.

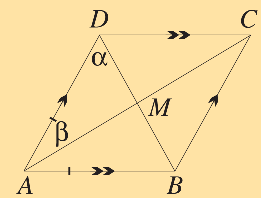
(a) Explain why $\angle ABM = \alpha$.

(b) Hence prove that $\angle CDM = \alpha$.

(c) Prove that $\angle DCM = \beta$.

(d) Use the angle sum of $\triangle DAC$ to prove that $\alpha + \beta = 90°$.

(e) Hence prove that $AC \perp BD$.

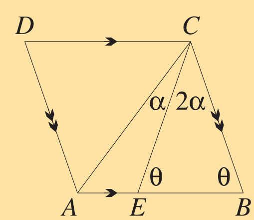


12. Let $ABCD$ be a rhombus and let E be a point on AB.
Suppose that $\angle CBE = \angle CEB = \theta$,
and that $\angle ACE = \alpha$ and $\angle BCE = 2\alpha$.

(a) Use the isosceles $\triangle BAC$ to show that $\angle CAB = 3\alpha$.

(b) Use an exterior angle of $\triangle ACE$ to show that $\theta = 4\alpha$.

(c) Use the angle sum of $\triangle CEB$ to find α and θ.



13. COURSE THEOREM: *If the diagonals of a quadrilateral are equal and bisect each other, then it is a rectangle.*

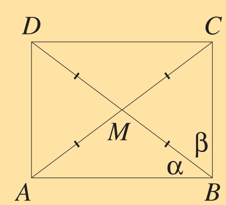
Let $ABCD$ be a quadrilateral with diagonals meeting at M.
Suppose that $AM = BM = CM = DM$.
Let $\alpha = \angle ABM$ and $\beta = \angle CBM$.

(a) Give a reason why $ABCD$ is a parallelogram.

(b) Explain why $\angle BAM = \alpha$.

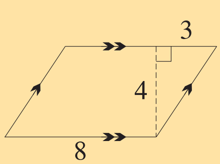
(c) Explain why $\angle MCB = \beta$.

(d) Using the angle sum of $\triangle ABC$, show that $\angle ABC = 90°$.

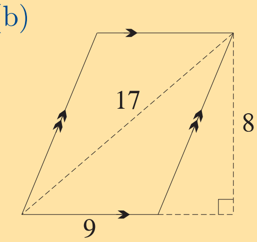


14. Find the area and perimeter of each figure, using Pythagoras' theorem where necessary.

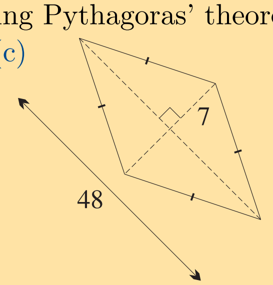
(a)



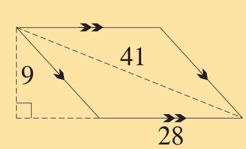
(b)



(c)



(d)



15. (a) Find the height of a rectangle with area $75\,\text{cm}^2$ and base $15\,\text{cm}$.

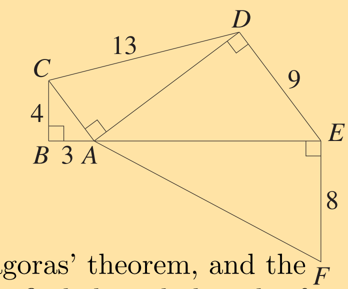
(b) Find the lengths of the side and diagonal of a square with area $2\frac{1}{4}\,\text{km}^2$.

(c) Find the distance between the parallel sides of a trapezium whose area is $84\,\text{cm}^2$ and whose parallel sides are $8\,\text{cm}$ and $16\,\text{cm}$.

(d) Find the perpendicular height of a triangle with area $45\,\text{cm}^2$ and base $12\,\text{cm}$.

(e) Find the other diagonal of a rhombus with area $20\,\text{m}^2$ and one diagonal 5 metres.

16. In the diagram to the right, all four triangles are right-angled as indicated.



(a) Apply Pythagoras' theorem successively in each of the triangles, starting with $\triangle ABC$, to find the length of AF.

(b) Show that $\triangle ABC \,|||\, \triangle EDA$.

(c) Hence show that $\angle BAE = 180°$.

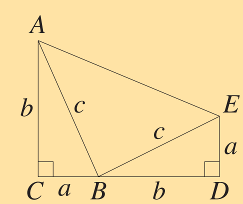
17. (a) The diagonals of a rhombus are $10\,\text{cm}$ and $24\,\text{cm}$. Use Pythagoras' theorem, and the fact that the diagonals of a rhombus meet at right angles, to find the side length of the rhombus.

(b) One diagonal of a rhombus is $12\,\text{cm}$ and its sides have length $8\,\text{cm}$. How long is the other diagonal?

(c) One diagonal of a rhombus is $14\,\text{cm}$ and its area is $336\,\text{cm}^2$.

(i) How long is the other diagonal? (ii) How long are its sides?

18. COURSE THEOREM: *An alternative proof of Pythagoras' theorem (attributed to President Garfield of the US).*



Let $\triangle ABC$ be a triangle right-angled at C.
Let $AB = c$, $BC = a$ and $CA = b$.
Construct another triangle $\triangle BDE$ congruent to $\triangle ABC$, so that CBD is a straight line, as shown in the diagram.

(a) Explain why $\angle ABE$ is a right angle.

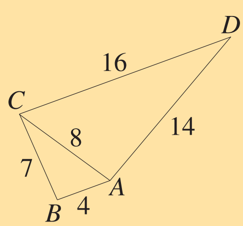
(b) Find the areas of each of the three triangles $\triangle ABC$, $\triangle ABE$ and $\triangle BDE$. By adding these areas, show that the figure $ACDE$ has area $ab + \frac{1}{2}c^2$.

(c) Explain why figure $ACDE$ is a trapezium.

(d) Use the formula for the area of a trapezium to show that $ACDE$ has area $\frac{1}{2}(a + b)^2$.

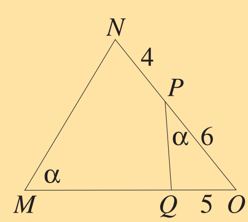
(e) By equating the answers to parts (b) and (d), show that $a^2 + b^2 = c^2$.

19. (a)



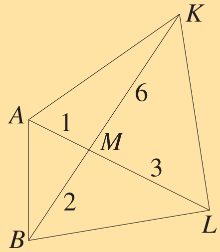
Show that $\triangle ADC \,|||\, \triangle BCA$.
Hence show that $AB \parallel DC$.

(b)



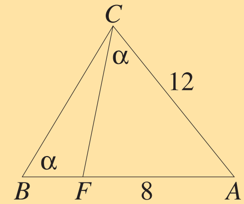
Show that $\triangle OPQ \,|||\, \triangle OMN$.
Hence find OM and QM.

(c)



Show that $\triangle AMB \;|||\; \triangle LMK$.
What type of quadrilateral is $ABLK$?

(d)



Show that $\triangle ABC \;|||\; \triangle ACF$.
Hence find AB and FB.

20. COURSE THEOREM: *The line through the midpoint of one side of a triangle and parallel to another side bisects the third side.*

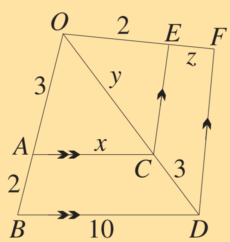
Let P be the midpoint of the side AB of $\triangle ABC$.
Let the line through P parallel to BC meet AC at Q.

(a) Prove that $\triangle APQ \;|||\; \triangle ABC$.

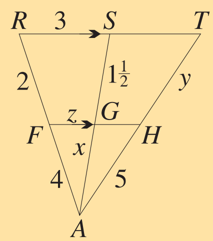
(b) Hence show that $AQ = \frac{1}{2}AC$.

21. Find x, y and z in the diagrams below, giving reasons.

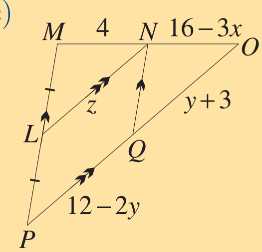
(a)



(b)

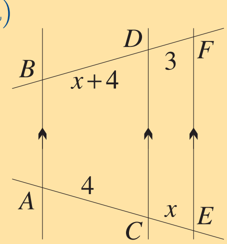


(c)

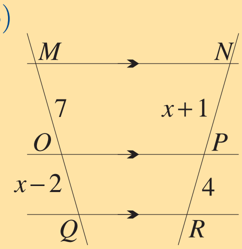


22. For each diagram, write down a quadratic equation for x, giving reasons. Then solve the equation to find the value or values of x in each case.

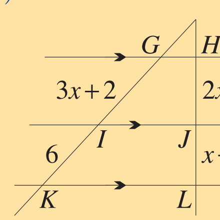
(a)



(b)



(c)



μηδεὶς ἀγεωμέτρητος εἰσίτω
'Let no-one enter who does not know geometry.'
(Inscribed over the doorway to Plato's Academy in Athens.)

CHAPTER **EIGHT**

Probability

Probability arises when one performs an experiment that has various possible outcomes, but for which there is insufficient information to predict precisely which of these outcomes will occur. The classic examples of this are tossing a coin, throwing a die and drawing a card from a pack. Probability, however, is involved in almost every experiment done in science and is fundamental to understanding statistics. This chapter will review some of the basic ideas of probability and develop a more systematic approach to solving probability questions.

8 A Probability and Sample Spaces

The first task is to develop a workable formula for probability that can serve as the foundation for the topic. This will be done by dividing the results of an experiment into equally likely possible outcomes.

Equally Likely Possible Outcomes: The idea of equally likely possible outcomes is well illustrated by the experiment of throwing a die. (A *die*, plural *dice*, is a cube with its corners and edges rounded so that it rolls easily, and with the numbers 1–6 printed on its six sides.) The outcome of this experiment is the number on the top face when the die stops rolling, giving six *possible outcomes* — 1, 2, 3, 4, 5, 6. This is a complete list of the possible outcomes, because each time the die is rolled, one and only one of these outcomes can occur.

Provided that the die is completely symmetric, that is, it is not *biased* in any way, there is no reason to expect that any one outcome is more likely to occur than any of the other five, and these six possible outcomes are called *equally likely possible outcomes*.

With the results of the experiment thus divided into six equally likely possible outcomes, the probability $\frac{1}{6}$ is assigned to each of these six outcomes. Notice that these six probabilities are equal and they all add up to 1.

The general case is as follows.

1 EQUALLY LIKELY POSSIBLE OUTCOMES: Suppose that the possible results of an experiment can be divided into n *equally likely possible outcomes* — meaning that one and only one of these n outcomes will occur, and that there is no reason to expect one outcome to be more likely than another.

Then the probability $\dfrac{1}{n}$ is assigned to each of these equally likely possible outcomes.

.

Randomness: Notice that it has been assumed that the terms 'more likely' and 'equally likely' already have a meaning in the mind of the reader. There are many ways of interpreting these words. In the case of a thrown die, one could interpret the phrase 'equally likely' as meaning that the die is perfectly symmetric. Alternatively, one could interpret it as saying that we lack entirely the knowledge to make any statement of preference for one outcome over another.

The word *random* can be used here. In the context of equally likely possible outcomes, saying that a die is thrown 'randomly' means that we are justified in assigning the same probability to each of the six possible outcomes. In a similar way, a card can be drawn 'at random' from a pack, or a queue of people can be formed in a 'random order'.

The Fundamental Formula for Probability: Suppose that a throw of at least 3 on a die is needed to win a game. Then getting at least 3 is called the particular *event* under discussion, the outcomes 3, 4, 5 and 6 are called *favourable* for this event, and the other two possible outcomes 1 and 2 are called *unfavourable*. The probability assigned to getting a score of at least 3 is then

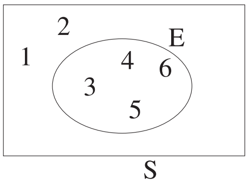
$$\begin{aligned}\mathcal{P}(\text{scoring at least 3}) &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}\\ &= \frac{4}{6}\\ &= \frac{2}{3}\,.\end{aligned}$$

In general, if the results of an experiment can be divided into a number of equally likely possible outcomes, some of which are favourable for a particular event and the others unfavourable, then:

2 THE FUNDAMENTAL FORMULA FOR PROBABILITY:

$$\mathcal{P}(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

The Sample Space and the Event Space: Venn diagrams and the language of sets make some of the theory of probability easier to explain. Section 8C will present a short account of sets and Venn diagrams, but at this stage the diagrams should be self-explanatory.



The Venn diagram to the right shows the six possible outcomes when a die is thrown. The set of all these outcomes is

$$S = \{\,1,\ 2,\ 3,\ 4,\ 5,\ 6\,\}.$$

This set is called the *sample space* and is represented by the outer rectangular box. The event 'scoring at least 3' is the set

$$E = \{\,3,\ 4,\ 5,\ 6\,\},$$

which is called the *event space* and is represented by the ellipse. In general, the set of all equally likely possible outcomes is called the *sample space* and the set of all favourable outcomes is called the *event space*. The basic probability formula can then be restated in set language.

3 THE SAMPLE SPACE AND THE EVENT SPACE:

Suppose that an event E has sample space S. Then

$$\mathcal{P}(E) = \frac{|E|}{|S|},$$

where the symbols $|E|$ and $|S|$ mean the number of members of E and S.

Probabilities Involving Playing Cards: So many questions in probability involve a pack of playing cards that any student of probability needs to be familiar with them — the reader is encouraged to acquire some cards and play some simple games with them. A pack of cards consists of 52 cards organised into four *suits*, each containing 13 cards. The four suits are

two black suits:	♣ clubs,	♠ spades,
two red suits:	♢ diamonds,	♡ hearts.

Each of the four suits contains 13 cards:

A (Ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (Jack), Q (Queen), K (King).

An ace can also be regarded as a 1. It is assumed that when a pack of cards is shuffled, the order is totally *random*, meaning that there is no reason to expect any one ordering of the cards to be more likely to occur than any other.

WORKED EXERCISE:

A card is drawn at random from a standard pack of 52 playing cards. Find the probability that the card is:

(a) the seven of hearts,

(b) any heart,

(c) any seven,

(d) any red card,

(e) any picture card (that is, a Jack, a Queen, or a King),

(f) any green card,

(g) any red or black card.

SOLUTION: In each case, there are 52 equally likely possible outcomes.

(a) Since there is 1 seven of hearts, $\mathcal{P}(7\heartsuit) = \frac{1}{52}$.

(b) Since there are 13 hearts, $\mathcal{P}(\text{heart}) = \frac{13}{52}$
$= \frac{1}{4}$.

(c) Since there are 4 sevens, $\mathcal{P}(\text{seven}) = \frac{4}{52}$
$= \frac{1}{13}$.

(d) Since there are 26 red cards, $\mathcal{P}(\text{red card}) = \frac{26}{52}$
$= \frac{1}{2}$.

(e) Since there are 12 picture cards, $\mathcal{P}(\text{picture card}) = \frac{12}{52}$
$= \frac{3}{13}$.

(f) Since no card is green, $\mathcal{P}(\text{green card}) = \frac{0}{52}$
$= 0$.

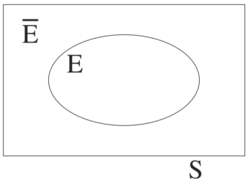
(g) Since all 52 cards are red or black, $\mathcal{P}(\text{red or black card}) = \frac{52}{52}$
$= 1$.

Impossible and Certain Events: Parts (f) and (g) of the previous worked exercise were intended to illustrate the probabilities of events that are impossible or certain. Since getting a green card is impossible, there are no favourable outcomes, so the probability is 0. Since all the cards are either red or black, and getting a red or black card is certain to happen, all possible outcomes are favourable outcomes, so the probability is 1. Notice that for the other five events, the probability lies between 0 and 1.

4

IMPOSSIBLE AND CERTAIN EVENTS:
- An event has probability 0 if and only if it cannot happen.
- An event has probability 1 if and only if it is certain to happen.
- For any other event, $0 < \mathcal{P}(\text{event}) < 1$.

Complementary Events and the Word 'Not': It is often easier to find the probability that an event does <u>not</u> occur than the probability that it does occur. The *complementary event* of an event E is the event 'E does <u>not</u> occur' and is written as $\overline{E}$. The complementary event $\overline{E}$ is represented by the region outside the circle in the Venn diagram to the right.



Since $|\overline{E}| = |S| - |E|$, it follows that $\mathcal{P}(\overline{E}) = 1 - \mathcal{P}(E)$.

5

COMPLEMENTARY EVENTS: Suppose that E is an event with sample space S. Define the *complementary event* $\overline{E}$ to be the event 'E does <u>not</u> occur'. Then

$$\mathcal{P}(\overline{E}) = 1 - \mathcal{P}(E)\,.$$

THE WORD 'NOT':
Consider using complementary events whenever the word 'not' occurs.

In Section 8C, the *complement* $\overline{E}$ of a set E will be defined to be the set of things in S but <u>not</u> in E. The notation $\overline{E}$ for complementary event is quite deliberately the same notation as that for the complement of a set.

WORKED EXERCISE:
A card is drawn at random from a pack of playing cards. Find the probability:
(a) that it is not a spade,
(b) that it does not have a picture on it,
(c) that it is neither a red two nor a black six.

SOLUTION:
(a) Thirteen cards are spades, so $\mathcal{P}(\text{spade}) = \frac{13}{52} = \frac{1}{4}$.

Hence, using complementary events, $\mathcal{P}(\text{not a spade}) = 1 - \frac{1}{4} = \frac{3}{4}$.

(b) Twelve cards have pictures on them, so $\mathcal{P}(\text{picture}) = \frac{12}{52} = \frac{3}{13}$.

Hence, using complementary events, $\mathcal{P}(\text{not a picture}) = 1 - \frac{3}{13} = \frac{10}{13}$.

(c) There are two red 2s and two black 6s, so $\mathcal{P}(\text{red 2 or black 6}) = \frac{4}{52}$
$= \frac{1}{13}$.

Hence, using complementary events, $\mathcal{P}(\text{neither}) = 1 - \frac{1}{13}$
$= \frac{12}{13}$.

Invalid Arguments: Arguments offered in probability theory can be invalid for all sorts of subtle reasons, and it is common for a question to ask for comment on a given argument. It is most important in such a situation that any fallacy in the given argument be explained — it is not sufficient only to offer an alternative argument with a different conclusion.

WORKED EXERCISE:
Comment on the validity of this argument.

> 'Brisbane is one of fourteen teams in the Rugby League, so the probability that Brisbane wins the premiership is $\frac{1}{14}$.'

SOLUTION:
[Identifying the fallacy] The division into fourteen possible outcomes is correct provided that one assumes that a tie for first place is impossible, but no reason has been offered as to why each team is equally likely to win, so the argument is invalid.

[Offering a replacement argument] What can be said with confidence is that if a team is selected at random from the fourteen teams, then the probability that it is the premiership-winning team is $\frac{1}{14}$.

NOTE: It is difficult to give a complete account of this situation. It is not clear that an exact probability can be assigned to the event 'Brisbane wins', although one can safely assume that those with knowledge of the game would have some idea of ranking the fourteen teams in order from most likely to win to least likely to win. If there is an organised system of betting, one may, or may not, agree to take this as an indication of the community's collective wisdom on the fourteen probabilities.

Experimental Probability: When a drawing pin is thrown, there are two possible outcomes, point-up and point-down. But these two outcomes are not equally likely and there seems to be no way to analyse the results of the experiment into equally likely possible outcomes. In the absence of any fancy arguments from physics about rotating pins falling on a smooth surface, however, some estimate of the two probabilities can be gained by performing the experiment a number of times.

The questions in the following worked example could raise difficult issues beyond the scope of this course, but the intention here is only that they be answered briefly in a common-sense manner.

WORKED EXERCISE:
A drawing pin is thrown 400 times and falls point-up 362 times.

(a) What probability does this experiment suggest for the result 'point-up'?

(b) A machine repeats the experiment 1 000 000 times and the pin falls point-up 916 203 times. Does this change your answer to part (a)?

SOLUTION:

(a) These results suggest that $\mathcal{P}(\text{point-up}) \doteqdot 0{\cdot}905$, but with only 400 trials, there would be little confidence in this result past the second, or even the first, decimal place, since different runs of the same experiment would be expected to differ by small numbers. The safest conclusion is that $\mathcal{P}(\text{point-up}) \doteqdot 0{\cdot}9$.

(b) The new results suggest that the estimate of the probability can now be refined to $\mathcal{P}(\text{point-up}) \doteqdot 0{\cdot}916$ — we can be reasonably sure that the rounding to $0{\cdot}9$ in part (a) was too low.

Exercise 8A

1. A coin is tossed. Write down the probability that it shows:
(a) a head, (b) a tail, (c) either a head or a tail, (d) neither a head nor a tail.

2. If a die is rolled, find the probability that the uppermost face is:
(a) a three,
(b) an even number,
(c) a number greater than four,
(d) a multiple of three.

3. A bag contains five red and seven blue marbles. If one marble is drawn from the bag at random, find the probability that it is: (a) red, (b) blue, (c) green.

4. A bag contains eight red balls, seven yellow balls and three green balls. A ball is selected at random. Find the probability that it is: (a) red, (b) yellow or green, (c) not yellow.

5. In a bag there are four red capsicums, three green capsicums, six red apples and five green apples. One item is chosen at random. Find the probability that it is:
(a) green,
(b) red,
(c) an apple,
(d) a capsicum,
(e) a red apple,
(f) a green capsicum.

6. A letter is chosen at random from the word TASMANIA. Find the probability that it is:
(a) an A, (b) a vowel, (c) a letter of the word HOBART.

7. A letter is randomly selected from the 26 letters in the English alphabet. Find the probability that the letter is:
(a) the letter S,
(b) a vowel,
(c) a consonant,
(d) the letter γ,
(e) either C, D or E,
(f) one of the letters of the word MATHS.
[NOTE: The letter Y is normally classified as a consonant.]

8. A student has a 22% chance of being chosen as a prefect. What is the chance that he will not be chosen as a prefect?

9. When breeding labradors, the probability of breeding a black dog is $\frac{3}{7}$.
(a) What is the probability of breeding a dog that is not black?
(b) If you bred 56 dogs, how many would you expect not to be black?

10. A box containing a light bulb has a chance of $\frac{1}{15}$ of holding a defective bulb.
(a) If 120 boxes were checked, how many would you expect to hold defective bulbs?
(b) What is the probability that the box holds a bulb that works?

DEVELOPMENT

11. A number is selected at random from the integers 1, 2, 3, ..., 19, 20. Find the probability of choosing:
(a) the number 4,
(b) a number greater than 15,
(c) an even number,
(d) an odd number,
(e) a prime number,
(f) a square number,
(g) a multiple of 4,
(h) the number e,
(i) a rational number.

12. From a regular pack of 52 cards, one card is drawn at random. Find the probability that:
(a) it is black,
(b) it is red,
(c) it is a king,
(d) it is the jack of hearts,
(e) it is a club,
(f) it is a picture card,
(g) it is a heart or a spade,
(h) it is a red five or a black seven,
(i) it is less than a four.

13. A book has 150 pages. The book is randomly opened at a page. Find the probability that the page number is:
(a) greater than 140,
(b) a multiple of 20,
(c) an odd number,
(d) a number less than 25,
(e) either 72 or 111,
(f) a three-digit number.

14. An integer x, where $1 \le x \le 200$, is chosen at random. Determine the probability that it:
(a) is divisible by 5,
(b) is a multiple of 13,
(c) has two digits,
(d) is a square number,
(e) is greater than 172,
(f) has three equal digits.

15. A bag contains three times as many yellow marbles as blue marbles. If a marble is chosen at random, find the probability that it is: (a) yellow, (b) blue.

16. Fifty tagged fish were released into a dam known to contain fish. Later a sample of 30 fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just prior to the sample of 30 being removed.

CHALLENGE

17. Comment on the following arguments. Identify precisely any fallacies in the arguments, and, if possible, give some indication of how to correct them.
(a) 'On every day of the year it either rains or it doesn't. Therefore the chance that it will rain tomorrow is $\frac{1}{2}$.
(b) 'When the Sydney Swans play Hawthorn, either Hawthorn wins, the Swans win or the game is a draw. Therefore the probability that the next game between these two teams results in a draw is $\frac{1}{3}$.'
(c) 'When answering a multiple-choice test in which there are four possible answers to each question, the chance that Peter answers a question correctly is $\frac{1}{4}$.'
(d) 'A bag contains a number of red, white and black beads. If you choose one bead at random from the bag, the probability that it is black is $\frac{1}{3}$.'
(e) 'Four players play in a knockout tennis tournament resulting in a single winner. A man with no knowledge of the game or the players declares that one particular player will win his semi-final, but lose the final. The probability that he is correct is $\frac{1}{4}$.'

18. A rectangular field is 60 metres long and 30 metres wide. A cow wanders randomly around the field. Find the probability that the cow is:
(a) more than 10 metres from the edge of the field,
(b) not more than 10 metres from a corner of the field.

8 B Sample Space Graphs and Tree Diagrams

Many experiments consist of several *stages*. For example, when a die is thrown twice, the two throws can be regarded as two separate stages of the one experiment. This section develops two approaches — graphing and tree diagrams — to investigating the sample space of a multi-stage experiment.

Graphing the Sample Space: The reason for using the word 'sample space' rather than 'sample set' is that the sample space of a multi-stage experiment takes on some of the characteristics of a space. In particular, the sample space of a two-stage experiment can be displayed on a two-dimensional graph, and the sample space of a three-stage experiment can be displayed in a three-dimensional graph.

The following worked example shows how a two-dimensional dot diagram can be used for calculations with the sample space of a die thrown twice.

WORKED EXERCISE:

A die is thrown twice. Find the probability that:

(a) the pair is a double,

(b) at least one number is four,

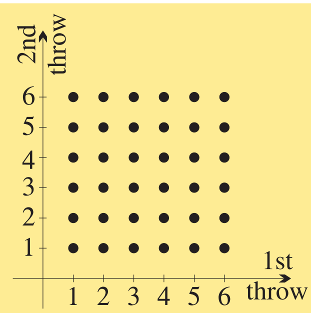
(c) both numbers are greater than four,

(d) both numbers are even,

(e) the sum of the two numbers is six,

(f) the sum is at most four.

SOLUTION:



The horizontal axis in the diagram to the right represents the six possible outcomes of the first throw, and the vertical axis represents the six possible outcomes of the second throw. The 36 dots therefore represent the 36 different possible outcomes of the two-stage experiment, all equally likely, which is the full sample space.

Parts (a)–(f) can now be answered by counting the dots representing the various event spaces.

(a) Since there are 6 doubles, $\mathcal{P}(\text{double}) = \frac{6}{36}$
$= \frac{1}{6}$.

(b) Since 11 pairs contain a 4, $\mathcal{P}(\text{at least one is a 4}) = \frac{11}{36}$.

(c) Since 4 pairs consist only of 5 or 6, $\mathcal{P}(\text{both greater than 4}) = \frac{4}{36}$
$= \frac{1}{9}$.

(d) Since 9 pairs have two even members, $\mathcal{P}(\text{both even}) = \frac{9}{36}$
$= \frac{1}{4}$.

(e) Since 5 pairs have sum 6, $\mathcal{P}(\text{sum is 6}) = \frac{5}{36}$.

(f) Since 6 pairs have sum 2, 3 or 4, $\mathcal{P}(\text{sum at most 4}) = \frac{6}{36}$
$= \frac{1}{6}$.

Complementary Events in Multi-stage Experiments: The following worked exercise continues the two-stage experiment in the previous worked exercise, but this time it involves working with complementary events.

WORKED EXERCISE:

A die is thrown twice.

(a) Find the probability of failing to throw a double six.

(b) Find the probability that the sum is not seven.

SOLUTION:

The same sample space can be used as in the previous worked exercise.

(a) The double six is one outcome amongst 36,

so $\mathcal{P}(\text{double six}) = \frac{1}{36}$.

Hence $\mathcal{P}(\text{not throwing a double six}) = 1 - \mathcal{P}(\text{double six})$

$= \frac{35}{36}$.

(b) The graph shows that there are six outcomes giving a sum of seven,

so $\mathcal{P}(\text{sum is seven}) = \frac{6}{36}$

$= \frac{1}{6}$.

Hence $\mathcal{P}(\text{sum is not seven}) = 1 - \mathcal{P}(\text{sum is seven})$

$= \frac{5}{6}$.

Tree Diagrams: Listing the sample space of a multi-stage experiment can be difficult and the dot diagrams of the previous paragraph are hard to draw in more than two dimensions. Tree diagrams provide a very useful alternative way to display the sample space. Such diagrams have a column for each stage, plus an initial column labelled 'Start' and a final column listing the possible outcomes.

WORKED EXERCISE:

A three-letter word is chosen in the following way. The first and last letters are chosen from the three vowels 'A', 'O' and 'U', with repetition not allowed, and the middle letter is chosen from 'L' and 'M'. List the sample space, then find the probability that:

(a) the word is 'ALO',

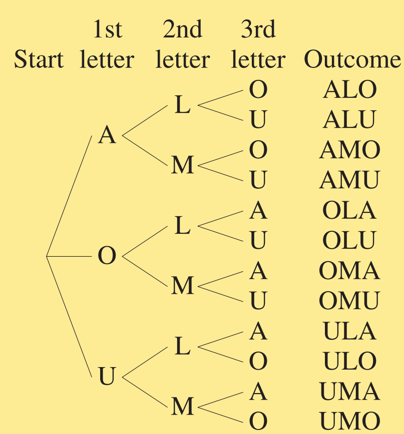
(b) the letter 'O' does not occur,

(c) 'M' and 'U' do not both occur,

(d) the letters are in alphabetical order.

SOLUTION:

The tree diagram to the right lists all 12 equally likely possible outcomes. The two vowels must be different, because repetition was not allowed.



(a) $\mathcal{P}(\text{‘ALO’}) = \frac{1}{12}$

(b) $\mathcal{P}(\text{no ‘O’}) = \frac{4}{12}$

$= \frac{1}{3}$

(c) $\mathcal{P}(\text{not both ‘M’ and ‘U’}) = \frac{8}{12}$

$= \frac{2}{3}$

(d) $\mathcal{P}(\text{alphabetical order}) = \frac{4}{12}$

$= \frac{1}{3}$

The Meaning of ‘Word’: In the worked exercise above and throughout this chapter, ‘word’ simply means an arrangement of letters — the arrangement doesn’t have to have any meaning or be a word in the dictionary. Thus ‘word’ simply becomes a convenient device for discussing arrangements of things in particular orders.

Invalid Arguments: The following worked exercise illustrates another invalid argument in probability. Notice again that the solution offers an explanation of the fallacy first, before offering an alternative argument with a different conclusion.

WORKED EXERCISE:

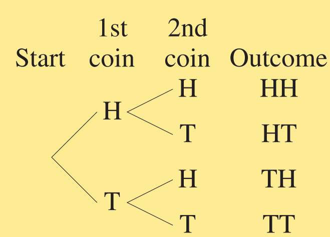
Comment on the validity of this argument.

> ‘When two coins are tossed together, there are three outcomes: two heads, two tails, and one of each. Hence the probability of getting one of each is $\frac{1}{3}$.’

SOLUTION:

[Identifying the fallacy] The division of the results into the three given outcomes is correct, but no reason is offered as to why these outcomes are equally likely.

[Supplying the correct argument] On the right is a diagram of the sample space that divides the results of the experiment into four *equally likely* possible outcomes; since two of these outcomes, HT and TH, are favourable to the event ‘one of each’, it follows that $\mathcal{P}(\text{one of each}) = \frac{2}{4} = \frac{1}{2}$.



Exercise 8B

1. From a group of four students, Anna, Bill, Charlie and David, two are chosen at random to be on the Student Representative Council. List the sample space, and hence find the probability that:

(a) Bill and David are chosen,
(b) Anna is chosen,
(c) Charlie is chosen but Bill is not,
(d) neither Anna nor David is selected.

2. A fair coin is tossed twice. Use a tree diagram to list the possible outcomes. Hence find the probability that the two tosses result in:

(a) two heads, (b) a head and a tail, (c) a head on the first toss and a tail on the second.

3. A die is thrown and a coin is tossed. Use a tree diagram to list all the possible outcomes. Hence find the probability of obtaining:

(a) a head and an even number,
(b) a tail and a number greater than four,
(c) a tail and a number less than four,
(d) a head and a prime number.

4. From the integers 2, 3, 8 and 9, two-digit numbers are formed in which no digit can be repeated in the same number.

(a) Use a tree diagram to list the possible outcomes.
(b) If one of the two-digit numbers is chosen at random, find the probability that it is:
  (i) the number 82,
  (ii) a number greater than 39,
  (iii) an even number,
  (iv) a multiple of 3,
  (v) a number ending in 2,
  (vi) a perfect square.

5. A captain and vice-captain of a cricket team are to be chosen from Amanda, Belinda, Carol, Dianne and Emma.

(a) Use a tree diagram to list the possible pairings, noting that order is important.

(b) Find the probability that:

(i) Carol is captain and Emma is vice-captain,

(ii) Belinda is either captain or vice-captain,

(iii) Amanda is not selected for either position,

(iv) Emma is vice-captain.

DEVELOPMENT

6. A coin is tossed three times. Draw a tree diagram to illustrate the possible outcomes. Then find the probability of obtaining:

(a) three heads,
(b) a head and two tails,
(c) at least two tails,
(d) at most one head,
(e) more heads than tails,
(f) a head on the second toss.

7. A green die and a red die are thrown simultaneously. List the set of 36 possible outcomes on a two-dimensional graph and hence find the probability of:

(a) obtaining a three on the green die,
(b) obtaining a four on the red die,
(c) a double five,
(d) a total score of seven,
(e) a total score greater than nine,
(f) an even number on both dice,
(g) at least one two,
(h) neither a one nor a four appearing,
(i) a five and a number greater than three,
(j) the same number on both dice.

8. Suppose that the births of boys and girls are equally likely.

(a) In a family of two children, determine the probability that there are:

(i) two girls, (ii) no girls, (iii) one boy and one girl.

(b) In a family of three children, determine the probability that there are:

(i) three boys, (ii) two girls and one boy, (iii) more boys than girls.

9. An unbiased coin is tossed four times. Find the probability of obtaining:

(a) four heads,
(b) exactly three tails,
(c) at least two heads,
(d) at most one head,
(e) two heads and two tails,
(f) more tails than heads.

10. A hand of five cards contains a ten, jack, queen, king and ace. From the hand, two cards are drawn in succession, the first card not being replaced before the second card is drawn. Find the probability that:

(a) the ace is drawn, (b) the king is not drawn, (c) the queen is the second card drawn.

CHALLENGE

11. Three-digit numbers are formed from the digits 2, 5, 7 and 8, without repetition.

(a) Use a tree diagram to list all the possible outcomes. How many are there?

(b) Hence find the probability that the number is:

(i) greater than 528, (ii) divisible by 3, (iii) divisible by 13, (iv) prime.

12. If a coin is tossed n times, where $n > 1$, find the probability of obtaining:

(a) n heads, (b) at least one head and at least one tail.

8 C Sets and Venn Diagrams

This section is a brief account of sets and Venn diagrams for those who have not met these ideas already when solving problems in Years 7 and 8. The three key ideas needed in probability are the intersection of sets, the union of sets and the complement of a set.

Logic is very close to the surface when we talk about sets and Venn diagrams. The three ideas of intersection, union and complement mentioned above correspond very precisely to the words 'and', 'or' and 'not'.

Listing Sets and Describing Sets: A *set* is a collection of things. When a set is specified, it needs to be made absolutely clear what things are its members. This can be done by *listing* the members inside curly brackets. For example:

$$S = \{1, 3, 5, 7, 9\},$$

which is read as 'S is the set whose members are 1, 3, 5, 7 and 9'. It can also be done by *writing a description* of the members inside curly brackets. For example,

$$T = \{\text{odd integers from } 0 \text{ to } 10\},$$

read as 'T is the set of odd integers from 0 to 10'.

Equal Sets: Two sets are called *equal* if they have exactly the same members. Hence the sets S and T in the previous paragraph are equal, which is written as $S = T$. The order in which the members are written doesn't matter at all, neither does repetition. For example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\} = \{1, 3, 1, 5, 1, 7, 9\}.$$

The Size of a Set: A set may be finite, like the set above of positive odd numbers less than 10, or infinite, like the set of all integers. Only finite sets are needed here.

If a set S is finite, then the symbol $|S|$ is used to mean the number of members in S. For example:

If $A = \{5, 6, 7, 8, 9, 10\}$, then $|A| = 6$.
If $B = \{\text{letters in the alphabet}\}$, then $|B| = 26$.
If $C = \{12\}$, then $|C| = 1$.
If $D = \{\text{odd numbers between } 1{\cdot}5 \text{ and } 2{\cdot}5\}$, then $|D| = 0$.

The Empty Set: The last set D above is called the *empty set*, because it has no members at all. The usual symbol for the empty set is $\emptyset$. There is only one empty set, because any two empty sets have the same members (that is, none at all) and so are equal.

Intersection and Union : There are two obvious ways of combining two sets A and B. The *intersection* $A \cap B$ of A and B is the set of everything belonging to A <u>and</u> B. The *union* $A \cup B$ of A and B is the set of everything belonging to A <u>or</u> B. For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 6\}$, then

$$A \cap B = \{1, 3\}$$
$$A \cup B = \{0, 1, 2, 3, 6\}$$

Two sets A and B are called *disjoint* if they have no elements in common, that is, if $A \cap B = \emptyset$. For example, if $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$, then

$$A \cap B = \emptyset, \text{ so } A \text{ and } B \text{ are disjoint.}$$

‘And’ means Intersection, ‘Or’ means Union: The important mathematical words ‘and’ and ‘or’ can be interpreted in terms of union and intersection:

$$A \cap B = \{\, \text{objects that are in } A \text{ \underline{and} in } B \,\}$$
$$A \cup B = \{\, \text{objects that are in } A \text{ \underline{or} in } B \,\}$$

NOTE: The word ‘or’ in mathematics always means ‘and/or’. Similarly, $A \cup B$ includes all the objects which are in both A and B.

Subsets of Sets: A set A is called a *subset* of a set B if every member of A is a member of B. This relation is written as $A \subset B$. For example,

$$\{\, \text{men in Australia} \,\} \subset \{\, \text{people in Australia} \,\}$$
$$\{\, 2, 3, 4 \,\} \subset \{\, 1, 2, 3, 4, 5 \,\}$$
$$\{\, \text{vowels} \,\} \subset \{\, \text{letters in the alphabet} \,\}$$

Because of the way subsets have been defined, every set is a subset of itself. Also, the empty set is a subset of every set. For example,

$$\{\, 1, 3, 5 \,\} \subset \{\, 1, 3, 5 \,\}, \text{ and}$$
$$\emptyset \subset \{\, 1, 3, 5 \,\}$$

The Universal Set and the Complement of a Set: A *universal set* is the set of everything under discussion in a particular situation. For example, if $A = \{\, 1, 3, 5, 7, 9 \,\}$, then possible universal sets could be the set of all positive integers less than 11, or the set of all integers.

Once a universal set E is fixed, then the *complement* $\overline{A}$ of any set A is the set of all members of that universal set which are not in A. For example,

$$\text{If } A = \{\, 1, 3, 5, 7, 9 \,\} \text{ and } E = \{\, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \,\},$$
$$\text{then } \overline{A} = \{\, 2, 4, 6, 8, 10 \,\}.$$

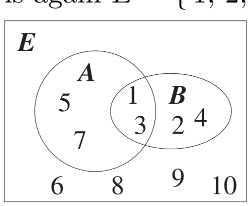
Notice that every member of the universal set is either in A or in $\overline{A}$, but never in both A and $\overline{A}$. This means that

$$A \cap \overline{A} = \emptyset, \text{ the empty set, and}$$
$$A \cup \overline{A} = E, \text{ the universal set.}$$

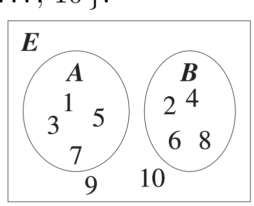
‘Not’ means Complement: As mentioned in Section 8A, the important mathematical word ‘not’ can be interpreted in terms of the complementary set:

$$\overline{A} = \{\, \text{members of } E \text{ that are \underline{not} members of } A \,\}$$

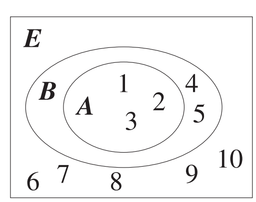
Venn Diagrams: A *Venn diagram* is a diagram used to represent the relationship between sets. For example, the four diagrams below represent the four different possible relationships between two sets A and B. In each case, the universal set is again $E = \{\, 1, 2, 3, \ldots, 10 \,\}$.



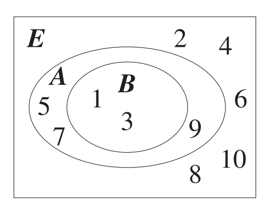
$A = \{\, 1, 3, 5, 7 \,\}$
$B = \{\, 1, 2, 3, 4 \,\}$



$A = \{\, 1, 3, 5, 7 \,\}$
$B = \{\, 2, 4, 6, 8 \,\}$



$A = \{\, 1, 2, 3 \,\}$
$B = \{\, 1, 2, 3, 4, 5 \,\}$

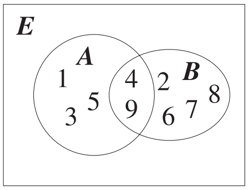


$A = \{\, 1, 3, 5, 7 ,9 \,\}$
$B = \{\, 1, 3 \,\}$

Sets such as $A \cup B$, $A \cap B$ and $\overline{A} \cap B$ can be visualised by shading regions of the Venn diagram, as is done in a question in the following exercise.

The Counting Rule for Sets: To calculate the size of the union $A \cup B$ of two sets, adding the sizes of A and of B will not do, because the members of the intersection $A \cap B$ would be counted twice. Hence $|A \cap B|$ needs to be subtracted again, and the rule is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



For example, the Venn diagram to the right shows the sets

$$A = \{1, 3, 4, 5, 9\} \quad \text{and} \quad B = \{2, 4, 6, 7, 8, 9\}.$$

From the diagram, $|A \cup B| = 9$, $|A| = 5$, $|B| = 6$ and $|A \cap B| = 2$, and the formula works because

$$9 = 5 + 6 - 2.$$

When two sets are disjoint, there is no overlap between A and B to cause any double counting. With $|A \cap B| = 0$, the counting rule becomes

$$|A \cup B| = |A| + |B|.$$

Problem Solving Using Venn Diagrams: A Venn diagram is often the most convenient way to sort out problems involving overlapping sets of things. Note that in the following exercise, the number of members of each region is written inside the region, rather than the members themselves.

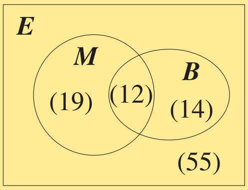
WORKED EXERCISE: 100 Sydneysiders were surveyed to find out how many of them had visited the cities of Melbourne and Brisbane. 31 people had visited Melbourne, 26 people had visited Brisbane and 12 people had visited both cities. Find how many people had visited:

(a) Melbourne or Brisbane,

(b) Brisbane but not Melbourne,

(c) only one of the two cities,

(d) neither city.



SOLUTION: Let M be the set of people who have visited Melbourne, let B be the set of people who have visited Brisbane, and let E be the universal set of all people surveyed. Calculations should begin with the 12 people in the intersection of the two regions. Then the numbers shown in the other three regions of the Venn diagram can easily be found.

(a) Number visiting Melbourne or Brisbane $= 19 + 14 + 12$
$= 45$

(b) Number visiting Brisbane but not Melbourne $= 14$

(c) Number visiting only one city $= 19 + 14$
$= 33$

(d) Number visiting neither city $= 100 - 45$
$= 55$

Exercise 8C

1. Find $A \cup B$ and $A \cap B$ for each pair of sets:
(a) $A = \{1, 3, 5\}$, $B = \{3, 5, 7\}$ (b) $A = \{1, 3, 4, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$
(c) $A = \{\text{h, o, b, a, r, t}\}$, $B = \{\text{b, i, c, h, e, n, o}\}$
(d) $A = \{\text{j, a, c, k}\}$, $B = \{\text{e, m, m, a}\}$
(e) $A = \{\text{prime numbers less than } 10\}$, $B = \{\text{odd numbers less than } 10\}$

2. If $A = \{1, 4, 7, 8\}$ and $B = \{1, 2, 4, 5, 7\}$, state whether the following statements are true or false:
(a) $A = B$ (c) $|B| = 6$ (e) $A \cup B = \{1, 2, 4, 5, 7, 8\}$
(b) $|A| = 4$ (d) $A \subset B$ (f) $A \cap B = \{1, 4, 7\}$

3. If $A = \{1, 3, 5\}$, $B = \{3, 4\}$ and $E = \{1, 2, 3, 4, 5\}$, find:
(a) $|A|$ (c) $A \cup B$ (e) $A \cap B$ (g) $\overline{A}$
(b) $|B|$ (d) $|A \cup B|$ (f) $|A \cap B|$ (h) $\overline{B}$

4. If $A = \{\text{students who study Japanese}\}$ and $B = \{\text{students who study History}\}$, carefully describe each of the following sets:
(a) $A \cap B$ (b) $A \cup B$

5. If $A = \{\text{students with blue eyes}\}$ and $B = \{\text{students with blonde hair}\}$, with universal set $E = \{\text{students at Clarence High School}\}$, carefully describe each of the following:
(a) $\overline{A}$ (b) $\overline{B}$ (c) $A \cup B$ (d) $A \cap B$

6. List all the subsets of each of these sets:
(a) $\{\text{a}\}$ (b) $\{\text{a, b}\}$ (c) $\{\text{a, b, c}\}$ (d) $\emptyset$

7. State in each case whether or not $A \subset B$ (that is, whether A is a subset of B):
(a) $A = \{2, 3, 5\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
(b) $A = \{3, 6, 9, 12\}$, $B = \{3, 5, 9, 11\}$
(c) $A = \{\text{d, a, n, c, e}\}$, $B = \{\text{e, d, u, c, a, t, i, o, n}\}$
(d) $A = \{\text{a, m, y}\}$, $B = \{\text{s, a, r, a, h}\}$
(e) $A = \emptyset$, $B = \{51, 52, 53, 54\}$

DEVELOPMENT

8. Let $A = \{1, 3, 7, 10\}$ and $B = \{4, 6, 7, 9\}$, and take the universal set to be the set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List the members of:
(a) $\overline{A}$ (c) $A \cap B$ (e) $A \cup B$
(b) $\overline{B}$ (d) $\overline{A \cap B}$ (f) $\overline{A \cup B}$

9. Let $A = \{1, 3, 6, 8\}$ and $B = \{3, 4, 6, 7, 10\}$, and take the universal set to be the set $E = \{1, 2, 3, \ldots, 10\}$. List the members of:
(a) $\overline{A}$ (c) $\overline{A} \cup \overline{B}$ (e) $A \cup B$ (g) $A \cap B$
(b) $\overline{B}$ (d) $\overline{A} \cap \overline{B}$ (f) $\overline{A \cup B}$ (h) $\overline{A \cap B}$

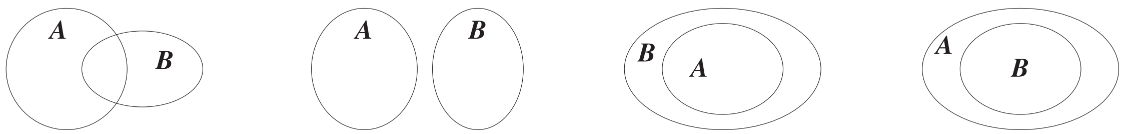
10. Answer true or false:
(a) If $A \subset B$ and $B \subset A$, then $A = B$.
(b) If $A \subset B$ and $B \subset C$, then $A \subset C$.

11. Copy and complete:

(a) If $P \subset Q$, then $P \cup Q = \ldots$

(b) If $P \subset Q$, then $P \cap Q = \ldots$

12. Select the Venn diagram that best shows the relationship between each pair of sets A and B:



I II III IV

(a) $A = \{\text{Tasmania}\}$, $B = \{\text{states of Australia}\}$

(b) $A = \{7, 1, 4, 8, 3, 5\}$, $B = \{2, 9, 0, 7\}$

(c) $A = \{\text{l, e, a, r, n}\}$, $B = \{\text{s, t, u, d, y}\}$

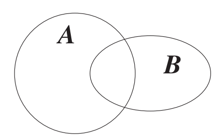
(d) $A = \{\text{politicians in Australia}\}$, $B = \{\text{politicians in NSW}\}$.

13. (a) Explain the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$ by making reference to the Venn diagram opposite.

(b) If $|A \cup B| = 17$, $|A| = 12$ and $|B| = 10$, find $|A \cap B|$.

(c) Show that the relationship in part (a) is satisfied when

$$A = \{3, 5, 6, 8, 9\} \quad \text{and} \quad B = \{2, 3, 5, 6, 7, 8\}.$$



14. Use a Venn diagram to solve each of these problems:

(a) In a group of 20 people, there are 8 who play the piano, 5 who play the violin and 3 who play both. How many people play neither?

(b) Each person in a group of 30 plays either tennis or golf. 17 play tennis, while 9 play both. How many play golf?

(c) In a class of 28 students, there are 19 who like geometry and 16 who like trigonometry. How many like both if there are 5 students who don't like either?

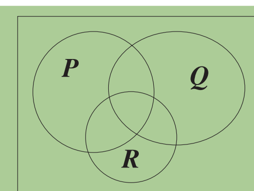
CHALLENGE

15. Shade each of the following regions on the given diagram (use a separate copy of the diagram for each part).

(a) $P \cap Q \cap R$

(b) $(P \cap R) \cup (Q \cap R)$

(c) $\overline{P} \cup \overline{Q} \cup \overline{R}$ (where $\overline{P}$ denotes the complement of P)



16. A group of 80 people was surveyed about their approaches to keeping fit.

It was found that 20 jog, 22 swim and 18 go to the gym on a regular basis.

Further questioning found that 10 people both jog and swim, 11 people both jog and go to the gym, and 6 people both swim and go to the gym.

Finally, 43 people do none of these activities.

How many of the people do all three?

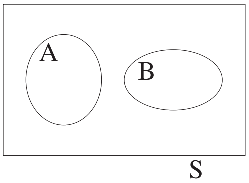
8 D Venn Diagrams and the Addition Theorem

In this section, Venn diagrams and the language of sets are used to visualise the sample space and the event space in situations where an event is described using the logical words 'and', 'or' and 'not'.

Mutually Exclusive Events and Disjoint Sets: Two events A and B with the same sample space S are called *mutually exclusive* if they cannot both occur. For example:

If a die is thrown, the events 'throwing a number less than three' and 'throwing a number greater than four' cannot both occur and so are mutually exclusive.

If a card is drawn at random from a pack, the events 'drawing a red card' and 'drawing a spade' cannot both occur and so are mutually exclusive.



In the Venn diagram of such a situation, the two events A and B are represented as disjoint sets (*disjoint* means that their intersection is empty). The event 'A and B' is impossible and therefore has probability zero.

On the other hand, the event 'A or B' is represented on the Venn diagram by the union $A \cup B$ of the two sets. Since $|A \cup B| = |A| + |B|$ for disjoint sets, it follows that

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B)\,.$$

6 MUTUALLY EXCLUSIVE EVENTS: Suppose that A and B are mutually exclusive events with sample space S. Then the event 'A or B' is represented by $A \cup B$, and

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B)\,.$$

The event 'A and B' cannot occur and has probability zero.

WORKED EXERCISE:

If a die is thrown, find the probability that it is less than three or greater than four.

SOLUTION:

The events 'throwing a number less than three' and 'throwing a number greater than four' are mutually exclusive,

so $\mathcal{P}(\text{less than three or greater than four}) = \mathcal{P}(\text{less than three}) + \mathcal{P}(\text{greater than four})$

$$= \tfrac{1}{3} + \tfrac{1}{3}$$
$$= \tfrac{2}{3}.$$

WORKED EXERCISE:

If a card is drawn at random from a pack, find the probability that it is a red card or a spade.

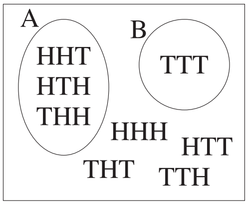
SOLUTION:

'Drawing a red card' and 'drawing a spade' are mutually exclusive,

so $\mathcal{P}(\text{a red card or a spade}) = \mathcal{P}(\text{a red card}) + \mathcal{P}(\text{a spade})$

$$= \tfrac{1}{2} + \tfrac{1}{4}$$
$$= \tfrac{3}{4}.$$

WORKED EXERCISE: [Mutually exclusive events in multi-stage experiments]
If three coins are tossed, find the probability of throwing an odd number of tails.



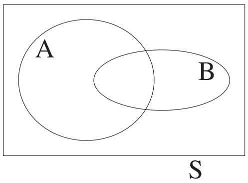
SOLUTION:
Let A be the event 'one tail' and B the event 'three tails'. Then A and B are mutually exclusive and

$$A = \{\,\text{HHT, HTH, THH}\,\} \quad \text{and} \quad B = \{\,\text{TTT}\,\}.$$

The full sample space has eight members altogether (question 6 in the previous exercise lists them all), so

$$\mathcal{P}(A \text{ or } B) = \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{1}{2}.$$

The Events 'A and B' and 'A or B' — The Addition Rule: More generally, suppose that A and B are any two events with the same sample space S, not necessarily mutually exclusive. The Venn diagram of the situation will now represent the two events A and B as overlapping sets within the same universal set S. The event 'A and B' will then be represented by the intersection $A \cap B$ of the two sets, and the event 'A or B' will be represented by the union $A \cup B$.



The general counting rule for sets is $|A \cup B| = |A| + |B| - |A \cap B|$, because the members of the intersection $A \cap B$ are counted in A and again in B, and so have to be subtracted. It follows then that $\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B)$ — this rule is often called the *addition rule* of probability.

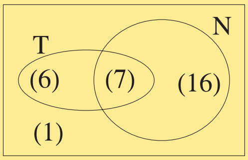
7 THE EVENTS 'A OR B' AND 'A AND B': Suppose that A and B are two events with sample space S. Then the event 'A and B' is represented by the intersection $A \cap B$ and the event 'A or B' is represented by the union $A \cup B$, and

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B).$$

As explained in the previous section, the word 'or' is closely linked with the union of sets, and the word 'and' is closely linked with the intersection of sets. For this reason, the event 'A or B' is often written as '$A \cup B$' and the event 'A and B' is often written as '$A \cap B$' or just 'AB'.

WORKED EXERCISE:
In a class of 30 girls, 13 play tennis and 23 play netball. If 7 girls play both sports, what is the probability that a girl chosen at random plays neither sport?

SOLUTION:
Let T be the event 'she plays tennis',
and let N be the event 'she plays netball'.



$$\begin{aligned}
\text{Then} \qquad \mathcal{P}(T) &= \tfrac{13}{30} \\
\mathcal{P}(N) &= \tfrac{23}{30} \\
\text{and} \qquad \mathcal{P}(N \text{ and } T) &= \tfrac{7}{30}. \\
\text{Hence} \qquad \mathcal{P}(N \text{ or } T) &= \tfrac{13}{30} + \tfrac{23}{30} - \tfrac{7}{30} \\
&= \tfrac{29}{30}, \\
\text{and} \qquad \mathcal{P}(\text{neither sport}) &= 1 - \mathcal{P}(N \text{ or } T) \\
&= \tfrac{1}{30}.
\end{aligned}$$

NOTE: An alternative approach is shown in the diagram. Starting with the 7 girls in the intersection, the numbers 6 and 16 can then be written into the respective regions 'tennis but not netball' and 'netball but not tennis'. Since these numbers add to 29, this leaves only one girl playing neither tennis nor netball.

Complementary Events and the Addition Rule: In the following worked exercise, the addition rule has to be applied in combination with the idea of complementary events. Some careful thinking is required when the words 'and' and 'or' are combined with 'not'.

WORKED EXERCISE:
A card is drawn at random from a pack.
(a) Find the probability that it is not an ace and not a two.
(b) Find the probability that it is an even number or a picture card or red.

NOTE: Remember that the word 'or' always means 'and/or' in logic and mathematics. Thus in part (b), the words 'or any two of these, or all three of these' are understood and need not be added.

SOLUTION:
(a) The complementary event $\overline{E}$ is drawing a card that is an ace or a two.
Since there are eight such cards, $\mathcal{P}(\text{ace or two}) = \frac{8}{52}$
$= \frac{2}{13}$.
Hence $\mathcal{P}(\text{not an ace and not a two}) = 1 - \frac{2}{13}$
$= \frac{11}{13}$.

(b) The complementary event $\overline{E}$ is drawing a card that is a black odd number less than ten. This complementary event has 10 members:

$$\overline{E} = \{\,A\clubsuit, 3\clubsuit,\ 5\clubsuit,\ 7\clubsuit,\ 9\clubsuit,\ A\spadesuit,\ 3\spadesuit,\ 5\spadesuit,\ 7\spadesuit,\ 9\spadesuit\,\}.$$

There are 52 possible cards to choose, so

$$\mathcal{P}\big(\overline{E}\big) = \tfrac{10}{52} = \tfrac{5}{26}.$$

Hence, using the complementary event formula,

$$\mathcal{P}(E) = 1 - \mathcal{P}\big(\overline{E}\big) = \tfrac{21}{26}.$$

Exercise 8D

1. A die is rolled. If n denotes the number on the uppermost face, find:
(a) $\mathcal{P}(n = 5)$
(b) $\mathcal{P}(n \neq 5)$
(c) $\mathcal{P}(n = 4 \text{ or } n = 5)$
(d) $\mathcal{P}(n = 4 \text{ and } n = 5)$
(e) $\mathcal{P}(n \text{ is even or odd})$
(f) $\mathcal{P}(n \text{ is neither even nor odd})$
(g) $\mathcal{P}(n \text{ is even and divisible by three})$
(h) $\mathcal{P}(n \text{ is even or divisible by three})$

2. A card is selected from a regular pack of 52 cards. Find the probability that the card:
(a) is a jack,
(b) is a ten,
(c) is a jack or a ten,
(d) is a jack and a ten,
(e) is neither a jack nor a ten,
(f) is black,
(g) is a picture card,
(h) is a black picture card,
(i) is black or a picture card,
(j) is neither black nor a picture card.

3. A die is thrown. Let A be the event that an even number appears. Let B be the event that a number greater than two appears.

(a) Are A and B mutually exclusive?

(b) Find: (i) $\mathcal{P}(A)$ (ii) $\mathcal{P}(B)$ (iii) $\mathcal{P}(A \text{ and } B)$ (iv) $\mathcal{P}(A \text{ or } B)$

4. Two dice are thrown. Let a and b denote the numbers rolled. Find:

(a) $\mathcal{P}(a \text{ is odd})$

(b) $\mathcal{P}(b \text{ is odd})$

(c) $\mathcal{P}(a \text{ and } b \text{ are odd})$

(d) $\mathcal{P}(a \text{ or } b \text{ is odd})$

(e) $\mathcal{P}(\text{neither } a \text{ nor } b \text{ is odd})$

(f) $\mathcal{P}(a = 1)$

(g) $\mathcal{P}(b = a)$

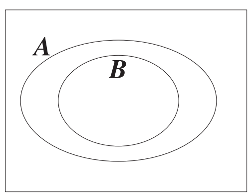
(h) $\mathcal{P}(a = 1 \text{ and } b = a)$

(i) $\mathcal{P}(a = 1 \text{ or } b = a)$

(j) $\mathcal{P}(a \neq 1 \text{ and } a \neq b)$

DEVELOPMENT

5. (a)



If $\mathcal{P}(A) = \frac{1}{2}$ and $\mathcal{P}(B) = \frac{1}{3}$, find:

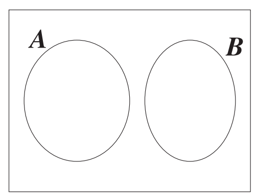
(i) $\mathcal{P}(\overline{A})$ (ii) $\mathcal{P}(\overline{B})$

(iii) $\mathcal{P}(A \text{ and } B)$

(iv) $\mathcal{P}(A \text{ or } B)$

(v) $\mathcal{P}(\text{neither } A \text{ nor } B)$

(b)



If $\mathcal{P}(A) = \frac{2}{5}$ and $\mathcal{P}(B) = \frac{1}{5}$, find:

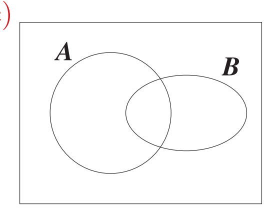
(i) $\mathcal{P}(\overline{A})$ (ii) $\mathcal{P}(\overline{B})$

(iii) $\mathcal{P}(A \text{ or } B)$

(iv) $\mathcal{P}(A \text{ and } B)$

(v) $\mathcal{P}(\text{not both } A \text{ and } B)$

(c)



If $\mathcal{P}(A) = \frac{1}{2}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ and } B) = \frac{1}{6}$, find:

(i) $\mathcal{P}(\overline{A})$ (ii) $\mathcal{P}(\overline{B})$

(iii) $\mathcal{P}(A \text{ or } B)$

(iv) $\mathcal{P}(\text{neither } A \text{ nor } B)$

(v) $\mathcal{P}(\text{not both } A \text{ and } B)$

6. Use the addition rule $\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B)$ to answer the following questions:

(a) If $\mathcal{P}(A) = \frac{1}{5}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ and } B) = \frac{1}{15}$, find $\mathcal{P}(A \text{ or } B)$.

(b) If $\mathcal{P}(A) = \frac{1}{2}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ or } B) = \frac{5}{6}$, find $\mathcal{P}(A \text{ and } B)$.

(c) If $\mathcal{P}(A \text{ or } B) = \frac{9}{10}$, $\mathcal{P}(A \text{ and } B) = \frac{1}{5}$ and $\mathcal{P}(A) = \frac{1}{2}$, find $\mathcal{P}(B)$.

(d) If A and B are mutually exclusive and $\mathcal{P}(A) = \frac{1}{7}$ and $\mathcal{P}(B) = \frac{4}{7}$, find $\mathcal{P}(A \text{ or } B)$.

7. An integer n is picked at random, where $1 \le n \le 20$. The events A, B, C and D are:

A: an even number is chosen, B: a number greater than 15 is chosen,

C: a multiple of 3 is chosen, D: a one-digit number is chosen.

(a) (i) Are the events A and B mutually exclusive?

(ii) Find $\mathcal{P}(A)$, $\mathcal{P}(B)$ and $\mathcal{P}(A \text{ and } B)$, and hence evaluate $\mathcal{P}(A \text{ or } B)$.

(b) (i) Are the events A and C mutually exclusive?

(ii) Find $\mathcal{P}(A)$, $\mathcal{P}(C)$ and $\mathcal{P}(A \text{ and } C)$, and hence evaluate $\mathcal{P}(A \text{ or } C)$.

(c) (i) Are the events B and D mutually exclusive?

(ii) Find $\mathcal{P}(B)$, $\mathcal{P}(D)$ and $\mathcal{P}(B \text{ and } D)$, and hence evaluate $\mathcal{P}(B \text{ or } D)$.

8. In a group of 50 students, there are 26 who study Latin and 15 who study Greek and 8 who take both languages. Draw a Venn diagram and find the probability that a student chosen at random:

(a) studies only Latin, (b) studies only Greek, (c) does not study either language.

9. During a game, all 21 members of an Australian Rules football team consume liquid. Some players drink only water, some players drink only Gatorade^{TM} and some players drink both. If there are 14 players who drink water and 17 players who drink Gatorade^{TM}:

(a) How many drink both water and Gatorade^{TM}?

(b) If one team member is selected at random, find the probability that:

(i) he drinks water but not Gatorade^{TM}, (ii) he drinks Gatorade^{TM} but not water.

10. Each student in a music class of 28 studies either the piano or the violin or both. It is known that 20 study the piano and 15 study the violin. Find the probability that a student selected at random studies both instruments.

CHALLENGE

11. List the 25 primes less than 100. A number is drawn at random from the integers from 1 to 100. Find the probability that:

(a) it is prime, (b) it has remainder 1 after division by 4,

(c) it is prime and it has remainder 1 after division by 4,

(d) it is either prime or it has remainder 1 after division by 4.

12. A group of 60 students was invited to try out for three sports: rugby, soccer and cross country — 32 tried out for rugby, 29 tried out for soccer, 15 tried out for cross country, 11 tried out for rugby and soccer, 9 tried out for soccer and cross country, 8 tried out for rugby and cross country, and 5 tried out for all three sports. Draw a Venn diagram and find the probability that a student chosen at random:

(a) tried out for only one sport, (c) tried out for at least two sports,

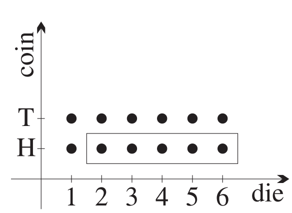
(b) tried out for exactly two sports, (d) did not try out for a sport.

8 E Multi-Stage Experiments and the Product Rule

This section deals with experiments that have a number of stages. The full sample space of such an experiment can quickly become too large to be conveniently listed, but instead a rule can be developed for multiplying together the probabilities associated with each stage.

Two-Stage Experiments — The Product Rule: Here is a simple question about a two-stage experiment:

'Throw a die, then toss a coin. What is the probability of obtaining at least two on the die followed by a head?'



Graphed to the right are the twelve possible outcomes of the experiment, all equally likely, with a box drawn around the five favourable outcomes. Thus

$$\mathcal{P}(\text{at least two and a head}) = \tfrac{5}{12}.$$

Now consider the two stages separately. The first stage is throwing a die and the desired outcome is A = 'getting at least two' — here there are six possible outcomes and five favourable outcomes, giving probability $\frac{5}{6}$. The second stage is tossing a coin and the desired outcome is B = 'tossing a head' — here there are two possible outcomes and one favourable outcome, giving probability $\frac{1}{2}$.

The full experiment then has $6 \times 2 = 12$ possible outcomes and there are $5 \times 1 = 5$ favourable outcomes. Hence

$$\mathcal{P}(AB) = \frac{5 \times 1}{6 \times 2} = \frac{5}{6} \times \frac{1}{2} = \mathcal{P}(A) \times \mathcal{P}(B)\,.$$

Thus the probability of the compound event 'getting at least two and a head' can be found by multiplying together the probabilities of the two stages. The argument here can easily be generalised to any two-stage experiment.

8 TWO-STAGE EXPERIMENTS: If A and B are independent events in successive stages of a two-stage experiment, then

$$\mathcal{P}(AB) = \mathcal{P}(A) \times \mathcal{P}(B)\,,$$

where the word 'independent' means that the outcome of one stage does not affect the outcome of the other stage.

Independent Events: The word 'independent' needs further discussion. In the example above, the throwing of the die clearly does not affect the tossing of the coin, so the two events are independent.

Here is a very common and important type of experiment where the two stages are not independent:

> 'Choose an elector at random from the NSW population. First note the elector's gender. Then ask the elector if he or she voted Labor or non-Labor in the last State election.'

In this example, one might suspect that the gender and the political opinion of a person may not be independent and that there is *correlation* between them. This is in fact the case, as almost every opinion poll has shown over the years. Correlation is beyond this course, but it is one of the things statisticians most commonly study in their routine work.

WORKED EXERCISE:

A pair of dice is thrown twice. What is the probability that the first throw is a double and the second throw gives a sum of at least four?

SOLUTION:

We saw in Section 8B that when two dice are thrown, there are 36 possible outcomes, graphed in the diagram to the right.

There are six doubles amongst the 36 possible outcomes,

so $\qquad \mathcal{P}(\text{double}) = \frac{6}{36}$

$\qquad\qquad = \frac{1}{6}$.

All but the pairs $(1,1)$, $(2,1)$ and $(1,2)$ give a sum at least four,

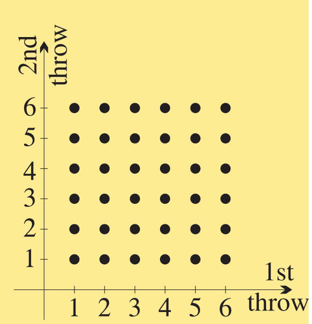
so $\qquad \mathcal{P}(\text{sum is at least four}) = \frac{33}{36}$

$\qquad\qquad = \frac{11}{12}$.

Since the two stages are independent,

$\qquad \mathcal{P}(\text{double, sum at least four}) = \frac{1}{6} \times \frac{11}{12}$

$\qquad\qquad = \frac{11}{72}$.



Multi-Stage Experiments — The Product Rule: The same arguments clearly apply to an experiment with any number of stages.

9 MULTI-STAGE EXPERIMENTS: If $A_1, A_2, \ldots, A_n$ are independent events, then

$$\mathcal{P}(A_1 A_2 \ldots A_n) = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n)\,.$$

WORKED EXERCISE:

A coin is tossed four times. Find the probability that:

(a) every toss is a head,

(b) no toss is a head,

(c) there is at least one head.

SOLUTION:

The four tosses are independent events.

(a) $\mathcal{P}(\text{HHHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.

(b) $\mathcal{P}(\text{no heads}) = \mathcal{P}(\text{TTTT}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.

(c) $\mathcal{P}(\text{at least one head}) = 1 - \mathcal{P}(\text{no heads}) = 1 - \frac{1}{16} = \frac{15}{16}$.

Sampling Without Replacement — An Extension of the Product Rule: The product rule can be extended to the following question, where the two stages of the experiment are not independent.

WORKED EXERCISE:

A box contains five discs numbered 1, 2, 3, 4 and 5.

Two numbers are drawn in succession, without replacement.

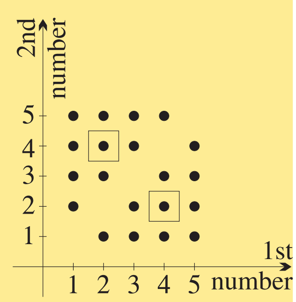
What is the probability that both are even?

SOLUTION:

The probability that the first number is even is $\frac{2}{5}$.

When this even number is removed, one even and three odd numbers remain, so the probability that the second number is also even is $\frac{1}{4}$.

Hence $\mathcal{P}(\text{both even}) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$.



NOTE: The graph to the right allows the calculation to be checked by examining its full sample space.

Because doubles are not allowed (that is, there is no replacement), there are only 20 possible outcomes.

The two boxed outcomes are the only outcomes that consist of two even numbers, giving the same probability of $\frac{2}{20} = \frac{1}{10}$.

Listing the Favourable Outcomes: The product rule is often combined with a listing of the favourable outcomes. A tree diagram may help in producing that listing, although this is hardly necessary in the straightforward worked exercise below, which continues an earlier example.

WORKED EXERCISE:

A coin is tossed four times. Find the probability that:

(a) the first three coins are heads,
(b) the middle two coins are tails,
(c) there are at least three heads,
(d) there are exactly two heads.

SOLUTION:

(a) $\mathcal{P}(\text{the first three coins are heads}) = \mathcal{P}(\text{HHHH}) + \mathcal{P}(\text{HHHT})$

(notice that the two events HHHH and HHHT are mutually exclusive)

$$= \tfrac{1}{16} + \tfrac{1}{16}$$

(since each of these two probabilities is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$)

$$= \tfrac{1}{8}.$$

(b) $$\begin{aligned}\mathcal{P}(\text{middle two are tails}) &= \mathcal{P}(\text{HTTH}) + \mathcal{P}(\text{HTTT}) + \mathcal{P}(\text{TTTH}) + \mathcal{P}(\text{TTTT}) \\ &= \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} \\ &= \tfrac{1}{4}.\end{aligned}$$

(c) $$\begin{aligned}\mathcal{P}(\text{at least 3 heads}) &= \mathcal{P}(\text{HHHH}) + \mathcal{P}(\text{HHHT}) + \mathcal{P}(\text{HHTH}) + \mathcal{P}(\text{HTHH}) + \mathcal{P}(\text{THHH}) \\ &= \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} \\ &= \tfrac{5}{16}.\end{aligned}$$

(d) $$\begin{aligned}\mathcal{P}(\text{exactly 2 heads}) &= \mathcal{P}(\text{HHTT}) + \mathcal{P}(\text{HTHT}) + \mathcal{P}(\text{THHT}) \\ &\quad + \mathcal{P}(\text{HTTH}) + \mathcal{P}(\text{THTH}) + \mathcal{P}(\text{TTHH})\end{aligned}$$

(since these are all the six possible orderings of H, H, T and T)

$$\begin{aligned}&= \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} \\ &= \tfrac{3}{8}.\end{aligned}$$

Describing the Experiment in a Different Way: Sometimes, the manner in which an experiment is described makes calculation difficult, but the experiment can be described in a different way so that the probabilities are the same but the calculations are much simpler.

WORKED EXERCISE:

Wes is sending Christmas cards to ten friends. He has two cards with angels, two with snow, two with reindeer, two with trumpets and two with Santa Claus.

What is the probability that Harry and Helmut get matching cards?

SOLUTION:

Describe the process in a different way as follows:

> 'Wes decides that he will choose Harry's card first and Helmut's card second. Then he will choose the cards for his remaining eight friends.'

All that matters now is whether the card that Wes chooses for Helmut is the same as the card that he has already chosen for Harry. After he chooses Harry's card, there are nine cards remaining, of which only one matches Harry's card. Thus the probability that Helmut's card matches is $\frac{1}{9}$.

Exercise 8E

1. Suppose that A, B, C and D are independent events, with $\mathcal{P}(A) = \frac{1}{8}$, $\mathcal{P}(B) = \frac{1}{3}$, $\mathcal{P}(C) = \frac{1}{4}$ and $\mathcal{P}(D) = \frac{2}{7}$. Use the product rule to find:
(a) $\mathcal{P}(AB)$ (b) $\mathcal{P}(AD)$ (c) $\mathcal{P}(BC)$ (d) $\mathcal{P}(ABC)$ (e) $\mathcal{P}(BCD)$ (f) $\mathcal{P}(ABCD)$

2. A coin and a die are tossed. Use the product rule to find the probability of obtaining:
(a) a three and a head,
(b) a six and a tail,
(c) an even number and a tail,
(d) a number less than five and a head.

3. One set of cards contains the numbers 1, 2, 3, 4 and 5, and another set contains the letters A, B, C, D and E. One card is drawn at random from each set. Use the product rule to find the probability of drawing:
(a) 4 and B,
(b) 2 or 5, then D,
(c) 1, then A or B or C,
(d) an odd number and C,
(e) an even number and a vowel,
(f) a number less than 3, and E,
(g) the number 4, followed by a letter from the word MATHS.

4. Two marbles are picked at random, one from a bag containing three red and four blue marbles, and the other from a bag containing five red and two blue marbles. Find the probability of drawing:
(a) two red marbles,
(b) two blue marbles,
(c) a red marble from the first bag and a blue marble from the second.

5. A box contains five light bulbs, two of which are faulty. Two bulbs are selected, one at a time without replacement. Find the probability that:
(a) both bulbs are faulty,
(b) neither bulb is faulty,
(c) the first bulb is faulty and the second one is not,
(d) the second bulb is faulty and the first one is not.

DEVELOPMENT

6. A die is rolled twice. Using the product rule, find the probability of throwing:
(a) a six and then a five,
(b) a one and then an odd number,
(c) a double six,
(d) two numbers greater than four,
(e) a number greater than four and then a number less than four.

7. A box contains twelve red and ten green discs. Three discs are selected, one at a time without replacement.
(a) What is the probability that the discs selected are red, green, red in that order?
(b) What is the probability of this event if the disc is replaced after each draw?

8. (a) From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:
(i) a spade and then a heart,
(ii) two clubs,
(iii) a jack and then a queen,
(iv) the king of diamonds and then the ace of clubs.
(b) Repeat the question if the first card is replaced before the second card is drawn.

9. A coin is weighted so that it is twice as likely to fall heads as it is tails.
(a) Write down the probabilities that the coin falls: (i) heads, (ii) tails.
(b) If you toss the coin three times, find the probability of:
(i) three heads, (ii) three tails, (iii) head, tail, head in that order.

10. If a coin is tossed repeatedly, find the probability of obtaining at least one head in:
(a) two tosses, (b) five tosses, (c) ten tosses.

11. [Valid and invalid arguments] Identify any fallacies in the following arguments. If possible, give some indication of how to correct them.
(a) 'The probability that a Year 12 student chosen at random likes classical music is 50%, and the probability that a student plays a classical instrument is 20%. Therefore the probability that a student chosen at random likes classical music and plays a classical instrument is 10%.'
(b) 'The probability of a die showing a prime is $\frac{1}{2}$, and the probability that it shows an odd number is $\frac{1}{2}$. Hence the probability that it shows an odd prime number is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.'
(c) 'I choose a team at random from an eight-team competition. The probability that it wins any game is $\frac{1}{2}$, so the probability that it defeats all of the other seven teams is $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$.'
(d) 'A normal coin is tossed and shows heads eight times. Nevertheless, the probability that it shows heads the next time is still $\frac{1}{2}$.'

CHALLENGE

12. One layer of tinting material on a window cuts out $\frac{1}{5}$ of the sun's UV rays.
(a) What fraction would be cut out by using two layers?
(b) How many layers would be required to cut out at least $\frac{9}{10}$ of the sun's UV rays?

13. A die is rolled twice. Using the product rule, find the probability of rolling:
(a) a double two,
(b) any double,
(c) a number greater than three, then an odd number,
(d) a one and then a four,
(e) a four and then a one,
(f) a one and a four in any order,
(g) an even number, then a five,
(h) a five and then an even number,
(i) an even number and a five in any order.

14. An archer fires three shots at a bullseye. He has a 90% chance of hitting the bullseye. Using H for hit and M for miss, list all eight possible outcomes. Then, assuming that successive shots are independent, use the product rule to find the probability that he will:
(a) hit the bullseye three times,
(b) miss the bullseye three times,
(c) hit the bullseye on the first shot only,
(d) hit the bullseye exactly once,
(e) miss the bullseye on the first shot only,
(f) miss the bullseye exactly once.
[HINT: Part (d) requires adding the probabilities of HMM, MHM and MMH, and part (f) requires a similar calculation.]

15. In a lottery, the probability of the jackpot prize being won in any draw is $\frac{1}{60}$.
(a) What is the probability that the jackpot prize will be won in each of four consecutive draws?
(b) How many consecutive draws need to be made for there to be a greater than 98% chance that at least one jackpot prize will have been won?

16. [This question is best done by retelling the story of the experiment, as explained in the notes above.] Nick has five different pairs of socks to last the week, and they are scattered loose in his drawer. Each morning, he gets up before light and chooses two socks at random. Find the probability that he wears a matching pair:
(a) on the first morning,
(b) on the last morning,
(c) on the third morning,
(d) the first two mornings,
(e) every morning,
(f) every morning but one.

17. [A notoriously confusing question] In a television game show, the host shows the contestant three doors, only one of which conceals the prize, and the game proceeds as follows. First, the contestant chooses a door. Secondly, the host opens one of the other two doors, showing the contestant that it is not the prize door. Thirdly, the host invites the contestant to change her choice, if she wishes. Analyse the game, and advise the contestant what to do.

8 F Probability Tree Diagrams

In more complicated problems, and particularly in unsymmetric situations, a *probability tree diagram* can be very useful in organising the various cases, in preparation for the application of the product rule.

Constructing a Probability Tree Diagram: A probability tree diagram differs from the tree diagrams used in Section 8B for counting possible outcomes, in that the relevant probabilities are written on the branches and then multiplied together in accordance with the product rule. An example will demonstrate the method. Notice that, as before, these diagrams have one column labelled 'Start', a column for each stage, and a column listing the outcomes, but there is now an extra column labelled 'Probability' at the end.

WORKED EXERCISE:

One card is drawn from each of two packs. Use a probability tree diagram to find the probability that:

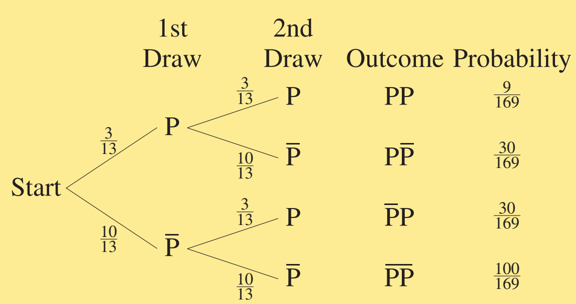
(a) both cards are picture cards,

(b) neither card is a picture card,

(c) one card is a picture card and the other is not.

SOLUTION:

In each suit, there are 12 picture cards out of 52 cards,
so $\mathcal{P}(\text{picture card}) = \frac{3}{13}$ and $\mathcal{P}(\text{not a picture card}) = \frac{10}{13}$.



Multiplying the probabilities along each arm and then adding the cases:

(a) $\mathcal{P}(\text{two picture cards}) = \frac{3}{13} \times \frac{3}{13}$
$= \frac{9}{169}$,

(b) $\mathcal{P}(\text{no picture cards}) = \frac{10}{13} \times \frac{10}{13}$
$= \frac{100}{169}$,

(c) $\mathcal{P}(\text{one picture card})$
$= \frac{3}{13} \times \frac{10}{13} + \frac{10}{13} \times \frac{3}{13}$
$= \frac{30}{169} + \frac{30}{169}$
$= \frac{60}{169}$.

NOTE: The four probabilities in the last column of the tree diagram add exactly to 1, which is a useful check on the working. The three answers also add to 1.

WORKED EXERCISE: [A more complicated experiment]
A bag contains six white marbles and four blue marbles. Three marbles are drawn in succession. At each draw, if the marble is white it is replaced, and if it is blue it is not replaced.

Find the probabilities of drawing:
(a) no blue marbles,
(b) one blue marble,
(c) two blue marbles,
(d) three blue marbles.

SOLUTION:
With the ten marbles all in the bag,

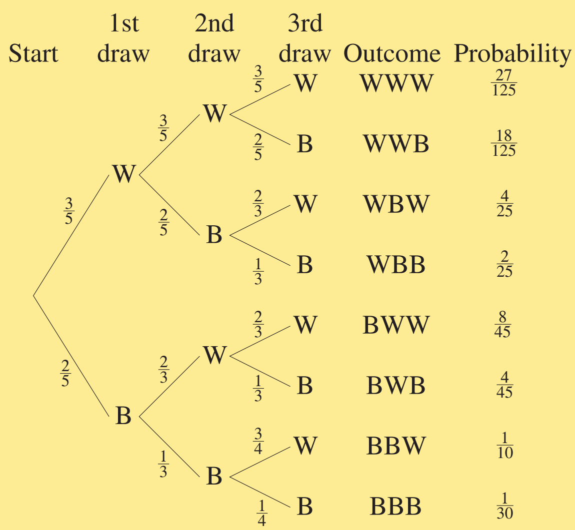
$$\mathcal{P}(\text{W}) = \tfrac{6}{10} = \tfrac{3}{5} \qquad \text{and} \qquad \mathcal{P}(\text{B}) = \tfrac{4}{10} = \tfrac{2}{5}.$$

If one blue marble is removed, there are six white and three blue marbles, so

$$\mathcal{P}(\text{W}) = \tfrac{6}{9} = \tfrac{2}{3} \qquad \text{and} \qquad \mathcal{P}(\text{B}) = \tfrac{3}{9} = \tfrac{1}{3}.$$

If two blue marbles are removed, there are six white and two blue marbles, so

$$\mathcal{P}(\text{W}) = \tfrac{6}{8} = \tfrac{3}{4} \qquad \text{and} \qquad \mathcal{P}(\text{B}) = \tfrac{2}{8} = \tfrac{1}{4}.$$



In each part, multiply the probabilities along each arm and then add the cases.

(a) $\mathcal{P}(\text{no blue marbles}) = \frac{27}{125}$.

(b) $\mathcal{P}(\text{one blue marble}) = \frac{18}{125} + \frac{4}{25} + \frac{8}{45}$
$= \frac{542}{1125}$.

(c) $\mathcal{P}(\text{two blue marbles}) = \frac{2}{25} + \frac{4}{45} + \frac{1}{10}$
$= \frac{121}{450}$.

(d) $\mathcal{P}(\text{three blue marbles}) = \frac{1}{30}$.

NOTE: Again, as a check on the working, your calculator will show that the eight probabilities in the last column of the diagram add exactly to 1 and that the four answers above also add to 1.

An Infinite Probability Tree Diagram: Some situations generate an infinite probability tree diagram. In the following, more difficult worked example, the limiting sum of a GP is used to evaluate the resulting sum.

WORKED EXERCISE:

Wes and Wilma toss a coin alternately, beginning with Wes. The first to toss heads wins. What probability of winning does Wes have?

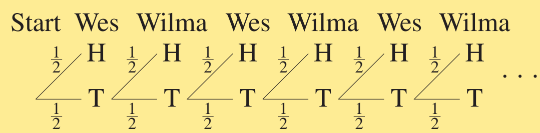
SOLUTION:

The branches on the tree diagram below terminate when a head is tossed, and the person who tosses that head wins the game. Limitations of space preclude writing in either the final outcome or the product of the probabilities!

From the diagram, $\mathcal{P}(\text{Wes wins}) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$.

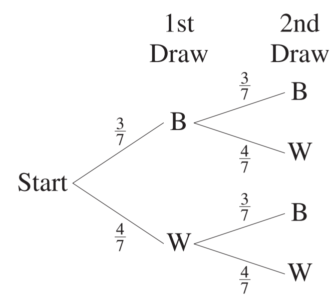
This is the limiting sum of the GP with $a = \frac{1}{2}$ and $r = \frac{1}{4}$,

so
$$\begin{aligned}\mathcal{P}(\text{Wes wins}) &= \frac{a}{1-r}\\ &= \tfrac{1}{2} \div \tfrac{3}{4}\\ &= \tfrac{2}{3}.\end{aligned}$$



Exercise 8F

1. A bag contains three black and four white discs. A disc is selected from the bag, its colour is noted, and it is then returned to the bag before a second disc is drawn.

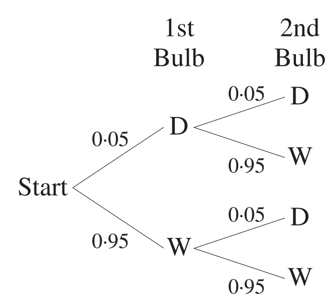


(a) By multiplying along the branches of the tree, find:
(i) $\mathcal{P}(\text{BB})$ (ii) $\mathcal{P}(\text{BW})$ (iii) $\mathcal{P}(\text{WB})$ (iv) $\mathcal{P}(\text{WW})$

(b) Hence, by adding, find the probability that:
(i) both discs drawn have the same colour,
(ii) the discs drawn have different colours.

(c) Draw your own tree diagram, and repeat part (b) if the first ball is not replaced before the second one is drawn.

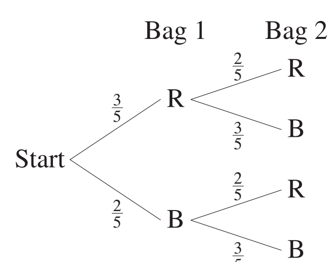
2. Two light bulbs are selected at random from a large batch of bulbs in which 5% are defective.



(a) By multiplying along the branches of the tree, find:
(i) $\mathcal{P}$(both bulbs work),
(ii) $\mathcal{P}$(the first works, but the second is defective),
(iii) $\mathcal{P}$(the first is defective, but the second works),
(iv) $\mathcal{P}$(both bulbs are defective).

(b) Hence find the probability that at least one bulb works.

3. One bag contains three red and two blue balls and another bag contains two red and three blue balls. A ball is drawn at random from each bag.



(a) By multiplying along the branches of the tree, find:
(i) $\mathcal{P}(\text{RR})$ (ii) $\mathcal{P}(\text{RB})$ (iii) $\mathcal{P}(\text{BR})$ (iv) $\mathcal{P}(\text{BB})$

(b) Hence, by adding, find the probability that:
(i) the balls have the same colour,
(ii) the balls have different colours.

4. In group A there are three girls and seven boys, and in group B there are six girls and four boys. One person is chosen at random from each group. Draw a probability tree diagram.

(a) By multiplying along the branches of the tree, find:

(i) $\mathcal{P}(\text{GG})$ (ii) $\mathcal{P}(\text{GB})$ (iii) $\mathcal{P}(\text{BG})$ (iv) $\mathcal{P}(\text{BB})$

(b) Hence, by adding, find the probability that:

(i) two of the same sex are chosen, (ii) one boy and one girl are chosen.

5. There is an 80% chance that Gary will pass his driving test and a 90% chance that Emma will pass hers. Draw a probability tree diagram, and find the chance that:

(a) Gary passes and Emma fails,

(b) Gary fails and Emma passes,

(c) only one of Gary and Emma passes,

(d) at least one fails.

DEVELOPMENT

6. The probability that a set of traffic lights will be green when you arrive at them is $\frac{3}{5}$. A motorist drives through two sets of lights. Assuming that the two sets of traffic lights are not synchronised, find the probability that:

(a) both sets of lights will be green,

(b) at least one set of lights will be green.

7. A factory assembles calculators. Each calculator requires a chip and a battery. It is known that 1% of chips and 4% of batteries are defective. Find the probability that a calculator selected at random will have at least one defective component.

8. The probability of a woman being alive at 80 years of age is 0·2, and the probability of her husband being alive at 80 years of age is 0·05. Find the probability that:

(a) they will both live to be 80 years of age,

(b) only one of them will live to be 80.

9. Alex and Julia are playing in a tennis tournament. They will play each other twice, and each has an equal chance of winning the first game. If Alex wins the first game, his probability of winning the second game is increased to 0·55. If he loses the first game, his probability of winning the second game is reduced to 0·25. Find the probability that Alex wins exactly one game.

10. One bag contains four red and three blue discs, and another bag contains two red and five blue discs. A bag is chosen at random and then a disc is drawn from it. Find the probability that the disc is blue.

11. In a raffle in which there are 200 tickets, the first prize is drawn and then the second prize is drawn without replacing the winning ticket. If you buy 15 tickets, find the probability that you win:

(a) both prizes, (b) at least one prize.

12. A box contains 10 chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Hugo randomly selects and eats three chocolates from the box. Find the probability that:

(a) the first chocolate Hugo eats is caramel,

(b) Hugo eats three mint chocolates,

(c) Hugo eats exactly one caramel chocolate.

13. In an aviary there are four canaries, five cockatoos and three budgerigars. If two birds are selected at random, find the probability that:

(a) both are canaries,

(b) neither is a canary,

(c) one is a canary and one is a cockatoo,

(d) at least one is a canary.

14. Max and Jack each throw a die. Find the probability that:

(a) they do not throw the same number,

(b) the number thrown by Max is greater than the number thrown by Jack,

(c) the numbers they throw differ by three.

15. In a large coeducational school, the population is 47% female and 53% male. Two students are selected from the school population at random. Find, correct to two decimal places, the probability that:

(a) both are male,

(b) they are of different sexes.

16. The numbers 1, 2, 3, 4 and 5 are each written on a card. The cards are shuffled and one card is drawn at random. The number is noted and the card is then returned to the pack. A second card is selected, and in this way a two-digit number is recorded. For example, a 2 on the first draw and a 3 on the second results in the number 23.

(a) What is the probability of the number 35 being recorded?

(b) What is the probability of an odd number being recorded?

17. A twenty-sided die has the numbers from 1 to 20 on its faces.

(a) If it is rolled twice, what is the probability that the same number appears on the uppermost face each time?

(b) If it is rolled three times, what is the probability that the number 15 appears on the uppermost face exactly twice?

18. Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.

19. An interviewer conducts a poll on the popularity of the prime minister in Sydney and Melbourne. In Sydney, 52% of the population approve of the prime minister, and in Melbourne her approval rating is 60%. If one of the two capital cities is selected at random and two electors are surveyed, find the probability that:

(a) both electors approve of the prime minister,

(b) at least one elector approves of the prime minister.

20. In a bag there are four green, three blue and five red discs.

(a) Two discs are drawn at random, and the first disc is not replaced before the second disc is drawn. Find the probability of drawing:

(i) two red discs

(ii) one red and one blue disc

(iii) at least one green disc

(iv) a blue disc on the first draw

(v) two discs of the same colour

(vi) two differently coloured discs

(b) Repeat part (a) if the first disc is replaced before the second disc is drawn.

21. In a game, two dice are rolled and the score given is the maximum of the two numbers on the uppermost faces. For example, if the dice show a three and a five, the score is a five.

(a) Find the probability that you score a one in a single throw of the two dice.

(b) What is the probability of scoring three consecutive ones in three rolls of the dice?

(c) Find the probability that you score a six in a single roll of the dice.

22. In each game of Sic Bo, three regular six-sided dice are thrown once.
(a) In a single game, what is the probability that all three dice show six?
(b) What is the probability that exactly two of the dice show six?
(c) What is the probability that exactly two of the dice show the same number?
(d) What is the probability of rolling three different numbers on the dice?

23. A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:
(a) both Bob's cards are jacks,
(b) at least one of Bob's cards is a jack,
(c) given that one of Bob's cards is a jack, the other one is also.

CHALLENGE

24. Shadia has invented a game for one person. She throws two dice repeatedly until the sum of the two numbers shown is either six or eight. If the sum is six, she wins. If the sum is eight, she loses. If the sum is any other number, she continues to throw until the sum is six or eight.
(a) What is the probability that she wins on the first throw?
(b) What is the probability that a second throw is needed?
(c) Find an expression for the probability that Shadia wins on her first, second or third throw.
(d) Calculate the probability that Shadia wins the game.

25. There are two white and three black discs in a bag. Two players A and B are playing a game in which they draw a disc from the bag and then replace it. Player A must draw a white disc to win and player B must draw a black disc. Player A goes first. Find the probability that:
(a) player A wins on the first draw,
(b) player B wins on her first draw,
(c) player A wins in less than four of her turns,
(d) player A wins the game.

26. A bag contains two green and two blue marbles. Marbles are drawn at random, one by one without replacement, until two green marbles have been drawn. What is the probability that exactly three draws will be required?

27. A coin is tossed continually until, for the first time, the same result appears twice in succession. That is, you continue tossing until you toss two heads or two tails in a row.
(a) Find the probability that the game ends before the sixth toss of the coin.
(b) Find the probability that an even number of tosses will be required.

8G Chapter Review Exercise

1. If a die is rolled, find the probability that the uppermost face is:
(a) a two,
(b) an odd number,
(c) a number less than two,
(d) a prime number.

2. A number is selected at random from the integers 1, 2, …, 10. Find the probability of choosing:
(a) the number three,
(b) an even number,
(c) a square number,
(d) a negative number,
(e) a number less than 20,
(f) a multiple of three.

3. From a regular pack of 52 cards, one card is drawn at random. Find the probability that the chosen card is:

(a) black,
(b) red,
(c) a queen,
(d) the ace of spades,
(e) a club or a diamond,
(f) not a seven.

4. A student has a 63% chance of passing his driving test. What is the chance that he does not pass?

5. A fair coin is tossed twice. Find the probability that the two tosses result in:

(a) two tails,
(b) a head followed by a tail,
(c) a head and a tail.

6. Two dice are thrown simultaneously. Find the probability of:

(a) a double three,
(b) a total score of five,
(c) a total greater than nine,
(d) at least one five,
(e) neither a four nor a five appearing,
(f) a two and a number greater than four,
(g) the same number on both dice,
(h) a two on at least one die.

7. In a group of 60 tourists, 31 visited Queenstown, 33 visited Strahan and 14 visited both places. Draw a Venn diagram and find the probability that a tourist:

(a) visited Queenstown only,
(b) visited Strahan only,
(c) did not visit either place.

8. A die is thrown. Let A be the event that an odd number appears. Let B be the event that a number less than five appears.

(a) Are A and B mutually exclusive?
(b) Find: (i) $\mathcal{P}(A)$ (ii) $\mathcal{P}(B)$ (iii) $\mathcal{P}(A \text{ and } B)$ (iv) $\mathcal{P}(A \text{ or } B)$

9. The events A, B and C are independent, with $\mathcal{P}(A) = \frac{1}{4}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(C) = \frac{3}{5}$. Use the product rule to find:

(a) $\mathcal{P}(AB)$
(b) $\mathcal{P}(BC)$
(c) $\mathcal{P}(AC)$
(d) $\mathcal{P}(ABC)$

10. (a) From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:

(i) a club then a diamond,
(ii) two hearts,
(iii) a seven then an ace,
(iv) the queen of hearts then the eight of diamonds.

(b) Repeat part (a) if the first card is replaced before the second card is drawn.

11. There is a 70% chance that Harold will be chosen for the boys' debating team, and an 80% chance that Grace will be chosen for the girls' team. Draw a probability tree diagram and find the chance that:

(a) Harold is chosen and Grace is not,
(b) Grace is chosen and Harold is not,
(c) only one of Harold and Grace is chosen,
(d) neither Harold nor Grace is chosen.

12. In a park there are four labradors, six German shepherds and five beagles. If two dogs are selected at random, find the probability that:

(a) both are beagles,
(b) neither is a labrador,
(c) at least one is a labrador,
(d) a beagle and German shepherd are chosen.

13. There are 500 tickets sold in a raffle. The winning ticket is drawn and then the ticket for second prize is drawn, without replacing the winning ticket. If you buy 20 tickets, find the probability that you win:

(a) both prizes,
(b) at least one prize.

14. Two dice are rolled. A five appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than nine.

Answers to Exercises

Chapter One

Exercise **1A** (Page 4)

1(a) 6 **(b)** 12 **(c)** 8 **(d)** 9 **(e)** 2 **(f)** $\frac{25}{2}$ **(g)** 6 **(h)** 20

2(a) 8 **(b)** 25 **(c)** 9 **(d)** 24 **(e)** 36 **(f)** 24 **(g)** 9 **(h)** 8

3(a) You should count approximately 133 squares. We shall see later that $\int_0^1 x^2\,dx = \frac{1}{3}$.

(b) The exact values are: **(i)** $\frac{1}{24}$ **(ii)** $\frac{7}{24}$

4(a) 15 **(b)** 15 **(c)** 25 **(d)** 40 **(e)** $\frac{25}{2}$ **(f)** 12 **(g)** 16 **(h)** 24 **(i)** 8 **(j)** 18 **(k)** 4 **(l)** 16 **(m)** 4 **(n)** 16 **(o)** $\frac{25}{2}$ **(p)** $\frac{25}{2}$

6(a) 8π **(b)** $\frac{25}{4}\pi$

7(b) $0{\cdot}79$ **(c)** $3{\cdot}16$

Exercise **1B** (Page 10)

1(a) 1 **(b)** 15 **(c)** 16 **(d)** 84 **(e)** 19 **(f)** 243 **(g)** 62 **(h)** 2 **(i)** 1

2(a)(i) 4 **(ii)** 25

(iii) 1 [NOTE: $\int_4^5 dx$ means $\int_4^5 1\,dx$.]

(b) Each function is a horizontal line, and so each integral is a rectangle.

3(a) 30 **(b)** 6 **(c)** 33 **(d)** 18 **(e)** 132 **(f)** 2 **(g)** 23 **(h)** 44 **(i)** 60

4(a) 2 **(b)** 2 **(c)** 9 **(d)** 7 **(e)** 30 **(f)** 96 **(g)** 208 **(h)** 77 **(i)** 10

5(a) $\frac{9}{2}$ **(b)** $\frac{27}{2}$ **(c)** $8\frac{2}{3}$ **(d)** $4\frac{2}{3}$ **(e)** $33\frac{3}{4}$ **(f)** $19\frac{1}{2}$ **(g)** 2 **(h)** $20\frac{5}{6}$ **(i)** 98

6(a) 24 **(b)** 18 **(c)** 4 **(d)** $7\frac{2}{3}$ **(e)** $\frac{8}{3}$ **(f)** $32\frac{2}{3}$ **(g)** 21 **(h)** 125 **(i)** $\frac{1}{4}$ **(j)** $7\frac{7}{12}$ **(k)** $\frac{8}{15}$ **(l)** $27\frac{1}{2}$

7(a) 42 **(b)** 14 **(c)** 62 **(d)** $\frac{25}{3}$ **(e)** $6\frac{2}{3}$ **(f)** 6

8(a) $\frac{1}{24}$ **(b)** $1\frac{7}{27}$ **(c)** $1\frac{5}{72}$

9(a)(i) $\frac{1}{10}$ **(ii)** $\frac{5}{36}$ **(iii)** 15 **(b)(i)** $\frac{1}{2}$ **(ii)** $\frac{15}{32}$ **(iii)** 7

10(a)(ii) 8 **(b)(ii)** 6

11(a) $1 + \frac{\pi}{2}$ **(b)** $2\frac{1}{2}$

12(a) $\frac{3}{2}$ **(b)** $\frac{5}{8}$ **(c)** $42\frac{1}{3}$

13(a) $13\frac{1}{3}$ **(b)** $52\frac{292}{405}$ **(c)** $34\frac{1}{6}$

14(a) x^2 is never negative.

(b) The function is discontinuous at $x = 0$, which lies in the given interval. Hence the use of the fundamental theorem is invalid.

Exercise **1C** (Page 17)

1(a) -4 **(b)** -9 **(c)** 15 **(d)** 0 **(e)** 27 **(f)** 9 **(g)** 2050 **(h)** $-16\frac{1}{4}$ **(i)** 0

2(a) -5 **(b)** -2 **(c)** 14 **(d)** -12 **(e)** 4 **(f)** -36 **(g)** 0 **(h)** 63 **(i)** 0 **(j)** -256 **(k)** $-2\frac{2}{3}$ **(l)** $40\frac{4}{5}$

3(a) -22 **(b)** 4 **(c)** -40 **(d)** $-\frac{2}{3}$ **(e)** $2\frac{2}{3}$ **(f)** $143\frac{3}{4}$

4(a) -8 **(b)** 40 **(c)** -14

5(a) $k = 1$ **(b)** $k = 4$ **(c)** $k = 8$ **(d)** $k = 3$ **(e)** $k = 3$ or $k = -5$ **(f)** $k = 2$ or $k = -\frac{8}{5}$

6(a)(i) 6 **(ii)** -6. The integrals are opposites because the bounds have been reversed.

(b)(i) 5 **(ii)** 5. The factor 20 can be taken out of the integral.

(c)(i) 45 **(ii)** 30 **(iii)** 15. An integral of a sum is the sum of the integrals. **(d)(i)** 48 **(ii)** 3 **(iii)** 45. The interval $0 \le x \le 2$ can be dissected into the intervals $0 \le x \le 1$ and $1 \le x \le 2$. **(e)(i)** 0 **(ii)** 0. An integral over an interval of zero width is zero.

7(a) 0. The interval has zero width.

(b) 0 The interval has zero width.

(c) 0. The integrand is odd.

(d) 0. The integrand is odd.

(e) 0. The integrand is odd.

(f) 0. The integrand is odd.

8(a) The curves meet at $(0, 0)$ and at $(1, 1)$.

(b) In the interval $0 \le x \le 1$, the curve $y = x^3$ is below the curve $y = x^2$. **(c)** $\frac{1}{4}$ and $\frac{1}{3}$

9(a) $\frac{\pi}{2} - \frac{1}{2}$ **(b)** $1 - \frac{\pi}{2}$

10(a) $1\frac{19}{32}$ **(b)** $3\frac{1}{12}$ **(c)** $\frac{1}{6}$

11(a) $-1\frac{19}{32}$ **(b)** $-3\frac{1}{12}$ **(c)** $-\frac{1}{6}$

12(a) false **(b)** true **(c)** false **(d)** false

Exercise 1D (Page 22)

1(a) $4x + C$ **(b)** $x + C$ **(c)** C **(d)** $-2x + C$
(e) $\frac{x^2}{2} + C$ **(f)** $\frac{x^3}{3} + C$ **(g)** $\frac{x^4}{4} + C$ **(h)** $\frac{x^8}{8} + C$
2(a) $x^2 + C$ **(b)** $2x^2 + C$ **(c)** $x^3 + C$ **(d)** $x^4 + C$
(e) $x^{10} + C$ **(f)** $\frac{x^4}{2} + C$ **(g)** $\frac{2x^6}{3} + C$ **(h)** $\frac{x^9}{3} + C$
3(a) $\frac{x^2}{2} + \frac{x^3}{3} + C$ **(b)** $\frac{x^5}{5} - \frac{x^4}{4} + C$
(c) $\frac{x^8}{8} + \frac{x^{11}}{11} + C$ **(d)** $x^2 + x^5 + C$ **(e)** $x^9 - 11x + C$
(f) $\frac{x^{14}}{2} + \frac{x^9}{3} + C$ **(g)** $4x - \frac{3x^2}{2} + C$
(h) $x - \frac{x^3}{3} + \frac{x^5}{5} + C$ **(i)** $x^3 - 2x^4 + \frac{7x^5}{5} + C$
4(a) $-x^{-1} + C$ **(b)** $-\frac{1}{2}x^{-2} + C$ **(c)** $-\frac{1}{7}x^{-7} + C$
(d) $-x^{-3} + C$ **(e)** $-x^{-9} + C$ **(f)** $-2x^{-5} + C$
5(a) $\frac{2}{3}x^{\frac{3}{2}} + C$ **(b)** $\frac{3}{4}x^{\frac{4}{3}} + C$ **(c)** $\frac{4}{5}x^{\frac{5}{4}} + C$ **(d)** $\frac{3}{5}x^{\frac{5}{3}} + C$
(e) $2x^{\frac{1}{2}} + C$ **(f)** $\frac{8}{3}x^{\frac{3}{2}} + C$
6(a) $\frac{1}{3}x^3 + x^2 + C$ **(b)** $2x^2 - \frac{1}{4}x^4 + C$
(c) $\frac{5}{3}x^3 - \frac{3}{4}x^4 + C$ **(d)** $\frac{1}{5}x^5 - \frac{5}{4}x^4 + C$
(e) $\frac{1}{3}x^3 - 3x^2 + 9x + C$ **(f)** $\frac{4}{3}x^3 + 2x^2 + x + C$
(g) $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C$ **(h)** $4x - 3x^3 + C$
(i) $\frac{1}{3}x^3 - \frac{1}{2}x^4 - 3x + 3x^2 + C$
7(a) $\frac{1}{2}x^2 + 2x + C$ **(b)** $\frac{1}{2}x^2 + \frac{1}{3}x^3 + C$
(c) $\frac{1}{6}x^3 - \frac{1}{16}x^4 + C$
8(a) $-\dfrac{1}{x} + C$ **(b)** $-\dfrac{1}{2x^2} + C$ **(c)** $-\dfrac{1}{4x^4} + C$
(d) $-\dfrac{1}{9x^9} + C$ **(e)** $-\dfrac{1}{x^3} + C$ **(f)** $-\dfrac{1}{x^5} + C$
(g) $-\dfrac{1}{x^7} + C$ **(h)** $-\dfrac{1}{3x} + C$ **(i)** $-\dfrac{1}{28x^4} + C$
(j) $\dfrac{1}{10x^2} + C$ **(k)** $\dfrac{1}{4x^4} - \dfrac{1}{x} + C$
(l) $-\dfrac{1}{2x^2} - \dfrac{1}{3x^3} + C$
9(a) $\frac{2}{3}x^{\frac{3}{2}} + C$ **(b)** $\frac{3}{4}x^{\frac{4}{3}} + C$ **(c)** $2\sqrt{x} + C$
(d) $\frac{3}{5}x^{\frac{5}{3}} + C$
10(a) 18 **(b)** 12 **(c)** 4 **(d)** $\frac{3}{5}$
11(a) $\frac{1}{6}(x+1)^6 + C$ **(b)** $\frac{1}{4}(x+2)^4 + C$
(c) $-\frac{1}{5}(4-x)^5 + C$ **(d)** $-\frac{1}{3}(3-x)^3 + C$
(e) $\frac{1}{15}(3x+1)^5 + C$ **(f)** $\frac{1}{32}(4x-3)^8 + C$
(g) $-\frac{1}{14}(5-2x)^7 + C$ **(h)** $-\frac{1}{40}(1-5x)^8 + C$
(i) $\frac{1}{24}(2x+9)^{12} + C$ **(j)** $\frac{3}{22}(2x-1)^{11} + C$
(k) $\frac{4}{35}(5x-4)^7 + C$ **(l)** $-\frac{7}{8}(3-2x)^4 + C$
12(a) $\frac{3}{5}(\frac{1}{3}x - 7)^5 + C$
(b) $\frac{4}{7}(\frac{1}{4}x - 7)^7 + C$ **(c)** $-\frac{5}{4}(1 - \frac{1}{5}x)^4 + C$
13(a) $-\dfrac{1}{2(x+1)^2} + C$ **(b)** $-\dfrac{1}{3(x-5)^3} + C$
(c) $-\dfrac{1}{3(3x-4)} + C$ **(d)** $\dfrac{1}{4(2-x)^4} + C$
(e) $-\dfrac{3}{5(x-7)^5} + C$ **(f)** $-\dfrac{1}{2(4x+1)^4} + C$
(g) $\dfrac{2}{15(3-5x)^3} + C$ **(h)** $\dfrac{1}{5-20x} + C$
(i) $-\dfrac{7}{96(3x+2)^4} + C$
14(a) $\frac{3}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + C$ **(b)** $\frac{1}{2}x^2 - 4x + C$
(c) $2x^2 - \frac{8}{3}x^{\frac{3}{2}} + x + C$
15(a)(i) $\frac{2}{3}$ **(ii)** 2 **(iii)** 12 **(b)(i)** $5\frac{1}{3}$ **(ii)** $96\frac{4}{5}$ **(iii)** 4
16(a) 2 **(b)** $-\frac{13}{6}$ **(c)** $12\frac{1}{6}$
17 $\displaystyle\int x^{-1}\,dx = \frac{x^0}{0} + C$ is meaningless. Chapter Three deals with the resolution of this problem.
18(a) $\frac{1}{3}(2x-1)^{\frac{3}{2}} + C$ **(b)** $-\frac{1}{6}(7-4x)^{\frac{3}{2}} + C$
(c) $\frac{3}{16}(4x-1)^{\frac{4}{3}} + C$ **(d)** $\frac{2}{3}\sqrt{3x+5} + C$
19(a) $\frac{243}{5}$ **(b)** 0 **(c)** $121\frac{1}{3}$ **(d)** 1 **(e)** $\frac{13}{6}$ **(f)** 2 **(g)** 0
(h) $\frac{112}{9}$ **(i)** $8\frac{2}{5}$
20(a)(i) $10x(x^2+1)^4$ **(ii)** $(x^2+1)^5 + C$
(b)(i) $12x^2(x^3+1)^3$ **(ii)** $(x^3+1)^4 + C$
(c)(i) $40x^4(x^5-7)^7$ **(ii)** $(x^5-7)^8 + C$
(d)(i) $6(2x+1)(x^2+x)^5$ **(ii)** $(x^2+x)^6 + C$
21(a)(i) $10x(x^2-3)^4$ **(ii)** $\frac{1}{10}(x^2-3)^5 + C$
(b)(i) $33x^2(x^3+1)^{10}$ **(ii)** $\frac{1}{33}(x^3+1)^{11} + C$
(c)(i) $28x^3(x^4+8)^6$ **(ii)** $\frac{1}{28}(x^4+8)^7 + C$
(d)(i) $6(x+1)(x^2+2x)^2$ **(ii)** $\frac{1}{6}(x^2+2x)^3 + C$

Exercise 1E (Page 29)

1(a) $4\,\text{u}^2$ **(b)** $26\,\text{u}^2$ **(c)** $81\,\text{u}^2$ **(d)** $12\,\text{u}^2$ **(e)** $9\,\text{u}^2$
(f) $6\frac{2}{3}\,\text{u}^2$ **(g)** $\frac{128}{3}\,\text{u}^2$ **(h)** $6\,\text{u}^2$ **(i)** $\frac{1}{4}\,\text{u}^2$ **(j)** $57\frac{1}{6}\,\text{u}^2$
(k) $36\,\text{u}^2$ **(l)** $60\,\text{u}^2$
2(a) $25\,\text{u}^2$ **(b)** $8\,\text{u}^2$ **(c)** $4\,\text{u}^2$ **(d)** $108\,\text{u}^2$ **(e)** $\frac{9}{2}\,\text{u}^2$
(f) $34\frac{2}{3}\,\text{u}^2$ **(g)** $18\,\text{u}^2$ **(h)** $2\,\text{u}^2$
3(a) $\frac{4}{3}\,\text{u}^2$ **(b)** $\frac{27}{2}\,\text{u}^2$ **(c)** $\frac{81}{4}\,\text{u}^2$ **(d)** $46\frac{2}{5}\,\text{u}^2$
4(a) $\frac{9}{2}\,\text{u}^2$ **(b)** $\frac{4}{3}\,\text{u}^2$ **(c)** $\frac{45}{4}\,\text{u}^2$ **(d)** $9\,\text{u}^2$
5(b) $4\frac{1}{2}\,\text{u}^2$ **(c)** $2\,\text{u}^2$ **(d)** $6\frac{1}{2}\,\text{u}^2$ **(e)** $2\frac{1}{2}$. This is the area above the x-axis minus the area below it.
6(b) $10\frac{2}{3}\,\text{u}^2$ **(c)** $2\frac{1}{3}\,\text{u}^2$ **(d)** $13\,\text{u}^2$ **(e)** $-8\frac{1}{3}$. This is the area above the x-axis minus the area below it.
7(b) $2\frac{2}{3}\,\text{u}^2$ **(c)** $\frac{5}{12}\,\text{u}^2$ **(d)** $3\frac{1}{12}\,\text{u}^2$ **(e)** $-2\frac{1}{4}$. This is the area above the x-axis minus the area below it.
8(a) $11\frac{2}{3}\,\text{u}^2$ **(b)** $128\frac{1}{2}\,\text{u}^2$ **(c)** $4\,\text{u}^2$ **(d)** $8\frac{1}{2}\,\text{u}^2$
(e) $32\frac{3}{4}\,\text{u}^2$ **(f)** $11\frac{1}{3}\,\text{u}^2$ **(g)** $6\frac{3}{4}\,\text{u}^2$ **(h)** $466\frac{13}{15}\,\text{u}^2$
9(a) $13\,\text{u}^2$ **(b)** $2\frac{1}{2}\,\text{u}^2$ **(c)** $9\frac{1}{3}\,\text{u}^2$ **(d)** $7\frac{1}{3}\,\text{u}^2$
10(a)(i) $64\,\text{u}^2$ **(ii)** $128\,\text{u}^2$ **(iii)** $64\frac{4}{5}\,\text{u}^2$
(b)(i) $50\,\text{u}^2$ **(ii)** $18\,\text{u}^2$ **(iii)** $\frac{32}{3}\,\text{u}^2$
11 $8\,\text{u}^2$
12(a) $(2,0)$, $(0,4\sqrt{2})$, $(0,-4\sqrt{2})$ **(b)** $\frac{16}{3}\sqrt{2}\,\text{u}^2$
13(a) $y = \frac{1}{3}x^3 - 2x^2 + 3x$ **(b)** The curve passes through the origin, $(1, 1\frac{1}{3})$ is a maximum turning point and $(3, 0)$ is a minimum turning point.

(c) $\frac{4}{3}\,\mathrm{u}^2$

Exercise 1F (Page 36)

1(a) $\frac{1}{6}\,\mathrm{u}^2$ **(b)** $\frac{1}{4}\,\mathrm{u}^2$ **(c)** $\frac{3}{10}\,\mathrm{u}^2$ **(d)** $\frac{1}{12}\,\mathrm{u}^2$ **(e)** $\frac{2}{35}\,\mathrm{u}^2$ **(f)** $20\frac{5}{6}\,\mathrm{u}^2$ **(g)** $36\,\mathrm{u}^2$ **(h)** $20\frac{5}{6}\,\mathrm{u}^2$
2(a) $\frac{4}{3}\,\mathrm{u}^2$ **(b)** $\frac{1}{6}\,\mathrm{u}^2$ **(c)** $\frac{4}{3}\,\mathrm{u}^2$ **(d)** $4\frac{1}{2}\,\mathrm{u}^2$
3(a) $16\frac{2}{3}\,\mathrm{u}^2$ **(b)** $9\frac{1}{3}\,\mathrm{u}^2$
4(a) $5\frac{1}{3}\,\mathrm{u}^2$ **(b)** $\frac{9}{4}\,\mathrm{u}^2$
5(c) $4\frac{1}{2}\,\mathrm{u}^2$
6(c) $\frac{4}{3}\,\mathrm{u}^2$
7(c) $36\,\mathrm{u}^2$
8(a) $\frac{4}{15}\,\mathrm{u}^2$ **(b)** $\frac{1}{32}\,\mathrm{u}^2$ **(c)** $20\frac{5}{6}\,\mathrm{u}^2$ **(d)** $57\frac{1}{6}\,\mathrm{u}^2$
9(c) $36\,\mathrm{u}^2$
10(c) $\frac{4}{3}\,\mathrm{u}^2$
11(a) $4\frac{1}{2}\,\mathrm{u}^2$ **(b)** $20\frac{5}{6}\,\mathrm{u}^2$ **(c)** $21\frac{1}{3}\,\mathrm{u}^2$ **(d)** $\frac{1}{2}\,\mathrm{u}^2$
12 $5\frac{5}{8}\,\mathrm{u}^2$
13(c) $\frac{1}{3}\,\mathrm{u}^2$
14(b) $y = x - 2$ **(c)** $5\frac{1}{3}\,\mathrm{u}^2$
15(c) $108\,\mathrm{u}^2$

Exercise 1G (Page 43)

1(a) x^2 **(b)** x^4 **(c)** $9x^4$ **(d)** $4x^{10}$ **(e)** $x^2 - 2x + 1$ **(f)** $x^2 + 10x + 25$ **(g)** $4x^2 - 12x + 9$ **(h)** x **(i)** $x - 4$
2(a) $16\pi\,\mathrm{u}^3$ **(b)** $75\pi\,\mathrm{u}^3$ **(c)** $9\pi\,\mathrm{u}^3$ **(d)** $81\pi\,\mathrm{u}^3$ **(e)** $\frac{32\pi}{5}\,\mathrm{u}^3$ **(f)** $\frac{\pi}{7}\,\mathrm{u}^3$ **(g)** $6\pi\,\mathrm{u}^3$ **(h)** $\frac{16\pi}{3}\,\mathrm{u}^3$ **(i)** $36\pi\,\mathrm{u}^3$ **(j)** $21\pi\,\mathrm{u}^3$ **(k)** $9\pi\,\mathrm{u}^3$ **(l)** $16\pi\,\mathrm{u}^3$
3(a) $3\pi\,\mathrm{u}^3$ **(b)** $20\pi\,\mathrm{u}^3$ **(c)** $\frac{256\pi}{3}\,\mathrm{u}^3$ **(d)** $57\pi\,\mathrm{u}^3$ **(e)** $\frac{3093\pi}{5}\,\mathrm{u}^3$ **(f)** $\frac{243\pi}{5}\,\mathrm{u}^3$ **(g)** $\frac{\pi}{2}\,\mathrm{u}^3$ **(h)** $\frac{256\pi}{3}\,\mathrm{u}^3$ **(i)** $\frac{16\pi}{3}\,\mathrm{u}^3$ **(j)** $\frac{8\pi}{3}\,\mathrm{u}^3$ **(k)** $31\pi\,\mathrm{u}^3$ **(l)** $\frac{16\pi}{15}\,\mathrm{u}^3$
4(b) $81\pi\,\mathrm{u}^3$
5(b) $36\pi\,\mathrm{u}^3$
6(a) $192\pi\,\mathrm{u}^3$ **(b)** $\frac{242\pi}{5}\,\mathrm{u}^3$ **(c)** $\frac{125\pi}{2}\,\mathrm{u}^3$ **(d)** $896\pi\,\mathrm{u}^3$
7(a) $\frac{26\pi}{3}\,\mathrm{u}^3$ **(b)** $\frac{211\pi}{5}\,\mathrm{u}^3$ **(c)** $\frac{25\pi}{2}\,\mathrm{u}^3$ **(d)** $624\pi\,\mathrm{u}^3$
8(a) $\frac{296\pi}{3}\,\mathrm{u}^3$ **(b)** $\frac{13\pi}{3}\,\mathrm{u}^3$ **(c)** $\frac{625\pi}{6}\,\mathrm{u}^3$ **(d)** $\frac{16\pi}{105}\,\mathrm{u}^3$
9(a) $\frac{\pi}{3}\,\mathrm{u}^3$ **(b)** $\frac{28\pi}{15}\,\mathrm{u}^3$ **(c)** $\frac{81\pi}{10}\,\mathrm{u}^3$ **(d)** $\frac{\pi}{2}\,\mathrm{u}^3$
10 $\frac{2048\pi}{3}\,\mathrm{u}^3$
11 $\frac{26\,352\pi}{5}\,\mathrm{u}^3$
12(c) $\frac{2\pi}{35}\,\mathrm{u}^3$
13(d) $\frac{64\pi}{3}\,\mathrm{u}^3$
14 $2\pi a^3\,\mathrm{u}^3$
16(a) $x \le 9,\ y \ge 0$ **(c)** $18\,\mathrm{u}^2$ **(d)(i)** $\frac{81\pi}{2}\,\mathrm{u}^3$ **(ii)** $\frac{648\pi}{5}\,\mathrm{u}^3$
17(b) $\frac{1024\pi}{5}\,\mathrm{u}^3$ **(c)** $256\pi\,\mathrm{u}^3$ **(d)** $128\pi\,\mathrm{u}^3$ **(e)** $128\pi\,\mathrm{u}^3$
18(a) $\frac{32\pi}{5}\,\mathrm{u}^3,\ 8\pi\,\mathrm{u}^3$ **(b)** $\frac{50\pi}{3}\,\mathrm{u}^3,\ \frac{5\pi}{3}\,\mathrm{u}^3$ **(c)** $8\pi\,\mathrm{u}^3,\ \frac{128\pi}{5}\,\mathrm{u}^3$ **(d)** $\frac{24\pi}{5}\,\mathrm{u}^3,\ \frac{\pi}{2}\,\mathrm{u}^3$
19(a) The curves intersect at $(0,0)$ and $(1,1)$. **(b)(i)** $\frac{3\pi}{10}\,\mathrm{u}^3$ **(ii)** $\frac{3\pi}{10}\,\mathrm{u}^3$ **(c)** The curves are symmetrical about the line $y = x$.

Exercise 1H (Page 49)

1(a) 40 **(b)** 22 **(c)** -26
2(a) $64,\ 100$ **(b)** 164
3(a) $12\frac{1}{2},\ 17\frac{1}{2}$ **(b)** 30
4(a) The curve is concave up, so the chord is above the curve, and the area under the chord will be greater than the area under the curve.
(b) The curve is concave down, so the chord is underneath the curve, and the area under the chord will be less than the area under the curve.
5(b)(i) $1\frac{1}{2}$ **(ii)** $3\frac{1}{2}$ **(iii)** $3\frac{1}{2}$ **(iv)** $1\frac{1}{2}$ **(c)** 10 **(d)** $10\frac{2}{3}$, the curve is concave down. **(e)** $6\frac{1}{4}\%$
6(b) $10\frac{1}{10}$ **(c)** y'' is positive in the interval $1 \le x \le 5$, so the curve is concave up.
7(b) $24{\cdot}7$ **(c)** $24\frac{2}{3}$. y'' is negative in the interval $9 \le x \le 16$, so the curve is concave down.
8(a) $0{\cdot}729$ **(b)** $4{\cdot}5$ **(c)** $3{\cdot}388$ **(d)** $36{\cdot}974$
9(a) $1{\cdot}117$ **(b)** $0{\cdot}705$ **(c)** $22{\cdot}925$ **(d)** $0{\cdot}167$
10 $9{\cdot}2$ metres
11 $550\,\mathrm{m}^2$
12(a) $4\pi\int_0^3 x^2\,dx$ **(b)** $38\pi\,\mathrm{u}^3$ **(c)** $36\pi\,\mathrm{u}^3,\ 5\frac{5}{9}\%$
13(a) $\pi\int_1^3 2^{2x+2}\,dx$ **(b)** $180\pi\,\mathrm{u}^3$

Exercise 1I (Page 53)

1(a) 11 **(b)** $7{\cdot}1$ **(c)** -9
2(a) $92,\ 150$ **(b)** 242
3(a) $30{\cdot}8,\ 42{\cdot}8$ **(b)** $73{\cdot}6$
4(a) $2,\ \frac{4}{3},\ 1,\ \frac{4}{5},\ \frac{2}{3}$ **(b)** $\frac{25}{18}$ **(c)** $\frac{73}{90}$ **(d)** $\frac{11}{5}$
5(b) $2{\cdot}80$
6(b) $14{\cdot}137$ **(d)(i)** $12{\cdot}294$ **(ii)** $13{\cdot}392$
7(b) $\frac{32}{3}$ **(c)** $\frac{32}{3}$. Simpson's rule gives the exact value for quadratic functions.
8(a) $\frac{7}{15}$ **(b)** $\frac{22}{9}$
9(a) $7{\cdot}740$ **(b)** $0{\cdot}9376$ **(c)** 660
10 $6\frac{19}{30}$ metres
11 $613\frac{1}{3}\,\mathrm{m}^2$
12(a) $0{\cdot}7709$ **(b)** $3{\cdot}084$
13(a) $\pi\int_1^3 3^{2x-2}\,dx$ **(b)** $115{\cdot}19\,\mathrm{u}^3$

Review Exercise **1J** (Page 55)

1(a) 1 **(b)** $\frac{3}{2}$ **(c)** 609 **(d)** $\frac{2}{5}$ **(e)** -12 **(f)** $8\frac{2}{3}$ **(g)** 8 **(h)** -10 **(i)** $21\frac{1}{3}$

2(a) $4\frac{2}{3}$ **(b)** $-1\frac{2}{3}$ **(c)** $\frac{1}{3}$

3(a) $-1\frac{1}{2}$ **(b)** 15 **(c)** $-6\frac{1}{6}$

4(a)(ii) $k = 6$ **(b)(ii)** $k = 3$

5(a) 0. The integral has zero width. **(b)** 0. The integrand is odd. **(c)** 0. The integrand is odd.

6(a) 8 **(b)** $\frac{3}{2}$

7(a) $\frac{x^2}{2} + 2x + C$ **(b)** $\frac{x^4}{4} + x^3 - \frac{5x^2}{2} + x + C$ **(c)** $\frac{x^3}{3} - \frac{x^2}{2} + C$ **(d)** $-\frac{x^3}{3} + \frac{5x^2}{2} - 6x + C$ **(e)** $-x^{-1} + C$ **(f)** $-\frac{x^{-6}}{6} + C$ **(g)** $\frac{2x^{\frac{3}{2}}}{3} + C$ **(h)** $\frac{1}{5}(x+1)^5 + C$ **(i)** $\frac{1}{12}(2x-3)^6 + C$

8(a) $9\frac{1}{3}\,\text{u}^2$ **(b)** $4\,\text{u}^2$ **(c)** $\frac{4}{3}\,\text{u}^2$ **(d)** $1\,\text{u}^2$ **(e)** $\frac{1}{6}\,\text{u}^2$ **(f)** $\frac{4}{15}\,\text{u}^2$ **(g)** $\frac{1}{6}\,\text{u}^2$ **(h)** $4\frac{1}{2}\,\text{u}^2$

9(a) $\frac{32\pi}{5}\,\text{u}^3$ **(b)** $\frac{32\pi}{3}\,\text{u}^3$ **(c)** $\frac{37\pi}{3}\,\text{u}^3$

10(a) $\frac{\pi}{5}\,\text{u}^3$ **(b)** $\frac{64\pi}{3}\,\text{u}^3$ **(c)** $\frac{9\pi}{2}\,\text{u}^3$

11(b) $\frac{4}{3}\,\text{u}^2$

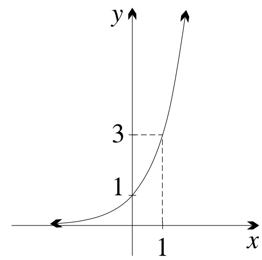
12(b) $\frac{2\pi}{9}\,\text{u}^3$

13(a) 9 **(b)** $0{\cdot}563$

Chapter Two

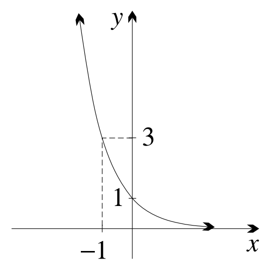
Exercise **2A** (Page 63)

1(b)



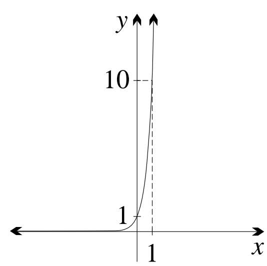
(c) domain: all real numbers, range: $y > 0$

2(b)



(c) domain: all real numbers, range: $y > 0$

3(b)



(c) domain: all real numbers, range: $y > 0$

4(a) 1 **(b)** $\frac{1}{2}$ **(c)** $1{\cdot}41$ **(d)** $2{\cdot}83$ **(e)** $0{\cdot}35$ **(f)** $1{\cdot}62$

5(a) 2^{10} **(b)** 6^7 **(c)** 10×3^5 **(d)** 8×5^{10}

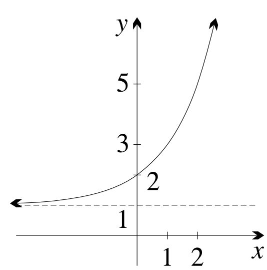
6(a) 2^2 **(b)** 6 **(c)** 5^{-6} **(d)** 10^{-3} **(e)** 2×6^3 **(f)** 3×5^{-4}

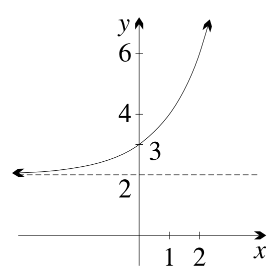
7(a) 1 **(b)** 1 **(c)** 1 **(d)** 1

8(a) $\frac{1}{2}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{1}{p}$ **(d)** $\frac{1}{x}$ **(e)** $\frac{1}{4}$ **(f)** $\frac{1}{9}$ **(g)** $\frac{1}{p^2}$ **(h)** $\frac{1}{x^2}$ **(i)** $\frac{1}{2^x}$ **(j)** $\frac{1}{3^x}$ **(k)** $\frac{1}{5^{2x}}$ **(l)** $\frac{1}{p^{2x}}$

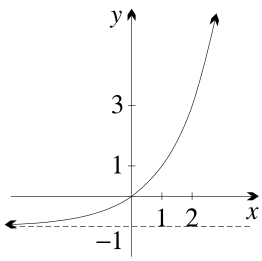
9(a) 4 **(b)** 5 **(c)** 27 **(d)** 8 **(e)** $\frac{1}{2}$ **(f)** $\frac{1}{3}$ **(g)** $\frac{1}{32}$ **(h)** $\frac{1}{27}$

10(a) $y > 1$ **(b)** $y > 2$

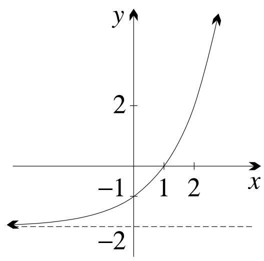




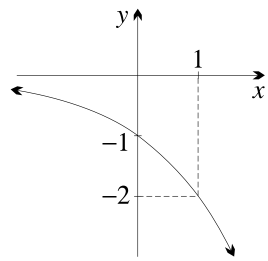
(c) $y > -1$



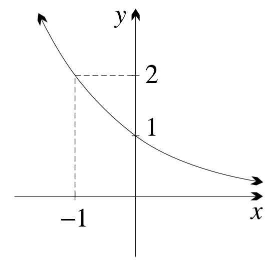
(d) $y > -2$



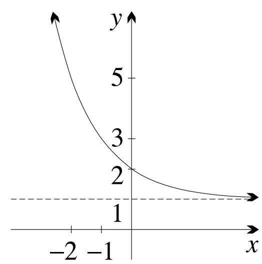
(e)



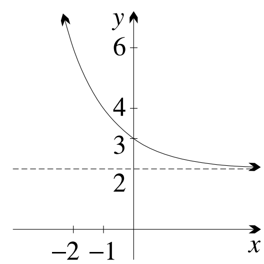
(f)



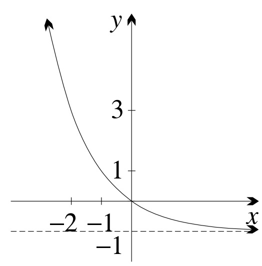
11(a) $y > 1$



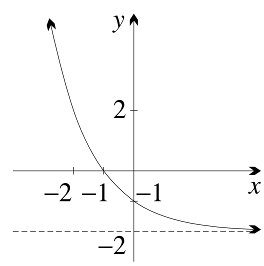
(b) $y > 2$



(c) $y > -1$



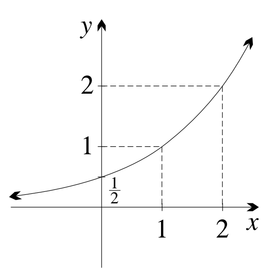
(d) $y > -2$



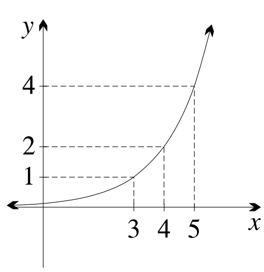
12(a) 3^{3x} **(b)** 2^{9x} **(c)** 10^{11x} **(d)** 14×3^{4x} **(e)** 20×7^{8x} **(f)** 5^{2x} **(g)** 2^{3x} **(h)** 3^{3-x} **(i)** 6^{x-1}

13(a) 3^x **(b)** 2^{2x} **(c)** 7^{2x} **(d)** 4×3^{3x} **(e)** 8×7^{-3x} **(f)** 3^{-x} **(g)** 4^{x-3} **(h)** 2^{x-1} **(i)** 5^{x+3}

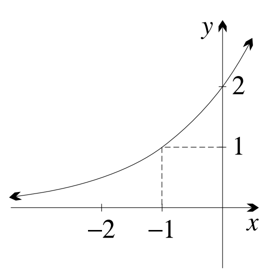
14(a)



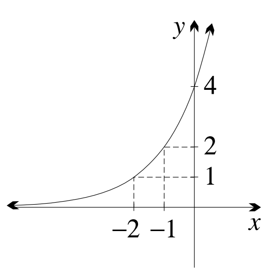
(b)



(c)



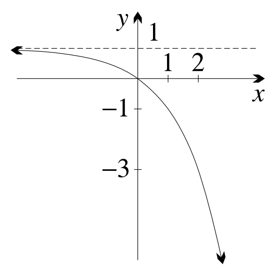
(d)



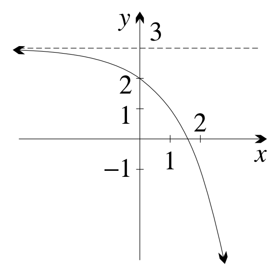
15 The graph of $y = a^x$ becomes steeper as a increases.

17 $0{\cdot}4771$

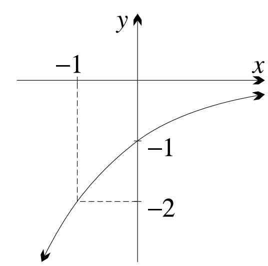
18(a) $y < 1$



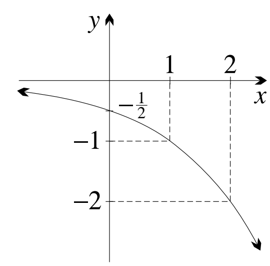
(b) $y < 3$



(c) $y < 0$



(d) $y < 0$



19(a) 2^{2x} **(b)** 2^{5x} **(c)** 2^{5x} **(d)** 2^{7x} **(e)** 2^{4x} **(f)** 2^{2x}

20(a) $4 \times 3^{\frac{1}{3}x}$ **(b)** 27×2^{9x} **(c)** $\frac{1}{216} \times 5^{-6x}$ **(d)** $\frac{1}{25} \times 3^{-8x}$

Exercise **2B** (Page 70)

1(a) $7{\cdot}3891$ **(b)** $20{\cdot}0855$ **(c)** $22\,026{\cdot}4658$ **(d)** $1{\cdot}0000$ **(e)** $2{\cdot}7183$ **(f)** $0{\cdot}3679$ **(g)** $0{\cdot}1353$ **(h)** $1{\cdot}6487$ **(i)** $0{\cdot}6065$

2(b) Both are equal to 1.

(c)

height y	$\frac{1}{2}$	1	2	3
gradient y'	$\frac{1}{2}$	1	2	3
$\dfrac{\text{gradient}}{\text{height}}$	1	1	1	1

(d) They are all equal to 1.

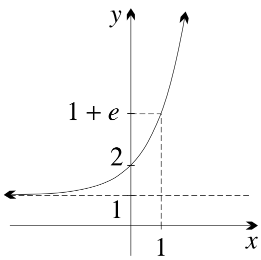
3(b) Both are equal to 1. **(c)** The values are: $0{\cdot}14$, $0{\cdot}37$, $2{\cdot}72$. **(d)** The x-intercept is always 1 unit to the left of the point of contact.

4(b)

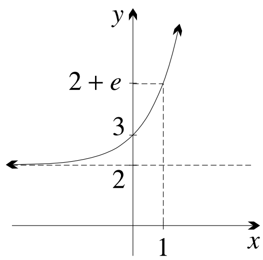
x	-2	-1	0	1	2
height y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
gradient y'	$0{\cdot}17$	$0{\cdot}35$	$0{\cdot}69$	$1{\cdot}39$	$2{\cdot}77$
$\dfrac{\text{gradient}}{\text{height}}$	$0{\cdot}69$	$0{\cdot}69$	$0{\cdot}69$	$0{\cdot}69$	$0{\cdot}69$

(c) They are all about $0{\cdot}69$.

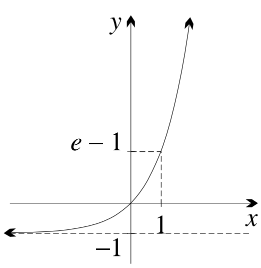
5(a) $y > 1$



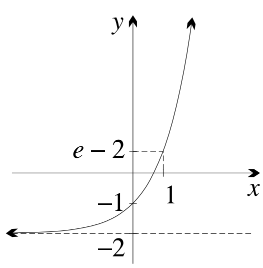
(b) $y > 2$



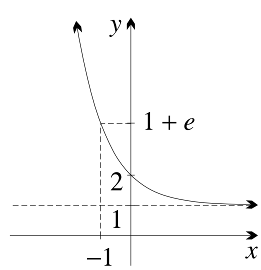
(c) $y > -1$



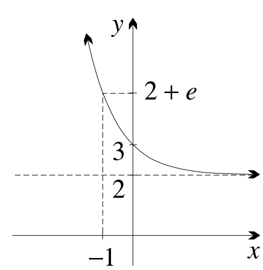
(d) $y > -2$



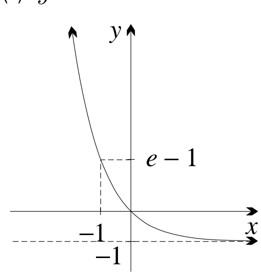
6(a) $y > 1$



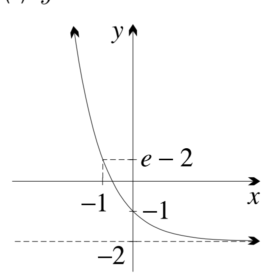
(b) $y > 2$



(c) $y > -1$



(d) $y > -2$



7(a) 1 **(b)** 1 **(c)** $y = x + 1$, -1

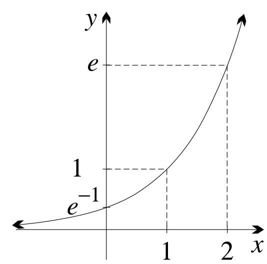
(d) For $x = -2$: $y = e^{-2}(x + 3)$, -3; for $x = -1$: $y = e^{-1}(x + 2)$, -2; for $x = 1$: $y = ex$, 0

(e) They are the same.

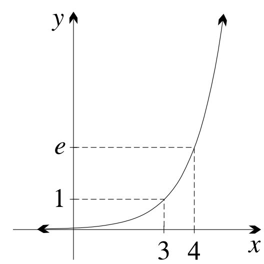
8(a) $y > 0$ **(b)** $y' = e^x > 0$ **(c)** $y'' = e^x > 0$

9(c) The gradient of $y = e^x$ is 1 at its y-intercept. The graph of $y = e^{-x}$ is obtained by reflecting the first graph in the y-axis. Hence its tangent has gradient -1, and the two are perpendicular.

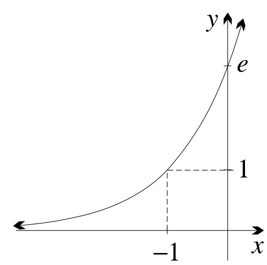
10(a)



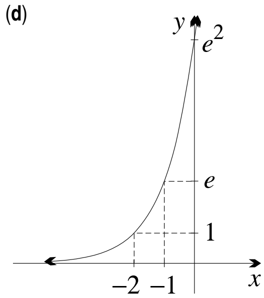
(b)



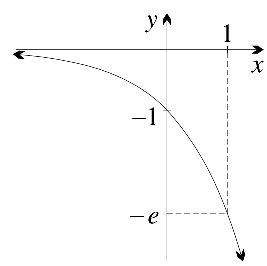
(c)



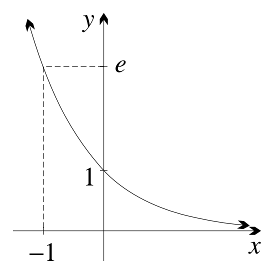
(d)



(e)

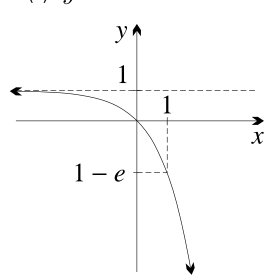


(f)

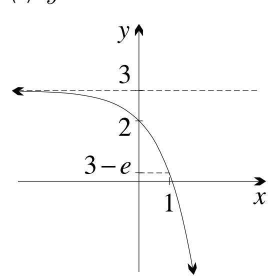


11(a) From left to right: $\frac{1}{2}$, 1 **(c)** $0{\cdot}69$

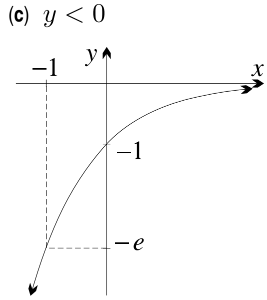
15(a) $y < 1$



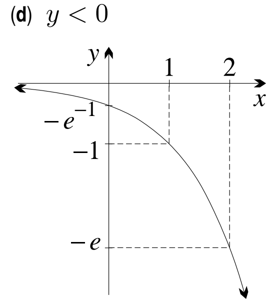
(b) $y < 3$



(c) $y < 0$



(d) $y < 0$



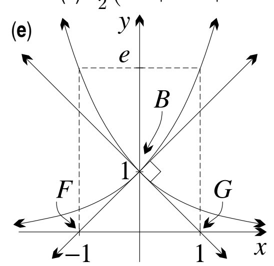
Exercise **2C** (Page 76)

1(a) $2\,e^{2x}$ **(b)** $7\,e^{7x}$ **(c)** $12\,e^{3x}$ **(d)** $-e^{-x}$ **(e)** $-5\,e^{5x}$
(f) $-10\,e^{2x}$ **(g)** $\frac{1}{2}\,e^{\frac{1}{2}x}$ **(h)** $2\,e^{\frac{1}{3}x}$ **(i)** $7\,e^{-7x}$
(j) $-\frac{1}{3}e^{-\frac{1}{3}x}$ **(k)** $e^{\frac{1}{5}x}$ **(l)** e^{-2x}
2(a) $y' = e^{x+2}$ **(b)** $y' = e^{x-3}$ **(c)** $y' = 3\,e^{3x+4}$
(d) $y' = 5\,e^{5x+1}$ **(e)** $y' = 2\,e^{2x-1}$ **(f)** $y' = 4\,e^{4x-3}$
(g) $y' = -4\,e^{-4x+1}$ **(h)** $y' = -3\,e^{-3x+4}$
(i) $y' = -2\,e^{-2x-7}$ **(j)** $y' = -3\,e^{-3x-6}$
(k) $y' = e^{\frac{1}{2}x+4}$ **(l)** $y' = e^{3x-2}$
3(a) $e^x - e^{-x}$ **(b)** $2\,e^{2x} + 3\,e^{-3x}$ **(c)** $\frac{e^x + e^{-x}}{2}$
(d) $\frac{e^x - e^{-x}}{3}$ **(e)** $e^{2x} + e^{3x}$ **(f)** $e^{4x} + e^{5x}$
4(a) $y' = 3\,e^{3x}$ **(b)** $y' = 2\,e^{2x}$ **(c)** $y' = 2\,e^{2x}$
(d) $y' = 6\,e^{6x}$ **(e)** $y' = 3\,e^{3x}$ **(f)** $y' = -e^{-x}$
(g) $y' = -3\,e^{-3x}$ **(h)** $y' = -5\,e^{-5x}$
5(a) $y' = e^x$, $y'(2) = e^2 \doteqdot 7{\cdot}39$
(b) $y' = 3\,e^{3x}$, $y'(2) = 3\,e^6 \doteqdot 1210{\cdot}29$
(c) $y' = -e^{-x}$, $y'(2) = -e^{-2} \doteqdot -0{\cdot}14$
(d) $y' = -2\,e^{-2x}$, $y'(2) = -2\,e^{-4} \doteqdot -0{\cdot}04$
(e) $y' = \frac{1}{2}\,e^{\frac{1}{2}x}$, $y'(2) = \frac{1}{2}e \doteqdot 1{\cdot}36$
(f) $y' = \frac{3}{2}\,e^{\frac{3}{2}x}$, $y'(2) = \frac{3}{2}\,e^3 \doteqdot 30{\cdot}13$
6(a)(i) $-e^{-x}$ **(ii)** e^{-x} **(iii)** $-e^{-x}$ **(iv)** e^{-x}
(b) Successive derivatives alternate in sign.
(More precisely, $f^{(n)}(x) = \begin{cases} e^{-x}, & \text{if } n \text{ is even,} \\ -e^{-x}, & \text{if } n \text{ is odd.)} \end{cases}$
7(a)(i) $2\,e^{2x}$ **(ii)** $4\,e^{2x}$ **(iii)** $8\,e^{2x}$ **(iv)** $16\,e^{2x}$
(b) Each derivative is twice the previous one.
(More precisely, $f^{(n)}(x) = 2^n\,e^{2x}$.)
8(a) $2\,e^{2x} + e^x$ **(b)** $2\,e^{2x} - e^x$ **(c)** $e^{-x} - 4\,e^{-2x}$
(d) $2\,e^{2x} + 2\,e^x$ **(e)** $2\,e^{2x} + 6\,e^x$ **(f)** $2\,e^{2x} - 2\,e^x$
(g) $2\,e^{2x} - 4\,e^x$ **(h)** $2(e^{2x} + e^{-2x})$
(i) $10(e^{10x} + e^{-10x})$
9(a) $2\,e^{2x+1}$ **(b)** $3\,e^{3x-5}$ **(c)** $-2\,e^{1-2x}$ **(d)** $-5\,e^{3-5x}$
(e) $2x\,e^{x^2}$ **(f)** $-2x\,e^{-x^2}$ **(g)** $2x\,e^{x^2+1}$
(h) $-2x\,e^{1-x^2}$ **(i)** $2(x+1)\,e^{x^2+2x}$
(j) $(1-2x)\,e^{6+x-x^2}$ **(k)** $(3x-1)\,e^{3x^2-2x+1}$
(l) $(x-1)\,e^{2x^2-4x+3}$
10(a) $(x+1)\,e^x$ **(b)** $(1-x)\,e^{-x}$ **(c)** xe^x
(d) $(3x+4)\,e^{3x-4}$ **(e)** $(2x-x^2)\,e^{-x}$ **(f)** $4x\,e^{2x}$
(g) $(x^2+2x-5)\,e^x$ **(h)** $x^2 e^{2x}(3+2x)$
11(a) $y' = \frac{x-1}{x^2}\,e^x$ **(b)** $y' = (1-x)\,e^{-x}$
(c) $y' = \frac{(x-2)\,e^x}{x^3}$ **(d)** $y' = (2x-x^2)\,e^{-x}$
(e) $y' = \frac{x}{(x+1)^2}\,e^x$ **(f)** $y' = -x\,e^{-x}$
(g) $y' = (7-2x)\,e^{-2x}$ **(h)** $y' = (x^2-2x-1)\,e^{-x}$
12(a) $2\,e^{2x} + 3\,e^x$ **(b)** $4\,e^{4x} + 2\,e^{2x}$
(c) $-2\,e^{-2x} - 6\,e^{-x}$ **(d)** $-6\,e^{-6x} + 18\,e^{-3x}$
(e) $3\,e^{3x} + 2\,e^{2x} + e^x$ **(f)** $12\,e^{3x} + 2\,e^{2x} + e^{-x}$
13(a) $-5\,e^x(1-e^x)^4$ **(b)** $16\,e^{4x}(e^{4x}-9)^3$
(c) $-\frac{e^x}{(e^x-1)^2}$ **(d)** $-\frac{6e^{3x}}{(e^{3x}+4)^3}$
15(a) $f'(x) = 2\,e^{2x+1}$, $f'(0) = 2e$, $f''(x) = 4\,e^{2x+1}$,
$f''(0) = 4e$ **(b)** $f'(x) = -3\,e^{-3x}$, $f'(1) = -3\,e^{-3}$,
$f''(x) = 9\,e^{-3x}$, $f''(1) = 9\,e^{-3}$
(c) $f'(x) = (1-x)\,e^{-x}$, $f'(2) = -e^{-2}$,
$f''(x) = (x-2)\,e^{-x}$, $f''(2) = 0$
(d) $f'(x) = -2x\,e^{-x^2}$, $f'(0) = 0$,
$f''(x) = (4x^2-2)\,e^{-x^2}$, $f''(0) = -2$
16(a) 1, 45° **(b)** e, $69^\circ 48'$ **(c)** $e{-}2$, $7^\circ 42'$
(d) e^5, $89^\circ 37'$
17(a) $y' = ae^{ax}$ **(b)** $y' = -ke^{-kx}$ **(c)** $y' = Ak\,e^{kx}$
(d) $y' = -B\ell\,e^{-\ell x}$ **(e)** $y' = p\,e^{px+q}$
(f) $y' = pCe^{px+q}$ **(g)** $y' = \frac{pe^{px} - qe^{-qx}}{r}$
(h) $e^{ax} - e^{-px}$
18(a) $3\,e^x(e^x + 1)^2$ **(b)** $4(e^x - e^{-x})(e^x + e^{-x})^3$
(c) $(1+2x+x^2)\,e^{1+x} = (1+x)^2\,e^{1+x}$
(d) $(2x^2-1)\,e^{2x-1}$ **(e)** $\frac{e^x}{(e^x+1)^2}$
(f) $-\frac{2\,e^x}{(e^x-1)^2}$
19(a) $y' = -e^{-x}$ **(b)** $y' = e^x$ **(c)** $y' = e^{-x} - 4\,e^{-2x}$
(d) $y' = -12\,e^{-4x} - 3\,e^{-3x}$ **(e)** $y' = e^x - 9\,e^{3x}$
(f) $y' = -2\,e^{-x} - 2\,e^{-2x}$
20(a) $y' = \frac{1}{2}\sqrt{e^x}$ **(b)** $y' = \frac{1}{3}\sqrt{e^x}$ **(c)** $y' = -\frac{1}{2\sqrt{e^x}}$
(d) $y' = -\frac{1}{3\sqrt[3]{e^x}}$
21(a) $\frac{1}{2\sqrt{x}}\,e^{\sqrt{x}}$ **(b)** $-\frac{1}{2\sqrt{x}}\,e^{-\sqrt{x}}$ **(c)** $-\frac{1}{x^2}\,e^{\frac{1}{x}}$
(d) $\frac{1}{x^2}\,e^{-\frac{1}{x}}$ **(e)** $(1 + \frac{1}{x^2})\,e^{x-\frac{1}{x}}$ **(f)** $\frac{2}{x^3}\,e^{3-\frac{1}{x^2}}$
24(a) -5 or 2 **(b)** $-\frac{1}{2}\left(1+\sqrt{5}\right)$ or $-\frac{1}{2}\left(1-\sqrt{5}\right)$

Exercise **2D** (Page 82)

1(a) e **(b)** $y = ex$
2(a) 1 **(b)** $y = x + 1$
3(a) $\frac{1}{e}$ **(b)** $y = \frac{1}{e}(x+2)$
4(a) $A = (\frac{1}{2}, 1)$ **(b)** $y' = 2\,e^{2x-1}$ **(c)** $y = 2x$
5(a) $R = (-\frac{1}{3}, 1)$
(b) $y' = 3\,e^{3x+1}$ **(c)** $-\frac{1}{3}$ **(d)** $3x + 9y - 8 = 0.$
6(a) $e - 1$ **(b)** $\dfrac{dy}{dx} = e^x$. When $x = 1$, $\dfrac{dy}{dx} = e$.
(c) $y = ex - 1$
7(a) $-e$ **(b)** $\frac{1}{e}$ **(c)** $x - ey + e^2 + 1 = 0$
(d) $x = -e^2 - 1$, $y = e + e^{-1}$ **(e)** $\frac{1}{2}(e^3 + 2e + e^{-1})$
8(a) 1
(b) $y = x + 1$
(c) -1 **(d)** $y = -x + 1$
(f) isosceles right triangle, 1 square unit

(e)



9(a) $1 - e$ **(b)** $y = (1-e)x$

10(a) $y = e^{-1}(x+3)$ **(b)** -3

11(b) The x-intercept is -1 and the y-intercept is e. **(c)** Area $= \frac{1}{2}e$

12(a) $y' = 3\,e^{3x-6}$ **(b)** $3\,e^{3x-6}$ is always positive. **(c)** $(2, 1)$ **(d)** $3\,e^{-6}$, $-\frac{1}{3}\,e^{6}$

13(a) $ex - 2y + (2e^{-1} - e) = 0$ **(b)** $1 - 2\,e^{-2}$

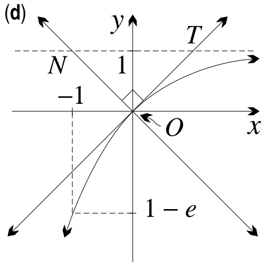
14(a) $y = e^{t}(x - t + 1)$

15(b) $y = -x$

(c) $y = 1$

(e) 1 square unit

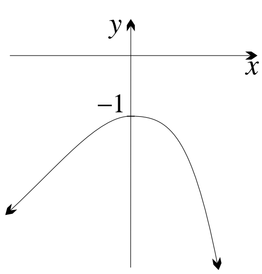
(d)



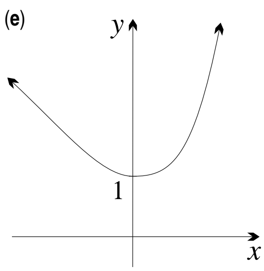
16(a) $y' = 1 - e^{x}$, $y'' = -e^{x}$

(c) maximum turning point at $(0, -1)$

(d) $y \le -1$

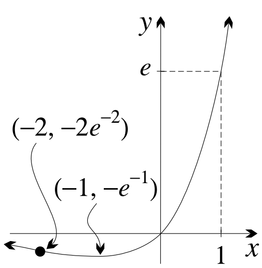


(e)

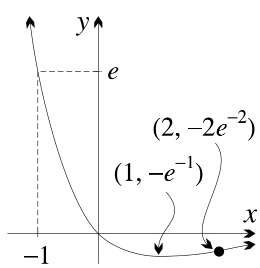


17(b) $\lim\limits_{x \to -\infty} x\,e^{x} = 0$ **(d)** $\lim\limits_{x \to \infty} x\,e^{-x} = 0$

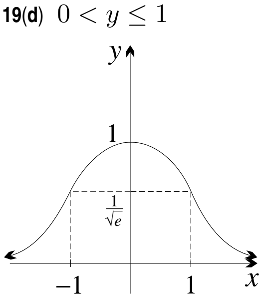
18(d) $y \ge -e^{-1}$



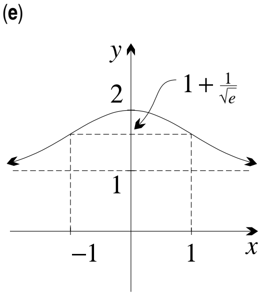
(e)



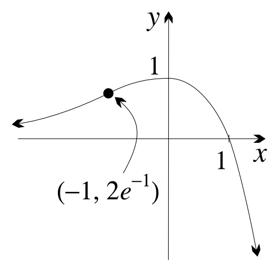
19(d) $0 < y \le 1$



(e)

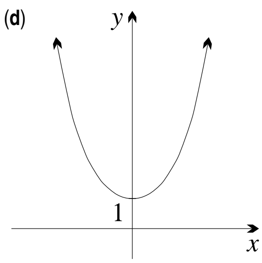


20(c) $y \le 1$

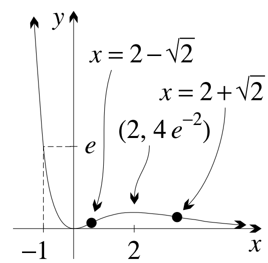


21(b) $y' = \frac{e^{x} - e^{-x}}{2}$

(d)



22(d) $y \ge 0$



Exercise **2E** (Page 87)

1(a) $\frac{1}{2}\,e^{2x} + C$ **(b)** $\frac{1}{3}\,e^{3x} + C$ **(c)** $3\,e^{\frac{1}{3}x} + C$

(d) $2\,e^{\frac{1}{2}x} + C$ **(e)** $5\,e^{2x} + C$ **(f)** $4\,e^{3x} + C$

2(a) $\frac{1}{4}\,e^{4x+5} + C$ **(b)** $\frac{1}{3}\,e^{3x+1} + C$

(c) $\frac{1}{4}\,e^{4x-2} + C$ **(d)** $e^{x-1} + C$ **(e)** $2\,e^{3x+2} + C$

(f) $3\,e^{5x+1} + C$ **(g)** $e^{2x-1} + C$ **(h)** $e^{4x+3} + C$

(i) $-e^{3-x} + C$ **(j)** $-\frac{1}{2}\,e^{7-2x} + C$ **(k)** $\frac{4}{5}\,e^{5x-1} + C$

(l) $-\frac{1}{6}\,e^{1-3x} + C$

3(a) $e - 1$ **(b)** $e^{2} - e$ **(c)** $e - e^{-3}$ **(d)** $e^{2} - 1$

(e) $\frac{1}{2}(e^{4} - 1)$ **(f)** $\frac{1}{3}(e^{3} - e^{-3})$ **(g)** $4(e^{5} - e^{-10})$

(h) $2(e^{12} - e^{-4})$ **(i)** $\frac{3}{2}(e^{18} - e^{-6})$

4(a) $e - e^{-1}$ **(b)** $\frac{1}{2}(e^{3} - e^{-1})$ **(c)** $\frac{1}{4}(e^{-3} - e^{-11})$

(d) $\frac{1}{3}(e^{-1} - e^{-4})$ **(e)** $\frac{e^{2}}{2}(e^{2} - 1)$ **(f)** $\frac{e}{3}(e^{2} - 1)$

(g) $2\,e^{4}(e^{3} - 1)$ **(h)** $3\,e^{3}(e^{4} - 1)$ **(i)** $4\,e^{2}(e^{3} - 1)$

5(a) $-e^{-x} + C$ **(b)** $-\frac{1}{2}\,e^{-2x} + C$

(c) $-\frac{1}{3}\,e^{-3x} + C$ **(d)** $e^{-3x} + C$ **(e)** $-3\,e^{-2x} + C$

(f) $4\,e^{2x} + C$

6(a) $f(x) = x + 2\,e^{x} - 1$, $f(1) = 2e$

(b) $f(x) = 2 + x - 3\,e^{x}$, $f(1) = 3 - 3e$

(c) $f(x) = 1 + 2x - e^{-x}$, $f(1) = 3 - e^{-1}$

(d) $f(x) = 1 + 4x + e^{-x}$, $f(1) = 5 + e^{-1}$

(e) $f(x) = \frac{1}{2}\,e^{2x-1} + \frac{5}{2}$, $f(1) = \frac{1}{2}(e + 5)$

(f) $f(x) = 1 - \frac{1}{3}\,e^{1-3x}$, $f(1) = 1 - \frac{1}{3}\,e^{-2}$

(g) $f(x) = 2\,e^{\frac{1}{2}x+1} - 6$, $f(1) = 2\,e^{\frac{3}{2}} - 6$

(h) $f(x) = 3\,e^{\frac{1}{3}x+2} - 1$, $f(1) = 3\,e^{\frac{7}{3}} - 1$

7(a) $\frac{1}{2}\,e^{2x} + e^{x} + C$ **(b)** $\frac{1}{2}\,e^{2x} - e^{x} + C$

(c) $e^{-x} - e^{-2x} + C$ **(d)** $\frac{1}{2}\,e^{2x} + 2\,e^{x} + x + C$

(e) $\frac{1}{2}\,e^{2x} + 6\,e^{x} + 9x + C$ **(f)** $\frac{1}{2}\,e^{2x} - 2\,e^{x} + x + C$

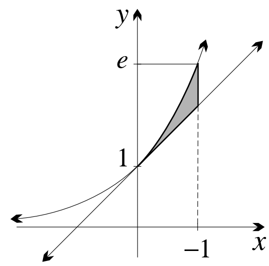
(g) $\frac{1}{2}\,e^{2x} - 4\,e^{x} + 4x + C$ **(h)** $\frac{1}{2}(e^{2x} + e^{-2x}) + C$

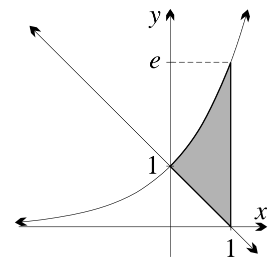
(i) $\frac{1}{10}(e^{10x} + e^{-10x}) + C$

8(a) $\frac{1}{2}\,e^{2x+b}+C$ **(b)** $\frac{1}{7}\,e^{7x+q}+C$ **(c)** $\frac{1}{3}\,e^{3x-k}+C$
(d) $\frac{1}{6}\,e^{6x-\lambda}+C$ **(e)** $\frac{1}{a}\,e^{ax+3}+C$ **(f)** $\frac{1}{s}\,e^{sx+1}+C$
(g) $\frac{1}{m}\,e^{mx-2}+C$ **(h)** $\frac{1}{k}\,e^{kx-1}+C$ **(i)** $e^{px+q}+C$
(j) $e^{mx+k}+C$ **(k)** $\frac{A}{s}\,e^{sx-t}+C$ **(l)** $\frac{B}{k}\,e^{kx-\ell}+C$
9(a) $-e^{1-x}+C$ **(b)** $-\frac{1}{3}\,e^{1-3x}+C$
(c) $-\frac{1}{2}\,e^{-2x-5}+C$ **(d)** $-2\,e^{1-2x}+C$ **(e)** $2\,e^{5x-2}+C$
(f) $-4\,e^{5-3x}+C$
10(a) $x-e^{-x}+C$ **(b)** $e^{x}-e^{-x}+C$
(c) $\frac{1}{2}\,e^{-2x}-e^{-x}+C$ **(d)** $e^{-3x}-\frac{1}{2}\,e^{-2x}+C$
(e) $e^{-3x}-e^{-2x}+C$ **(f)** $e^{-x}-e^{-2x}+C$
11(a) $y=e^{x-1},\ y=e^{-1}$ **(b)** $y=e^{2}+1-e^{2-x}$, $y=e^{2}+1$ **(c)** $f(x)=e^{x}+\frac{x}{e}-1,\ f(0)=0$
(d) $f(x)=e^{x}-e^{-x}-2x$
12(a) $e^{2}-e$ **(b)** $\frac{1}{2}(e^{2}-e^{-2})+4(e-e^{-1})+8$
(c) $e+e^{-1}-2$ **(d)** $\frac{1}{4}(e^{4}-e^{-4})+\frac{1}{2}(e^{-2}-e^{2})$
(e) $e-e^{-1}$ **(f)** $e-e^{-1}+\frac{1}{2}(e^{-2}-e^{2})$
13(a)(i) $2x\,e^{x^2+3}$ **(ii)** $e^{x^2+3}+C$
(b)(i) $2(x-1)\,e^{x^2-2x+3}$ **(ii)** $\frac{1}{2}\,e^{x^2-2x+3}+C$
(c)(i) $(6x+4)\,e^{3x^2+4x+1}$ **(ii)** $\frac{1}{2}\,e^{3x^2+4x+1}+C$
(d)(i) $3x^{2}\,e^{x^3}$ **(ii)** $\frac{1}{3}(1-e^{-1})$
14(a) $-\frac{1}{2}\,e^{-2x}+C$ **(b)** $-\frac{1}{3}\,e^{-3x}+C$ **(c)** $2\,e^{\frac{1}{2}x}+C$
(d) $3\,e^{\frac{1}{3}x}+C$ **(e)** $-2\,e^{-\frac{1}{2}x}+C$ **(f)** $-3\,e^{-\frac{1}{3}x}+C$
15(a)(i) $y'=x\,e^{x}$ **(ii)** $e^{2}+1$
(b)(i) $y'=-x\,e^{-x}$ **(ii)** $-1-e^{2}$
16(a) $2\,e^{\frac{1}{2}x}+\frac{2}{3}\,e^{-\frac{3}{2}x}+C$ **(b)** $\frac{3}{2}\,e^{\frac{2}{3}x}-\frac{3}{4}\,e^{-\frac{4}{3}x}+C$
17(b) 0

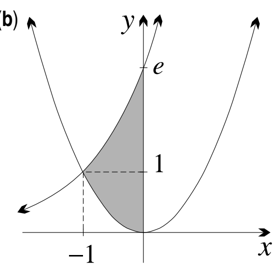
Exercise 2F (Page 92)

1(a)(i) $e-1\doteqdot 1{\cdot}72$ **(ii)** $1-e^{-1}\doteqdot 0{\cdot}63$
(iii) $1-e^{-2}\doteqdot 0{\cdot}86$ **(iv)** $1-e^{-3}\doteqdot 0{\cdot}95$
(d) The total area is exactly 1.
2(a) $(1-e^{-1})$ square units
(b) $e(e^{2}-1)$ square units **(c)** $(e-e^{-1})$ square units
(d) $e-e^{-2}$ square units
3(a)(i) $\frac{1}{2}(e^{6}-1)\doteqdot 201{\cdot}2$ square units
(ii) $\frac{1}{2}(1-e^{-6})\doteqdot 0{\cdot}4988$ square units
(b)(i) $1-e^{-1}\doteqdot 0{\cdot}6321$ square units
(ii) $e-1\doteqdot 1{\cdot}718$ square units
(c)(i) $3(e-1)\doteqdot 5{\cdot}155$ square units
(ii) $3(1-e^{-1})\doteqdot 1{\cdot}896$ square units
4(a) $e(e^{2}-1)\,\mathrm{u}^2$ **(b)** $e(e^{2}-1)\,\mathrm{u}^2$ **(c)** $\frac{1}{2}(e-e^{-1})\,\mathrm{u}^2$
(d) $\frac{1}{3}(e-e^{-2})\,\mathrm{u}^2$ **(e)** $(e^{2}-1)\,\mathrm{u}^2$ **(f)** $\frac{1}{2}e(e^{2}-1)\,\mathrm{u}^2$
(g) $3e^{2}(e-1)\,\mathrm{u}^2$ **(h)** $2(1-e^{-2})\,\mathrm{u}^2$
5(a) $(e^{2}-1)\,\mathrm{u}^2$ **(b)** $2(e-e^{-\frac{1}{2}})\,\mathrm{u}^2$ **(c)** $(1-e^{-1})\,\mathrm{u}^2$
(d) $2(e^{\frac{1}{2}}-e^{-1})\,\mathrm{u}^2$
6(a) $(3-e^{-2})\doteqdot 2{\cdot}865\,\mathrm{u}^2$ **(b)** $e^{-1}\doteqdot 0{\cdot}3679\,\mathrm{u}^2$
(c) $2(e^{2}-e^{-2})\doteqdot 14{\cdot}51\,\mathrm{u}^2$
(d) $18+e^{3}-e^{-3}\doteqdot 38{\cdot}04\,\mathrm{u}^2$
7(a) $(1+e^{-2})\,\mathrm{u}^2$ **(b)** $1\,\mathrm{u}^2$ **(c)** $e^{-1}\,\mathrm{u}^2$ **(d)** $(3+e^{-2})\,\mathrm{u}^2$
(e) $1\,\mathrm{u}^2$ **(f)** $(9+e^{-2}-e)\,\mathrm{u}^2$
8(a) $\displaystyle\int_0^1 (e^{x}-1-x)\,dx = (e-2\tfrac{1}{2})\,\mathrm{u}^2$ **(b)** $\displaystyle\int_0^1 (e^{x}-1+x)\,dx = (e-1\tfrac{1}{2})\,\mathrm{u}^2$

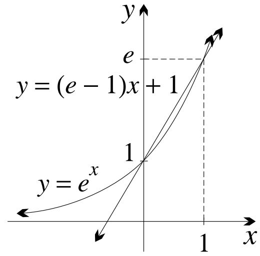




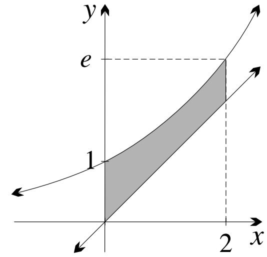
9(a) The region is symmetric, so the area is twice the area in the first quadrant.
(b) $2-\frac{2}{e}$ square units
10(a) The region is symmetric, so the area is twice the area in the first quadrant.
(b) 2 square units
11(b) 0 **(c)** The region is symmetric, so the area is twice the area in the first quadrant.
(d) $2(e^{3}+e^{-3}-2)$ square units
12(b)



(c) $(e-1\frac{1}{3})\,\mathrm{u}^2$
13 $\pi\displaystyle\int_0^1 (e^{x})^2\,dx$, $\frac{\pi}{2}(e^{2}-1)$ cubic units
14 $\frac{\pi}{2}(e-e^{-1}-2)$ cubic units
15 $\pi\left(2+2e^{-1}-2e^{-3}+\frac{1}{2}e^{-2}-\frac{1}{2}e^{-6}\right)\doteqdot 8{\cdot}491$ cubic units
16(b) $\frac{1}{2}(3-e)\,\mathrm{u}^2$



17 $(e^{2}-3)$ square units



18(a) $e-1\doteqdot 1{\cdot}7183$ **(b)** $1{\cdot}7539$ **(c)** $1{\cdot}7189$
19(a) $0{\cdot}8863$ square units **(b)** $0{\cdot}8362$ square units
20(a) $3{\cdot}5726$ square units **(b)** $3{\cdot}5283$ square units
21(a)(i) $1-e^{N}$ **(ii)** 1 **(b)(i)** $1-e^{-N}$ **(ii)** 1
22(a) $-e^{-x^2}$ **(b)** $\pi(1-e^{-4})\doteqdot 3{\cdot}084\,\mathrm{mL}$

Review Exercise **2G** (Page 96)

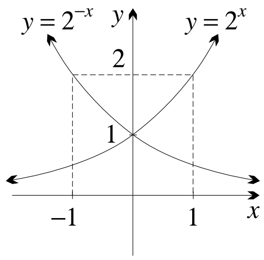
1(a) 3^9 **(b)** 3^{12} **(c)** 3^5 **(d)** 6^5

2(a) $\frac{1}{5}$ **(b)** $\frac{1}{100}$ **(c)** $\frac{1}{x^3}$ **(d)** $\frac{1}{3^x}$

3(a) 3 **(b)** 3 **(c)** 4 **(d)** $\frac{1}{4}$ **(e)** $\frac{1}{9}$ **(f)** $\frac{1}{1000}$

4(a) 2^{3x} **(b)** 2^{4x} **(c)** 2^{6x} **(d)** 10^x **(e)** 2^{2x+3} **(f)** 2^{2x-1}

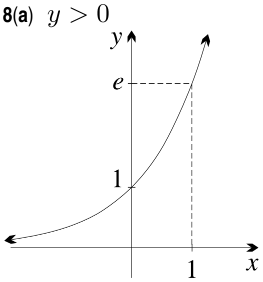
5 Each graph is reflected onto the other graph in the line $x = 0$.



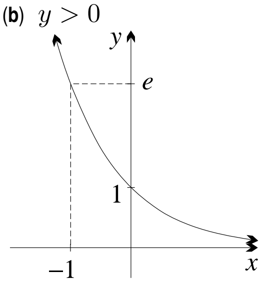
6(a) $2{\cdot}718$ **(b)** $54{\cdot}60$ **(c)** $0{\cdot}1353$ **(d)** $4{\cdot}482$

7(a) e^{5x} **(b)** e^{6x} **(c)** e^{-4x} **(d)** e^{9x}

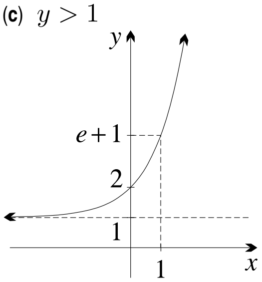
8(a) $y > 0$



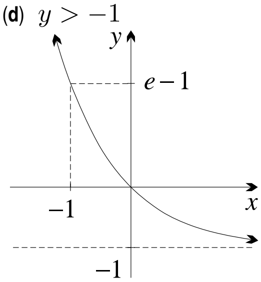
(b) $y > 0$



(c) $y > 1$



(d) $y > -1$



9(a) e^x **(b)** $3e^{3x}$ **(c)** e^{x+3} **(d)** $2e^{2x+3}$ **(e)** $-e^{-x}$ **(f)** $-3e^{-3x}$ **(g)** $-2e^{3-2x}$ **(h)** $6e^{2x+5}$ **(i)** $2e^{\frac{1}{2}x}$ **(j)** $3x^2e^{x^3}$ **(k)** $(2x-3)e^{x^2-3x}$ **(l)** $4e^{6x-5}$

10(a) $5e^{5x}$ **(b)** $4e^{4x}$ **(c)** $-3e^{-3x}$ **(d)** $-6e^{-6x}$

11(a) $e^{2x} + 2xe^{2x} = e^{2x}(1 + 2x)$ **(b)** $6e^{2x}(e^{2x} + 1)^2$ **(c)** $\frac{e^{3x}(3x-1)}{x^2}$ **(d)** $2xe^{x^2}(1 + x^2)$ **(e)** $5(e^x + e^{-x})(e^x - e^{-x})^4$ **(f)** $\frac{4xe^{2x}}{(2x+1)^2}$

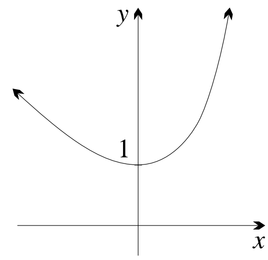
12(a) $y' = 2e^{2x+1}$, $y'' = 4e^{2x+1}$ **(b)** $y' = 2xe^{x^2+1}$, $y'' = 2e^{x^2+1}(2x^2 + 1)$

13 2

14 $y = e^2x - e^2$, x-intercept 1, y-intercept $-e^2$.

15(a) $\frac{1}{3}$ **(b)** When $x = 0$, $y'' = 9$, so the curve is concave up there.

16(a) $y' = e^x - 1$, $y'' = e^x$ **(b)** $(0, 1)$ is a minimum turning point. **(c)** $y'' = e^x$, which is positive for all x. **(d)** Range: $y \geq 1$



17 $(\frac{1}{2}, \frac{1}{2e})$ is a maximum turning point.

18(a) $\frac{1}{5}e^{5x} + C$ **(b)** $\frac{1}{5}e^{5x+3} + C$ **(c)** $-e^{-5x} + C$ **(d)** $-2e^{2-5x} + C$ **(e)** $5e^{\frac{1}{5}x} + C$ **(f)** $\frac{3}{5}e^{5x-4} + C$

19(a) $e^2 - 1$ **(b)** $\frac{1}{2}(e^2 - 1)$ **(c)** $e - 1$ **(d)** $\frac{1}{3}(e^2 - 1)$ **(e)** $\frac{1}{2}e^2(e - 1)$ **(f)** $4(e - 1)$

20(a) $-\frac{1}{5}e^{-5x} + C$ **(b)** $\frac{1}{4}e^{4x} + C$ **(c)** $-2e^{-3x} + C$ **(d)** $\frac{1}{6}e^{6x} + C$ **(e)** $-\frac{1}{2}e^{-2x} + C$ **(f)** $e^x - \frac{1}{2}e^{-2x} + C$ **(g)** $\frac{1}{3}e^{3x} + e^x + C$ **(h)** $x - 2e^{-x} - \frac{1}{2}e^{-2x} + C$

21(a) $2 - e^{-1}$ **(b)** $\frac{1}{2}(e^4 + 3)$ **(c)** $2(1 - e^{-1})$ **(d)** $\frac{1}{3}(e - 2)$ **(e)** $e - e^{-1}$ **(f)** $\frac{1}{2}(e^2 + 4e - 3)$

22 $f(x) = e^x + e^{-x} - x + 1$, $f(1) = e + e^{-1}$

23(a) $3x^2e^{x^3}$ **(b)** $\frac{1}{3}(e - 1)$

24(a) $3{\cdot}19\,\text{u}^2$ **(b)** $0{\cdot}368\,\text{u}^2$

25(a) $\pi(e - e^{-1})\,\text{u}^3$ **(b)** $\frac{\pi}{2}(e - e^{-1} + 2)\,\text{u}^3$

26(a) $\frac{1}{2}(1 + e^{-2})\,\text{u}^2$ **(b)** $\frac{1}{2}(3 - e)\,\text{u}^2$

Chapter Three

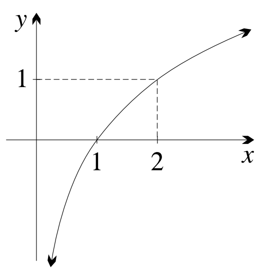
Exercise 3A (Page 103)

1(a) 0 **(b)** $0{\cdot}3010$ **(c)** 1 **(d)** $1{\cdot}1761$ **(e)** $-0{\cdot}3010$ **(f)** -1 **(g)** $-1{\cdot}1761$ **(h)** -2

2(a) $3^x = 9$, $x = 2$ **(b)** $2^x = 8$, $x = 3$ **(c)** $6^x = 36$, $x = 2$ **(d)** $3^x = 81$, $x = 4$ **(e)** $2^x = \frac{1}{32}$, $x = -5$ **(f)** $3^x = \frac{1}{27}$, $x = -3$ **(g)** $7^x = \frac{1}{49}$, $x = -2$ **(h)** $10^x = \frac{1}{10}$, $x = -1$

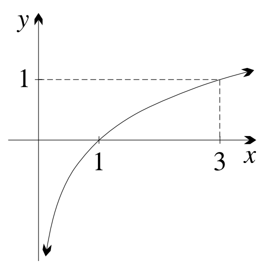
3(a) 0 **(b)** 0 **(c)** -1 **(d)** -1 **(e)** 1 **(f)** 1 **(g)** $\frac{1}{2}$ **(h)** $\frac{1}{2}$

4(b)



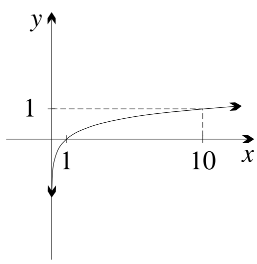
(c) domain: $x > 0$, range: all real numbers

5(b)



(c) domain: $x > 0$, range: all real numbers

6(b)

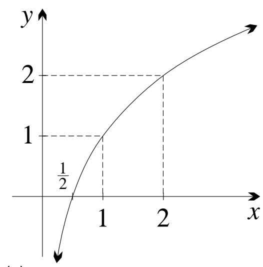


(c) domain: $x > 0$, range: all real numbers

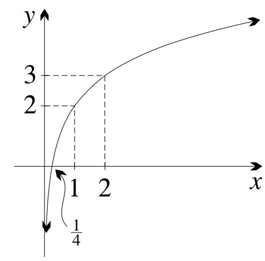
7(a) $x = \log_{10} 3 \doteqdot 0{\cdot}4771$ **(b)** $x = \log_{10} 8 \doteqdot 0{\cdot}9031$ **(c)** $x = \log_{10} 25 \doteqdot 1{\cdot}3979$ **(d)** $x = \log_{10} 150 \doteqdot 2{\cdot}1761$ **(e)** $x = \log_{10} 3000 \doteqdot 3{\cdot}4771$ **(f)** $x = \log_{10} 0{\cdot}2 \doteqdot -0{\cdot}6990$ **(g)** $x = \log_{10} 0{\cdot}05 \doteqdot -1{\cdot}3010$ **(h)** $x = \log_{10} 0{\cdot}006\,25 \doteqdot -2{\cdot}2041$

10(a) $\log_2 5 + \log_2 x$ **(b)** $\log_2 3 + \log_2 x$ **(c)** $1 + \log_2 x$ **(d)** $2 + \log_2 x$ **(e)** $\log_2 x - \log_2 7$ **(f)** $\log_2 x - \log_2 3$ **(g)** $-\log_2 x$ **(h)** $\log_2 12 - \log_2 x$ **(i)** $2\log_2 x$ **(j)** $\log_2 5 + 2\log_2 x$ **(k)** $\log_2 5 - 2\log_2 x$ **(l)** $\frac{1}{2}\log_2 x$

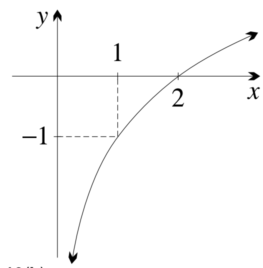
11(a)



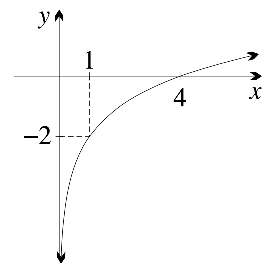
(b)



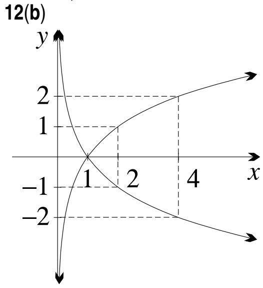
(c)



(d)



12(b)

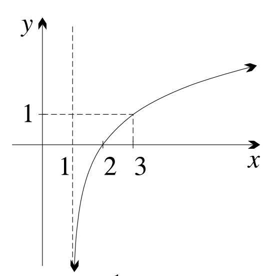


13(a) $\log_2(x+1) + \log_2(x+2)$ **(b)** $\log_2 x + \log_2(x-1)$ **(c)** $\log_2(x+3) - \log_2(x-1)$ **(d)** $\log_2(x-2) - \log_2(x+5)$

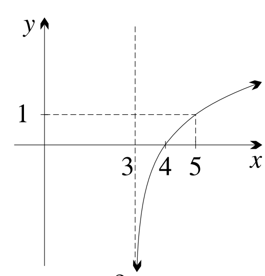
14(a) $2{\cdot}3219$ **(b)** $0{\cdot}4307$ **(c)** $3{\cdot}5046$ **(d)** $2{\cdot}6334$

15(a) $0{\cdot}4771$ **(b)** $3{\cdot}322$ **(c)** $3{\cdot}113$ **(d)** $4{\cdot}096$

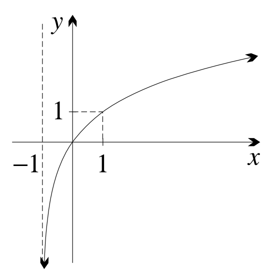
16(a) $x > 1$



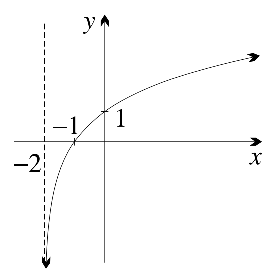
(b) $x > 3$



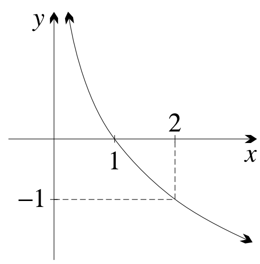
(c) $x > -1$



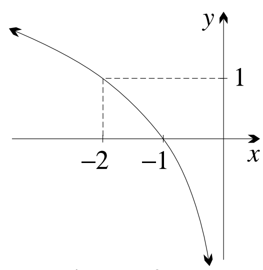
(d) $x > -2$



(e) $x > 0$



(f) $x < 0$



20(a) The domain of $y = \log_{10} x$ is $x > 0$.
(b) $\frac{1}{8}$ is less than 1. **(c)** The index is negative.
(d) $0 < x < 1$ **(e)(i)** The output is never zero.
(ii) $\log_{10} 1 = 0$ **(f)(i)** $100\,000$ **(ii)** $0{\cdot}01$ **(iii)** $10^{3\cdot5}$
(iv) $10^{-1\cdot7}$ **(g)(i)** 2 **(ii)** -3 **(iii)** $\log_{10} 60$ **(iv)** $\log_{10} 0{\cdot}3$
21(a) $\log_{10} 100 = 2$ and $\log_{10} 1000 = 3$. Since 274 is between 100 and 1000, $\log_{10} 274$ lies between 2 and 3. **(b)** Since $1000 < 4783 < 10\,000$, $3 < \log_{10} 4783 < 4$. **(c)** $5 < \log_{10} 516\,287 < 6$ **(d)(i)** $238 < 500 \log_{10} 3 < 239$ so there are 239 digits. **(ii)** 603 **(iii)** 254 **(iv)** 450
22(a) $x = b^y$

Exercise **3B** (Page 109)

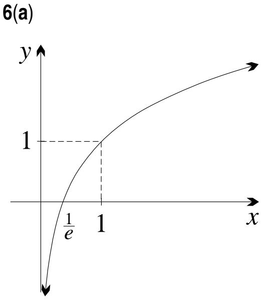
1(a) 0 **(b)** $0{\cdot}6931$ **(c)** $1{\cdot}0986$ **(d)** $2{\cdot}0794$
(e) $-0{\cdot}6931$ **(f)** $-1{\cdot}0986$ **(g)** $-2{\cdot}0794$ **(h)** $-2{\cdot}3026$
4(b) Both are equal to 1.

(c)

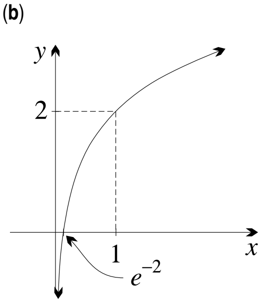
x	$\frac{1}{e}$	$\frac{1}{2}$	1	2	e
gradient y'	e	2	1	$\frac{1}{2}$	$\frac{1}{e}$
$\dfrac{1}{x}$	e	2	1	$\frac{1}{2}$	$\frac{1}{e}$

(d) The y-intercept is 1 unit below the point of contact.
5(d) The y-intercept is 1 unit below the point of contact.

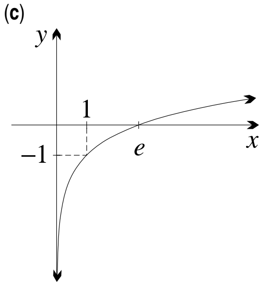
6(a)



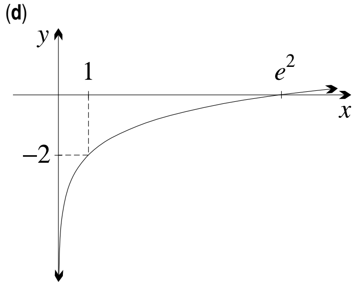
(b)



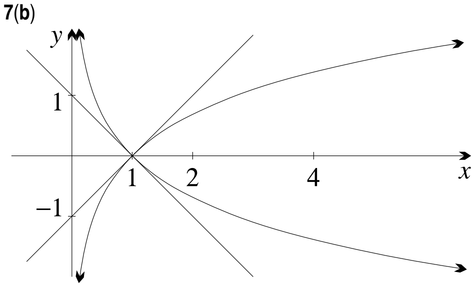
(c)



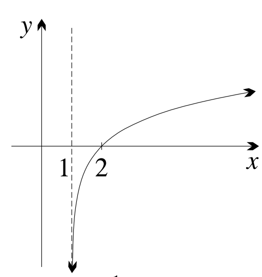
(d)



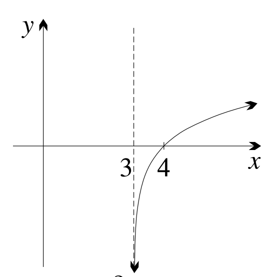
7(b)



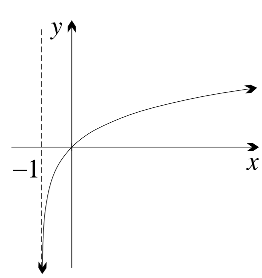
(c) The gradient of $y = \log_e x$ at its x-intercept is 1. The graph of $y = -\log_e x$ is obtained by reflecting the first in the x-axis. Hence its tangent has gradient -1, and the two are perpendicular.
8(a) e **(b)** $-\frac{1}{e}$ **(c)** 6 **(d)** $\frac{1}{2}$ **(e)** $2e$ **(f)** 0 **(g)** e
(h) 1 **(i)** 0
9(a) $\log_e 6$ **(b)** $\log_e 4$ **(c)** $2\log_e 2$ **(d)** $3\log_e 3$
10(a) $x = 1$ **(b)** 1 **(c)** $y = x - 1$, -1
(d) for $y = -1$: $y = ex - 2$, -2;
for $y = 1$: $y = \frac{1}{e}x$, 0; for $y = 2$: $y = \frac{1}{e^2}x + 1$, 1
(e) They are the same.
11(a) $x > 0$ **(b)** $y' = \frac{1}{x} > 0$ **(c)** $y'' = -\frac{1}{x^2} < 0$
12(a) $x > 1$



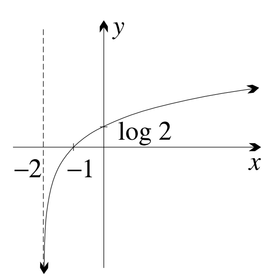
(b) $x > 3$



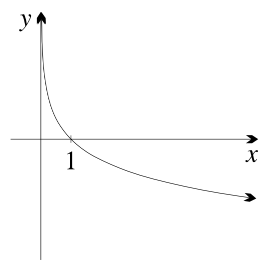
(c) $x > -1$



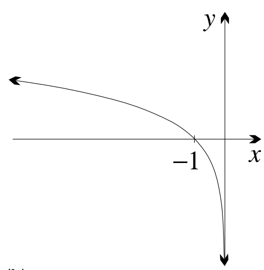
(d) $x > -2$



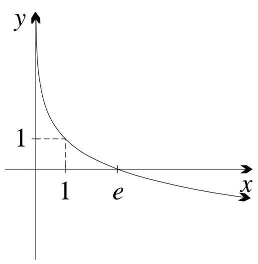
(e) $x > 0$



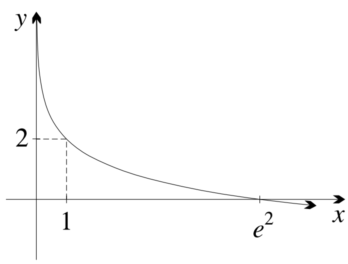
(f) $x < 0$



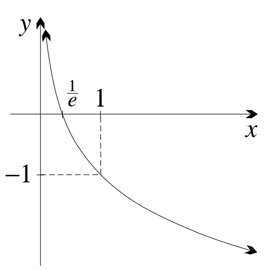
13(a)



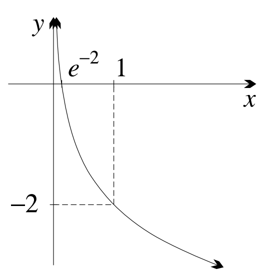
(b)



(c)



(d)



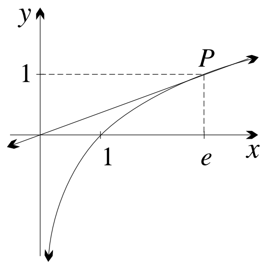
Exercise 3C (Page 114)

1(a) $y' = \frac{1}{x+2}$ **(b)** $y' = \frac{1}{x-3}$ **(c)** $y' = \frac{3}{3x+4}$
(d) $y' = \frac{5}{5x+1}$ **(e)** $y' = \frac{2}{2x-1}$ **(f)** $y' = \frac{4}{4x-3}$
(g) $y' = \frac{-4}{-4x+1}$ **(h)** $y' = \frac{-3}{-3x+4}$
(i) $y' = \frac{-2}{-2x-7} = \frac{2}{2x+7}$ **(j)** $y' = \frac{-3}{-3x-6} = \frac{1}{x+2}$
(k) $y' = \frac{6}{2x+4} = \frac{3}{x+2}$ **(l)** $y' = \frac{15}{3x-2}$
2(a) $y = \log_e 2 + \log_e x,\ y' = \frac{1}{x}$
(b) $y = \log_e 5 + \log_e x,\ y' = \frac{1}{x}$ **(c)** $\frac{1}{x}$ **(d)** $\frac{1}{x}$ **(e)** $\frac{4}{x}$
(f) $\frac{3}{x}$ **(g)** $\frac{4}{x}$ **(h)** $\frac{3}{x}$
3(a) $y' = \frac{1}{x+1},\ y'(3) = \frac{1}{4}$ **(b)** $y' = \frac{2}{2x-1},\ y'(3) = \frac{2}{5}$
(c) $y' = \frac{2}{2x-5},\ y'(3) = 2$ **(d)** $y' = \frac{4}{4x+3},\ y'(3) = \frac{4}{15}$
(e) $y' = \frac{5}{x+1},\ y'(3) = \frac{5}{4}$ **(f)** $y' = \frac{12}{2x+9},\ y'(3) = \frac{4}{5}$
4(a) $\frac{1}{x}$ **(b)** $-\frac{1}{x}$ **(c)** $\frac{-1}{x+1}$ **(d)** $1 + \frac{4}{x}$ **(e)** $2 + \frac{3}{x}$
(f) $\frac{2}{2x-1} + 6x$ **(g)** $-\frac{1}{3-x} + 2x + 3$ **(h)** $8x^3 - \frac{1}{x}$
(i) $3x^2 - 3 + \frac{5}{5x-7}$
5(a) $y = 3\log x,\ y' = \frac{3}{x}$ **(b)** $y = 2\log x,\ y' = \frac{2}{x}$
(c) $y = -3\log x,\ y' = -\frac{3}{x}$
(d) $y = -2\log x,\ y' = -\frac{2}{x}$ **(e)** $y = \frac{1}{2}\log x,\ y' = \frac{1}{2x}$
(f) $y = \frac{1}{2}\log(x+1),\ y' = \frac{1}{2(x+1)}$
6(a) $\frac{1}{x}$ **(b)** $\frac{1}{x}$ **(c)** $\frac{3}{x}$ **(d)** $-\frac{6}{x}$ **(e)** $1 + \frac{1}{x}$ **(f)** $12x^2 - \frac{1}{x}$
7(a) $\frac{2x}{x^2+1}$ **(b)** $\frac{2x+3}{x^2+3x+2}$ **(c)** $\frac{-2x}{2-x^2}$ **(d)** $\frac{6x^2}{1+2x^3}$
(e) $\frac{e^x}{1+e^x}$ **(f)** $\frac{e^x}{e^x-2}$ **(g)** $1 - \frac{2x+1}{x^2+x}$ **(h)** $2x + \frac{3x^2-1}{x^3-x}$
(i) $12x^2 - 10x + \frac{4x-3}{2x^2-3x+1}$
8(a) $1,\ 45°$ **(b)** $\frac{1}{3},\ 18°26'$ **(c)** $2,\ 63°26'$
(d) $\frac{1}{4},\ 14°2'$
9(a) $1 + \log x$ **(b)** $\frac{2x}{2x+1} + \log(2x+1)$
(c) $\frac{2x+1}{x} + 2\log x$ **(d)** $x^3(1 + 4\log x)$
(e) $\log(x+3) + 1$ **(f)** $\frac{2(x-1)}{2x+7} + \log(2x+7)$
(g) $e^x(\frac{1}{x} + \log x)$ **(h)** $e^{-x}(\frac{1}{x} - \log x)$
10(a) $\frac{1-\log x}{x^2}$ **(b)** $\frac{1-2\log x}{x^3}$ **(c)** $\frac{\log x - 1}{(\log x)^2}$ **(d)** $\frac{x(2\log x - 1)}{(\log x)^2}$
(e) $\frac{1 - x\log x}{xe^x}$ **(f)** $\frac{e^x(x\log x - 1)}{x(\log x)^2}$
11(a) $\frac{3}{x}$ **(b)** $\frac{4}{x}$ **(c)** $\frac{1}{3x}$ **(d)** $\frac{1}{4x}$ **(e)** $-\frac{1}{x}$ **(f)** $-\frac{1}{x}$
(g) $\frac{1}{4-2x}$ **(h)** $\frac{5}{10x+4}$
12(a) $f'(x) = \frac{1}{x-1},\ f'(3) = \frac{1}{2},\ f''(x) = -\frac{1}{(x-1)^2},$
$f''(3) = -\frac{1}{4}$ **(b)** $f'(x) = \frac{2}{2x+1},\ f'(0) = 2,$
$f''(x) = -\frac{4}{(2x+1)^2},\ f''(0) = -4$ **(c)** $f'(x) = \frac{2}{x},$
$f'(2) = 1,\ f''(x) = -\frac{2}{x^2},\ f''(2) = -\frac{1}{2}$
(d) $f'(x) = 1 + \log x,\ f'(e) = 2,\ f''(x) = \frac{1}{x},$
$f''(e) = \frac{1}{e}$
13(a) $\log x,\ x = 1$ **(b)** $x(1 + 2\log x),\ x = e^{-\frac{1}{2}}$
(c) $\frac{1-\log x}{x^2},\ x = e$ **(d)** $\frac{2\log x}{x},\ x = 1$
(e) $\frac{4(\log x)^3}{x},\ x = 1$ **(f)** $\frac{-1}{x(1+\log x)^2}$ is never zero.
(g) $\frac{8}{x}(2\log x - 3)^3,\ x = e^{\frac{3}{2}}$
(h) $\frac{-1}{x(\log x)^2}$ is never zero. **(i)** $\frac{1}{x\log x}$ is never zero.
14(a) $(\frac{1}{e}, -\frac{1}{e})$ **(b)** $(1, 1)$
15(a) $y' = \dfrac{\log x - 1}{(\log x)^2}$
16(a) $\frac{1}{x+2} + \frac{1}{x+1}$ **(b)** $\frac{1}{x+5} + \frac{3}{3x-4}$ **(c)** $\frac{1}{1+x} + \frac{1}{1-x}$
(d) $\frac{3}{3x-1} - \frac{1}{x+2}$ **(e)** $\frac{2}{x-4} - \frac{3}{3x+1}$ **(f)** $\frac{1}{x} + \frac{1}{2(x+1)}$
17(a) $y = x\log_e 2,\ y' = \log_e 2$ **(b)** $y = x,\ y' = 1$
(c) $y = x\log_e x,\ y' = 1 + \log_e x$

Exercise 3D (Page 119)

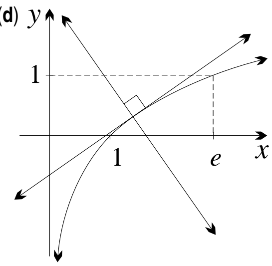
1(a) $\frac{1}{e}$ **(b)** $y = \frac{1}{e}x$
2(a) 1 **(b)** $y = x - 1$
3(a) e **(b)** $y = ex - 2$
4(a) 1
(c) $y = -x + 1$. When $x = 0$, $y = 1$.
5(a) $y = 4x - 4,\ y = -\frac{1}{4}x + \frac{1}{4}$ **(b)** $y = x + 2,$
$y = -x + 4$ **(c)** $y = 2x - 4,\ y = -\frac{1}{2}x - 1\frac{1}{2}$
(d) $y = -3x + 4,\ y = \frac{1}{3}x + \frac{2}{3}$
6(b) $3,\ -\frac{1}{3}$ **(c)** $y = 3x - 3,\ -3,\ y = -\frac{1}{3}x + \frac{1}{3},\ \frac{1}{3}$
(d) $\frac{5}{3}$ square units

7 As P moves left along the curve, the tangent becomes steeper, and so it cannot pass through the origin.
As P moves right, the angle of the tangent becomes less steep, hence it cannot pass through the origin.

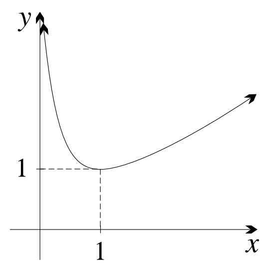


8(a) $\frac{1}{2}$ **(b)** $y = \frac{1}{2}x$
9(a) $y = -\log 2 \times (x - 2)$ **(b)** $2 \log 2$
10(a) $x > 0$, $y = \frac{1}{x}$
(b) $\frac{1}{x}$ is always positive in the domain.
(c) $-x$ is always negative in the domain.
(e) $y'' = -\frac{1}{x^2}$. It is always concave down.

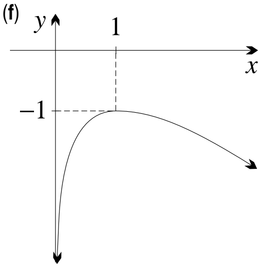
(d)



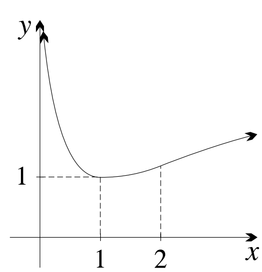
11(a) $(2, \log_e 2)$, $y = \frac{1}{2}x - 1 + \log_e 2$, $y = -2x + 4 + \log_e 2$ **(b)** $(\frac{1}{2}, -\log_e 2)$, $y = 2x - 1 - \log_e 2$, $y = -\frac{1}{2}x + \frac{1}{4} - \log_e 2$
12(a) $x > 0$ **(b)** $y' = 1 - \frac{1}{x}$, $y'' = \frac{1}{x^2}$
(c) $y'' > 0$, for all x **(d)** $(1, 1)$ **(e)** $y \geq 1$



(f)

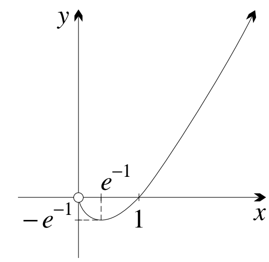


13(a) $x > 0$ **(d)** $y \geq 1$

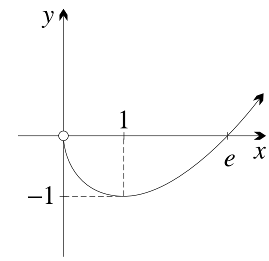


14(a) $\displaystyle\lim_{x\to\infty} \frac{\log x}{x} = 0$
(b) $\displaystyle\lim_{x\to 0^+} x \log x = 0$

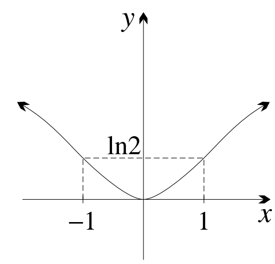
15(a) $x > 0$, $(1, 0)$
(b) $y'' = \frac{1}{x}$
(c) $(e^{-1}, -e^{-1})$ is a minimum turning point.
(d) $y > -e^{-1}$



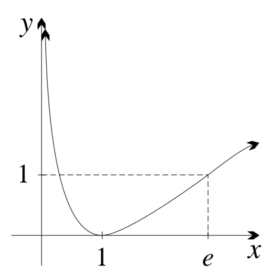
16(a) $x > 0$, $(e, 0)$
(b) $y'' = \dfrac{1}{x}$
(c) $(1, -1)$ is a minimum turning point.
(d) $y > -1$



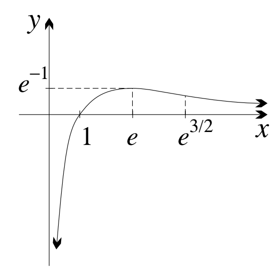
17(a) all real x
(d) $(1, \log_e 2)$ and $(-1, \log_e 2)$
(e) $y \geq 0$



18(a) $x > 0$
(b) $y' = \frac{2}{x} \ln x$
(d) $y \geq 0$



19(a) $x > 0$
(d) $(e^{\frac{3}{2}}, \frac{3}{2}e^{-\frac{3}{2}})$
(e) $y \leq e^{-1}$



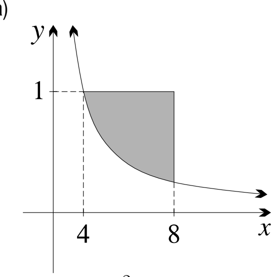
Exercise **3E** (Page 125)

1(a) $2 \log_e x + C$ **(b)** $5 \log_e x + C$ **(c)** $\frac{1}{2} \log_e x + C$
(d) $\frac{1}{3} \log_e x + C$
(e) $\frac{4}{5} \log_e x + C$ **(f)** $\frac{3}{2} \log_e x + C$
2(a) $\frac{1}{4} \log_e(4x + 1) + C$ **(b)** $\frac{1}{2} \log_e(2x + 1) + C$
(c) $\frac{1}{5} \log_e(5x - 3) + C$ **(d)** $\frac{1}{7} \log_e(7x - 2) + C$
(e) $2 \log_e(3x + 2) + C$ **(f)** $3 \log_e(5x + 1) + C$
(g) $\log_e(2x - 1) + C$ **(h)** $\log_e(4x + 3) + C$
(i) $-\log_e(3 - x) + C$ **(j)** $-\frac{1}{2} \log_e(7 - 2x) + C$
(k) $\frac{4}{5} \log_e(5x - 1) + C$ **(l)** $-4 \log_e(1 - 3x) + C$
3(a) $\log_e 5$ **(b)** $\log_e 3$ **(c)** $\log_e 8 - \log_e 2 = 2 \log_e 2$
(d) $\log_e 9 - \log_e 3 = \log_e 3$
(e) $\frac{1}{2}(\log_e 8 - \log_e 2) = \log_e 2$
(f) $\frac{1}{5}(\log_e 75 - \log_e 25) = \frac{1}{5} \log_e 3$

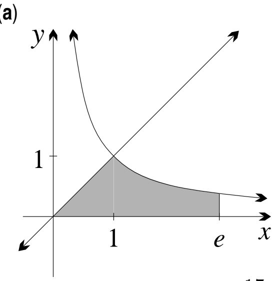
4(a) $\log_e 2 \doteqdot 0{\cdot}6931$ **(b)** $\log_e 5 - \log_e 3 \doteqdot 0{\cdot}5108$
(c) $3\log_e 2 \doteqdot 2{\cdot}079$ **(d)** $\frac{2}{3}\log_e 2 \doteqdot 0{\cdot}4621$
(e) $\frac{1}{2}\log_e 7 \doteqdot 0{\cdot}9730$ **(f)** $\frac{3}{2}\log_e 3 \doteqdot 1{\cdot}648$
(g) $\log_e \frac{5}{2} \doteqdot 0{\cdot}9163$ **(h)** $\frac{3}{2}\log_e 5 \doteqdot 2{\cdot}414$
(i) $\frac{5}{2}\log_e 3 \doteqdot 2{\cdot}747$
5(a) 1 **(b)** 2 **(c)** 3 **(d)** $\frac{1}{2}$
6(a) $x + \log_e x + C$ **(b)** $\frac{1}{5}x + \frac{3}{5}\log_e x + C$
(c) $\frac{2}{3}\log_e x - \frac{1}{3}x + C$ **(d)** $\frac{1}{9}\log_e x - \frac{8}{9}x + C$
(e) $3x - 2\log_e x + C$ **(f)** $x^2 + x - 4\log_e x + C$
(g) $\frac{3}{2}x^2 + 4\log_e x + \frac{1}{x} + C$ **(h)** $\frac{1}{3}x^3 - \log_e x - \frac{2}{x} + C$
7(a) $\log_e(x^2 - 9) + C$ **(b)** $\log_e(3x^2 + x) + C$
(c) $\log_e(x^2 + x - 3) + C$ **(d)** $\log_e(2 + 5x - 3x^2) + C$
(e) $\frac{1}{2}\log_e(x^2 + 6x - 1) + C$
(f) $\frac{1}{4}\log_e(12x - 3 - 2x^2) + C$ **(g)** $\log_e(1 + e^x) + C$
(h) $-\log_e(1 + e^{-x}) + C$ **(i)** $\log_e(e^x + e^{-x}) + C$
8(a) $f(x) = x + 2\log x,\ f(2) = 2 + 2\log 2$
(b) $f(x) = x^2 + \frac{1}{3}\log x + 1,\ f(2) = 5 + \frac{1}{3}\log 2$
(c) $f(x) = 3x + \frac{5}{2}\log(2x - 1) - 3,\ f(2) = 3 + \frac{5}{2}\log 3$
(d) $f(x) = 2x^3 + 5\log(3x + 2) - 2,\ f(2) = 14 + 5\log 8$
9(a) $y = \frac{1}{4}(\log x + 2),\ x = e^{-2}$
(b) $y = 2\log(x + 1) + 1$
(c) $y = \log\left(\frac{x^2 + 5x + 4}{10}\right) + 1,\ y(0) = \log\frac{4}{10} + 1$
(d) $y = 2\log x + x + C,\ y = 2\log x + x,$
$y(2) = \log 4 + 2$
(e) $f(x) = 2 + x - \log x,\ f(e) = e + 1$
10(a) $\frac{1}{2}\log(2x + b) + C$ **(b)** $\frac{1}{3}\log(3x - k) + C$
(c) $\frac{1}{a}\log(ax + 3) + C$ **(d)** $\frac{1}{m}\log(mx - 2) + C$
(e) $\log(px + q) + C$ **(f)** $\frac{A}{s}\log(sx - t) + C$
11(a) $\log(x^3 - 5) + C$ **(b)** $\log(x^4 + x - 5) + C$
(c) $\frac{1}{4}\log(x^4 - 6x^2) + C$ **(d)** $\frac{1}{2}\log(5x^4 - 7x^2 + 8) + C$
(e) $2\log 2$ **(f)** $\log\frac{4(e+1)}{e+2}$
12(a) $f(x) = x + \log x + \frac{1}{2}x^2$
(b) $g(x) = x^2 - 3\log x + \frac{4}{x} - 6$
13(a) $a = e^5$ **(b)** $a = e^{-4}$
14 $\frac{1}{3}(e^3 - e^{-3}) + 2$
15(a) $y' = \log_x$ **(b)(i)** $x\log_e x - x + C$ **(ii)** $\frac{\sqrt{e}}{2}$
16(b) $\frac{1}{2}x^2\log x - \frac{1}{4}x^2$ **(c)** $2\log 2 - 1 - \frac{e^2}{4}$
17(a) $\frac{2\log_e x}{x}$ **(b)** $\frac{3}{8}$
18 $\log(\log x) + C$
19 Applying the log laws to the second solution, $\frac{1}{5}(\log 5x + 5C_2) = \frac{1}{5}(\log x + \log 5 + 5C_2)$, which is the same as the first solution with $C_1 = \frac{1}{5}(\log 5 + 5C_2)$.
20(a) $\log(1 + \sqrt{2})$ **(b)** $\log(2 + \sqrt{3})$

Exercise **3F** (Page 129)

1(b) $e \doteqdot 2{\cdot}7$
2(a)(i) $1\,\mathrm{u}^2$ **(ii)** $\log_e 5 \doteqdot 1{\cdot}609\,\mathrm{u}^2$
(b)(i) $1\,\mathrm{u}^2$ **(ii)** $2\log_e 2 \doteqdot 1{\cdot}386\,\mathrm{u}^2$
(c)(i) $14\,\mathrm{u}^2$ **(ii)** $14\log_e 5 \doteqdot 22{\cdot}53\,\mathrm{u}^2$
3(a) $\log_e 2$ square units
(b) $(\log_e 3 - \log_e 2)$ square units
(c) $-\log_e \frac{1}{3} = \log_e 3$ square units
(d) $\log_e 2 - \log_e \frac{1}{2} = 2\log_e 2$ square units
4(a)(i) $\frac{1}{2}(\log_e 11 - \log_e 5) \doteqdot 0{\cdot}3942\,\mathrm{u}^2$
(ii) $\frac{1}{2}\log_e 3 \doteqdot 0{\cdot}5493\,\mathrm{u}^2$
(b)(i) $\frac{1}{3}(\log_e 5 - \log_e 2) \doteqdot 0{\cdot}3054\,\mathrm{u}^2$
(ii) $\frac{1}{3}\log_e 10 \doteqdot 0{\cdot}7675\,\mathrm{u}^2$
(c)(i) $\frac{1}{2}\log_e 3 \doteqdot 0{\cdot}5493\,\mathrm{u}^2$ **(ii)** $\log_e 3 \doteqdot 1{\cdot}099\,\mathrm{u}^2$
(d)(i) $9\,\mathrm{u}^2$ **(ii)** $3(\log_e 11 - \log_e 2) \doteqdot 5{\cdot}114\,\mathrm{u}^2$
5(a) $\log 2 + 1\,\mathrm{u}^2$ **(b)** $2\log 2 + \frac{15}{8}\,\mathrm{u}^2$ **(c)** $\log 3 + 8\frac{2}{3}\,\mathrm{u}^2$
6(a) $(6 - 3\log_e 3)\,\mathrm{u}^2$ **(b)** $(4 - \log_e 3)\,\mathrm{u}^2$
(c) $(3\frac{3}{4} - 2\log_e 4)\,\mathrm{u}^2$ **(d)** $(3\frac{3}{4} - 2\log_e 4)\,\mathrm{u}^2$
7(a)



(b) $4(1 - \log_e 2)\,\mathrm{u}^2$
8(a) $2\log_e 2\,\mathrm{u}^2$ **(b)** $(1 - \log_e 2)\,\mathrm{u}^2$
9(a) $(\log_e 4)\,\mathrm{u}^2$ **(b)** $(6 - 3\log_e 3)\,\mathrm{u}^2$ **(c)** $\frac{1}{2}\,\mathrm{u}^2$
(d) $2 - \frac{1}{2}\log_e 3\,\mathrm{u}^2$
10(a) $(\frac{1}{3}, 3)$ and $(1, 1)$ **(b)** $(\frac{4}{3} - \log 3)\,\mathrm{u}^2$
11(a) $2x$ **(b)** $\frac{1}{2}\log_e 5 \doteqdot 0{\cdot}805\,\mathrm{u}^2$
12(a) $2(x + 1)$ **(b)** $\frac{1}{2}\log 2\,\mathrm{u}^2$
13 $\pi\log 6\,\mathrm{u}^3$
14(a)

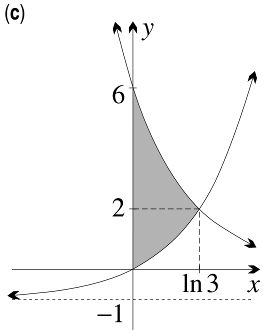


(b) $\frac{3}{2}\,\mathrm{u}^2$
15(a) $\log 2 \doteqdot 0{\cdot}693$ **(b)** $\frac{17}{24} \doteqdot 0{\cdot}708$ **(c)** $\frac{25}{36} \doteqdot 0{\cdot}694$
16(a) $\pi\log 2\,\mathrm{u}^3$ **(b)** $\pi\log 16\,\mathrm{u}^3$ **(c)** $\pi(\frac{25}{6} + \log 36)\,\mathrm{u}^3$
17(a) $\log 3 \doteqdot 1{\cdot}0986$ square units
(b) $1{\cdot}1667$ square units **(c)** $1{\cdot}1111$ square units
18(a) $3{\cdot}9828$ square units **(b)** $4{\cdot}0415$ square units
19(b) $(e - 1)\,\mathrm{u}^2$ **(c)** $1\,\mathrm{u}^2$
20(a) $\pi\left(\frac{15}{2} - \log 4\right)\mathrm{u}^3$

(b) $\pi\left(\frac{3}{2}+\log 4\right)\text{u}^3$.
There is a difference because $(a-b)^2 \neq a^2-b^2$.

21(b) $(\log 3, 2)$

(c)



(d) $(2+\log 3)\,\text{u}^2$

Exercise 3G (Page 138)

1(a) $1{\cdot}58$ **(b)** $2{\cdot}32$ **(c)** $3{\cdot}32$ **(d)** $1{\cdot}72$ **(e)** $2{\cdot}02$
(f) $1{\cdot}89$ **(g)** $3{\cdot}50$ **(h)** $2{\cdot}63$

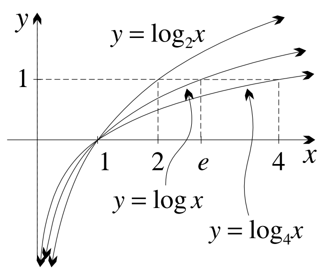
2(a) $\frac{1}{x\log_e 2}$ **(b)** $\frac{1}{x\log_e 10}$ **(c)** $\frac{1}{x\log_e 5}$ **(d)** $\frac{1}{x\log_e 3}$

3(a) $3^x\ \log_e 3$ **(b)** $4^x\ \log_e 4$ **(c)** $2^x\ \log_e 2$
(d) $10^x\ \log_e 10$

4(a) $\frac{2^x}{\log_e 2}+C$ **(b)** $\frac{6^x}{\log_e 6}+C$ **(c)** $\frac{7^x}{\log_e 7}+C$
(d) $\frac{3^x}{\log_e 3}+C$

5(a) $\frac{1}{\log_e 2} \doteqdot 1{\cdot}443$ **(b)** $\frac{2}{\log_e 3} \doteqdot 1{\cdot}820$
(c) $\frac{24}{5\log_e 5} \doteqdot 2{\cdot}982$ **(d)** $\frac{15}{\log_e 4} \doteqdot 10{\cdot}82$

6(b)



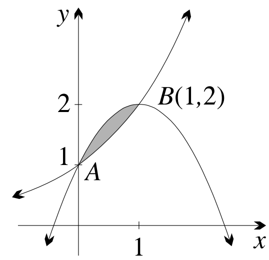
7(a) $\frac{1}{\log_e 2}$ **(b)** $y=\frac{1}{\log_e 2}(x-1)$
(c)(i) $y=\frac{1}{\log_e 3}(x-1)$ **(ii)** $y=\frac{1}{\log_e 5}(x-1)$

8(a) $\frac{1}{\log_e 2} \doteqdot 1{\cdot}4427$ **(b)** $2+\frac{8}{3\log_e 3} \doteqdot 4{\cdot}4273$
(c) $\frac{99}{\log_e 10}-20 \doteqdot 32{\cdot}9952$

9 $y=\frac{\log_e x}{\log_e 10}$, $y'=\frac{1}{x\log_e 10}$ **(a)** $\frac{1}{10\log_e 10}$
(b) $x-10y\log_e 10+10(\log_e 10-1)=0$
(c) $x=\frac{1}{\log_e 10}$

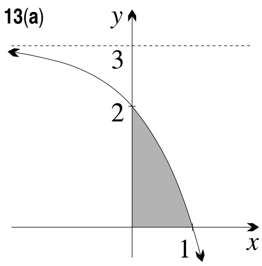
10(a) $y=\frac{1}{\log_e 2}(\frac{x}{3}-1+\log_e 3)$, $y=\frac{x}{3}-1+\log_e 3$,
$y=\frac{1}{\log_e 4}(\frac{x}{3}-1+\log_e 3)$
(b) They all meet the x-axis at $(3-3\log_e 3, 0)$

11(b) $(\frac{5}{3}-\frac{1}{\log_e 2})\,\text{u}^2$



12 intercepts $(0,7)$ and $(3,0)$, area $(24-\frac{7}{\log_e 2})$ square units

13(a)



(b) $(3-\frac{2}{\log_e 3})\,\text{u}^2$
(c) $\pi(9-\frac{8}{\log_e 3})\,\text{u}^3$

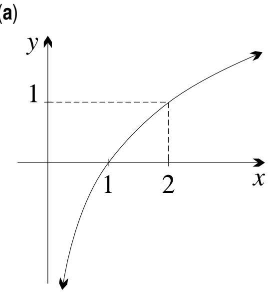
14(b) $\displaystyle\int_{-\frac{1}{2}}^{0} x+1-4^x\,dx$ **(c)** $\frac{3}{8}-\frac{1}{2\log_e 4}$

16(a) $x\log_e x-x+C$ **(b)** $10-\frac{9}{\log_e 10}$

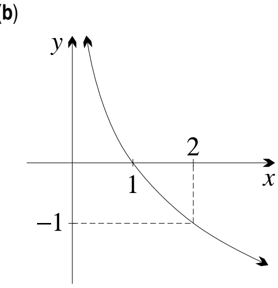
Review Exercise 3H (Page 139)

1(a) $1{\cdot}4314$ **(b)** $-0{\cdot}3010$ **(c)** $0{\cdot}6931$ **(d)** $2{\cdot}6391$
2(a) $1{\cdot}1761$ **(b)** $0{\cdot}4771$ **(c)** $1{\cdot}9459$ **(d)** $-1{\cdot}0986$
3(a) $2{\cdot}402$ **(b)** $5{\cdot}672$ **(c)** $5{\cdot}197$ **(d)** $3{\cdot}034$

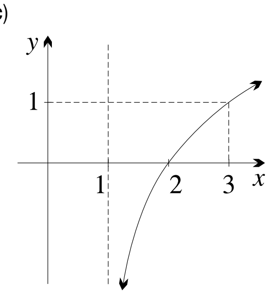
4(a)



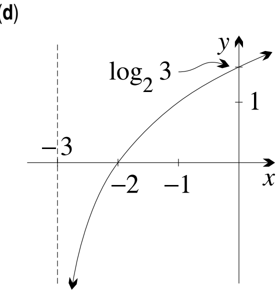
(b)



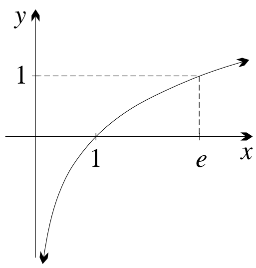
(c)



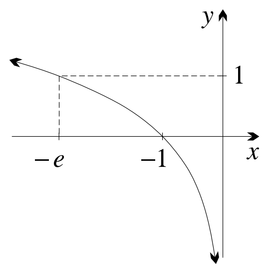
(d)



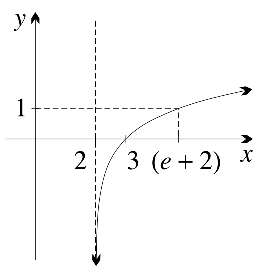
5(a)



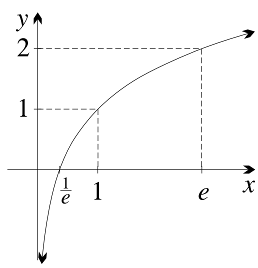
(b)



(c)



(d)



6(a) e **(b)** 3 **(c)** -1 **(d)** e

7(a) $\dfrac{1}{x}$ **(b)** $\dfrac{1}{x}$ **(c)** $\dfrac{1}{x+4}$ **(d)** $\dfrac{2}{2x-5}$ **(e)** $\dfrac{10}{5x-1}$

(f) $1+\dfrac{1}{x}$ **(g)** $\dfrac{2x-5}{x^2-5x+2}$ **(h)** $\dfrac{15x^4}{1+3x^5}$

(i) $8x-24x^2+\dfrac{2x}{x^2-2}$

8(a) $\dfrac{3}{x}$ **(b)** $\dfrac{1}{2x}$ **(c)** $\dfrac{1}{x}+\dfrac{1}{x+2}$ **(d)** $\dfrac{1}{x}-\dfrac{1}{x-1}$

9(a) $1+\log x$ **(b)** $\dfrac{e^x}{x}+e^x\log x$ **(c)** $\dfrac{\log x-1}{(\log x)^2}$

(d) $\dfrac{1-2\log x}{x^3}$

10 $y=3x+1$

12(a) $\log_e x+C$ **(b)** $3\log_e x+C$

(c) $\frac{1}{5}\log_e x+C$ **(d)** $\log_e(x+7)+C$

(e) $\frac{1}{2}\log_e(2x-1)+C$ **(f)** $-\frac{1}{3}\log_e(2-3x)+C$

(g) $\log_e(2x+9)+C$ **(h)** $-2\log_e(1-4x)+C$

13(a) $\log_e\frac{3}{2}$ **(b)** $\frac{1}{4}\log_e 13$ **(c)** 1 **(d)** 1

14(a) $\log_e(x^2+4)+C$ **(b)** $\log_e(x^3-5x+7)+C$

(c) $\frac{1}{2}\log_e(x^2-3)+C$ **(d)** $\frac{1}{4}\log_e(x^4-4x)+C$

15 $\log_e 2\,\mathrm{u}^2$

16(b) $12-5\log_e 5\,\mathrm{u}^2$

17 $\pi\log_e 2\,\mathrm{u}^3$

18(a) e^x **(b)** $2^x\log_e 2$ **(c)** $3^x\log_e 3$ **(d)** $5^x\log_e 5$

19(a) e^x+C **(b)** $\dfrac{2^x}{\log_e 2}+C$ **(c)** $\dfrac{3^x}{\log_e 3}+C$

(d) $\dfrac{5^x}{\log_e 5}+C$

Chapter Four

Exercise 4A (Page 145)

1(a) $\frac{\pi}{2}$ **(b)** $\frac{\pi}{4}$ **(c)** $\frac{\pi}{6}$ **(d)** $\frac{\pi}{3}$ **(e)** $\frac{2\pi}{3}$ **(f)** $\frac{5\pi}{6}$ **(g)** $\frac{3\pi}{4}$

(h) $\frac{5\pi}{4}$ **(i)** 2π **(j)** $\frac{5\pi}{3}$ **(k)** $\frac{3\pi}{2}$ **(l)** $\frac{7\pi}{6}$

2(a) 180° **(b)** 360° **(c)** 720° **(d)** 90° **(e)** 60°

(f) 45° **(g)** 120° **(h)** 150° **(i)** 135° **(j)** 270°

(k) 240° **(l)** 315° **(m)** 330°

3(a) $1{\cdot}274$ **(b)** $0{\cdot}244$ **(c)** $2{\cdot}932$ **(d)** $0{\cdot}377$ **(e)** $1{\cdot}663$

(f) $3{\cdot}686$

4(a) $114^\circ35'$ **(b)** $17^\circ11'$ **(c)** $82^\circ30'$ **(d)** $7^\circ3'$

(e) $183^\circ16'$ **(f)** $323^\circ36'$

5(a) $0{\cdot}91$ **(b)** $-0{\cdot}80$ **(c)** $0{\cdot}07$ **(d)** $1{\cdot}55$ **(e)** $2{\cdot}99$

(f) $-0{\cdot}97$

6(a) $\frac{1}{2}$ **(b)** $\frac{1}{\sqrt{2}}$ **(c)** $\frac{\sqrt{3}}{2}$ **(d)** $\sqrt{3}$ **(e)** 1 **(f)** $\frac{1}{2}$ **(g)** $\sqrt{2}$

(h) $\frac{1}{\sqrt{3}}$

7(a) $x=\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **(b)** $x=\frac{2\pi}{3}$ or $\frac{4\pi}{3}$

(c) $x=\frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **(d)** $x=\frac{\pi}{2}$ **(e)** $x=\frac{\pi}{6}$ or $\frac{11\pi}{6}$

(f) $x=\frac{\pi}{6}$ or $\frac{7\pi}{6}$ **(g)** $x=\pi$ **(h)** $x=\frac{5\pi}{4}$ or $\frac{7\pi}{4}$

(i) $x=\frac{\pi}{4}$ or $\frac{5\pi}{4}$

8(a) $\frac{\pi}{9}$ **(b)** $\frac{\pi}{8}$ **(c)** $\frac{\pi}{5}$ **(d)** $\frac{5\pi}{9}$ **(e)** $\frac{5\pi}{8}$ **(f)** $\frac{7\pi}{5}$

9(a) 15° **(b)** 72° **(c)** 400° **(d)** $247{\cdot}5^\circ$ **(e)** 306°

(f) 276°

10(a) $\frac{\pi}{3}$ **(b)** $\frac{5\pi}{6}$

11 $\frac{4\pi}{9}$

12(a) $\frac{\sqrt{3}}{2}$ **(b)** $-\frac{1}{2}$ **(c)** $-\frac{\sqrt{3}}{2}$ **(d)** $\sqrt{3}$ **(e)** -1 **(f)** $\frac{1}{2}$

(g) $-\frac{1}{\sqrt{2}}$ **(h)** $\frac{1}{\sqrt{3}}$

13(a) $0{\cdot}283$ **(b)** $0{\cdot}819$

14(a) $0{\cdot}841,\ 0{\cdot}997,\ 0{\cdot}909$ **(b)** $1{\cdot}0$

16(a) $0{\cdot}733$ **(b)** $0{\cdot}349$

17(a) 3π **(c)** $\frac{2\pi}{5},\ \frac{\pi}{2},\ \frac{3\pi}{5},\ \frac{7\pi}{10},\ \frac{4\pi}{5}$

Exercise 4B (Page 151)

1(a) 3π **(b)** $\frac{5\pi}{2}$ **(c)** $7{\cdot}5$ **(d)** 24 **(e)** $\frac{\pi}{4}$ **(f)** 2π

2(a) 2π **(b)** $\frac{4\pi}{3}$ **(c)** 2 **(d)** $\frac{2\pi}{3}$ **(e)** 6 **(f)** 20

3(a) $\frac{1}{2}(\frac{\pi}{2}-1)=\frac{1}{4}(\pi-2)$ **(b)** $18(\frac{\pi}{6}-\frac{1}{2})=3(\pi-3)$

4(a) $12\,\mathrm{cm}$ **(b)** $3\,\mathrm{cm}$ **(c)** $2\pi\,\mathrm{cm}$ **(d)** $\frac{3\pi}{2}\,\mathrm{cm}$

5(a) $32\,\mathrm{cm}^2$ **(b)** $96\,\mathrm{cm}^2$ **(c)** $8\pi\,\mathrm{cm}^2$ **(d)** $12\pi\,\mathrm{cm}^2$

6 $4\,\mathrm{cm}$

7 $1{\cdot}5$ radians

8(a) $2{\cdot}4\,\mathrm{cm}$ **(b)** $4{\cdot}4\,\mathrm{cm}$

9 $8727\,\mathrm{m}^2$

10(a) $8\pi\,\mathrm{cm}$ **(b)** $16\pi\,\mathrm{cm}^2$

11 84°

12 $11{\cdot}6\,\mathrm{cm}$

13(a) $6\pi\,\mathrm{cm}^2$ **(b)** $9\sqrt{3}\,\mathrm{cm}^2$ **(c)** $3\left(2\pi-3\sqrt{3}\right)\mathrm{cm}^2$

14(a) $\frac{4}{3}(5\pi-3)$ **(b)** $\frac{4}{3}(7\pi+3)$

15 $15\,\text{cm}^2$

16(a) $4(\pi+2)$ cm **(b)** $8\pi\,\text{cm}^2$

17(a) $\frac{25\pi}{2}\,\text{cm}^2$ **(b)** $\frac{25(4-\pi)}{2}\,\text{cm}^2$

18(a) $\frac{4\pi}{3}$ **(b)** 5 cm

19(a) 1·38 radians **(b)** $10\,\text{cm}^2$

20(a) $\frac{2\pi}{3}$ cm **(b)** $\frac{2\pi}{3}\,\text{cm}^2$ **(c)** 2π cm

(d) $\sqrt{3}\,\text{cm}^2$, $2\left(\pi-\sqrt{3}\right)\text{cm}^2$

21(a) For $\sin\theta = 0$, that is, for $\theta = 0,\ \pi,\ -\pi,\ 2\pi,\ -2\pi,\ \ldots$. **(b)** For $\sin\theta < 0$, that is, for θ in quadrants 3 and 4. **(c)** The area of a minor segment is less than that of the corresponding minor sector, but the area of a major segment is greater than that of the corresponding major sector.

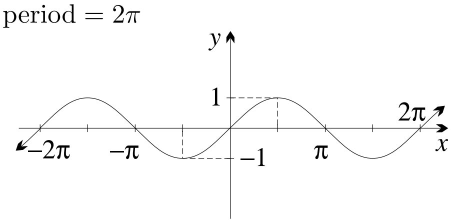
22 $\frac{4}{3}\left(4\pi-3\sqrt{3}\right)\text{cm}^2$

23(c) $3\sqrt{55}\,\pi\,\text{cm}^3$ **(d)** $24\pi\,\text{cm}^2$

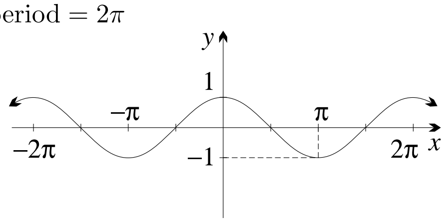
24(b) $9\,\text{cm}^2$

Exercise **4C** (Page 157)

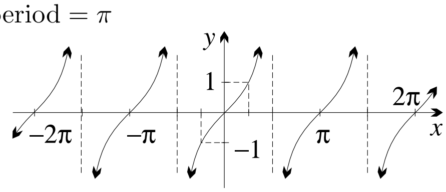
1(a) period $= 2\pi$



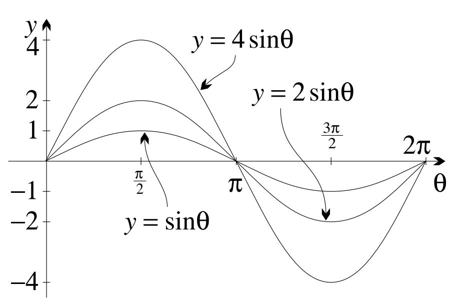
(b) period $= 2\pi$



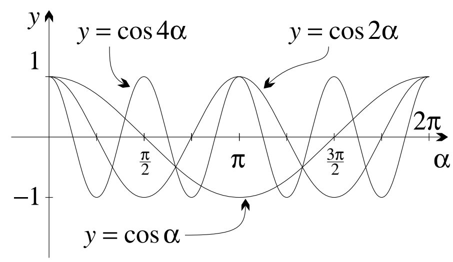
(c) period $= \pi$



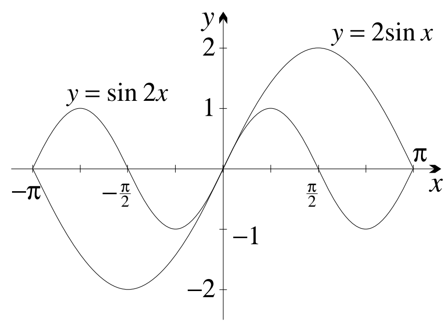
2



3



4



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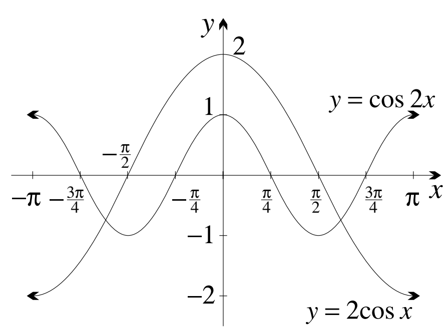

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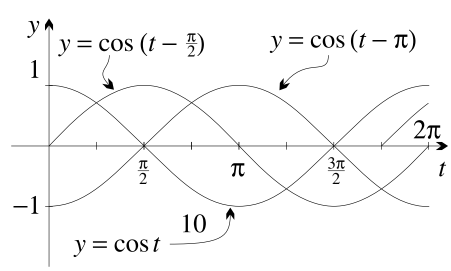
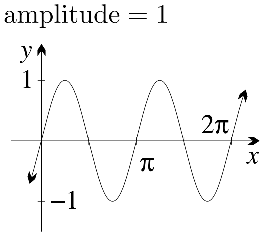
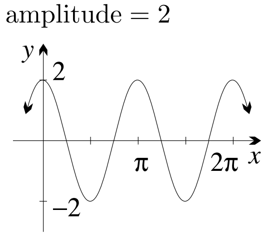


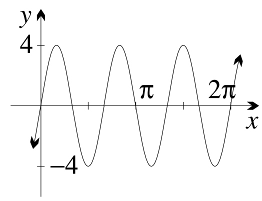
7(a) period $= \pi$, amplitude $= 1$



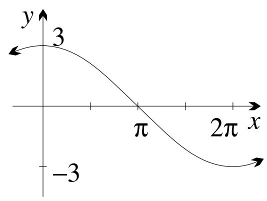
(b) period $= \pi$, amplitude $= 2$



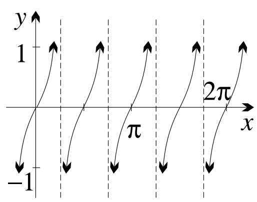
(c) period $= \frac{2\pi}{3}$, amplitude $= 4$



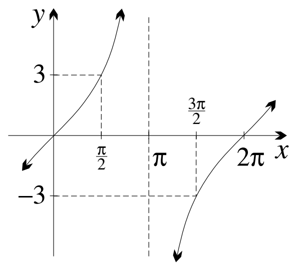
(d) period $= 4\pi$, amplitude $= 3$



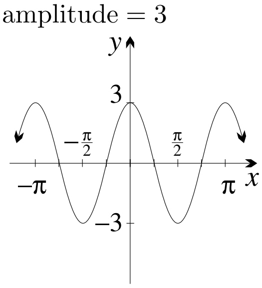
(e) period $= \frac{\pi}{2}$, no amplitude



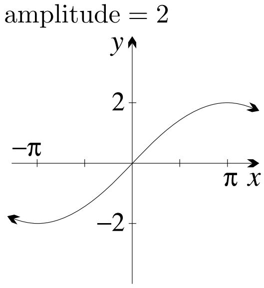
(f) period $= 2\pi$, no amplitude



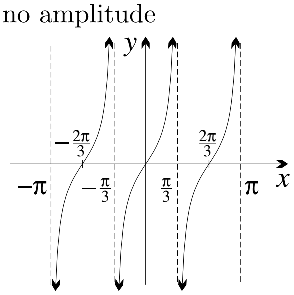
8(a) period $= \pi$, amplitude $= 3$



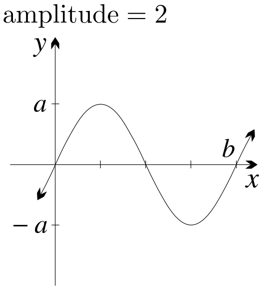
(b) period $= 4\pi$, amplitude $= 2$



(c) period $= \frac{2\pi}{3}$, no amplitude



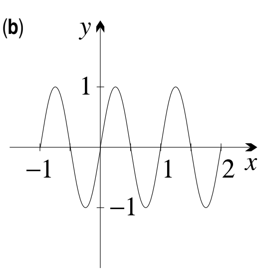
(d) period $= \frac{2\pi}{3}$, amplitude $= 2$



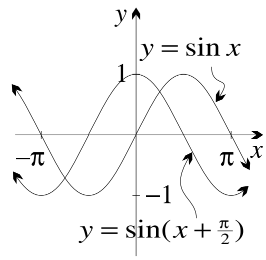
9(a) $a = 1$, $b = 2\pi$ **(b)** $a = 3$, $b = \pi$ **(c)** $a = 2$, $b = \frac{2\pi}{3}$ **(d)** $a = 4$, $b = \frac{\pi}{2}$

10(a) amplitude $= 1$, period $= 1$

(b)



11



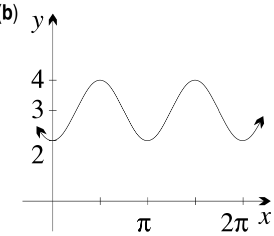
(c) $\sin(x + \frac{\pi}{2}) = \cos x$

12(a) 3 **(b)** 3 solutions, 1 positive solution

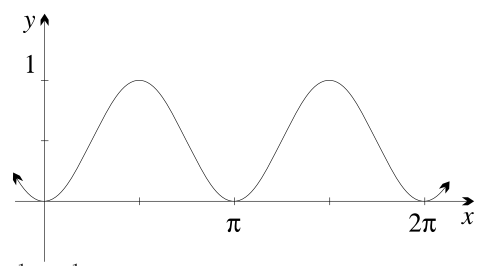
(c) Outside this domain the line is beyond the range of the sine curve.

13 $x \doteqdot 1{\cdot}9$, $x \doteqdot -1{\cdot}9$ or $x = 0$

14(b)



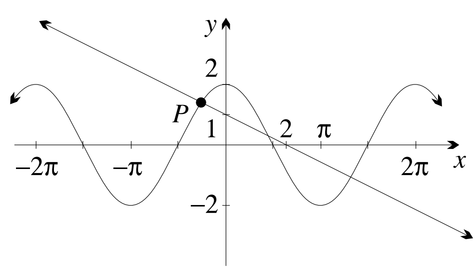
15



(a) $0, \frac{1}{2}, 1, \frac{1}{2}, 0$ **(b)** $\sin x$ has values from -1 to 1, so $\sin^2 x$ has values from 0 to 1.

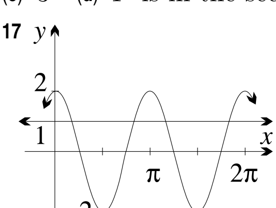
(c) period $= \pi$, amplitude $= \frac{1}{2}$

16



(c) 3 **(d)** P is in the second quadrant.

17



(c) 4 **(d)** $x = \frac{\pi}{6}$

(e) $\frac{\pi}{6} < x < \frac{5\pi}{6}$ or $\frac{7\pi}{6} < x < \frac{11\pi}{6}$

18(a) $x = -2\pi$, $x = -\pi$, $x = 0$, $x = \pi$, $x = 2\pi$

(b) each of its x-intercepts **(c)** translations to the right or left by 2π or by integer multiples of 2π

(d) translation right or left by π (e) translation to the right by $\frac{\pi}{2}$ or to the left by $\frac{3\pi}{2}$
(f) $x = \frac{\pi}{4}$, $x = -\frac{3\pi}{4}$
19(a)(ii) $2{\cdot}55$ (b) 146° (c)(ii) 205°

Exercise 4D (Page 162)

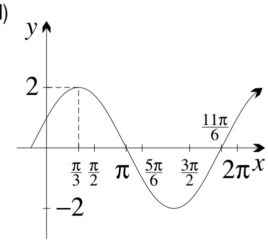
1(a) The entries under $0{\cdot}2$ are $0{\cdot}198\,669$, $0{\cdot}993\,347$, $0{\cdot}202\,710$, $1{\cdot}013\,550$, $0{\cdot}980\,067$. (b) 1 and 1
3(a) $\frac{\pi}{90}$ (b) $\sin 2^\circ = \sin\frac{\pi}{90} \doteqdot \frac{\pi}{90}$ (c) $0{\cdot}0349$
4(a) The entries under 5° are $0{\cdot}087\,27$, $0{\cdot}087\,16$, $0{\cdot}9987$, $0{\cdot}087\,49$, $1{\cdot}003$, $0{\cdot}9962$.
(b) $\sin x < x < \tan x$ (c)(i) 1 (ii) 1
(d) $x \le 0{\cdot}0774$ (correct to four decimal places), that is, $x \le 4^\circ 26'$.
6 87 metres
7 $26'$
12(a) $AB^2 = 2r^2(1-\cos x)$, arc $AB = rx$
(b) The arc is longer than the chord, so $\cos x$ is larger than the approximation.

Exercise 4E (Page 170)

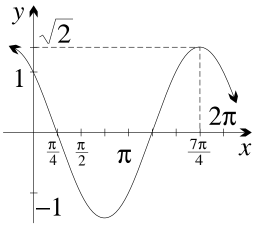
1(a) $\cos x$ (b) $-\sin x$ (c) $\sec^2 x$ (d) $2\cos x$
(e) $2\cos 2x$ (f) $-3\sin x$ (g) $-3\sin 3x$ (h) $4\sec^2 4x$
(i) $4\sec^2 x$ (j) $6\cos 3x$ (k) $4\sec^2 2x$ (l) $-8\sin 2x$
(m) $-2\cos 2x$ (n) $2\sin 2x$ (o) $-2\sec^2 2x$
(p) $\frac{1}{2}\sec^2\frac{1}{2}x$ (q) $-\frac{1}{2}\sin\frac{1}{2}x$ (r) $\frac{1}{2}\cos\frac{x}{2}$ (s) $\sec^2\frac{1}{5}x$
(t) $-2\sin\frac{x}{3}$ (u) $4\cos\frac{x}{4}$
2(a) $2\cos 2x,\ -4\sin 2x,\ -8\cos 2x,\ 16\sin 2x$
(b) $-10\sin 10x,\ -100\cos 10x,\ 1000\sin 10x,\ 10\,000\cos 10x$
(c) $\frac{1}{2}\cos\frac{1}{2}x,\ -\frac{1}{4}\sin\frac{1}{2}x,\ -\frac{1}{8}\cos\frac{1}{2}x,\ \frac{1}{16}\sin\frac{1}{2}x$
(d) $-\frac{1}{3}\sin\frac{1}{3}x,\ -\frac{1}{9}\cos\frac{1}{3}x,\ \frac{1}{27}\sin\frac{1}{3}x,\ \frac{1}{81}\cos\frac{1}{3}x$
3 $-2\sin 2x$ (a) 0 (b) -1 (c) $-\sqrt{3}$ (d) -2
4 $\frac{1}{4}\cos(\frac{1}{4}x+\frac{\pi}{2})$ (a) 0 (b) $-\frac{1}{4}$ (c) $\frac{1}{8}\sqrt{2}$ (d) $-\frac{1}{8}\sqrt{2}$
5(a) $x\cos x + \sin x$ (b) $2(\tan 2x + 2x\sec^2 2x)$
(c) $2x(\cos 2x - x\sin 2x)$ (d) $3x^2(\sin 3x + x\cos 3x)$
6(a) $\frac{x\cos x - \sin x}{x^2}$ (b) $\frac{-x\sin x - \cos x}{x^2}$
(c) $\frac{x(2\cos x + x\sin x)}{\cos^2 x}$ (d) $\frac{1+\sin x - x\cos x}{(1+\sin x)^2}$
7(a) $2x\cos(x^2)$ (b) $-2x\cos(1-x^2)$
(c) $-3x^2\sin(x^3+1)$ (d) $-\frac{1}{x^2}\cos(\frac{1}{x})$
(e) $-2\cos x\sin x$ (f) $3\sin^2 x\cos x$ (g) $2\tan x\sec^2 x$
(h) $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$
8(a) $2\pi\cos 2\pi x$ (b) $\frac{\pi}{2}\sec^2\frac{\pi}{2}x$ (c) $3\cos x - 5\sin 5x$
(d) $4\pi\cos\pi x - 3\pi\sin\pi x$ (e) $2\cos(2x-1)$
(f) $3\sec^2(1+3x)$ (g) $2\sin(1-x)$
(h) $-5\sin(5x+4)$ (i) $-21\cos(2-3x)$
(j) $-10\sec^2(10-x)$ (k) $3\cos(\frac{x+1}{2})$
(l) $-6\sin(\frac{2x+1}{5})$
9(d) $y = \cos x$
11(a) $e^{\tan x}\sec^2 x$ (b) $2e^{\sin 2x}\cos 2x$
(c) $2e^{2x}\cos(e^{2x})$ (d) $-\tan x$ (e) $\cot x$
(f) $-4\tan 4x$
12(a) $\cos^2 x - \sin^2 x$
(b) $14\sin 7x\cos 7x$ (c) $-15\cos^4 3x\sin 3x$
(d) $9\sin 3x(1-\cos 3x)^2$
(e) $2(\cos 2x\sin 4x + 2\sin 2x\cos 4x)$
(f) $15\tan^2(5x-4)\sec^2(5x-4)$
13(a) $\frac{-\cos x}{(1+\sin x)^2}$ (b) $\frac{1}{1+\cos x}$ (c) $\frac{-1}{1+\sin x}$
(d) $\frac{-1}{(\cos x+\sin x)^2}$
14(c)(i) The graphs are reflections of each other in the x-axis. (ii) The graphs are identical.
16(a) $y' = e^x\sin x + e^x\cos x,\ y'' = 2e^x\cos x$
(b) $y' = -e^{-x}\cos x - e^{-x}\sin x,\ y'' = 2e^{-x}\sin x$
18(a) $\log_b P - \log_b Q$

Exercise 4F (Page 176)

1(a) 1 (b) -1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$ (e) $\frac{1}{\sqrt{2}}$ (f) 1 (g) 2
(h) -2 (i) $\frac{\sqrt{3}}{4}$ (j) $\frac{1}{4}$ (k) 8 (l) $\sqrt{3}$
3(a) $y = -x + \pi$ (b) $2x - y = \frac{\pi}{2} - 1$
(c) $x + 2y = \frac{\pi}{6} + \sqrt{3}$ (d) $y = -2x + \frac{\pi}{2}$
(e) $x + y = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$
(f) $y = -\pi x + \pi^2$
4(a) $\frac{\pi}{2}, \frac{3\pi}{2}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$ (c) $\frac{\pi}{6}, \frac{5\pi}{6}$ (d) $\frac{5\pi}{6}, \frac{7\pi}{6}$
5(a) $12\sqrt{3}\,x - 6y = 2\sqrt{3}\,\pi - 3$, $6x + 12\sqrt{3}\,y = \pi + 6\sqrt{3}$
6(b) 1 and -1 (c) $x - y = \frac{\pi}{4} - \frac{1}{2}$, $x + y = \frac{\pi}{4} + \frac{1}{2}$
(d) $\frac{\pi^2-4}{32}$ units^2
7(b) $\frac{\pi}{2}, \frac{3\pi}{2}$
8(b) $0,\ \pi,\ 2\pi$
9(a) $y' = -\sin x + \sqrt{3}\cos x,\ y'' = -\cos x - \sqrt{3}\sin x$
(b) maximum turning point $(\frac{\pi}{3}, 2)$, minimum turning point $(\frac{4\pi}{3}, -2)$
(c) $(\frac{5\pi}{6}, 0)$, $(\frac{11\pi}{6}, 0)$ (d)

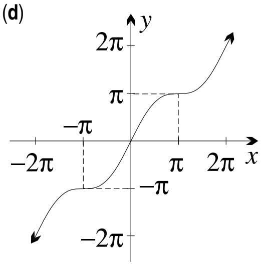


10(a) $y' = -\sin x - \cos x,\ y'' = -\cos x + \sin x$, minimum turning point $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$, maximum turning point $\left(\frac{7\pi}{4}, \sqrt{2}\right)$, points of inflexion $(\frac{\pi}{4}, 0)$, $(\frac{5\pi}{4}, 0)$

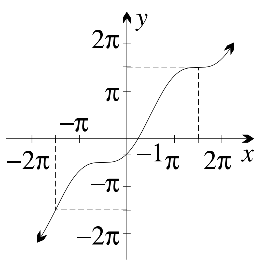


11(a) $y' = 1 + \cos x$ **(b)** $(-\pi, -\pi)$ and (π, π) are horizontal points of inflexion. **(c)** $(0, 0)$

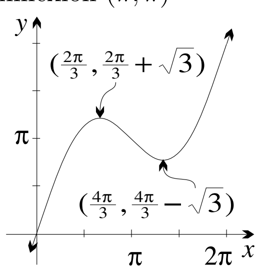
(d)



12 $y' = 1 + \sin x$, $y'' = \cos x$,
horizontal points of inflexion $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{3\pi}{2})$,
points of inflexion $(-\frac{3\pi}{2}, -\frac{3\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{2})$



13 maximum turning point $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right)$,
minimum turning point $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$,
inflexion (π, π)



15(b) minimum $\sqrt{3}$ when $\theta = \frac{\pi}{6}$,
maximum 2 when $\theta = 0$

Exercise 4G (Page 183)

1(a) $\tan x + C$ **(b)** $\sin x + C$ **(c)** $-\cos x + C$
(d) $\cos x + C$ **(e)** $2\sin x + C$ **(f)** $\frac{1}{2}\sin 2x + C$
(g) $\frac{1}{2}\sin x + C$ **(h)** $2\sin\frac{1}{2}x + C$ **(i)** $-\frac{1}{2}\cos 2x + C$
(j) $\frac{1}{5}\tan 5x + C$ **(k)** $\frac{1}{3}\sin 3x + C$ **(l)** $3\tan\frac{1}{3}x + C$
(m) $-2\cos\frac{x}{2} + C$ **(n)** $-5\sin\frac{1}{5}x + C$ **(o)** $2\cos 2x + C$
(p) $-\cos\frac{1}{4}x + C$ **(q)** $-36\tan\frac{1}{3}x + C$
(r) $6\sin\frac{x}{3} + C$

2(a) 1 **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{\sqrt{2}}$ **(d)** $\sqrt{3}$ **(e)** 1 **(f)** $\frac{3}{4}$ **(g)** 2
(h) 1 **(i)** 4
3(a) $y = 1 - \cos x$ **(b)** $y = \sin x + \cos 2x - 1$
(c) $y = -\cos x + \sin x - 3$
6(a) $\sin(x+2) + C$ **(b)** $\frac{1}{2}\sin(2x+1) + C$
(c) $-\cos(x+2) + C$ **(d)** $-\frac{1}{2}\cos(2x+1) + C$
(e) $\frac{1}{3}\sin(3x-2) + C$ **(f)** $\frac{1}{5}\cos(7-5x) + C$
(g) $-\tan(4-x) + C$ **(h)** $-3\tan(\frac{1-x}{3}) + C$
(i) $3\cos(\frac{1-x}{3}) + C$
7(a) $2\sin 3x + 8\cos\frac{1}{2}x + C$
(b) $4\tan 2x - 40\sin\frac{1}{4}x - 36\cos\frac{1}{3}x + C$
8(a) $f(x) = \sin\pi x,\ f(\frac{1}{3}) = \frac{1}{2}\sqrt{3}$
(b) $f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sin\pi x,\ f(\frac{1}{6}) = \frac{1}{\pi}$
(c) $f(x) = -2\cos 3x + x + (1 - \frac{\pi}{2})$
9(a) $-\cos(ax+b) + C$ **(b)** $\pi\sin\pi x + C$
(c) $\frac{1}{u^2}\tan(v+ux) + C$ **(d)** $\tan ax + C$
10(a) $1 + \tan^2 x = \sec^2 x,\ \tan x - x + C$
(b) $1 - \sin^2 x = \cos^2 x,\ 2\sqrt{3}$
11(a) $\log_e f(x) + C$
12(a) $\displaystyle\int \tan x = -\ln\cos x + C$
(b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx = \Big[\log\sin x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \log 2$
13(a)(i) $2x\cos x^2$ **(ii)** $\sin x^2 + C$
(b)(i) $-3x^2\sin x^3$ **(ii)** $-\frac{1}{3}\cos x^3 + C$
(c)(i) $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$ **(ii)** $2\tan\sqrt{x} + C$
14(a) 1 **(b)** $\frac{5}{24}$
15(a) $5\sin^4 x\cos x,\ \frac{1}{5}\sin^5 x + C$
(b) $3\tan^2 x\sec^2 x,\ \frac{1}{3}\tan^3 x + C$
16(a) $\cos x e^{\sin x},\ e - 1$ **(b)** $e^{\tan x} + C,\ e - 1$
17 $\sin 2x + 2x\cos 2x,\ \frac{\pi - 2}{8}$
18(a) $\frac{1}{8}(\pi + 2)$ **(b)** $\frac{1}{3}\log_e\left(1 + \sqrt{2}\right)$

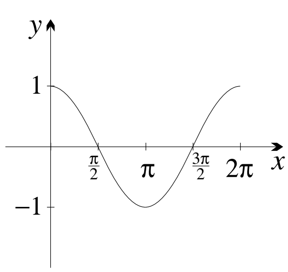
Exercise 4H (Page 189)

1(a) 1 square unit **(b)** $\frac{1}{2}$ square unit
2(a) 1 square unit **(b)** $\sqrt{3}$ square units
3(a) $1 - \frac{1}{\sqrt{2}}$ square units **(b)** $1 - \frac{\sqrt{3}}{2}$ square units
4(a) $\frac{1}{2}\sqrt{3}\,\text{u}^2$ **(b)** $\frac{1}{2}\sqrt{3}\,\text{u}^2$
5(a) $\frac{1}{2}\,\text{u}^2$ **(b)** $\frac{1}{2}\,\text{u}^2$ **(c)** $1 - \frac{\sqrt{3}}{2} = \frac{1}{2}(2 - \sqrt{3})\,\text{u}^2$
(d) $\frac{1}{3}(1 - \frac{1}{\sqrt{2}}) = \frac{1}{6}(2 - \sqrt{2})\,\text{u}^2$ **(e)** $\frac{2}{3}\sqrt{3}\,\text{u}^2$ **(f)** $4\,\text{u}^2$
6(a) $\left(\sqrt{2} - 1\right)\text{u}^2$ **(b)** $\frac{1}{4}\,\text{u}^2$ **(c)** $\left(\frac{\pi^2}{8} - 1\right)\text{u}^2$
(d) $(\pi - 2)\,\text{u}^2$
7(a) $\left(2 - \sqrt{2}\right)\text{u}^2$ **(b)** $1\frac{1}{2}\,\text{u}^2$
8(a) $\pi\sqrt{3}\,\text{u}^3$ **(b)** $\frac{\pi}{4}\,\text{u}^3$ **(c)** $\frac{\pi}{4}(\pi + 2)\,\text{u}^3$
9(a) $2\,\text{u}^2$ **(b)** $1\,\text{u}^2$

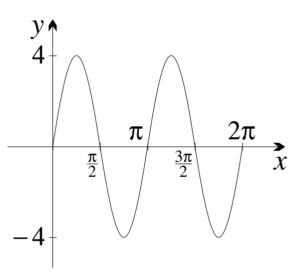
10(a) $2\,\mathrm{u}^2$ **(b)** $\sqrt{2}\,\mathrm{u}^2$ **(c)** $2\,\mathrm{u}^2$ **(d)** $\frac{1}{2}\,\mathrm{u}^2$ **(e)** $4\,\mathrm{u}^2$
(f) $1\,\mathrm{u}^2$
11(b) $\frac{4}{\pi}\,\mathrm{u}^2$
12 $3{\cdot}8\,\mathrm{m}^2$
13 $4\,\mathrm{u}^2$
14(a) $\ln 2\,\mathrm{u}^2$ **(b)** $\frac{\pi}{3}\left(3\sqrt{3}-\pi\right)\mathrm{u}^3$
15(b) $\frac{1}{2}\left(3+\sqrt{3}\right)\mathrm{u}^2$
16(c) $\frac{3}{4}\sqrt{3}\,\mathrm{u}^2$
17(b) $\frac{1}{6}\left(1+2\sqrt{2}\right)$
18(b) They are all $4\,\mathrm{u}^2$.
19(b) The curve is below $y=1$ just as much as it is above $y=1$, so the area is equal to the area of a rectangle n units long and 1 unit high.
20(b) $(\frac{\pi}{4}-\frac{1}{2}\ln 2)\,\mathrm{u}^2$ **(c)** $\pi(1-\ln 2)\,\mathrm{u}^3$

Review Exercise 4I (Page 192)

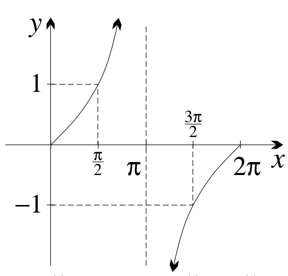
1(a) π **(b)** $\frac{\pi}{9}$ **(c)** $\frac{4\pi}{3}$ **(d)** $\frac{7\pi}{4}$
2(a) $30°$ **(b)** $108°$ **(c)** $540°$ **(d)** $300°$
3(a) $\frac{\sqrt{3}}{2}$ **(b)** $-\frac{1}{\sqrt{3}}$
4(a) $x=\frac{\pi}{4}$ or $\frac{7\pi}{4}$ **(b)** $x=\frac{2\pi}{3}$ or $\frac{5\pi}{3}$
5(a) $3\pi\,\mathrm{cm}$ **(b)** $24\pi\,\mathrm{cm}^2$
6 $16(\pi-2)\doteqdot 18{\cdot}3\,\mathrm{cm}^2$
7 $148°58'$
8(a) amplitude $=1$, period $=2\pi$



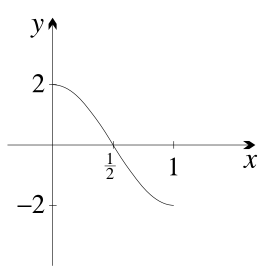
(b) amplitude $=4$, period $=\pi$



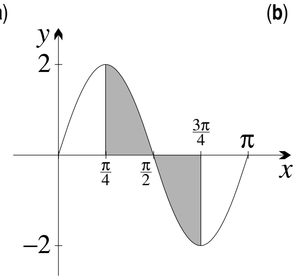
(c) no amplitude, period $=2\pi$



9 amplitude $=2$, period $=2$



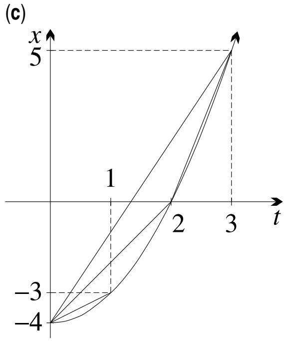
10(a) $5\cos x$ **(b)** $5\cos 5x$ **(c)** $-25\sin 5x$
(d) $5\sec^2(5x-4)$ **(e)** $\sin 5x + 5x\cos 5x$
(f) $\frac{-5x\sin 5x-\cos 5x}{x^2}$ **(g)** $5\sin^4 x\cos x$
(h) $5x^4\sec^2(x^5)$ **(i)** $-5\sin 5x e^{\cos 5x}$
(j) $\frac{5\cos 5x}{\sin 5x}=5\cot 5x$
11 $-\sqrt{3}$
12(a) $y=4x+\sqrt{3}-\frac{4\pi}{3}$ **(b)** $y=-\frac{\pi}{2}x+\frac{\pi^2}{4}$
13(a) $4\sin x+C$ **(b)** $-\frac{1}{4}\cos 4x+C$ **(c)** $4\tan\frac{1}{4}x+C$
14(a) $\sqrt{3}-1$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{2}$
15 $0{\cdot}089$
16 $y=2\sin\frac{1}{2}x-1$
17(a)

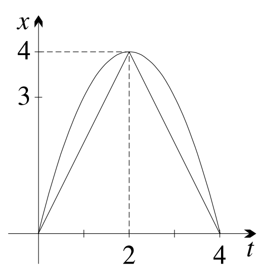


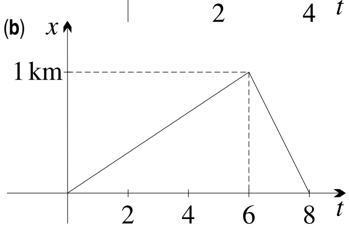
(b) $2\,\mathrm{u}^2$
18(a) $\frac{1}{2}\,\mathrm{u}^2$ **(b)** $\frac{3\sqrt{3}}{4}\,\mathrm{u}^2$
19(a) $\tan^2 x=\sec^2 x-1$ **(b)** $\pi(1-\frac{\pi}{4})\,\mathrm{u}^3$

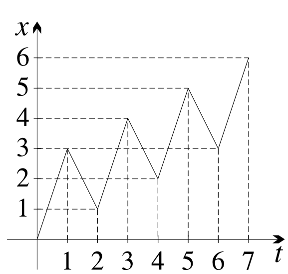
Chapter Five

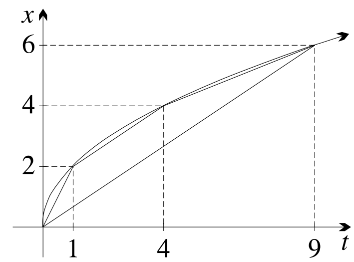
Exercise 5A (Page 199)

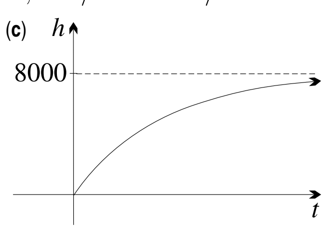
1(a) $x = 2$ **(b)** $x = 18$ **(c)** $4\,\text{m/s}$
2(a) $x = 0,\ 20;\ 10\,\text{cm/s}$ **(b)** $x = 4,\ 0;\ -2\,\text{cm/s}$
(c) $x = 3,\ 3;\ 0\,\text{cm/s}$ **(d)** $x = 1,\ 4;\ 1\frac{1}{2}\,\text{cm/s}$
3(a) $x = 0,\ -3,\ 0,\ 15,\ 48$
(b)(i) $-3\,\text{cm/s}$ **(ii)** $3\,\text{cm/s}$ **(iii)** $15\,\text{cm/s}$ **(iv)** $33\,\text{cm/s}$
4(a) $x = -4,\ -3,\ 0,\ 5$ **(c)**


(b)(i) $1\,\text{m/s}$
(ii) $2\,\text{m/s}$
(iii) $3\,\text{m/s}$
(iv) $5\,\text{m/s}$
5(a) $x = 0,\ 120,\ 72,\ 0$ **(b)** 240 metres
(c) [20 m/s **(d)(i)** $30\,\text{m/s}$ **(ii)** $-15\,\text{m/s}$ **(iii)** $0\,\text{m/s}$
6(a) $x = 0,\ 3,\ 4,\ 3,\ 0$ **(e)**


(c) The total distance travelled is 8 metres. The average speed is $2\,\text{m/s}$.
(d)(i) $2\,\text{m/s}$ **(ii)** $-2\,\text{m/s}$
(iii) $0\,\text{m/s}$
7(a)(i) 6 minutes **(b)**


(ii) 2 minutes
(c) $15\,\text{km/hr}$
(d) $20\,\text{km/hr}$
8(a) $x = 0,\ 3,\ 1,\ 4,\ 2,\ 5,\ 3,\ 6$ **(b)**


(c) 7 hours
(d) 18 metres, $2\frac{4}{7}\,\text{m/hr}$
(e) $\frac{6}{7}\,\text{m/hr}$ **(f)** those between 1 and 2 metres high or between 4 and 5 metres high
9(a) $t = 0,\ 1,\ 4,\ 9,\ 16$ **(c)**


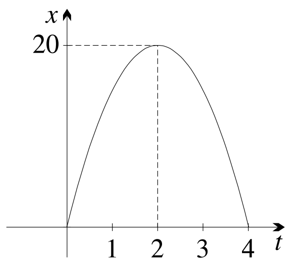
(b)(i) $2\,\text{cm/s}$
(ii) $\frac{2}{3}\,\text{cm/s}$
(iii) $\frac{2}{5}\,\text{cm/s}$
(iv) $\frac{2}{3}\,\text{cm/s}$
(c) They are parallel.
10(a)(i) $-1\,\text{m/s}$ **(ii)** $4\,\text{m/s}$ **(iii)** $-2\,\text{m/s}$
(b) 40 metres, $1\frac{1}{3}\,\text{m/s}$ **(c)** 0 metres, $0\,\text{m/s}$
(d) $2\frac{2}{19}\,\text{m/s}$
11(a)(i) once **(ii)** three times **(iii)** twice
(b)(i) when $t = 4$ and when $t = 14$
(ii) when $0 \le t < 4$ and when $4 < t < 14$
(c) It rises 2 metres, at $t = 8$.
(d) It sinks 1 metre, at $t = 17$.
(e) As $t \to \infty$, $x \to 0$, meaning that eventually it ends up at the surface.
(f)(i) $-1\,\text{m/s}$ **(ii)** $\frac{1}{2}\,\text{m/s}$ **(iii)** $-\frac{1}{3}\,\text{m/s}$
(g)(i) 4 metres **(ii)** 6 metres **(iii)** 9 metres
(iv) 10 metres
(h)(i) $1\,\text{m/s}$ **(ii)** $\frac{3}{4}\,\text{m/s}$ **(iii)** $\frac{9}{17}\,\text{m/s}$
12(b) $x = 3$ and $x = -3$ **(c)** $t = 4$, $t = 20$
(d) $t = 8$, $t = 16$ **(e)** $8 < t < 16$
(f) $12\,\text{cm}$, $\frac{3}{4}\,\text{cm/s}$
13(a) amplitude: 4 metres, period: 12 seconds
(b) 10 times **(c)** $t = 3,\ 15,\ 27,\ 39,\ 51$
(d) It travels 16 metres with average speed $1\frac{1}{3}\,\text{m/s}$.
(e) $x = 0$, $x = 2$ and $x = 4$, $2\,\text{m/s}$ and $1\,\text{m/s}$
14(a) When $t = 0$, $h = 0$. As $t \to \infty$, $h \to 8000$. **(c)**


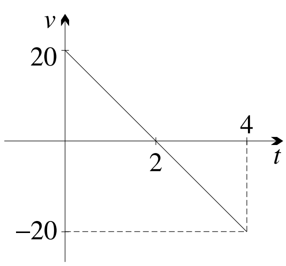
(b) 0, 3610, 5590, 6678
(d) $361\,\text{m/min}$, $198\,\text{m/min}$, $109\,\text{m/min}$

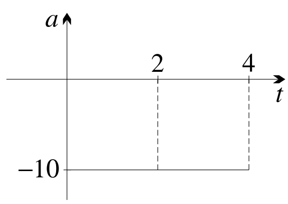
Exercise 5B (Page 209)

1(a) $v = -2t$ **(b)** $a = -2$
(c) $x = 11$ metres, $v = -6\,\text{m/s}$, $a = -2\,\text{m/s}^2$
(d) distance from origin: 11 metres, speed: $6\,\text{m/s}$
2(a) $v = 10t - 10$, $a = 10$. When $t = 1$, $x = -5$ metres, $v = 0\,\text{m/s}$, $a = 10\,\text{m/s}^2$.
(b) $v = 3 - 6t^2$, $a = -12t$. When $t = 1$, $x = 1$ metre, $v = -3\,\text{m/s}$, $a = -12\,\text{m/s}^2$.
(c) $v = 4t^3 - 2t$, $a = 12t^2 - 2$. When $t = 1$, $x = 4$ metres, $v = 2\,\text{m/s}$, $a = 10\,\text{m/s}^2$.
3(a) $v = 2t - 10$

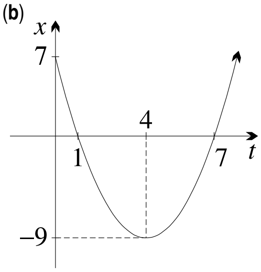
(b) displacement: -21 cm, distance from origin: 21 cm, velocity: $v=-4$ cm/s, speed: $|v|=4$ cm/s **(c)** When $v=0$, $t=5$ and $x=-25$.
4(a) $v=3t^2-12t$, $a=6t-12$ **(b)** When $t=0$, $x=0$ cm, $|v|=0$ cm/s and $a=-12$ cm/s^2. **(c)** left ($x=-27$ cm) **(d)** left ($v=-9$ cm/s) **(e)** right ($a=6$ cm/s^2) **(f)** When $t=4$, $v=0$ cm/s and $x=-32$ cm. **(g)** When $t=6$, $x=0$, $v=36$ cm/s and $|v|=36$ cm/s.
5(a) $v=\cos t$, $a=-\sin t$, 1 cm, 0 cm/s, -1 cm/s^2
(b) $v=-\sin t$, $a=-\cos t$, 0 cm, -1 cm/s, 0 cm/s^2
6(a) $v=e^t$, $a=e^t$, e metres, e m/s, e m/s^2
(b) $v=-e^{-t}$, $a=e^{-t}$, $1/e$ metres, $-1/e$ m/s, $1/e$ m/s^2
7(a) $x=5t(4-t)$
$v=20-10t$
$a=-10$

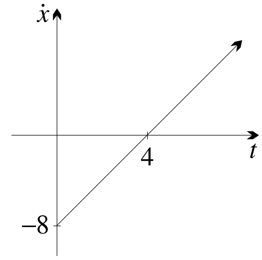


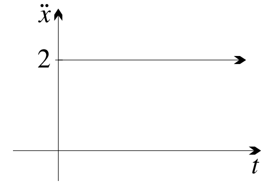




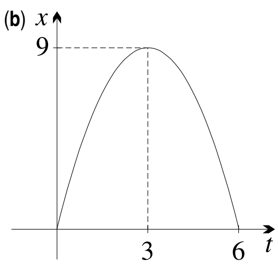
(b) 20 m/s **(c)** It returns at $t=4$; both speeds are 20 m/s. **(d)** 20 metres after 2 seconds
(e) -10 m/s^2. Although the ball is stationary, its velocity is changing, meaning that its acceleration is nonzero.
8 $\dot{x}=-4e^{-4t}$, $\ddot{x}=16e^{-4t}$ **(a)** e^{-4t} is positive, for all t, so $\dot{x}$ is always negative and $\ddot{x}$ is always positive. **(b)(i)** $x=1$ **(ii)** $x=0$ **(c)(i)** $\dot{x}=-4$, $\ddot{x}=16$ **(ii)** $\dot{x}=0$, $\ddot{x}=0$
9 $v=2\pi\cos\pi t$, $a=-2\pi^2\sin\pi t$
(a) When $t=1$, $x=0$, $v=-2\pi$ and $a=0$.
(b)(i) right ($v=\pi$) **(ii)** left ($a=-\pi^2\sqrt{3}$)
10(a) $x=(t-7)(t-1)$
$\dot{x}=2(t-4)$
$\ddot{x}=2$
(b)

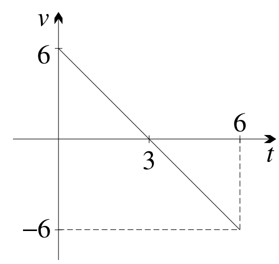




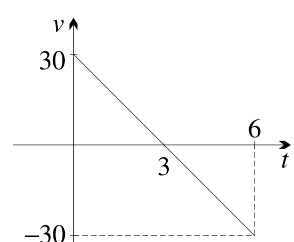


(c)(i) $t=1$ and $t=7$ **(ii)** $t=4$ **(d)(i)** 7 metres when $t=0$ **(ii)** 9 metres when $t=4$ **(iii)** 27 metres when $t=10$ **(e)** -1 m/s, $t=3\frac{1}{2}$, $x=-8\frac{3}{4}$
(f) 25 metres, $3\frac{4}{7}$ m/s
11(a) $x=t(6-t)$, $v=2(3-t)$, $a=-2$
(b)



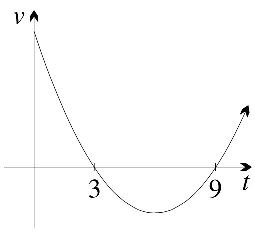


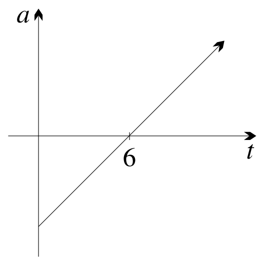
(c)(i) When $t=2$, it is moving upwards and accelerating downwards. **(ii)** When $t=4$, it is moving downwards and accelerating downwards.
(d) $v=0$ when $t=3$. It is stationary for zero time, 9 metres up the plane, and is accelerating downwards at 2 m/s^2.
(e) 4 m/s. When $v=4$, $t=1$ and $x=5$.
(f) All three average speeds are 3 m/s.
12(a) 45 metres, 3 seconds, 15 m/s
(b) 30 m/s, 20, 10, 0, -10, -20, -30
(c) 0 seconds
(d) The acceleration was always negative.



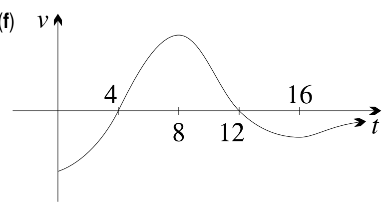
(e) The velocity was decreasing at a constant rate of 10 m/s every second.
13(a) 8 metres, when $t=3$
(b)(i) when $t=3$ and $t=9$ (because the gradient is zero) **(ii)** when $0\le t<3$ and when $t>9$ (because the gradient is positive) **(iii)** when $3<t<9$ (because the gradient is negative)
(c) $x=0$ again when $t=9$. Then $v=0$ (because the gradient is zero) and it is accelerating forwards (because the concavity is upwards).
(d) at $t=6$ (at the point of inflexion the second derivative is zero), $x=4$, moving backwards
(e) $0\le t<6$ **(f)(i)** $t\doteqdot 4, 12$ **(ii)** $t\doteqdot 10$

(g)





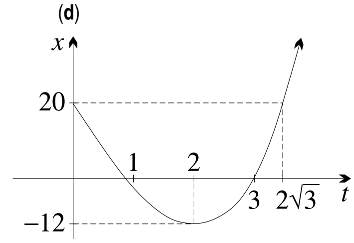
14(a) $x = 4\cos\frac{\pi}{4}t$, $v = -\pi\sin\frac{\pi}{4}t$, $a = -\frac{1}{4}\pi^2\cos\frac{\pi}{4}t$
(b) maximum displacement: $x = 4$ when $t = 0$ or $t = 8$, maximum velocity: π m/s when $t = 6$, maximum acceleration: $\frac{1}{4}\pi^2$ m/s^2 when $t = 4$
(c) 40 metres, 2 m/s **(d)** $1\frac{1}{3} < t < 6\frac{2}{3}$
(e)(i) $t = 4$ and $t = 8$ **(ii)** $4 < t < 8$
15(a)(i) $0 \le t < 8$ **(ii)** $0 \le t < 4$ and $t > 12$
(iii) roughly $8 < t < 16$ **(b)** roughly $t = 8$
(c)(i) $t \doteqdot 5,\ 11,\ 13$ **(ii)** $t \doteqdot 13,\ 20$ **(d)** twice
(e) 17 units
(f)



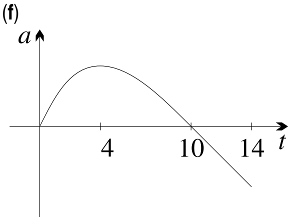
16(b)(i) downwards (Downwards is positive here.)
(ii) upwards **(c)** The velocity and acceleration tend to zero and the position tends to 12 metres below ground level. **(d)** $t = 2\log 2$ minutes. The speed then is 3 m/min (half the initial speed of 6 m/min) and the acceleration is $-1\frac{1}{2}$ m/min^2 (half the initial acceleration of -3 m/min^2).
(e) When $t = 18$, $x \doteqdot 11{\cdot}9985$ metres.
When $t = 19$, $x \doteqdot 11{\cdot}9991$ metres.

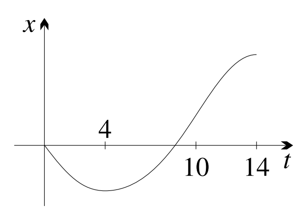
Exercise 5C (Page 218)

1(a) $x = t^3 - 3t^2 + 4$
(b) When $t = 2$, $x = 0$ metres and $v = 0$ m/s.
(c) $a = 6t - 6$ **(d)** When $t = 1$, $a = 0$ m/s^2 and $x = 2$ metres.
2(a) $v = -3t^2$ **(b)** When $t = 5$, $v = -75$ cm/s and $|v| = 75$ cm/s. **(c)** $x = -t^3 + 8$ **(d)** When $t = 2$, $x = 0$ cm and $a = -12$ cm/s^2.
3(a) $v = 8t - 16$ **(b)** $x = 4t^2 - 16t + 16$
(c) When $t = 0$, $x = 16$, $v = -16$ and $|v| = 16$.
4(a) $v = 6t - 30$ **(b)** 5 seconds **(c)** $a = 0$
5(a) $v = 2t - 20$ **(b)** $x = t^2 - 20t$ **(c)** When $v = 0$, $t = 10$ and $x = -100$. **(d)** When $x = 0$, $t = 0$ or 20. When $t = 20$, $v = 20$ m/s.
6(a) $v = 10t$, $x = 5t^2$
(b) When $t = 4$, $x = 80$ metres, which is at the bottom, and $v = 40$ m/s. **(c)** After 2 seconds, it has fallen 20 metres and its speed is 20 m/s.
(d) When $t = 2\sqrt{2}$, $x = 40$ metres, which is halfway down, and $v = 20\sqrt{2}$ m/s.
7(a) $a = -10$, $v = -10t - 25$, $x = -5t^2 - 25t + 120$
(b) 3 seconds **(c)** 55 m/s **(d)** 40 m/s
8(a) $\dot{x} = -4t$, $x = -2t^2$ **(b)** $\dot{x} = 3t^2$, $x = t^3$
(c) $\dot{x} = 2e^{\frac{1}{2}t} - 2$, $x = 4e^{\frac{1}{2}t} - 2t - 4$
(d) $\dot{x} = -\frac{1}{3}e^{-3t} + \frac{1}{3}$, $x = \frac{1}{9}e^{-3t} + \frac{1}{3}t - \frac{1}{9}$
(e) $\dot{x} = -4\cos 2t + 4$, $x = -2\sin 2t + 4t$
(f) $\dot{x} = \frac{1}{\pi}\sin\pi t$, $x = -\frac{1}{\pi^2}\cos\pi t + \frac{1}{\pi^2}$
(g) $\dot{x} = \frac{2}{3}t^{\frac{3}{2}}$, $x = \frac{4}{15}t^{\frac{5}{2}}$
(h) $\dot{x} = -12(t+1)^{-1} + 12$, $x = -12\log(t+1) + 12t$
9(a) $a = 0$, $x = -4t - 2$ **(b)** $a = 6$, $x = 3t^2 - 2$
(c) $a = \frac{1}{2}e^{\frac{1}{2}t}$, $x = 2e^{\frac{1}{2}t} - 4$
(d) $a = -3e^{-3t}$, $x = -\frac{1}{3}e^{-3t} - 1\frac{2}{3}$
(e) $a = 16\cos 2t$, $x = -4\cos 2t + 2$
(f) $a = -\pi\sin\pi t$, $x = \frac{1}{\pi}\sin\pi t - 2$
(g) $a = \frac{1}{2}t^{-\frac{1}{2}}$, $x = \frac{2}{3}t^{\frac{3}{2}} - 2$
(h) $a = -24(t+1)^{-3}$, $x = -12(t+1)^{-1} + 10$
10(a) $\dot{x} = 6t^2 - 24$, $x = 2t^3 - 24t + 20$
(b) $t = 2\sqrt{3}$, speed: 48 m/s
(c) $x = -12$ when $t = 2$.
(d)

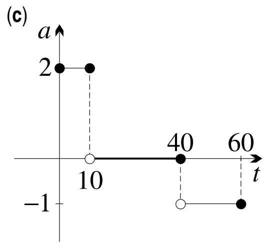


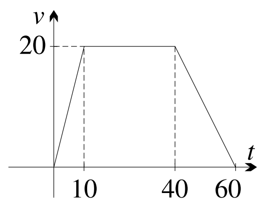
11(a) $k = 6$ and $C = -9$, hence $a = 6t$ and $v = 3t^2 - 9$. **(b)(i)** $x = t^3 - 9t + 2$
(ii) at $t = 3$ seconds (Put $x = 2$ and solve for t.)
12(a) $4 < t < 14$ **(b)** $0 < t < 10$ **(c)** $t = 14$
(d) $t = 4$ **(e)** $t \doteqdot 8$
(f)

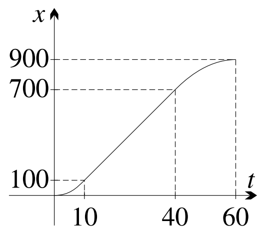




13(a) 20 m/s
(c)







14(a) $\ddot{x} = -4$, $x = 16t - 2t^2 + C$ **(b)** $x = C$ after 8 seconds, when the speed is 16 cm/s. **(c)** $\dot{x} = 0$ when $t = 4$. Maximum distance right is 32 cm when $t = 4$, maximum distance left is 40 cm when $t = 10$. The acceleration is $-4\,\text{cm/s}^2$ at all times.
(d) 104 cm, 10·4 cm/s

15(a) $x = \log(t+1) - 1$, $a = -\dfrac{1}{(t+1)^2}$
(b) $e - 1$ seconds, $v = 1/e$, $a = -1/e^2$
(c) The velocity and acceleration approach zero, but the particle moves to infinity.

16(a) $\dot{x} = -5 + 20e^{-2t}$, $x = -5t + 10 - 10e^{-2t}$, $t = \log 2$ seconds **(b)** It rises $7\frac{1}{2} - 5\log 2$ metres, when the acceleration is $10\,\text{m/s}^2$ downwards.
(c) The velocity approaches a limit of 5 m/s downwards, called the *terminal velocity*.

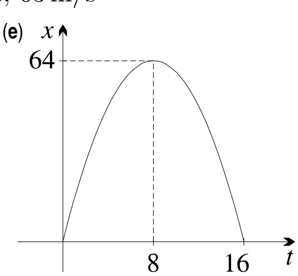
17(a) $v = 1 - 2\sin t$, $x = t + 2\cos t$
(b) $\frac{\pi}{2} < t < \frac{3\pi}{2}$ **(c)** $t = \frac{\pi}{6}$ when $x = \frac{\pi}{6} + \sqrt{3}$, and $\frac{5\pi}{6}$ when $x = \frac{5\pi}{6} - \sqrt{3}$.
(d) 3 m/s when $t = \frac{3\pi}{2}$, and -1 m/s when $t = \frac{\pi}{2}$

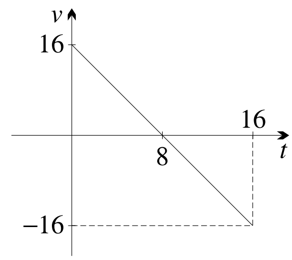
Review Exercise **5D** (Page 220)

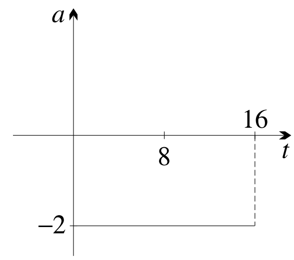
1(a) $x = 24$, $x = 36$, 6 cm/s
(b) $x = 16$, $x = 36$, 10 cm/s
(c) $x = -8$, $x = -8$, 0 cm/s
(d) $x = 9$, $x = 81$, 36 cm/s

2(a) $v = 40 - 2t$, $a = -2$, 175 m, 30 m/s, $-2\,\text{m/s}^2$
(b) $v = 3t^2 - 25$, $a = 6t$, 0 m, 50 m/s, $30\,\text{m/s}^2$
(c) $v = 8(t-3)$, $a = 8$, 16 m, 16 m/s, $8\,\text{m/s}^2$
(d) $v = -4t^3$, $a = -12t^2$, -575 m, -500 m/s, $-300\,\text{m/s}^2$ **(e)** $v = 4\pi\cos\pi t$, $a = -4\pi^2\sin\pi t$, 0 m, -4π m/s, $0\,\text{m/s}^2$ **(f)** $v = 21\,e^{3t-15}$, $a = 63\,e^{3t-15}$, 7 m, 21 m/s, $63\,\text{m/s}^2$

3(a) $v = 16 - 2t$, $a = -2$
(b) 60 m, -4 m/s, 4 m/s, $-2\,\text{m/s}^2$
(c) $t = 16$ s, $v = -16$ m/s
(d) $t = 8$ s, $x = 64$ m
(e)







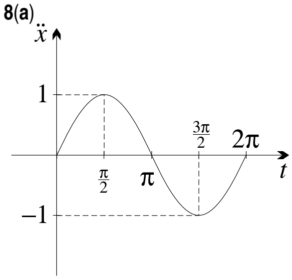
4(a) $a = 0$, $x = 7t + 4$ **(b)** $a = -18t$, $x = 4t - 3t^3 + 4$
(c) $a = 2(t-1)$, $x = \frac{1}{3}(t-1)^3 + 4\frac{1}{3}$
(d) $a = 0$, $x = 4$ **(e)** $a = -24\sin 2t$, $x = 4 + 6\sin 2t$
(f) $a = -36\,e^{-3t}$, $x = 8 - 4\,e^{-3t}$

5(a) $v = 3t^2 + 2t$, $x = t^3 + t^2 + 2$ **(b)** $v = -8t$, $x = -4t^2 + 2$ **(c)** $v = 12t^3 - 4t$, $x = 3t^4 - 2t^2 + 2$
(d) $v = 0$, $x = 2$ **(e)** $v = 5\sin t$, $x = 7 - 5\cos t$
(f) $v = 7\,e^t - 7$, $x = 7\,e^t - 7t - 5$

6(a) $\dot{x} = 3t^2 - 12$, $x = t^3 - 12t$ **(b)** When $t = 2$, $\dot{x} = 0$. **(c)** 16 cm **(d)** $2\sqrt{3}$ seconds, 24 cm/s, $12\sqrt{3}\,\text{cm/s}^2$ **(e)** As $t \to \infty$, $x \to \infty$ and $v \to \infty$.

7(a) The acceleration is $10\,\text{m/s}^2$ downwards.
(b) $v = -10t + 40$, $x = -5t^2 + 40t + 45$
(c) 4 seconds, 125 metres **(d)** When $t = 9$, $x = 0$.
(e) 50 m/s **(f)** 80 metres, 105 metres **(g)** 25 m/s

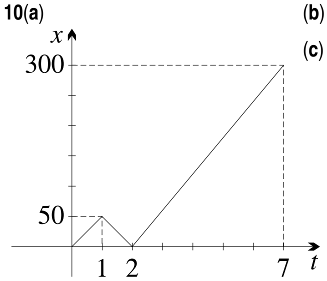
8(a)



(b) $t = \pi$ and $t = 2\pi$
(c) $\dot{x} = -\cos t$
(d) $t = \frac{\pi}{2}$
(e)(i) $x = 5 - \sin t$
(ii) $x = 4$

9(a) $v = 20$ m/s **(b)** $20\,e^{-t}$ is always positive.
(c) $a = -20\,e^{-t}$ **(d)** $-20\,\text{m/s}^2$ **(e)** $x = 20 - 20\,e^{-t}$
(f) As $t \to \infty$, $a \to 0$, $v \to 0$ and $x \to 20$.

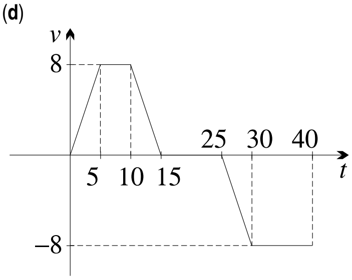
10(a)



(b) 400 km
(c) $57\frac{1}{7}$ km/hr

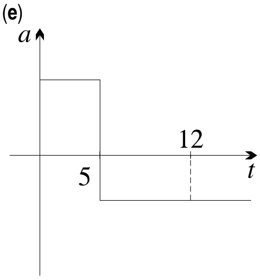
11(a) $x = 20$ m, $v = 0$
(b)(i) 8 m/s **(ii)** 0 **(iii)** -8 m/s
(c)(i) north (The graph is concave up.)
(ii) south (The graph is concave down.)
(iii) south (The graph is concave down.)

(d)

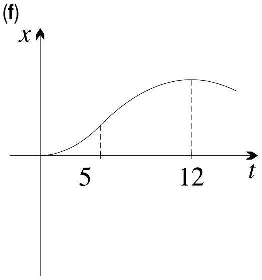


12(a) at $t = 5$ **(b)** at $t = 12$, $0 < t < 12$, $t > 12$
(c) $0 < t < 5$, $t > 5$
(d) at $t = 12$, when the velocity was zero

(e)



(f)



Chapter Six

Exercise **6A** (Page 228)

1(a) $T_3 - T_2 = T_2 - T_1 = 7$ **(b)** $a = 8,\ d = 7$
(c) 358 **(d)** 1490
2(a) $a = 2,\ d = 2$ **(b)** $250\,500$
3(a) $\frac{T_3}{T_2} = \frac{T_2}{T_1} = 2$ **(b)** $a = 5, r = 2$ **(c)** 320 **(d)** 635
4(a) $\frac{T_3}{T_2} = \frac{T_2}{T_1} = \frac{1}{2}$ **(b)** $a = 96,\ r = \frac{1}{2}$ **(c)** $\frac{3}{4}$
(d) $191\frac{1}{4}$ **(e)** $-1 < r < 1,\ S_\infty = 192$
5(a)(i) $a = 52,\ d = 6$ **(ii)** 14 **(iii)** 1274
(b)(i) 125 **(ii)** $-\frac{25}{49}$
(c)(i) $d = -3$ **(ii)** $T_{35} = -2$
(iii) $S_n = \frac{1}{2}n(203 - 3n)$
6(a)(i) $1{\cdot}01$ **(ii)** $T_{20} = 100 \times 1{\cdot}01^{19} \doteqdot 120{\cdot}81$
(iii) $2201{\cdot}90$ **(b)(i)** $\frac{3}{2}$ **(ii)** $26\,375$ **(iii)** $|r| = \frac{3}{2} > 1$
(c)(i) $\frac{1}{3}$ **(ii)** $|r| = \frac{1}{3} < 1,\ S_\infty = 27$
7(a) \$48 000, \$390 000 **(b)** the 7th year
8(a) $r = 1{\cdot}05$ **(b)** \$62 053, \$503 116
9(a) \$25 000, \$27 500, \$30 000, $d =$ \$2500
(b) \$20 000, \$23 000, \$26 450, $r = 1{\cdot}15$
(c) For Lawrence $T_5 =$ \$35 000 and $T_6 =$ \$37 500. For Julian $T_5 \doteqdot$ \$34 980.12 and $T_6 \doteqdot$ \$40 227.14. The difference in T_6 is about \$2727.
10(a)(i) $T_n = 47\,000 + 3000n$ **(ii)** the 18th year
(b) \$71 166
11(a) 12 metres, 22 metres, 32 metres **(b)** $10n + 2$
(c)(i) 6 **(ii)** 222 metres
12(a) 18 times **(b)** 1089 **(c)** Monday
13 27 000 litres
14(a) 85 000 **(b)** 40 000
15(a) $D = 3200$ **(b)** $D = 3800$ **(c)** the 15th year
(d) $S_{13} =$ \$546 000, $S_{14} = 602\,000$
16(a) 40 m, 20 m, 10 m, $a = 40,\ r = \frac{1}{2}$ **(b)** 80 m
(c) The GP has ratio $r = 2$ and hence does not converge. Thus Stewart would never stop running.
17(a) $\left(\frac{1}{2}\right)^{\frac{1}{4}}$ **(b)** $S_\infty = \dfrac{F}{1 - \left(\frac{1}{2}\right)^{\frac{1}{4}}} \doteqdot 6{\cdot}29F$
18(a)(i) $r = \cos^2 x$ **(ii)** $x = 0, \pi, 2\pi$ **(b)(i)** $r = \sin^2 x$
(ii) $x = \frac{\pi}{2}, \frac{3\pi}{2}$
19(b) at $x = 16$ **(c)(i)** at $x = 18$, halfway between the original positions **(ii)** 36 metres, the original distance between the bulldozers

Exercise **6B** (Page 237)

1(a) 5 **(b)** 14 **(c)** 23 **(d)** 3 **(e)** 9 **(f)** 15 **(g)** 4 **(h)** 8 **(i)** 14 **(j)** 2 **(k)** 5 **(l)** 11

2(a) $\frac{T_3}{T_2} = \frac{T_2}{T_1} = 1{\cdot}1$ **(b)** $a = 10,\ r = 1{\cdot}1$

(c) $T_{15} = 10 \times 1{\cdot}1^{14} \doteqdot 37{\cdot}97$ **(d)** 19

3(a) $r = 1{\cdot}05$ **(b)** \$62 053, \$503 116

(c) the 13th year

4 the 19th year

7(a) SC50: 50%, SC75: 25%, SC90: 10% **(c)** 4

(d) at least 7

8(a) $T_n = 3 \times (\frac{2}{3})^{n-1}$ **(b)** 4·5 metres **(c)(ii)** 16

9(a) the 10th year **(b)** the 7th year

10(a) $S_n = \dfrac{3(1-(\frac{2}{3})^n)}{1-\frac{2}{3}} = 9(1-(\frac{2}{3})^n)$ **(b)** The common ratio is less than 1. $S = 9$ **(c)** $n = 17$

Exercise **6C** (Page 241)

1(a)(i) \$900 **(ii)** \$5900 **(b)(i)** \$120 **(ii)** \$420

(c)(i) \$3750 **(ii)** \$13 750 **(d)(i)** \$5166 **(ii)** \$17 166

2(a)(i) \$5955.08 **(ii)** \$955.08 **(b)(i)** \$443.24

(ii) \$143.24 **(c)(i)** \$14 356.29 **(ii)** \$4356.29

(d)(i) \$18 223.06 **(ii)** \$6223.06

3(a)(i) \$4152.92 **(ii)** \$847.08 **(b)(i)** \$199.03

(ii) \$100.97 **(c)(i)** \$6771.87 **(ii)** \$3228.13

(d)(i) \$7695.22 **(ii)** \$4304.78

4(a) \$507.89 **(b)** \$1485.95 **(c)** \$1005.07

(d) \$10 754.61

5(a) \$6050 **(b)** \$25 600 **(c)** 11 **(d)** 5·5%

6(a) $A_n = 10\,000(1 + 0{\cdot}065 \times n)$

(b) $A_{15} = \$19\,750,\ A_{16} = \$20\,400$

7(a) \$101 608.52 **(b)** \$127 391.48

8(a) Howard — his is \$21 350 and hers is \$21 320.

(b) Juno — hers is now \$21 360.67 so is better by \$10.67.

9(a) \$1120 **(b)** \$1123.60 **(c)** \$1125.51

(d) \$1126.83

10(a) \$8000 **(b)** \$12 000 **(c)** \$20 000

11 \$19 990

12(a) \$7678.41 **(b)** \$1678.41 **(c)** 9·32% per annum

13(a) \$12 209.97 **(b)** 4·4% per annum

14 \$1 110 000

15(a) 24 **(b)** 14 **(c)** 7 **(d)** 9

16 $A_n = 6000 \times 1{\cdot}12^n$ **(a)** 7 years **(b)** 10 years

(c) 13 years **(d)** 21 years

17 8 years and 9 months

18 7·0%

19(a) \$5250 **(b)** \$20 250 **(c)** 6·19% per annum

20(a) \$40 988 **(b)** \$42 000

21(b) 3 years

Exercise **6D** (Page 247)

1(a)(i) \$732.05 **(ii)** \$665.50 **(iii)** \$605 **(iv)** \$550

(v) \$2552.55 **(b)(i)** \$550, \$605, \$665.50, \$732.05

(ii) $a = 550,\ r = 1{\cdot}1,\ n = 4$ **(iii)** \$2552.55

2(a)(i) \$1531.54 **(ii)** \$1458.61

(iii) \$1389.15, \$1323, \$1260 **(iv)** \$6962.30

(b)(i) \$1260, \$1323, \$1389.15, \$1458.61, \$1531.54

(ii) $a = 1260,\ r = 1{\cdot}05,\ n = 5$ **(iii)** \$6962.30

3(a)(i) $\$1500 \times 1{\cdot}07^{15}$ **(ii)** $\$1500 \times 1{\cdot}07^{14}$

(iii) $\$1500 \times 1{\cdot}07$ **(iv)** $A_{15} = (1500 \times 1{\cdot}07) + (1500 \times 1{\cdot}07^2) + \cdots + (1500 \times 1{\cdot}07^{15})$ **(b)** \$40 332

4(a)(i) $\$250 \times 1{\cdot}005^{24}$ **(ii)** $\$250 \times 1{\cdot}005^{23}$

(iii) $\$250 \times 1{\cdot}005$ **(iv)** $A_{24} = (250 \times 1{\cdot}005) + (250 \times 1{\cdot}005^2) + \cdots + (250 \times 1{\cdot}005^{24})$ **(b)** \$6390

5(a)(i) $\$3000 \times 1{\cdot}065^{25}$ **(ii)** $\$3000 \times 1{\cdot}065^{24}$

(iii) $\$3000 \times 1{\cdot}065$

(iv) $A_{25} = (3000 \times 1{\cdot}065) + (3000 \times 1{\cdot}065^2) + \cdots + (3000 \times 1{\cdot}065^{25})$ **(c)** \$188 146 and \$75 000

6(b) \$669 174.36 **(c)** \$429 174.36 **(e)** \$17 932.55

7(c)(iii) 18

8(a) \$200 000 **(b)** \$67 275 **(c)** \$630 025

(d)(i) $A_n = 100\,000 \times 1{\cdot}1 \times ((1{\cdot}1)^n - 1)$ **(iii)** 25

(e) $\frac{1\,000\,000}{630\,025} \times 10\,000 \doteqdot \$15\,872$

9(a) \$360 **(b)** \$970.27

10(a) \$31 680 **(b)** \$394 772 **(c)** \$1 398 905

11(a) \$67 168.92 **(b)** \$154 640.32

12 \$3086

13(a) \$286 593 **(b)(i)** \$107 355 **(ii)** \$152 165

14(a) \$27 943.29 **(b)** the 19th year

15(a) 18

16 The function `FV` calculates the value just after the last premium has been paid, not at the end of that year.

17(c) $A_2 = 1{\cdot}01\,M + 1{\cdot}01^2\,M$,
$A_3 = 1{\cdot}01\,M + 1{\cdot}01^2\,M + 1{\cdot}01^3\,M$,
$A_n = 1{\cdot}01\,M + 1{\cdot}01^2\,M + \cdots + 1{\cdot}01^n\,M$

(e) \$4350.76 **(f)** \$363.70

18(b) $A_2 = 1{\cdot}002 \times 100 + 1{\cdot}002^2 \times 100$,
$A_3 = 1{\cdot}002 \times 100 + 1{\cdot}002^2 \times 100 + 1{\cdot}002^3 \times 100$,
$A_n = 1{\cdot}002 \times 100 + 1{\cdot}002^2 \times 100 + \cdots + 1{\cdot}002^n \times 100$

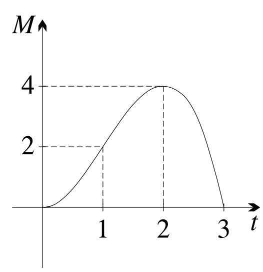
(d) about 549 weeks

Exercise **6E** (Page 255)

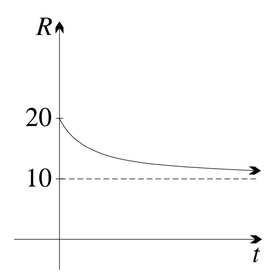
1(b)(i) \$210.36 **(ii)** \$191.24 **(iii)** \$173.86 **(iv)** \$158.05 **(v)** \$733.51
(c)(i) \$158.05, \$173.86, \$191.24, \$210.36
(ii) $a = 158.05$, $r = 1{\cdot}1$, $n = 4$ **(iii)** \$733.51
2(b)(i) \$1572.21 **(ii)** \$1497.34 **(iii)** \$1426.04, \$1358.13, \$1293.46 **(iv)** \$7147.18 **(c)(i)** \$1293.46, \$1358.13, \$1426.04, \$1497.34, \$1572.21
(ii) $a = 1293.46$, $r = 1{\cdot}05$, $n = 5$ **(iii)** \$7147.18
3(a)(ii) $\$1646.92 \times 1{\cdot}07^{14}$ **(iii)** $\$1646.92 \times 1{\cdot}07^{13}$
(iv) $\$1646.92 \times 1{\cdot}07$ **(v)** \$1646.92
(vi) $A_{15} = 15\,000 \times (1{\cdot}07)^{15} - \Big(1646{\cdot}92 + 1646{\cdot}92 \times 1{\cdot}07 + \cdots + 1646{\cdot}92 \times (1{\cdot}07)^{13} + 1646{\cdot}92 \times (1{\cdot}07)^{14}\Big)$
(c) \$0
4(a)(i) $100\,000 \times 1{\cdot}005^{240}$ **(ii)** $M \times 1{\cdot}005^{239}$
(iii) $M \times 1{\cdot}005^{238}$ and M
(iv) $A_{240} = 100\,000 \times 1{\cdot}005^{240} - (M + 1{\cdot}005M + 1{\cdot}005^2 M + \cdots + 1{\cdot}005^{239} M)$
(c) The loan is repaid. **(d)** \$716.43
(e) \$171 943.20
5(a)(i) $10\,000 \times 1{\cdot}015^{60}$ **(ii)** $M \times 1{\cdot}015^{59}$
(iii) $M \times 1{\cdot}015^{58}$ and M **(iv)** $A_{60} = 10\,000 \times 1{\cdot}015^n - (M + 1{\cdot}015M + 1{\cdot}015^2 M + \cdots + 1{\cdot}015^{59} M)$
(c) \$254
6(a) $A_{180} = 165\,000 \times 1{\cdot}0075^{180} - (1700 + 1700 \times 1{\cdot}0075 + 1700 \times 1{\cdot}0075^2 + \cdots + 1700 \times 1{\cdot}0075^{179})$
(c) $-\$10\,012.67$
7(a) $A_n = 250\,000 \times 1{\cdot}006^n - (2000 + 2000 \times 1{\cdot}006 + 2000 \times 1{\cdot}006^2 + \cdots + 2000 \times 1{\cdot}006^{n-1})$
(c) \$162 498, which is more than half.
(d) $-\$16\,881$ **(f)** 8 months
8(c) It will take 57 months, but the final payment will only be \$5466.50 .
9(a) The loan is repaid in 25 years. **(c)** \$1226.64
(d) \$367 993 **(e)** \$187 993 and 4·2% pa
10(b) \$345
11(a) \$4202 **(b)** $A_{10} = \$6.65$ **(c)** Each instalment is approximately 48 cents short because of rounding.
12(b) \$216 511
13(a) \$2915.90 **(b)** \$84.10
14(a) \$160 131.55 **(b)** $\$1633.21 < \$1650$, so the couple can afford the loan.
15(b) zero balance after 20 years **(c)** \$2054.25
16 \$44 131.77
17(b) 57
18(c) $A_2 = 1{\cdot}005^2\,P - M - 1{\cdot}005\,M$,
$A_3 = 1{\cdot}005^3\,P - M - 1{\cdot}005\,M - 1{\cdot}005^2\,M$,
$A_n = 1{\cdot}005^n\,P - M - 1{\cdot}005\,M - \cdots - 1{\cdot}005^{n-1}\,M$
(e) \$1074.65 **(f)** \$34 489.78
19(b) $A_2 = 1{\cdot}008^2\,P - M - 1{\cdot}008\,M$,
$A_3 = 1{\cdot}008^3\,P - M - 1{\cdot}008\,M - 1{\cdot}008^2\,M$,
$A_n = 1{\cdot}008^n\,P - M - 1{\cdot}008\,M - \cdots - 1{\cdot}008^{n-1}\,M$
(d) \$136 262
(e) $n = \log_{1{\cdot}008} \dfrac{125M}{125M - P}$, 202 months

Exercise **6F** (Page 265)

1(a) $y = 3t - 1$ **(b)** $y = 2 + t - t^2$ **(c)** $y = \sin t + 1$
(d) $y = e^t - 1$
2(a) 180 ml **(b)** When $t = 0$, $V = 0$. **(c)** 300 ml
(d) 60 ml/s
3(b) 15 min
4(a) 80 000 litres **(b)** 35 000 litres **(c)** 20 min
(d) 2000 litres/min
5(a) 25 minutes **(c)** 3145 litres
6(a)(i) $3\,\text{cm}^3/\text{min}$ **(ii)** $13\,\text{cm}^3/\text{min}$ **(b)** $E = \frac{1}{2}t^2 + 3t$
(c)(i) $80\,\text{cm}^3$ **(ii)** $180\,\text{cm}^3$
7(a) \$2 **(b)** \$5·60 **(c)** \$2·40 per annum
(e) At the start of 1980.
8(b) $t = 4$ **(c)** 57 **(d)** $t = 2$
9(a)



(b) $t = 2$ **(c)** $t = 1$
10(a)(i) 12 kg/min
(ii) $10\frac{2}{3}$ kg/min
(b) 10 kg/min
(c)



(d) This requires integrating the rate.
11(a) $P = 6{\cdot}8 - 2\log(t + 1)$
(b) approximately 29 days
12(b) $k = \frac{5}{24}$
13(a) 0 **(b)** 250 m/s
(c) $x = 1450 - 250(5e^{-0{\cdot}2t} + t)$
14 The scheme appears to have worked initially and the level of pollution decreased, but the rate at which the pollution decreased gradually slowed down and was almost zero in 2000. A new scheme

would have been required to remove the remaining pollution.

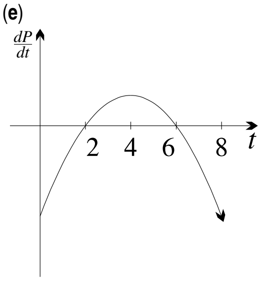
15(a) It is at a maximum on 1st July and at a minimum on 1st March.

(b) It is increasing from 1st March to 1st July. It is decreasing from 1st January to 1st March and after 1st July.

(c) on 1st May

(d) from 1st March to 1st May

(e)



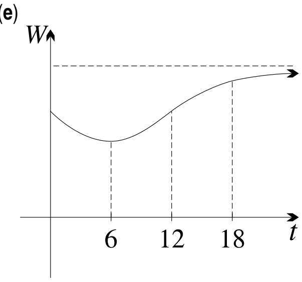
16(a) It was decreasing for the first 6 months and increasing thereafter.

(b) after 6 months

(c) after 12 months

(d) It appears to have stabilised, increasing towards a limiting value.

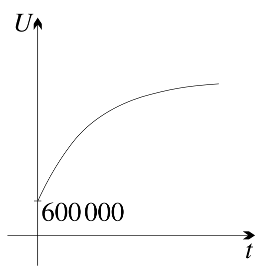
(e)



17(a) Unemployment was increasing.

(b) The rate of increase was decreasing.

(c)



18(a) $-2\,\text{m}^3/\text{s}$ **(b)** $20\,\text{s}$ **(c)** $V = 520 - 2t + \frac{1}{20}t^2$ **(d)** $20\,\text{m}^3$ **(e)** 2 minutes and 20 seconds

19(a) $A = 9 \times 10^5$ **(b)** $N(1) = 380\,087$ **(c)** When t is large, N is close to $4{\cdot}5 \times 10^5$. **(d)** $N' = \frac{9\times10^5 e^{-t}}{(2+e^{-t})^2}$

20(a) $V = \frac{1}{5}t^2 - 20t + 500$ **(c)** $t = 50 - 25\sqrt{2} \doteqdot 15$ seconds. Discard the other answer $t = 50 + 25\sqrt{2}$ because after 50 seconds the bottle is empty.

21(a) $I = 18\,000 - 5t + \frac{48}{\pi} \sin \frac{\pi}{12} t$

(b) $\dfrac{dI}{dt}$ has a maximum of -1, so it is always negative. **(c)** There will be 3600 tonnes left.

Exercise **6G** (Page 273)

1(a) 7·39 **(b)** 1·49 **(c)** 33·78

2(b)(i) 2·68 **(ii)** 11·86

3(a) 4034 **(b)** 2·3 **(d)** 113·4 **(e)** 603

4(a) 0·2695 **(b)** 2·77 **(d)** $-12{\cdot}5$ **(e)** $-2{\cdot}7$

5(a) 20 **(b)** 66 **(c)** 24th **(d)** 5 rabbits per month

6(a) 100 kg **(b)** 67 kg **(c)** 45 kg **(d)** 75

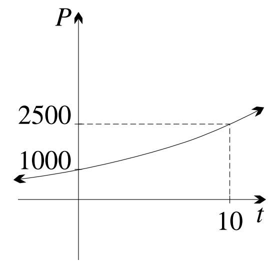
(e) 0·8 kg/year **(f)** 2 kg/year

7(a) $k = \frac{1}{10} \log \frac{5}{2}$ $\doteqdot 0{\cdot}092$

(c) 8230

(d) during 2000

(e) $\dfrac{dP}{dt} = kP$ $\doteqdot 916$



8(b) 1350 **(c)** 135 per hour **(d)** 23 hours

9(c) 6·30 grams, 1·46 grams per minute

(d) 6 minutes 58 seconds

(e) 20 g, $20\,e^{-k} \doteqdot 15{\cdot}87$ g, $20\,e^{-2k} \doteqdot 12{\cdot}60$ g, $20\,e^{-3k} = 10$ g, $r = e^{-k} = 2^{-\frac{1}{3}} \doteqdot 0{\cdot}7937$

10(b) $-\frac{1}{5} \log \frac{7}{10}$ **(c)** 10 290 **(d)** At $t \doteqdot 8{\cdot}8$, that is, some time in the fourth year from now.

11(b) 25 **(c)** $k = \frac{1}{2} \log \frac{5}{3}$ (or $-\frac{1}{2} \log \frac{3}{5}$)

(d) 6 hours 18 minutes

12(b) $h_0 = 100$ **(c)** $k = -\frac{1}{5} \log \frac{2}{5} \doteqdot 0{\cdot}18$ **(d)** 6·4° C

13(b) $k = \frac{\log 2}{5750} \doteqdot 1{\cdot}21 \times 10^{-4}$

(c) $t = \frac{1}{k} \log \frac{100}{15} \doteqdot 16\,000$ years, correct to the nearest 1000 years.

14(b) 8 more years

15(b) 30 **(c)(i)** 26 **(ii)** $\frac{1}{5} \log \frac{15}{13}$ (or $-\frac{1}{5} \log \frac{13}{15}$)

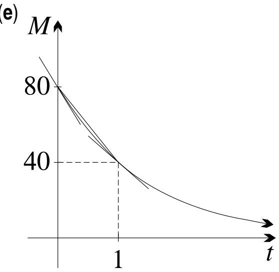
16(a) 80 g, 40 g, 20 g, 10 g

(b) 40 g, 20 g, 10 g. During each hour, the average mass loss is 50%.

(c) $M_0 = 80$, $k = \log 2 \doteqdot 0{\cdot}693$

(d) 55·45 g/hr, 27·73 g/hr, 13·86 g/hr, 6·93 g/hr

(e)



17(a) $C = C_0 \times 1{\cdot}01^t$ **(i)** $1{\cdot}01^{12} - 1 \doteqdot 12{\cdot}68\%$

(ii) $\log_{1\cdot01} 2 \doteqdot 69{\cdot}66$ months **(b)** $k = \log 1{\cdot}01$

(i) $e^{12k} - 1 \doteqdot 12{\cdot}68\%$ **(ii)** $\frac{1}{k} \log 2 \doteqdot 69{\cdot}66$ months

18(a) 72% **(b)** 37% **(c)** 7%

19(a) $k = \frac{\log 2}{1690} \doteqdot 4{\cdot}10 \times 10^{-4}$

(b) $t = \frac{\log 5}{k} \doteqdot 3924$ years

20(b) $\mu_1 = 1{\cdot}21 \times 10^{-4}$ **(c)** $\mu_2 = 1.16 \times 10^{-4}$

(d) The values of μ differ so the data are inconsistent. **(e)(i)** 625·5 millibars

(ii) 1143·1 millibars **(iii)** 19 205 metres

21(a) 34 minutes **(b)** 2·5%

22(b) $C_0 = 20\,000$, $k = \frac{1}{5} \log \frac{9}{8} \doteqdot 0{\cdot}024$

(c) 64 946 ppm **(d)(i)** 330 metres from the cylinder

(ii) If it had been rounded down, then the concentration would be above the safe level.

23(a) $p = \frac{1}{10}\log\frac{5}{4} \doteqdot 0{\cdot}022$, $q = \frac{1}{10}\log\frac{6}{5} \doteqdot 0{\cdot}018$ **(b)** $t = \frac{\log 2}{p+q} \doteqdot 17{\cdot}10$ years, that is during 1997.

Review Exercise 6H (Page 278)

1(a) $a = 31$, $d = 13$ **(b)** 16 **(c)** 2056
2(a) $\frac{1}{2}$ **(b)** $|r| = \frac{1}{2} < 1$ **(c)** $S_\infty = 48$
3(a) $r = 1{\cdot}04$ **(b)** \$49 816, \$420 214
4(a) $T_n = 43\,000 + 4000n$ **(b)** 2017
5 2029
6(a) $A_n = 15\,000 + 705n$
7 \$15 090
8(a) \$15 593.19 **(b)** \$3593.19 **(c)** 5·99%
9(b) \$224 617.94 **(c)** \$104 617.94 **(d)** The value is \$277 419.10, with contributions of \$136 000.00.
11(a) $A_{180} = 159\,000 \times 1{\cdot}005625^{180} - (1410 + 1410 \times 1{\cdot}005625 + 1410 \times 1{\cdot}005625^2 + \cdots + 1410 \times 1{\cdot}005625^{179})$ **(c)** $-\$928.62$ **(d)** \$1407.01
12(a) $A_n = 1\,700\,000 \times 1{\cdot}00375^n - (18\,000 + 18\,000 \times 1{\cdot}00375 + 18\,000 \times 1{\cdot}00375^2 + \cdots + 18\,000 \times 1{\cdot}00375^{n-1})$
(c) \$919 433, which is more than half.
(d) $-\$57\,677.61$ **(f)** 3 months
13(a) $\dfrac{dQ}{dt} = 8\,e^{\frac{1}{5}t}$
(b)(i) $40\,e^{1\cdot 4} \doteqdot 162{\cdot}2$ **(ii)** $8\,e^2 \doteqdot 59{\cdot}11$
(iii) $5\log_e 10 \doteqdot 11{\cdot}51$ **(iv)** $5\log_e 2{\cdot}5 \doteqdot 4{\cdot}581$
14(a) $\dfrac{dM}{dt} = -500k\,e^{-kt}$ **(b)(i)** $500\,e^{-3} \doteqdot 24{\cdot}89$
(ii) $\frac{1}{12}\log_e 50 \doteqdot 0{\cdot}3260$ **(iii)** $\frac{10}{3}\log_e 5 \doteqdot 5{\cdot}365$
(iv) $-50\,e^{-0\cdot 5} \doteqdot -30{\cdot}33$ **(v)** $\frac{1}{2}\log_e 5 \doteqdot 0{\cdot}8047$
15(b) 3664 **(c)** 183 per hour **(d)** 14 hours
16(b) $-\frac{1}{5}\log_e\frac{5}{8} \doteqdot 0{\cdot}094$ **(c)** 3125
(d) At $t \doteqdot 11{\cdot}555$, that is, some time in the seventh year from now.
17(a) after 25 minutes **(b)(ii)** 3135 litres
18(a) $A = 7000$ **(b)** $N(1) = 1602$
(d) 134 seals per year
(e) When t is large, N is close to 1750.

Chapter Seven

Exercise 7A (Page 287)

1(a) 70° **(b)** 45° **(c)** 60° **(d)** 50° **(e)** 22°
(f) $\alpha = 153°$, $\beta = 27°$ **(g)** 34°
(h) $\alpha = 70°$, $\beta = 70°$
2(a) 35° **(b)** 43° **(c)** 60° **(d)** $\alpha = 130°$, $\beta = 50°$
(e) $\alpha = 123°$, $\beta = 123°$ **(f)** 60°
(g) $\alpha = 65°$, $\beta = 65°$ **(h)** $\alpha = 90°$, $\beta = 90°$
3(a) The alternate angles are equal.
(b) The corresponding angles are equal.
(c) The co-interior angles are supplementary.
(d) The co-interior angles are supplementary.
5(a) $\alpha = 52°$, $\beta = 38°$ **(b)** $\alpha = 30°$, $\beta = 60°$
(c) 24° **(d)** 36° **(e)** $\alpha = 15°$, $\beta = 105°$, $\gamma = 60°$, $\delta = 105°$ **(f)** 24° **(g)** 15° **(h)** 22°
6(a) $\alpha = 75°$, $\beta = 105°$ **(b)** $\alpha = 252°$, $\beta = 72°$
(c) 32° **(d)** 62° **(e)** 60° **(f)** 135° **(g)** 48° **(h)** 35°
7(a) The adjacent angles add to 90°.
(c) The adjacent angles add to 180°.
8(a) The corresponding angles are not equal.
(b) The alternate angles are not equal.
(c) The co-interior angles are not supplementary.
(d) The alternate angles are not equal.
9(a) The adjacent angles do not add to 90°.
(c) The adjacent angles do not add to 180°.
10(a) $\theta = 58°$ **(b)** $\theta = 37°$, $\phi = 15°$ **(c)** $\theta = 10°$
(d) $\theta = 12°$, $\phi = 41°$
11(a) $\angle DOB$ and $\angle COE$ are straight angles; $\angle BOC$ and $\angle DOE$ are vertically opposite angles and so are $\angle BOE$ and $\angle COD$.
(b) $GA \parallel BD$ (alternate angles are equal)
(c) $\angle BOE = 90°$
12(a) $\alpha = 60°$ **(b)** $\alpha = 90°$ **(c)** $\alpha = 105°$
14 $\angle EBF = \angle EBD + \angle FBD$
$= \frac{1}{2}(\angle ABD + \angle CBD) = \frac{1}{2} \times 180° = 90°$
16(a) two walls and the ceiling of a room
(b) three pages of a book
(c) the floors of a multi-storey building
(d) the sides of a simple tent and the ground
(e) the floor, ceiling and one wall of a room
(f) a curtain rod in front of a window pane
(g) the corner post of a soccer field

Exercise 7B (Page 295)

1(a) 55° **(b)** 55° **(c)** 52° **(d)** 70° **(e)** 60° **(f)** 30° **(g)** 18° **(h)** 20°

2(a) 108° **(b)** 129° **(c)** 24° **(d)** 74°

3(a) 99° **(b)** 138° **(c)** 65° **(d)** 56° **(e)** 60° **(f)** 80° **(g)** 36° **(h)** 24°

4(a) $\alpha = 59^\circ$, $\beta = 108^\circ$ **(b)** $\alpha = 45^\circ$, $\beta = 60^\circ$ **(c)** $\alpha = 76^\circ$, $\beta = 106^\circ$ **(d)** $\alpha = 110^\circ$, $\beta = 50^\circ$ **(e)** $\alpha = 106^\circ$, $\beta = 40^\circ$ **(f)** $\alpha = 51^\circ$, $\beta = 43^\circ$ **(g)** $\alpha = 104^\circ$, $\beta = 48^\circ$ **(h)** $\alpha = 87^\circ$

5(a) 50° **(b)** $\theta = 35^\circ$ $\phi = 40^\circ$ **(c)** $\theta = 40^\circ$ $\phi = 50^\circ$ **(d)** $\theta = 108^\circ$ $\phi = 144^\circ$

6(a)(i) 108° **(ii)** 72° **(b)(i)** 120° **(ii)** 60° **(c)(i)** 135° **(ii)** 45° **(d)(i)** 140° **(ii)** 40° **(e)(i)** 144° **(ii)** 36° **(f)(i)** 150° **(ii)** 30°

7(a)(i) 8 **(ii)** 10 **(iii)** 45 **(iv)** 180 **(b)(i)** 5 **(ii)** 9 **(iii)** 20 **(iv)** 720

(c) Solving for n does not give an integer value.

(d) Solving for n does not give an integer value.

10(a) 23° **(b)** 17° **(c)** 22° **(d)** 31° **(e)** 44° **(f)** 38° **(g)** 60° **(h)** 45°

11(a) 140° **(b)** 68° **(c)** 47° **(d)** 50°

12(a) $2\alpha + 2\beta = 180^\circ$ (angle sum of $\triangle ABC$)

(b) $\alpha^2 + \beta^2 - 2\alpha\beta = 0$ (exterior angle of $\triangle ABC$), so $(\alpha-\beta)^2 = 0$ and thus $\alpha = \beta$. **(c)** $2\alpha+\beta = 180^\circ$ and $\alpha + 3\beta = 180^\circ$ (co-interior angles, $AD \parallel CB$)

(d) $2\alpha + 2\beta = 360^\circ$ (angle sum of quadrilateral $ABCD$), so $\alpha+\beta = 180$, so $AB \parallel CD$ (co-interior angles are supplementary)

13(a) 5 **(b)** 9 **(c)** 20

15(b)(i) No, because $n = 3\frac{1}{3}$, which is not an integer.

(ii) Yes, $n = 9$ and the polygon is a nonagon.

16(a) angle sum of $\triangle ABC$,
exterior angle of $\triangle DBC$

17(b) 108°

Exercise 7C (Page 303)

1(a) $\triangle ABC \equiv \triangle RQP$ (AAS)

(b) $\triangle ABC \equiv \triangle CDA$ (SSS)

(c) $\triangle ABC \equiv \triangle CDE$ (RHS)

(d) $\triangle PQR \equiv \triangle GEF$ (SAS)

2(a) $\triangle ABC \equiv \triangle DEF$ (AAS), $x = 4$

(b) $\triangle GHI \equiv \triangle LKJ$ (RHS), $x = 20$

(c) $\triangle QRS \equiv \triangle UTV$ (SAS), $x = \sqrt{61}$

(d) $\triangle MLN \equiv \triangle MPN$ (AAS), $x = 12$

3(a) $\triangle ABC \equiv \triangle FDE$ (SSS), $\theta = 67^\circ$

(b) $\triangle XYZ \equiv \triangle XVW$ (SAS), $\theta = 86^\circ$

(c) $\triangle ABC \equiv \triangle BAD$ (SSS), $\theta = 49^\circ$

(d) $\triangle PQR \equiv \triangle HIG$ (RHS), $\theta = 71^\circ$

4(a) $\theta = 64^\circ$ **(b)** $\theta = 69^\circ$ **(c)** $\theta = 36^\circ$ **(d)** $\theta = 84^\circ$ **(e)** $\theta = 64^\circ$ **(f)** $\theta = 90^\circ$ **(g)** $\theta = 45^\circ$ **(h)** $\theta = 120^\circ$

5(a) The diagram does not show a pair of equal sides. The correct reason is AAS.

(b) The diagram does not show a pair of equal hypotenuses. The correct reason is SAS.

6(a) $\triangle AXB \equiv \triangle CXD$ (SAS)

(b) $\triangle ABD \equiv \triangle CBD$ (SSS)

(c) $\triangle ABC \equiv \triangle ADC$ (RHS)

(d) $\triangle ABF \equiv \triangle DEC$ (AAS)

7 In both cases, two sides are given but not the included angle.

8(a) Use the SAS test.

9(a) Use the RHS test.

10(a) Use the SAS test.

11(a) Use the RHS test.

12(a) Use the SSS test.

13(a) The three sides are radii and so are all equal.

(b) All angles of an equilateral triangle are 60°.

14(a) Use the SSS test.

15(b) SAS

16(a) corresponding angles, $DE \parallel BC$ **(b)** SAS

(c) $\angle CBE = \angle BED$ (alternate angles, $DE \parallel BC$) $\angle BED = \angle AED$ (matching angles), $\angle AED = \angle BCE$ (corresponding angles, $DE \parallel BC$)

17(a) SSS **(b)** The base angles are equal.

(c) $CX = AC - AX = BD - BX = DX$

(d) The alternate angles are equal.

18(b) $EB = ED$ (opposite angles in $\triangle EBD$ are equal)

19(a) SAS **(b)** $\angle ADB = \angle ABD$ (matching angles of congruent triangles), $AB = AD$ (opposite angles in $\triangle ABD$ are equal)

20(a) RHS

(b) The base angles $\angle CAB$ and $\angle CBA$ are equal.

21(a) 66° **(b)** 24°

22(a) Use exterior angles of $\triangle ABX$ and $\triangle AYC$.

(b) The base angles are equal.

23(a) Two equal radii form two sides of each triangle. **(b)** SSS **(c)** AAS or SAS **(d)** matching sides and matching angles, $\triangle AMO \equiv \triangle BMO$

Exercise 7D (Page 309)

1(a) $\alpha = 115°$, $\beta = 72°$ **(b)** $\alpha = 128°$, $\beta = 52°$ **(c)** $\alpha = 90°$, $\beta = 102°$ **(d)** $\alpha = 47°$, $\beta = 133°$
2(a) $\alpha = 27°$, $\beta = 99°$ **(b)** $\alpha = 41°$, $\beta = 57°$ **(c)** $\alpha = 40°$, $\beta = 100°$ **(d)** $\alpha = 30°$, $\beta = 150°$
3 Test for a parallelogram: Both pairs of opposite sides are equal and parallel.
4 Test for a parallelogram: The diagonals bisect each other.
5(a) The co-interior angles are supplementary.
(b)(i) AAS **(ii)** matching sides of congruent triangles **(c)(i)** AAS **(ii)** matching sides of congruent triangles
6(a)(i) angle sum of a quadrilateral
(ii) The co-interior angles are supplementary.
(b)(i) SSS **(ii)** matching angles of congruent triangles **(iii)** The alternate angles are equal.
(c)(i) SAS **(ii)** matching angles of congruent triangles **(iii)** The alternate angles are equal.
(d)(i) SAS **(ii)** matching sides and angles of congruent triangles
7 No. It could be a trapezium with a pair of equal but non-parallel sides.
8(a) Properties of a parallelogram: Opposite angles are equal. **(b)** Properties of a parallelogram: Opposite sides are equal. **(c)** SAS
(d) A quadrilateral with equal opposite sides is a parallelogram.
9(a) SAS **(b)** SAS **(c)** Test for a parallelogram: Opposite sides are equal. Alternatively, use the equality of alternate angles to prove that the opposite sides are parallel.
10 $\angle AED = \theta$ (base angles of isosceles $\triangle ADE$), $\angle CDE = \theta$ (alternate angles, $AB \parallel DC$)
11(a) SAS **(b)** matching angles, $\triangle BAD \equiv \triangle ABC$
(d) The co-interior angles are supplementary.

Exercise 7E (Page 314)

1(a) $45°$ **(b)** $76°$ **(c)** $15°$ **(d)** $9°$
2(a) $\alpha = 15°$, $\phi = 105°$
3 The diagonals bisect each other at right angles.
4 The diagonals are equal and bisect each other.
5 By parts (a) and (b), $ABCD$ is both a rectangle and a rhombus.
6(a) Test for a rhombus: All sides are equal.
(b) Property of a rhombus: The diagonals bisect the vertex angles.
7(a) base angles of isosceles triangle ABD
(b) alternate angles, $AB \parallel DC$
8(a)(i) Test for a parallelogram: The diagonals bisect each other. **(ii)** SAS **(b)(i)** half the angle sum of a quadrilateral **(iii)** Test for a parallelogram: The opposite angles are equal. **(iv)** The base angles of $\triangle ABD$ are equal.
9(b) SAS
10(b)(i) Test for a parallelogram: The diagonals bisect each other.
(ii) base angles of isosceles triangle ABM
(iii) base angles of isosceles triangle BCM
11(a) Test for a rhombus: All sides are equal.
(b) Properties of a rhombus: The diagonals bisect each other at right angles.
12(a)(i) SAS **(b)(i)** SAS
13(a)(i) SAS **(ii)** Test for a rhombus: All sides are equal. **(b)(i)** The opposite sides are parallel by construction. **(ii)** Test for a rhombus: The diagonals bisect the vertex angles.
14(a) SAS
15(a) Property of a rhombus: The diagonals bisect the vertex angles. **(b)** SAS
(c) alternate angles, $AD \parallel BC$ **(d)** $90°$
16(a) alternate angles, $BC \parallel AR$ **(b)** The diagonals of a rectangle are equal and bisect each other.
17(a)(i) SSS **(ii)** SAS **(b)(i)** SAS

Exercise 7F (Page 319)

1(a) 33 **(b)** 50 **(c)** 28 **(d)** 72
2(a) $A = 36$, $P = 24$ **(b)** $A = 18$, $P = 12\sqrt{2}$ **(c)** $A = 60$, $P = 32$ **(d)** $A = 48$, $P = 28$
3(a) $A = 54$, $P = 18 + 4\sqrt{13}$, diagonals: $\sqrt{205}$ and $\sqrt{61}$ **(b)** $A = 264$, $P = 72$, diagonals: 30 and $4\sqrt{37}$ **(c)** $A = 120$, $P = 52$, diagonals: 24 and 10 **(d)** $A = 600$, $P = 100$, diagonals: 40 and 30
4(a) $24\,\text{m}^2$ **(b)** $20\frac{1}{4}\,\text{m}^2$ **(c)** $12\frac{1}{2}\,\text{cm}^2$ **(d)** $42\,\text{cm}^2$ **(e)** $168\,\text{cm}^2$ **(f)** $6\,\text{km}^2$ **(g)** $60\,\text{cm}^2$ **(h)** $15\,\text{cm}^2$ **(i)** $12\sqrt{3}\,\text{cm}^2$
5(a) $7\frac{1}{2}$ cm **(b)** side: $4\sqrt{2}$ km, diagonal: 8 km **(c)** 5 cm **(d)** $2\frac{1}{2}$ cm **(e)** 6 metres
6(a) A square is a rhombus, so the result follows from the area formula for a rhombus.
(b) Area of square $= (\sqrt{ab}\,)^2 = ab$.
7(a) The altitude from A to BC gives the perpendicular height of both triangles.

8(a) Both triangles have the same base and altitude — the distance between the parallel lines. **(b)** $\triangle BCX = \triangle ABC - \triangle ABX$
$= \triangle ABD - \triangle ABX = \triangle ADX$
9(a) $SP = \sqrt{20}$ metres, area $= 20\,\text{m}^2$ **(b)** $76\,\text{m}^2$
10(b) Any two adjacent triangles have the same height and equal bases. They will all be congruent when the parallelogram is also a rhombus.
11(b) $\frac{1}{2}\,\text{m}^2$ when $x = \frac{1}{2}$ metre.

Exercise 7G (Page 322)

1 a, c, d
2(a) $c = 13$ **(b)** $c = \sqrt{41}$ **(c)** $a = 5\sqrt{7}$
(d) $b = 2\sqrt{10}$
3(a) 5 metres **(b)** $\sqrt{41}$ metres and $4\sqrt{5}$ metres
4(a) 60 minutes **(b)** 56 minutes
5(a) $17\,\text{cm}$ **(b)** $4\sqrt{11}\,\text{cm}$
(c)(i) $10\,\text{cm}$ **(ii)** $5\sqrt{5}\,\text{cm}$
6(a) $a^2 = s^2 - b^2$ **(b)(i)** $108\,\text{cm}^2$ **(ii)** $40\sqrt{14}\,\text{cm}^2$
(c) This is an equilateral triangle with $a = b\sqrt{3}$ and area $= b^2\sqrt{3}$.
7(b) $x = 3$ or 4, so the diagonals are $6\,\text{cm}$ and $8\,\text{cm}$.
8 Here $C = 90^\circ$ and $\cos 90^\circ = 0$, so the third term $-2ab\cos C$ of the cosine rule disappears, giving Pythagoras' theorem.
9(b)(i) c^2 **(ii)** $(b-a)^2$ **(iii)** Each is $\frac{1}{2}ab$.
10(a)(i) 3, 4, 5 **(ii)** 5, 12, 13 **(iii)** 7, 24, 25
(iv) 33, 56, 65 **(b)(ii)** $10\,\text{cm}$ when $t = 3$, and $17\,\text{cm}$ when $t = 4$.
12 9, 12 and 15
13(a) $\angle PRS = 15^\circ$
14(a) $x^2 + y^2 = 25$, $(x+10)^2 + y^2 = 169$
(b) $x = \frac{11}{5}$, $\cos\alpha = \frac{61}{65}$
15 4

Exercise 7H (Page 328)

1(a) $\triangle ABC \,|||\, \triangle QPR$ (AA similarity test)
(b) $\triangle ABC \,|||\, \triangle CAD$ (SSS similarity test)
(c) $\triangle ABD \,|||\, \triangle DBC$ (RHS similarity test)
(d) $\triangle ABC \,|||\, \triangle ACD$ (SAS similarity test)
2(a) $\triangle ABC \,|||\, \triangle DEF$ (AA similarity test), $x = 4\frac{4}{5}$
(b) $\triangle GHI \,|||\, \triangle LKJ$ (RHS similarity test), $x = 9$
(c) $\triangle QRS \,|||\, \triangle UTV$ (SAS similarity test), $x = 61$
(d) $\triangle LMN \,|||\, \triangle LPM$ (AA similarity test), $x = 18$
3(a) $\triangle ABC \,|||\, \triangle FDE$ (SSS similarity test), $\theta = 67^\circ$
(b) $\triangle XYZ \,|||\, \triangle XVW$ (SAS similarity test), $\theta = 86^\circ$, $UW \parallel ZY$ because alternate angles are equal.
(c) $\triangle PQR \,|||\, \triangle PRS$ (SSS similarity test), $\theta = 52^\circ$
(d) $\triangle PQR \,|||\, \triangle HIG$ (RHS similarity test), $\theta = 71^\circ$
4(a) SAS similarity test **(b)** AA similarity test **(c)** RHS similarity test **(d)** AA similarity test
5(a) 64 metres. Use the AA similarity test.
(b)(i) $1\,\text{km}$ **(ii)** $15\,\text{km}$
6(a) $5\,\text{cm}$, $15\,\text{cm}^2$, $15\,\text{cm}^3$ **(b)(i)** $3:1$ **(ii)** $9:1$ **(iii)** $3:1$ **(iv)** $27:1$ **(v)** $9:1$ **(vi)** $9:1$ **(c)** $\sqrt{2}:1$ **(d)** $\sqrt[3]{2}:1$ **(e)** $1\,\text{cm}$
7(a) SSS similarity test. Alternate angles $\angle BAC$ and $\angle ACD$ are equal. **(b)** AA similarity test, $ON = 21$, $PN = 17$ **(c)** SAS similarity test, trapezium with $AB \parallel KL$ (alternate angles $\angle BAL$ and $\angle ALK$ are equal)
(d) AA similarity test, $AB = 16$, $FB = 7$ **(e)** AA similarity test, $FQ = 6$, $GQ = 8$, $PQ = 3\sqrt{5}$, $RQ = 4\sqrt{5}$ **(f)** AA similarity test, $RL = 6$
8(a) Use the AA similarity test.
9(a) Use the SAS similarity test. Then $\angle QPB = \angle CAB$ (matching angles of similar triangles), so $PQ \parallel AC$ (corresponding angles are equal).
10(a) AA similarity test, $AD = 15$, $DC = 20$, $BC = 16$ **(b)** $AM = 12$, $BM = 16$, $DM = 9$
11(a) Use the AA similarity test.
12(a) Yes, the similarity factor is the ratio of their side lengths. **(b)** No, the ratio of side lengths may differ in the two rectangles. **(c)** No, the ratio of diagonals may differ in the two rhombuses. **(d)** No, the ratio of adjacent sides may differ in the two parallelograms. **(e)** No, the ratio of parallel sides may differ in the two trapeziums. **(f)** Yes, the similarity factor is the ratio of their side lengths. **(g)** No, the ratio of leg to base may differ in the two triangles. **(h)** Yes, the similarity factor is the ratio of their radii. **(i)** Yes, the similarity factor is the ratio of their side lengths. **(j)** No, the two hexagons could have quite different shapes.

13(a) AA similarity test
(b) similar triangles in the ratio of $2:1$ **(c)** $20x^2$

Exercise 7I (Page 334)

1(a) $x = 7\frac{1}{2}$ **(b)** $x = 11$ **(c)** $x = 15$, $y = 7$
(d) $x = 5$, $y = 18$, $z = 12$
2(a) $x = 7$ **(b)** $x = 5$ **(c)** $x = 2$ **(d)** $x = 6$
(e) $x = 12\frac{1}{2}$ **(f)** $x = 10\frac{1}{2}$ **(g)** $x = 4$ **(h)** $x = 13$
3(a) $x = 6$, $y = 4\frac{1}{2}$, $z = \frac{2}{3}$
(b) $x = 2\frac{2}{3}$, $y = 4\frac{1}{2}$, $z = 2$
(c) $x = 7\frac{1}{2}$, $y = 15$, $z = 3\frac{1}{2}$
(d) $x = 7$, $y = 6$
4(a) $x = 12$ **(b)** $x = 2$ **(c)** $x = 1$, $y = 2\frac{1}{4}$, $z = 7\frac{1}{2}$
(d) $x = 1$, $y = 1\frac{2}{3}$
5(a) $x = 2$ **(b)** $x = 4$ **(c)** $x = 5$ **(d)** $x = 1 + \sqrt{22}$
6(a) $x = 12$, $y = 6\frac{2}{5}$, $z = 9\frac{3}{5}$ **(b)** $1:2$
(c) $ARPQ$ is a parallelogram, because the opposite sides are parallel by intercepts. $1:2$
7(a) SAS similarity test
(b) $PQ \parallel BC$ (corresponding angles are equal)
8(a) AA similarity test
9(a) A line parallel to the base divides the other two sides in the same ratio. Since $AB = AC$, it follows that $DB = EC$. **(b)** SAS congruence test
10(b) 6 metres above the ground **(c)** The height is unchanged when the distance apart changes.
11(a) The base angles are equal.

Review Exercise 7J (Page 336)

1(a) $\alpha = 70°$, $\beta = 20°$ **(b)** $\alpha = 297°$, $\beta = 63°$
(c) $\alpha = 72°$, $\beta = 72°$ **(d)** $\alpha = 130°$, $\beta = 50°$
2(a) $\alpha = 65°$ **(b)** $\alpha = 75°$ **(c)** $\alpha = 120°$
3(a) $\theta = 42°$ **(b)** $\theta = 70°$ **(c)** $\theta = 42°$, $\phi = 48°$
(d) $\theta = 140°$, $\phi = 120°$
4 See Box 6.
5(a) $\triangle ABD \equiv \triangle CDB$ (AAS)
(b) $\triangle CBA \equiv \triangle CDA$ (SSS)
(c) $\triangle ABX \equiv \triangle CDX$ (SAS)
6(a) Use the SSS test.
7(a) Use the RHS test.
8(a) $8\alpha + 4° = 180°$, $\alpha = 22°$, $\beta = 92°$
(b) $8\alpha = 3\alpha + 75°$, $\alpha = 15°$, $\beta = 60°$
(c) $7\alpha + 5° = 180°$, $\alpha = 25°$, $\beta = 130°$
(d) $10\alpha = 180°$, $\alpha = 18°$, $\beta = 162°$
9(a) AAS
10(a) SAS **(b)** $\angle DAC = \angle BCA$ from the congruence. These are alternate angles, so $AD \parallel BC$.
11(a) base angles of isosceles $\triangle ABD$
(b) alternate angles, $AB \parallel DC$
(c) base angles of isosceles $\triangle DAC$
(e) angle sum of $\triangle ADM$
12(c) $\alpha = 18°$, $\theta = 72°$
13(a) Test for a parallelogram: The diagonals bisect each other.
(b) base angles of isosceles triangle ABM
(c) base angles of isosceles triangle BCM
14(a) $A = 32$, $P = 26$ **(b)** $A = 72$, $P = 38$
(c) $A = 336$, $P = 100$ **(d)** $A = 252$, $P = 86$
15(a) 5 cm **(b)** side: $1\frac{1}{2}$ km, diagonal: $\frac{3\sqrt{2}}{2}$ km
(c) 7 cm **(d)** $7\frac{1}{2}$ cm **(e)** 8 metres
16(a) $AC = 5$, $AD = 12$, $AE = 15$, $AF = 17$
(b) SAS or SSS similarity test
17(a) 13 cm **(b)** $4\sqrt{7}$ cm
(c)(i) 48 cm **(ii)** 25 cm
18(c) The co-interior angles $\angle ACD$ and $\angle CDE$ are supplementary, hence $AC \parallel ED$.
19(a) SSS similarity test. The alternate angles $\angle BAC$ and $\angle ACD$ are equal.
(b) AA similarity test, $OM = 12$, $QM = 7$
(c) SAS similarity test, trapezium with $AB \parallel KL$ (alternate angles $\angle BAL$ and $\angle ALK$ are equal)
(d) AA similarity test, $AB = 18$, $FB = 10$
20(a) Use the AA similarity test. **(b)** See Box 37.
21(a) $x = 6$, $y = 4\frac{1}{2}$, $z = 1\frac{1}{3}$
(b) $x = 3$, $y = 2\frac{1}{2}$, $z = 2$
(c) $x = 4$, $y = 3$, $z = 6$
22(a) $x = 2$ **(b)** $x = 6$ **(c)** $x = \frac{1}{3}$ or 2

Chapter Eight

Exercise **8A** (Page 346)

1(a) $\frac{1}{2}$ **(b)** $\frac{1}{2}$ **(c)** 1 **(d)** 0
2(a) $\frac{1}{6}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{3}$
3(a) $\frac{5}{12}$ **(b)** $\frac{7}{12}$ **(c)** 0
4(a) $\frac{4}{9}$ **(b)** $\frac{5}{9}$ **(c)** $\frac{11}{18}$
5(a) $\frac{4}{9}$ **(b)** $\frac{5}{9}$ **(c)** $\frac{11}{18}$ **(d)** $\frac{7}{18}$ **(e)** $\frac{1}{3}$ **(f)** $\frac{1}{6}$
6(a) $\frac{3}{8}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{2}$
7(a) $\frac{1}{26}$ **(b)** $\frac{5}{26}$ **(c)** $\frac{21}{26}$ **(d)** 0 **(e)** $\frac{3}{26}$ **(f)** $\frac{5}{26}$
8 78%
9(a) $\frac{4}{7}$ **(b)** 32
10(a) 8 **(b)** $\frac{14}{15}$
11(a) $\frac{1}{20}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{2}$ **(e)** $\frac{2}{5}$ **(f)** $\frac{1}{5}$ **(g)** $\frac{1}{4}$ **(h)** 0 **(i)** 1
12(a) $\frac{1}{2}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{13}$ **(d)** $\frac{1}{52}$ **(e)** $\frac{1}{4}$ **(f)** $\frac{3}{13}$ **(g)** $\frac{1}{2}$ **(h)** $\frac{1}{13}$ **(i)** $\frac{3}{13}$ (counting an ace as a one)
13(a) $\frac{1}{15}$ **(b)** $\frac{7}{150}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{4}{25}$ **(e)** $\frac{1}{75}$ **(f)** $\frac{17}{50}$
14(a) $\frac{1}{5}$ **(b)** $\frac{3}{40}$ **(c)** $\frac{9}{20}$ **(d)** $\frac{7}{100}$ **(e)** $\frac{7}{50}$ **(f)** $\frac{1}{200}$
15(a) $\frac{3}{4}$ **(b)** $\frac{1}{4}$
16 187 or 188
17(a) The argument is invalid, because on any one day the two outcomes are not equally likely. The argument really can't be corrected.
(b) The argument is invalid. One team may be significantly better than the other, the game may be played in conditions that suit one particular team, and so on. Even when the teams are evenly matched, the high-scoring nature of the game makes a draw an unlikely event. The three outcomes are not equally likely. The argument really can't be corrected.
(c) The argument is invalid, because we would presume that Peter has some knowledge of the subject, and is therefore more likely to choose one answer than another. The argument would be valid if the questions were answered at random.
(d) The argument is only valid if there are equal numbers of red, white and black beads, otherwise the three outcomes are not equally likely.
(e) This argument is valid. He is as likely to pick the actual loser of the semi-final as he is to pick any of the other three players.
18(a) $\frac{2}{9}$ **(b)** $\frac{\pi}{18}$

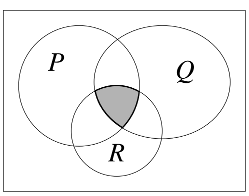
Exercise **8B** (Page 350)

1 AB, AC, AD, BC, BD, CD **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{6}$
2 HH, HT, TH, TT **(a)** $\frac{1}{4}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{4}$
3(a) $\frac{1}{4}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{4}$
4(a) 23, 32, 28, 82, 29, 92, 38, 83, 39, 93, 89, 98 **(b)(i)** $\frac{1}{12}$ **(ii)** $\frac{1}{2}$ **(iii)** $\frac{1}{2}$ **(iv)** $\frac{1}{6}$ **(v)** $\frac{1}{4}$ **(vi)** 0
5(a) The captain is listed first and the vice-captain second: AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, BA, CA, DA, EA, CB, DB, EB, DC, EC, ED **(b)(i)** $\frac{1}{20}$ **(ii)** $\frac{2}{5}$ **(iii)** $\frac{3}{5}$ **(iv)** $\frac{1}{5}$
6 HHH, HHT, HTH, HTT, THH, THT, TTH, TTT **(a)** $\frac{1}{8}$ **(b)** $\frac{3}{8}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{2}$ **(e)** $\frac{1}{2}$ **(f)** $\frac{1}{2}$
7 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66 **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{1}{36}$ **(d)** $\frac{1}{6}$ **(e)** $\frac{1}{6}$ **(f)** $\frac{1}{4}$ **(g)** $\frac{11}{36}$ **(h)** $\frac{4}{9}$ **(i)** $\frac{5}{36}$ **(j)** $\frac{1}{6}$
8(a)(i) $\frac{1}{4}$ **(ii)** $\frac{1}{4}$ **(iii)** $\frac{1}{2}$ **(b)(i)** $\frac{1}{8}$ **(ii)** $\frac{3}{8}$ **(iii)** $\frac{1}{2}$
9(a) $\frac{1}{16}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{11}{16}$ **(d)** $\frac{5}{16}$ **(e)** $\frac{3}{8}$ **(f)** $\frac{5}{16}$
10(a) $\frac{2}{5}$ **(b)** $\frac{3}{5}$ **(c)** $\frac{1}{5}$
11(a) 24 **(b)(i)** $\frac{2}{3}$ **(ii)** $\frac{1}{4}$ **(iii)** $\frac{1}{12}$ **(iv)** $\frac{1}{6}$
12(a) $\dfrac{1}{2^n}$ **(b)** $1 - 2^{1-n}$

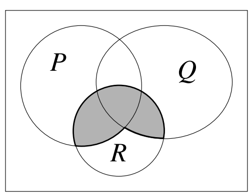
Exercise **8C** (Page 355)

1(a) $A \cup B = \{1, 3, 5, 7\}$, $A \cap B = \{3, 5\}$
(b) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A \cap B = \{4, 9\}$
(c) $A \cup B = \{\text{h, o, b, a, r, t, i, c, e, n}\}$, $A \cap B = \{\text{h, o, b}\}$
(d) $A \cup B = \{\text{j, a, c, k, e, m}\}$, $A \cap B = \{\text{a}\}$
(e) $A \cup B = \{1, 2, 3, 5, 7, 9\}$, $A \cap B = \{3, 5, 7\}$
2(a) false **(b)** true **(c)** false **(d)** false **(e)** true **(f)** false
3(a) 3 **(b)** 2 **(c)** $\{1, 3, 4, 5\}$ **(d)** 4 **(e)** $\{3\}$ **(f)** 1 **(g)** $\{2, 4\}$ **(h)** $\{1, 2, 5\}$
4(a) students who study both Japanese and History **(b)** students who study either Japanese or History or both
5(a) students at Clarence High School who do not have blue eyes **(b)** students at Clarence High School who do not have blonde hair **(c)** students at Clarence High School who have blue eyes or blonde hair or both **(d)** students at Clarence High School who have blue eyes and blonde hair

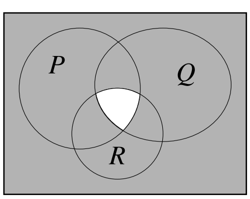
6(a) $\emptyset, \{a\}$ **(b)** $\emptyset, \{a\}, \{b\}, \{a, b\}$ **(c)** $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$
(d) $\emptyset$
7(a) true **(b)** false **(c)** true **(d)** false **(e)** true
8(a) $\{2, 4, 5, 6, 8, 9\}$ **(b)** $\{1, 2, 3, 5, 8, 10\}$
(c) $\{7\}$ **(d)** $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$
(e) $\{1, 3, 4, 6, 7, 9, 10\}$ **(f)** $\{2, 5, 8\}$
9(a) $\{2, 4, 5, 7, 9, 10\}$ **(b)** $\{1, 2, 5, 8, 9\}$
(c) $\{1, 2, 4, 5, 7, 8, 9, 10\}$
(d) $\{2, 5, 9\}$ **(e)** $\{1, 3, 4, 6, 7, 8, 10\}$
(f) $\{2, 5, 9\}$ **(g)** $\{3, 6\}$
(h) $\{1, 2, 4, 5, 7, 8, 9, 10\}$
10(a) true **(b)** true
11(a) Q **(b)** P
12(a) III **(b)** I **(c)** II **(d)** IV
13(a) $|A \cap B|$ is subtracted so that it is not counted twice. **(b)** 5 **(c)** LHS $= 7$, RHS $= 5 + 6 - 4 = 7$
14(a) 10 **(b)** 22 **(c)** 12
15(a)



(b)



(c)



16 4

Exercise **8D** (Page 359)

1(a) $\frac{1}{6}$ **(b)** $\frac{5}{6}$ **(c)** $\frac{1}{3}$ **(d)** 0 **(e)** 1 **(f)** 0 **(g)** $\frac{1}{6}$ **(h)** $\frac{2}{3}$
2(a) $\frac{1}{13}$ **(b)** $\frac{1}{13}$ **(c)** $\frac{2}{13}$ **(d)** 0 **(e)** $\frac{11}{13}$ **(f)** $\frac{1}{2}$ **(g)** $\frac{3}{13}$
(h) $\frac{3}{26}$ **(i)** $\frac{8}{13}$ **(j)** $\frac{5}{13}$
3(a) no **(b)(i)** $\frac{1}{2}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{1}{3}$ **(iv)** $\frac{5}{6}$
4(a) $\frac{1}{2}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{3}{4}$ **(e)** $\frac{1}{4}$ **(f)** $\frac{1}{6}$ **(g)** $\frac{1}{6}$
(h) $\frac{1}{36}$ **(i)** $\frac{11}{36}$ **(j)** $\frac{25}{36}$
5(a)(i) $\frac{1}{2}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{1}{3}$ **(iv)** $\frac{1}{2}$ **(v)** $\frac{1}{2}$
(b)(i) $\frac{3}{5}$ **(ii)** $\frac{4}{5}$ **(iii)** $\frac{3}{5}$ **(iv)** 0 **(v)** 1
(c)(i) $\frac{1}{2}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{2}{3}$ **(iv)** $\frac{1}{3}$ **(v)** $\frac{5}{6}$
6(a) $\frac{7}{15}$ **(b)** 0 **(c)** $\frac{3}{5}$ **(d)** $\frac{5}{7}$
7(a)(i) no **(ii)** $\frac{1}{2}, \frac{1}{4}, \frac{3}{20}, \frac{3}{5}$ **(b)(i)** no **(ii)** $\frac{1}{2}, \frac{3}{10}, \frac{3}{20}, \frac{13}{20}$ **(c)(i)** yes **(ii)** $\frac{1}{4}, \frac{9}{20}, 0, \frac{7}{10}$
8(a) $\frac{9}{25}$ **(b)** $\frac{7}{50}$ **(c)** $\frac{17}{50}$
9(a) 10 **(b)(i)** $\frac{4}{21}$ **(ii)** $\frac{1}{3}$
10 $\frac{1}{4}$
11 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
(a) $\frac{1}{4}$ **(b)** $\frac{6}{25}$ **(c)** $\frac{11}{100}$ **(d)** $\frac{19}{50}$
12(a) $\frac{7}{12}$ **(b)** $\frac{13}{60}$ **(c)** $\frac{3}{10}$ **(d)** $\frac{7}{60}$

Exercise **8E** (Page 365)

1(a) $\frac{1}{24}$ **(b)** $\frac{1}{28}$ **(c)** $\frac{1}{12}$ **(d)** $\frac{1}{96}$ **(e)** $\frac{1}{42}$ **(f)** $\frac{1}{336}$
2(a) $\frac{1}{12}$ **(b)** $\frac{1}{12}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{3}$
3(a) $\frac{1}{25}$ **(b)** $\frac{2}{25}$ **(c)** $\frac{3}{25}$ **(d)** $\frac{3}{25}$ **(e)** $\frac{4}{25}$ **(f)** $\frac{2}{25}$ **(g)** $\frac{1}{25}$
4(a) $\frac{15}{49}$ **(b)** $\frac{8}{49}$ **(c)** $\frac{6}{49}$
5(a) $\frac{1}{10}$ **(b)** $\frac{3}{10}$ **(c)** $\frac{3}{10}$ **(d)** $\frac{3}{10}$
6(a) $\frac{1}{36}$ **(b)** $\frac{1}{12}$ **(c)** $\frac{1}{36}$ **(d)** $\frac{1}{9}$ **(e)** $\frac{1}{6}$
7(a) $\frac{1}{7}$ **(b)** $\frac{180}{1331}$
8(a)(i) $\frac{13}{204}$ **(ii)** $\frac{1}{17}$ **(iii)** $\frac{4}{663}$ **(iv)** $\frac{1}{2652}$
(b) $\frac{1}{16}, \frac{1}{16}, \frac{1}{169}, \frac{1}{2\,704}$
9(a)(i) $\frac{2}{3}$ **(ii)** $\frac{1}{3}$ **(b)(i)** $\frac{8}{27}$ **(ii)** $\frac{1}{27}$ **(iii)** $\frac{4}{27}$
10(a) $\frac{3}{4}$ **(b)** $\frac{31}{32}$ **(c)** $\frac{1023}{1024}$
11(a) The argument is invalid, because the events 'liking classical music' and 'playing a classical instrument' are not independent. One would expect that most of those playing a classical instrument would like classical music, whereas a smaller proportion of those not playing a classical instrument would like classical music. The probability that a student does both cannot be discovered from the given data — one would have to go back and do another survey. **(b)** The argument is invalid, because the events 'being prime' and 'being odd' are not independent — two out of the three odd numbers less than 7 are prime, but only one out of the three such even numbers is prime. The correct argument is that the odd prime numbers amongst the numbers 1, 2, 3, 4, 5 and 6 are 3 and 5, hence the probability that the die shows an odd prime number is $\frac{2}{6} = \frac{1}{3}$. **(c)** The teams in the competition may not be of equal ability, and factors such as home-ground advantage may also affect the outcome of a game, hence assigning a probability of $\frac{1}{2}$ to winning each of the seven games is unjustified. Also, the outcomes of successive games are not independent — the confidence gained after winning a game may improve a team's chances in the next one, a loss may adversely affect their chances, or a team may receive injuries in one game leading to a depleted team

in the next. The argument really can't be corrected. **(d)** This argument is valid. The coin is normal, not biased, and tossed coins do not remember their previous history, so the next toss is completely unaffected by the previous string of heads.

12(a) $\frac{9}{25}$ **(b)** 11

13(a) $\frac{1}{36}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{36}$ **(e)** $\frac{1}{36}$ **(f)** $\frac{1}{18}$ **(g)** $\frac{1}{12}$ **(h)** $\frac{1}{12}$ **(i)** $\frac{1}{6}$

14 HHH, HHM, HMH, MHH, HMM, MHM, MMH, MMM **(a)** $p\,(\text{HHH}) = 0{\cdot}9^3 = 0{\cdot}729$ **(b)** $0{\cdot}001$ **(c)** $p\,(\text{HMM}) = 0{\cdot}9 \times 0{\cdot}1^2 = 0{\cdot}009$ **(d)** $p\,(\text{HMM}) + p\,(\text{MHM}) + p\,(\text{MMH}) = 3 \times 0{\cdot}009 = 0{\cdot}027$ **(e)** $0{\cdot}081$ **(f)** $0{\cdot}243$

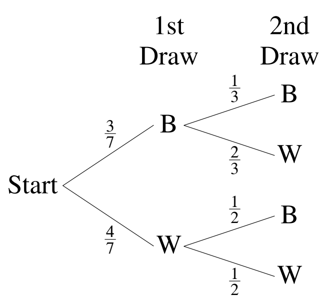
15(a) $\frac{1}{12\,960\,000}$ **(b)** 233

16(a) $\frac{1}{9}$ **(b)** $\frac{1}{9}$. Retell as 'Nick begins by picking out two socks for the last morning and setting them aside'. **(c)** $\frac{1}{9}$. Retell as 'Nick begins by picking out two socks for the third morning and setting them aside'. **(d)** $\frac{1}{63}$ **(e)** $\frac{1}{9\times7\times5\times3}$ **(f)** zero

17 Suppose first that the contestant changes her choice. If her original choice was correct, she loses, otherwise she wins, so her chance of winning is $\frac{2}{3}$. Suppose now that the contestant does not change her choice. If her original choice was correct, she wins, otherwise she loses, so her chance of winning is $\frac{1}{3}$. Thus the strategy of changing will double her chance of winning.

Exercise **8F** (Page 369)

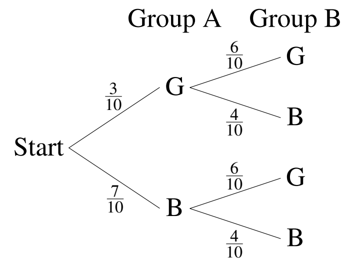
1(a)(i) $\frac{9}{49}$ **(ii)** $\frac{12}{49}$ **(iii)** $\frac{12}{49}$ **(iv)** $\frac{16}{49}$ **(b)(i)** $\frac{25}{49}$ **(ii)** $\frac{24}{49}$ **(c)(i)** $\frac{3}{7}$ **(ii)** $\frac{4}{7}$



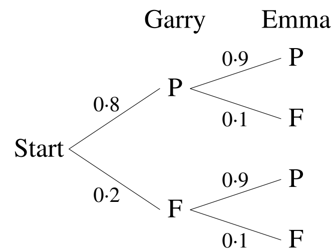
2(a)(i) $90{\cdot}25\%$ **(ii)** $4{\cdot}75\%$ **(iii)** $4{\cdot}75\%$ **(iv)** $0{\cdot}25\%$ **(b)** $99{\cdot}75\%$

3(a)(i) $\frac{6}{25}$ **(ii)** $\frac{9}{25}$ **(iii)** $\frac{4}{25}$ **(iv)** $\frac{6}{25}$ **(b)(i)** $\frac{12}{25}$ **(ii)** $\frac{13}{25}$

4(a)(i) $\frac{9}{50}$ **(ii)** $\frac{3}{25}$ **(iii)** $\frac{21}{50}$ **(iv)** $\frac{7}{25}$ **(b)(i)** $\frac{23}{50}$ **(ii)** $\frac{27}{50}$



5(a) 8% **(b)** 18% **(c)** 26% **(d)** 28%



6(a) $\frac{9}{25}$ **(b)** $\frac{21}{25}$

7 $4{\cdot}96\%$

8(a) $0{\cdot}01$ **(b)** $0{\cdot}23$

9 $0{\cdot}35$

10 $\frac{4}{7}$

11(a) $\frac{21}{3980}$ **(b)** $\frac{144}{995}$

12(a) $\frac{3}{10}$ **(b)** $\frac{7}{24}$ **(c)** $\frac{21}{40}$

13(a) $\frac{1}{11}$ **(b)** $\frac{14}{33}$ **(c)** $\frac{10}{33}$ **(d)** $\frac{19}{33}$

14(a) $\frac{5}{6}$ **(b)** $\frac{5}{12}$ **(c)** $\frac{1}{6}$

15(a) $0{\cdot}28$ **(b)** $0{\cdot}50$

16(a) $\frac{1}{25}$ **(b)** $\frac{3}{5}$

17(a) $\frac{1}{20}$ **(b)** $\frac{57}{8000}$

18 $\frac{4}{11}$

19(a) $31{\cdot}52\%$ **(b)** $80{\cdot}48\%$

20(a)(i) $\frac{5}{33}$ **(ii)** $\frac{5}{22}$ **(iii)** $\frac{19}{33}$ **(iv)** $\frac{1}{4}$ **(v)** $\frac{19}{66}$ **(vi)** $\frac{47}{66}$ **(b)** $\frac{25}{144}$, $\frac{5}{24}$, $\frac{5}{9}$, $\frac{1}{4}$, $\frac{25}{72}$, $\frac{47}{72}$

21(a) $\frac{1}{36}$ **(b)** $\frac{1}{46656}$ **(c)** $\frac{11}{36}$

22(a) $\frac{1}{216}$ **(b)** $\frac{5}{72}$ **(c)** $\frac{5}{12}$ **(d)** $\frac{5}{9}$

23(a) $\frac{1}{6}$ **(b)** $\frac{5}{6}$ **(c)** $\frac{1}{5}$

24(a) $\frac{5}{36}$ **(b)** $\frac{13}{18}$ **(c)** $\frac{5}{36} + \frac{13}{18} \times \frac{5}{36} + \frac{13}{18}^2 \times \frac{5}{36}$ **(d)** $\frac{1}{2}$

25(a) $\frac{2}{5}$ **(b)** $\frac{9}{25}$ **(c)** $\frac{1622}{3125}$ **(d)** $\frac{10}{19}$

26 $\frac{1}{3}$

27(a) $\frac{15}{16}$ **(b)** $\frac{2}{3}$

Review Exercise **8G** (Page 372)

1(a) $\frac{1}{6}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{6}$ **(d)** $\frac{1}{2}$

2(a) $\frac{1}{10}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{3}{10}$ **(d)** 0 **(e)** 1 **(f)** $\frac{3}{10}$

3(a) $\frac{1}{2}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{13}$ **(d)** $\frac{1}{52}$ **(e)** $\frac{1}{2}$ **(f)** $\frac{12}{13}$

4 37%

5(a) $\frac{1}{4}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{2}$

6(a) $\frac{1}{36}$ **(b)** $\frac{1}{9}$ **(c)** $\frac{1}{6}$ **(d)** $\frac{11}{36}$ **(e)** $\frac{4}{9}$ **(f)** $\frac{1}{9}$ **(g)** $\frac{1}{6}$
(h) $\frac{11}{36}$
7(a) $\frac{17}{60}$ **(b)** $\frac{19}{60}$ **(c)** $\frac{1}{6}$
8(a) No **(b)(i)** $\frac{1}{2}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{1}{3}$ **(iv)** $\frac{5}{6}$
9(a) $\frac{1}{12}$ **(b)** $\frac{1}{5}$ **(c)** $\frac{3}{20}$ **(d)** $\frac{1}{20}$
10(a)(i) $\frac{13}{204}$ **(ii)** $\frac{1}{17}$ **(iii)** $\frac{4}{663}$ **(iv)** $\frac{1}{2652}$
(b)(i) $\frac{1}{16}$ **(ii)** $\frac{1}{16}$ **(iii)** $\frac{1}{169}$ **(iv)** $\frac{1}{2704}$
11(a) 14% **(b)** 24% **(c)** 38% **(d)** 6%
12(a) $\frac{2}{21}$ **(b)** $\frac{11}{21}$ **(c)** $\frac{10}{21}$ **(d)** $\frac{2}{7}$
13(a) $\frac{19}{12\,475}$ **(b)** $\frac{979}{12\,475}$
14 $\frac{3}{11}$

Index